On bounded solutions of semilinear fractional order differential inclusions in Hilbert spaces

Mikhail Kamenskii[1], Sergey Kornev[2], Valeri Obukhovskii[3], and Jen-Chih Yao[*4]

[1] Mikhail Kamenskii,
Department of Mathematics,
Voronezh State University,
Voronezh, 394018, Russia
e-mail: mikhailkamenski@mail.ru

[2] Sergey Kornev,
Department of Physics and Mathematics,
Voronezh State Pedagogical University,
Voronezh, 394043, Russia
e-mail: kornev_vrn@rambler.ru

[3] Valeri Obukhovskii,
Department of Physics and Mathematics,
Voronezh State Pedagogical University,
Voronezh, 394043, Russia
e-mail: valerio-ob2000@mail.ru

[*]The corresponding author,
[4] Jen-Chih Yao,
Center for General Education
Kaohsiung Medical University, Taiwan
e-mail: yaojc@kmu.edu.tw

Abstract

We prove a result on a priori estimate for mild solutions to the initial value problem for a semilinear fractional-order differential inclusion in a separable Hilbert space under assumptions that the linear part of the inclusion is presented by a unbounded strictly negatively defined operator and the multivalued nonlinearity satisfies an one-sided estimate. To prove this result, we use approximation methods, based, in particular, on Yosida approximations of the linear part of the inclusion. The obtained result is applied to justify the existence of a mild solution to the initial value problem on each finite interval and to prove the existence of mild solutions bounded on the semi-axis.

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1 Introduction

It is well known (see, for example, [1], [2], [3]) that in the case of differential equations the condition of smoothness of right-hand sides does not provide the existence of solutions to the initial value problem on a unbounded from the right interval. For the existence of such solutions, in the theory of ordinary differential equations, the method of one-sided estimates is widely applied (see, for example [1]). For a differential equation in a Hilbert space H of the form

$$x' = f(t,x)$$

one of the simplest estimations of this type can be presented by the inequality

$$\langle f(t,x), x\rangle \leq a\|x\|^2 + b. \tag{1.1}$$

To obtain the existence of solutions on unbounded intervals in the case of fractional-order equations and inclusions usually the condition of a sublinear growth of the right-hand side is used (see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]). It is evident that this condition is more strong than (1.1).

In the present paper, we prove the existence of mild solutions of such type for a semilinear fractional-order differential inclusion in a separable Hilbert space under assumptions that the linear part of the inclusion is presented by a unbounded strictly negatively defined operator and the multivalued nonlinearity satisfies the estimate of type (1.1). Notice that our construction is essentially based on the following property of the Caputo fractional derivative of a function $x(t)$ in a Hilbert space:

$${}^C D_0^q \|x(t)\|^2 \leq \langle x(t), {}^C D_0^q x(t)\rangle,$$

which was studied in the works [19], [20], [21].

The paper is organized in the following way. In the next section we present some preliminaries from the fractional analysis and theory of condensing multivalued maps. In the third section we give a result on a priori estimate for solutions to the initial value problem for a semilinear fractional differential nclusion in a Hilbert space under assumption that the linear part of the inclusion is negatively defined and the multivalued nonlinearity satisfies an one-sided estimate. To prove this result, we use approximation methods, based, in particular, on Yosida approximations of the linear part of the inclusion. In the last sections, we apply this result to prove the existence of a mild solution to our initial value problem on each finite interval and to verify the existence of mild solutions bounded on the semi-axis.

2 Preliminaries

2.1 Fractional derivative

In this section we will recall some notions and definition which we will need in the sequel (details can be found, e.g., in [22], [23], [24], [25]).

Let E be a real Banach space.

Definition 1. *The Riemann–Liouville fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,a] \to E$ is the function $D_0^q g$ of the following form:*

$$D_0^q g(t) = \frac{1}{\Gamma(1-q)}\,\frac{d}{dt}\int_0^t (t-s)^{-q} g(s)\,ds$$

provided the right-hand side of this equality is well defined.

Here Γ is the Euler gamma-function

$$\Gamma(r) = \int_0^\infty s^{r-1} e^{-s} ds.$$

Definition 2. *The Caputo fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,a] \to E$ is the function ${}^C D_0^q g$ defined in the following way:*

$${}^C D_0^q g(t) = \Big(D^q (g(\cdot) - g(0)) \Big)(t)$$

provided the right-hand side of this equality is well defined.

Definition 3. *A function of the form*

$$E_{q,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(qn+\beta)}, \quad q, \beta > 0,\ z \in \mathbb{C}$$

is called the Mittag–Leffler function.

The Mittag–Leffler function has the following asymptotic representation as $z \to \infty$ (see, e.g., [22]):

$$E_{q,\beta}(z) = \begin{cases} \frac{1}{q} z^{\frac{1-\beta}{q}} e^{z^{\frac{1}{q}}} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - qn)} + O(|z|^{-N}), & |arg z| \le \frac{1}{2}\pi q, \\ -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - qn)} + O(|z|^{-N}), & |arg(-z)| \le (1 - \frac{1}{2}q)\pi. \end{cases} \tag{2.1}$$

Denote $E_{q,1}$ by E_q. Notice that the second of the above formulae implies that in the case $z = \tau < 0$ and $0 < q < 1$ we have

$$E_q(\tau) \to 0 \ \text{ as } \ \tau \to -\infty. \tag{2.2}$$

Notice that from the relations (see, e.g., [26]):

$$E_q(-z) = \int_0^\infty \xi_q(\theta) e^{-z\theta} d\theta$$

and

$$E_{q,q}(-z) = \int_0^\infty q\theta \xi_q(\theta) e^{-z\theta} d\theta,$$

where

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \Psi_q(\theta^{-1/q}), \tag{2.3}$$

$$\Psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in \mathbb{R}_+, \tag{2.4}$$

it follows that

$$E_q(\tau) > 0,\ E_{q,q}(\tau) > 0 \text{ for } \tau < 0. \tag{2.5}$$

Remark 4. *(See, e.g. [17])* $\int_0^\infty \theta \xi_q(\theta)\, d\theta = \frac{1}{\Gamma(q+1)},\ \int_0^\infty \xi_q(\theta)\, d\theta = 1,\ \xi_q(\theta) \ge 0.$

Consider a scalar equation of the form

$$^{C}D^{q}x(t) = \lambda x(t) + f(t), \quad t \in [0,T] \tag{2.6}$$

with the initial condition

$$x(0) = x_0, \tag{2.7}$$

where $\lambda \in \mathbb{R}$, $f : [0,T] \to \mathbb{R}$ is a continuous function. By a solution of this problem we mean a continuous function $x : [0,T] \to \mathbb{R}$ satisfying condition (2.7) whose fractional derivative $^{C}D^{q}x$ is also continuous and satisfies equation (2.6). It is known (see [23], Example 4.9) that the unique solution of this equation has the form

$$x(t) = E_q(\lambda t^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)\,ds. \tag{2.8}$$

We will need the following auxiliary assertion which is an analogue of the known Gronwall lemma on integral inequalities (see [10]).

Lemma 5. *Let a bounded measurable function* $\omega : [0,T] \to \mathbb{R}$ *satisfy the integral inequality*

$$\omega(t) \le E_q(-\eta t^q)\omega(0) + \int_0^t (t-s)^{q-1}E_{q,q}(-\eta(t-s)^q)\Big(K + m\,\omega(s)\Big)\,ds \tag{2.9}$$

where $K \ge 0,\ 0 < m < \eta$. *Then*

$$\omega(t) \le E_q\Big((-\eta+m)t^q\Big)\omega(0) + K\int_0^t (t-s)^{q}E_{q,q}\Big((-\eta+m)(t-s)^q\Big)\,ds.$$

2.2 Measures of noncompactness and condensing multivalued maps

Let us recall some notions and facts (details can be found, for example, in [27], [28]).

Let $\mathcal{E}$ be a Banach space. Introduce the following notation:

- $Pb(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing \text{ is bounded }\}$;
- $Pv(\mathcal{E}) = \{A \in Pb(\mathcal{E}) : A \text{ is convex}\}$;
- $K(\mathcal{E}) = \{A \in Pb(\mathcal{E}) : A \text{ is compact}\}$;
- $Kv(\mathcal{E}) = Pv(\mathcal{E}) \cap K(\mathcal{E})$.

Definition 6. Let $(\mathcal{A}, \ge)$ be a partially ordered set. A function $\beta : \mathrm{Pb}(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in \mathrm{Pb}(\mathcal{E})$ we have:

$$\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega),$$

where $\overline{\mathrm{co}}\,\Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) *monotone* if for each $\Omega_0, \Omega_1 \in \mathrm{Pb}(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;

2) *nonsingular* if for each $a \in \mathcal{E}$ and each $\Omega \in \mathrm{Pb}(\mathcal{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$;

If $\mathcal{A}$ is a cone in a Banach space, the MNC β is called:

4) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in \mathrm{Pb}(\mathcal{E})$;

5) *real* if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering;

6) *algebraically semiadditive* if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathrm{Pb}(\mathcal{E})$.

It should be mentioned that the Hausdorff MNC obeys all above properties. Another examples can be presented by the following measures of noncompactness defined on $Pb(C([0,a];E))$, where $C([0,a];E)$ is the space of continuous functions with the values in a separable Banach space E:

(i) *the modulus of fiber noncompactness*

$$\varphi(\Omega) = \sup_{t\in[0,a]} e^{-pt}\chi_E(\Omega(t)),$$

where $p > 0$, χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) *the modulus of equicontinuity* defined as

$$\mathrm{mod}_C\left(\Omega\right) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\leq\delta} \left\|y\left(t_1\right) - y\left(t_2\right)\right\| .$$

Notice that these MNCs satisfy all above-mentioned properties except regularity. To obtain a regular MNC in the space $C([0,a];E)$ we can consider the MNC

$$\nu(\Omega) = \Big(\varphi(\Omega), \mathrm{mod}_C(\Omega)\Big)$$

with the values in the cone $\mathbb{R}^2$ with the natural partial order.

Definition 7. *Let $X \subseteq \mathcal{E}$ be a closed subset; a multivalued map (multimap) $\mathcal{F}\colon X \to K(\mathcal{E})$ is called upper semicontinuous (u.s.c.) if the pre-image*

$$\mathcal{F}^{-1}(V) = \{x \in X \colon \mathcal{F}(x) \subset V\}$$

of each open set $V \subset \mathcal{E}$ is open in X.

Definition 8. *A u.s.c. multimap $\mathcal{F} : X \to K(\mathcal{E})$ is called condensing with respect to a MNC β (or β-condensing) if for every bounded set $\Omega \subseteq X$ that is not relatively compact we have*

$$\beta\left(\mathcal{F}\left(\Omega\right)\right) \ngeq \beta\left(\Omega\right).$$

More generally, given a metric space Λ of parameters, we will say that a u.s.c. multimap $\Gamma : \Lambda\times X \to K(\mathcal{E})$ is *a condensing family with respect to a MNC β (or β-condensing family)* if for every bounded set $\Omega \subseteq X$ that is not relatively compact we have

$$\beta\left(\Gamma\left(\Lambda \times \Omega\right)\right) \ngeq \beta\left(\Omega\right).$$

Let $V \subset \mathcal{E}$ be a bounded open set, β a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to Kv(\mathcal{E})$ a β-condensing multimap such that $x \notin \mathcal{F}\left(x\right)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the closure and the boundary of the set V.

In such a setting, *the topological degree*

$$\deg\left(i - \mathcal{F}, \overline{V}\right)$$

of the corresponding vector multifield $i-\mathcal{F}$ satisfying the standard properties is defined, where i is the identity map on $\mathcal{E}$. In particular, the condition

$$\deg\left(i-\mathcal{F},\overline{V}\right)\neq 0$$

implies that *the fixed point set* $Fix\mathcal{F}=\{x : x\in\mathcal{F}(x)\}$ is a nonempty subset of V.

To describe the next property, let us introduce the following notion.

Definition 9. *Suppose that β-condensing multimaps $\mathcal{F}_0$, $\mathcal{F}_1 : \overline{V}\to Kv(\mathcal{E})$ have no fixed points on the boundary ∂V and there exists a β-condensing family $\mathfrak{H}\colon [0,1]\times\overline{V}\to K(\mathcal{E})$ such that:*

(i) $x\notin\mathfrak{H}(\lambda,x)$ *for all* $(\lambda,x)\in[0,1]\times\partial V$;

(ii) $\mathfrak{H}(0,\cdot)=\mathcal{F}_0$; $\mathfrak{H}(1,\cdot)=\mathcal{F}_1$.

Then the multifields $\Phi_0=i-\mathcal{F}_0$ and $\Phi_1=i-\mathcal{F}_1$ are called homotopic:

$$\Phi_0\sim\Phi_1.$$

The homotopy invariance property of the topological degree asserts that if $\Phi_0\sim\Phi_1$ then $\deg\left(i-\mathcal{F}_0,\overline{V}\right)=\deg\left(i-\mathcal{F}_1,\overline{V}\right)$.

Let us mention also the following property of the topological degree which we will need in the sequel.

The normalization property. If $\mathcal{F}(x)\equiv A\in K(\mathcal{E})$, then

$$deg\left(i-\mathcal{F},\overline{V}\right)=\begin{cases} 1 & \text{if} \quad A\subset V,\\ 0 & \text{if} \quad A\cap\overline{V}=\emptyset.\end{cases}$$

3 A priori estimates of solutions

Let H be a separable Hilbert space. We will consider the Cauchy problem for a semilinear fractional order differential inclusion in H :

$$^{C}D_0^q x(t)\in Ax(t)+F(t,x(t)),\quad t\in[0,T] \tag{3.1}$$

$$x(0)=x_0, \tag{3.2}$$

where $0<q<1$ and the linear operator A satisfies the following condition:

(A) $A : D(A)\subseteq H\to H$ is a linear closed (not necessarily bounded) operator generating a bounded C_0-semigroup $\{U_A(t)\}_{t\geq 0}$ of linear operators in H and such that

$$\langle Ax,x\rangle\leq -d\,\|x\|^2,\quad \forall x\in D(A)$$

for some $d>0$.

It will be assumed that the multimap $F\colon[0,T]\times H\to Kv(H)$ obeys the following conditions:

$F1)$ the multifunction $F(\cdot,x)\colon[0,T]\to Kv(H)$ admits a measurable selection for each $T>0$ and $x\in H$, i.e., there exists a measurable function $f\colon[0,T]\to H$ such that $f(t)\in F(t,x)$ for a.e. $t\in[0,T]$;

$F2)$ the multimap $F(t,\cdot)\colon \ \to Kv(H)$ is u.s.c. for each $T>0$ and a.e. $t\in[0,T]$;

$F3)$ for each $R>0$ and $T>0$ there exists a function $m_R\in L^\infty[0,T]$ such that $x\in H$, $\|x\|\le R$ implies
$$\|F(t,x)\|\le m_R(t) \ \text{ for a.e. } \ t\in[0,T];$$

$F4)$ for each $T>0$ there exists $\kappa\in L^\infty[0,T]$ such that for every bounded set $\Omega\subset H$ it holds:
$$\chi(t,\Omega))\le\kappa(t)\chi(\Omega),$$
where χ denotes the Hausdorff MNC in the space H.

$F5)$ there exists $a\ge 0$ such that
$$\sup_{y\in F(t,x)}\langle y,x\rangle\le a\,\|x\|^2+G(t),\quad t\in[0,T],$$
where $G\colon\mathbb{R}_+\to\mathbb{R}_+$ is a locally L^∞-function, i.e., $G|_{[0,T]}\in L^\infty[0,T]$ for each $T>0$.

From conditions $(F1)-(F3)$ it follows that for each $T>0$ the superposition multioperator $\mathcal{P}_F\colon C([0,T];H)\multimap L^\infty((0,T);H)$ is defined by the formula
$$\mathcal{P}_F(x)=\{f\in L^\infty((0,T);H)\colon f(s)\in F(s,x(s)) \text{ for a.e. } s\in[0,T]\}\tag{3.3}$$
(see, for example, [27], [28]).

Let us recall (see, for example, [5], [9], [10], [11], [12], [13]) that a mild solution to problem (3.1), (3.2) is a function $x\in C([0,T],H)$ of the form
$$x(t)=\mathcal{G}_A(t)x_0+\int_0^t(t-s)^{q-1}\mathcal{T}_A(t-s)f(s)ds,\tag{3.4}$$
where
$$\mathcal{G}_A(t)=\int_0^\infty\xi_q(\theta)U_A(t^q\theta)d\theta,\qquad \mathcal{T}_A(t)=q\int_0^\infty\theta\xi_q(\theta)U_A(t^q\theta)d\theta,$$
$f\in\mathcal{P}_F(x)$ and the function ξ_q is defined by formula (2.3).

Lemma 10. *(See [17], Lemma 3.4.) The operator functions $\mathcal{G}_A$ and $\mathcal{T}_A$ possess the following properties:*

1) *for each $t\in[0,T]$, $\mathcal{G}_A(t)$ and $\mathcal{T}_A(t)$ are linear bounded operators, more precisely, for each $x\in E$ we have*
$$\|\mathcal{G}_A(t)x\|_H\le M\,\|x\|_H\,,\tag{3.5}$$
$$\|\mathcal{T}_A(t)x\|_H\le\frac{qM}{\Gamma(1+q)}\,\|x\|_H\,,\tag{3.6}$$
where
$$M=\sup_{t\in[0,+\infty)}\|U_A(t)\|.\tag{3.7}$$

2) *the operator functions $\mathcal{G}_A(\cdot)$ and $\mathcal{T}_A(\cdot)$ are strongly continuous, i.e., functions $t\in[0,T]\to\mathcal{G}_A(t)x$ and $t\in[0,T]\to\mathcal{T}_A(t)x$ are continuous for each $x\in H$.*

Remark 11. *Notice, that if A is a bounded operator, then the solution defined by formula* (3.4) *satisfies the following differential equation (see [25])*

$$^{C}D_0^q x(t) = Ax(t) + f(t).$$

Now, suppose that $x \in C([0,T];H)$ be any mild solution to problem (3.1), (3.2). Take a selection $f \in \mathcal{P}_F(x)$ satisfying (3.4). Then condition $(F3)$ implies that

$$\|f(t)\| \leq \omega_R(t) \text{ for a.e. } t \in [0,T], \tag{3.8}$$

where $R = \|x\|_{C([0,T];H)}$ and $\omega_R \in L^\infty(0,T)$.

Then the following assertion holds true.

Lemma 12. *For each $\varepsilon > 0$ there exists a set $m_\varepsilon \subset [0,T]$ of a Lebesgue measure $\mu(m_\varepsilon) < \varepsilon$ and a piecewise linear function $g_\varepsilon \colon [0,T] \to H$ with a finite number of nodes belonging to $D(A)$ such that*

$$\|f(t) - g_\varepsilon(t)\| < \varepsilon, \quad t \in [0,T] \setminus m_\varepsilon. \tag{3.9}$$

Proof. Notice that we can assume, without loss of generality, that the selection f is a continuous function. In fact, consider the functions $f_\gamma \colon [0,T] \to H$ defined by the formula

$$f_\gamma(t) = \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \overline{f}(s)ds,$$

where

$$\overline{f}(s) = \begin{cases} f(s), & \text{for } s \in [0,T], \\ 0, & \text{for } t \notin [0,T]. \end{cases}$$

Then $f_\gamma(t) \to f(t)$ for a.e. $t \in [0,T]$ as $\gamma \to 0$ since for a measurable function the Lebesgue points form a complete measure space (see [29]). Notice that functions f_γ are continuous and

$$\|f_\gamma(t)\| \leq \|f\|_{L^\infty([0,T],H)} \text{ for } t \in [0,T]. \tag{3.10}$$

Hence each function f_γ may be approximated with an arbitrary degree of accuracy in the space $C([0,T];H)$ by piecewise linear functions g_δ with a finite number of nodes belonging to $D(A)$.

Take a sequence $\gamma_k \to 0$. applying to functions f_{γ_k} the Egorov theorem (see [30]), for a given $\varepsilon > 0$ we may find $m_\varepsilon \subset [0,T]$ such that $\mu(m_\varepsilon) < \varepsilon$ and the sequence $\{f_{\gamma_k}\}$ uniformly converges to f on $[0,T] \backslash m_\varepsilon$. So, we will have for a sufficiently large k

$$\|f(t) - f_{\gamma_k}(t)\| < \frac{\varepsilon}{2} \text{ for } t \in [0,T] \backslash m_\varepsilon.$$

Taking now a piecewise linear function g_ε satisfying

$$\|f_{\gamma_k} - g_\varepsilon\|_{C([0,T],H)} < \frac{\varepsilon}{2} \tag{3.11}$$

we will get the desired. □

Now, taking a piecewise linear function g_ε satisfying conditions of Lemma 12, consider the function

$$x^\varepsilon(t) = \mathcal{G}_A(t)x_0^\varepsilon + \int_0^t (t-s)^{q-1}\mathcal{T}_A(t-s)g_\varepsilon(s)ds, \tag{3.12}$$

where $x_0^\varepsilon \in D(A)$ and $x_0^\varepsilon \to x_0$ as $\varepsilon \to 0$.

Lemma 13. *The expression*

$$I(\varepsilon)(t) := \int_0^t (t-s)^{q-1}\|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds$$

tends to zero as $\varepsilon \to 0$ *uniformly on* $[0,T]$.

Proof. Denoting

$$N = \frac{qM}{\Gamma(1+q)}$$

we get from (3.6)

$$\|\mathcal{T}_A(t)\| \le N, \quad t \in [0,T].$$

For a given $\gamma > 0$ choose $\sigma > 0$ such that

$$\frac{\sigma^q}{q} N\Big(2\|\omega_R\|_{L^\infty} + 1\Big) < \frac{\gamma}{2}.$$

From the construction of the function g_ε (see relations (3.10) and (3.11)) it follows that for a sufficiently small $\varepsilon > 0$ we get

$$\|g_\varepsilon(t)\| \le \|\omega_R\|_{L^\infty} + 1, \quad t \in [0,T].$$

Then for the case $t \le \sigma \le T$ we have

$$\int_0^t (t-s)^{q-1}\|\mathcal{T}(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds \le \int_0^\sigma (t-s)^{q-1} N\Big(\|g_\varepsilon(s)\| + \|f(s)\|\Big) ds$$

$$\le N\Big(2\|\omega_R\|_{L^\infty} + 1\Big) \int_0^\sigma (t-s)^{q-1} ds = \frac{\sigma^q}{q} N\Big(2\|\omega_R\|_{L^\infty} + 1\Big) < \frac{\gamma}{2}.$$

If $\sigma < t$ we get

$$I(\varepsilon)(t) = \int_{t-\sigma}^t (t-s)^{q-1}\|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds$$

$$+ \int_0^{t-\sigma} (t-s)^{q-1}\|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds = I_1(\varepsilon)(t) + I_2(\varepsilon)(t).$$

For $I_1(\varepsilon)$ the following estimate holds:

$$I_1(\varepsilon)(t) \le \frac{\sigma^q}{q} N\Big(2\|\omega_R\|_{L^\infty} + 1\Big) < \frac{\gamma}{2}.$$

For $I_2(\varepsilon)$ we have

$$I_2(\varepsilon)(t) = \int_{[0,t-\sigma]\setminus m_\varepsilon} (t-s)^{q-1}\|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds$$

$$+ \int_{[0,t-\sigma]\cap m_\varepsilon} (t-s)^{q-1}\|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds = I_{21}(\varepsilon)(t) + I_{22}(\varepsilon)(t).$$

Set

$$N_1 = \max_{s \in [0,t-\sigma]} (t-s)^{q-1}\|\mathcal{T}_A(t-s)\|.$$

Since $\|g_\varepsilon(s) - f(s)\| < \varepsilon$ for $s \in [0, t-\sigma] \setminus m_\varepsilon$ we obtain the estimate

$$I_{21}(\varepsilon)(t) \leq N_1 \varepsilon (t-\sigma) < N_1 \varepsilon T.$$

For $I_{22}(\varepsilon)$ we have

$$I_{22}(\varepsilon)(t) \leq N_1 \Big(2\|\omega_R\|_{L^\infty} + 1\Big) \mu([0, t-\sigma] \cap m_\varepsilon) \leq N_1 \Big(2\|\omega_R\|_{L^\infty} + 1\Big) \varepsilon.$$

Now, choosing ε so that

$$N_1(T + 2\|\omega_R\|_{L^\infty} + 1)\varepsilon < \frac{\gamma}{2}$$

we get the desired. □

Corollary 14. *The expression $\|x^\varepsilon - x\|_{C([0,T];H)}$ tends to zero as $\varepsilon \to 0$.*

Proof. We have the estimate

$$\|x(t) - x^\varepsilon(t)\| \leq \|\mathcal{G}_A(t)(x_0 - x_0^\varepsilon)\| + \int_0^t (t-s)^{q-1} \|\mathcal{T}_A(t-s)\| \cdot \|g_\varepsilon(s) - f(s)\| ds$$

Since the operator function $\mathcal{G}_A(t)$ is strongly continuous and $x_0^\varepsilon \to x_0$ as $\varepsilon \to 0$, the first term in this sum tends to zero uniformly on $[0, T]$. The second term uniformly tends to zero due to Lemma 13. □

Remark 15. *If in Lemma 13 we will replace $\|\mathcal{T}_A(t-s)\|$ with the Mittag–Leffler function $E_{q,q}((-d+a)t)$ then, repeating the above reasonings, we get*

$$\int_0^t (t-s)^{q-1} E_{q,q}((-d+a)(t-s)) \cdot \|g_\varepsilon(s) - f(s)\| ds \to 0$$

as $\varepsilon \to 0$ uniformly on $[0, T]$.

Now consider Yosida approximations for the operator A :

$$A_n = nA(nI - A)^{-1}, \quad n \geq 1.$$

It is known (see e.g. [31], [30]) that A_n are bounded, mutually commuting operators, A_n converges to A pointwise on $D(A)$ and each A_n generates the uniformly continuous contraction semigroup U_{A_n}.

Introduce the approximations x_n^ε by the formulas

$$x_n^\varepsilon(t) = \mathcal{G}_{A_n}(t) x_0^\varepsilon + \int_0^t (t-s)^{q-1} \mathcal{T}_{A_n}(t-s) g_\varepsilon(s) ds, \tag{3.13}$$

Lemma 16. *For a fixed $\varepsilon > 0$ the sequence x_n^ε converges to x^ε as $n \to \infty$ uniformly on $[0, T]$.*

Proof. Since for each fixed $x \in H$ we have

$$U_{A_n}(t)x \to U_A(t)x$$

uniformly with respect to $t \in [0, T]$ (see [31]) we also have the uniform convergence

$$\mathcal{G}_{A_n}(t)x \to \mathcal{G}_A(t)x,$$

$$\mathcal{T}_{A_n}(t)x \to \mathcal{T}_A(t)x$$

that implies for a fixed $\varepsilon > 0$ the desired convergence. □

Notice that due to the closedness of the operator A we have for $x \in D(A)$:

$$AU_A(t)x = U_A(t)Ax,$$

$$AA_n x = A_n Ax,$$

$$AU_{A_n}(t)x = U_{A_n}(t)Ax.$$

By the definition of the operator functions $\mathcal{G}_A(t)$ and $\mathcal{T}_A(t)$ we have for $x \in D(A)$

$$A\mathcal{G}_A(t)x = \mathcal{G}_A(t)Ax;$$

$$A\mathcal{T}_A(t)x = \mathcal{T}_A(t)Ax;$$

$$A\mathcal{G}_{A_n}(t)x = \mathcal{G}_{A_n}(t)Ax;$$

$$A\mathcal{T}_{A_n}(t)x = \mathcal{T}_{A_n}(t)Ax.$$

Since for a given piecewise linear function g_ε the set $\{Ag_\varepsilon(s) : s \in [0,T]\}$ is compact in H it follows that the range of the function g_ε lies in $D(A)$ and therefore $\{Ax_n^\varepsilon(t) : t \in [0,T]\}$ is a compact set.

Lemma 17. *For a fixed $\varepsilon > 0$ we have*

$$(n(nI - A)^{-1} - I)Ax_n^\varepsilon(t) \to 0$$

as $n \to \infty$ uniformly with respect to $t \in [0,T]$.

Proof. For $x \in D(A)$ it holds that

$$\langle (nA(nI - A)^{-1} - I)x(t), x(t)\rangle = \langle (n(nI - A)^{-1} - I)Ax(t), x(t)\rangle =$$

$$= \langle Ax(t), x(t)\rangle + \langle (n(nI - A)^{-1} - I)Ax(t), x(t)\rangle.$$

Since $(n(nI - A)^{-1} - I)y \to y$ for each fixed $y \in H$ (see [31]) we have

$$(n(nI - A)^{-1} - I)Ax \to 0$$

uniformly with respect to $x \in \{g_\varepsilon(t) : t \in [0,T]\}$.

Since x_n^ε can be expressed through g_ε by the formula (3.13) we get the desired. □

Now we are in position to present the main result of this section.

Theorem 18. *Under above conditions, there exists a continuous function $\mathcal{C}\colon [0,+\infty) \to [0,+\infty)$ such that for every solution x of problem (3.1)-(3.2) defined on an interval $[0,T]$ the following a priori estimate holds true:*

$$\|x\|_{C([0,T],H)} \leq \mathcal{C}(T).$$

Proof. Take a sequence of positive numbers $\theta_k \to 0$ and choose a sequence of approximations $\{x^{\varepsilon_k}\}$ so that

$$\|x^{\varepsilon_k} - x\|_{C([0,T],H)} < \theta_k, \tag{3.14}$$

Further, according to Lemma 13 find $n^1(\varepsilon_k)$ such that for $n \geq n^1(\varepsilon_k)$ the following holds:

$$\|x_n^{\varepsilon_k} - x^{\varepsilon_k}\| < \theta_k.$$

Since x_n^ε lie in $D(A)$ and Ax_n^ε for each fixed ε is uniformly bounded in n (see Lemma 17) we may indicate $n^2(\varepsilon_k)$ such that for $n \geq n^2(\varepsilon_k)$ we have

$$\sup_{t\in[0,T]} \langle (n(nI-A)^{-1}-I)Ax_n^\varepsilon, x_n^\varepsilon\rangle < \theta_k.$$

Take $n_k = \max(n^1(\varepsilon_k), n^2(\varepsilon_k))$.

Then we get

$$\|x_{n_k}^{\varepsilon_k} - x^{\varepsilon_k}\|_{C([0,T],H)} < \theta_k, \tag{3.15}$$

$$\sup_{t\in[0,T]} \langle (n_k(n_kI+A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle < \theta_k. \tag{3.16}$$

Notice that simultaneously we construct the corresponding sequences of functions $\{g^{\varepsilon_k}\}$ and sets m_{ε_k}.

By virtue of Remark 11 we have

$${}^C D_0^q x_{n_k}^{\varepsilon_k}(t) = A_n x_{n_k}^{\varepsilon_k}(t) + g_{\varepsilon_k}(t).$$

From [19], [20] it follows that

$${}^C D_0^q \|x_{n_k}^{\varepsilon_k}(t)\|^2 \leq \langle A_n x_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \langle g_{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle. \tag{3.17}$$

Now let us estimate the right hand side of inequality (3.17).

$$\begin{gathered}
\langle A_{n_k} x_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \langle g_{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle \leq \langle A_{n_k} x_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \langle (n_k(n_kI-A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \\
+\langle g_{\varepsilon_k}(t)-f(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \langle f(t), x(t)\rangle + \langle f(t), x_{n_k}^{\varepsilon_k}(t)-x(t)\rangle + \langle (n_k(n_kI-A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle \leq \\
\leq -d\|x_{n_k}^{\varepsilon_k}(t)\|^2 + a\|x(t)\|^2 + \langle g_{\varepsilon_k}(t)-f(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \langle f(t), x_{n_k}^{\varepsilon_k}(t)-x(t)\rangle + \\
\langle (n_k(n_kI-A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + G(t) + \langle ((n_k(n_kI-A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle \leq \\
\leq (-d+a)\|x_{n_k}^{\varepsilon_k}(t)\|^2 + |a|\|x-x_{n_k}^{\varepsilon_k}\|(\|x\|+\|x_{n_k}^{\varepsilon_k}\|) + \|g_{\varepsilon_k}(t)-f(t)\|\|x_{n_k}^{\varepsilon_k}(t)\| + \|f(t)\|\|x_{n_k}^{\varepsilon_k}(t)-x(t)\| + \\
+\langle (n_k(n_kI-A)^{-1}-I)Ax_{n_k}^{\varepsilon_k}(t), x_{n_k}^{\varepsilon_k}(t)\rangle + \|G(t)\|.
\end{gathered}$$

We get the inequality

$$\begin{gathered}
{}^C D_0^q \|x_{n_k}^{\varepsilon_k}(t)\|^2 \leq (-d+a)\|x_{n_k}^{\varepsilon_k}(t)\|^2 + \theta_k 2M + \theta_k\|f(t)\| + \theta_k \\
+M\|g_{\varepsilon_k}(t)-f(t)\| + G(t).
\end{gathered}$$

By virtue of the analog of Lemma 5 the following inequality holds true

$$\begin{aligned}
\|x_{n_k}^{\varepsilon_k}\|^2 \leq E_q(t)\|x_0^{\varepsilon_k}\|^2 &+ \int_0^t (t-s)^{q-1}E_{q,q}((-d+a)(t-s)^q)\|G(s)\|ds + \\
&+\theta_k \int_0^t (t-s)^{q-1}E_{q,q}((-d+a)(t-s)^q)(2M+\omega_R(s))ds + \\
&+M\int_0^t (t-s)^{q-1}E_{q,q}((-d+a)(t-s)^q)\|g_{\varepsilon_k}(s)-f(s)\|ds.
\end{aligned} \tag{3.18}$$

Notice that the third and forth terms are tending to zero as $k \to \infty$. In fact, in the third term the integral is uniformly bounded on $[0,T]$ and we can apply Remark 15 to the forth term.

Passing in (3.18) to the limit as $k \to \infty$ we get

$$\|x(t)\|^2 \leq E_q(t)\|x_0\| + \int_0^t (t-s)^{q-1}E_{q,q}((-d+a)(t-s)^q)G(s)ds \tag{3.19}$$

Therefore, the right hand side determines the function of a priori estimate $\mathcal{C}$ on the interval $[0,T]$. □

4 Existence result

From Theorem 18 we can obtain the following result on the existence of a solution to problem (3.1)-(3.2) on an arbitrary interval $[0, T]$.

Theorem 19. *Under above conditions, problem (3.1)-(3.2) has a mild solution on* $[0, T]$ *for each* $T > 0$.

Proof. Consider the family of multivalued integral operators $\mathfrak{F}\colon C([0,T];H) \times [0,1] \multimap C([0,T];H)$ defined in the following way:

$$\mathfrak{F}(x,\lambda) = \{z = \mathcal{G}_A(t)x_0 + \lambda \int_0^t (t-s)^{q-1}\mathcal{T}_A(t-s)f(s)ds : f \in \mathcal{P}_F(x)\}, \tag{4.1}$$

where $\mathcal{P}_F$ is the superposition multioperator defined by (3.3).

It is clear that each fixed point $x_\lambda \in C([0,T];H)$ of the multimap $\mathfrak{F}(\cdot,\lambda)$, $\lambda \in [0,1]$ is a mild solution to the problem

$$^C D_0^q x(t) \in Ax(t) + \lambda F(t, x(t)), \quad t \in [0,T] \tag{4.2}$$

$$x(0) = x_0. \tag{4.3}$$

Moreover, it is known (see [5], [7], [9], [10], [11], [12], [13]) that the family (4.1) has compact convex values and is condensing with respect to the MNC ν in $C([0,T];H)$ (see Section 2). Since the multioperators λF satisfy conditions $(F1)$–$(F5)$ independently on λ, by applying Theorem 18, we conclude that there exists a constant $\mathcal{C}(T)$ such that all solutions to problem (4.2)–(4.3) satisfy the a priori estimate

$$\|x_\lambda\| \leq \mathcal{C}(T).$$

So, the multioperators $\mathfrak{F}(\cdot,\lambda)$ from family (4.1) are fixed point free on the boundary of the ball $\mathfrak{B}$ of the space $C([0,T];H)$ centered at zero of the radius $\mathcal{C}(T)+1$. Notice that the range of the multioperator $\mathfrak{F}(\cdot,0)$ consists of single function $y(t) = \mathcal{G}_A(t)x_0$ being its fixed point.

Now, applying the homotopy and normalization properties of the topological degree we obtain

$$deg(i - \mathfrak{F}(\cdot,1), \mathfrak{B}) = deg(i - \mathfrak{F}(\cdot,0), \mathfrak{B}) = 1$$

that yields, by the existence property of the topological degree, the desired result. □

5 Bounded solutions on the semi-axis

Obtained a priori estimate (3.19) allows to prove the following assertion about the existence of solutions to problem (3.1)–(3.2) bounded on $[0,\infty)$.

Theorem 20. *Suppose that* $d > a$, *conditions* $(F1)$–$(F5)$ *hold true and the function* G *in condition* $(F5)$ *belongs to the space* $L^r[0,\infty)$, *where* $r > \frac{1}{q}$. *Then each solution to problem (3.1)– (3.2) is bounded on* $[0,\infty)$.

Proof. By virtue of Theorem 18 all solutions to problem (3.1)–(3.2) are defined on $[0,\infty)$ and satisfy estimate (3.19). Since

$$E_q((-d+a)t)\|x_0\| \to 0 \quad \text{as} \quad t \to \infty,$$

to prove the theorem it is sufficient to demonstrate that the expression

$$\int_0^t (t-s)^{q-1} E_{q,q}\big((-d+a)(t-s)^q\big) G(s) ds \tag{5.1}$$

is bounded while $t \in (0, \infty)$.

Let for $\tau > T$ the following asymptotic for the function $E_{q,q}(\tau)$ holds true (see [22])

$$0 \leq E_{q,q}\big((-d+a)\tau\big) < \frac{c}{\tau},$$

where c is a constant. Then for $\tau > T^{\frac{1}{q}}$ we have the next estimate:

$$0 \leq E_{q,q}\big((-d+a)\tau^q\big)| < \frac{c}{\tau^q}$$

For $\tau \leq T^{\frac{1}{q}}$ we have the estimate $E_{q,q}(\tau^q) \leq M$.

Estimate, for large t, expression (5.1):

$$\Big|\int_0^t (t-s)^{q-1} E_{q,q}((-d+a)(t-s)^q) G(s) ds\Big| \leq \int_{t-T^{\frac{1}{q}}}^t (t-s)^{q-1} E_{q,q}\big((-d+a)(t-s)^q\big)\big|G(s)\big| ds +$$

$$\int_0^{t-T^{\frac{1}{q}}} (t-s)^{q-1} E_{q,q}\big((-d+a)(t-s)^q\big)\big|G(s)\big| ds = I_1(t) + I_2(t)$$

Let $\frac{1}{p} + \frac{1}{r} = 1$, then by the Holder inequality

$$I_1(t) \leq \Big(\int_{t-T^{\frac{1}{q}}}^t (t-s)^{p(q-1)} M^p ds\Big)^{\frac{1}{p}} \Big(\int_{t-T^{\frac{1}{q}}}^t |G(s)|^r ds\Big)^{\frac{1}{r}}.$$

Since $r > \frac{1}{q}$ implies the inequality

$$p(q-1) + 1 > 1,$$

we get

$$I_1(t) \leq \Big(\frac{T^{\frac{p(q-1)+1}{q}}}{p(q-1)+1}\Big)^{\frac{1}{p}} \|G\|_{L^r[0,\infty)}.$$

For I_2 we have the estimate

$$I_2(t) \leq \int_{T^{\frac{1}{q}}}^t \tau^{q-1} \frac{c}{\tau^q} \big|G(t-\tau)\big| d\tau \leq c\Big(\int_{T^{\frac{1}{q}}}^t \tau^{-p} d\tau\Big)^{\frac{1}{p}} \Big(\int_{T^{\frac{1}{q}}}^t G(t-\tau) d\tau\Big)^{\frac{1}{r}} \leq$$

$$\leq c\Big(\frac{T^{\frac{1}{q}(1-p)}}{p-1}\Big)^{\frac{1}{p}} \|G\|_{L^r[0,\infty)}$$

□

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