- Inverse Gaussian distribution / Also known as Wald distribution is a two parameter family of continuous probability distribution with support on $(0,\infty)$. Its pdf is given by

$$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \quad \text{for } x>0 \text{ where } \mu>0 \text{ is}$$

the mean & $\lambda>0$ is the shape factor.

$E(X)=\mu$ (mean) & $\operatorname{var}(X)=\frac{\mu^3}{\lambda}$

Chapter 7 Functions of random variables.

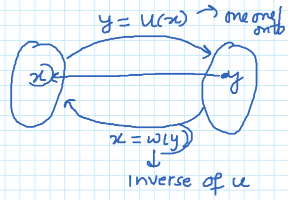
Problem Statement

Let X be a r.v. with pdf $f(x)$.

and $Y=u(X)$ is a one-one/onto function of X. or One one mapping

therefore Y is another random variable.

Question is what is the pdf of Y in terms of $f(x)$.



If X is discrete

Then pdf of $Y=u(X)$ is

$$\boxed{\begin{aligned} g(y) &= P(Y=y)\\ &= P(X=w(y)) = f(w(y))\end{aligned}}$$

If X is continuous.

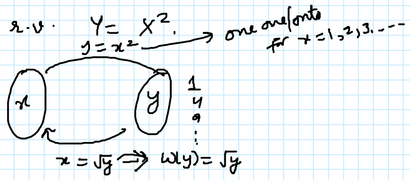
The pdf of $Y=u(X)$ will be

$$\boxed{g(y) = f(w(y))\,w'(y)}$$

Que Let X be a discrete geometric r.v. with pdf $f(x)=\frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$ $x=1,2,3,\ldots$

Find the pdf of r.v. $Y=X^2$.

Soln



$\therefore$ pdf of $Y = f(w(y))$

$= f(\sqrt{y})$

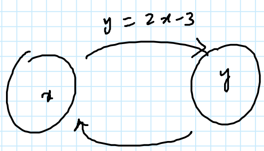
$$=\begin{cases}\frac{3}{4}\left(\frac{1}{4}\right)^{\sqrt{y}-1} & \text{for } y=1,4,9,\ldots\\ 0 & \text{otherwise}\end{cases}$$

Que Let X be a continuous r.v. with pdf

$$f(x)=\begin{cases}\frac{x}{12} & 1<x<5\\ 0 & \text{otherwise}\end{cases}$$

Find the pdf of $Y=2X-3$.

Soln



$x=\frac{y+3}{2} \Rightarrow w(y)=\frac{y+3}{2}$ → $w'(y)=\frac{1}{2}$

$$\text{pdf of } Y = g(y) = f(w(y))\,w'(y) = \begin{cases}\frac{w(y)}{12}*\frac{1}{2} & 1<w(y)<5\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{\frac{y+3}{2}}{12}\times\frac{1}{2} & 1<\frac{y+3}{2}<5\\ 0 & \text{otherwise}\end{cases}$$

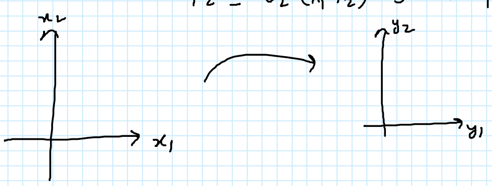
$$\boxed{g(y)=\begin{cases}\frac{y+3}{48} & -1<y<7\\ 0 & \text{elsewhere}\end{cases}}$$

Next

Let X_1 & X_2 be two r.v.

with joint pdf $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$, $Y_2 = u_2(X_1, X_2)$ } one one & onto mapping $\Rightarrow$ inverse exists $\Rightarrow$ $X_1 = \omega_1(Y_1\, Y_2)$, $X_2 = \omega_2(Y_1\, Y_2)$



If X_1 & X_2 are discrete: $f\left(\omega_1(y_1 y_2), \omega_2(y_1, y_2)\right)$

Ques what is the joint pdf of Y_1 & Y_2 in terms of $f(x_1, x_2)$.

If X_1 & X_2 are continuous: $f(\omega_1(y_1 y_2)\ \omega_2(y_1 y_2)) * J$

where $J \to$ Jacobian $= \begin{vmatrix} \frac{\partial \omega_1}{\partial y_1} & \frac{\partial \omega_1}{\partial y_2} \\ \frac{\partial \omega_2}{\partial y_1} & \frac{\partial \omega_2}{\partial y_2} \end{vmatrix}$

Que Let X_1 & X_2 be two continuous r.v. with joint pdf

$$f(x_1, x_2) = \begin{cases} 4 x_1 x_2 & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the joint pdf of $Y_1 = X_1^2$ & $Y_2 = X_1 X_2$

Soln

$$y_1 = x_1^2, \quad y_2 = x_1 x_2 \Rightarrow x_1 = \sqrt{y_1}, \quad x_2 = \frac{y_2}{x_1} = y_2/\sqrt{y_1} \Rightarrow x_1 = \sqrt{y_1} \to \omega_1(y_1 y_2), \quad x_2 = \frac{y_2}{\sqrt{y_1}} \to \omega_2(y_1 y_2)$$

$\therefore$ Joint pdf of Y_1 & Y_2 will be $f(\omega_1(y_1 y_2)\ \omega_2(y_1 y_2))\ J$

$$J = \begin{vmatrix} \frac{\partial \omega_1}{\partial y_1} & \frac{\partial \omega_1}{\partial y_2} \\ \frac{\partial \omega_2}{\partial y_1} & \frac{\partial \omega_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ -\frac{1}{2}\frac{y_2}{(y_1)^{3/2}} & \frac{1}{\sqrt{y_1}} \end{vmatrix} = \frac{1}{2y_1} = J$$

$$\therefore g(y_1, y_2) = f\left(\sqrt{y_1}, \frac{y_2}{\sqrt{y_1}}\right) * \frac{1}{2y_1} = \begin{cases} 4\sqrt{y_1} * \frac{y_2}{\sqrt{y_1}} * \frac{1}{2y_1} & \text{for } 0 < \sqrt{y_1} < 1,\ 0 < \frac{y_2}{\sqrt{y_1}} < 1 \\ 0 & \text{otherwise} \end{cases}$$

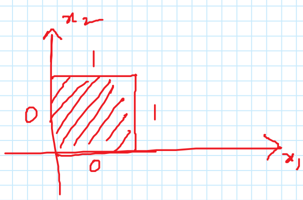
$$= \begin{cases} \frac{2y_2}{y_1} & \text{for } y_1 y_2 \text{ in } D \\ 0 & \text{otherwise} \end{cases}$$

$0 < y_1 < 1$

$0 < y_2 < \sqrt{y_1}$

$0 < x_1 < 1$

$0 < x_2 < 1$



$0 < x_1 < 1$

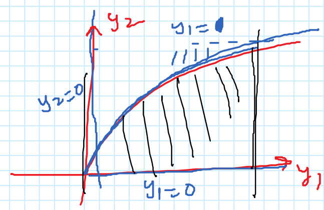
$0 < x_2 < 1$

$x_1 = 0 \longrightarrow \sqrt{y_1} = 0 \Rightarrow y_1 = 0$

$x_1 = 1 \longrightarrow \sqrt{y_1} = 1 \Rightarrow y_1 = 1$

$x_2 = 0 \longrightarrow \frac{y_2}{\sqrt{y_1}} = 0 \Rightarrow y_2 = 0$

$x_2 = 1 \longrightarrow y_2/\sqrt{y_1} = 1 \Rightarrow y_2^2 = y_1$



Que Let X_1 & X_2 be two independent r.v. having Poisson distributions with parameters μ_1 & μ_2 respectively. Find the distribution of the r.v. $Y_1 = X_1 + X_2$.

(Solved example in book)

Happy Diwali!