GAGA

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I took it as a project to read the minimal necessary background to fully understand a proof of GAGA. This is meant to be a self-contained write-up of that proof, including the definitions of every notion that is used. I tried to fill a lot of the details myself, so there may be mistakes. Let me know if you find any.

1. Ringed Space Theory

A ringed space $(X, \mathcal{O}_X)$ consists of a topological space X with a sheaf of rings $\mathcal{O}_X$. Morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consist of a continuous map $f : X \to Y$ and a sheaf-of-rings morphism $f^{\#} : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$.

A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ so that the stalks $\mathcal{O}_{X,x}$ are local rings. A morphism between locally ringed spaces is a ringed space morphism so that the induced $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are local (meaning, the maximal ideal maps into the maximal ideal).

1.1. Immersions.

Given a ringed space $(X, \mathcal{O}_X)$ and an open $U \subseteq X$, $(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$ is an open immersion. It represents the maps into X with topological image in U.

Definition 1.1. *$i : Z \to X$ is a closed immersion if it is a closed immersion topologically and $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*

Given an ideal sheaf $\mathcal{I} \lhd \mathcal{O}_X$, we let $i : Z \subseteq X$ be its locus of vanishing $(\operatorname{supp}(\mathcal{O}_X/\mathcal{I}))$. Then $(Z, i^{-1}(\mathcal{O}_X/\mathcal{I})) \to (X, \mathcal{O}_X)$ is the closed immersion corresponding to $\mathcal{I}$.

Proposition 1.2. *Let $\mathcal{I} \lhd \mathcal{O}_X$ be an ideal sheaf. Then the closed immersion $Z \to X$ corresponding to it represents the maps into X which annihilate $\mathcal{I}$.*

Proof. Let $(f, f^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ annihilate $\mathcal{I}$. Then $\mathcal{O}_{X,f(y)} \neq \mathcal{I}_{f(y)}$, meaning $f(y) \in Z$. Suppose $f = ig$. Then $f^*\mathcal{O}_X \to \mathcal{O}_Y$ becomes $f^*(\mathcal{O}_X/\mathcal{I}) \to \mathcal{O}_Y$, which becomes $i^{-1}(\mathcal{O}_X/\mathcal{I}) \to g_*\mathcal{O}_Y$ □

This establishes an equivalence of ideal sheaves and closed immersions. If X is a scheme, then for $\mathcal{I}$ to define a scheme as its corresponding closed immersion, it should be locally generated by sections.

1.2. Quasi-Coherent sheaves.

An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if it has a local presentation, i.e. an exact $\mathcal{O}_X^{\oplus I}|_U \to \mathcal{O}_X^{\oplus J}|_U \to \mathcal{F}|_U \to 0$ locally. Note that a quasi-coherent sheaf that vanishes at a stalk vanishes in a neighborhood of the stalk.

If X is a ringed space and M is an $R = \Gamma(\mathcal{O}_X)$-module, we can define a sheaf $\widetilde{M}$ by sheafifying the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$. Then $M \mapsto \widetilde{M}$ is a right exact functor, and it takes a free R-module to a free $\mathcal{O}$-module. It follows that $\widetilde{M}$ is quasi-coherent. In fact, $\widetilde{-} \vdash \Gamma$ between the categories of R-modules and quasi-coherent sheaves.

Theorem 1.3. *Let X be a ringed space and $\mathcal{F}$ a quasi-coherent sheaf on X. Let $x \in X$ be a point with fundamental system of compact open neighborhoods. Then in a neighborhood of x, $\mathcal{F}$ has the form $\widetilde{M}$.*

Proof. Locally, in a compact neighborhood, $\mathcal{F}$ has a presentation $\mathcal{O}_X^{\oplus J} \to \mathcal{O}_X^{\oplus I} \to \mathcal{F} \to 0$. For every $j \in J$, there is a finite covering on which the image of e_j lives in finitely many components. Thus, its image lives in finitely many components globally in the neighborhood. Then we can define M as the cokernel of $R^{\oplus J} \to R^{\oplus I}$ □

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On general ringed spaces the notion of quasi-coherence is not very good. It is interesting mostly in the scheme case.

1.3. **Coherent Sheaves.** Let X be a ringed space. We say an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if it is of finite type, i.e. $\mathcal{O}_X^n|_U \to \mathcal{F}_U \to 0$ locally, and if each map $\mathcal{O}_X^{\oplus m}|_U \to \mathcal{F}|_U$ has a kernel of finite type.

Theorem 1.4. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. Then, if two of them are coherent, so is the third.*

Proof. Suppose $\mathcal{F}, \mathcal{G}$ are coherent. Then $\mathcal{H}$ has finite type, and consider $\ker(\phi : \mathcal{O}_X^n|_U \to \mathcal{H}|_U)$. Locally ϕ factors though $\mathcal{G}$. Then our kernel is the kernel of this map to $\mathcal{G}$ (which has finite type), direct sum with the image of $\mathcal{F}$ in $\mathcal{G}$ which also has finite type.

Suppose $\mathcal{G}, \mathcal{H}$ are coherent. We only need to show $\mathcal{F}$ has finite type. Locally we have a surjective $\mathcal{O}_X^n|_U \to \mathcal{G}|_U$, and the kernel into $\mathcal{H}|_U$ is surjective on $\mathcal{F}$. This kernel has finite type by the coherence of $\mathcal{G}$.

Suppose $\mathcal{F}, \mathcal{H}$ are coherent. For $\mathcal{G}$ to be finite type, use the local-finality of $\mathcal{F}, \mathcal{H}$ and recall there are local lifts to the sections generating $\mathcal{H}$. Now take $\phi : \mathcal{O}_X^n|_U \to \mathcal{G}|_U$. Then, moving to a smaller U, $\ker(\mathcal{O}_X^n \to \mathcal{H}|_U)$ is finitely generated by $(f_1^i, \dots, f_n^i)$. These elements, under the map to $\mathcal{G}$, can be lifted to $\mathcal{F}$, in which the collection of relations on them is finitely generated. Those relations form the generators for the kernel into $\mathcal{G}$ $\square$

Consequences from this:

(1) kernels and cokernels between coherent sheaves are coherent
(2) If a sequence of coherent sheaves is exact at a point, it is exact in a neighborhood (this uses the fact a quasi-coherent vanishing in a point vanishes in a neighborhood)
(3) Suppose $\mathcal{O}$ is coherent over itself. Then coherence over $\mathcal{O}$ is equivalent to local finite presentation ($\mathcal{O}_X^m|_U \to \mathcal{O}_X^n|_U \to \mathcal{F} \to 0$).

Theorem 1.5 (Extension Principle)**.** *If $\mathcal{O}$ is coherent over itself and $\mathcal{I} \triangleleft \mathcal{O}$ is a finite type ideal sheaf, then an $\mathcal{O}/\mathcal{I}$-module is $\mathcal{O}/\mathcal{I}$-coherent iff it is $\mathcal{O}$-coherent.*

Proof. the equivalence for finite type is clear. If $\mathcal{F}$ is $\mathcal{O}$-coherent, Given $(\mathcal{O}_X/\mathcal{I})^n|_U \to \mathcal{F}$, we get $\mathcal{O}_X^n|_U \to \mathcal{F}$ whose kernel is locally finitely generated, and modulo $\mathcal{I}$ generates the kernel for $\mathcal{O}_X/\mathcal{I}$. If $\mathcal{F}$ is $\mathcal{O}_X/\mathcal{I}$-coherent, then given $\mathcal{O}_X^n|_U \to \mathcal{F}$, we know that the relations are finitely generated over $\mathcal{O}_X/\mathcal{I}$. But by the finite type of $\mathcal{I}$, we can use its generators directly for the kernel and thus finish $\square$

Theorem 1.6. *Let X be a ringed space. Let $\mathcal{F}$ be a locally finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_X$-module.*

Proof. Locally $F_1 \to F_0 \to \mathcal{F} \to 0$ is exact. We get $0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(F_0, \mathcal{G}) \to \mathcal{H}om(F_1, \mathcal{G})$ exact, which finishes the proof $\square$

Theorem 1.7. *If $\mathcal{F}$ is locally finitely presented, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$ is an isomorphism.*

Proof. Locally we know that $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \ker(\mathcal{G}^n \to \mathcal{G}^m)$ and $\mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x) \cong \ker(\mathcal{G}_x^n \to \mathcal{G}_x^m)$ $\square$

2. Commutative Algebra

Theorem 2.1 (Krull intersection)**.** *Let (A, m) be a Noetherian local ring. Then $\bigcap m^n = 0$.*

Proof. Let $I = \bigcap m^n$. Let $x \in I$. Then x is a polynomial of degree n in the finite collection of generators of m. From the Hilbert basis theorem, these polynomials eventually become dependent, which implies $x \in xm$. This implies $x = 0$ from Nakayama $\square$

Theorem 2.2. *Let A be a Noetherian ring and $m \triangleleft A$. Then $A \to \hat{A}_m$ is flat.*

Proof. consider $M \subseteq N$ an inclusion of A-modules, then $\bigoplus_n m^n N$ is finitely generated, so the same goes for $\bigoplus_n (m^n N \cap M)$, implying that $\hat{M} \subseteq \hat{N}$ $\square$

Theorem 2.3 (Syzygy)**.** *Let (R,m) be a local Noetherian domain with m generated by n elements. Then $cd(R) \leq n$, meaning that every finitely generated R-module M has a syzygy $0 \to R^{r_n} \to \cdots \to R^{r_0} \to M \to 0$. In fact, any resolution $R^{r_{n-1}} \to \cdots \to R^{r_0} \to M \to 0$ has a free kernel.*

Proof. Let $x_1, \ldots, x_n \in m$ generate m. Write a resolution $R^{r_n} \to \cdots \to R^{r_0} \to M \to 0$. Let $K_j \subseteq R^{r_j}$ be the kernel of the boundary map.

We show inductively that $K_j \cap \mathfrak{m}_i R^{k_j} = \mathfrak{m}_i K_j$, for K_j the kernel in the R^{r_j}, and $\mathfrak{m}_i = (x_1, \ldots, x_i)$, and for $j \geq i$. Indeed, it is trivial for $i = 1$, and elsewhere, taking $f \in K_j \cap \mathfrak{m}_i R^{r_j}$, write $f = x_1 f_1 + \cdots + x_i f_i$, then we get $x_i \partial(f_i) \in K_{j-1} \cap \mathfrak{m}_{i-1} R^{r_{j-1}}$, thus $x_i \partial(f_i) = x_1 g_1 + \cdots + x_{i-1} g_{i-1}$ for $\partial g_k = 0$. So all g's are liftable, meaning we can alter the original representation of f to make $\partial(x_i f_i) = 0$ and thus $\partial(f_i) = 0$. This finishes, again by induction.

Now, if an element of K_n was not in $\mathfrak{m}_n R^{r_n}$, we would be able to decrease r_n, so wlog that is not the case. This implies $K_n = \mathfrak{m}_n K_n$. By Nakayama (and Noetherianity of R), we find $K_n = 0$ □

This theorem applies for $k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$, and $k[[x_1, \ldots, x_n]]_{(x_1, \ldots, x_n)}$, and $k\{x_1, \ldots, x_n\}_{(x_1, \ldots, x_n)}$ (the ring of converging power series), if we show those are Noetherian.

3. Random Cohomology Stuff

Theorem 3.1. *Flasque sheaves are acyclic.*

Proof. Injective sheaves are flasque. Thus, for $\mathcal{F}$ flasque, we have an exact $\mathcal{F} \to \mathcal{I} \to \mathcal{G}$ with $\mathcal{I}$ injective, and it follows that $\mathcal{G}$ is flasque (this uses axiom of choice). From flasqueness we see $\Gamma(\mathcal{I}) \to \Gamma(\mathcal{G})$ is surjective, and it follows that all higher cohomologies of flasque sheaves vanish. □

Theorem 3.2 (Serre Vanishing)**.** *Suppose $\mathcal{F}$ is a sheaf on X so that $H^i(U, \mathcal{F})$ vanishes for all U in some cover and $1 \leq i < n$. Then each $\mathbb{A} \in H^n(X; \mathcal{F})$ vanishes in $H^n(X; \mathcal{F}|_U)$ for U in a cover.*

Proof. Take an exact $\mathcal{F} \to \mathcal{G} \to \mathcal{K}$ with $\mathcal{G}$ flasque. First we do the case of $n > 1$. Then locally $H^i(X, \mathcal{F}|_U) = 0$ for $1 \leq i < n$, so in particular $0 \to \mathcal{F}|_U \to \mathcal{G}|_U \to \mathcal{K}|_U \to 0$ is exact. This implies that $H^n(X; \mathcal{F}|_U) \cong H^{n-1}(X; \mathcal{K}|_U)$ and $H^i(U; \mathcal{F}) \cong H^{i-1}(U, \mathcal{K})$, which proves the claim inductively. Thus it is sufficient to consider the case of $n = 1$.

We know that $H^0(X, \mathcal{K}) \to H^1(X, \mathcal{F})$ is surjective, so $\mathbb{A}$ comes from $\beta \in H^0(X, \mathcal{K})$. Then it suffices to show that β, when restricted to $H^0(X, \mathcal{K}|_U)$, lifts to $H^0(X, \mathcal{G}|_U)$. But this is clear from the surjectivity of $\mathcal{G} \to \mathcal{K}$ □

Theorem 3.3 (Cech-to-cohomology spectral sequence)**.** *Let $\mathcal{F}$ be a sheaf on X and $\mathfrak{U}$ a cover. Define the presheaf $\mathcal{H}^i(\mathcal{F})$ as $U \mapsto H^i(U, \mathcal{F}|_U)$. Then there is a spectral sequence $H^i(\mathfrak{U}, \mathcal{H}^j(\mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F})$.*

Proof. This is the Grothendieck spectral sequence for the composition $Sh(X) \to Psh(X) \xrightarrow{H^0(\mathfrak{U}, -)} \mathrm{Ab}$ □

As a corollary, we see that if $\mathcal{F}$ is acyclic on every finite intersection of the opens in $\mathfrak{U}$, then $H^*(\mathfrak{U}, \mathcal{F}) \cong H^*(X, \mathcal{F})$.

4. Algebraic Theory

Schemes are locally ringed spaces that are locally isomorphic to $\mathrm{Spec}(R)$.

Closed immersions corresponding to an ideal sheaf $\mathcal{I}$ in a scheme X are schemes only when $\mathcal{I}$ is locally generated by sections.

4.1. Quasi-Coherent Sheaves.

Theorem 4.1. *Suppose $\mathcal{F}$ is a quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$. Then $\mathcal{F} \cong \widetilde{M}$ for some R-module M.*

Proof. Clearly, points in schemes have fundamental systems of compact neighborhoods, so we know the claim holds locally. Suppose $\mathcal{F}|_{\mathrm{Spec}(R_{f_i})} \cong \widetilde{M_i}$. Then $(M_i)_{f_j} \cong (M_j)_{f_i}$ in an isomorphism that is good on triplets. Consider M, the R-module of simultaneous sections of all M_i's that agree in the intersections. There is an R_{f_i}-module map $M_{f_i} \to M_i$, easily seen to be an isomorphism from exactness of localization and the gluing property of the M_j's. Thus, we see that the map $\widetilde{M} \to \mathcal{F}$ induces an isomorphism on the open cover $\mathrm{Spec}(R_{f_i})$, so it is an isomorphism □

Theorem 4.2. *Quasi-coherent sheaves over* $\mathrm{Spec}\, R$ *are in correspondence with R-modules. Under this correspondence, for R coherent (meaning $\mathcal{O}$ is coherent on* $\mathrm{Spec}(R)$*, e.g. when R is Noetherian), the coherent sheaves correspond to the finitely presented R-modules.*

Proof. There is a map $M \to \Gamma(\widetilde{M})$. Injectivity is easy. For surjectivity, there are $f_1, \ldots, f_n$ generating R with $\frac{m_i}{f_i^{n_i}}$ on each $\mathrm{Spec}\, R_{f_i}$, which glue on intersections. Then we know that $(f_i f_j)^{N_{ij}}(f_j^{n_j} m_i - f_i^{n_i} m_j) = 0$. By replacing the f by their powers we may assume $N_{ij} = 0, n_i = 1$. Then, taking $\sum a_i f_i = 1$, we see that $\sum a_i m_i$ is a global section restricting to all the $\frac{m_i}{f_i}$. So, $M \cong \Gamma(\widetilde{M})$. Since we just saw every quasi-coherent sheaf $\mathcal{F}$ has the form $\widetilde{M}$, we see that $\widetilde{\Gamma(\mathcal{F})} \to \mathcal{F}$ is an isomorphism.

For coherency on affine schemes over a coherent ring: the claim we just proved (together with the 3-lemma on coherent sheaves) shows that coherent sheaves are precisely those with a global finite presentation. Thus they correspond to the the finitely presented modules □

Theorem 4.3. *Quasi-coherent sheaves on an affine scheme are acyclic.*

Proof. Suppose we showed this for $0 < i < n$. Serre Vanishing shows that every $\mathbb{A} \in H^n(X; \mathcal{F})$ would vanish on some cover. But if the cover is the affine V_i, then $\mathcal{F} \to \bigoplus \mathcal{F}|_{V_i} \to \mathcal{G}$ is exact, and we get that $H^{n-1}(\mathcal{G}) \to H^n(\mathcal{F})$ lifts $\mathbb{A}$. This finishes for $n > 1$. If $n = 1$, we finish by exactness of $\bigoplus \Gamma(\mathcal{F}_{V_i}) \to \Gamma(\mathcal{G}) \to 0$ (here we used affinness) □

4.2. **Cohomology in Projective Space.** Let k be an infinite field. Recall the twists $\mathcal{O}(n)$ as coherent sheaves on $\mathbb{P}^n_k$.

Theorem 4.4. *$H^i(\mathbb{P}^n, \mathcal{O}(m))$ is $k[T_0, \ldots, T_n]_m$ for $i = 0$, $k[T_0, \ldots, T_n]^\vee_{-n-m-1}$ for $i = n$, and 0 otherwise.*

Proof. This is done via a Cech calculation, separating monomials into their multi-degree components □

Lemma 4.5. *Let X be a quasi-compact scheme, where intersections of affines are quasi-compact. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and $\mathcal{L}$ a line bundle. Then for every $s \in \Gamma(\mathcal{L})$, the map $\Gamma(X, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$ is an isomorphism. Here $\Gamma(X, \mathcal{F})_{(s)} = \lim \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ and the map is restriction and tensor by $(s^{-1})^{\otimes n}$.*

Proof. For injectivity, we can restrict to local neighborhoods to assume X is affine and $\mathcal{L}$ trivial (this uses quasi-compactness). Then both sides are isomorphic to M_s. For surjectivity, take $\sigma \in \Gamma(X_s, \mathcal{F})$. Then locally it lifts after enough multiplications by s. Now there are errors on the intersections, which would vanish after enough $\otimes s$ given that the intersections are quasi-compact, which we assume they are. This finishes the proof □

Theorem 4.6. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Then, after enough twists, the higher cohomology of $\mathcal{F}$ vanishes, and $\mathcal{F}$ is generated by global sections.*

Proof. The last lemma shows that any section on X_s comes from a global map from $\mathcal{O}(-d)$. Thus there is a surjective $\mathcal{O}(-d)^{\oplus n} \to \mathcal{F}$, or $\mathcal{O}^n \to \mathcal{F}(d)$. Continuing like this we get a resolution $\mathcal{O}^{r_{n-1}} \to \cdots \to \mathcal{O}^{r_0} \to \mathcal{F}(N) \to 0$, and the syzygy theorem implies that the kernel if free, so we get a finite syzygy for $\mathcal{F}(N)$. This implies everything we needed, given that $\mathcal{O}$ has no higher cohomologies (which we have shown) □

5. Analytic Theory

5.1. Standard Theory.

5.1.1. *One variable.* A smooth function f on a domain $U \subseteq \mathbb{C}$ is holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$.

Theorem 5.1 (Cauchy). *Let $f : U \to \mathbb{C}$ be smooth, for $U \subseteq \mathbb{C}$ open. then $f(z) = \frac{1}{\tau i} \int_{\partial\Delta} \frac{f(w)dw}{w-z} + \frac{1}{\tau i} \int_{\Delta} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z}$. In particular for f holomorphic, $f(z) = \frac{1}{\tau i} \int_{\partial\Delta} \frac{f(w)dw}{w-z}$.*

Proof. Apply Stokes for $\frac{f(w)dw}{w-z}$, on a ring around z □

It follows that functions are holomorphic iff they are analytic. When we refer to a function on a compact set, what we mean is a function in some neighborhood of that set.

Theorem 5.2 (Runge). *Let $K \subseteq \mathbb{C}$ be compact, and let D be a discrete set containing one point in each connected component of $\mathbb{C} - K$ (except for the infinite one where we take ∞). Then every holomorphic function on K can be approximated by rational functions with poles only in D.*

Proof. Let γ be a curve going closely around K. Then $f(z) = \int_\gamma \frac{f(w)}{w-z} dw$ on K, which can be approximated by its Riemann sums. Thus, it suffices to prove the claim for $f(z) = \frac{1}{w-z}$, for $w \notin K$. Then, I should show that if w, w' are close enough, then $\frac{1}{z-w}$ can be approximated by rational functions with pole only in w'. Wlog $w' = 0$,Then $\frac{1}{z-w} = \frac{1}{z} \cdot \frac{1}{1-\frac{w}{z}} = \frac{1}{z} \cdot \sum_n (\frac{w}{z})^n$, and the partial sums are an approximation □

5.1.2. *many variables.* A smooth $f : U \to \mathbb{C}$ for $U \in \mathbb{C}^n$ is holomorphic if $\frac{\partial f}{\partial \bar{z}_i} = 0$ for all i, analytic around $(a_1, \ldots, a_n)$ if it is locally a power series in monomials in $z_i - a_i$, which converges absolutely and uniformly, and analytic in a domain if it is analytic around any point. This is equivalent to holomorphicity: If f is analytic, it is clearly holomorphic, and if it is holomorphic, then iterated Cauchy formula together with Fubini gives $f(z) = (\frac{1}{\tau i})^n \int_\Delta \frac{f(w)dw_1 \wedge \cdots \wedge dw_n}{\prod (w_i - z_i)}$, leading to the standard proof of analyticy by expressing this as a power series.

$f : U \to \mathbb{C}^n$ is holomorphic if all its components are. Holomorphic functions are closed under addition, multiplication, inversion where they don't vanish, and composition.

Note that $dz \wedge d\bar{z} = 2idx \wedge dy$. Now, given f a holomorphic function between two open subsets of $\mathbb{C}^n$, we know that $f^*(dx_1 \wedge dy_1 \wedge \ldots) = J_{f,\mathbb{R}} dx_1 \wedge dy_1 \wedge \ldots$, thus $df_1 \wedge d\overline{f_1} \wedge \cdots = f^*(dz_1 \wedge d\overline{z_1} \wedge \ldots) = J_{f,\mathbb{R}} dz_1 \wedge d\overline{z_1} \wedge \ldots$. But $df_i = \sum \frac{\partial f}{\partial z_i} dz_i$, and we conclude $J_{f,\mathbb{R}} = |J_{f,\mathbb{C}}|^2$. Then it follows that if f has nonvanishing complex Jacobian, it has an inverse and the inverse is holomorphic.

Much of the standard theory follows, including e.g. the maximum principle.

A polydisk is a product of disks and copies of $\mathbb{C}$. By the Riemann mapping theorem, this is equivalent to a product of simply connected domains. A compact polydisk is the closure of a product of disks. A holomorphic function on a compact polydisk is a holomorphic function defined in a neighborhood of it (and later the same terminology is used for sections of sheaves on a compact polydisk).

Theorem 5.3 (Vitali). *Let D be a domain in $\mathbb{C}^n$. Consider a sequence of holomorphic functions f_n on D which are uniformly bounded by some M. Then there is a subsequence which converges uniformly in every compact subset of D.*

Proof. We can cover D by countably many compact polydisks Δ, and then a diagonalization argument shows it suffices to consider the case $D = \Delta$ with the f_n defined in a common neighborhood. Suppose $\Delta = \overline{B_r(w)}$ and suppose the f_n are defined in $B_{r+\delta}(w)$. Then any point of $z \in \Delta$ satisfies

$$\frac{\partial f}{\partial z_i}(z) = \int_{|\zeta_j - w_j| = r+\delta} \frac{f(\zeta) dz_1 \ldots dz_n}{(\zeta_1 - w_1) \ldots (\zeta_i - w_i)^2 \ldots (\zeta_n - w_n)} \leq \frac{M(r+\delta)^n}{\delta^{n+1}}$$

Showing our collection of functions is equicontinuous on Δ. It follows any sequence has a converging subsequence on every Δ, and we finish by diagonalization □

5.1.3. *The Riemann Mapping theorem.*

Theorem 5.4 (Hurewitz)**.** *Suppose f_n is a sequence of holomorphic functions in a connected domain, that converges uniformly on compact subsets to a holomorphic f. If all f_n are injective, then f is injective or constant.*

Proof. Suppose $f(a) = f(b) = w$. Take a neighborhood of b where f attains the value w only once. In that neighborhood, no f_n attains the value $f_n(a)$. Thus $\frac{1}{\tau i}\int_{B_\varepsilon(b)} \frac{f_n'(\zeta)}{f_n(\zeta)-f_n(a)} d\zeta = 0$. Hence $\frac{1}{\tau i}\int_{B_\varepsilon(b)} \frac{f'(\zeta)}{f(\zeta)-f(a)} d\zeta = 0$, a contradiction ☐

Theorem 5.5 (Riemann Mapping)**.** *Any simply connected open proper subset $\circ \subsetneq \mathbb{C}$ is biholomorphic to the disk.*

Proof. Choosing $z_0 \in \circ$, I claim there is a unique biholomorphism $F : \circ \to \Delta$ so that $F(z_0) = 0, F'(z_0) > 0$. The Schwartz lemma implies such an F would be unique. Consider the space $\mathcal{F}$ of holomorphic functions $f : \circ \to \Delta$ which are injective and send $z_0 \mapsto 0$. By taking some $a \in \mathbb{C} - \circ$ and considering a branch of $l(z) = \log(z - a)$, we see $\mathcal{F}$ is nonempty. Let $\lambda = \sup_{f\in\mathcal{F}} |f'(z_0)|$. This is positive (otherwise we would get an injective function with a double root). If $(f_n) \subseteq \mathcal{F}$ is a sequence so that $|f_n'(z_0)| \to \lambda$, then by Vitali's theorem we get that it has a converging subsequence. So suppose $f_n \to f$. If we would have $f \in \mathcal{F}$ then it would follow that $|f'(z_0)| = \lambda$ and this will be a good candidate for the biholomorphism. We just need to show f is injective. This follows from Hurewitz' theorem.

Obviously, we may assume $f'(z_0) = \lambda$. Now we need to prove f is surjective. Suppose there is $\mathbb{A} \in \Delta$ not in the image of f. Let $\psi_a(w) = \frac{a-z}{1-\bar{a}z}$ and $g(z) = \psi_{\sqrt{\mathbb{A}}}(\sqrt{\psi_{\mathbb{A}}(f(z))})$. This function is in $\mathcal{F}$. Note that $f(z) = \Phi(g(z))$ for $\Phi = \psi_{\mathbb{A}}^{-1} \circ (-)^2 \circ \psi_{\sqrt{\mathbb{A}}}^{-1}$, so it suffices to show $|\Phi'(0)| < 1$. By the Schwartz lemma, the only other option is $\Phi(z) = az$ for $|a| = 1$, which is obviously not true ☐

5.2. **The Ring of Holomorphic Germs.** Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the local ring of holomorphic germs in $\mathbb{C}^n$ around 0. We say that $f \in \mathcal{O}_{\mathbb{C}^n,0}$ is z_n-regular of order m if $\frac{\partial^m f}{\partial z_n^m}(0) \neq 0$ and $\frac{\partial^k f}{\partial z_n^k}(0) = 0$ for all $k < m$. We say it is z_n-regular if it is with some finite m.

Theorem 5.6 (Weierstrass Division Theorem)**.** *Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$, a function of $(z_1, \dots, z_n) = (z', z_n)$, z_n-regular of order m. Then any $f \in \mathcal{O}_{\mathbb{C}^n,0}$ has a unique expression as $qg + r$ with $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_n]$ of degree $< m$ (in other words, $\mathcal{O}_{\mathbb{C}^n,0}/g \cong \mathcal{O}^m_{\mathbb{C}^{n-1},0}$).*

Proof. This is actually a Banach space argument. Let $g = uz_n^m + s$, with u unit and $\deg_{z_n}(s) < m$, $s(0, z_n) = 0$. Consider the operator T_g of multiplying by g, truncating the small z_n-powers and dividing by z_n^m. Our claim is that T_g is invertible. Note that $T_g = M_u + T_s$.

Indeed, consider the subspace of functions converging absolutely within a given radius ρ, preserved by M_g. Define the sub-multiplicative norm $\| \sum c \cdot z^{\mathbb{A}} \|_\rho = \sum |c| \cdot |\rho|^{\mathbb{A}}$.

The norm of M_u is $|u(\rho)|$. Now consider substituting the value $(\varepsilon\rho, \dots, \varepsilon\rho, \rho)$ for $\varepsilon \to 0$. The norm of M_u converges to $|u(0)| > 0$, and the norm of M_s converges to 0 (using the fact the last ρ is fixed). Then for some ε, we find that the operator $M_u + M_s$ must be invertible (noting, of course, that M_u is invertible), finishing the proof (this used completeness). Uniqueness is easy ☐

There is a stronger form of this: if $f_n \to f$ uniformly in a neighborhood of 0, and $f_i = g_i q + r_i, f = gq + r$, then $g_i \to g$ uniformly on some neighborhood of 0. Indeed, this proof just constructed an inverse operator, which is continuous. A corollary of this is that when we have a sequence (f_n) which converges uniformly to f in a neighborhood U of z_0, then there is a smaller neighborhood in which a finite number of the f_n generate f.

Theorem 5.7 (Weierstrass Preparation)**.** *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be m-regular with respect to z_n. Then f is uniquely a product gh where $h(0) \neq 0$ and $g = z_n^m + a_{m-1}(z_1, \dots, z_{n-1})z_n^{m-1} + \cdots + a_0(z_1, \dots, z_{n-1})$ with $a_i(0) = 0$.*

Proof. This is a direct consequence of Weierstrass division. Here's a cute idea I can't seem to work out: denote $z' = (z_1, \dots, z_{n-1})$, and let $\lambda_i(z')$ be the roots of $f(z', -)$ in a small enough neighborhood of 0. Then $\frac{1}{\tau i}\int_{\partial\Delta} w^j \frac{\frac{\partial f}{\partial w}(z',w)}{f(z',w)} dw = \sum \lambda_i(z')^j$. This is a holomorphic function in z', which is integral and thus constant for $j = 0$. This shows the amount of roots is fixed, say m. Let $\sigma_i(z')$ be the symmetric functions in the $\lambda_i(z')$. Then $\sigma_i(z')$ are holomorphic! now $g = \prod(z_n - \lambda_i(z'))$ is our candidate. Certainly, on each z'-fiber, the quotient $\frac{f}{g}$ is holomorphic and nonvanishing. But why is it a globally holomorphic function? Away from the roots of g it is fine, so by Morera we should just bound the quotient around the roots. Then we should be fine from the regularity at z_n, but I'm not sure I quite see it [???] □

Theorem 5.8. *$\mathcal{O}_{\mathbb{C}^n,0}$ is Noetherian.*

Proof. Suppose we have some ideal $\mathcal{I}$. Take some $f \in \mathcal{I}$, and if needed change coordinates to make it regular in z_n. Then use Weierstrass preparation to turn f into a polynomial in z_n, and use Weierstrass division to reduce every other element of $\mathcal{I}$ modulo f. This got us rid of one variable, and we can continue inductively □

It now follows, from the syzygy theorem, that every finitely generated $\mathcal{O}_{\mathbb{C}^n,0}$-module has a resolution of length $n + 1$.

Theorem 5.9. *Let $\mathcal{I} \lhd \mathcal{O}_{\mathbb{C}^n,0}$. Then, after a change of coordinates, $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}$ is a finite $\mathcal{O}_d$-algebra and $\mathcal{O}_{\mathbb{C}^d,0} \cap \mathcal{I} = 0$ for some d.*

Proof. Consider a series of $\mathcal{O}_{\mathbb{C}^0,0} \subseteq \mathcal{O}_{\mathbb{C}^1,0} \subseteq \dots$, constructed so that $\mathcal{O}_i \cap \mathcal{I}$ vanishes for $i < d$ and contains a z_i-regular function for $i \geq d$ (the construction goes backwards). Then $\mathcal{O}_{\mathbb{C}^i,0}/(\mathcal{O}_{\mathbb{C}^i,0} \cap \mathcal{I})$ is a finite $\mathcal{O}_{\mathbb{C}^{i-1},0}$-algebra for $i > d$, by Weierstrass preparation, and this finishes the proof inductively. □

Theorem 5.10 (Rückert's nullstellensatz)**.** *Suppose $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on the locus $Z(\mathcal{I})$ of an ideal $\mathcal{I} \lhd \mathcal{O}_{\mathbb{C}^n,0}$. Then $f \in \sqrt{\mathcal{I}}$.*

Proof. Wlog, $\mathcal{I}$ is radical. A radical ideal is the intersection of all primes containing it, thus $Z(\mathcal{I}) = \bigcup_{\mathcal{I}\subseteq\mathcal{P}} Z(\mathcal{P})$, so wlog assume $\mathcal{I} = \mathcal{P}$ is prime. Now $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{I}$ is a finite $\mathcal{O}_{\mathbb{C}^d,0}$-algebra, so there is $P \in \mathcal{O}_{\mathbb{C}^d,0}[t]$ with $P(f) \in \mathcal{I}$. If $P = a_0 + \dots + a_k t^k$, it follows that $a_0|_{Z(\mathcal{I})} = 0$. If we would show that this implies $a_0 = 0$, then primality of $\mathcal{I}$ would allow us to divide by f and reduce the degree of P, and we would be done. Now, a_0 does not vanish for a generic set in the $(z_1, \dots, z_d)$. All we have to do, then, is to show that generically there is an extension from $\mathbb{A} = (\alpha_1, \dots, \alpha_d)$ to a point in $Z(\mathcal{I})$. Indeed, otherwise, by standard nullstellensatz, there would be an explicit combination of the generators of $\mathcal{I}$ that would not vanish on $\mathbb{A}$ (killing all the formal variable coefficients). The combination can be taken with the coefficients as polynomials in the original coefficients of the generators of $\mathcal{I}$. But this means that this combination can be taken globally, resulting in a nonzero element of $\mathcal{I} \cap \mathcal{O}_{\mathbb{C}^d,0}$, a contradiction □

5.3. **Complex Manifolds and Analytic Spaces.** A complex manifold of dimension n is a locally ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a subsheaf of the sheaf of continuous maps $X \to \mathbb{C}$, which is locally isomorphic to the locally ringed space of $\mathbb{C}^n$ with the sheaf of holomorphic functions. This can also be seen as a pair $(X, C(X,\mathbb{C})) \to (X, \mathcal{O}_X)$ satisfying some conditions, which should clarify things categorically, as the morphisms are required to commute with the pair maps, and the categorical constructions look nice now.

Since $J_{f,\mathbb{R}} \geq 0$ for all holomorphic functions f, it follows that complex manifolds have a canonical orientation.

The following is not standard terminology, but I use it here.

An affine complex analytic space is a locally ringed space with closed immersion into $(\Delta, \mathcal{O}_\Delta)$ for Δ a polydisk in $\mathbb{C}^n$, defined by an ideal $\mathcal{I} \lhd \mathcal{O}_\Delta$ which is locally of finite type.

An Analytic space is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to some affine complex analytic space, again with $\mathcal{O}_X$ a subsheaf of the sheaf of continuous functions to $\mathbb{C}$.

The local rings of an analytic space are Noetherian by previous results.

5.4. **Dolbeault Cohomology.**

Theorem 5.11. *Any smooth function g on a compact disk $\overline{\Delta} \subseteq \mathbb{C}$ is of the form $\frac{\partial f}{\partial \overline{w}}$.*

Recall Cauchy's theorem, $f(z) = \frac{1}{\tau i}\int_{\partial\Delta}\frac{f(w)dw}{w-z} + \frac{1}{\tau i}\int_{\Delta}\frac{\partial f}{\partial \overline{w}}\frac{dw\wedge d\overline{w}}{w-z}$. Clearly $\int_{\partial\Delta}\frac{f(w)dw}{w-z}$ is a holomorphic function in z, and it follows that $f(z) = \frac{1}{\tau i}\int_{\Delta} g\frac{dw\wedge d\overline{w}}{w-z}$ satisfies $\frac{\partial f}{\partial \overline{w}} = g$ for any smooth g.

Theorem 5.12. *Let $U \subseteq \mathbb{R}^n$ be a domain. Let $\mathcal{F}$ be a sheaf of C^∞-modules on U. Then $H^i(U, \mathcal{F}) = 0$ for $i > 0$.*

Proof. We will show that Γ is an exact functor in the category of C^∞-modules (and note that there are enough flasque C^∞-modules). Suppose $\mathcal{F} \to \mathcal{G}$ is surjective, so any section of $\mathcal{G}$ has local lifts; We can patch these together using a partition of unity □

Lemma 5.13 (Dolbeault)**.** *If $\overline{\Delta} \subseteq \mathbb{C}^n$ is a compact polydisk, and ω a smooth (p, q)-form on a neighborhood of $\overline{\Delta}$, and $q > 0$ and $\overline{\partial}\omega = 0$, then we get $\omega = \overline{\partial}\eta$ for η a $(p, q-1)$-form.*

Proof. Write $\omega = \mathbb{A} \wedge d\overline{z}_r + \beta$, for $\mathbb{A}, \beta$ involving only the conjugate differentials below r. Then the condition implies $\mathbb{A}$ is holomorphic in $z_{r+1}, \dots, z_n$. By a previous claim, we can find γ with $\mathbb{A} = \frac{\partial\gamma}{\partial \overline{z}_r}$. Then using γ to lift of $\mathbb{A}$, we reduce the r and finish inductively □

It seems that this only works for polydisks because to find γ we need to make an integration on a triangle in every fiber, and for this computation to result in a holomorphic function, the triangle should change holomorphically, which is hard unless it is constant.

As a consequence, we get an exact sequence of sheaves $0 \to \mathcal{O} \to \circ^{0,0} \to \circ^{0,1} \to \cdots \to \circ^{0,n} \to 0$. All of those are C^∞-modules and thus acyclic, so this resolution can be used to compute cohomologies of $\mathcal{O}$.

For D a region in $\mathbb{C}^n$, we let $H^{p,q}(D)$ be the space of $\overline{\partial}$-closed (p, q)-forms on D modulo the $\overline{\partial}$-exact forms.

Theorem 5.14. *The Dolbeault cohomologies of a polydisk Δ are the following: $H^{p,q}(\Delta) = 0$ for $q > 0$, and $H^{p,0}(\Delta)$ is the space of $(p, 0)$-forms with holomorphic coefficients.*

Proof. For the $H^{p,0}$, this is clear. For the $H^{p,q}$ and $q \geq 2$, choose an increasing sequence of compact polydisks $K_n \subseteq \Delta$, and then for each find a form lifting our form (by Dolbeault Lemma), and fix the disagreement with the previous K_{n-1} be Dolbeault again together with some extension theorem for smooth functions. For $q = 1$, this doesn't quite work, but we can make the difference on K_{n-1} be a holomorphic $(p-1, 0)$-form whose coefficients are smaller than 2^{-n} in sup norm. We do this by adding a polynomial eating an increasingly large part of the power series expansion. Then the limit works □

Theorem 5.15. *Let Δ be a polydisk, then $\mathcal{O}$ is acyclic on Δ.*

Proof. There is a resolution $\mathcal{O} \to \circ^{0,0} \to \circ^{0,1} \to \cdots \to \circ^{0,n} \to 0$, and all the $\circ^{p,q}$ are C^∞-modules, so they are acyclic by a partition of unity. Thus the cohomologies of $\mathcal{O}$ can be computed by the $\circ^{0,i}$-resolution evaluated on $\mathcal{O}$, and this is just the Dolbeault cohomology □

In particular, every module which has a finite free resolution on a polydisk is acyclic.

5.5. **Bundles on the polydisk.**

Let D be a compact region in $\mathbb{C}^n$, and consider the group of invertible holomorphic matrices on D of rank m (i.e., matrices that are defined in a neighborhood of D and have a holomorphic inverse in that neighborhood). Then those form a topological group $\mathrm{GL}_m(\mathcal{O}_D)$ via the norm $\|A\| = \sup_{z\in D}\|A(z)\|_{\text{operator}}$. This group is connected (take $A(z) \mapsto A(tz)$ and then $A(0) \mapsto I$), so every element is a finite product of elements F with $\|1 - F\| < 1$.

Theorem 5.16. *Let $\Delta \subseteq \mathbb{C}^n$ be a compact polydisk. The map $\mathrm{GL}_m(\mathcal{O}_{\mathbb{C}^n}) \to \mathrm{GL}_m(\mathcal{O}_\Delta)$ has dense image.*

Proof. It suffices to show elements with $\|I - F\| < 1$ can be approximated. Then $G = \log F = -\sum_{i\geq 1} \frac{(I-F)^i}{i}$ is defined and we have $F = \exp G$. As exponents are always invertible, we just need to approximate G by functions defined on all $\mathbb{C}$. Runge's theorem implies we can do this with polynomials. $\square$

Claim 5.17. *If $1 + F_i \in \mathrm{GL}_n(\mathcal{O}_\Delta)$, then the infinite product $\prod(1 + F_i)$ is in $\mathrm{GL}_n(\mathcal{O}_\Delta)$, assuming that $\sum \|F_i\|$ converges.*

Proof. The convergence of the product is easy. We need to bound the product of the determinants from below. For this, we have $\prod \|1 + F_i\| \geq \prod(1 - \|F_i\|) > 0$ by the triangle inequality $\square$

From now, we let K_1, K_2 be two compact polyrectangles of the form $K_1 = ([a_1, a_3] \times I) \times D_2 \times \ldots, D_n$ and $K_2 = ([a_2, a_4] \times I) \times D_2 \times \ldots, D_n$, with $a_1 < a_2 < a_3 < a_4$, in $\mathbb{C}^n$.

Theorem 5.18 (Stable H^0)**.** *If f, g are holomorphic functions on K_1, K_2, then there is a holomorphic function f on $K_1 \cup K_2$, so that $\|f - f_1\|_{K_1}, \|f - f_2\|_{K_2} < C\|f_1 - f_2\|_{K_1\cap K_2}$.*

Proof. Take a curve γ going closely in a loop around $[a_1, a_4] \times I$. express it as $\gamma_1\gamma_2$, with the intersection points over and below $[a_2, a_3]$. Then let $f(z) = \int_{\gamma_1} \frac{f_1(w)}{z-w} dw + \int_{\gamma_2} \frac{f_2(w)}{z-w} dw$, where the integrals are taken over the right fiber. This function is holomorphic in $K_1 \cup K_2$, and it is easy to see that the difference from f_1, f_2 is expressed by $\int_\delta \frac{f_1(w)-f_2(w)}{z-w} dw$ where δ is some curve, internal to $K_1 \cap K_2$, connecting the two intersection points of γ_1, γ_2; For every z there is such a curve δ at least some constant distance away from z and of bounded length, so this f works $\square$

Theorem 5.19 (Stable H^1)**.** *If f is a holomorphic function on $K_1 \cap K_2$, then it is the difference $f_1 - f_2$ for f_1, f_2 on K_1, K_2, such that $\|f_1\|_{K_1}, \|f_2\|_{K_2} < C\|f\|_{K_1\cap K_2}$.*

Proof. We know it is the difference of some f_1, f_2, as we have shown $H^1(\Delta, \mathcal{O}) = 0$ for a polydisk Δ. From the computation of the stable H^0, we conclude that f_1, f_2 can be fixed to satisfy what we need $\square$

Theorem 5.20. *The multiplication map $\mathrm{GL}_m(\mathcal{O}_{K_1}) \times \mathrm{GL}_m(\mathcal{O}_{K_2}) \overset{mul}{\to} \mathrm{GL}_m(\mathcal{O}_{K_1\cap K_2})$ is surjective.*

Proof. As any $F \in \mathrm{GL}_m(K_1 \cap K_2)$ can be approximated by global invertible holomorphic matrices, we may assume $\|I - F\| \ll 1$. Now, express $F - I$ as $G_1 + G_2$ for holomorphic functions on K_1, K_2, bounded by $C\|F - I\|_{K_1\cap K_2}$.

In particular (choosing the original F close enough to I) we know that $I + G_1, I + G_2$ are invertible. Now let $F' = (I + G_1)^{-1}F(I + G_2)^{-1}$. Then we have

$$\|I - F'\|_{K_1\cap K_2} = \|(I + G_1)^{-1}G_1G_2(I + G_2)^{-1}\| < C^2\delta^2(1 + C\delta)^2$$

for $\delta = \|F - I\|$. So for small enough δ, we get a F' with much smaller δ. Continuing inductively, we get $F = \prod_{i<n}(I + G_i)F'_n \prod_{i<n}(I + G_{n-i})$, with $\|I - F'_n\| < \frac{\delta}{2^n}$ and $\|G_k\| < \frac{\delta}{2^k}$. It follows that we can take the limit product here, resulting in the required $F = F_1F_2$ for invertible F_1, F_2 on K_1, K_2 $\square$

Theorem 5.21. *Any holomorphic bundle on a polydisk Δ is trivial.*

Proof. We begin by showing this on a compact polydisk, that we also call Δ (this is a weaker statement). By the Riemann Mapping Theorem, the polydisk is equivalent to a product of rectangles, and we partition every rectangle into many smaller rectangles, until the bundle is trivial on every mini-cube. Then we need to take two mini-cubes and glue their trivializations together. In other words, we are given two cubes, that intersect in a face, and we have an invertible holomorphic matrix around the intersection, and we need to express it as a product of an invertible holomorphic matrix on one cube times an invertible holomorphic matrix on the other cube, which is exactly the content of the previous claim.

Now, in a general polydisk Δ, we have a sequence of open polydisks Δ_n which converge to Δ, and a trivialization on each one. This means that we have an $\mathcal{O}^m$ on every Δ_n, with invertible holomorphic maps $F_n : \mathcal{O}^m_{\Delta_{n+1}}|_{\Delta_n} \cong \mathcal{O}^m_{\Delta_n}$, and our bundle is the inverse limit. We can change basis to $\mathcal{O}^m_{\Delta_{n+1}}$ to make the matrix F_n really close to I, as $\mathrm{GL}(\mathcal{O}_{\mathbb{C}^n}) \to \mathrm{GL}(\mathcal{O}_D)$ has dense image (this may require shrinking our open sets slightly, but they still converge to Δ). Now, we can use every F_n to update the basis choice on every smaller polydisk, thus making the identification maps trivial between Δ_{n+1} and the smaller polydisks. If the F_n were chosen to converge to I fast enough, which is possible, this sequence of base-changes would converge, and we would thus get a global trivialization of our bundle ☐

5.6. **Oka's and Cartan's Theorems.**

Theorem 5.22 (Oka)**.** *The structure sheaf of any complex analytic space is coherent over itself.*

Proof. Let us begin by showing this on a polydisk Δ. Let $\phi : \mathcal{O}^n_U \xrightarrow{g} \mathcal{O}_U$ be some map, with $g = (g_1, \ldots, g_n)$. Let π denote quotient by g_1. Then, if we knew that $\mathcal{O}_U/g_1\mathcal{O}_U$ was a coherent $\mathcal{O}_U$-module, then $\ker(\pi\phi) = \ker(\phi\pi^n)$ would be a quotient of $\mathcal{O}^N_U$, but $\ker(\phi)$ is a quotient of this (for every $a \in \ker(\pi\phi)$, let $\phi(a) = bg_1$, then b is uniquely determined and we can map $a \mapsto a - (b, 0, \ldots, 0)$), which is surjective. Thus, all we need is for $\mathcal{O}_U/g\mathcal{O}_U$ to be coherent over $\mathcal{O}_U$, which by the Extension Principle is equivalent to $\mathcal{O}/g\mathcal{O}$ being coherent over itself. We know, by induction and Weierstrass preparation, that $\mathcal{O}/g\mathcal{O} \cong \mathcal{O}^{\oplus m}_{n-1}$ and that $\mathcal{O}_{n-1}$ is coherent over itself. Then the claim follows.

In the general case, As affine analytic space was defined with an ideal $\mathcal{I} \lhd \mathcal{O}_\Delta$ which is locally of finite type. Thus $\mathcal{I}$ is coherent over $\mathcal{O}$, thus $\mathcal{O}/\mathcal{I}$ is also, and thus $\mathcal{O}/\mathcal{I}$ is coherent over itself ☐

Note also that the pushforward from a closed analytic subvariety preserves sheaf coherence.

Lemma 5.23 (Amalgamation of syzygies)**.** *Let $\mathcal{F}$ be an analytic sheaf on a polydisk Δ. Then any two finite resolutions $(\mathcal{O}^{p_i})^\circ \Rightarrow \mathcal{F}, (\mathcal{O}^{q_i})^\circ \Rightarrow \mathcal{F}$ differ by a finite amount of trivial modifications, i.e. adding a trivial $\cdots \to 0 \to \mathcal{O} \to \mathcal{O} \to 0 \to \ldots$.*

Proof. The resolutions begin by two maps $\mathcal{O}^p \to \mathcal{F}, \mathcal{O}^q \to \mathcal{F}$. Then there are connecting homomorphisms $F : \mathcal{O}^p \to \mathcal{O}^q, G : \mathcal{O}^q \to \mathcal{O}^p$ between them, because of the vanishing of cohomologies of $\mathcal{O}$ on Δ. Make the trivial modification to make them $\mathcal{O}^{p+q}, \mathcal{O}^{q+p}$, and then use the alternating maps defined by $(u^p, v^q) \mapsto (v^q - FGv^q + Fu^p, u^p - Gv^p)$ and the symmetric one. These two maps are truly interchanging, and their compositions are the identities. Thus we shorten the resolutions by one, and thus it only remains to prove the case with a resolution of length 0, i.e. that $\mathcal{O}^p \cong \mathcal{O}^q$ implies $p = q$, which is easy (say via dividing by $(z_1, \ldots, z_n)$ and taking the stalk at 0) ☐

We now state and prove a weaker version for Cartan's theorems A and B, that is sufficient to our purposes.

Proposition 5.24 (Pre-Cartan)**.** *Let $\mathcal{F}$ be an analytic coherent sheaf defined in a compact polydisk Δ. Then $\mathcal{F}$ has a finite syzygy on Δ, meaning that there is an exact sequence $0 \to \mathcal{O}^{r_n} \to \ldots \mathcal{O}^{n_0} \to \mathcal{F} \to 0$.*

Proof. Δ is covered by $S_1 \times \ldots S_n$, with S_i rectangles in $\mathbb{C}$ (this uses the Riemann mapping theorem - specifically, that a rectangle is biholomorphic to the disk). Partition each rectangle to many smaller rectangles, so that $\mathcal{F}$ on each mini-cube is generated by global sections. The kernel is coherent, using Oka's theorem, and we can do the same process for it, and so on. In the end, we get that locally there is a resolution $\mathcal{O}^{p_{n-1}} \to \mathcal{O}^{p_{n-2}} \to \cdots \to \mathcal{O}^{p_0} \to \mathcal{F} \to 0$, at which point the syzygy theorem, together with the knowledge that the stalks of $\mathcal{O}$ are Noetherian, and that holomorphic bundles on Δ are trivial, gives us a local syzygy for $\mathcal{F}$. Now, our polydisk is equivalent to a polyrectangle by the Riemann mapping theorem, and so, we may divide Δ into a finite collection of polyrectangles, so that $\mathcal{F}$ has a finite syzygy on each. Then the remaining problem is to take two touching polyrectangles, which we can assume are of the form of K_1, K_2, on which $\mathcal{F}$ has finite syzygies, and extend those syzygies to a syzygy on the union. We know that on the intersection, the two syzygies are equivalent up to trivial modifications, and those modifications can be carried out across the entire K_1, K_2. Thus, we get a finite resolution $0 \to \mathcal{V}_n \to \mathcal{V}_{n-1} \to \cdots \to \mathcal{V}_0 \to \mathcal{F} \to 0$, where the $\mathcal{V}_i$ are holomorphic vector bundles. But as holomorphic bundles on a polydisk are trivial we are done ☐

Theorem 5.25. *(Cartan B): Let $\mathcal{F}$ be a coherent sheaf on a polydisk Δ. Then $H^i(\Delta, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Cover Δ with a standard (infinite) cover by rectangles (if Δ is not $\mathbb{C}^n$, this uses the Riemann mapping theorem). Then we know the higher cohomologies vanish on each rectangle, so the Cech-to-cohomology spectral sequence shows we can compute the Cech cohomology of this cover to find $H^i(\Delta, \mathcal{F})$. Suppose we are given some cycle in the Cech cohomology. We know that every finite union of rectangles has no cohomologies, so that the lifting problem is locally solvable. We can write an increasing sequence of solutions, each on a larger collection of rectangles, that eventually converges to all of $\mathbb{C}^n$. We would like some sort of convergence.

Whenever we increase our solution, say from K_m to K_{m+1}, we see a difference in K_m from the previous solution. This difference can be lifted to an earlier level, then this lift can be thought of as a chain in K_{m+1}, whose boundary we subtract from our solution in K_m, and win!

The only problem is when the lift of the difference on K_m cannot be lifted to all K_{m+1}, which is a problem that only happens in the computation of H^1. In this case we really take some approximation of the relevant function which extends to K_{m+1} (which exists from Runge's theorem), and we make the approximations increasingly good in order for the convergence to really occur ☐

Theorem 5.26 (Cartan A)**.** *Let $\mathcal{F}$ be a coherent sheaf on any polydisk $\Delta \subseteq \mathbb{C}^n$. Then it is generated (not necessarily finitely) by global sections.*

Proof. Let $\mathcal{I} \triangleleft \mathcal{O}_\Delta$ be the ideal of functions vanishing at a point. Taking the long cohomology sequence associated with $\mathcal{I}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F}$, together with Cartan B, shows the stalk of $\mathcal{F}$ at the point is generated by global sections ☐

Note: all those statements pass automatically to coherent sheaves on analytic subvarieties $Z(\mathcal{I})$ of Δ, as the closed immersion $i : Z(\mathcal{I}) \to \Delta$ preserves coherence and cohomologies under pushforward.

(This argument surprised me, as it implied that $\mathcal{O}$ is acyclic on $\mathbb{C} - \{0\}$, which can be embedded as a closed subvariety of a polydisk. But this simply is true. Another surprise is that this shows there are nontrivial holomorphic functions in $\mathbb{C}$, or Δ^1, vanishing in any discrete set of points)

5.7. **Cohomology in Complex Projective Space.** For any $C \geq 1$, let $U_i = \{x \in \mathbb{C}\mathrm{P}^n : \forall j, |x_j| < C|x_i|\}$, and consider the cover $\mathfrak{U}_C = \{U_1, \ldots, U_n\}$. Note that all finite intersections in $\mathfrak{U}_C$ are equivalent, by virtue of maps of the form $\frac{x_i}{x_j}$, to closed analytic subvarieties of polydisks. Thus, Cartan's theorem B implies every coherent sheaf on $\mathbb{C}\mathrm{P}^n$ is acyclic on the finite intersections of $\mathfrak{U}_C$. From the Cech-to-cohomology spectral sequence, we find that $H^i(\mathbb{C}\mathrm{P}^n, \mathcal{F}) \cong H^i(\mathfrak{U}_C, \mathcal{F})$.

This section assumes the result of the appendix on functional analysis. In particular, it uses the language of Fréchet spaces.

Theorem 5.27. *A coherent analytic sheaf on $\mathbb{C}P^n$ has finite-dimensional cohomologies. [comment: this should be true on more compact analytic spaces]*

Proof. By the previous remark, we can see this from Cech cohomology on the $\mathfrak{U}_C$. We will need results from the section on functional analysis. Note that for every Riemann surface X, the function space $\Gamma(X, \mathcal{O}_X)$ with the compact-open topology is a Fréchet space. We are going to define a topology on $\Gamma(U, \mathcal{F})$ for U an intersection of sets of some $\mathfrak{U}_C$, making it also a Fréchet space.

On those sets U, we have a surjection $\mathcal{O}^n \to \mathcal{F}$ [why n finite ???] which becomes a surjection $\mathcal{O}(U)^n \to \mathcal{F}(U)$. Define the topology on $\mathcal{F}(U)$ as the quotient topology coming from this surjection. Note that this is well defined as there are interchanging maps between any two projections $\mathcal{O}^n \to \mathcal{F}$. If $\mathcal{K} = \ker(\mathcal{O}_X^n|_U \to \mathcal{F}|_U)$, then the map is just the topological quotient by $\mathcal{K}(U)$.

I claim $\mathcal{K}(U) \subseteq \mathcal{O}_X^n(U)$ to be closed, which would imply $\mathcal{F}(U)$ is also Fréchet. For $n = 1$ it means that if a sequence of functions converges on compact subsets, then it generates the limit locally (on the stalks). We claim a slightly stronger thing, that there are finitely many elements which generate the sequence and the limit with coefficients forming convergent sequences. This follows inductively from the stronger form of Weierstrass division. The case for general n is similar.

Let us now choose two covers, $\mathfrak{U}_C, \mathfrak{U}_{C'}$, for $C < C'$. Then every finite intersections of sets, V, in $\mathfrak{U}_C$, is a precompact subset of a corresponding finite intersection U from $\mathfrak{U}_{C'}$.

Now consider the map $\mathcal{F}(U) \to \mathcal{F}(V)$, where we note that $V \subseteq \overline{V} \subseteq U$. Then Vitali's Theorem implies it is a compact operator.

Now, if we consider maps between the Fréchet spaces $Z^i(\mathcal{U}, \mathcal{F}) \oplus C^{i-1}(\mathcal{U}', \mathcal{F})$ and $Z^i(\mathcal{U}', \mathcal{F})$, we see that res $+\delta$ is surjective, and that res is compact. So by the main claim proven in the functional analysis section, their difference δ is an operator with finite-dimensional cokernel. But its cokernel is precisely the Cech cohomology, which is precisely $H^i(\mathbb{C}\mathrm{P}^n, \mathcal{F})$ □

6. GAGA

6.1. **The Analytification Functor.** Let X be an affine scheme of finite type over $\mathbb{C}$. Then it has a closed immersion $i : X \to \mathbb{A}^n$, with different such maps being equivalent under polynomial maps of the $\mathbb{A}^n$. This implies we can grant X a canonical structure as an affine complex analytic space, inherited from $\mathbb{A}^n$ as $i^{-1}(\mathcal{O}_{\mathbb{C}^n})$. This defines an Analytification functor $(-)^{an}$ from affine schemes of finite type over $\mathbb{C}$ to analytic spaces. Note that those are two categories of locally ringed spaces.

Theorem 6.1. *Analytification preserves open immersions (of affine schemes of finite type over $\mathbb{C}$).*

Proof. Suppose $U \subseteq X$ is an open affine subspace, and consider closed immersions $U \to \mathbb{A}^n, X \to \mathbb{A}^m$. Then U inherits two analytic structure sheaves that we should show are canonically isomorphic. It suffices to do so for $U = X_f$ (because a cover $U = \cup X_f$ means also $U = \cup U_f$!), in which case we can choose the maps $(x, y) \mapsto x, x \mapsto (x, \frac{1}{f(x)})$ as algebraic isomorphisms between Zariski neighborhoods of U in the two ambient spaces, implying that the induced analytic structure is identical as required □

It follows that analytification extends as a functor from schemes locally of finite type over $\mathbb{C}$ to analytic spaces.

Theorem 6.2. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then the local stalk map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{an}$ becomes isomorphism in the completions, and is thus flat.*

Proof. The map $A \to \hat{A}$ is always flat for A Noetherian (see the section on commutative algebra), which shows the implication. Since $\widehat{A/I} \cong \widehat{A}/\widehat{I}$ (which also follows from flatness), By definition of $\mathcal{O}_X^{an}$ it suffices to prove this in $\mathbb{C}^n$, where it is obvious □

Theorem 6.3. *Analytification commutes with closed immersions. The subscheme defined by $\mathcal{I}$ maps to the one defined by $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{an}$.*

Proof. We should show $(\mathcal{O}_X/\mathcal{I}) \otimes \mathcal{O}_X^{an} \cong \mathcal{O}_X^{an}/(\mathcal{I} \otimes \mathcal{O}_X^{an})$, but this is a result of $\mathcal{O}_X^{an}$ being flat over $\mathcal{O}_X$ □

Theorem 6.4 (Universal Property). *Let X be a scheme locally of finite type over $\mathbb{C}$. Let Y be an analytic space. Then there is a natural isomorphism* $\mathrm{Hom}_{LRS}(Y, X^{an}) \cong \mathrm{Hom}_{LRS}(Y, X)$.

Proof. Consider the family of X satisfying this. If $\mathfrak{U}$ is a basis for X, then X satisfies this iff all of $\mathfrak{U}$ does. Thus it suffices to prove the theorem for X affine. Now $i : X \to \mathbb{A}^n$ is a closed immersion, defined by ideal sheaf $\mathcal{I}$. We know that $\mathrm{Hom}_{LRS}(Y, X)$ is equivalent to maps $Y \to \mathbb{A}^n$ annihilating $\mathcal{I}$, which we may assume is equivalent to maps $Y \to \mathbb{C}^n$ annihilating $\mathcal{I} \otimes \mathcal{O}_{\mathbb{C}^n}$ (as later we deal with the case $X = \mathbb{A}^n$), and we know that $\mathrm{Hom}_{LRS}(Y, X^{an})$ also represents the same thing (as $X^{an} \to \mathbb{C}^n$ is a closed immersion with ideal sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{C}^n}$).

The case $X = \mathbb{A}^n$ easily reduces to $X = \mathbb{A}^1$. Now, $\mathrm{Hom}(Y, \mathbb{A}^1_{\mathbb{C}}) \cong \Gamma(Y, \mathcal{O}_Y)$ (this is true for all locally ringed spaces of functions to $\mathbb{C}$). $\mathrm{Hom}(Y, \mathbb{C})$ represents continuous topological maps $f : Y \to \mathbb{C}$ which pull holomorphic functions on $\mathbb{C}$ to regular functions on Y. Those are equivalent, and we are done $\square$

Theorem 6.5. *The analytification of a reduced variety is reduced.*

Proof. We should show that $\mathcal{I}_0 \otimes \mathcal{O}^{an}_{X,0} \triangleleft \mathcal{O}^{an}_{X,0}$ is radical whenever $\mathcal{I}_0 \triangleleft \mathcal{O}_{X,0}$ is. For this, we use that $\mathcal{I}_0\hat{\mathcal{O}}_{X,0}$ is radical [???] and then, from $\mathcal{O}^{an}_{X,0}$ being local and Noetherian, the Krull intersection theorem gives us that $\mathcal{I}_0\mathcal{O}^{an}_{X,0} = \mathcal{I}_0\hat{\mathcal{O}}_{X,0} \cap \mathcal{O}^{an}_{X,0}$ which finishes the proof. $\square$

I don't see why completion preserves radicals in the polynomial ring. If you do, tell me. But anyways, this doesn't seem to be required for GAGA.

We define the analytification of a sheaf by $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{an}}$.

Theorem 6.6. *Analytification is an exact functor from coherent algebraic sheaves to coherent analytic sheaves.*

Proof. Exactness follows from flatness. As both algebraic and analytic structure sheaves are coherent (Oka's theorem), it follows that coherence is equivalent to local finite presentation, so exactness finishes the proof $\square$

6.2. **Proof of GAGA.**

Theorem 6.7 (GAGA, part one)**.** *Let X be a closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$. Then analytification preserves coherent sheaf cohomology on X.*

Proof. It suffices to prove this for $X = \mathbb{P}^n_{\mathbb{C}}$, as i_* preserves cohomologies.

As any coherent algebraic sheaf $\mathcal{F}$ is a quotient of some $\mathcal{O}(d)^n$, and we can continue this syzygy, it suffices to prove the claim for $\mathcal{O}(d)$. By the exact sequence $0 \to \mathcal{O}(d-1) \to \mathcal{O}(d) \to i_*\mathcal{O}(d) \to 0$ and the 5-lemma, it suffices to prove the claim for $n = 0$ (trivial) and for $d = 0$. We already know the cohomologies $H^*(\mathbb{P}^n; \mathcal{O})$, which are $\mathbb{C}$ for $i = 0$ and 0 otherwise, so we just need to prove the same for $H^*(\mathbb{C}\mathrm{P}^n; \mathcal{O}^{an})$.

Note that the cohomologies of $\mathcal{O}^{an}$ vanish on the intersections of sets in the standard cover of $\mathbb{C}\mathrm{P}^n$, by Cartan B and their closed immersion into $\mathbb{C}^m$'s.

Note that analytic functions on $\mathbb{C}^n$ minus the loci of $z_1 z_2 \ldots z_k = 0$ are the same as Laurent series with negative powers of $z_1, \ldots, z_k$.

Now, we have a Cech cycle. Separately multi-degree-wise, this is a cycle of rational functions, so it lifts. We need to show that the lift converges as a Laurent series. It suffices to show that a cycle in a given multi-degree with $O(1)$ coefficients is lifted to $O(1)$ coefficients. And this is easy by looking at the explicit cycles $\square$

Theorem 6.8 (GAGA, part 2)**.** *Let X be a closed subscheme of $\mathbb{P}^r_{\mathbb{C}}$. Then analytification is a fully faithful between algebraic coherent sheaves on X and analytic coherent sheaves on X.*

Proof. Let $\mathcal{F}, \mathcal{G}$ be algebraic coherent sheaves on X. Recall that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent (in general). There is a map $\mathcal{H}om(\mathcal{F}, \mathcal{G})^{an} \to \mathcal{H}om(\mathcal{F}^{an}, \mathcal{G}^{an})$. Recall that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$ for $\mathcal{F}$ locally finitely presented, so when we pass to stalks the map becomes $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}_x, \mathcal{G}_x) \otimes \mathcal{O}^{an}_x \to \mathrm{Hom}(\mathcal{F}_x \otimes \mathcal{O}^{an}_x, \mathcal{G}_x \otimes \mathcal{O}^{an}_x)$. Flatness of $\mathcal{O}^{an}_X$ shows we may replace $\mathcal{F}_x$ by $\mathcal{O}_x$, and then $\mathcal{G}_x$ by $\mathcal{O}_x$, making the claim trivial. Thus we have an isomorphism $\mathcal{H}om(\mathcal{F}, \mathcal{G})^{an} \cong \mathcal{H}om(\mathcal{F}^{an}, \mathcal{G}^{an})$. Now,

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \cong H^0(\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})) \cong H^0(\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})^{an}) \cong H^0(\mathcal{H}om_{\mathcal{O}^{an}}(\mathcal{F}^{an}, \mathcal{G}^{an})) \cong \mathrm{Hom}_{\mathcal{O}^{an}}(\mathcal{F}^{an}, \mathcal{G}^{an})$$

Theorem 6.9 (GAGA, part 3)**.** *Let X be a closed subscheme of $\mathbb{P}^r_{\mathbb{C}}$. Then every coherent $\mathcal{O}^{an}$-module on X is the analytification of a coherent $\mathcal{O}$-module.*

Proof. Finally, we should show that any coherent analytic sheaf $\mathcal{F}$ on X^{an} is the analytification of an algebraic one. Wlog, $X = \mathbb{P}^r_{\mathbb{C}}$, as $\mathcal{O}_X \to \mathcal{O}_X^{an}$ is flat and injective. Let us induct on r. If we show a twist of $\mathcal{F}$ is generated by global sections, then applying this twice results in $\mathcal{O}(-d_2)^n \to \mathcal{O}(-d_1)^m \to \mathcal{F} \to 0$ which means $\mathcal{F}$ is algebraic as a cokernel of algebraic sheaves.

Write an exact sequence $0 \to \mathcal{K} \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{C} \to 0$. Then $\mathcal{C}, \mathcal{K}$ are supported on $\mathbb{P}^{r-1}$ and thus algebraic. It follows that a large enough twist of them has no higher cohomologies and is generated by global sections. Replace all sheaves of the sequence by a large twist. The spectral sequence of the exact sequence gives the exact sequence $0 \to H^0(\mathcal{K}) \to H^0(\mathcal{F}(-1)) \to H^0(\mathcal{F}) \to H^0(\mathcal{C}) \to H^1(\mathcal{F}(-1)) \to H^1(\mathcal{F}) \to 0$, and $H^i(\mathcal{F}(-1)) \cong H^i(\mathcal{F})$ for $i \geq 2$.

Note that we find $H^1(\mathcal{F}(n-1)) \to H^1(\mathcal{F}(n))$ is always a surjection. From the finite-dimensionality of cohomologies of coherent sheaves on projective space, there is some for n from which this stabilizes.

Then, we find that $H^0(\mathcal{F}) \to H^0(\mathcal{C})$ is surjective, and the latter is generated by global sections, whose lifts must generate $\mathcal{F}$ from Nakayama.

Thus, we get a surjective map $\mathcal{O}^n_{X^{an}}(m) \to \mathcal{F}$. We can apply the same for the kernel to get a presentation for $\mathcal{F}$. But then $\mathcal{F}$ is the cokernel of algebraic sheaves, so it is algebraic itself! □

To conclude, we found that analytification induces an equivalence of categories between algebraic coherent sheaves and analytic coherent sheaves on a projective scheme over $\mathbb{C}$, which preserves cohomologies. Victory!

7. Appendix - Functional Analysis

This section is here for a single purpose - to show that the sum of a surjective and a compact operator has finite-dimensional cokernel. This is used to show the finite-dimensionality of cohomology groups of $\mathbb{C}\mathrm{P}^n$.

7.1. **Fréchet Spaces.**

A Fréchet Space is a complex vector space X with a sequence $(\|\cdot\|_n)$ of pseudonorms (may have $\|a\| = 0$ for $a \neq 0$), so that any vector which vanishes in all norms is zero, and any sequence of vectors which is Cauchy with respect to any norm converges to a (single) vector in all norms. A Banach space is a Fréchet space with a single norm. We see that Fréchet spaces are metrizable (say by $\sum_n \frac{1}{2^n} \frac{\|u-v\|_n}{\|u-v\|_n+1}$).

If X is a Fréchet space and Z a closed subspace, then Z is clearly a Fréchet space. What is less obvious is the following:

Theorem 7.1. *If X is a Fréchet space and $Z \leq X$ is a closed subspace, then X/Z is a Fréchet space.*

Proof. As norm we will take, for every $\|\cdot\|_N$ a finite sum of the standard norms on X, the norm $\|x + Z\| = \inf \|x + Z\|_M$. Then it is easy to verify $x + Z$ vanishes in all of those iff $x \in Z$. For the completeness part, note that convergence of simultaneous Cauchy sequences is equivalent to convergence of simultaneous absolutely-convergent infinite sums. For those, the claim is easy. We also see that if X is Banach, then X/Z is Banach □

Theorem 7.2 (Open Mapping)**.** *A continuous surjection $f : X \to Y$ between Fréchet spaces is open.*

Proof. Let B be the unit ball. Then $X = \bigcup_n nB$, so $Y = \bigcup_n nf(B)$, so by the Baire category theorem for complete metric spaces we know $\overline{f(B)}$ has interior, and thus contains a neighborhood of zero. Let $y \in \overline{f(B)}$. Now letting $B_1 \subseteq B$ be a small ball, we see that $y - f(B)$ intersects $\overline{f(B_1)}$. Continuing in this fashion we see that $y \in f(\overline{B})$, meaning $f(\overline{B})$ itself contains an open neighborhood of 0. Since balls contain the closures of smaller balls, we are done □

Theorem 7.3. *Every finite-dimensional Hausdorff vector space has the usual topology.*

Proof. The natural map $\mathbb{C}^n \to V$ (given a basis) is continuous, and now it suffices to show there is an open bounded neighborhood of 0 in V. By compactness of S^{2n-1} and Hausdorff, we can find a neighborhood of 0 which is disjoint from S^{2n-1}. Then, by continuity of $\mathbb{C} \times V \to V$, we see that this implies some sub-neighborhood is completely contained in the unit disk $\square$

Theorem 7.4. *A locally compact Hausdorff vector space X is finite-dimensional*

Proof. Let K be a compact neighborhood of 0. Then $K \subseteq S + \frac{1}{2}K$ for a finite S. Thus $K \subseteq \operatorname{Span} S + \frac{1}{2^n}K$ for all n. If U is a neighborhood of 0, then eventually $\frac{1}{2^n}K \subseteq U$ (as this is true for every $v \in K$, and is an open condition). Thus $K \subseteq \operatorname{Span} S + U$ for every $0 \in U$, meaning $K \subseteq \overline{\operatorname{Span} S}$. From the previous theorem, we find $K \subseteq \operatorname{Span} S$, so that X is finite-dimensional $\square$

7.2. The Statement for Banach Spaces.

If E is a Banach space, then the space of bounded operators $E \to \mathbb{C}$, denoted E^*, is Banach with the operator norm. A bounded $T : E \to F$ becomes a bounded $T^* : F^* \to E^*$.

Theorem 7.5. *T has closed image iff T^* has closed image.*

Suppose T has closed image. Suppose that $T^*\phi_n$ converge to ψ. This means that $\sup_{\|u\|=1} \|\phi_n(Tu) - \psi(u)\| \to 0$. If $Tv = 0$ we get that also $\psi(v) = 0$, so $\psi = \phi \circ T$ for some linear map ϕ, defined on the image of T. As the image of T is closed, we just need ϕ to be bounded on that image. And this is true because T can not take very large vectors to very small vectors on its image by the open mapping theorem.

Suppose that T^* has closed image. Then so does T^{**}. And from the commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E^{**} \arrow[d] \\ F \arrow[r] & F^{**} \end{tikzcd}$$

With the horizontal arrows being closed injections, we are done $\square$

We say a continuous operator $f : X \to Y$ is compact if $\overline{f(B)}$ is compact for B the unit disk.

Theorem 7.6. *If T is compact, then T^* is compact.*

Proof. Suppose we have a sequence $T^*\phi_n = \phi_n T$ for $\|\phi_n\| \leq 1$. We know that T sends the unit disk to a compact set, which wlog is contained in the unit disk. So the ϕ_n induce maps from $\overline{T(B)}$ to $\overline{B_0(1)}$, and I would like for a subsequence to converge. The domain is separable as a compact metric space, and so we can assume all the points in a dense subset converge pointwise. Consider some point with a sequence from the dense subset converging to it; From uniform boundedness of the operators, we find that the $\phi_n T$ also converge on this point. Thus they converge everywhere to some bounded linear operator $\square$

Theorem 7.7. *Let E, F be Banach spaces and $T, K : E \to F$ a surjective and a compact operator. Then $T + K$ has closed image.*

Proof. Taking the dual, we need to prove that a closed injection T plus a compact K has closed image. The kernel of $T + K$ is finite dimensional as clearly every bounded sequence in it converges. Thus it is closed, so it has a complement, and the T, K from the complement are also injective with closed image and compact. So we may assume $T + K$ is injective. Now suppose $(T + K)x_n \to z$. If the x_n are not bounded we get $(T + K)u_n \to 0$ for unit vectors u_n, but then Ku_n converges and thus Tu_n does which is impossible. Thus the x_n are bounded and Kx_n, Tx_n both converge, which means the x_n have a partial limit $\square$

7.3. **The statement for Fréchet spaces.** We say an operator T of Fréchet spaces is compact if $\overline{T(B)}$ is compact for B some neighborhood of 0.

Theorem 7.8. *The sum of a surjective operator and a compact operator, between Fréchet spaces, has finite cokernel.*

Proof. Denote the operators by $T, K : X \to Y$. I claim it suffices to show $T + K$ has closed image. Indeed, in that case, the space $Y/(T+K)(X)$ would be locally compact: for every sequence of small (y_n), we would find small (x_n) with $Tx_n = y_n \equiv -Kx_n$. This implies the y_n have a partial limit and we are done.

To prove $T + K$ has closed image: Let $\| \cdot \|_n$ be a norm on Y. From continuity of T, we must have that $\|T(x)\|_n \leq C\|x\|_M$ for some large enough norm M on X. From openness, every $y \in Y$ has $\|x\|_M \leq C\|y\|_n$ so that $T(x) = y$. If we define $Z_k = \overline{Z/\ker \| \cdot \|_k}$, it follows that T reduces as a map $T_n : X_M \to Y_n$, which is also continuous and open (thus surjective). For M large enough we also get that K reduces as a continuous $K_n : X_M \to Y_n$, and this K_n is compact because the map $X \to X_M$ takes a neighborhood of zero to a dense set around zero. So, from the statement for Banach spaces, we know that the local maps $T_n + K_n : X_M \to Y_n$ have closed image. Thus, it suffices to prove the following lemma:

Lemma 7.9. *Suppose there is a map $\phi : X \to Y$ of Fréchet spaces, and suppose $\phi_n : X_n \to Y_n$ have closed images for a complete set of pseudonorms on X, Y. Then ϕ has closed image. By the open mapping theorem, there are constants C_n so that every $y \in \phi_n(X_n)$ is attained by $\|x\|_n \leq C_n\|y_n\|_n$. We may assume that the C_n are strictly increasing.*

Let $y \in \overline{\phi(X)}$. Form a sequence of $a_n \in X$, so that $\|a_n\|_{n-1} < 2^{-n} + 2^{-n-1}$ and $\|y - \sum_{i \leq n} \phi(a_i)\|_n < (2^n C_n)^{-1}$. This can be done inductively: pick $a''_n \in X_n$ with the second condition, fix him into a'_n to get the first condition, and approximate him with $a_n \in X$. Then we find $y = \phi(\sum a_i) \in \phi(X)$, as required □

8. References

(There were probably more references than this)

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