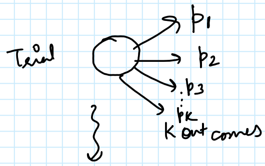
Lecture 18

13 September 2024 13:06

$\rightarrow$ We are done till Chapter 5.

Multinomial distribution :- Instead of two categories as in Binomial, you have many categories (say $k \rightarrow$ classes) .

$$P(X \in i^{th} \text{ class}) = p_i$$



$$\sum_{i=1}^{k} p_i = 1$$

n such independent trials

X_1 = no. of outcomes in class 1

X_2 = " " " " "

$\vdots$

X_k = " " " " class k

We look for

$$P(X_1 = x_1, X_2 = x_2, \ldots X_k = x_k) = {}^{n}C_{x_1} (p_1)^{x_1} \; {}^{n-x_1}C_{x_2} (p_2)^{x_2} \ldots {}^{n-x_1-x_2-\ldots-x_{k-1}}C_{x_k} \, p_k^{x_k}$$

$\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} p_i = 1$

$$= \frac{n!}{x_1!\, x_2! \ldots x_k!} \prod_{i=1}^{k} p_i^{x_i}$$

Que Suppose on a particular day, prob of being sunny = 0.5

rainy = 0.3

cloudy = 0.2

What is the prob. that in next 30 days, we will have 15 cloudy, 10 rainy & 5 sunny days?

$$P(X_1 = 5, X_2 = 10, X_3 = 15) = \frac{30!}{5!\,10!\,15!} (0.5)^5 (0.3)^{10} (0.2)^{15}$$

Chapter 5 over

One topic from chapter 7

Moments and moment generating function (mgf)

- Given a random variable X

$$E(X^r) \text{ is called } r^{th} \text{ moment of X.} = \sum x^r f(x) \text{ or } \int x^r f(x)\, dx$$

$E(X^0) = E(1) = 1 \rightarrow$ zeroth moment of X.

$E(X)$ = mean $\longrightarrow$ 1^{st} moment

$E(X^2)$ = second moment

$\vdots$

- $\text{Mgf}(X) = E(e^{tX}) = \int e^{tx} f(x)\, dx$ It will be a function of t.

$M_X(t)$ or $\sum e^{tx} f(x)$

Result $\left.\dfrac{d\, M_X(t)}{dt}\right|_{t=0} = E(X) \rightarrow$ first moment of X

Result

$$\left.\frac{d\, M_X(t)}{dt}\right|_{t=0} = E(X) \rightarrow \text{first moment of } X$$

$$\left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = E(X^2) \rightarrow \text{second moment of } X$$

$$\vdots$$

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = E(X^r) \rightarrow r^{th} \text{ moment of } X$$

Que Use concept of mgf to find mean & variance of binomial r.v. X with parameters n & p.

Sol^n

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=0}^{n} e^{tx} f(x) \quad f(x) \rightarrow {}^nC_x\, p^x q^{n-x}$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x\, q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x \left(pe^t\right)^x q^{n-x}$$

Binomial expansion.

$$M_X(t) = \left(pe^t + q\right)^n$$

first moment

$$E(X) = \text{mean} = \left.\frac{d\, M_X(t)}{dt}\right|_{t=0} = \left.\frac{d}{dt}\left(pe^t+q\right)^n\right|_{t=0} = \left. n\left(pe^t+q\right)^{n-1}\left(pe^t\right)\right|_{t=0}$$

$$= n\underbrace{(p+q)^{n-1}}_{=1}(p.1)$$

$$= \boxed{np}$$

Second moment

$$E(X^2) = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = np\left[\underbrace{e^t}_{1}\underbrace{\left(pe^t+q\right)^{n-1}}_{1} + (n-1)\underbrace{\left(pe^t+q\right)^{n-2}}\left(pe^t\right)\left(e^t\right)\right]_{t=0}$$

$$= np\left[1 + (n-1)p\right]$$

$$Var = E(X^2) - \left(E(X)\right)^2 = \left(np + n^2p^2 - np^2\right) - (np)^2 = np - np^2$$

$$= np(1-p) = npq$$

Some properties of Mgf

(1) $M_{X+a}(t) = e^{at} M_X(t)$

$$M_{X+a}(t) = E\left(e^{t(X+a)}\right) = E\left(e^{tX} \cdot e^{ta}\right) = e^{ta} E\left(e^{tX}\right)$$

$$= e^{ta} M_X(t)$$

(2) $M_{aX}(t) = M_X(at)$

$$M_{aX}(t) = E\left(e^{t(aX)}\right) = E\left(e^{(ta)X}\right) = M_X(ta)$$

(3) If $X_1, X_2, \ldots X_n$ are independent r.v. with mgf $M_{X_1}(t), M_{X_2}(t), \ldots M_{X_n}(t)$

then mgf of $Y = X_1 + X_2 + \ldots + X_n$ is

$$\boxed{M_Y(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)}$$

then mgf of $Y = X_1 + X_2 + \dots + X_n$ is

$$\boxed{M_Y(t) = M_{X_1}(t)\, M_{X_2}(t) \dots M_{X_n}(t)}$$

Que If mgf of a distribution is $e^t \sin t$, find its mean & variance.