Midterm Practice Set Solution

October 19, 2025

Problem 1: Given vectors $\vec{u}$ and $\vec{v}$, prove the triangle inequality:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Hint: Use the dot product to express the magnitude of a vector.

Solution:
We start by expressing the square of the magnitude using the dot product:

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}).$$

Expanding this, we get:

$$\|\vec{u} + \vec{v}\|^2 = \vec{u} \cdot \vec{u} + 2\,\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}.$$

By definition of the dot product,

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\,(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.$$

Using the Cauchy–Schwarz inequality,

$$\vec{u} \cdot \vec{v} \leq \|\vec{u}\|\,\|\vec{v}\|.$$

Therefore,

$$\|\vec{u} + \vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2.$$

Since both sides are nonnegative, taking square roots yields:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Equality condition: Equality holds if and only if $\vec{u}$ and $\vec{v}$ are in the same direction, i.e., one is a nonnegative scalar multiple of the other.

Problem 2: Assume we scale a 3D model by x on the x-axis, y on the y-axis, and z on the z-axis. The model is bounded by a cuboid whose body diagonal from the origin to the opposite vertex is $\vec{v} = \langle a, b, c \rangle$. Find the perimeter (sum of all 12 edge lengths) of the scaled cuboid.

Solution:

Since $\vec{v} = \langle a, b, c \rangle$ is the body diagonal of an axis-aligned cuboid, the original edge lengths are

$$L_0 = a, \qquad W_0 = b, \qquad H_0 = c, \qquad (a, b, c > 0).$$

Scaling by factors x, y, z along the coordinate axes maps these to

$$L = |x|\, a, \qquad W = |y|\, b, \qquad H = |z|\, c,$$

where absolute values account for the possibility of reflections if a scaling factor is negative.

A cuboid has 4 edges of each length, so the total perimeter (sum of all 12 edges) is

$$P \;=\; 4(L + W + H) \;=\; 4\big(|x|a + |y|b + |z|c\big).$$

Special case (usual positive scaling): If $x, y, z > 0$, then

$$P \;=\; 4(xa + yb + zc).$$

Problem.3 Let L be a map between vector spaces (over the same field, e.g. $\mathbb{R}$). Show that the pair of conditions

$$\begin{aligned} &(1) \quad L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}), \\ &(2) \quad L(c\,\mathbf{v}) = c\,L(\mathbf{v}) \qquad \text{(for all scalars } c) \end{aligned}$$

is equivalent to the single condition

$$(3) \qquad L(r\,\mathbf{u} + s\,\mathbf{v}) = r\,L(\mathbf{u}) + s\,L(\mathbf{v}) \quad \text{(for all scalars } r, s).$$

Part A: $(1), (2) \Rightarrow (3)$. Using (1) and then (2),

$$L(r\,\mathbf{u} + s\,\mathbf{v}) = L(r\,\mathbf{u}) + L(s\,\mathbf{v}) = r\,L(\mathbf{u}) + s\,L(\mathbf{v}),$$

which is exactly (3).

Part B: $(3) \Rightarrow (1), (2)$.

- *Additivity (1):* Set $r = s = 1$ in (3):

$$L(\mathbf{u} + \mathbf{v}) = L(1 \cdot \mathbf{u} + 1 \cdot \mathbf{v}) = 1 \cdot L(\mathbf{u}) + 1 \cdot L(\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}).$$

- *Homogeneity (2):* First note from (3) with $r = s = 0$ that

$$L(\mathbf{0}) = L(0 \cdot \mathbf{u} + 0 \cdot \mathbf{v}) = 0 \cdot L(\mathbf{u}) + 0 \cdot L(\mathbf{v}) = \mathbf{0},$$

  so $L(\mathbf{0}) = \mathbf{0}$. Now fix any vector $\mathbf{w}$ and scalar c and take $r = c$, $s = 0$ in (3):

$$L(c\,\mathbf{w}) = L(c\,\mathbf{w} + 0 \cdot \mathbf{v}) = c\,L(\mathbf{w}) + 0 \cdot L(\mathbf{v}) = c\,L(\mathbf{w}),$$

  which is (2) (renaming $\mathbf{w}$ as $\mathbf{v}$).

Thus, (1) and (2) together are equivalent to (3).

Problem 4: Let $\vec{u}$ and $\vec{v}$ be two arbitrary vectors in a 3-dimensional vector space. Prove that

$$\vec{u} \times \vec{v} = -\,\vec{v} \times \vec{u}.$$

Solution:
Let

$$\vec{u} = \langle u_1, u_2, u_3 \rangle, \qquad \vec{v} = \langle v_1, v_2, v_3 \rangle.$$

The cross product is defined by the determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Expanding this determinant gives:

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\,\hat{\imath} - (u_1 v_3 - u_3 v_1)\,\hat{\jmath} + (u_1 v_2 - u_2 v_1)\,\hat{k}.$$

Now compute $\vec{v} \times \vec{u}$:

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (v_2 u_3 - v_3 u_2)\,\hat{\imath} - (v_1 u_3 - v_3 u_1)\,\hat{\jmath} + (v_1 u_2 - v_2 u_1)\,\hat{k}.$$

Observe that each term in $\vec{v} \times \vec{u}$ is the negative of the corresponding term in $\vec{u} \times \vec{v}$:

$$v_2 u_3 - v_3 u_2 = -(u_2 v_3 - u_3 v_2),$$

and similarly for the other components.

Hence,

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}),$$

or equivalently,

$$\boxed{\vec{u} \times \vec{v} = -\,\vec{v} \times \vec{u}}.$$

Geometric interpretation: The cross product $\vec{u} \times \vec{v}$ gives a vector perpendicular to both $\vec{u}$ and $\vec{v}$. Swapping $\vec{u}$ and $\vec{v}$ reverses the orientation of the right-hand rule, hence reversing the direction of the resulting vector.

Problem.5 Let $\vec{v} = \langle x, y, z\rangle$ and $\vec{w} = \langle z, x, y\rangle$ with $x + y + z = 0$. Show that the cosine of the angle θ between $\vec{v}$ and $\vec{w}$ is always $-1/2$.

Solution. By the dot product formula,

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}.$$

Compute

$$\vec{v} \cdot \vec{w} = xz + yx + zy = xy + yz + zx.$$

Let $S := x^2 + y^2 + z^2$. Using the constraint $x + y + z = 0$, we square both sides:

$$0 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = S + 2(xy + yz + zx).$$

Hence

$$xy + yz + zx = -\frac{S}{2}.$$

Next, observe that

$$\|\vec{v}\|^2 = x^2 + y^2 + z^2 = S \quad \text{and} \quad \|\vec{w}\|^2 = z^2 + x^2 + y^2 = S,$$

so $\|\vec{v}\| = \|\vec{w}\| = \sqrt{S}$ (provided not all of x, y, z are zero).

Therefore,

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{-\frac{S}{2}}{\sqrt{S}\,\sqrt{S}} = -\frac{1}{2}.$$

Problem.6

Consider three vectors

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

Answer the following questions:

(a) Are these three vectors linearly independent or dependent? Argue by finding a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3 = \mathbf{0}$.

(b) Characterize the subspace of $\mathbb{R}^3$ these 3 vectors lie in.

(c) Is the matrix W with columns $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ invertible?

(d) The rows of that matrix W produce three vectors:

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}.$$

From your conclusion about invertibility of W, what can you say about the subspace of $\mathbb{R}^3$ these 3 vectors lie in?

(e) Are there many combinations with $y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + y_3\mathbf{r}_3 = 0$? If yes, find two such sets of y's. If no, argue why not.

Given.

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

(a) Linear dependence. Observe

$$\mathbf{w}_1 - 2\mathbf{w}_2 + \mathbf{w}_3 = \begin{pmatrix} 1-8+7 \\ 2-10+8 \\ 3-12+9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly dependent. Equivalently, $\mathbf{w}_3 = 2\mathbf{w}_2 - \mathbf{w}_1$.

(b) Subspace spanned by the three vectors. Since $\mathbf{w}_3 \in \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, we have

$$\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\},$$

a 2-dimensional subspace (a plane through the origin). An implicit description is

$$\mathcal{P} = \{(x, y, z)^\mathsf{T} \in \mathbb{R}^3 : \; x - 2y + z = 0\}.$$

Indeed, the normal vector $\mathbf{n} = (1, -2, 1)^\mathsf{T}$ satisfies $\mathbf{n} \cdot \mathbf{w}_i = 0$ for $i = 1, 2, 3$.

(c) Invertibility of W. The columns are linearly dependent, so $\operatorname{rank}(W) = 2 < 3$ and $\det(W) = 0$. Thus W is *not* invertible.

(d) Row vectors and their subspace. The row vectors of W (written as column vectors here) are

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}.$$

Since $\operatorname{rank}(W) = \operatorname{rank}(W^\mathsf{T}) = 2$, the rows are also linearly dependent and span a 2-dimensional subspace (a plane) in $\mathbb{R}^3$. Indeed,

$$\mathbf{r}_1 - 2\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}.$$

(e) Combinations $y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + y_3\mathbf{r}_3 = \mathbf{0}$. Because the row space has dimension 2 (with 3 rows total), the space of linear dependencies has dimension 1. Hence there are infinitely many triples (y_1, y_2, y_3), all proportional to a single nonzero relation. One such relation is

$$(y_1, y_2, y_3) = (1, -2, 1),$$

and therefore any scalar multiple works, e.g.

$$(1, -2, 1), \quad (2, -4, 2) \quad (\text{or } (-1, 2, -1)).$$

Problem. 7 Suppose A is an $m \times m$ matrix and $x \in \mathbb{R}^m$. There are two ways to compute A^3x:

$$\text{Method 1: } (AAA)\,x, \qquad \text{Method 2: } A\big(A(Ax)\big).$$

Which method is faster? (Count only scalar multiplications; ignore additions.)

Solution. A naive $m \times m$ by $m \times m$ matrix multiplication costs m^3 scalar multiplications, and an $m \times m$ by $m \times 1$ matrix–vector multiplication costs m^2.

Method 1: Compute $B = AA$ (m^3 mults), then $C = BA$ (m^3 mults), then Cx (m^2 mults). Total:

$$m^3 + m^3 + m^2 \;=\; 2m^3 + m^2.$$

Method 2: Compute $y_1 = Ax$ (m^2), $y_2 = Ay_1$ (m^2), $y_3 = Ay_2$ (m^2). Total:

$$m^2 + m^2 + m^2 \;=\; 3m^2.$$

Since $2m^3 + m^2 \gg 3m^2$ for $m \geq 2$, **Method 2** is faster (they tie when $m = 1$).

Problem. 8 Let P_n be the space of degree-n polynomials in the variable t. Suppose $L : P_2 \to P_3$ is linear and satisfies

$$L(1) = 4, \qquad L(t) = t^3, \qquad L(t^2) = t - 1.$$

(a) Find $L(1 + t + t^2)$.

(b) Find $L(a + bt + ct^2)$.

(c) Find all a, b, c such that $L(a + bt + ct^2) = 1 + 3t + 2t^3$.

Solution. Since $\{1, t, t^2\}$ is a basis of P_2 and L is linear, for any $p(t) = a + bt + ct^2$ we have

$$L(p) = a\,L(1) + b\,L(t) + c\,L(t^2).$$

(a) Using linearity,

$$L(1+t+t^2) = L(1) + L(t) + L(t^2) = 4 + t^3 + (t-1) = t^3 + t + 3.$$

(b) For $p(t) = a + bt + ct^2$,

$$L(a+bt+ct^2) = a \cdot 4 + b \cdot t^3 + c \cdot (t-1) = (4a-c) + c\,t + b\,t^3.$$

(c) Set $L(a+bt+ct^2) = 1 + 3t + 2t^3$ and equate coefficients:

$$\begin{cases} \text{const:} & 4a - c = 1, \\ t: & c = 3, \\ t^3: & b = 2. \end{cases}$$

From $c = 3$ and $4a - c = 1$ we get $4a - 3 = 1 \Rightarrow a = 1$. Thus

$$\boxed{a = 1, \quad b = 2, \quad c = 3}.$$

Problem. 9 Consider the 4×4 difference equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

(a) Express x_1, x_2, x_3, x_4 in terms of b_1, b_2, b_3, b_4.

(b) Write the solution as $\mathbf{x} = S\mathbf{b}$ and hence find $S = A^{-1}$.

Solution. Writing $A\mathbf{x} = \mathbf{b}$ component wise,

$$\begin{aligned} x_1 &= b_1, \\ -x_1 + x_2 &= b_2, \\ -x_2 + x_3 &= b_3, \\ -x_3 + x_4 &= b_4. \end{aligned}$$

Back substitution gives

$$\begin{aligned} &x_1 = b_1, \\ &x_2 = b_2 + x_1 = b_1 + b_2, \\ &x_3 = b_3 + x_2 = b_1 + b_2 + b_3, \\ &x_4 = b_4 + x_3 = b_1 + b_2 + b_3 + b_4. \end{aligned}$$

Thus $\mathbf{x} = S\mathbf{b}$ with

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

One readily checks that $AS = I_4$ (or $SA = I_4$), hence $S = A^{-1}$.

Problem. 10 If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). Conclude that for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

if the rows are dependent then the columns are dependent.

Solution. Assume the two row vectors are proportional:

$$(a, b) = k\,(c, d) \qquad \text{for some } k \in \mathbb{R}.$$

Then

$$a = kc, \qquad b = kd.$$

Form the column vectors:

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} kc \\ c \end{pmatrix} = c \begin{pmatrix} k \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} kd \\ d \end{pmatrix} = d \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

Since $d \neq 0$, we can write

$$\begin{pmatrix} a \\ c \end{pmatrix} = \frac{c}{d} \begin{pmatrix} b \\ d \end{pmatrix}.$$

Thus (a, c) is a scalar multiple of (b, d), as required. Hence the columns of A are linearly dependent whenever the rows are linearly dependent.

Problem. 11 Let

$$\vec{p} = \langle 1, 2, -3 \rangle, \quad \vec{q} = \langle 2, -3, 4 \rangle, \quad \vec{r} = \langle a, 0, -1 \rangle.$$

The vector set $(\vec{p}, \vec{q}, \vec{r})$ is linearly dependent for what value of a?

Solution.
Three vectors in $\mathbb{R}^3$ are linearly dependent if the determinant of the matrix formed by using them as columns (or rows) is zero.

$$A = \begin{pmatrix} 1 & 2 & a \\ 2 & -3 & 0 \\ -3 & 4 & -1 \end{pmatrix}.$$

Compute the determinant:

$$\det(A) = 1\begin{vmatrix} -3 & 0 \\ 4 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} + a\begin{vmatrix} 2 & -3 \\ -3 & 4 \end{vmatrix}.$$

Now calculate each 2×2 determinant:

$$\begin{vmatrix} -3 & 0 \\ 4 & -1 \end{vmatrix} = (-3)(-1) - (0)(4) = 3,$$

$$\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} = (2)(-1) - (0)(-3) = -2,$$

$$\begin{vmatrix} 2 & -3 \\ -3 & 4 \end{vmatrix} = (2)(4) - (-3)(-3) = 8 - 9 = -1.$$

Substitute these into the determinant expansion:

$$\det(A) = 1(3) - 2(-2) + a(-1) = 3 + 4 - a = 7 - a.$$

For linear dependence, $\det(A) = 0$, so:

$$7 - a = 0 \quad \Rightarrow \quad \boxed{a = 7.}$$

Conclusion: The vectors $\vec{p}, \vec{q}, \vec{r}$ are linearly dependent when $a = 7$.

Problem. 12 Consider the subspace $S = \{\langle x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ and two bases

$$B = \{\langle 2, 0, -2\rangle,\ \langle 0, 3, -3\rangle\}, \qquad C = \{\langle 1, 1, -2\rangle,\ \langle -2, 1, 1\rangle\}.$$

Given $\vec{u} = \langle -2, 5, -3\rangle$ and $\vec{v} = \langle 3, 3, -6\rangle$ in S:

(a) Compute $\text{Coords}(5\vec{u} - 3\vec{v}, B)$ and $\text{Coords}(5\vec{u} - 3\vec{v}, C)$.

(b) An orthogonal basis for S is $O = \{\langle 1, -1, 0\rangle,\ \langle 1, 1, -2\rangle\}$. Form an orthonormal basis N from O and compute $\text{Coords}(\vec{u}, N)$ and $\text{Coords}(\vec{v}, N)$.

Solution.
(a) Coordinates in B and C. First,

$$5\vec{u} - 3\vec{v} = 5\langle -2, 5, -3\rangle - 3\langle 3, 3, -6\rangle = \langle -10, 25, -15\rangle - \langle 9, 9, -18\rangle = \langle -19, 16, 3\rangle.$$

In basis B: find α, β with

$$\alpha\langle 2, 0, -2\rangle + \beta\langle 0, 3, -3\rangle = \langle -19, 16, 3\rangle.$$

From components: $2\alpha = -19 \Rightarrow \alpha = -\frac{19}{2}$, $3\beta = 16 \Rightarrow \beta = \frac{16}{3}$, and the z-entry checks: $-2\alpha - 3\beta = 19 - 16 = 3$. Thus

$$\boxed{\text{Coords}(5\vec{u} - 3\vec{v}, B) = \begin{pmatrix} -\frac{19}{2} \\ \frac{16}{3} \end{pmatrix}}.$$

In basis C: find γ, δ with

$$\gamma\langle 1, 1, -2\rangle + \delta\langle -2, 1, 1\rangle = \langle -19, 16, 3\rangle,$$

i.e.

$$\begin{cases} \gamma - 2\delta = -19, \\ \gamma + \delta = 16, \\ -2\gamma + \delta = 3. \end{cases}$$

Solving: $\delta = \frac{35}{3}$, $\gamma = \frac{13}{3}$ (and the third equation is satisfied). Hence

$$\boxed{\text{Coords}(5\vec{u} - 3\vec{v}, C) = \begin{pmatrix} \frac{13}{3} \\ \frac{35}{3} \end{pmatrix}}.$$

(b) Orthonormal basis and coordinates. The given orthogonal basis is

$$\vec{o}_1 = \langle 1, -1, 0\rangle, \qquad \vec{o}_2 = \langle 1, 1, -2\rangle,$$

with norms $\|\vec{o}_1\| = \sqrt{2}$ and $\|\vec{o}_2\| = \sqrt{6}$. Thus an orthonormal basis is

$$N = \left\{ \vec{n}_1 = \frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle, \quad \vec{n}_2 = \frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle \right\}.$$

For an orthonormal basis, coordinates are dot products:

$$\text{Coords}(\vec{u}, N) = \begin{pmatrix} \vec{u}\cdot\vec{n}_1 \\ \vec{u}\cdot\vec{n}_2 \end{pmatrix}, \qquad \text{Coords}(\vec{v}, N) = \begin{pmatrix} \vec{v}\cdot\vec{n}_1 \\ \vec{v}\cdot\vec{n}_2 \end{pmatrix}.$$

Compute:

$$\vec{u}\cdot\vec{n}_1 = \frac{-2-5}{\sqrt{2}} = -\frac{7}{\sqrt{2}}, \qquad \vec{u}\cdot\vec{n}_2 = \frac{-2+5+6}{\sqrt{6}} = \frac{9}{\sqrt{6}},$$

$$\vec{v}\cdot\vec{n}_1 = \frac{3-3}{\sqrt{2}} = 0, \qquad \vec{v}\cdot\vec{n}_2 = \frac{3+3+12}{\sqrt{6}} = \frac{18}{\sqrt{6}} = 3\sqrt{6}.$$

Therefore,

$$\boxed{\text{Coords}(\vec{u}, N) = \begin{pmatrix} -\dfrac{7}{\sqrt{2}} \\ \dfrac{9}{\sqrt{6}} \end{pmatrix}, \qquad \text{Coords}(\vec{v}, N) = \begin{pmatrix} 0 \\ 3\sqrt{6} \end{pmatrix}}.$$

Problem. 13 Let $V = \{\vec{u}, \vec{w}\}$, where

$$\vec{u} = \langle 1, 2, 0 \rangle, \qquad \vec{w} = \langle 0, -3, 1 \rangle.$$

(a) Describe $\text{Span}(V)$.

(b) Determine whether $\langle 2, 1, 1 \rangle$ belongs to $\text{Span}(V)$, and justify your answer.

Solution.
(a) Description of $\text{Span}(V)$**.**
The span of V consists of all linear combinations of $\vec{u}$ and $\vec{w}$:

$$\text{Span}(V) = \{\, \alpha\vec{u}+\beta\vec{w} \mid \alpha, \beta \in \mathbb{R} \,\} = \{\alpha\langle 1, 2, 0\rangle + \beta\langle 0, -3, 1\rangle = \langle \alpha,\, 2\alpha - 3\beta,\, \beta \rangle \mid \alpha, \beta \in \mathbb{R}\}.$$

Hence

$$\text{Span}(V) = \{\langle x, y, z\rangle \in \mathbb{R}^3 \mid y = 2x - 3z\},$$

which is a plane through the origin.

(b) Test for membership.
We ask whether $\langle 2, 1, 1 \rangle$ satisfies $y = 2x - 3z$:

$$2x - 3z = 2(2) - 3(1) = 4 - 3 = 1,$$

which matches the y-coordinate. Therefore $\langle 2, 1, 1 \rangle \in \text{Span}(V)$.

Alternatively, we can explicitly find α, β:

$$\langle 2, 1, 1 \rangle = \alpha\langle 1, 2, 0 \rangle + \beta\langle 0, -3, 1 \rangle \Rightarrow \begin{cases} \alpha = 2, \\ 2\alpha - 3\beta = 1, \\ \beta = 1. \end{cases}$$

The second equation is satisfied: $2(2) - 3(1) = 1$, confirming the solution $\alpha = 2, \beta = 1$.

Conclusion.

$$\boxed{\text{Span}(V) = \{\langle x, y, z\rangle \mid y = 2x - 3z\}, \quad \langle 2, 1, 1\rangle \in \text{Span}(V).}$$

Problem. 14 Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} A \\ 2A \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 1 \\ 2 & 4 \\ 8 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} A & 2A \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 4 & 1 & 8 & 2 \end{pmatrix}.$$

Describe $\mathcal{N}(B)$ and $\mathcal{N}(C)$.

Solution.
First note

$$\det(A) = 1 \cdot 1 - 2 \cdot 4 = -7 \neq 0 \Rightarrow A \text{ is invertible.}$$

Null space of B. For $x \in \mathbb{R}^2$,

$$Bx = \begin{pmatrix} A \\ 2A \end{pmatrix} x = \begin{pmatrix} Ax \\ 2Ax \end{pmatrix} = 0 \iff Ax = 0.$$

Since A is invertible, $Ax = 0 \Rightarrow x = 0$. Hence

$$\boxed{\mathcal{N}(B) = \{0\}} \qquad \text{(nullity 0).}$$

Null space of C. For $x \in \mathbb{R}^4$, write $x = \binom{u}{v}$ with $u, v \in \mathbb{R}^2$. Then

$$Cx = Au + 2Av = A(u + 2v) = 0.$$

Invertibility of A gives $u + 2v = 0$, i.e. $u = -2v$. Thus

$$\mathcal{N}(C) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^4 : \ u = -2v \right\} = \left\{ \begin{pmatrix} -2v \\ v \end{pmatrix} \ : \ v \in \mathbb{R}^2 \right\}.$$

A convenient basis is obtained by taking $v = e_1 = (1, 0)^\mathsf{T}$ and $v = e_2 = (0, 1)^\mathsf{T}$:

$$\boxed{\mathcal{N}(C) = \operatorname{span} \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}} \qquad \text{(nullity 2).}$$

Problem. 15 Find a basis for the null space of

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{pmatrix}.$$

Solution. Given

$$A = \begin{pmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

write the system:

$$\begin{cases} x_1 - 2x_2 + 4x_4 = 0 \quad \text{(I)} \\ 3x_1 + x_2 + x_3 = 0 \qquad \text{(II)} \\ -x_1 - 5x_2 - x_3 + 8x_4 = 0 \quad \text{(III)} \end{cases}$$

From (I): $x_1 = 2x_2 - 4x_4$. Substitute into (II):

$$3(2x_2 - 4x_4) + x_2 + x_3 = 0 \implies 7x_2 + x_3 - 12x_4 = 0 \implies x_2 = \frac{-x_3 + 12x_4}{7}.$$

Then

$$x_1 = 2x_2 - 4x_4 = 2 \cdot \frac{-x_3 + 12x_4}{7} - 4x_4 = \frac{-2x_3 - 4x_4}{7}.$$

(Equation (III) is automatically satisfied by these expressions, so no further constraint.)

Let $x_3 = s$ and $x_4 = t$ be free. Then

$$x_1 = -\frac{2}{7}s - \frac{4}{7}t, \qquad x_2 = -\frac{1}{7}s + \frac{12}{7}t, \qquad x_3 = s, \qquad x_4 = t.$$

Thus

$$\mathbf{x} = s \begin{pmatrix} -\frac{2}{7} \\ -\frac{1}{7} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{4}{7} \\ \frac{12}{7} \\ 0 \\ 1 \end{pmatrix}.$$

Multiplying by 7 for an integer basis, a basis for the null space is

$$\boxed{\mathcal{N}(A) = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 12 \\ 0 \\ 7 \end{pmatrix} \right\}.}$$

Problem. 16 Suppose we have a system of equations

$$A\vec{x} = \vec{b}$$

for some $n \times d$ matrix A. We multiply both sides by a nonzero $m \times n$ matrix B, giving the new system

$$BA\vec{x} = B\vec{b}.$$

Provide:

(i) An example showing that the two systems need not have identical solution sets.

(ii) A general description of how their solution sets are related.

(iii) A sufficient condition on B under which the solution sets are identical.

Solution.
(i) Example where solution sets differ.
Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Then

$$A\vec{x} = \vec{b} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x_1 = 1, \text{ with } x_2 \text{ arbitrary.}$$

So the solution set of $A\vec{x} = \vec{b}$ is

$$S_1 = \left\{ \begin{pmatrix} 1 \\ t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Now, multiplying by B gives

$$BA\vec{x} = B\vec{b} \quad \Rightarrow \quad (0,1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} = (0,1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

That is,

$$(0,1) \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0 \Rightarrow 0 = 0,$$

so every $\vec{x} \in \mathbb{R}^2$ satisfies the equation. Thus the new system's solution set is

$$S_2 = \mathbb{R}^2.$$

Hence $S_1 \subsetneq S_2$; the solution sets differ.

(ii) General relationship between the solution sets.
The equation $BA\vec{x} = B\vec{b}$ is obtained by applying B to both sides of $A\vec{x} = \vec{b}$. If $\vec{x}$ satisfies $A\vec{x} = \vec{b}$, then it automatically satisfies $BA\vec{x} = B\vec{b}$. Thus:

$$\boxed{\text{All solutions of } A\vec{x} = \vec{b} \text{ are also solutions of } BA\vec{x} = B\vec{b},}$$

i.e. the original solution set S_1 is always contained in the new one S_2:

$$S_1 \subseteq S_2.$$

But the reverse inclusion need not hold, since multiplying by B can *destroy* information (e.g. if B has nontrivial null space).

(iii) Sufficient condition for identical solution sets.
If B is *invertible* (in particular, if $m = n$ and $\det(B) \neq 0$), then multiplying by B is a one-to-one operation:

$$BA\vec{x} = B\vec{b} \iff A\vec{x} = B^{-1}B\vec{b} = \vec{b}.$$

Thus in this case the two systems are equivalent, and their solution sets are identical.

More generally, the solution sets coincide if $\text{null}(B) = \{\vec{0}\}$, i.e. B has full column rank.

Summary.

$$\boxed{\begin{array}{l}\text{Always: } S(A,\vec{b}) \subseteq S(BA, B\vec{b}),\\ \text{Identical if } B \text{ has full column rank (e.g. is invertible).}\end{array}}$$

Problem. 17 Use Gaussian elimination to solve

$$\begin{cases} x + 3y + 2z = 2, \\ 2x + 7y + 7z = -1, \\ 2x + 5y + 2z = 7. \end{cases}$$

Solution. Write the augmented matrix and eliminate:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 2 & 7 & 7 & -1 \\ 2 & 5 & 2 & 7 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right].$$

Next eliminate in the third row:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -2 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Back substitution gives $z = -2$. From the second row, $y + 3z = -5 \Rightarrow y + 3(-2) = -5 \Rightarrow y = 1$. From the first row, $x + 3y + 2z = 2 \Rightarrow x + 3(1) + 2(-2) = 2 \Rightarrow x = 3$.

Optionally, continue to reduced form:

$$R_2 \leftarrow R_2 - 3R_3, \quad R_1 \leftarrow R_1 - 2R_3, \quad R_1 \leftarrow R_1 - 3R_2 \ \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

$$\boxed{(x, y, z) = (3,\ 1,\ -2).}$$

Problem. 18 A consistent linear system $A\vec{x} = \vec{b}$ (with $A \in \mathbb{R}^{n\times d}$) has at least one solution. Assume A has linearly independent rows (i.e., $\mathrm{rank}(A) = n \le d$). Let $\vec{v}$ be any solution to $A\vec{x} = \vec{b}$, and let the *right-inverse* solution be

$$\vec{v}_r \; = \; A^\top (AA^\top)^{-1} \vec{b},$$

so that $AA^\top$ is invertible and $AA^\top(AA^\top)^{-1} = I_n$. Show that

$$\|\vec{v}\|^2 = \|\vec{v} - \vec{v}_r\|^2 + \|\vec{v}_r\|^2 + 2\,\vec{v}_r^\top(\vec{v} - \vec{v}_r) \; \ge \; \|\vec{v}_r\|^2 + 2\,\vec{v}_r^\top(\vec{v} - \vec{v}_r),$$

and then prove $\vec{v}_r^\top(\vec{v} - \vec{v}_r) = 0$, hence $\|\vec{v}\|^2 \ge \|\vec{v}_r\|^2$.

Solution.

Step 1: Norm identity and inequality. Decompose $\vec{v} = (\vec{v} - \vec{v}_r) + \vec{v}_r$ and expand:

$$\|\vec{v}\|^2 = \|(\vec{v} - \vec{v}_r) + \vec{v}_r\|^2 = \|\vec{v} - \vec{v}_r\|^2 + \|\vec{v}_r\|^2 + 2\,\vec{v}_r^\top(\vec{v} - \vec{v}_r).$$

Since $\|\vec{v} - \vec{v}_r\|^2 \ge 0$, this implies

$$\|\vec{v}\|^2 \; \ge \; \|\vec{v}_r\|^2 + 2\,\vec{v}_r^\top(\vec{v} - \vec{v}_r).$$

Step 2: Show $\vec{v}_r^\top(\vec{v} - \vec{v}_r) = 0$. First, because $B := AA^\top$ is invertible and $\vec{v}_r = A^\top B^{-1}\vec{b}$, we have

$$A\vec{v}_r \; = \; AA^\top B^{-1}\vec{b} \; = \; \vec{b},$$

so $\vec{v}_r$ is a solution to $A\vec{x} = \vec{b}$. Hence for any solution $\vec{v}$,

$$A(\vec{v} - \vec{v}_r) \; = \; A\vec{v} - A\vec{v}_r \; = \; \vec{b} - \vec{b} \; = \; \vec{0},$$

so $\vec{v} - \vec{v}_r \in \mathcal{N}(A)$ (the null space of A).

Next, by construction $\vec{v}_r \in \mathrm{col}(A^\top)$ (the column space of $A^\top$), i.e., the *row space* of A. For any matrix A, the orthogonal decomposition theorem gives

$$\mathrm{col}(A^\top) \; = \; \mathcal{N}(A)^\perp.$$

Therefore every vector in $\mathrm{col}(A^\top)$ is orthogonal to every vector in $\mathcal{N}(A)$. Consequently,

$$\vec{v}_r^\top(\vec{v} - \vec{v}_r) = 0.$$

Step 3: Minimal norm among all solutions. Plugging the orthogonality into the identity of Step 1 yields

$$\|\vec{v}\|^2 = \|\vec{v} - \vec{v}_r\|^2 + \|\vec{v}_r\|^2 \; \ge \; \|\vec{v}_r\|^2,$$

with equality iff $\vec{v} = \vec{v}_r$. Thus among all solutions of $A\vec{x} = \vec{b}$, the right-inverse solution $\vec{v}_r$ has minimal Euclidean norm.

Problem. 19 For the augmented matrix

$$[\, A \mid \mathbf{b}\,] = \begin{pmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & e \\ 0 & 0 & d & c \end{pmatrix},$$

determine for which values of a, e, c, d the system has (i) no solution, (ii) infinitely many solutions.

Solution. This is already in (upper) row-echelon form, corresponding to

$$\begin{cases} x_1 + 2x_2 + 3x_3 = a, \\ 4x_2 + 5x_3 = e, \\ d\,x_3 = c. \end{cases}$$

- If $d \neq 0$, then $x_3 = \dfrac{c}{d}$, hence x_2 and x_1 are determined uniquely from the first two equations.

$$\Rightarrow \text{Unique solution for all } a, e, c \text{ when } d \neq 0.$$

- If $d = 0$, the last equation becomes $0 = c$.
  - If $c \neq 0$, we get $0 = c$ (impossible) $\Rightarrow$ **no solution**.
  - If $c = 0$, the last equation imposes no constraint on x_3 (free variable). The first two equations (with nonzero pivots 1 and 4) determine x_2 and x_1 in terms of x_3.

$$\Rightarrow \textbf{infinitely many solutions}.$$

Answer.

$$\boxed{\text{No solution} \iff d = 0 \text{ and } c \neq 0; \qquad \text{Infinitely many solutions} \iff d = 0 \text{ and } c = 0.}$$

(The values of a and e do not affect consistency, since the pivots in the first two rows are nonzero.)

Problem. 20 Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 + 10^{-10} & 1 - 10^{-10} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 - 10^{10} & 10^{10} \\ 1 + 10^{10} & -10^{10} \end{pmatrix}.$$

(a) Compute $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$.

(b) Compute the condition number $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$.

(c) Decide if A is ill–conditioned (i.e. $\kappa_\infty(A) > 1$).

(d) For $A\vec{x} = \vec{b}$ with $\vec{b} = \langle b_1, b_1\rangle$, and perturbed system $A(\vec{x} + \Delta\vec{x}) = \vec{b} + \Delta\vec{b}$ with $\Delta\vec{b} = \langle \Delta b_1, \Delta b_2\rangle$, express $\Delta\vec{x}$ in terms of $\Delta\vec{b}$.

(e) Take $\vec{b} = \langle 1, 1\rangle$ and $\Delta\vec{b} = \langle 10^{-5}, -10^{-5}\rangle$. Compute $\Delta\vec{x}$.

(f) Verify the standard perturbation bound

$$\frac{\|\Delta\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \;\le\; \kappa_\infty(A)\,\frac{\|\Delta\vec{b}\|_\infty}{\|\vec{b}\|_\infty}.$$

Solution.
(a) $\|A\|_\infty$. Row sums of A:

$$\text{row}_1: \ \tfrac{1}{2}(|1| + |1|) = 1, \qquad \text{row}_2: \ \tfrac{1}{2}\big(|1 + 10^{-10}| + |1 - 10^{-10}|\big) = \tfrac{1}{2}(2) = 1.$$

Hence $\boxed{\|A\|_\infty = 1}$.

(b) $\|A^{-1}\|_\infty$ and $\kappa_\infty(A)$. Row sums of A^{-1}:

$$\text{row}_1: \ |1 - 10^{10}| + |10^{10}| = (10^{10} - 1) + 10^{10} = 2\cdot 10^{10} - 1,$$

$$\text{row}_2: \ |1 + 10^{10}| + |-10^{10}| = (10^{10} + 1) + 10^{10} = 2\cdot 10^{10} + 1.$$

Thus

$$\boxed{\|A^{-1}\|_\infty = 2\cdot 10^{10} + 1}, \qquad \boxed{\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2\cdot 10^{10} + 1}.$$

(c) Ill–conditioning. Since $\kappa_\infty(A) = 2\cdot 10^{10} + 1 \gg 1$, $\boxed{A \text{ is ill–conditioned.}}$

(d) Perturbation $\Delta\vec{x}$ in terms of $\Delta\vec{b}$. From $A(\vec{x} + \Delta\vec{x}) = \vec{b} + \Delta\vec{b}$ and $A\vec{x} = \vec{b}$,

$$A\,\Delta\vec{x} = \Delta\vec{b} \quad\Rightarrow\quad \boxed{\Delta\vec{x} = A^{-1}\Delta\vec{b}}.$$

With the given A^{-1} and $\Delta\vec{b} = \binom{\Delta b_1}{\Delta b_2}$,

$$\Delta x_1 = (1 - 10^{10})\,\Delta b_1 + 10^{10}\,\Delta b_2, \qquad \Delta x_2 = (1 + 10^{10})\,\Delta b_1 - 10^{10}\,\Delta b_2.$$

(e) Numerical perturbation. For $\vec{b} = \binom{1}{1}$, the (unperturbed) solution is

$$\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} (1 - 10^{10}) + 10^{10} \\ (1 + 10^{10}) - 10^{10} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \|\vec{x}\|_\infty = 1.$$

With $\Delta\vec{b} = \binom{10^{-5}}{-10^{-5}}$,

$\Delta x_1 = (1 - 10^{10}) \cdot 10^{-5} + 10^{10} \cdot (-10^{-5}) = 10^{-5} - 10^5 - 10^5 = -2 \cdot 10^5 + 10^{-5}$,

$\Delta x_2 = (1 + 10^{10}) \cdot 10^{-5} - 10^{10} \cdot (-10^{-5}) = 10^{-5} + 10^5 + 10^5 = 2 \cdot 10^5 + 10^{-5}$.

Therefore

$$\boxed{\Delta\vec{x} = \begin{pmatrix} -200000 + 10^{-5} \\ 200000 + 10^{-5} \end{pmatrix}}, \qquad \|\Delta\vec{x}\|_\infty = 200000 + 10^{-5}.$$

Also $\|\Delta\vec{b}\|_\infty = 10^{-5}$ and $\|\vec{b}\|_\infty = 1$.

(f) Verify the perturbation bound.

$$\frac{\|\Delta\vec{x}\|_\infty}{\|\vec{x}\|_\infty} = 200000+10^{-5} \quad \text{and} \quad \kappa_\infty(A)\,\frac{\|\Delta\vec{b}\|_\infty}{\|\vec{b}\|_\infty} = (2\cdot 10^{10}+1)\cdot 10^{-5} = 200000+10^{-5}.$$

Hence

$$\boxed{\frac{\|\Delta\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \;\le\; \kappa_\infty(A)\,\frac{\|\Delta\vec{b}\|_\infty}{\|\vec{b}\|_\infty}}$$

holds with equality in this example.

Problem. 21 Let T_θ denote counterclockwise rotation about the origin by angle θ in $\mathbb{R}^2$, and likewise T_ϕ by ϕ. Geometrically, $T_\theta \circ T_\phi = T_{\theta+\phi}$. Using the standard rotation matrices, prove:

(a) $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$, (b) $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Solution. The matrix of rotation by α is

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Composition corresponds to matrix multiplication, so

$$R(\theta)\,R(\phi) = R(\theta+\phi).$$

Compute the left-hand side:

$$R(\theta)\,R(\phi) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}.$$

On the other hand,

$$R(\theta+\phi) = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}.$$

By equality of matrices, corresponding entries agree, giving

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$

which are exactly the desired identities.

Problem. 22 In $\mathbb{R}^3$, find the intersection of the plane

$$\{\vec{x} \mid \vec{x}\cdot\langle 2,1,2\rangle = 7\}$$

with the line

$$\{\ \langle 5,3,2\rangle + t\langle 1,1,1\rangle \mid t\in\mathbb{R}\ \}.$$

Solution. A generic point on the line is

$$\vec{x}(t) = \langle 5+t,\ 3+t,\ 2+t\rangle.$$

Impose the plane condition $\vec{x}(t)\cdot\langle 2,1,2\rangle = 7$:

$$(5+t,\ 3+t,\ 2+t)\cdot(2,1,2) = 2(5+t)+1(3+t)+2(2+t) = 10+2t+3+t+4+2t = 17+5t.$$

Set equal to 7:

$$17+5t = 7 \ \Rightarrow\ 5t = -10 \ \Rightarrow\ t = -2.$$

Therefore the intersection point is

$$\vec{x}(-2) = \langle 5-2,\ 3-2,\ 2-2\rangle = \boxed{\langle 3,1,0\rangle}.$$

Problem. 23 Let $c = (2,5)$ and $d = (5,5)$. Rotate d counterclockwise by 165° about c.

(a) Natural (Cartesian) coordinates via three transformations.
A rotation about c is achieved by the sequence:

(i) Translate by $-c$, (ii) Rotate by 165° about the origin, (iii) Translate by $+c$.

Apply to d:

(i) Translate to the origin:

$$d - c = (5,5) - (2,5) = (3,0).$$

(ii) Rotate $(3,0)$ by 165°:

$$R_{165^\circ}\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}3\cos 165^\circ\\3\sin 165^\circ\end{pmatrix}.$$

(iii) Translate back by c:

$$d_{\text{new}} = c + R_{165^\circ}(d-c) = \big(2 + 3\cos 165^\circ,\ 5 + 3\sin 165^\circ\big).$$

Using exact values

$$\cos 165^\circ = \cos(180^\circ - 15^\circ) = -\cos 15^\circ = -\frac{\sqrt{6}+\sqrt{2}}{4}, \qquad \sin 165^\circ = \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4},$$

we obtain

$$\boxed{d_{\text{new}} = \left(2 - \frac{3(\sqrt{6}+\sqrt{2})}{4},\ 5 + \frac{3(\sqrt{6}-\sqrt{2})}{4}\right).}$$

(Numerically $\approx (-0.8978,\ 5.7765)$.)

(b) Homogeneous coordinates as one matrix.
In homogeneous coordinates (column vectors), the translation by (t_x, t_y) is

$$T(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The overall transformation (translate to origin, rotate, translate back) is

$$M \;=\; T(2,5)\,R(165^\circ)\,T(-2,-5).$$

Multiplying,

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & -2\cos\theta + 5\sin\theta + 2 \\ \sin\theta & \cos\theta & -2\sin\theta - 5\cos\theta + 5 \\ 0 & 0 & 1 \end{pmatrix}_{\theta = 165^\circ}.$$

Apply M to d in homogeneous form $\tilde{d} = (5, 5, 1)^\mathsf{T}$:

$$\tilde{d}_{\text{new}} = M\,\tilde{d} = \begin{pmatrix} 2 + 3\cos 165^\circ \\ 5 + 3\sin 165^\circ \\ 1 \end{pmatrix}.$$

Thus the homogeneous coordinates of the new location are

$$\boxed{\tilde{d}_{\text{new}} = \big(2 + 3\cos 165^\circ,\ 5 + 3\sin 165^\circ,\ 1\big)^\mathsf{T}}$$

which agrees with the result from part (a).

Problem. 24 Are the four points

$$P_1 = \langle 1,3,4\rangle, \quad P_2 = \langle 0,1,2\rangle, \quad P_3 = \langle 2,1,3\rangle, \quad P_4 = \langle 7,6,1\rangle$$

coplanar? Justify your answer.

Solution. Four points in $\mathbb{R}^3$ are coplanar iff the three vectors from one point to the others are linearly dependent, equivalently iff their scalar triple product is zero.

Take P_1 as the base point. Form

$$\vec{v}_2 = P_2 - P_1 = \langle -1,-2,-2\rangle, \quad \vec{v}_3 = P_3 - P_1 = \langle 1,-2,-1\rangle, \quad \vec{v}_4 = P_4 - P_1 = \langle 6,3,-3\rangle.$$

Compute the scalar triple product (determinant):

$$\det\begin{pmatrix} -1 & 1 & 6 \\ -2 & -2 & 3 \\ -2 & -1 & -3 \end{pmatrix} = -33 \neq 0.$$

Since the scalar triple product is nonzero, the three vectors are linearly independent, so the four points are *not* coplanar.

$$\boxed{\text{The points are not coplanar.}}$$

Problem. 25 Let C be a Cartesian coordinate system. Let $p = (4,7)$ and $q = (2,5)$ in C. Let D be an orthogonal coordinate system with the same unit length as C, with origin at q, and whose axes are obtained by rotating the axes of C counterclockwise by $\theta = 0.3$ radians. Find the coordinates of p with respect to D.

Solution. To express p in the frame D:

1. Translate so that q becomes the origin: $p - q = (4-2,\ 7-5) = (2,2)$.
2. Since D's axes are rotated by $+\theta$ relative to C, to get D-coordinates we rotate the vector by the inverse rotation $-\theta$:

$$R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus

$$[p]_D = R(-\theta)\,(p-q) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(\cos\theta + \sin\theta) \\ 2(\cos\theta - \sin\theta) \end{pmatrix}.$$

With $\theta = 0.3$ (radians),

$$\boxed{[p]_D = \big(2(\cos 0.3 + \sin 0.3),\ 2(\cos 0.3 - \sin 0.3)\big).}$$

Numerically (to 4 d.p.): $\cos 0.3 \approx 0.9553$, $\sin 0.3 \approx 0.2955$, hence

$$\boxed{[p]_D \approx (2.5017,\ 1.3196).}$$

Problem. 26 Compute the determinant of

$$A = \begin{pmatrix} 4 & 4 & -1 \\ 2 & -3 & 0 \\ -1 & 2 & -1 \end{pmatrix}.$$

Solution.
We expand along the first row:

$$\det(A) = 4 \begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 4 \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}.$$

Compute each 2×2 determinant:

$$\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} = (-3)(-1) - (0)(2) = 3,$$

$$\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = (2)(-1) - (0)(-1) = -2,$$

$$\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-3)(-1) = 4 - 3 = 1.$$

Substitute these into the expansion:

$$\det(A) = 4(3) - 4(-2) + (-1)(1) = 12 + 8 - 1 = 19.$$

Answer.

$$\boxed{\det(A) = 19.}$$

Problem. 27 Using Cramer's Rule, find y for

$$\begin{cases} 2x - 37 + 4z = 1, \\ 4x + 9y - 0z = 0, \\ 7x - 2y + 5z = 0. \end{cases}$$

Solution. Write $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 4 & 9 & 0 \\ 7 & -2 & 5 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 38 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Cramer's Rule gives $y = \dfrac{\det A_y}{\det A}$, where A_y is A with its second column replaced by $\vec{b}$:

$$A_y = \begin{pmatrix} 2 & 38 & 4 \\ 4 & 0 & 0 \\ 7 & 0 & 5 \end{pmatrix}.$$

Compute $\det A$*:*

$$\det A = 2\begin{vmatrix} 9 & 0 \\ -2 & 5 \end{vmatrix} + 4\begin{vmatrix} 4 & 9 \\ 7 & -2 \end{vmatrix} = 2(45) + 4(-71) = 90 - 284 = -194.$$

Compute $\det A_y$*:* expand along row 2,

$$\det A_y = 4 \cdot (-1)^{2+1} \begin{vmatrix} 38 & 4 \\ 0 & 5 \end{vmatrix} = -4(38 \cdot 5 - 4 \cdot 0) = -4 \cdot 190 = -760.$$

Therefore

$$\boxed{y = \frac{\det A_y}{\det A} = \frac{-760}{-194} = \frac{380}{97} \approx 3.9175.}$$

Problem. 28 Compute A^{-1} using the cofactor/adjugate formula for

$$A = \begin{pmatrix} 6 & 2 & -1 \\ -5 & 3 & 7 \\ 4 & 6 & -8 \end{pmatrix}.$$

Solution. The inverse is given by

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A), \qquad \mathrm{adj}(A) = \big(C_{ij}\big)^{\top},$$

where $C_{ij} = (-1)^{i+j} \det M_{ij}$ are cofactors and M_{ij} are the 2×2 minors.

Step 1: Determinant.

$$\det A = 6\begin{vmatrix} 3 & 7 \\ 6 & -8 \end{vmatrix} - 2\begin{vmatrix} -5 & 7 \\ 4 & -8 \end{vmatrix} - 1\begin{vmatrix} -5 & 3 \\ 4 & 6 \end{vmatrix} = 6(-66) - 2(4) - 1(-42) = -378.$$

Step 2: Cofactor matrix $C = (C_{ij})$. Compute representative cofactors (all others similarly):

$$C_{11} = (+)\det\begin{vmatrix}3 & 7\\ 6 & -8\end{vmatrix} = -66, \quad C_{12} = (-)\det\begin{vmatrix}-5 & 7\\ 4 & -8\end{vmatrix} = -12, \quad C_{13} = (+)\det\begin{vmatrix}-5 & 3\\ 4 & 6\end{vmatrix} = -42,$$

$$C_{21} = (-)\det\begin{vmatrix}2 & -1\\ 6 & -8\end{vmatrix} = 10, \qquad C_{22} = (+)\det\begin{vmatrix}6 & -1\\ 4 & -8\end{vmatrix} = -44, \qquad C_{23} = (-)\det\begin{vmatrix}6 & 2\\ 4 & 6\end{vmatrix} = -28,$$

$$C_{31} = (+)\det\begin{vmatrix}2 & -1\\ 3 & 7\end{vmatrix} = 17, \qquad C_{32} = (-)\det\begin{vmatrix}6 & -1\\ -5 & 7\end{vmatrix} = -37, \quad C_{33} = (+)\det\begin{vmatrix}6 & 2\\ -5 & 3\end{vmatrix} = 28.$$

Hence

$$C = \begin{pmatrix} -66 & -12 & -42\\ 10 & -44 & -28\\ 17 & -37 & 28\end{pmatrix}.$$

Step 3: Adjugate and inverse.

$$\operatorname{adj}(A) = C^{\top} = \begin{pmatrix} -66 & 10 & 17\\ -12 & -44 & -37\\ -42 & -28 & 28\end{pmatrix}, \qquad A^{-1} = \frac{1}{-378}\operatorname{adj}(A).$$

Thus

$$A^{-1} = \begin{pmatrix} \frac{11}{63} & -\frac{5}{189} & -\frac{17}{378}\\ \frac{2}{63} & \frac{22}{189} & \frac{37}{378}\\ \frac{1}{9} & \frac{2}{27} & -\frac{2}{27}\end{pmatrix}.$$

Answer. $A^{-1} = \begin{pmatrix} 11/63 & -5/189 & -17/378\\ 2/63 & 22/189 & 37/378\\ 1/9 & 2/27 & -2/27\end{pmatrix}.$

Problem. 29 Compute $\det A$ using cofactors for

$$A = \begin{pmatrix} -1 & 0 & -2 & 1\\ -2 & 3 & 0 & 0\\ 1 & 1 & 0 & 1\\ 0 & -1 & 0 & -3\end{pmatrix}.$$

Solution. Expand along the third column (which has three zeros):

$$\det A = a_{13}\,C_{13} = (-2)\cdot(-1)^{1+3}\,\det M_{13} = (-2)\cdot\det M_{13},$$

where M_{13} is obtained by deleting row 1 and column 3:

$$M_{13} = \begin{pmatrix} -2 & 3 & 0\\ 1 & 1 & 1\\ 0 & -1 & -3\end{pmatrix}.$$

Compute $\det M_{13}$ by expanding along its first row:

$$\det M_{13} = (-2)\det\begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} - 3\det\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} + 0\cdot(\cdots).$$

Evaluate the 2×2 determinants:

$$\det\begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} = 1\cdot(-3) - 1\cdot(-1) = -3+1 = -2, \qquad \det\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} = 1\cdot(-3) - 1\cdot 0 = -3.$$

Thus

$$\det M_{13} = (-2)(-2) - 3(-3) = 4 + 9 = 13.$$

Finally,

$$\boxed{\det A = (-2)\cdot 13 = -26.}$$

Problem. 30 For what value of x will the determinant of the matrix B be zero?

$$B = \begin{pmatrix} 7 & 6 & 0 & 1 \\ 5 & 4 & x & 0 \\ 8 & 7 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Solution.
We compute $\det(B)$ by expanding along the last row (which has many zeros):

$$\det(B) = 1\cdot(-1)^{4+3}\begin{vmatrix} 7 & 6 & 1 \\ 5 & 4 & 0 \\ 8 & 7 & 1 \end{vmatrix} + 1\cdot(-1)^{4+4}\begin{vmatrix} 7 & 6 & 0 \\ 5 & 4 & x \\ 8 & 7 & 0 \end{vmatrix}.$$

That is,

$$\det(B) = -\begin{vmatrix} 7 & 6 & 1 \\ 5 & 4 & 0 \\ 8 & 7 & 1 \end{vmatrix} + \begin{vmatrix} 7 & 6 & 0 \\ 5 & 4 & x \\ 8 & 7 & 0 \end{vmatrix}.$$

Compute each 3×3 determinant.
First determinant:

$$\begin{vmatrix} 7 & 6 & 1 \\ 5 & 4 & 0 \\ 8 & 7 & 1 \end{vmatrix} = 7(4\cdot1-0\cdot7) - 6(5\cdot1-0\cdot8) + 1(5\cdot7-4\cdot8) = 7(4) - 6(5) + (35-32) = 28-30+3 = 1.$$

Second determinant:

$$\begin{vmatrix} 7 & 6 & 0 \\ 5 & 4 & x \\ 8 & 7 & 0 \end{vmatrix} = 7(4\cdot0-x\cdot7) - 6(5\cdot0-x\cdot8) + 0(\cdots) = 7(-7x) - 6(-8x) = -49x+48x = -x.$$

Therefore

$$\det(B) = -1 + (-x) = -(1+x).$$

$$\boxed{\det(B) = -(x+1).}$$

Hence, the determinant is zero when

$$\boxed{x = -1.}$$

Problem. 31 Find the change–of–basis matrix P from the basis

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\} \quad \text{to} \quad \beta = \left\{ \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \right\}.$$

Solution. Let A and B be the matrices whose columns are the basis vectors of α and β, respectively:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 0 \end{pmatrix}.$$

For any vector $v \in \mathbb{R}^3$, we have $v = A\,[v]_\alpha = B\,[v]_\beta$, hence

$$[v]_\beta = B^{-1}A\,[v]_\alpha.$$

Therefore, the change–of–basis matrix from α to β is

$$\boxed{P = B^{-1}A}.$$

Since $\det(B) = -49 \neq 0$, B is invertible, and a direct computation gives

$$P = \begin{pmatrix} -\dfrac{3}{49} & \dfrac{23}{49} & -\dfrac{26}{49} \\ -\dfrac{40}{49} & -\dfrac{20}{49} & -\dfrac{69}{49} \\ \dfrac{16}{49} & \dfrac{8}{49} & \dfrac{8}{49} \end{pmatrix}.$$

Answer. $P = B^{-1}A = \begin{pmatrix} -3/49 & 23/49 & -26/49 \\ -40/49 & -20/49 & -69/49 \\ 16/49 & 8/49 & 8/49 \end{pmatrix}$, so that $[v]_\beta = P\,[v]_\alpha$ for all v.

Problem. 32 Find the change–of–basis matrix P from the basis

$$\alpha = \{\, x^2+x+1,\ x^2+1,\ x-1 \,\} \quad \text{to} \quad \beta = \{\, 2x^2+3x+1,\ 2x^2+2x+1,\ -x^2-2 \,\}$$

in P_2. Use this to find the change–of–basis matrix from β to α. If a polynomial $p(x)$ has β–coordinates $\langle 1, 2, 3 \rangle$, find its α–coordinates.

Solution. Work in the standard basis $\{x^2, x, 1\}$. Form the matrices with columns equal to the coordinates of the basis vectors:

$$A = \big[\, [x^2+x+1]_{\mathrm{std}}\ \ [x^2+1]_{\mathrm{std}}\ \ [x-1]_{\mathrm{std}} \,\big] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \qquad B = \big[\, [2x^2+3x+1]_{\mathrm{std}}\ \ [2x^2+2x+1]_{\mathrm{std}}\ \ [-x^2-2]_{\mathrm{st}}$$

For any $p \in P_2$, $p = A[p]_\alpha = B[p]_\beta$. Hence

$$[p]_\beta = B^{-1}A\,[p]_\alpha.$$

Therefore the change–of–basis matrix from α to β is

$$\boxed{P_{\beta\leftarrow\alpha} = B^{-1}A = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}}\,.$$

Since $P_{\alpha\leftarrow\beta} = (P_{\beta\leftarrow\alpha})^{-1}$, we can also write

$$[p]_\alpha = A^{-1}B\,[p]_\beta,$$

so the change–of–basis matrix from β to α is

$$\boxed{P_{\alpha\leftarrow\beta} = A^{-1}B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}\,.$$

Finally, if $[p]_\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then

$$[p]_\alpha = P_{\alpha\leftarrow\beta}\,[p]_\beta = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}}\,.$$