Asymptotic Notations

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RAM: The Random-Access Machine

- RAM stands for <u>R</u>andom <u>A</u>ccess <u>M</u>achine, which is a very useful model of computation.
  - can access any memory location in constant time
  - each basic instruction requires one unit of time
  - running time proportional to the number of operations

- We will study functions $f(n)$ that asymptotically nonnegative, i.e., there exists constant N such that $f(n) \geq 0$ for all $\boldsymbol{n} \geq N$.

Four important summations

- $1+2+3+4+...+n = n(n+1)/2$

- $1+1/2+1/4+1/8+...+(1/2)^n = (1 - (1/2)^{n+1})/(1-1/2)$
- $1+x+x^2+x^3+...+x^n = (1 - x^{n+1})/(1-x)$, provided that $|x|<1$

- $1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1)/6$

- $1+1/2+1/3+1/4+...+1/n \in (\ln(n) + 1/n, \ln(n) + 1)$

Proofs

Theorem 1 (arithmetic sequence): For any positive integer n, we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

Proof. We prove this using mathematical induction.
Base Case: For $n = 1$, the left-hand-side (LHS) of (1) is 1, and the right-hand-side (RHS) of (1) is also 1. This shows that (1) is true for $n = 1$.
Induction Step: Assume that (1) is true for $n = k$, where k is some positive integer. We will show that if (1) is also true for $n = k + 1$.

Since (1) is true for $n = k$, we have

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \tag{2}$$

Therefore we have

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)[\frac{k}{2} + 1] \tag{3}$$

$$= (k+1)\frac{k+2}{2} = \frac{(k+1)[(k+1)+1]}{2}. \tag{4}$$

Hence (1) is also true for $n = k + 1$. By the principle of mathematical induction, we have proved that (1) is true for all integer $n \geq 1$.

Proofs

Theorem 2 (geometric sequence): For any real number x such that $x \neq 1$ and any positive integer n, we have

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}. \tag{1}$$

Proof. We have

$$(1 - x) \times \sum_{i=0}^{n} x^i = 1 - x^{n+1}. \tag{2}$$

Since $x \neq 1$, we can divide both sides of (2) by $(1 - x)$ to get (1).

Theorem 3 (sum of squares): For any positive integer n, we have

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

The proof is similar to that of Theorem 1.

Proofs

Theorem 4 (harmonic sequence): For any integer $n \geq 2$, we have

$$\ln(n) + \frac{1}{n} < \sum_{i=1}^{n} \frac{1}{i} < \ln(n) + 1. \tag{1}$$

Proof. For any integer $i \geq 1$ and $x \in (i, i+1)$, we have

$$\frac{1}{i+1} < \frac{1}{x} < \frac{1}{i}. \tag{2}$$

Therefore, we have

$$\frac{1}{i+1} < \int_{i}^{i+1} \frac{1}{x} dx < \frac{1}{i}. \tag{3}$$

Using the left inequality of (3), we have

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \sum_{i=2}^{n} \frac{1}{i} = 1 + \sum_{i=1}^{n-1} \frac{1}{i+1} < 1 + \sum_{i=1}^{n-1} \int_{i}^{i+1} \frac{1}{x} dx \tag{4}$$

$$= 1 + \int_{1}^{n} \frac{1}{x} dx = 1 + \ln(n). \tag{5}$$

Using the right inequality of (3), we have

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i} + \frac{1}{n} > \sum_{i=1}^{n-1} \int_{i}^{i+1} \frac{1}{x} dx + \frac{1}{n} \tag{6}$$

$$= \int_{1}^{n} \frac{1}{x} dx + \frac{1}{n} = \ln(n) + \frac{1}{n}. \tag{7}$$

This proves the theorem.

Takeaways

- $1+2+3+4+...+n = n(n+1)/2$
- $1+x+x^2+x^3+...+x^n = (1 - x^{n+1})/(1-x)$, provided $|x|<1$
- $1^2+2^2+3^2+...+n^2 = n(n+1)(2n+1)/6$
- $1+1/2+1/3+1/4+...+1/n \in (\ln(n) + 1/n, \ln(n) + 1)$

- We will use these frequently in this course.
- Know them (and their variants) by heart.

Definitions of Big-O, Big-Ω, and Big-Θ

Let $g(n)$ be an asymptotically non-negative function of variable n.

$O(g(n)) = \{f(n) | \exists c_f > 0 \text{ and } N_f > 0 \ s.t. \ f(n) \leq c_f \times g(n) \text{ for all } n \geq N_f\}$.

$\Omega(g(n)) = \{f(n) | \exists c_f > 0 \text{ and } N_f > 0 \ s.t. \ f(n) \geq c_f \times g(n) \text{ for all } n \geq N_f\}$.

$\Theta(g(n)) = \{f(n) | \exists c_f > 0, d_f > 0, \text{ and } N_f > 0 \ s.t. \ c_f \times g(n) \leq f(n) \leq d_f \times g(n) \text{ for all } n \geq N_f\}$.

Note that c_f, d_f, and N_f are constants, for the given function $f(n)$.

Note that $O(g(n)), \Omega(g(n)), \Theta(g(n))$ are sets of functions, for each given $g(n)$.

- **Big-O: asymptotic upper bound**
- **Big-Ω : asymptotic lower bound**
- **Commonly uses choices of g(n)**
  - **1, log(n), n, n log(n), n^2, n^3, 2^n.**

Proof example for Big-O

Prove that $f(n) = 10n^2 + 20000$ is an element of $O(n^2)$.
Proof Idea: We need to find constants $c > 0$ and $N > 0$ such that

$$10n^2 + 20000 \leq c \times n^2, \forall n \geq N. \tag{1}$$

Since

$$10n^2 + 20000 > 10n^2, \tag{2}$$

we have to choose $c > 10$ for inequality (1) to hold for some integer $N \geq 1$. Let's try $c = 11$. In this case, (1) is equivalent to

$$11n^2 \geq 10n^2 + 20000, \forall n \geq N. \tag{3}$$

(3) is equivalent to

$$n^2 \geq 20000, \forall n \geq N, \tag{4}$$

which is equivalent to

$$n \geq \sqrt{20000} \approx 141.42, \forall n \geq N. \tag{5}$$

Now we know that we can pick $c_f = 11$ and $N_f = 142$ in our proof.
Proof. We pick $c_f = 11$ and $N_f = 142$. Then for all $n \geq N_f$, we have

$$\begin{aligned} f(n) &= 10 \times n^2 + 20000 && (6) \\ &\leq c_f \times n^2 + (20000 - n^2) && (7) \\ &\leq c_f \times n^2. && (8) \end{aligned}$$

This proves that $f(n) \in O(n^2)$.

Example: O(1), O(5), O(1000000)

- $f(n) = 1$, for all $n \geq 1$.
- $f(n) = 0$, for all $n \geq 1$.
- $f(n) = 1 + 1/n$, for all $n \geq 1$.
- $f(n) = (1 + 1/n)^n$, for all $n \geq 1$.
- $f(n) = 10000000000$, for all $n \geq 1$.

- What about $f(n) = n$?
- What about $f(n) = \log(n)$?
- Is O(1) equal to O(5)?

Example: O(log(n))

- **If $f(n) \in O(1)$, then $f(n)$ $O(\log(n))$.**
  - **$f(n) = 1 + ½ + 1/3 + ¼ + ... + 1/n$, for all $n \geq 1$.**

  - **What about $f(n) = n$?**
  - **What about $f(n) = \log(n)^2$?**

Example: $O(n)$, $O(2n+5)$, $O(3n+1000000)$

- **If $f(n) \in O(1)$, then $f(n)$ $O(n)$.**
  - $f(n) = n$, for all $n \geq 1$.
  - $f(n) = 8n + 10000$, for all $n \geq 1$.
  - What about $f(n) = n^2$?
  - What about $f(n) = n \log(n)$?
  - Is $O(n)$ equal to $O(2n+5)$?

Example: $O(n^2)$

- $f(n) = 1 + 2 + 3 + 4 + \ldots + n$, for all $n \geq 1$.

- What about $f(n) = n^3$?
- What about $f(n) = n^2 \log(n)$?

Example: $O(n^3)$

- $f(n) = 1^2 + 2^2 + 3^2 \quad + \ldots + n^2$, for all $n \geq 1$.

- What about $f(n) = n^4$?
- What about $f(n) = n^3 \log(n)$?

Proof example for Big-Ω

Prove that $f(n) = 0.01n^2 + 20000$ is an element of $\Omega(n^2)$.
Proof Idea: We need to find constants $c > 0$ and $N > 0$ such that

$$0.01n^2 + 20000 \geq c \times n^2, \forall n \geq N. \tag{1}$$

Since

$$\lim_{n\to\infty} n^2 = \infty, \tag{2}$$

we have to choose $c \leq 0.01$ for inequality (1) to hold for some integer $N \geq 1$. Let's try $c = 0.01$. In this case, (1) is equivalent to

$$0.01n^2 + 20000 \geq 0.01n^2, \forall n \geq N. \tag{3}$$

(3) is always true for any $N \geq 1$. Now we know that we can pick $c_f = 0.01$ and $N_f = 1$ in our proof.
Proof. We pick $c_f = 0.01$ and $N_f = 1$. Then for all $n \geq N_f$, we have

$$\begin{aligned} f(n) &= 0.01 \times n^2 + 20000 && (4)\\ &\geq c_f \times n^2 + 20000 && (5)\\ &\geq c_f \times n^2. && (6) \end{aligned}$$

This proves that $f(n) \in \Omega(n^2)$.

Example: $\Omega(1)$, $\Omega(5)$, $\Omega(0.0001)$

- $f(n) = 1$, for all $n \geq 1$.
- $f(n) = \log(n)$, for all $n \geq 1$.
- $f(n) = n$, for all $n \geq 1$.
- $f(n) = n^2$, for all $n \geq 1$.
- $f(n) = 2^n$, for all $n \geq 1$.

- What about $f(n) = 0$?
- What about $f(n) = 1/n$?
- Is $\Omega(1)$ equal to $\Omega(5)$?

Example: $\Omega(\log(n))$

- $f(n) = 1 + \frac{1}{2} + 1/3 + \frac{1}{4} + \ldots + 1/n$, for all $n \geq 1$.

- What about $f(n) = 10000$?

Example: $\Omega(n)$, $\Omega(2n+5)$, $\Omega(3n+1000000)$

- $f(n) = n$, for all $n \geq 1$.
- $f(n) = 0.0001n - 10000$, for all $n \geq 1$.

- What about $f(n) = \log(n)$?
- Is $\Omega(n)$ equal to $\Omega(2n+5)$?

Example: $\Theta(1)$, $\Theta(5)$

- $f(n) = 1$, for all $n \geq 1$.
- $f(n) = (1+1/n)^n$, for all $n \geq 1$.
- $f(n) = 0.01$, for all $n \geq 1$.
- $f(n) = 1000$, for all $n \geq 1$.
- If $f(n) \in O(1)$ and $f(n) \in \Omega(1)$, then $f(n) \in \Theta(1)$
- If $f(n) \in \Theta(1)$, then $f(n) \in O(1)$ and $f(n) \in \Omega(1)$.

Example: $\Theta(n)$, $\Theta(5n)$

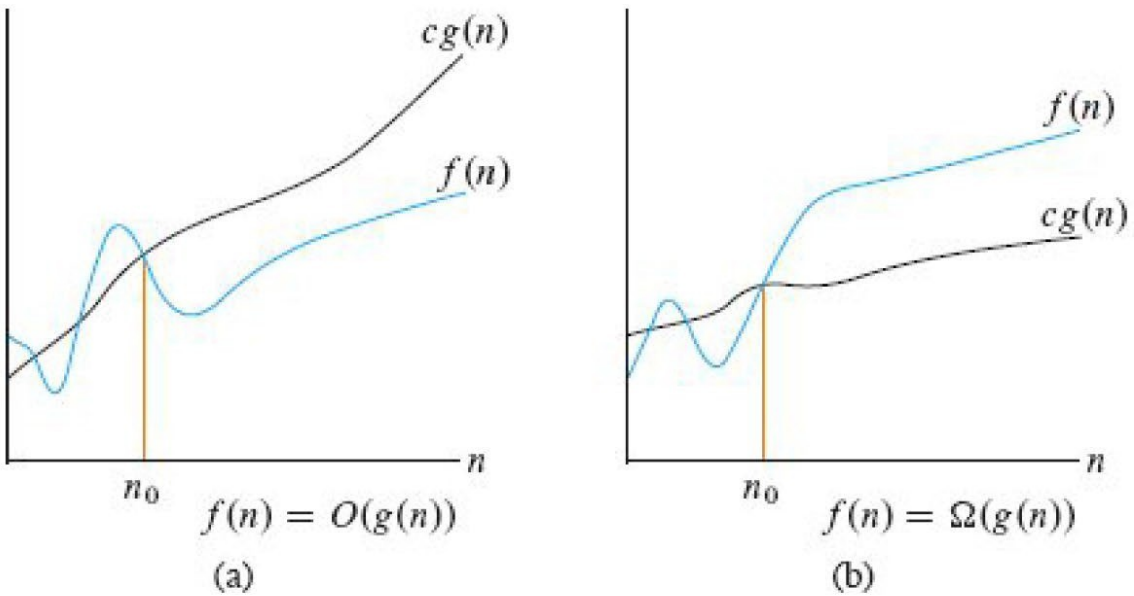
- If $f(n) \in O(n)$ and $f(n) \in \Omega(n)$, then $f(n) \in \Theta(n)$
- If $f(n) \in \Theta(n)$, then $f(n) \in O(n)$ and $f(n) \in \Omega(n)$.

Takeaways

- To prove $f(n) \in O(g(n))$, you need to find the constants c_f and N_f, and prove using the definition.
- The pair of constants are not unique. You just need to find a pair that works.

- To prove $f(n) \in \Omega(g(n))$, you need to find the constants c_f and N_f, and prove using the definition.
- To prove $f(n) \in \Theta(g(n))$, you need to find the constants c_f, d_f, N_f, and prove using the definition.

Asymptotic bounds



Asymptotic bounds

- $f(n) \le g(n)$ implies $f(n) \in O(g(n))$.
- $f(n) \in O(g(n))$ does not imply $f(n) \le g(n)$.

- Example
- $f(n) = 1000n$ for all n, $g(n) = n$ for all n.
- We have $f(n) \in O(g(n))$.

- Similarly, $f(n) \in \Omega(g(n))$ does not imply $f(n) \ge g(n)$.

Mathematical Proofs for Asymptotic Notations

- There are two main methods to analyze asymptotic notations
- Method 1: Use definitions
- Method 2: Use limit on the ratio of the two functions
  - If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c > 0$, then f(n) ∈ Θ(g(n))
  - If $\lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$, then f(n) ∈ O(g(n))
  - If $\lim_{n \to \infty} \frac{g(n)}{f(n)} < \infty$, then f(n) ∈ Ω(g(n))

Some Facts

- $O(1) \subseteq O(\ln (n)) \subseteq O(n) \subseteq O(n \ln(n)) \subseteq O(n^2)$
  - $\Omega(n^2) \subseteq \Omega(n \ln (n)) \subseteq \Omega(n) \subseteq \Omega(\ln(n)) \subseteq \Omega(1)$

Commonly used (bad) notations

- When we read $f(n) = O(g(n))$, read it as $f(n) \in O(g(n))$
- When we read $f(n) = \Omega(g(n))$, read it as $f(n) \in \Omega(g(n))$

- When we read $O(n) = O(n^2)$, read it as $O(n) \subseteq O(n^2)$.
- When we read $\Omega(n^2) = \Omega(n)$, read it as $\Omega(n^2) \subseteq \Omega(n)$.

Properties

Transitivity:

$$
\begin{aligned}
f(n) &= \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \quad \text{imply} \quad f(n) = \Theta(h(n)) , \\
f(n) &= O(g(n)) \text{ and } g(n) = O(h(n)) \quad \text{imply} \quad f(n) = O(h(n)) , \\
f(n) &= \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \quad \text{imply} \quad f(n) = \Omega(h(n)) , \\
f(n) &= o(g(n)) \text{ and } g(n) = o(h(n)) \quad \text{imply} \quad f(n) = o(h(n)) , \\
f(n) &= \omega(g(n)) \text{ and } g(n) = \omega(h(n)) \quad \text{imply} \quad f(n) = \omega(h(n)) .
\end{aligned}
$$

Reflexivity:

$$
\begin{aligned}
f(n) &= \Theta(f(n)) , \\
f(n) &= O(f(n)) , \\
f(n) &= \Omega(f(n)) .
\end{aligned}
$$

Properties

Symmetry:

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)) .$$

Transpose symmetry:

$$f(n) = O(g(n)) \text{ if and only if } g(n) = \Omega(f(n)) ,$$
$$f(n) = o(g(n)) \text{ if and only if } g(n) = \omega(f(n)) .$$

Example Proof

FACT: $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ implies $f(n) \in O(h(n))$.
Proof. Since $f(n) \in O(g(n))$, there exist constants $c_1 > 0$ and $N_1 \geq 1$ such that

$$f(n) \leq c_1 \times g(n), \forall n \geq N_1. \tag{1}$$

Since $g(n) \in O(h(n))$, there exist constants $c_2 > 0$ and $N_2 \geq 1$ such that

$$g(n) \leq c_2 \times h(n), \forall n \geq N_2. \tag{2}$$

Let $c = c_1 \times c_2$ and $N = \max\{N_1, N_2\}$. Then we have

$$f(n) \leq c_1 \times g(n), \ \forall n \geq N, \tag{3}$$

$$g(n) \leq c_2 \times h(n), \ \forall n \geq N. \tag{4}$$

Therefore for any $n \geq N$, we have

$$\begin{aligned} f(n) &\leq c_1 \times g(n) && (5)\\ &\leq c_1 \times c_2 \times h(n) && (6)\\ &= c \times h(n). && (7) \end{aligned}$$

This proves that $f(n) \in h(n)$.

Summary

- The Random Access Machine
- Four commonly used formulas
- Big-O, Big-Ω, Big-Θ notations