Shortest Paths

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Shortest Paths: Part 1

- Shortest Path Problem
- Dijkstra’s Shortest Path Algorithm
- Analysis

Definitions

- A path in a directed graph is a sequence of vertices $p = \langle v_0, v_1, \ldots, v_k \rangle$ s.t. there is an edge connecting v_i to v_{i+1} for all i

- A path in an undirected graph is a sequence of vertices $p = \langle v_0, v_1, \ldots, v_k \rangle$ s.t. there is an edge connecting v_i and v_{i+1} for all i

Definitions

- The weight of a path p = <v_0, v_1, ..., v_k> is

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- The shortest-path weight from vertex u to vertex v is

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & (\exists p)(u \ldots v) \\ \infty & otherwise \end{cases}$$

Applications: Navigation System

- Suppose you want to find the distance or travel time from one POI (point of interest) to another POI, you can ask Google:

- https://www.google.com/maps

- If you need to know how to go to an address, you can use an app on your phone.

- Why doesn't your phone ask for the source?

Vertex attributes, Initialization

- For each node, v, we have two attributes: $v.d$, $v.\pi$ representing the current distance (from the source) and the current predecessor of vertex v.

- $\boldsymbol{v.d}$ will not increase in the process.

- We initialize distance from the source to ∞

Initialize-Single-Source(G, s)

1. for each vertex v ∈ V[G] do
2. v.d = ∞
3. v.p = nil
4. s.d = 0

Relax(u, v, w)

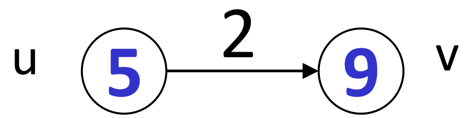
We adjust (if it reduces) the distance at vertex v whenever we traverse an edge (u, v).

Relax(u, v, w)

1. if (v.d > u.d + w(u, v)) then
2. v.d = u.d + w(u, v)
3. v.p = u

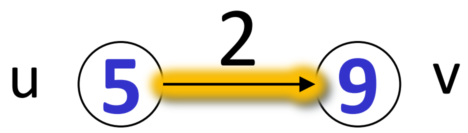
Relax (u, v, w): Illustration

Before:



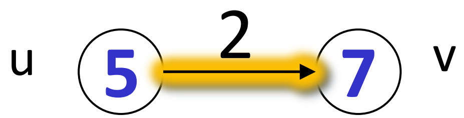
Relax (u, v, w): Illustration

Relax:



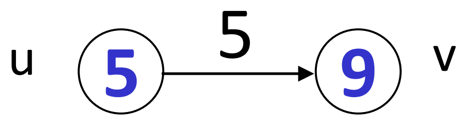
Relax (u, v, w): Illustration

Result:



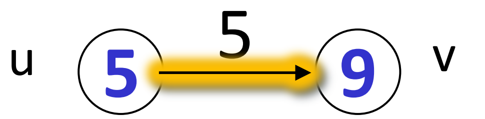
Relax (u, v, w): Illustration

Before:



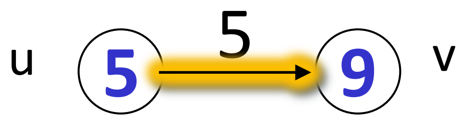
Relax (u, v, w): Illustration

Relax:



Relax (u, v, w): Illustration

Result:



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Shortest Paths: Part 2

- Shortest Path Problem
- Dijkstra’s Shortest Path Algorithm
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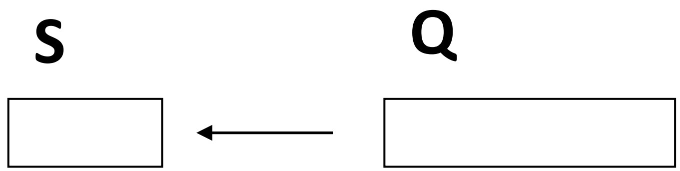
Dijkstra's Algorithm

The idea of the algorithm is the following:

- Use a greedy algorithm
- Use a priority queue Q, where the key of vertex v is the currently computed distance from the source.
- Processed vertices are removed from Q and stored in a set S

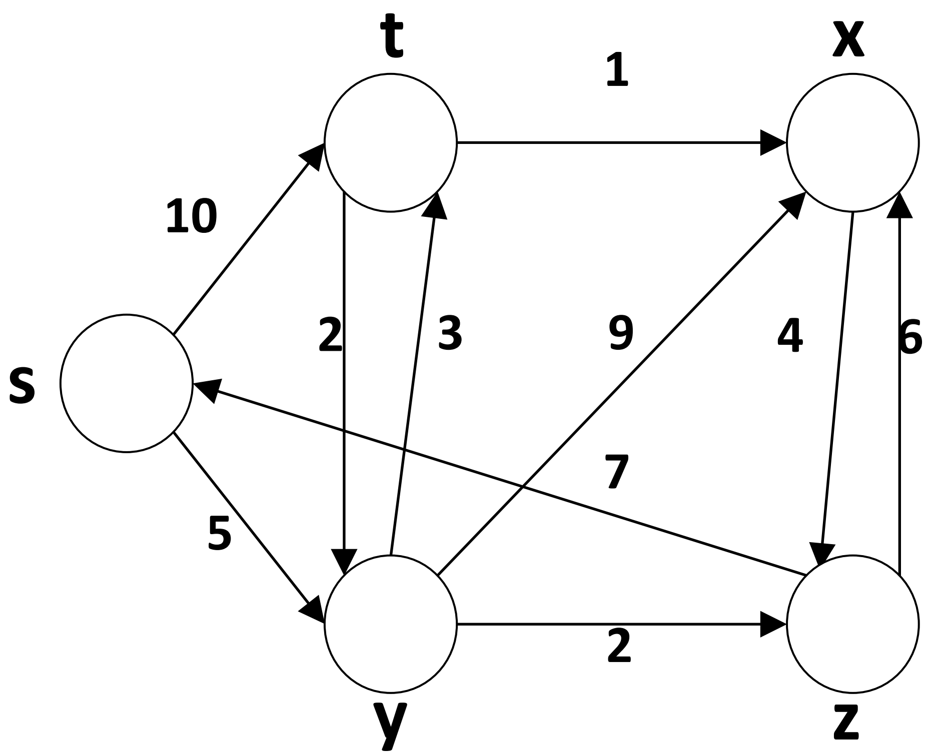


Dijkstra's Algorithm

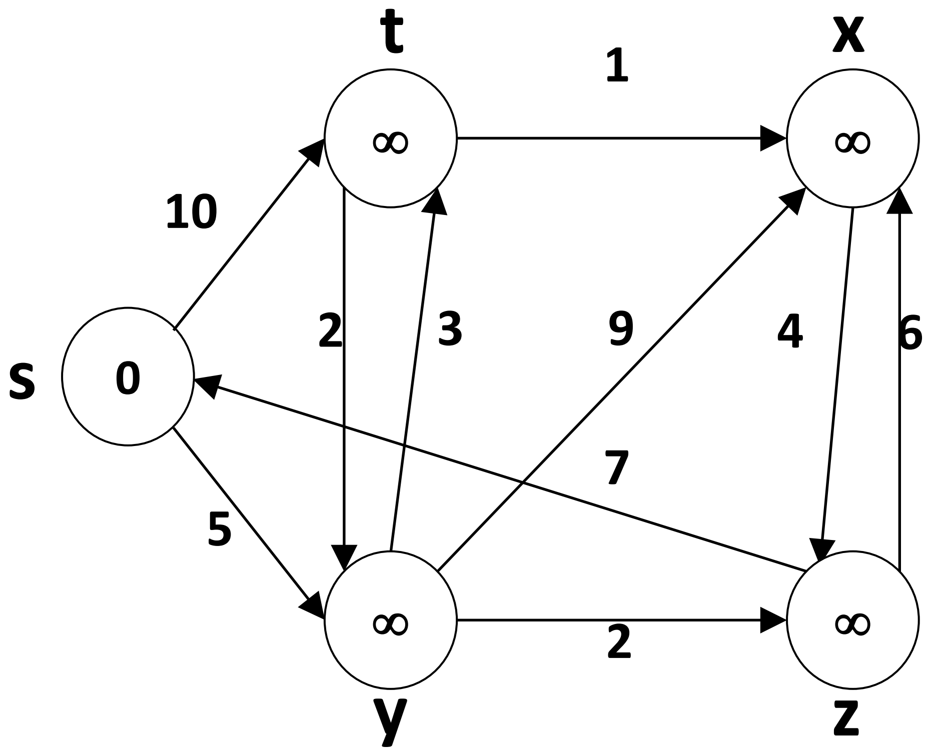
Dijkstra(G, w, s)

```
Initialize-Single-Source(G, s)
S = ϕ
Q = ϕ
for each vertex u ∈ G.V
    Insert(Q, u)
while Q ≠ ϕ do
    u = Extract-Min(Q)
    S = S ∪ {u}
    for each vertex v ∈ u.adj do
        Relax(u, v, w)
        if the call of Relax decreases v.d
            Decrease-Key(Q, v, v.d)
```

Running Example



Running Example



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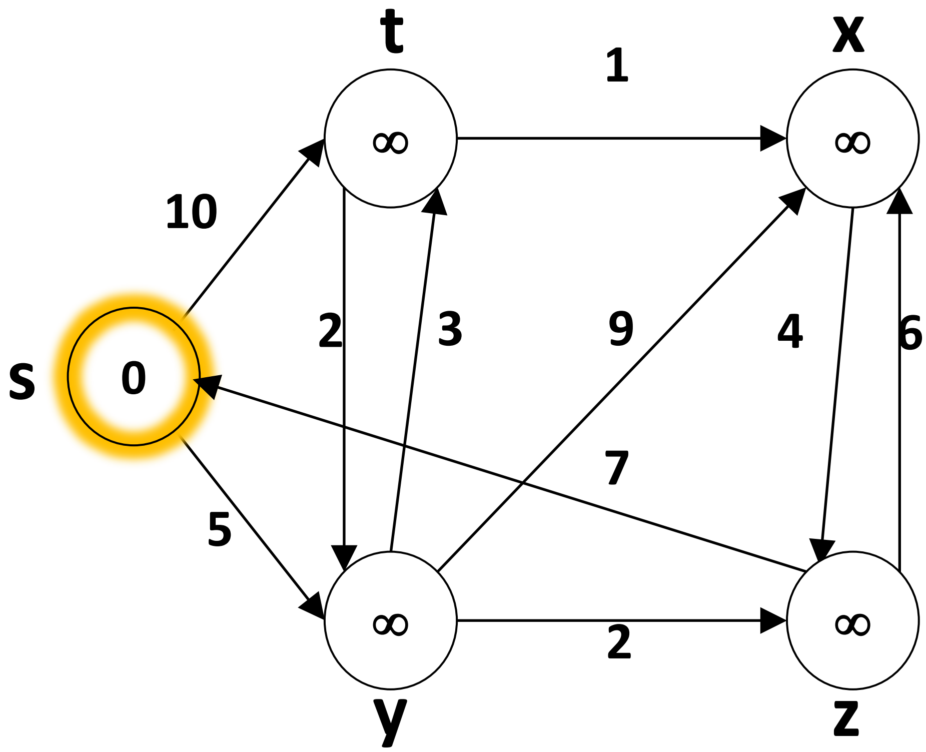

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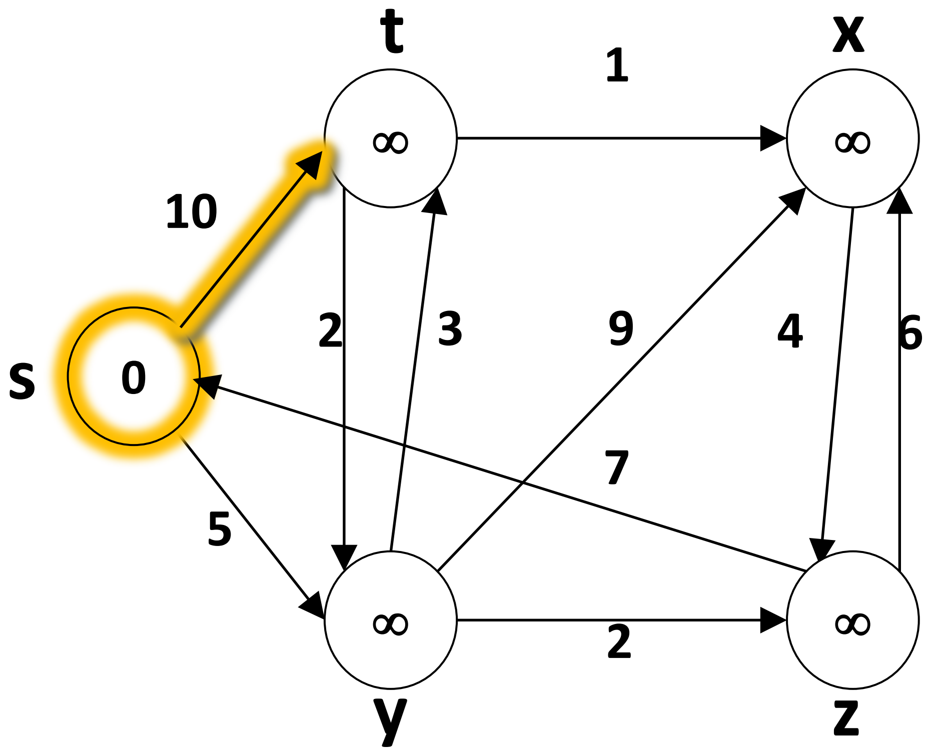

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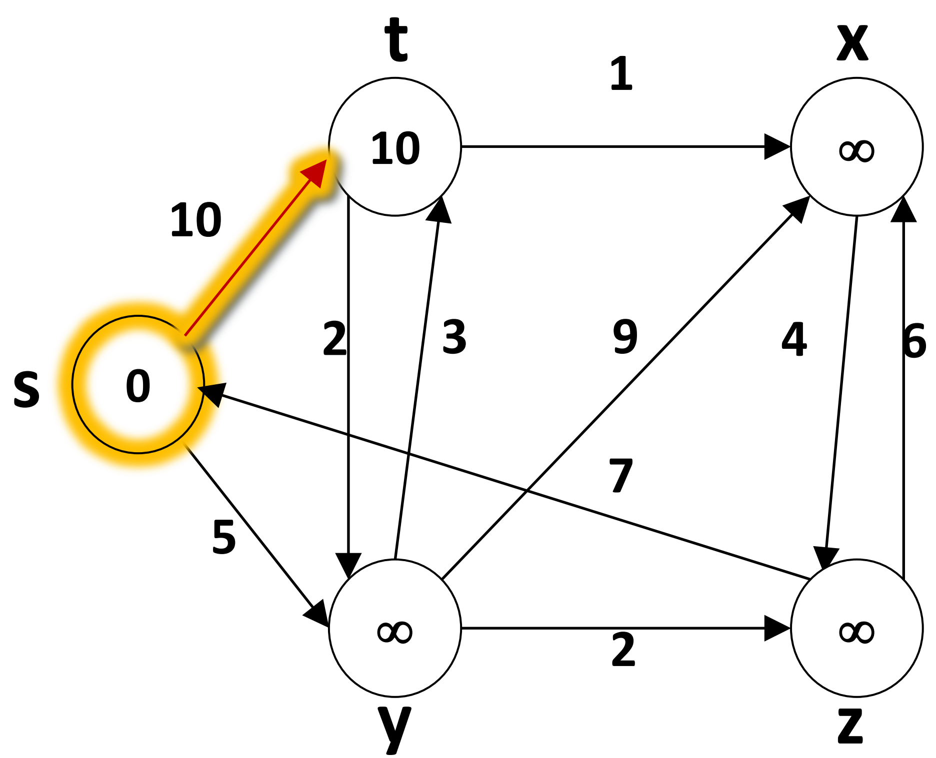

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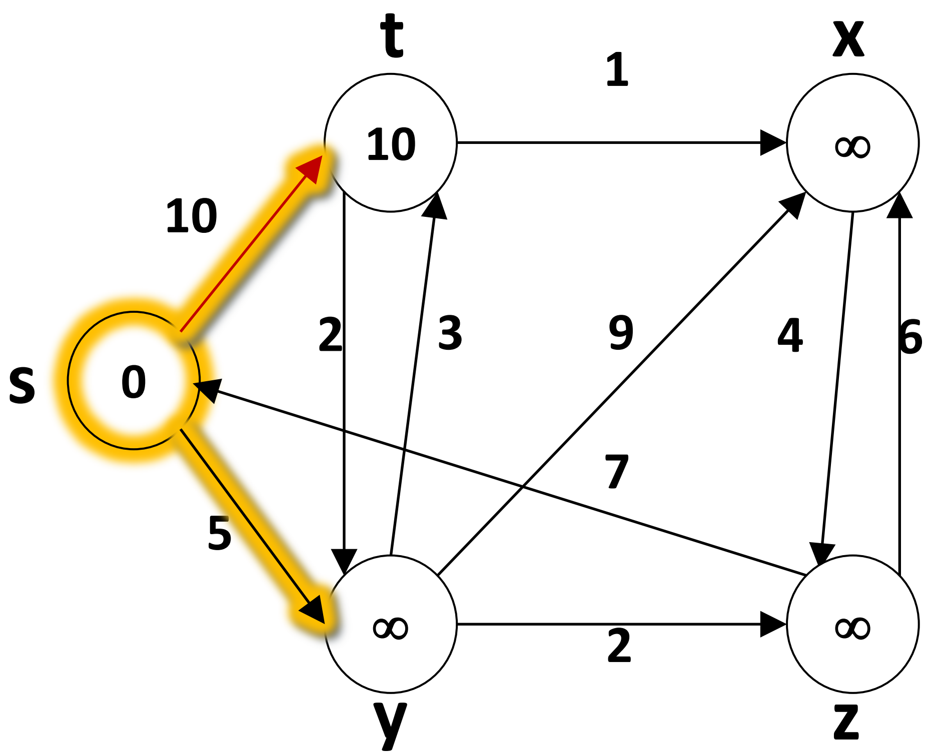

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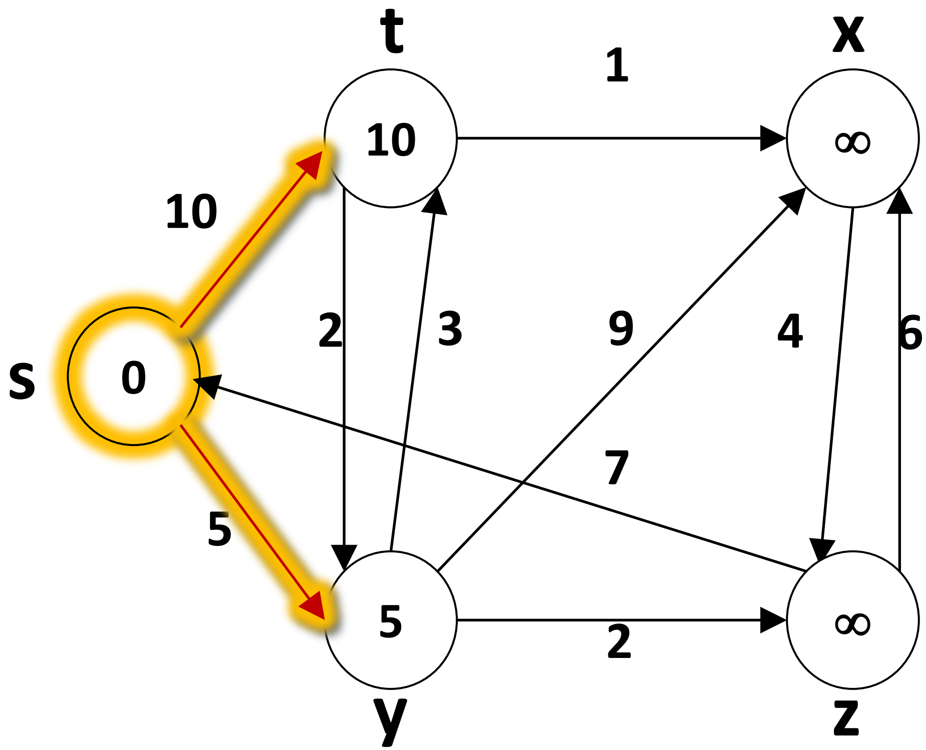

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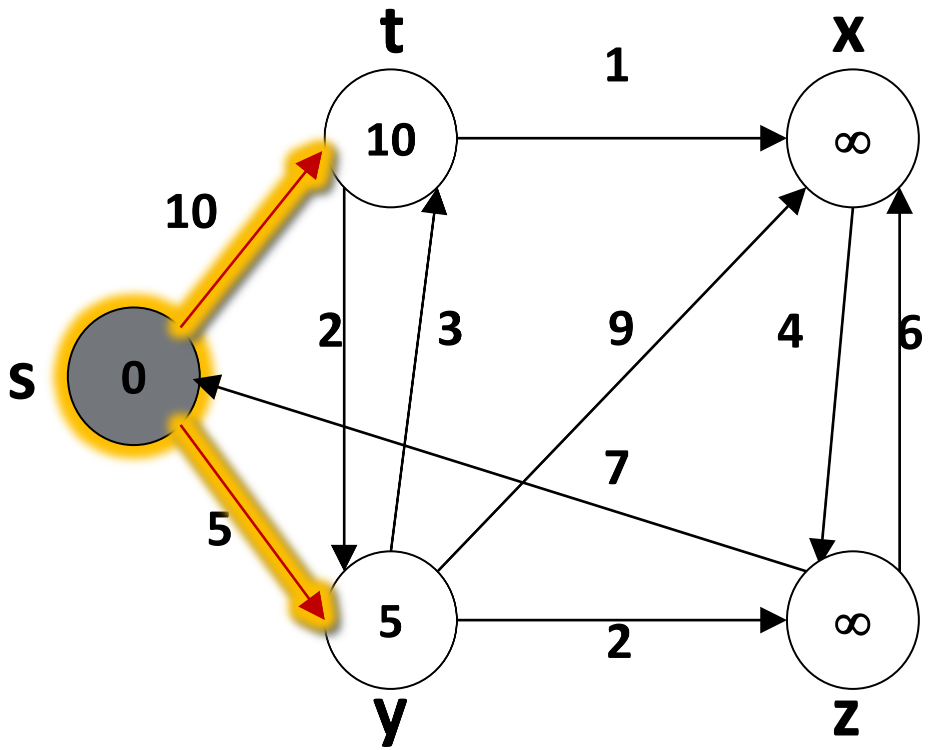

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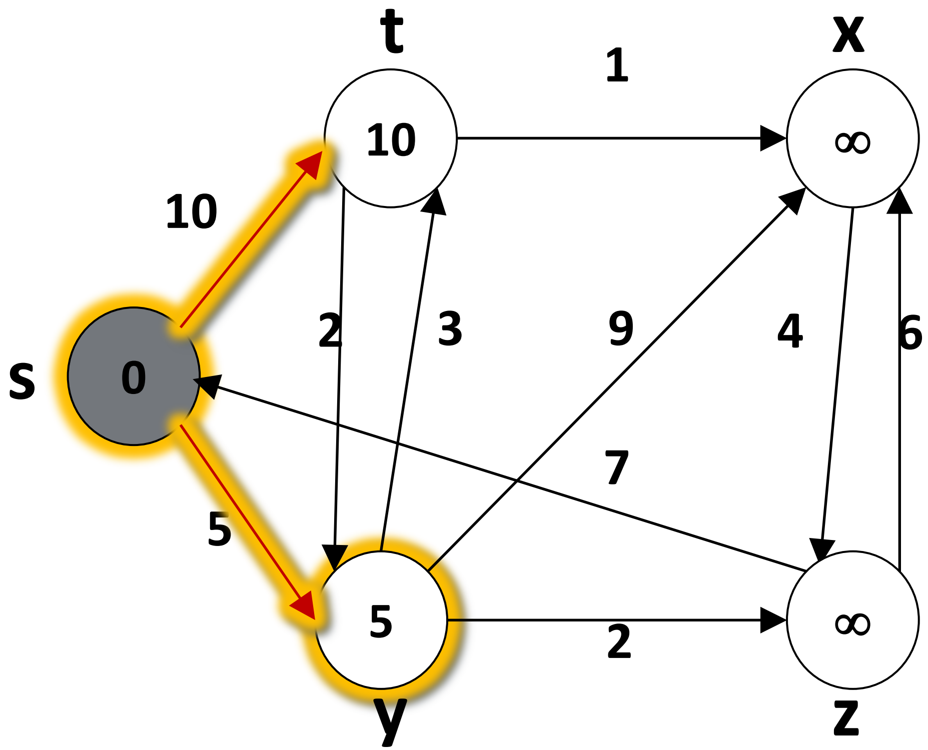

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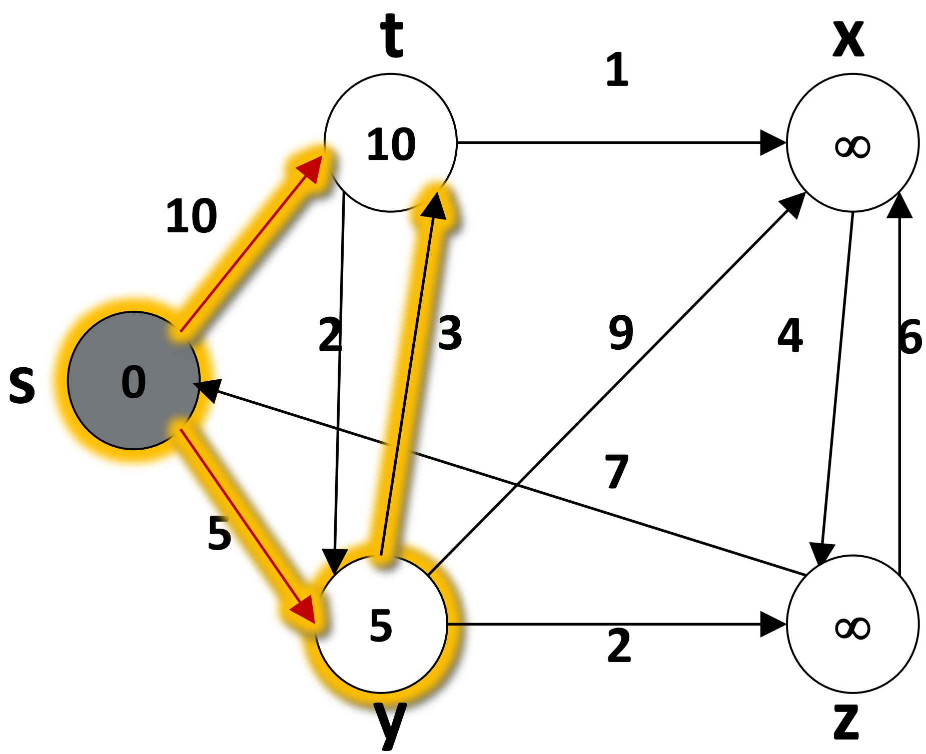

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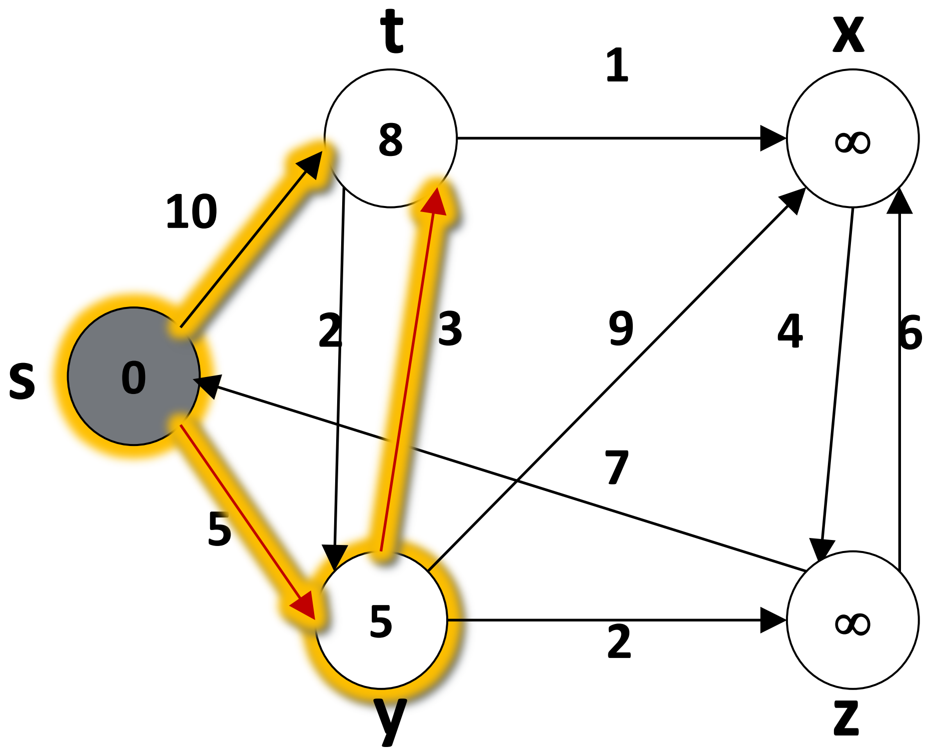

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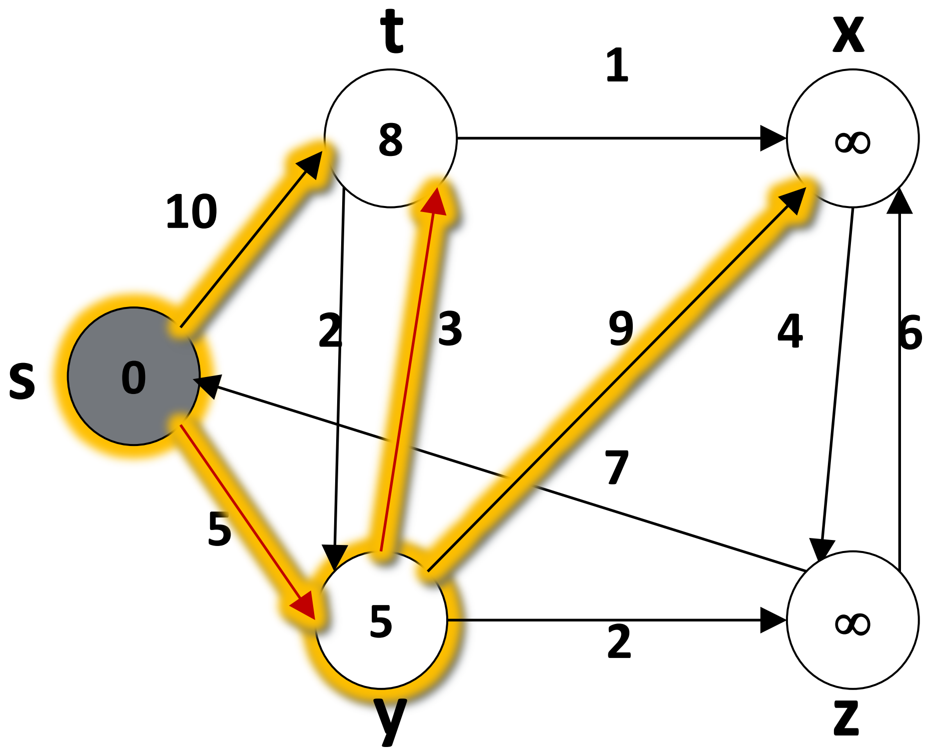

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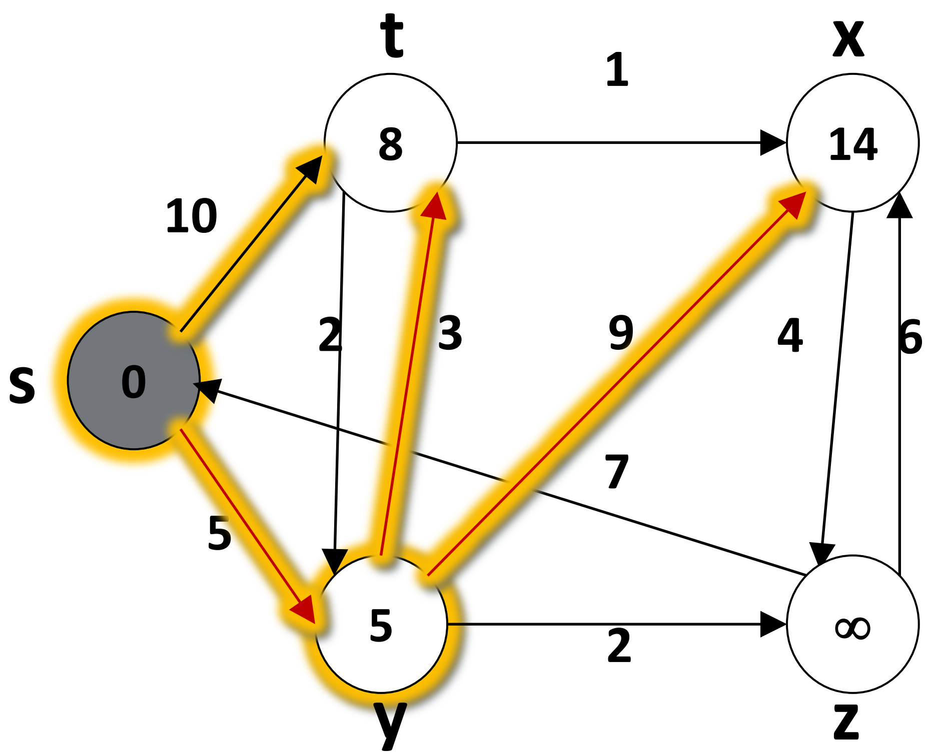

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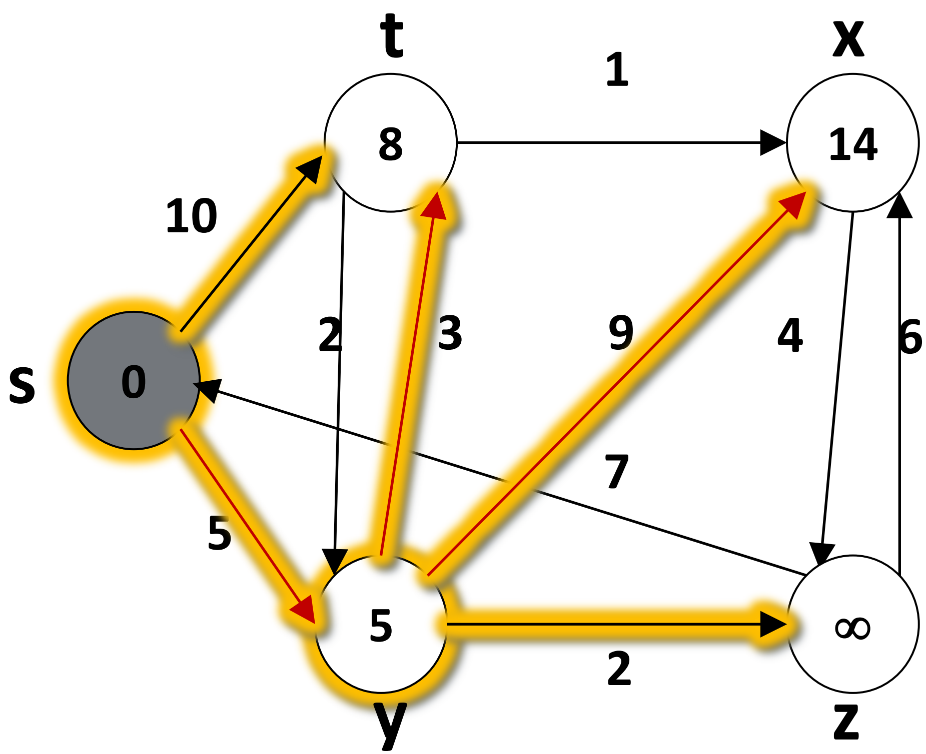

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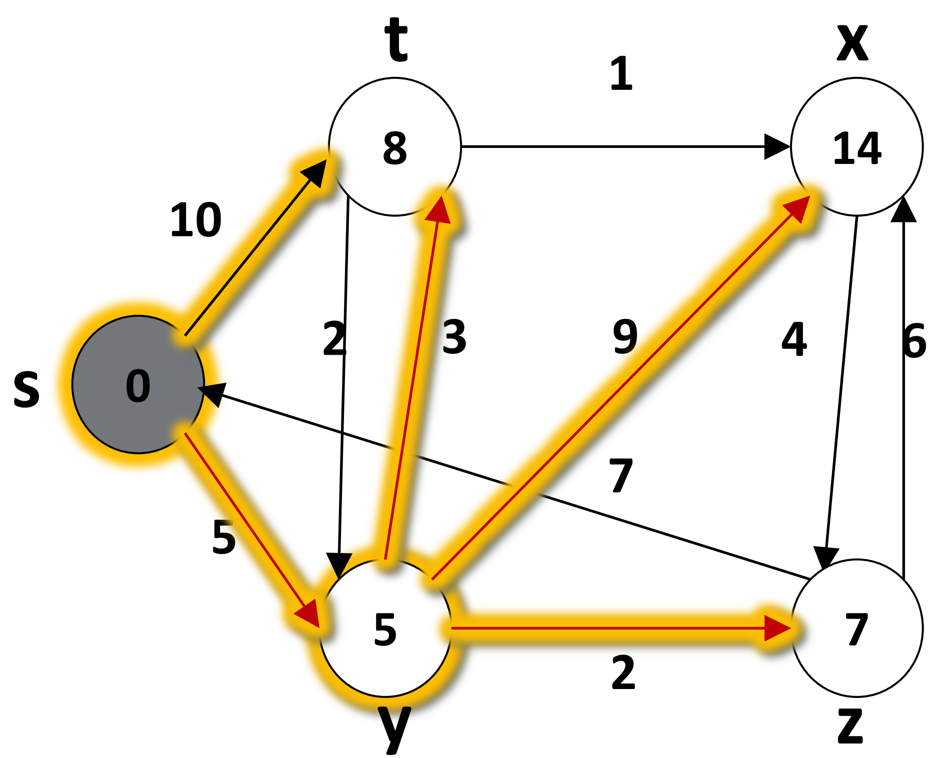

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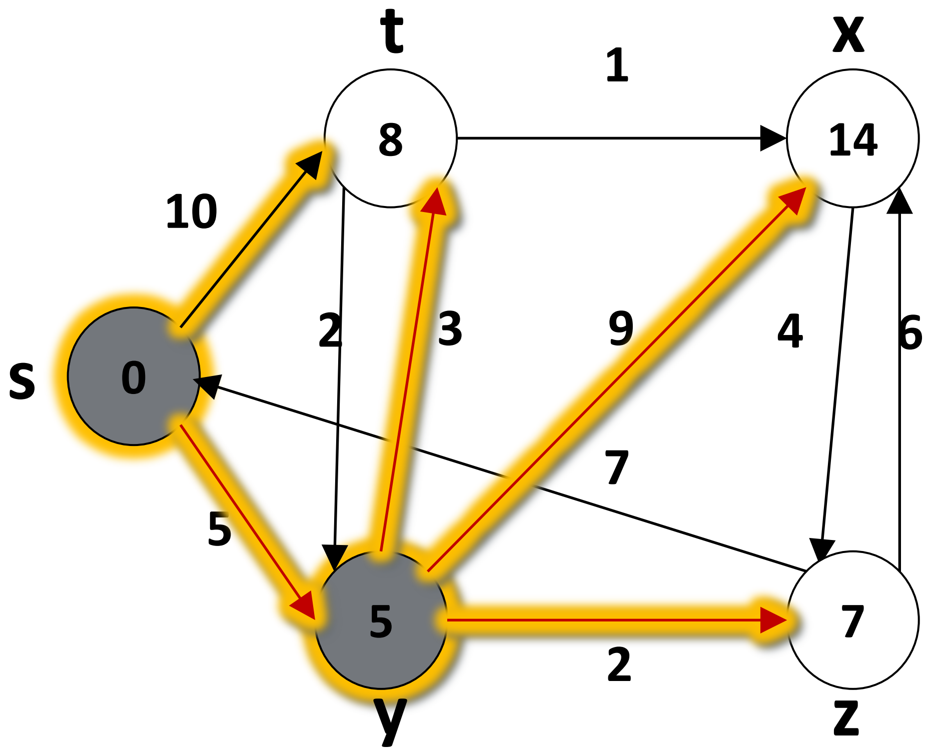

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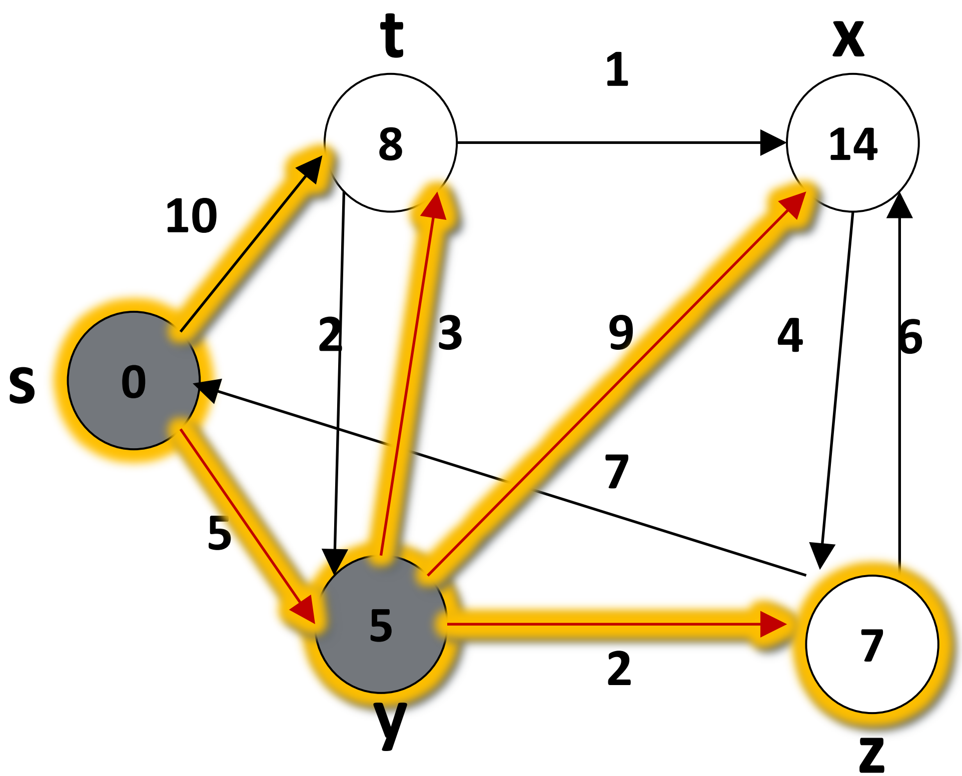

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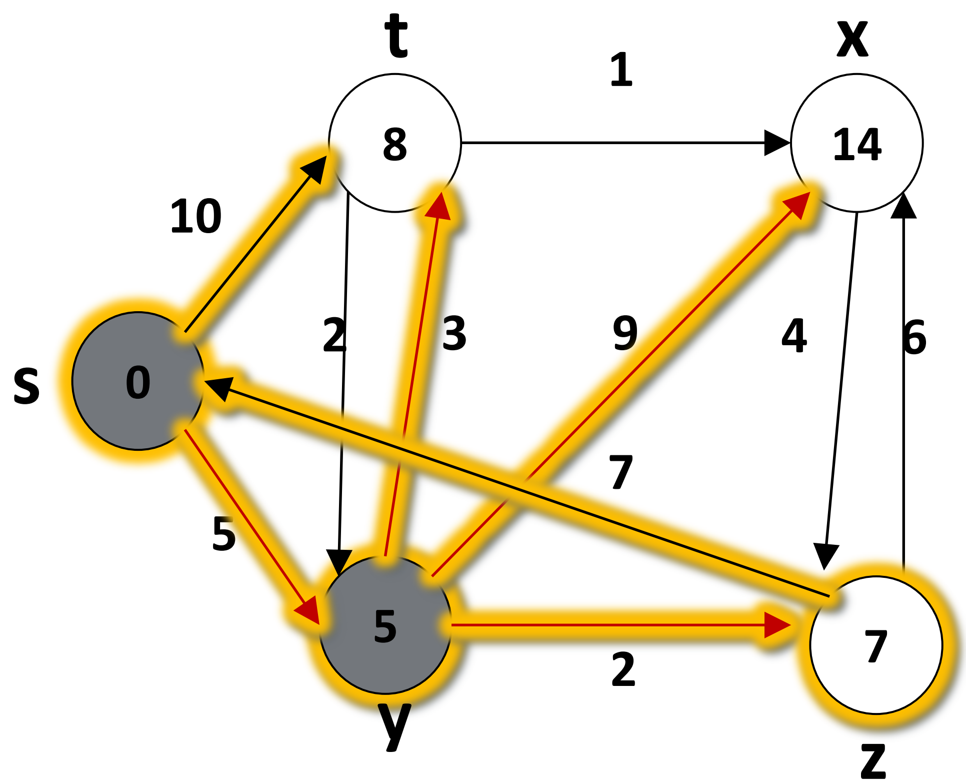

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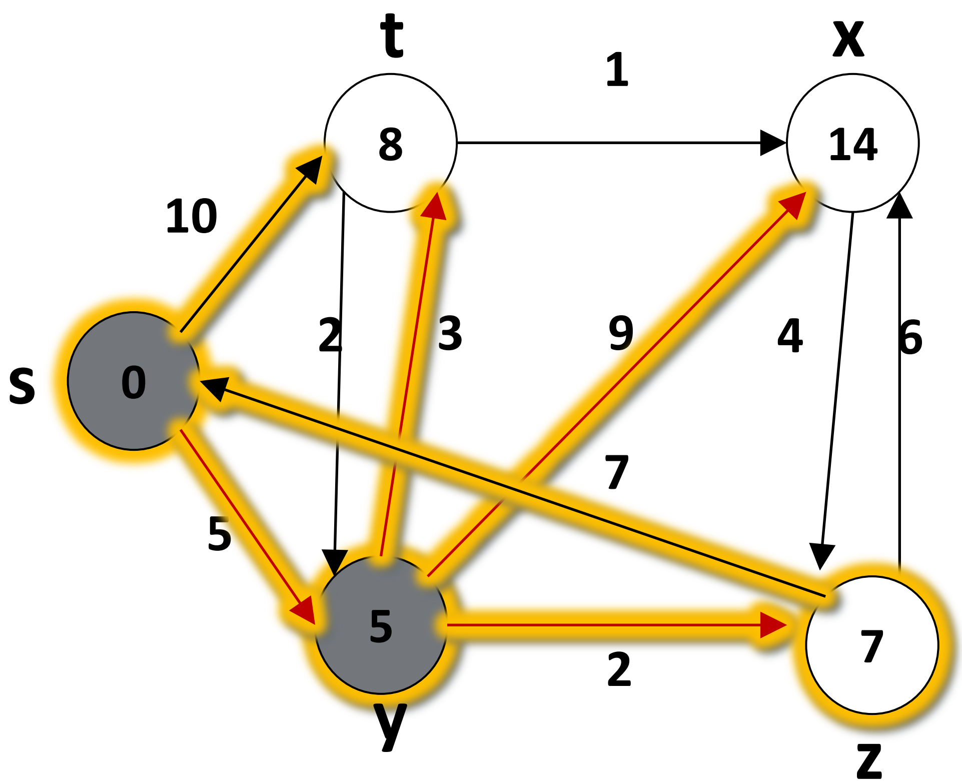

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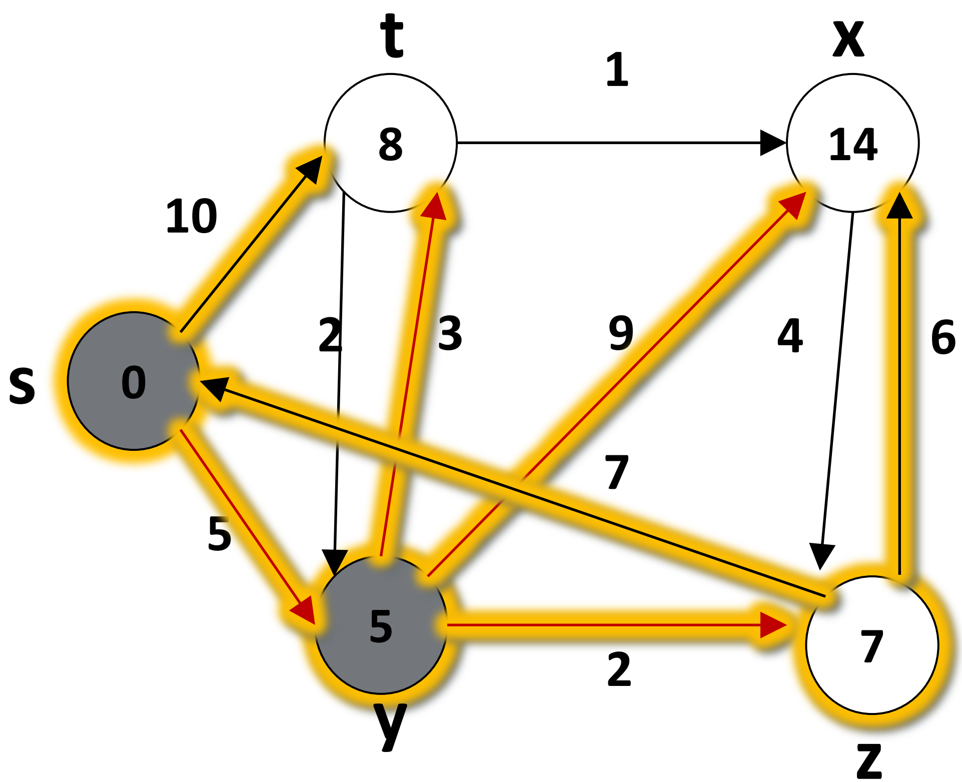

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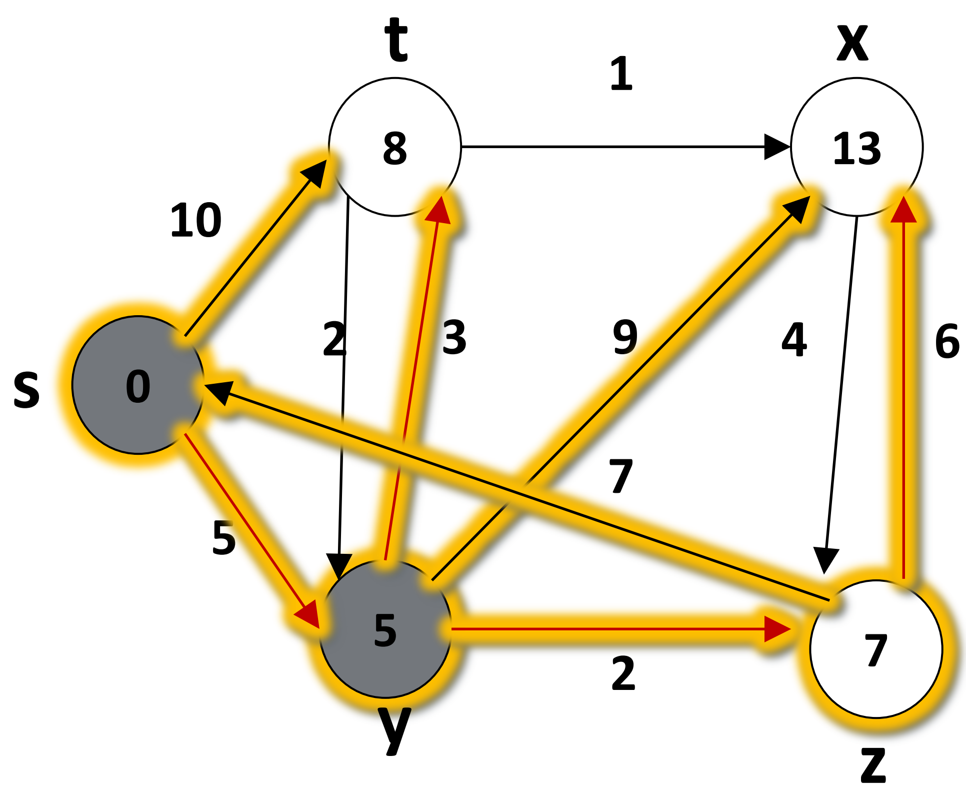

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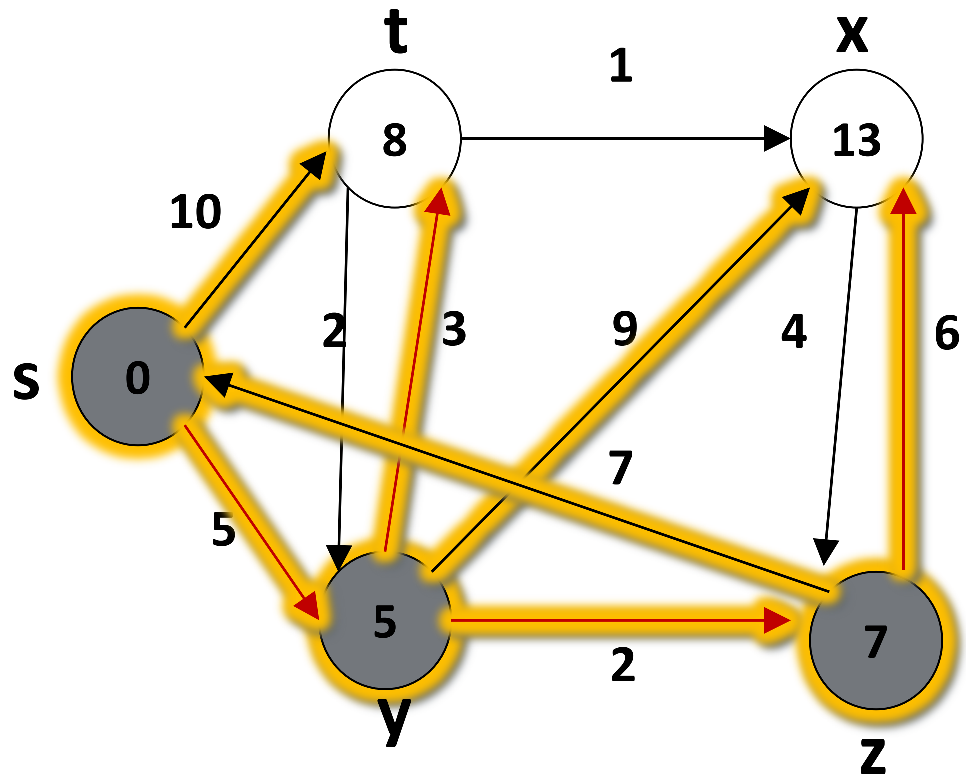

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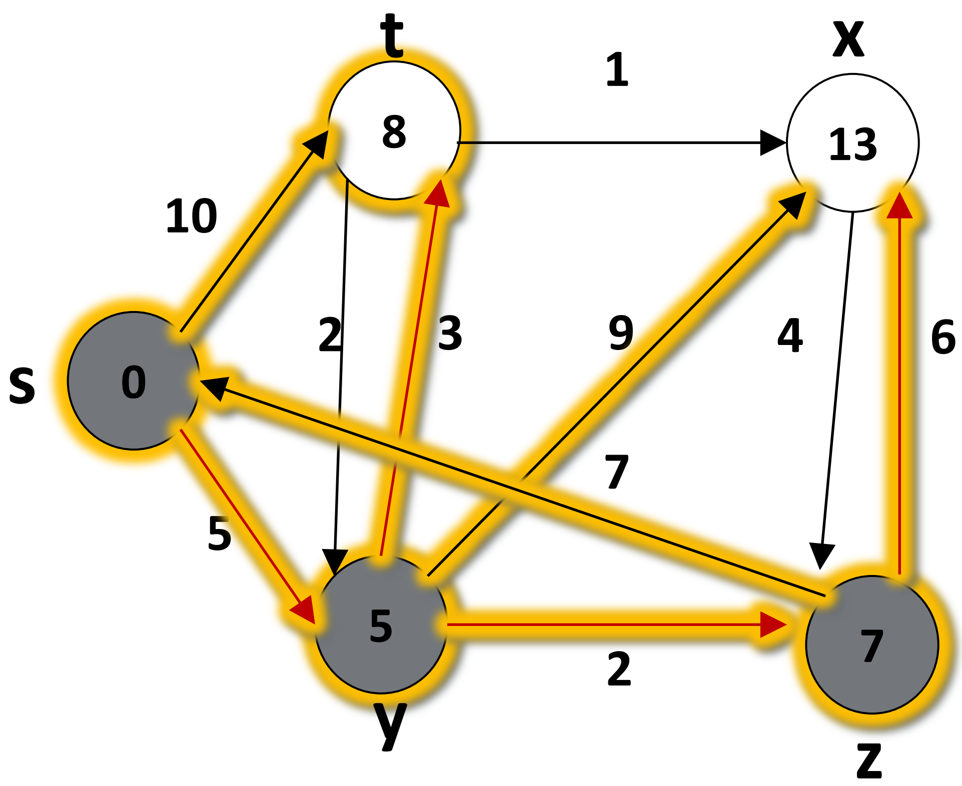

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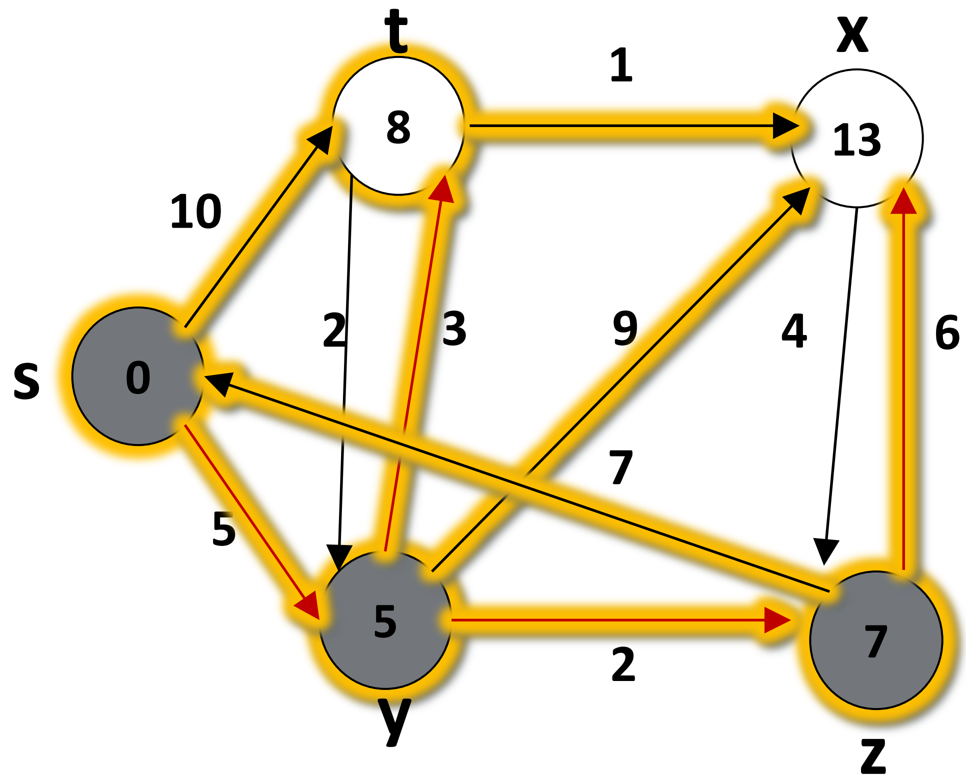

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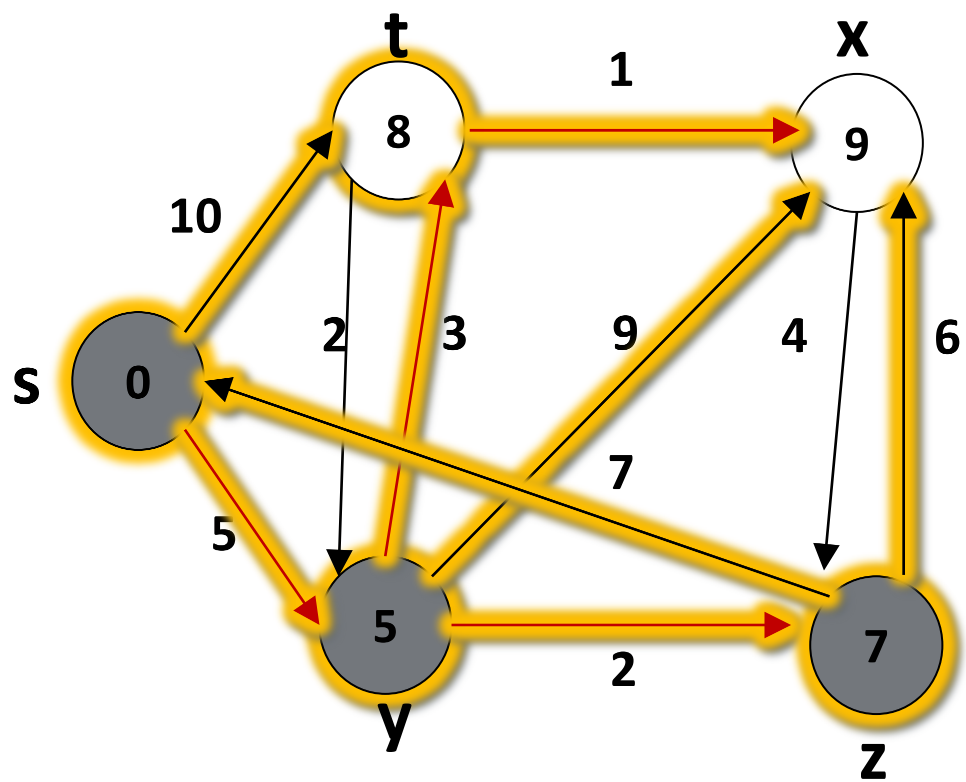

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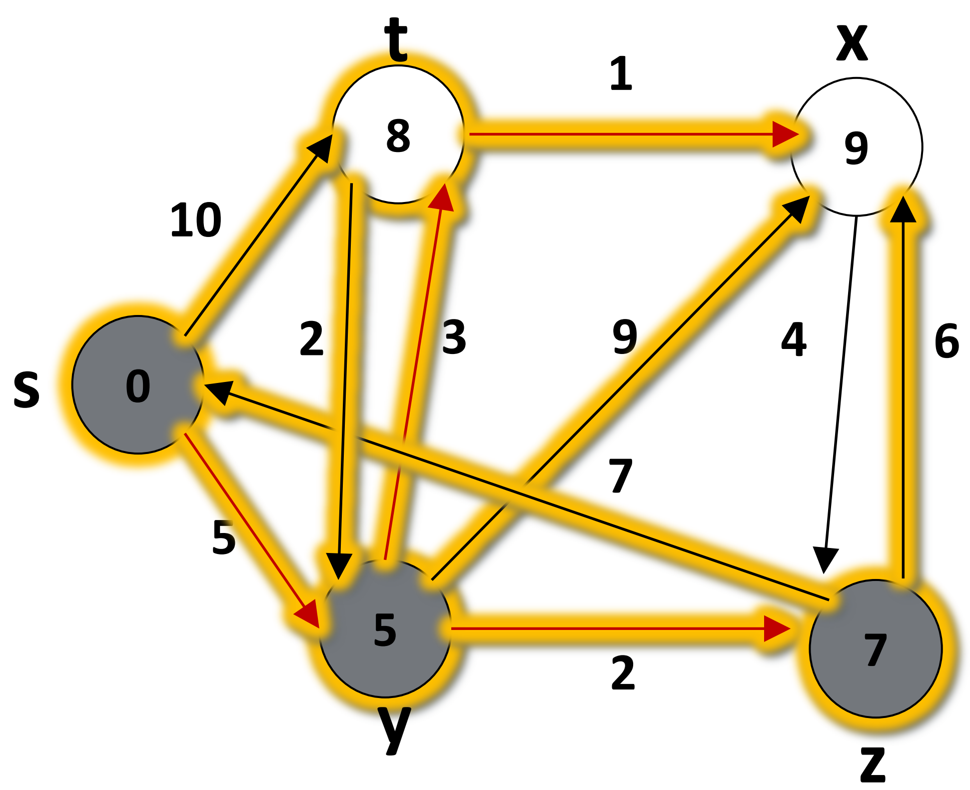

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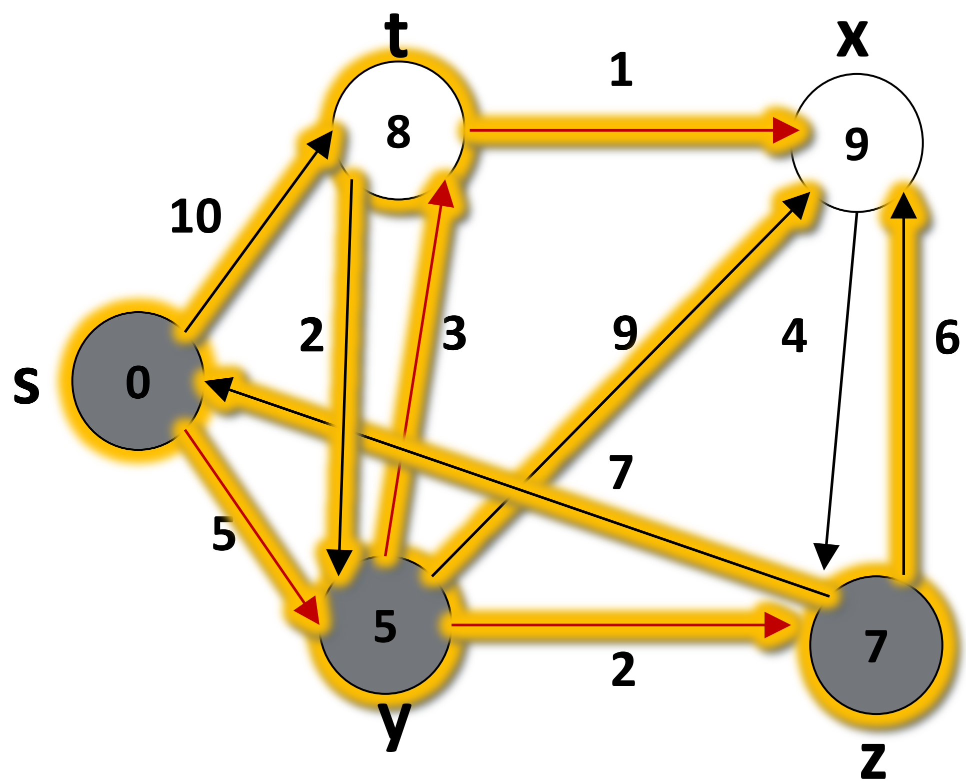

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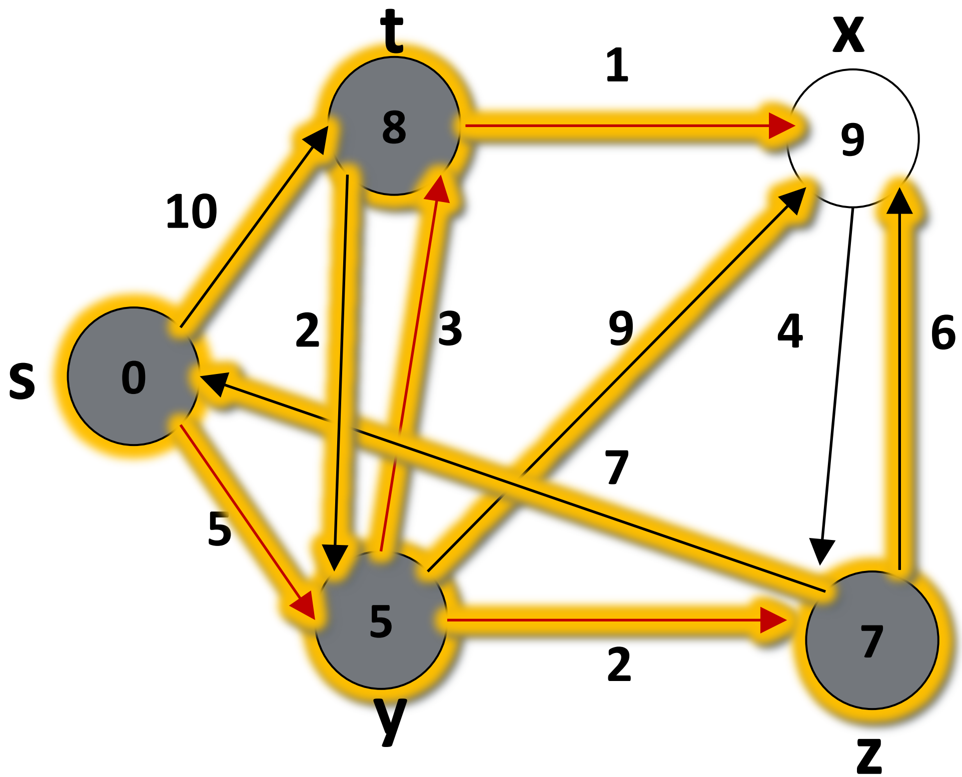

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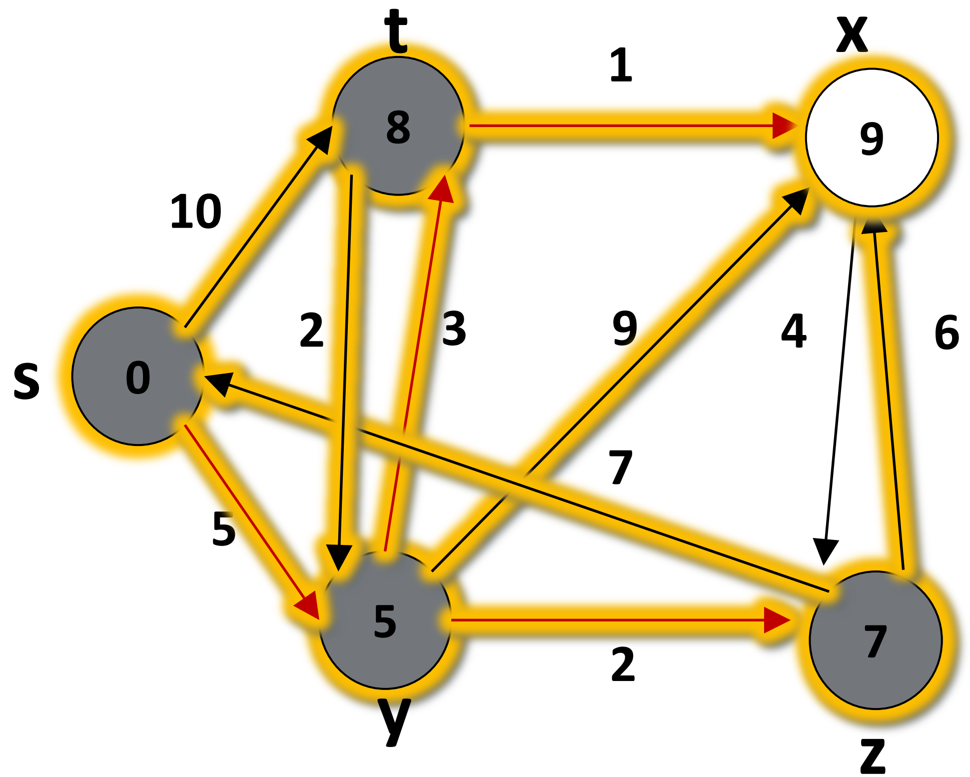

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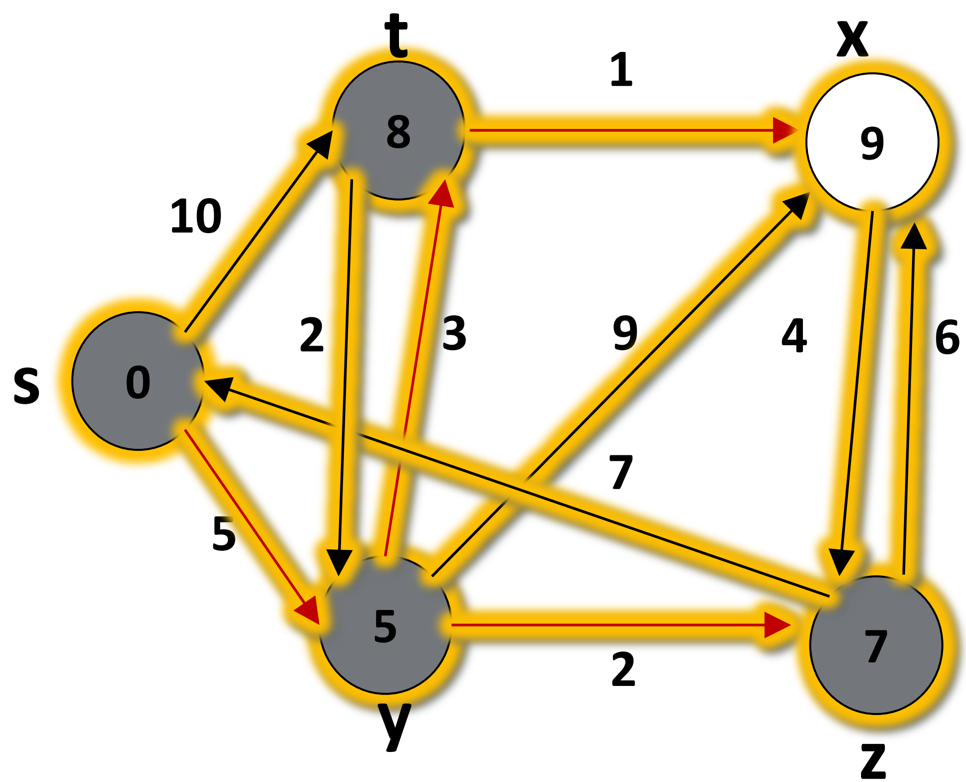

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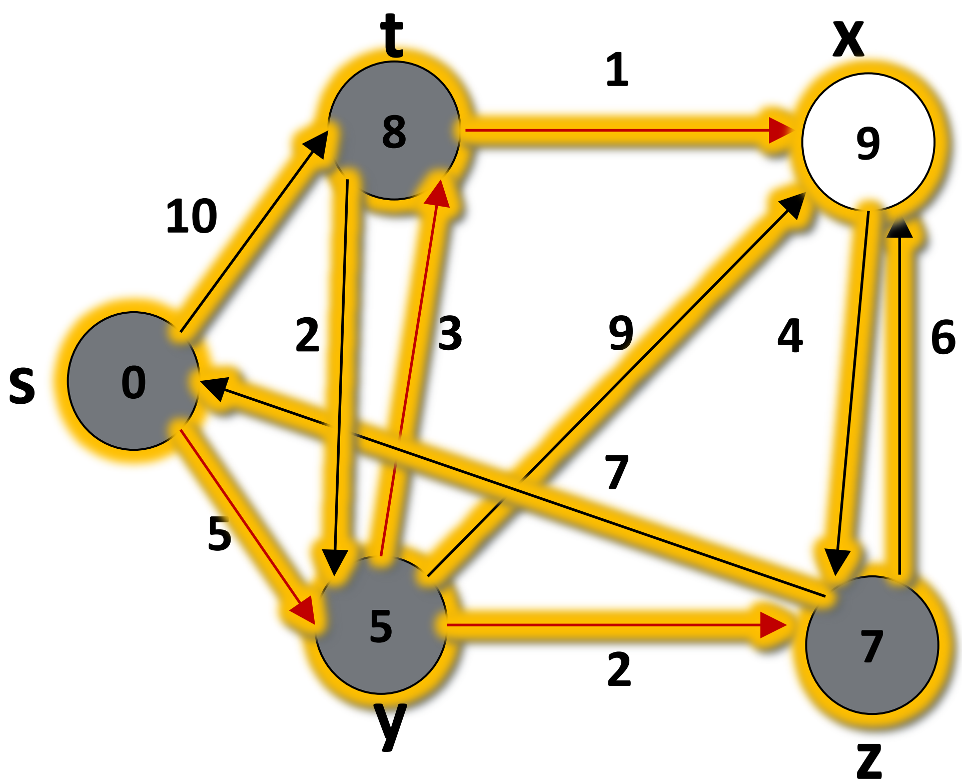

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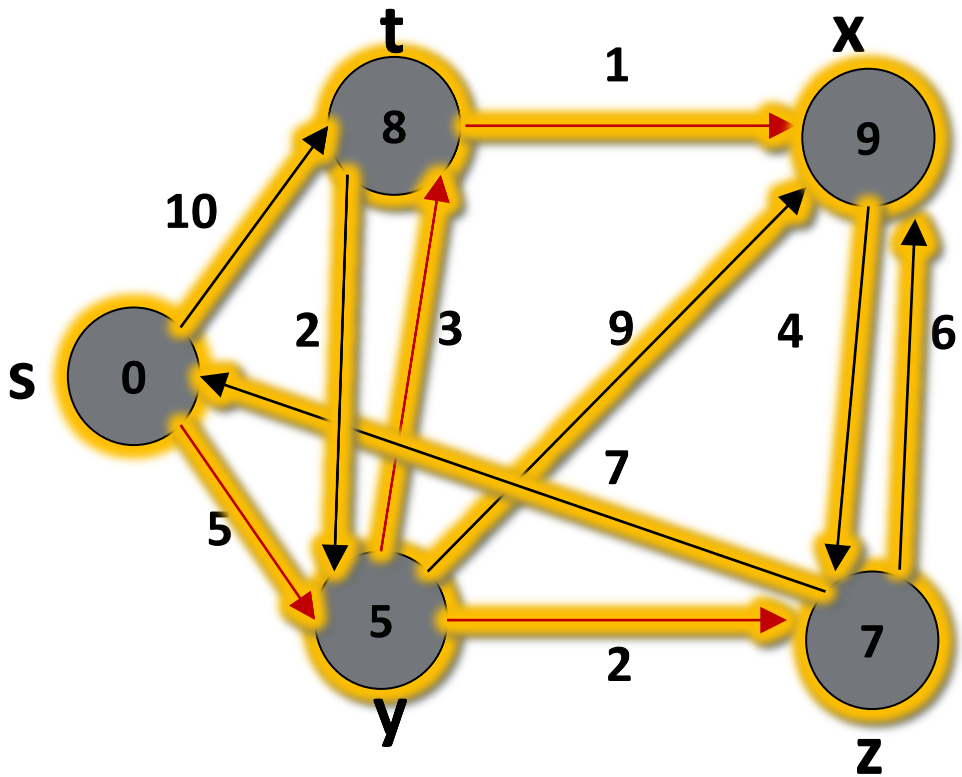

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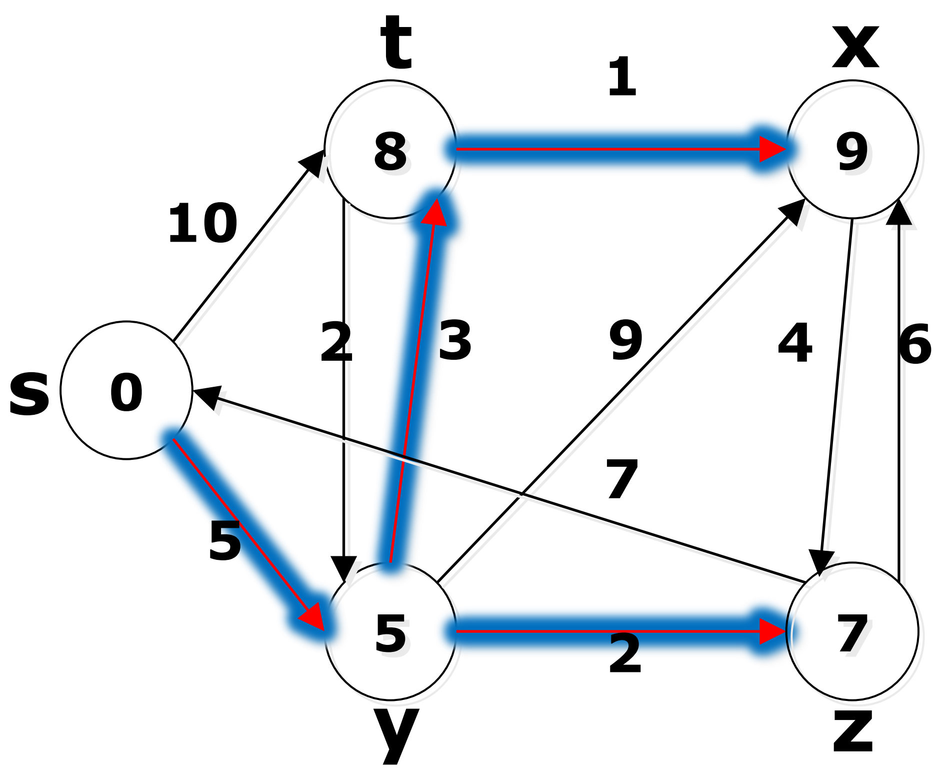

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Shortest Paths: Part 3

- Shortest Path Problem
- Dijkstra’s Shortest Path Algorithm
- Analysis

Running Time of Dijkstra's Algorithm

- **The running time depends on how we implement Q**
- **If we use a binary heap to implement the priority Q**
  - Insertion takes $O(\log|V|)$
  - Extract-Min takes $O(\log|V|)$
  - v.d := u.d+w(u, v) in Relax decrease-key: $O(\log|V|)$
  - The total running time is $O((|E|+|V|)\cdot\log|V|)$
    $= O(|E|\cdot\log|V|)$ if the graph is connected. If we use a
- **If we use Fibonacci heap to implement the priority Q**
  - The total running time is $O(|V|\cdot\log|V| + |E|)$

Fibonacci Heap Data Structure

- Fredman, Michael Lawrence; Tarjan, Robert E. "Fibonacci heaps and their uses in improved network optimization algorithms" (PDF). Journal of the Association for Computing Machinery. 34 (3): 596–615.

Correctness of Dijkstra's Algorithm

The algorithm produces correct output.

Partial Correctness

- If the preconditions are satisfied and
- if the program terminates,
- then the postconditions are satisfied.
- Core: loop invariant and induction

Termination

- If the preconditions are satisfied
- then the algorithm terminates in finite steps
- strictly decreasing order on finite set

Correctness of Dijkstra's Algorithm

Partial correctness and termination

Precondition:

weighted graph $G = (V, E, W)$ and $s \in V$ and $w \geq 0$

Postcondition:

$d[u] = \delta(s, u)$

Loop-invariant:

$(\forall u)\ (u \in S \Rightarrow d[u] = \delta(s, u))$

$S = V$ when exiting the loop

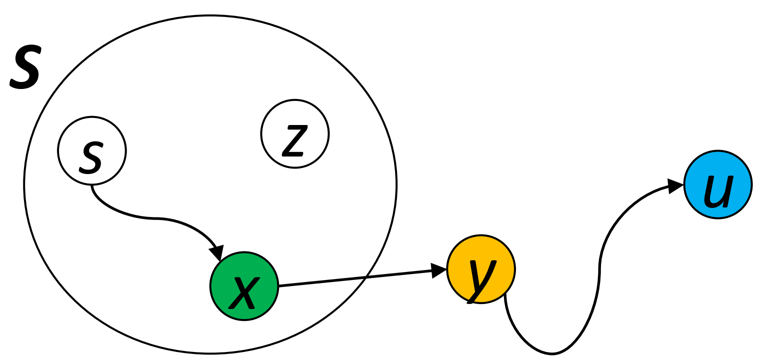
Therefore the postcondition is true.

Correctness of Dijkstra's Algorithm

- Use mathematical induction:
- Step 1: For iteration 1, $S = \{s\}$ and $d[u] = \delta(s, u)$
- Step 2: Assume that it is true when S has k elements, and we wish to prove that it is still true when another element u is added to S.
- Assume that $d[u] \neq \delta(s, u)$. We will derive a contradiction.
- Since $d[s] = \delta(s, s)=0$, $u \neq s$.
- Since u is added to S, there is a shortest path p from s to u.
- Let y be the first vertex on path p that is NOT in S.

Correctness of Dijkstra's Algorithm

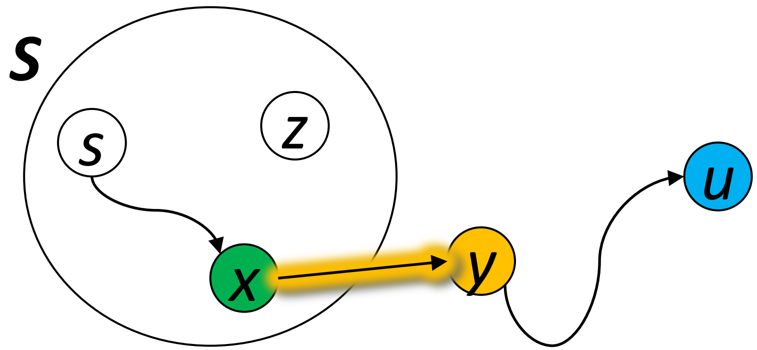
- Let y be the first vertex on path p that is not in S.
- We claim that $d[y] = \delta(s, y)$ when we add u to S.



Shortest path from s to u.

Correctness of Dijkstra's Algorithm

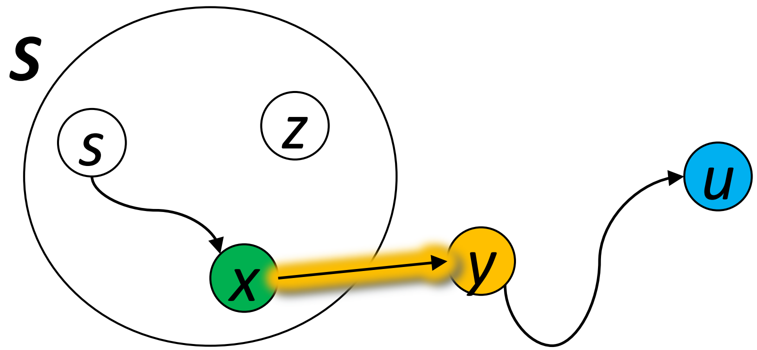
- Let y be the first vertex on path p that is not in S.
- We claim that $d[y] = \delta(s, y)$ when we add u to S.
- Let x be the predecessor of y on path p. Since y is the first vertex on p that is not in S, x is in S. At the time x is added to S, we have relaxed along edge (x, y). Hence $d[y]= \delta(s, x)+w(x, y)= \delta(s, y)$. If $y=u$, we have a contradiction. In the rest, we consider the case where $y \neq u$.



Shortest path from s to u.

Correctness of Dijkstra's Algorithm

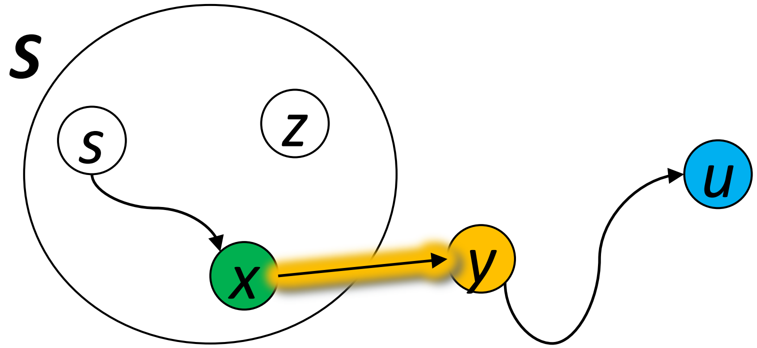
- Let y be the first vertex on path p that is not in S.
- We proved that $d[y]=\delta(s, y)$when we add u to S.
- Since y is ahead of u on path p, we have $\delta(s, y)\leq\delta(s, u)$.



Shortest path from s to u.

Correctness of Dijkstra's Algorithm

- Let y be the first vertex on path p that is not in S.
- We proved that $d[y]=\delta(s, y)$when we add u to S.
- Since y is ahead of u on path p, we have $\delta(s, y)\leq\delta(s, u)$.
- Since u is added to S before y, $d[u]\leq d[y]=\delta(s, y)$.
- Now we have $d[u]\leq d[y]=\delta(s,y)\leq\delta(s, u)\leq d[u]$.
- Therefore, **all the above inequalities must be equalities**. Hence, we have $d[u]=\delta(s, u)$. <u>This is a contradiction</u>.



Shortest path from s to u.

Dijkstra’s Algorithm on Undirected Graphs

- The algorithm works equally well on directed graphs and undirected graphs.
- Same Asymptotic time complexity.

Single-Pair Shortest Paths

- When you use a navigation system, you ask for a single-pair shortest paths, also known as s-t shortest paths, where s is the source (your current location) and t is the destination.
- We can modify Dijkstra’s algorithm slightly to compute s-t shortest paths.
- The key in this modification is: STOP as soon as the vertex t is deleted from the min-Heap.

Dijkstra’s Algorithm

Dijkstra-ST(G, w, s)

```
Initialize-Single-Source(G, s)
S = ϕ
Q = ϕ
for each vertex u ∈ G.V
    Insert(Q, u)
while Q ≠ ϕ do
        u = Extract-Min(Q)
        if (u==t) STOP
        S = S ∪ {u}
        for each vertex v ∈ u.adj do
            Relax(u, v, w)
            if the call of Relax decreases v.d
                Decrease-Key(Q, v, v.d)
```

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