Lecture 15(Gr 25-27)

31 August 2022 09:42

Recap

$X \to$ r.v. pdf $f(x)$

- $\mu = E(X) = \sum_x x f(x)$ or $\int x f(x) dx$
- $E(g(X)) = \sum_x g(x) f(x)$ or $\int g(x) f(x) dx$
- $\sigma_X^2 = E((X-\mu)^2) = \sum_x (x-\mu)^2 f(x)$ or $\int (x-\mu)^2 f(x) dx$
- $\sigma_X^2 = E(X^2) - (E(X))^2$

S.D = $\pm\sqrt{\text{variance}}$

X & Y are two r.v. with $f(x,y)$ joint pdf

$E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y)$

Today's work

X is a r.v. with pdf $f(x)$ and $g(X)$ is any function

$$\sigma^2_{g(X)} = E\left[\left(g(X) - \mu_{g(X)}\right)^2\right]$$

$$= \sum \left(g(x) - \mu_{g(x)}\right)^2 f(x)$$

Que Calculate the variance of $g(X) = 2X+3$, where X is a r.v. with pdf

x	0	1	2	3
$f(x)$	$1/4$	$1/8$	$1/2$	$1/8$

Sol^n

$$\mu_{g(X)} = \sum_x g(x) f(x) = \sum_x (2x+3) f(x)$$

$$= (2*0+3)\tfrac{1}{4} + (2*1+3)\tfrac{1}{8} + (2*2+3)*\tfrac{1}{2} + (2*3+3)*\tfrac{1}{8} = \boxed{6}$$

mean

$$\sigma^2_{g(X)} = E\left[(g(X) - 6)^2\right]$$

$$= \sum_x (g(x)-6)^2 f(x) = (2*0+3-6)^2 * \tfrac{1}{4} + (2*1+3-6)^2 * \tfrac{1}{8} + (2*2+3-6)^2 * \tfrac{1}{2} + (2*3+3-6)^2 * \tfrac{1}{8} = \boxed{4}$$

Que X is a continuous r.v. with pdf $f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$

Find variance of $g(X) = 4X+3$

Sol^n

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx = \int_{-1}^{2} (4x+3)\frac{x^2}{3}\,dx = 8$$

$$\sigma^2_{g(X)} = \int_{-\infty}^{\infty} \left(g(x) - 8\right)^2 f(x)\,dx = \int_{-1}^{2} \left(4x+3-8\right)^2 \frac{x^2}{3}\,dx$$

$$= \frac{51}{5} \text{ Ans.}$$

Next

Covariance

We have two r.v. X and Y

Suppose Y depends on X

We would like to know the kind of dependence.

$\underbrace{\text{kind of dependence}}_{\text{will be measured by covariance.}}$

Definition given two r.v. X & Y with joint pdf $f(x,y)$, their covariance denoted by $\text{cov}(X,Y)$ is defined as

$$E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

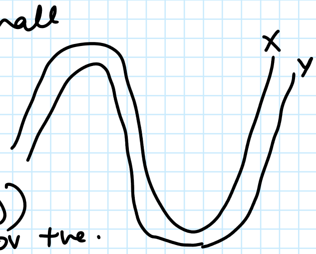
If X and Y are together large or together small

X is large $\quad X-\mu_X \to$ positive

Y is large $\quad Y-\mu_Y \to$ positive

$\Big\}$ $(X-\mu_X)(Y-\mu_Y)$ positive

$E\left((X-\mu_X)(Y-\mu_Y)\right)$ +ve $\Rightarrow$ cov +ve.



X is small $\quad X-\mu_X \to$ -ve

Y is small $\quad Y-\mu_Y \to$ -ve

$\Big\}$ product +ve

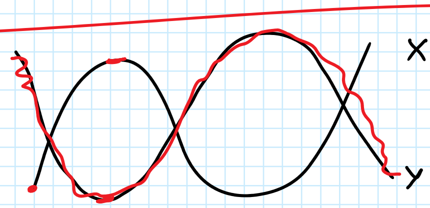
cov $\to$ +ve

X large $\to X-\mu_X$ +ve

Y small $\to Y-\mu_Y$ -ve

$\Big\}$ product -ve

$\therefore$ cov $\to$ -ve



X small $\to X-\mu_X$ -ve

Y large $\to Y-\mu_Y$ +ve

$\Big\}$ product -ve

$\therefore$ cov $\to$ -ve.

~~$E(XY)-E(X)E(Y)$~~

- If X and Y are independent $\Rightarrow \text{cov}(X,Y) = 0$
- However if $\text{cov}(X,Y) = 0 \not\Rightarrow X$ and Y are independent

• However if $cov(X,Y) = 0 \not\Rightarrow$ X and Y are independent.

COV only captures the linear relationship b/w X & Y

$$\left(\boxed{Y = \alpha + \beta X} + \gamma X^2 + \ldots\right)$$
$\downarrow$ linear

• Notation $\sigma_{XY} = cov(X,Y)$

Result $\sigma_{XY} = E(XY) - E(X)E(Y)$

$\sigma_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$

Proof

$$\sigma_{XY} = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

$$= \sum_x \sum_y (xy - \mu_Y x - \mu_X y + \mu_X \mu_Y) f(x,y)$$

$$= \sum_x \sum_y xy\, f(x,y) - \sum_x \sum_y \mu_Y x\, f(x,y) - \sum_x \sum_y \mu_X y\, f(x,y) + \sum_x \sum_y \mu_X \mu_Y f(x,y)$$

$$= \sum_x \sum_y xy\, f(x,y) - \mu_Y \sum_y \sum_x x\, f(x,y) - \mu_X \sum_x \sum_y y\, f(x,y) + \mu_X \mu_Y \underbrace{\sum_x \sum_y f(x,y)}_{=1}$$

$$= E(XY) - \mu_Y \underbrace{E(X)}_{\mu_X} - \cancel{\mu_X} \underbrace{E(Y)}_{\mu_Y} + \cancel{\mu_X}\mu_Y$$

$$\boxed{\sigma_{XY} = E(XY) - \mu_X \mu_Y}$$

Covariance $\leftarrow$ $\boxed{\sigma_{XY} = E(XY) - E(X)E(Y)}$ ✓

~~correlation~~ co correlation coefficient

$$\boxed{\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$

$$\boxed{-1 \le \rho_{XY} \le 1}$$

$\downarrow$
Not only tells type of relation b/w X & Y
it also tells you how strongly they are related.

Que For X and Y with joint pdf $f(x,y)$ given below find σ_{XY} and ρ_{XY}

y \ x	0	1	2	h(y)
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	3/7
2	[illegible]	0	0	1/28

$$\mu_X = \sum \sum x\, f(x,y) = \sum_x x\, g(x)$$

$$= 0 * 5/14 + 1 * \frac{15}{28} + 2 * \frac{3}{28}$$

$$= \boxed{3/4}$$

1	3/14	3/14	0	3/14
2	1/28	0	0	1/28
$g(x)$	5/14	15/28	3/28	

$= \boxed{3/4}$

$\mu_y = \sum_y y\,h(y) = 0 * \frac{15}{28} + 1 * \frac{3}{7} + 2 * \frac{1}{28}$

$= \boxed{1/2}$

$E[XY] = \sum_x \sum_y xy\, f(x,y) = (0*0) f(0,0) + (1*0)\, 9/28 \quad (2*0)\, 3/28$

$+ (0*1) * 3/14 + (1*1) * 3/14 + (0*2) * 1/28 = 3/14$

$\sigma_{XY} = E(XY) - E(X)E(Y) = 3/14 - 3/4 * \frac{1}{2} = -\frac{9}{56}$

σ_X^2

$\sigma_Y^2 =$
$E(Y^2) - (E(Y))^2$
$= \frac{4}{7} - (1/2)^2 = \frac{9}{28}$

$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

$= \frac{-9/56}{\sqrt{\frac{45}{112}}\ \sqrt{9/28}}$

$= -1/\sqrt{5}$

$\sigma_X^2 = E(X^2) - (E(X))^2$

$E(X^2) = \sum x^2 g(x) = 0^2 * 5/14 + 1^2 * 15/28 + 2^2 * 3/28$

$= \frac{27}{28}$

$\sigma_X^2 = \frac{27}{28} - (3/4)^2 = \frac{45}{112}$