Lecture 29

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Recap

Chapter 7

- X is r.v. with pdf $f(x)$ then if $g(X)$ is a function of X, then what is the pdf of $g(X)$.

Today's class

Moment generating function

What are moments? If X is a r.v. with pdf $f(x)$.

$\int$ or $\sum x f(x) \leftarrow E(X) \rightarrow$ first moment

$\int$ or $\sum x^2 f(x) \leftarrow E(X^2) \rightarrow$ second moment

⋮

$\int$ or $\sum x^r f(x) \leftarrow E(X^r) \rightarrow$ r^{th} moment of r.v. X $\rightarrow$ Notation μ'_r

To get rid of individual calculations, we are looking for a function which can give us all the moments.

↳ MOMENT GENERATING FUNCTION.

Definition Given a r.v. X, with pdf $f(x)$, a function

$$\text{function of } t \leftarrow E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

This function is called moment generating function for r.v. X denoted by $M_X(t)$

Theorem (without proof) If $M_X(t)$ is moment generating function, then

$$\text{first moment} = \left.\frac{d}{dt} M_X(t)\right|_{t=0}$$

$$\text{Second moment} = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0}$$

⋮

$$r^{th} \text{ moment} = \left.\frac{d^r}{dt^r} M_X(t)\right|_{t=0}$$

Que If $M_X(t) = t^3 + 3t^2 + 2$ is moment generating function of a

Que If $M_X(t) = t^3 + 3t^2 + 2$ is moment generating function of a r.v. X, then find its mean & variance.

Sol^n For mean $\mu \rightarrow$ first moment $= \left.\frac{d}{dt} M_X(t)\right|_{t=0}$

$= \left.(3t^2 + 6t)\right|_{t=0}$

$= 0$

Variance = Second moment − (Ist moment)^2

$= \left.(6t + 6)\right|_{t=0} - (0)^2 = \boxed{6}$ Ans.

Que Find the moment generating function of a binomial r.v. X and use it to verify that $\mu = np$ & $\sigma^2 = npq$.

Sol^n $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} f(x)$ → pdf

$f(x) = {}^nC_x\, p^x q^{n-x}$

$M_X(t) = \sum_{x=0}^{n} {}^nC_x\, (\underbrace{pe^t}_{a})^x \underbrace{q}_{b}^{n-x}$

$$\boxed{M_X(t) = (pe^t + q)^n}$$ Binomial expansion

mean $= \left.\frac{d}{dt} M_X(t)\right|_{t=0} = \left. n(pe^t + q)^{n-1}(pe^t)\right|_{t=0}$

$= n(\underbrace{p+q}_{})^{n-1}(pe^0)$

$= \boxed{np}$ Ans.

For Variance

μ_2' : Second moment $= \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = np\left[e^t(pe^t+q)^{n-1} + (n-1)(pe^t+q)^{n-2}\, pe^{2t}\right]_{t=0}$

$= np[1 + (n-1)p]$

$\sigma^2 = \mu_2' - (\mu_1')^2 = np[1 + (n-1)p] - n^2p^2 = np(1-p)$

$= npq.$

$np + \cancel{n^2}p^2 - np^2 - \cancel{n^2}p^2$

$np(1-p)$

Que Show that the moment generating function of a r.v. X having a normal probability distribution with mean μ & variance σ^2 is given by

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Sol^n

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\, pdf(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 + tx}\, dx$$

→ Try to complete the square

↓ Integrate it.

$$\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Some important properties

① $M_{X+a}(t) = e^{at} M_X(t)$.

Proof

$$M_{X+a}(t) = E\left[e^{t(X+a)}\right]$$

$$= E\left[e^{tX} \cdot e^{at}\right]$$

$$= e^{at} E\left[e^{tX}\right] = e^{at} M_X(t)$$

② $M_{aX}(t) = M_X(at)$

Proof → $E\left[e^{t(aX)}\right] = E\left[e^{(at)X}\right] = M_X(at)$

③ If $X_1, X_2, \ldots X_n$ are n independent r.v. with moment generating functions $M_{X_1}, M_{X_2} \ldots M_{X_n}$ then the r.v. $Y = X_1 + X_2 + \cdots + X_n$ has moment generating function

$$= M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$$

product.

Proof $E\left[e^{tY}\right] = E\left[e^{t(X_1 + X_2 \cdots + X_n)}\right]$

$$= E\left[e^{tX_1} e^{tX_2} \cdots e^{tX_n}\right]$$ independence

$$= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_n}\right]$$

$$= M_{X_1} M_{X_2} \cdots M_{X_n}$$

Chapter 7 ends here