Lecture 16 (Gr 25-27)

01 September 2022 10:37

Sessional for 10 marks
Thursday 15/09/2022
6:10 pm

3 chapters

Till pdf cdf
Joint pdf
Marginal pdf etc.

Syllabus for MST First 5 chapters.

Chapter 4
Mathematical expectation

$E(X)$ $E(g(X))$ $E(g(X,Y))$

$\sigma_X^2 \longrightarrow E(X^2) - (E(X))^2$

$E(X^n) \rightarrow n^{th}$ moment

Covariance $\sigma_{XY} = E\left((X-\mu_X)(Y-\mu_Y)\right) = E(XY) - E(X)\cdot E(Y)$

Some useful results

① If a and b are two constants & X is a r.v. then

$$E(aX+b) = aE(X)+b$$

Proof

$$E(aX+b) = \sum_x (ax+b) f(x) \quad \text{where } f(x) \text{ is pdf of } X$$

$$= \sum_x a x f(x) + \sum_x b f(x)$$

$$= a \sum_x x f(x) + b \sum_x f(x)$$

$$= a E(X) + b \cdot 1$$

$$\boxed{E(aX+b) = aE(X)+b}$$

Result

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Proof

$$E[g(X) \pm h(X)] = \sum_x (g(x) \pm h(x)) f(x), \quad \text{where } f(x) \text{ is pdf of } X$$

$$= \sum_x g(x) f(x) \pm \sum_x h(x) f(x)$$

$$= E[g(X)] \pm E[h(X)]$$

Que X is a r.v. with pdf

x	0	1	2	3
$f(x)$	1/3	1/2	0	1/6

Find $E((X-1)^2)$.

Sol^n

$$E((X-1)^2) = \sum_x (x-1)^2 f(x)$$

$$E((X-1)^2) = E(X^2-2X+1) = E(X^2) - 2E(X) + 1$$

$$= \underbrace{\left(0^2*\tfrac{1}{3} + 1^2*\tfrac{1}{2} + 2^2*0 + 3^2*\tfrac{1}{6}\right)}$$

$$-2\left[0*\tfrac{1}{3} + 1*\tfrac{1}{2} + 2*0 + 3*\tfrac{1}{6}\right]$$

$$+1$$

$$= 1$$

Result

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

Proof

$$E[g(X,Y) \pm h(X,Y)] = \sum_x \sum_y (g(x,y) \pm h(x,y)) f(x,y) \quad \text{where } f(x,y) \text{ is joint pdf}$$

$$= \sum_x \sum_y g(x,y) f(x,y) \pm \sum_x \sum_y h(x,y) f(x,y)$$

$$= E[g(X,Y)] \pm E[h(X,Y)]$$

Result If X and Y are two independent random variables, then

$$E(XY) = E(X)E(Y).$$

Proof If $f(x,y)$ is joint pdf of X & Y

$g(x)$ is marginal pdf of X.

$h(y)$ is " " " Y.

Then X & Y are independent $\Rightarrow$ $\boxed{f(x,y) = g(x)\,h(y)}$ *

Now $E(XY) = \sum_x \sum_y (xy) f(x,y)$ using *

$= \sum_x \sum_y x y\, g(x)\, h(y) = \left(\sum_x x\, g(x)\right)\left(\sum_y y\, h(y)\right)$

$$\boxed{E(XY) = E(X)\,E(Y)}$$

Result If X & Y are two independent random variance then $\sigma_{XY} = 0$.

Proof $\sigma_{XY} = E(XY) - E(X)E(Y)$

Now if X & Y are independent then $E(XY) = E(X)E(Y)$

$\therefore \sigma_{XY} = \cancel{E(X)E(Y)} - \cancel{E(X)E(Y)}$

$$\boxed{\sigma_{XY} = 0}$$

Result If X and Y are two r.v., a, b, c are constants then

$$\sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$$

Proof $\sigma^2_{aX+bY+c} = E\left[\left((aX+bY+c) - E(aX+bY+c)\right)^2\right]$

$= E\left[\left((aX+bY+c) - (aE(X) + bE(Y) + c)\right)^2\right]$, with $E(X) = \mu_X$, $E(Y) = \mu_Y$

$= E\left[\left(a(X-\mu_X) + b(Y-\mu_Y)\right)^2\right]$

$= E\left[a^2(X-\mu_X)^2 + b^2(Y-\mu_Y)^2 + 2ab(X-\mu_X)(Y-\mu_Y)\right]$

$= a^2 E\left[(X-\mu_X)^2\right] + b^2 E\left[(Y-\mu_Y)^2\right] + 2ab\, E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$$\sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$$

Corollary

① $\sigma^2_{aX+b} = a^2\sigma_X^2$

② $\sigma^2_{X+b} = \sigma_X^2$

Last result

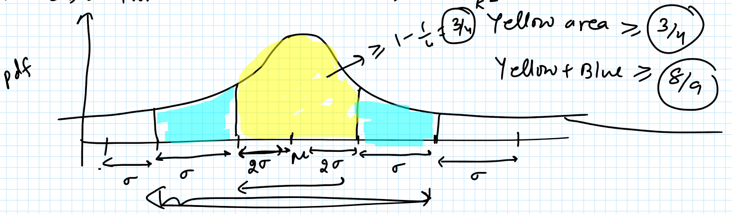
Chebyshev's theorem:-

If X is a r.v. with mean μ & standard deviation σ

then $P(\mu - k\sigma \le X \le \mu + k\sigma) \ge 1 - \frac{1}{k^2}$

i.e. Probability that the r.v. assumes a value which is k S.D.s from the mean is $\ge 1 - \frac{1}{k^2}$.

$\int pdf(x)\,dx = 1$



$1 - \frac{1}{9} = \frac{8}{9}$

$P(\mu - \sigma < X < \mu + \sigma)$

$\geq 1 - 1$

≥ 0

Que A r.v. has mean $\mu = 8$ & variance $\sigma^2 = 9$ & an unknown probability distribution. Find

$P(-4 < X < 20)$

$20 = 8 + k * 3$

Sol^n

$S.D. = \sigma = \sqrt{9} = 3$

$\mu + k\sigma$

$\mu - k\sigma$

$\mu = 8$

Chebyshev

$P(8 - 4(3) < X < 8 + 4(3)) \geq 1 - \frac{1}{4^2} = \frac{15}{16}$

$\mu - 4\sigma < X < \mu + 4\sigma$

$-4 = 8 - k * 3$

$\boxed{k = 4}$

$20 = 8 + k\,3$

$k = \frac{12}{3} = 4$

(b) $\boxed{P(|X - 8| \geq 6)} = 1 - P(|X - 8| < 6) \leq 1 + \left(-\frac{3}{4}\right)$

$6 = 2 \cdot \sigma$

$\leq \frac{1}{4}$

X is 2 S.D. from mean

$8 - 2 * 3 < X < 8 + 2 * 3$

Chebyshev's

$\boxed{P(|X - 8| < 6) \geq 1 - \frac{1}{2^2} = \frac{3}{4}}$

$-P(|X - 8| < 6) \leq -\frac{3}{4}$