Recap

- Mathematical expectation / mean
- Given $X \rightarrow$ a r.v. with pdf $f(x)$

$$E(X) = \sum_x x f(x) \quad \text{, in case } X \text{ is discrete}$$

$$\int x f(x)\,dx \quad X \text{ cts.}$$

- X $\quad f(x) \rightarrow$ pdf $\quad Y = g(X)$

$$E[Y] = E[g(X)] = \sum_x g(x) f(x) \text{ or } \int g(x) f(x)\,dx$$

Today's lecture

When we have X and Y two r.v. with joint pdf $f(x,y)$

then
$$E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$$

$$\iint g(x,y) f(x,y)\,dx\,dy$$

example
$$E[X+Y] = \sum_x \sum_y (x+y) f(x,y) \qquad E[X^2Y] = \sum_x \sum_y x^2 y\, f(x,y)$$

Note

If X & Y are two r.v. with joint pdf $f(x,y)$

then
$$E(X) = \sum_x \underbrace{\sum_y x\, f(x,y)}$$ $\rightarrow g(x) \rightarrow$ marginal pdf of X

$$= \sum_x x\, g(x)$$

$$E(Y) = \sum_x \sum_y y\, f(x,y)$$ $\rightarrow h(y) \rightarrow$ marginal pdf of Y

$$= \sum_y y\, h(y)$$

Que X & Y are two r.v. with the following joint pdf

$f(x,y)$		$X=x$: 0	1	2	$h(y)$
$Y=y$	0	3/28	9/28	3/28	15/28
	1	3/14	3/14	0	3/7
	2	1/28	0	0	1/28
	$g(x)$	5/14	15/28	3/28	

$E(XY) = ?$

$E(X+Y) = ?$

$E(X) = ?$

$= 0 * 5/14 + 1 * 15/28 + 2 * 3/28$

Sol^n $E[XY] = \sum_x \sum_y xy\, f(x,y)$

$$\left\{ \begin{array}{l} 0*0\, f(0,0) + 0*1\, f(0,1) + 0*2\, f(0,2) \\ + 1*0\, f(1,0) + 1*1\, f(1,1) + 1*2\, f(1,2) \\ + 2*0\, f(2,0) + 2*1\, \underbrace{f(2,1)}_{=0} + 2*2\, \underbrace{f(2,2)}_{=0} \end{array} \right.$$

$3/14$

Que (H.W) X & Y are two continuous r.v. with joint pdf

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4} & 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E\left(\frac{Y}{X}\right) = ?$

$\boxed{\text{Ans} = 3/8}$

$\rightarrow \int_x \int_y \frac{y}{x} f(x,y)\, dx\, dy$

X pdf Modeling

mean/Expected value) analysis

next thing variance.

Variance $\rightarrow$ a measure of deviation from mean.

$$\sigma^2 = \frac{\sum (x-\mu)^2}{N}$$

$\mu \rightarrow$ mean

Suppose x takes following 8 values

$\sigma^2 = \dfrac{\sum_x (x-\mu)}{N}$ $\qquad \mu \to$ mean

Suppose x takes following 8 values

$x \to \quad x_1 \; x_1 \; x_1 \quad x_2 \; x_2 \quad x_3 \; x_3 \; x_3$

$$= \frac{(x_1-\mu)^2 * 3 + (x_2-\mu)^2 * 2 + (x_3-\mu)^2 * 3}{8}$$

$P(x=x_1)$ $\quad P(x=x_2)$ $\quad P(x=x_3)$

$$= \sum_{x_i} (x_i-\mu)^2 \, P(x=x_i)$$

$P(x=x_i) \to$ pdf

If X is a r.v. $\quad \boxed{\sigma^2 = \sum_x (x-\mu)^2 f(x)} \to E\left[(x-\mu)^2\right]$

variable $\quad$ fixed number

$E \to$ expected value $\quad (x-\mu)^2 \to$ deviation from mean

$$\sigma^2 = \sum_x (x-E(X))^2 f(x) \quad \text{or} \quad \int (x-E(X))^2 f(x)\,dx$$

Que Let X be r.v. with pdf $\qquad \sigma_X^2 = ?$

x	1	2	3
$f(x)$	0.3	0.4	0.3

Soln $E(X) = 1*0.3 + 2*0.4 + 3*0.3 = 2 \qquad \sum_x x f(x)$

$$\sigma_X^2 = (1-2)^2 * 0.3 + (2-2)^2 * 0.4 + (3-2)^2 * 0.3 = \sum (x-\mu)^2 f(x)$$
$$= 0.6$$

Result $\sigma^2 = E\left[(X-\mu)^2\right] \qquad \mu = E(X) \qquad E[g(X)]$

$= \underbrace{E(X^2) - \mu^2}$ $\qquad = \sum g(x) f(x)$

Proof

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x)$$
$$\sum_x (x^2+\mu^2-2x\mu) f(x)$$
$$\sum_x x^2 f(x) + \sum_x \mu^2 f(x) - \sum_x 2x\mu f(x)$$
$$= E(X^2) + \mu^2 \sum_x f(x) - 2\mu \sum_x x f(x)$$

$\sum_x f(x) = 1 \qquad \sum_x x f(x) = E(X)$

$$= E(x^2) + \mu^2 \underbrace{\left(\sum_x f(x)\right)}_{1} - 2\mu \underbrace{\left(\sum_x x f(x)\right)}_{E(X)=\mu}$$

$$= E(x^2) + \mu^2 - 2\mu^2$$

$$\boxed{\sigma_X^2 = E(x^2) - \mu^2} = E(x^2) - (E(x))^2$$

Note $E(X^n) \rightarrow$ n^{th} moment of r.v. X

$E(X) \rightarrow$ First moment

$E(x^2) \rightarrow$ second moment & so on

Variance = second moment − (Ist moment)^2

Que

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Find σ_x^2

Sol^n $E(X) = 0*0.51 + 1*0.38 + 2*0.10 + 3*0.01 = 0.61$

$E(X^2) = 0^2 * 0.51 + 1^2 * 0.38 + 2^2 * 0.10 + 3^2 * 0.01 = 0.87$

$\sigma_X^2 = 0.87 - (0.61)^2 = 0.4979$

S.D. $= \sqrt{\text{variance}} = \sqrt{0.4979}$

Que X is a continuous r.v. with pdf $f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

Find σ_X^2

Sol^n $E(X^2) = \int x^2 f(x)\,dx = \int_1^2 x^2\, 2(x-1)\,dx = 17/6$

$E(X) = \int x f(x)\,dx = \int_1^2 x\, 2(x-1)\,dx = 5/3$

$E(X^2) - (E(X))^2$

$\sigma_X^2 = \boxed{\frac{17}{6} - \left(5/3\right)^2} = 1/18$ Ans.

$$\sigma_X^2 = \int (x - E(X))^2 f(x)\,dx$$