GROTHENDIECK SPECTRAL SEQUENCE

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1. Construction

Lemma 1.1. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in an abelian category with enough injectives, and let I_A°, I_C° be injective resolutions of A, C respectively. Then there is an injective resolution I_B° of B that fits into a short exact sequence of injective resolutions $0 \to I_A^\circ \to I_B^\circ \to I_C^\circ \to 0$, extending the original exact sequence. Further, each I_B^i can be taken to be the direct sum $I_A^i \oplus I_C^i$, such that the exact seqeuences $0 \to I_A^\circ \to I_B^\circ \to I_C^\circ \to 0$ are given by the natural inclusion and projection maps.*

Remark 1.2. *The boundary maps on I_B° will not in general be the direct sum of those on I_A° and I_C°.*

Proof. Let $I_B^i = I_A^i \oplus I_C^i$ (as individual objects, not yet as a complex). The proof goes by inductively filling the bonudary maps ∂_B^i in the following diagram:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & I_A^2 \arrow[r] & I_B^2 \arrow[r] & I_C^2 \arrow[r] & 0 \\
0 \arrow[r] & I_A^1 \arrow[r] \arrow[u, "\partial_A^2"] & I_B^1 \arrow[r] \arrow[u, dotted, "\partial_B^2"] & I_C^1 \arrow[r] \arrow[u, "\partial_C^2"] & 0 \\
0 \arrow[r] & I_A^0 \arrow[r] \arrow[u, "\partial_A^1"] & I_B^0 \arrow[r] \arrow[u, dotted, "\partial_B^1"] & I_C^0 \arrow[r] \arrow[u, "\partial_C^1"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[u, "\partial_A^0"] & B \arrow[r, "g"] \arrow[u, dotted, "\partial_B^0"] & C \arrow[r] \arrow[u, "\partial_C^0"] & 0
\end{tikzcd}$$

such that the diagram commutes, and the middle column is exact.

For the induction base, we need to define an injective function $\partial_B^0 : B \to I_A^0 \oplus I_C^0$ compatible with the maps around it. Since I_A^0 is injective, there is a map $\alpha : B \to I_A^0$ such that $\alpha \circ f = \partial_A^0$. Define $\partial_B^0(b) = (\alpha(b), \partial_C^0(g(b)))$. This is an injection, and makes the lower two squares commute.

The induction step is very similiar. suppose we have defined ∂_B^j for $j \leq i$. We need to define $\partial_B^{i+1} : I_A^i \oplus I_C^i \to I_A^{i+1} \oplus I_C^{i+1}$. By the induction hypothesis there is an injection $I_A^i / \operatorname{im}(\partial_A^{i-1}) \to I_B^i / \operatorname{im}(\partial_B^{i-1})$. Therefore, the map $I_A^i / \operatorname{im}(\partial_A^{i-1}) \xrightarrow{\overline{\partial}_B^i} I_A^{i+1}$ extends to a map $I_B^i / \operatorname{im}(\partial_B^{i-1}) \to I_A^{i+1}$. We can then define $\partial_B^{i+1} : I_B^i \to I_A^{i+1} \oplus I_C^{i+1}$ as the direct sum of the composition

$$I_B^i \to I_B^i / \operatorname{im}(\partial_B^{i-1}) \to I_A^{i+1}$$

with the composition

$$I_B^i \to I_C^i \xrightarrow{\partial_C^{i+1}} I_C^{i+1}.$$

This can be checked to satisfy all the required properties. □

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Remark 1.3. *The resolution of C does not need to be injective for this argument to work.*

Let C° be a complex in an abelian category $\mathcal{C}$. A double complex I_C^{ij} such that each column $I_C^{i\circ}$ is an injective resolution of C^i, in a manner compatible with the boundary maps in C°, is called an injective resolution of the complex C°.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots \\
C^2 \arrow[u] \arrow[r] & I_C^{2,0} \arrow[u] \arrow[r] & I_C^{2,1} \arrow[u] \arrow[r] & \cdots \\
C^1 \arrow[u] \arrow[r] & I_C^{1,0} \arrow[u] \arrow[r] & I_C^{1,1} \arrow[u] \arrow[r] & \cdots \\
C^0 \arrow[u] \arrow[r] & I_C^{0,0} \arrow[u] \arrow[r] & I_C^{0,1} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 1.4. *Let $\mathcal{C}$ be an abelian category with enough injectives, and let C° be a complex in $\mathcal{C}$. Denote $H^i = H^i(C^\circ)$ and let $H^i \to I_H^{i\circ}$ be injective resolutions for each H^i. Then there exists an injective resolution I_C^{ij} of C° such that for each fixed j, the complex $I_C^{\circ j}$ is the direct sum of a split exact complex and the trivial complex $I_H^{\circ j}$ with vanishing bonudary maps.*

Proof. Denote the cocycle and coboundary subgroups of C^i by Z^i and B^i respectively. Let $B^i \to I_B^{i\circ}$ be injective resolutions of all the coboundary subgroups. Applying Lemma 1.1 for the exact sequence $0 \to B^i \to Z^i \to H^i \to 0$, we find that there is an injective resolution $Z^i \to I_Z^{i\circ}$ such that $I_Z^{ij} \cong I_B^{ij} \oplus I_H^{ij}$ for all i, j, in a manner compatible with the maps $B^i \to Z^i \to H^i$. Applying Lemma 1.1 again for $0 \to Z^i \to C^i \to B^{i+1} \to 0$, we find that there is an injective resolution $C^i \to I_C^{i\circ}$ such that $I_C^{ij} \cong I_Z^{ij} \oplus I_B^{i+1,j}$ for all i, j, in a manner compatible with the maps $Z^i \to C^i \to B^{i+1}$. Since the map $C^i \to C^{i+1}$ is equal to the composition

$$C^i \to B^{i+1} \to Z^{i+1} \to C^{i+1}$$

it follows that $I_C^{ij} \cong I_B^{ij} \oplus I_H^{ij} \oplus I_B^{i+1,j}$, and that the maps $I_C^{ij} \to I_C^{i,j+1}$ defined by identity on $I_B^{i+1,j} \to I_B^{i+1,j+1}$ and zero on the other components induce a double complex I_C^{ij}. It is then clear that the row complex $I_C^{\circ j}$ is the direct sum of the split exact sequence $I_B^{\circ j} \oplus I_B^{\circ+1,j}$ with the trivial complex $I_H^{\circ j}$ with vanishing boundary maps. $\square$

Given left-exact functors of abelian groups $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G}\to \mathcal{E}$, with the additional assumption that F takes injective objects of $\mathcal{C}$ to G-acyclic objects of $\mathcal{D}$, the Grothendieck spectral sequence for $X \in \mathcal{C}$ has the terms $G^j(F^i(X))$ in its second page and converges to $(G \circ F)^{i+j}(X)$. We write this as $G^j(F^i(X)) \Rightarrow (G \circ F)^{i+j}(X)$. The formal statement is as follows.

Theorem 1.5. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be abelian categories with enough injectives, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be left exact functors, such that F takes injective objects of $\mathcal{C}$ to G-acyclic objects of $\mathcal{D}$. Let X be an object of $\mathcal{C}$ with injective resolution $X \to I_X^\circ$. Let J^{ij} be an injective resolution of the complex $F(I_X^\circ)$ in $\mathcal{D}$ arising from the construction in Lemma 1.4. Then the double complex $G(J^{ij})$ induces a spectral sequence whose second page i,jth term has the form $G^j(F^i(X))$ and which converges to $(G \circ F)^{i+j}(X)$.*

Proof. Consider the double complex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
G(J^{2,0}) & \longrightarrow & G(J^{2,1}) & \longrightarrow & G(J^{2,2}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
G(J^{1,0}) & \longrightarrow & G(J^{1,1}) & \longrightarrow & G(J^{1,2}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
G(J^{0,0}) & \longrightarrow & G(J^{0,1}) & \longrightarrow & G(J^{0,2}) & \longrightarrow & \cdots .
\end{array}$$

Deriving through the horizontal maps, we get the terms $G^j(F(I_X^i))$. However, these vanish for $j \neq 0$ by the assumption on F. Therefore we are left with a single row of $G(F(I_X^i))$. When we derive it through the vertical maps, we get the single term $(G \circ F)^i(X)$ on the ith diagonal, and the spectral sequence terminates.

On the other hand, when we derive through the vertical maps first, we obtain the terms $G(I_H^{ij})$ by the assumption on the columns of the injective resolution of $F(I_X^\circ)$; here $I_H^{i\circ}$ is an injective resolution for $H^i(F(I_X^\circ)) \cong F^i(X)$. Next, when deriving theough the horizontal maps, we obtain the terms $G^j(F^i(X))$. This finishes the proof. □

2. Examples

2.1. **The Cech Cohomology spectral sequence.** Let X be a topological space. Let $PSh(X)$ and $Sh(X)$ denote the categories of presheaves and sheaves on X. Let $\mathfrak{U}$ be an open cover of X. Recall that the Cech cohomologies $H^*(\mathfrak{U}, \mathcal{F})$ are functors $PSh(X) \to \text{Ab}$, defined as the cohomologies of the Cech complex $\prod_{U \in \mathfrak{U}} \mathcal{F}(U) \to \prod_{U,V \in \mathfrak{U}} \mathcal{F}(U \cap V) \to \ldots$. Consider the composition

$$Sh(X) \xrightarrow{F} Psh(X) \xrightarrow{G} \text{Ab}$$

where F is the forgetful functor and $G = H^0(\mathfrak{U}, -)$.

For a sheaf $\mathcal{F}$, we have that $F^i(\mathcal{F})$ is the presheaf $\mathcal{H}^i(X, \mathcal{F})$ defined by

$$U \mapsto H^i(U, \mathcal{F}|_U)$$

while $(G \circ F)^i(\mathcal{F})$ is the sheaf cohomology $H^i(X, \mathcal{F})$. Also, for a presheaf $\mathcal{P}$, we have $G^i(\mathcal{P}) = H^i(\mathfrak{U}, \mathcal{P})$.

We therefore obtain a Grothendieck spectral sequence

$$H^j(\mathfrak{U}, \mathcal{H}^i(X, \mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F}).$$

If the restriction of $\mathcal{F}$ to finite intersections of $\mathfrak{U}$ is acyclic, it follows that Cech cohomology is equal to standard sheaf cohomology.

2.2. **The Group Cohomology Spectral Sequence.** Let G be a finite group and let $N \triangleleft G$ be a normal subgroup. Consider the composition of functors

$$\text{Mod}_G \xrightarrow{(-)^N} \text{Mod}_{G/N} \xrightarrow{(-)^{G/N}} \text{Ab}\,.$$

Since the composition has derived functors $H^i(G, -)$, the functor $(-)^N$ has derived functors $H^i(N, -)$ (where those cohomologies are regarded as G/N-modules), and the functor $(-)^{G/N}$ has derived functors $H^i(G/N, -)$, we obtain a Grothendieck spectral sequence

$$H^j(G/N, H^i(N, M)) \Rightarrow H^{i+j}(G, M).$$