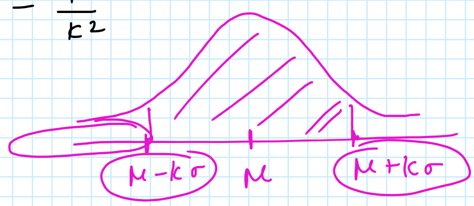
Chebyshev Theorem

Let X be a r.v. with mean $= \mu$ & S.D. $= \sigma$, then

$$P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$$

Let X be a r.v. with pdf $f(x)$



Proof

$$\sigma^2 = \sigma_X^2 = E\left(\left(X - E(X)\right)^2\right)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

(a positive quantity)

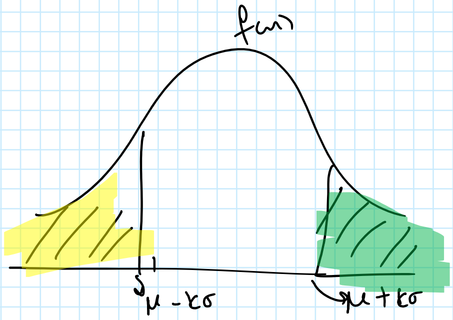
$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\ge \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$x \le \mu - k\sigma$ $\qquad$ $x \ge \mu + k\sigma$

$x - \mu \le -k\sigma$ $\qquad$ $x - \mu \ge k\sigma$

$$\underbrace{\qquad\qquad\qquad\qquad}_{|x-\mu| \ge k\sigma}$$



$$\ge \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$$

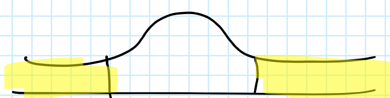
$$= k^2\sigma^2 \left(\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx \right)$$

$$= k^2\sigma^2 \left(P(X \le \mu - k\sigma) + P(X \ge \mu + k\sigma) \right)$$

$$= k^2\sigma^2 \, P(|X-\mu| \ge k\sigma)$$

I get $\quad \sigma^2 \ge k^2\sigma^2 P(|X-\mu| \ge k\sigma)$

$$\boxed{\frac{1}{k^2} \ge P(|X-\mu| \ge k\sigma)} \longrightarrow \boxed{P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2}}$$



$$P(|X-\mu| \le k\sigma) = 1 - P(|X-\mu| > k\sigma)$$

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$$P(|X-\mu| > k\sigma) \ge \frac{1}{k^2}$$

$$\boxed{P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}}$$

Hence proved.

Chapter 4 is over