Recap

- Normal distribution

Normal approximation to binomial $\to p \sim 1/2$

or when n is very large & $p \not\to 0$ and $p \sim 1$

- When np as well as $n(p-1)$ both are greater than 5

$$b(x; n, p) \sim n(x, np, \sqrt{np(p-1)})$$

Gamma distribution

Gamma function

$$\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \quad \text{for } \alpha > 0$$

Properties of gamma function :-

1. $\Gamma\alpha = (\alpha-1)\Gamma\alpha-1$
2. $\Gamma n = (n-1)!$ when $n > 0$ is an integer
3. $\Gamma(1) = 1$
4. $\Gamma 1/2 = \sqrt{\pi}$

Gamma distribution

A continuous r.v. X is said to have gamma distribution with parameters α & β if its pdf looks like

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Its mean $\mu = \alpha\beta$ & variance $\sigma^2 = \alpha\beta^2$

when $\alpha = 1$ I will get a distribution whose pdf is

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is called exponential distribution / negative exponential distribution

Mean $\mu = \beta$

Variance $\sigma^2 = \beta^2$

This exponential distribution has many applications

Application of exponential distribution

Consider a Poisson process

$\lambda \to$ average no. of occurence in time t

$X \to$ no. of occurences in time t

pdf $\quad P(X = x) = \dfrac{(\lambda t)^x e^{-\lambda t}}{x!}$

$t=0$ — 1st occurence — 2nd occurence - -

What is the probability that there is no occurence till time t.

Put $x = 0$

$$\frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

Put $x=0$

$$\frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

Now I define

T = Time for first occurence

$$P(T>t) = P(\text{No. occurrence till time } t) = e^{-\lambda t}$$

cdf

$$P(T \le t) = 1 - P(T>t) = 1 - e^{-\lambda t} \quad \text{for } t>0$$

↓ differentiate w.r.t. t

$$\text{pdf} = \begin{cases} \lambda e^{-\lambda t} & t>0 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ exponential distribution with $\lambda = \frac{1}{\beta}$

Conclusion :- Exponential distribution is useful to model time to first occurence in Poisson process with $\beta = \frac{1}{\lambda}$

$\beta \to$ mean time to failure.

Que Suppose that a system contains a certain type of component whose time to failure in years is given by T. T is an exponential r.v. with $\beta = 5$.

If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years.

$b(x; 5, 0.2)$

$b(x; 5, \overset{?}{p})$

Sol^n

S after 8 years

F after 8 years

→ Note that it is a binomial process

$X \to$ no. of Survivals after 8 years

we are looking for

$P(X \ge 2) = ?$

$$\sum_{x=2}^{5} b(x; 5, 0.2)$$

$$1 - \sum_{x=0}^{1} b(x; 5, 0.2)$$

↓ Table

$= 1 - 0.7373$

$= 0.2627$ Ans

Now let us compute p?

$T \to$ time to failure which is exponential distribution with $\beta = 5$

we are looking for

$$p = P(T>8)$$

$$= \int_8^\infty \text{pdf}(t)\,dt = \int_8^\infty \frac{1}{5} e^{-t/5}\,dt$$

$$\frac{1}{\cancel{5}}\left[\frac{e^{-t/5}}{-1/\cancel{5}}\right]_8^\infty$$

$$-\left[\lim_{t\to\infty} e^{-t/5} - e^{-8/5}\right]$$

$\to 0$

$$e^{-8/5} = \boxed{0.2}$$

$e^{-\infty/5}$

$f(x) = x^3$

$f(\infty)$

Very important note :-

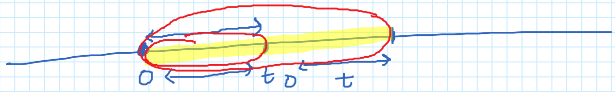
Memoryless property of Poisson process :-

No. of occurences in disjoint time intervals has nothing to do with each other.

⇓

This property is transformed into exponential process as follows

$$P(T \geq t_0 + t) = P(T \geq t_0 + t \mid T \geq t_0)$$



So it means that we can apply exponential distribution only in the cases where there is no continuous decay involved.

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