HOMOTOPIC TOPOLOGY

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These notes are based mostly on the book Homotopic Topology by Fomenko & Fuchs. There were also additional sources I used to complete some of the proofs.

1. Eilenberg-Zilber

Theorem 1.1. *There is a natural equivalence of complexes $\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$, with X, Y topological spaces (up to a natural homotopy).*

Proof. There is a special property of the functor $C_p \otimes C_q : \mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$. Namely, if K is another functor $\mathrm{Top} \times \mathrm{Top} \to \mathrm{Ab}$, then natural transformations $C_p \otimes C_q \to K$ are equivalent to elements of $K(\Delta_p, \Delta_q)$ (the image of $\Delta_p \otimes \Delta_q \in F(\Delta_p, \Delta_q)$).

Let $\mathcal{C}$ be the category of complexes of abelian groups.

The fact on $C_p \otimes C_q$, together with a diagram chase, proves a special property of the functor $C_* \otimes C_* : \mathrm{Top} \times \mathrm{Top} \to \mathcal{C}$ (which contains $\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$ in degree n): namely, if K_* is another functor $\mathrm{Top} \times \mathrm{Top} \to \mathcal{C}$ such that $K_*(\Delta_p, \Delta_q)$ is acyclic for all p, q, then natural transformations $C_* \otimes C_* \to K_*$ up to natural homotopies are equivalent to maps $C_0(*) \otimes C_0(*) \to K_0(*, *)$, i.e elements of $K_0(*, *)$.

In a similar fashion, we can show a similar property for the functor $C_n(- \times -) : \mathrm{Top} \times \mathrm{Top} \to \mathcal{C}$, this time using the special elements $\Delta_n \times \Delta_n \in C_n(\Delta_n \times \Delta_n)$. Namely, we claim that if $K_* : \mathrm{Top} \times \mathrm{Top} \to \mathcal{C}$ is such that $K_*(\Delta_n, \Delta_n)$ is acyclic for all n, then natural transformations $C_*(- \times -) \to K_*$ up to natural homotopies are equivalent to maps $C_0(* \times *) \to K_0(*, *)$, i.e. to elements of $K_0(*, *)$.

Since all the complexes $C_*(\Delta_p \times \Delta_q)$ and $C_*(\Delta_p) \otimes C_*(\Delta_q)$ are acyclic, we can utilize this for maps between either of $C_* \otimes C_*$ and $C_*(- \times -)$. This proves that those functors are equivalent up to a natural homotopy. □

This map can be extended to $\times : C_*(X, R) \otimes C_*(Y, S) \xrightarrow{\sim} C_*(X \times Y, R \otimes S)$ and to $\times : C^*(X, R) \otimes C^*(Y, S) \to C^*(X \times Y, R \otimes S)$.

The uniqueness of $\times$ in the proof allows us to collect a few properties (all true up to homotopy):

1. $\beta \times \alpha = (-1)^{|\alpha| \cdot |\beta|} \cdot \alpha \times \beta$ (under the trivial identification $X \times Y \cong Y \times X$).
2. $\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$.
3. $1 \times \alpha = \alpha \times 1 = \alpha$ (where 1 is the generator of $C_0(*)$)
4. $\partial(\alpha \times \beta) = \partial\alpha \times \beta + (-1)^{|\alpha|} \alpha \times \partial\beta$ (this simply follows from $\times$ being a morphism of complexes)

We can define the cup product using $\times$, via $\alpha \cup \beta = \Delta^*(p_0^* \alpha \times p_1^* \beta)$. The previous properties prove the well-known properties of cup product. It turns out that $\alpha \times \beta = p_X^*(\alpha) \cup p_Y^*(\beta)$ (using the fact $p_X^*(\alpha) = \alpha \times 1$).

A natural transformation $C_*(X) \to C_*(X) \otimes C_*(X)$ extending the trivial maps in degree 0 is called a diagonal approximation; by the previous results, a diagonal approximation exists and is unique up to homotopy.

2. Spectral Sequences

2.1. Algebraic Construction.

A filtered differential abelian group is an abelian group C together with an increasing filtration of subgroups $F_p \subseteq F_{p+1}$, called the filtra, whose union is C and intersection is zero, and a homomorphism $\partial : C \to C$ such that $\partial(F_p) \subseteq F_p$ and $\partial^2 = 0$.

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Let C be a filtered differential abelian group with filtra $F_p \subseteq F_{p+1}$. Define

$$E_p^r = \frac{F_p \cap \partial^{-1}(F_{p-r})}{F_{p-1} \cap \partial^{-1}(F_{p-r}) + F_p \cap \partial F_{p+r-1}}$$

for $r \geq 0$. Then ∂ induces maps $\partial : E_p^r \to E_{p-r}^r$, and if we consider the complex $E_{p+r}^r \xrightarrow{\partial} E_p^r \xrightarrow{\partial} E_{p-r}^r$, we can compute the homology in the middle as E_p^{r+1}. The limit terms E_p^∞ form the successive quotients of the induced filtration of the homology group $H(C)$ (defined as the homology of $C \xrightarrow{\partial} C \xrightarrow{\partial} C$).

Note that $E_p^0 = F_p/F_{p-1}$ and $E_p^\infty = F_pH(C)/F_{p-1}H(C)$. We write this as $E^0 = \operatorname{Gr} C, E^\infty = \operatorname{Gr} H(C)$, and we say that E is a spectral sequence that starts with $\operatorname{Gr} C$ and converges to $\operatorname{Gr} H(C)$ - an approximation to computing $H(C)$ from C.

If C is also graded, meaning $C = \bigoplus_n C_n$ and the grading is compatible with the filtration and differential (meaning that $F_p = \bigoplus_n F_p \cap C_n$ and $\partial C_n \subseteq C_{n-1}$), then we denote $E_{p,q}^r$ for the $p+q$-grading of E_p^r and get the maps $\partial : E_{pq}^r \to E_{p-r,q+r-1}^r$. A standard example is a double complex, where one dimension is used for the grading and another for the filtration.

If C also has a cup product (which is bilinear and satisfies $C_nC_m \subseteq C_{n+m}$, $F_pF_q \subseteq F_{p+q}$, and the Leibnitz rule $\partial(ab) = (\partial a)b + (-1)^{|a|} a\partial b,$), then it induces a cup product $E_{pq}^r \otimes E_{p'q'}^r \to E_{p+p',q+q'}^r$. In the limit terms E_{pq}^∞, this is compatible with the induced product on $\operatorname{Gr} H(C)$.

The construction of a spectral sequence is functorial - a homomorphism $C \to C'$ induces a homomorphism between the spectral sequences, $E_p^r(C) \to E_p^r(C')$, that is compatible with everything.

Given two spectral sequences induced by C and C', there are maps $E_p^r(C) \otimes E_q^r(C') \to E_{p+q}^r(C \otimes C')$ that preserve everything.

If $s : C \to C'$ is a map that increases filtration by at most k, $\partial s + s\partial$ vanishes on E^{k+1} and onwards.

For the case of a cohomological spectral sequence, everything is the same, we just take a decreasing filtration F^p and use $E_r^p = \frac{F^p \cap \partial^{-1}(F^{p+r})}{F^{p+1} + \partial F^{p-r+1}}$.

2.2. **Homology in Local Systems.** A Local System on a space X is a collection of abelian groups $\{G_x\}_{x \in X}$, together with isomorphisms $\phi_p : G_x \to G_y$ for every path homotopy class p from x to y, which is compatible with concatenations. Let $C_n(X; \mathcal{G})$ be the free group over simplices in X, with a simplex having coefficients in the G_x of its center, and boundary maps $C_n(X; \mathcal{G}) \to C_{n-1}(X; \mathcal{G})$ defined by the isomorphisms induced by fixed straight paths from the center of the simplex to the centers of its faces.

The dual group is $C^n(X; \mathcal{G})$, the group of maps from a simplex to the G_x of its center, with coboundary defined through the same paths. These define the homology and cohomology in the local system, $H_n(X; \mathcal{G})$ and $H^n(X; \mathcal{G})$.

2.3. **Serre Spectral Sequence.** Given a Serre fibration $\xi = (E, B)$, a homotopy class of paths in B determines a unique-up-to-homotopy homotopy equivalence between the inverse images of its endpoints. Thus, groups G_x depending only on the homotopy type of $p^{-1}(x)$ define local systems on B.

Given a filtered space $\emptyset = X_0 \subseteq \ldots \subseteq X_n = X$ with local system $\mathcal{G}$, we may consider (all coefficients in $\mathcal{G}$) the group $\bigoplus C_n(X)$ filtered by $C_n(X_p)$, or $\bigoplus C^n(X)$ filtered by $C^n(X, X_{p-1})$, making for homological and cohomological spectral sequences. These have $E_0^{pq} = C_n(X_p, X_{p-1}), E_1^{pq} = H_n(X_p, X_{p-1})$, and $E_{pq}^0 = C^n(X_p, X_{p-1}), E_{pq}^1 = H^n(X_p, X_{p-1})$. They converge to the gradings of $H_n(X), H^n(X)$. If we apply this to a CW complex filtered by its skeletons, we get the standard cellular homology and cohomology theorems (whose "elementary" proofs are quite unmotivated in comparison).

Let $E \to B$ be a fibration, with B a cellular complex whose skeletons B_p pull to a filtration E_p. The spectral sequences arising from this filtration are the Serre's Spectral Sequences.

2.3.1. *Homological.*

$$E^1_{pq} = H_n(p^{-1}B_p, p^{-1}B_{p-1}; \mathcal{G}) = \bigoplus_{D\in B_p} H_n(p^{-1}D, p^{-1}\partial D; \mathcal{G}) = \bigoplus_{D\in B_p} H_q(p^{-1}(*); \mathcal{G})$$

Thus $E_2^{pq} = H_p\left(B; H_q\left(p^{-1}(*), \mathcal{G}\right)\right)$. Note that $H_q(p^{-1}(*), \mathcal{G})$ is a local system.

We also have a tensor map $E^r_{p,n-p}(\xi_1) \otimes E^r_{q,m-q}(\xi_2) \to E^r_{p+q,n+m-p-q}(\xi_1 \times \xi_2)$, via $\alpha\otimes\beta \to \alpha\times\beta$. This map commutes with the above isomorphism.

2.3.2. *Cohomological.*

$$E_1^{pq} = H^n(p^{-1}B_p, p^{-1}B_{p-1}; \mathcal{G}) = \prod_{D\in B_p} H^n(p^{-1}D, p^{-1}\partial D; \mathcal{G}) = \prod_{D\in B_p} H^q(p^{-1}(*); \mathcal{G})$$

Thus $E^2_{pq} = H^p\left(B; H^q\left(p^{-1}(*); \mathcal{G}\right)\right)$.

Again we get $E_r^{p,n-p}(\xi_1) \otimes E_r^{q,m-q}(\xi_2) \to E_r^{p+q,n+m-p-q}(\xi_1 \times \xi_2)$ that commutes with it. This time, it gives a product structure: consider the lift of a diagonal cellular approximation

$$\begin{tikzcd}
X \arrow[r, "\widetilde{\Delta}"] \arrow[d] & X \times X \arrow[d] \\
B \arrow[r, "\widetilde{\Delta}"] & B \times B
\end{tikzcd}$$

and use it to map $E(\xi\times\xi) \to E(\xi)$. Suppose we took another approximation $\widetilde{\widetilde{\Delta}}$. As all cellular diagonal approximations $C_*(X) \to C_*(X)\otimes C_*(X)$ are homotopically equivalent (see Eilenberg-Zilber), their difference is some $s'\delta+\delta s'$ with $s : C_n(X_p) \to C_{n+1}((X\times X)_{p+1})$. Thus $s' : C^{n+1}(X\times X, (X\times X)_{p+1}) \to C^n(X, X_p)$ decreases filtration by at most 1 and $s'\delta + \delta s'$ must vanish starting from E_2, proving the product is well defined. It can also be seen from $H(B; H(F))$, but this way is cooler.

2.4. **Transgression.** The lowest row of the Serre spectral sequence is $H(B)$. The left most column is $H(F)$. But what is the sub-quotient map $H_n(B) \to H_{n-1}(F)$ or $H^{n-1}(F) \to H^n(B)$? For the homology case, I claim it comes from $H_n(B) \leftarrow H_n(E, F) \to H_{n-1}(F)$. Indeed, the elements left in E_p^{p0} are chains in E with boundary in F, and they are mapped via boundary. Similarly for cohomology it comes from $H^{n-1}(F) \to H^n(E; F) \leftarrow H^n(B)$.

2.5. **Applications.** We say that a homomorphism of graded groups is n-connected if it induces isomorphism on degrees $< n$ and an epimorphism on degree n. We say that a continuous map between topological spaces is n-connected if it induces an n-connected map on homotopy groups. We say that X is n-connected if $* \to X$ is. If there is an injection instead of surjection, we say nnected, as a joke on "co-connected".

In this language, the relative Hurewicz theorem implies that for simply connected spaces and $n \geq 2$, the map $A \to X$ is n-connected if and only if the map $H_*(A) \to H_*(X)$ is n-connected.

- The Serre Spectral Sequence can be used to compute cohomologies of Lie groups, using their fibrations inductively.
- Let X be an n-connected space, $n \geq 1$. Consider the fibration $\Omega X \to PX \to X$. Consider the diagram

$$\begin{tikzcd}
H_{r-1}(\Omega X) \arrow[r, "\sim"] & \widetilde{H}_r(\Sigma\Omega X) \arrow[r] & \widetilde{H}_r(X) \\
 & H_r(C\Omega X, \Omega X) \arrow[ul, "\sim"'] \arrow[u, "\sim"] \arrow[ur] \arrow[d, "\sim"] & \\
 & H_r(PX, \Omega X) \arrow[uul, "\sim"] \arrow[uur] &
\end{tikzcd}$$

The bottom composition is an inverse to the sub-quotient "map" of transgression. However, homologies of the base X start from $n+1$ and homologies of the fibre ΩX start from n, and since the spectral sequence converges to zero, we get that the bottom-right map is an isomorphism for $r < 2n+1$ and surjective for $r = 2n+1$. It follows that $\widetilde{H}_*(\Sigma\Omega X) \to \widetilde{H}_*(X)$, or just $\Sigma\Omega X \to X$, is $2n+1$-connected.

- Let X be an n-connected space, $n \geq 1$. Then $\widetilde{H}_*(\Sigma\Omega\Sigma X) \to \widetilde{H}_*(\Sigma X)$ is $2n+3$-connected, thus $\widetilde{H}_*(\Sigma X) \to \widetilde{H}_*(\Sigma\Omega\Sigma X)$ is $2n+3$-nnected, thus $\widetilde{H}_*(X) \to \widetilde{H}_*(\Omega\Sigma X)$ is $2n+2$-nnected, and from Hurewicz we see that $X \to \Omega\Sigma X$ is $2n+1$-connected. This map induces the suspension $\pi_*(X) \to \pi_{*+1}(\Sigma X)$. This is a proof for Freudenthal's suspension theorem.

2.6. **Killing Spaces.** Given X with minimal homotopy group $\pi_n(X) = \pi$, there is a fibration $X|_{n+1} \to X \to K(\pi, n)$. It can be used to compute homotopies.

Example 2.1. *The first stable homotopy group. There is a fibration $S^3|_4 \to S^3 \to K(\mathbb{Z}, 3)$, which can be transformed into $K(\mathbb{Z}, 2) \to S^3|_4 \to S^3$, i.e. $\mathbb{C}P^\infty \to S^3|_4 \to S^3$.*

Considering the spectral sequence more explicitly, and using the cup product, we get $H^q(S^3|_4) = \begin{cases} \mathbb{Z}/m & q = 2m+1 \\ 0 & \text{else} \end{cases}$. The universal coefficient theorem

$$0 \to \mathrm{Ext}^1(H_{k-1}(S^3|_4), \mathbb{Z}) \to H^k(S^3|_4) \to \mathrm{Hom}(H_k(S^3|_4), \mathbb{Z}) \to 0$$

implies that $H_n(S^3|_4) = \begin{cases} \mathbb{Z}/m & n = 2m \\ 0 & \text{else} \end{cases}$ (note those are finitely generated by the spectral sequence). By Hurewicz, this means $\pi_4(S^3) \cong \pi_4(S^3|_4) \cong H_4(S^3|_4) \cong \mathbb{Z}/2$. By Freudenthal, this is the first stable homotopy group.

2.6.1. *Preserved Properties.* Let $\mathcal{C}$ be a class of finitely generated abelian groups, closed under subquotients and gluing. This basically means either all finitely generated abelian groups, or only those of the form $\bigoplus \mathbb{Z}/p^n$ where the primes p come from a specific set.

Lemma 2.2. *Then $H^q(K(\pi, n), G) \in \mathcal{C}$ whenever $n, q > 0, \pi \in \mathcal{C}$, and G is finitely generated.*

Proof. express π as a product of cyclic groups, and use Kunneth, to assume $\pi = \mathbb{Z}$ or $\mathbb{Z}/p$ (noting $\mathcal{C}$ is closed under tensor). For $n = 1$, $K(\mathbb{Z}, 1) = S^1, K(\mathbb{Z}/p, 1) = S^\infty/e^{\frac{2\pi i}{p}}$, the first having cohomology $G \in \mathcal{C}$ (as $\mathbb{Z} \in \mathcal{C}$) and the second having the fibration $S^1 \to X \to S^\infty/S^1 \cong \mathbb{C}\mathrm{P}^\infty$, showing $H^q(X; G)$ is $\mathrm{Tor}(G, p)$ for odd q and G/p for even q, all in $\mathcal{C}$. For $n > 1$, the fibration $K(\pi, n) \to * \to K(\pi, n-1)$ does the job. $\square$

Theorem 2.3. *If X is simply connected and $H^*(X, \mathbb{Z}) \in \mathcal{C}$ then $\pi_*(X) \in \mathcal{C}$.*

Proof. By Hurewicz, the minimal homotopy $\pi = \pi_n(X)$ is in $\mathcal{C}$. Consider the fibration $K(\pi, n-1) \to X|_{n+1} \to X$. Its spectral sequence proves that $H^*(X|_{n+1}) \in \mathcal{C}$. We then apply this argument recursively on $X|_{n+1}$ $\square$

Theorem 2.4 (Relative Hurewicz)**.** *If X is simply connected and $\mathcal{C}$-n-connected (meaning $\pi_k(X) \in \mathcal{C}$ for $k \leq n$), the Hurewicz map $\pi_n(X) \to H_n(X; \mathbb{Z})$ is a $\mathcal{C}$-isomorphism (meaning its kernel and cokernel lie in $\mathcal{C}$).*

Proof. If X is n-connected this is clear. If it is $r < n$-connected, consider $X|_{r+1} \to X \to K(H_r(X), r)$. Consider the following square:

$$\begin{tikzcd} \pi_n\left(X|_{r+1}\right) \arrow[r] \arrow[d] & \pi_n(X) \arrow[d] \\ H_n\left(X|_{r+1}\right) \arrow[r] & H_n(X) \end{tikzcd}$$

The left map is a $\mathcal{C}$-isomorphism by induction, the top is an isomorphism by long exact sequence, and the bottom map is a $\mathcal{C}$-isomorphism by the spectral sequence ☐

Recalling that $H_n(S^3|_4) = \begin{cases} \mathbb{Z}/m & n = 2m \\ 0 & \text{else} \end{cases}$, this proves that all homotopies of $S^3|_4$, and thus all homotopies of S^3 except for π_3, are finite.

2.6.2. *Rational Cohomologies of* $K(\pi, n)$. From a previous proof, we can deduce that $H^q(K(\pi, n), \mathbb{Q}) = 0$ for finite abelian π ($\mathbb{Q}$ is not finitely generated, but it has no p-torsion or p-quotients, so the proof works). Thus we just need to know $H^*(K(\mathbb{Z}, n), \mathbb{Q})$. By an easy spectral sequence, this is the free skew-commutative graded $\mathbb{Q}$-algebra generated by an element of order n.

2.6.3. *Rank of homotopy groups.* As $H^*(X, \mathbb{Q}) = H^*(X) \otimes \mathbb{Q}$, it follows by a spectral sequence that $\pi_n(S^n), \pi_{4n-1}(S^{2n})$ have rank one and all other homotopies have rank 0.

3. Obstruction Theory

Let $\xi = (F, E, B)$ be a locally trivial fibration. Assume that B is a CW complex, and that both F and ξ are homotopically simple (meaning the action of $\pi_1(F)$ on $\pi_*(F)$ is trivial, and also for any $\gamma : S^1 \to B$, the pullback $\gamma^{-1}\xi$ is trivial). This induces a natural isomorphism between the π_n of all fibers.

Consider a section $s : B^{n-1} \to E$. Take an n-cell D of B. The pullback of ξ to D is trivial, so s defines a map $\partial D \to D \times F$, providing a value in $\pi_{n-1}(F)$ (note that we use the homotopical simplicity of F, ξ to make this well defined).

Overall, we get a cochain $c_s \in C^n_{\mathrm{CW}}(B; \pi_{n-1}(F)) = \mathrm{Hom}(H_n(B^n, B^{n-1}), \pi_{n-1}(F))$ that vanishes if and only if s can be extended to B^n.

Note that this works in the relative case as well (meaning a subcomplex $A \subseteq B$ on which the section is fixed). Also, if X is a CW complex, an attempt to lift $f : X^{n-1} \to Y$ into $f : X^n \to Y$ is equivalent to lifting a section of the trivial fibration $X \times Y \to X$, so the theory of extending maps is a special case of the theory of extending sections.

Given a pullback diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

witrh f cellular, and supposing s, s' are compatible sections on $B^{n-1}, (B')^{n-1}$, we have $c_s = f^* c_{s'}$.

Theorem 3.1. $\delta c_s = 0$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} H_{n+1}(B^{n+1}, B^n) \arrow[d, "\partial"] & \pi_{n+1}(B^{n+1}, B^n) \arrow[l, "\cong"'] \arrow[d, "\partial"] \\ H_n(B^n) \arrow[d] & \pi_n(B^n) \arrow[l] \arrow[d] \\ H_n(B^n, B^{n-1}) \arrow[dr, "c_s"] & \pi_n(B^n, B^{n-1}) \arrow[l, "\cong"'] \arrow[d] \\ & \pi_{n-1}(F) \end{tikzcd}$$

So it is required to prove that $\pi_{n+1}(B^{n+1}, B^n) \to \pi_{n-1}(F)$ vanishes. Using the long exact homotopy sequence of ξ, this vanishing is equivalent to the image of $\pi_{n+1}(B^{n+1}, B^n) \to \pi_n(B^n)$ lying inside the image of $\pi_n(E^n) \to \pi_n(B^n)$.

A generator of $\pi_{n+1}(B^{n+1}, B^n)$ can be represented by an $n+1$-cell D of B^{n+1}. Note that the pullback of ξ to D, and thus also to ∂D, is trivial. Thus the diagram

$$\begin{tikzcd}
 & \pi_{n+1}(D, \partial D) \arrow[d] \arrow[dr] & \\
\pi_n(\partial D \times F) \arrow[r] \arrow[dr] & \pi_n(\partial D) \arrow[dr] & \pi_{n+1}(B^{n+1}, B^n) \arrow[d] \\
 & \pi_n(E^n) \arrow[r] & \pi_n(B^n)
\end{tikzcd}$$

finishes the proof □

Now, suppose there are two sections $s, t : B^n \to E$ together with a homotopy h between them over B^{n-1}. What is the obstruction to extending h to B^n? Consider the fibration $E \times I \to B \times I$. We have a section on $(B \times I)^n$ that defines an obstruction cocycle $\widetilde{c}_{st} \in C^{n+1}_{\mathrm{CW}}(B \times I, \pi_n(F))$. We now define $c_{st} \in C^n_{\mathrm{CW}}(B, \pi_n(F))$ via $c_{st}(D) = \widetilde{c}_{st}(D \times I)$, so that $c_{st} = 0$ if and only if h can be extended to B^n.

Since $\widetilde{c}_{st}$ consists of c_{st}, c_s, c_t and has boundary zero, we can conclude that $\delta c_{st} = c_s - c_t$.

Theorem 3.2. *Suppose we have a section $s : B^{n-1} \to E$. Then there is $s' : B^n \to E$ agreeing with s on B^{n-2} iff c_s is a boundary.*

Proof. if such s' exists, we may restrict it to B^{n-1}, and have $\delta c_{ss'} = c_s - c_{s'} = c_s$. Conversely, If $c_s = \delta\alpha$, pick $s' : B^{n-1} \to E$ agreeing with s on B^{n-2} so that $c_{ss'} = \alpha$ i.e. $\delta c_{ss'} = c_s$ i.e. $c_{s'} = 0$ □

Theorem 3.3. *Suppose we have sections $s, t : B^{n+1} \to E$ and a homotopy h between them over B^{n-1}. Then there is a homotopy between them on B^n that agrees with h on B^{n-2} if and only if c_{st} is a boundary.*

Proof. the homotopy h is equivalent to a section $(B^{n+1} \times \partial I) \cup (B \times I)^n \to E \times I$, and the homotopy h' is equivalent to a section $(B \times I)^{n+1} \to E \times I$ that agrees with the previous one over $B^{n+1} \times \partial I$ and $(B \times I)^{n-2}$. By the previous theorem, h' exists if and only if the obstruction in $C^{n+1}_{\mathrm{CW}}(B \times I, B^{n+1} \times \partial I; \pi_n(F))$ is a boundary. But this obstruction is just c_{st} □

Now consider the case that F is $n-1$-connected. There are no obstructions to extending a section until B^n, so the first obstruction arises in $H^{n+1}(B, \pi_n(F))$. Since a homotopy of a subsection can always be extended to a homotopy of the section (by cell induction), and since the obstruction is invariant to homotopies of the section, we conclude that any two sections of B^n can be homotoped to be equal on B^{n-1}. Then their c_{st} is defined, and so $c_s - c_t = \delta c_{st}$ vanishes in cohomology. This means that we get a well-defined class $C_\xi \in H^{n+1}(B, \pi_n(F))$, called the characteristic class.

Given a pullback diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E' \arrow[d] \\
B \arrow[r, "f"] & B'
\end{tikzcd}$$

we have that $C_\xi = f^* C_{\xi'}$.

In the particular case of the tangent bundle (taking only nonzero vectors) of a connected closed oriented manifold X, $C_\xi = \chi(X) \cdot [X]$, following from the fact $\chi(X)$ is the number of zeros of a generic vector field. Thus it makes sense to call C_ξ the Euler class $e(\xi) \in H^n(B; \mathbb{Z})$.

In the case of $\Omega X \to EX \to X$ and $\pi_1(X) = \cdots = \pi_{n-1}(X) = 0$, $e(\xi) \in H^n(X; \pi_n(X))$ is the natural class.

3.1. **Maps into** $K(\pi, n)$**.** There is a natural $c \in C^n_{CW}(K(\pi, n), \pi)$ (seen when we take a specific CW construction of $K(\pi, n)$; later we will see it is independent of this). For an $n+1$-cell D with boundary $\sum k_i\sigma_i$, $\delta c(D) = c(\sum k_i\sigma_i) = \sum k_i\sigma_i = 0 \in \pi$, defining the fundamental class $F_\pi \in H^n(K(\pi, n), \pi)$.

Theorem 3.4. *There is a natural bijection for CW complexes* $[X, K(\pi, n)] \cong H^n(X, \pi)$

Proof. For the correspondence, given a map $X \to K(\pi, n)$, pull F_π into X.

- Surjectivity: Let $\gamma \in H^n(X, \pi)$, represented by $c \in C^n_{CW}(X, \pi)$. It defines a map $X^n \to K(\pi, n)$ by $\sigma \to c(\sigma)$. Extend this arbitrarily as $\delta c = 0$ and higher homotopies of $K(\pi, n)$ vanish.
- Injectivity: Suppose maps $f, g : X \to K(\pi, n)$ pull F_π to the same element. I wish to homotope them together. They agree (are constant) on X_{n-1}, and we only need one more step because then the homotopies of $K(\pi, n)$ vanish. So, we need $c_{fg} = 0$. And indeed, $c_{fg} = f^*(F_\pi) - g^*(F_\pi)$

$\square$

This shows that $K(\pi, n)$ are (weakly) homotopically unique and that F_π is independent of the CW structure.

Now, $H^n(X, \pi)$ is an abelian group, and $K(\pi, n)$ is a topological group (when π is abelian, at least). To show the bijection is actually a homomorphism, it suffices to show that $K(\pi, n) \times K(\pi, n) \to K(\pi, n)$ pulls F_π into $F_\pi \otimes 1 + 1 \otimes F_\pi$. This is trivial, however, because nothing of lower dimension exists.

Corollary 3.5. *for n-dimensional CW-complexes,* $[X, S^n] \cong H^n(X, \mathbb{Z})$.

Application 3.6. *Suppose X is an $n+1$-dimensional CW complex and let $S^n \subseteq X$. Then there is a retraction $X \to S^n$ iff $H^n(X) \to H^n(S^n)$ is surjective. Indeed, one direction is trivial, and if we take $a \in C^n(X)$ reducing to a generator of $H^n(S^n)$, then we can define a map $f : X_n \to S^n$ via $X_{n-1} \to *$ and an n-cell going to a cell representing its value in a. Now, $0 = \delta a = c_f$, implying that f can be extended to $X_{n+1} = X$.*

3.2. **Stiefel Whitney.** Given a real oriented n-bundle ξ, we can consider the fibration E_k given by k-frames of ξ. This fibration is homotopically simple by the orientability, and its fibers are $V(n, k)$, the space of k-frames in n-dimensional space.

Theorem 3.7. *Assume $1 \le k < n$. The first homotopy of $V(n, k)$ is*

$$\pi_{n-k}(V(n, k)) = \begin{cases} \mathbb{Z} & k = 1 \text{ or } n-k \text{ even} \\ \mathbb{Z}/2 & \text{else} \end{cases}$$

Proof. For $k = 1$ this is trivial. For $k = 2$ consider the fibration $S^{n-2} \to V(n, 2) \to S^{n-1}$; It can be seen that the map $\pi_{n-1}(S^{n-1}) \to \pi_{n-2}(S^{n-2})$ is 0 for n even and multiplication by 2 when n is odd, proving the case $k = 2$. The other cases follow trivially from the fibration $V(n-1, k-1) \to V(n, k) \to S^{n-1}$ $\square$

The characteristic classes are then $w_{n-k+1}(\xi) \in H^{n-k+1}(B; \pi_{n-k}(V(n, k)))$ for $k = 1, \ldots, n-1$. Clearly, the natural extension for This means w_n and w_{odd} have coefficients in $\mathbb{Z}$, and the others have coefficients in $\mathbb{Z}/2$. We extend the definition by having $w_0(\xi) = 1, w_1(\xi) = 0$, and have all the others be zero.

A more standard definition, however, mods all the coefficients by 2. In that case, homotopical simplicity is actually not needed (there can be no confusion with $\pi_{n-k}(F) = \mathbb{Z}/2$) so this works for all bundles. In this case there is a natural extension to the definition of $w_1(\xi) \in H^1(B, \mathbb{Z}/2)$, marking all the edges that change orientation. This vanishes exactly when the original bundle is oriented, so this is compatible with the previous definition.

A similar, but simpler, theory can be developed in the case of complex bundles. In that case we have $\pi_{2(n-k)+1}(V_{\mathbb{C}}(n, k)) = \mathbb{Z}$ as the first homotopy, providing the chern classes $c_j(\xi) \in H^{2j}(B; \mathbb{Z})$ for $j = 1, \ldots, n$.

3.3. **Characteristic classes as cohomology elements of the Grassmannian.** Note that vector bundles are equivalent to maps into G_n. This means that the Stiefel-Whitney classes can be defined as the pullbacks of their specific instances on the Grassmannian. Let us prove some properties of them, and compute all the cohomologies of the Grassmannian, to see that all of them are generated by the Stiefel-Whitney.

Theorem 3.8. $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) w_j(\eta)$.

Proof. We need to show that for $E \to X, E' \to X'$, $w_k(E \times E') = \sum_{i+j=k} w_i(E) \times w_j(E')$. Since

$$H^k(X \times X'; \mathbb{Z}/2) \cong \bigoplus_{i+j=k} H^i(X) \otimes H^j(X') \subseteq \bigoplus_{i+j=k} H^i(X^i) \otimes H^j(X'^j) \cong \bigoplus_{i+j=k} H^k(X^i \times X'^j; \mathbb{Z}/2)$$

We may reduce to the skeletons X_i, X'_j. So we need to prove $w_{p+q}(E \times E') = w_p(E) \times w_q(E')$ when $\dim B = p, \dim B' = q$. Adding a trivial bundle leaves the w_i invariant, so we can subtract trivial parts and stay with $\dim E = p, \dim E' = q$. Then, lifts with some isolated zeros for E, E' multiply to a lift for $E \times E'$, and counting the zeros we get the result. This argument further shows $e(\xi \oplus \eta) = e(\xi)e(\eta)$ $\square$

It then follows that the w_j for $(\mathbb{RP}^\infty)^n$ are the symmetric polynomials in x_i, as $w(\eta^n) = \prod(1 + x_i)$. We can consider the notion of a characteristic class of a bundle, which is a natural cohomological class associated to it, i.e. an element of $H^q(G_n)$. As the symmetric polynomials are independent, the amount of order q characteristic classes is at least the amount of partitions $q = r_1 + 2r_2 + \cdots + nr_n$. This is also the amount of q-cells of G_n (its points are $n \times \infty$ matrices, that can be put in a reduced form). The upper bound is thus equal to the lower bound, and we see that every characteristic class is a polynomial in Stiefel-Whitney.

Note that we can derive formulas for the w of $\xi^{\otimes k}, S^k \xi, \bigwedge^k \xi$ by finding out which symmetric polynomial they define on $(\mathbb{RP}^\infty)^n$, and using the fact that $w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$ for line bundles (which comes from constructing the sections by hand). This gives us

$$\begin{aligned} w\left(\xi^{\otimes k}\right) &= \prod_{j_1,\dots,j_k} (1 + x_{j_1} + \cdots + x_{j_k}) \\ w\left(S^k \xi\right) &= \prod_{j_1 \le \cdots \le j_k} (1 + x_{j_1} + \cdots + x_{j_k}) \\ w\left(\bigwedge^k \xi\right) &= \prod_{j_1 < \cdots < j_k} (1 + x_{j_1} + \cdots + x_{j_k}) \end{aligned}$$

As $\bigwedge^2 (\xi \oplus \eta) \cong \bigwedge^2 \xi \oplus \bigwedge^2 \eta \oplus \xi \otimes \eta$, we find $w(\xi \otimes \eta) = \frac{w(\bigwedge^2(\xi \oplus \eta))}{w(\bigwedge^2 \xi) w(\bigwedge^2 \eta)}$. Expressing all of these as polynomials in e_i, we find their Stiefel-Whitney classes.

Pretty much everything extends to the Chern classes too, now with $\eta_{\mathbb{C}} \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

We define the Chern character $\mathrm{ch}(\xi) \in H^{\mathrm{even}}(B; \mathbb{Q})$ as the class induced by $e^{x_1} + \cdots + e^{x_n} \in H^{\mathrm{even}}((\mathbb{CP}^\infty)^n; \mathbb{Q})$. Then $\mathrm{ch}(\xi \oplus \eta) = \mathrm{ch}(\xi) + \mathrm{ch}(\eta)$ and $\mathrm{ch}(\xi \otimes \eta) = \mathrm{ch}(\xi)\,\mathrm{ch}(\eta)$. The evenness is from working over $\mathbb{C}$.

4. Cohomology Operations

Cohomology operations are natural transformations $H^n(X; \pi) \to H^q(X, G)$. They are equivalent to elements of $H^q(K(\pi, n), G)$, or maps $[K(\pi, n), K(G, q)]$. Thus, a cohomology operation may not reduce dimension, and if they preserve it they come from a homomorphism $\pi \to G$. It is possible to classify operations increasing the dimension by 1.

4.1. **Stable Cohomology Operation.** A stable cohomology operation is a collection of $H^n(-,\pi) \to H^{n+r}(,G)$ that commutes with suspension. in the $K(\pi,n)$ perspective, this means a commutative square

$$\begin{tikzcd} \Sigma K(\pi,n) \arrow[r] \arrow[d, "\Sigma\phi_n"] & K(\pi,n+1) \arrow[d, "\phi_{n+1}"] \\ \Sigma K(G,n+r) \arrow[r] & K(G,n+r+1) \end{tikzcd}$$

But because of the adjunction $\Sigma \vdash \Omega$, this is equivalent to $\phi_n = \Omega\phi_{n+1}$, so stable operations are naturally an inverse limit with respect to the loop space operation.

Theorem 4.1. *a stable operation commutes with the long exact sequence, i.e. the following is commutative:*

$$\begin{tikzcd} H^n(A;\pi) \arrow[r] \arrow[d] & H^{n+1}(X,A;\pi) \arrow[d] \\ H^{n+r}(A;G) \arrow[r] & H^{n+r+1}(X,A;G) \end{tikzcd}$$

Proof. Expand the diagram into

$$\begin{tikzcd} {[A,K(\pi,n)]} \arrow[r] \arrow[d] & {[\Sigma A,\Sigma K(\pi,n)]} \arrow[r] \arrow[d] & {[\Sigma A,K(\pi,n+1)]} \arrow[r] \arrow[d] & {[X\cup CA,K(\pi,n+1)]} \arrow[d] \\ {[A,K(G,n+r)]} \arrow[r] & {[\Sigma A,\Sigma K(G,n+r)]} \arrow[r] & {[\Sigma A,K(G,n+r+1)]} \arrow[r] & {[X\cup CA,K(G,n+r+1)]} \end{tikzcd}$$

$\square$

Example 4.2. *the Bockstein is a stable operation of type* $(1,\mathbb{Z}/p,\mathbb{Z})$.

4.1.1. *Stable Operations and the Transgression.*

Theorem 4.3. *Stable cohomology operations commute with transgression. That is, elements α of the column that are killed by all differentials but the transgression τ, are passed to other such element by a stable operation ϕ, and $[\tau(\phi(\alpha))] \supset \phi([\tau(a)])$.*

Proof. This should follow from the definition of transgression: $H^q(F) \to H^{q+1}(E,F) \leftarrow H^{q+1}(B)$ $\square$

Consider the compositions $PX \cup C\Omega X \to \Sigma\Omega X \to X$ and $PX \cup C\Omega X \to PX/\Omega X \to X$. Writing them explicitly, one can see they are homotopic (making sure to choose the right direction of contraction for the first map). This implies the commutative diagram

$$\begin{tikzcd} H^{r+1}(X) \arrow[r] \arrow[dr] & H^{r+1}(\Sigma\Omega X) \arrow[r, "\cong"] & H^r(\Omega X) \arrow[dl, "\cong"] \\ & H^{r+1}(PX,\Omega X) & \end{tikzcd}$$

and plugging $X = K_n = K(\pi,n)$, it follows that $\Omega : H^{r+1}(K_n) \to H^r(K_{n-1})$ is an inverse to the transgression of the spectral sequence of $K_{n-1} \to * \to K_n$. This transgression stabilizes to an isomorphism, and thus so does the inverse limit defining the stable operations.

4.2. **The Steenrod Algebra.** Stable operations on $H^*(-;\mathbb{F}_p)$ form an associative graded $\mathbb{F}_p$-algebra under composition, that we call $\mathbb{A}_p$. This makes $H^*(X;\mathbb{F}_p)$ a graded $\mathbb{A}_p$-module. The isomorphism $H^*(X \times Y;\mathbb{F}_p) \cong H^*(X;\mathbb{F}_p) \otimes H^*(Y;\mathbb{F}_p)$ implies $\mathbb{A}_p$ may be a Hopf algebra, but we need to find a good formula for $\phi(\alpha \times \beta)$, for $\phi \in \mathbb{A}_p$, first.

4.2.1. *Construction of Operations.* All cohomologies are taken with coefficients in $\mathbb{F}_p$. Let $\pi = \mathbb{Z}/p \leq S_p$ and E_π a contractible CW complex with a free π-action (e.g. $C^i(E_\pi) = \mathbb{Z}[\pi]$ with boundaries $\sigma - 1$ and N), with the projection $E_\pi \to B_\pi$.

Quick computation of $H^*(B_\pi)$: By a spectral sequence, it is freely generated by α_1, α_2, with $\alpha_1^2 = \alpha_2$ in the case $p = 2$. We can further see that $\beta\alpha_1 = \alpha_2$, because the beginning of B_π can be constructed with a single S^1 and a 2-cell wrapping around it p times.

For a space X, we have a cyclic permutation action of π on $X^{\wedge p}$, and we would like to divide by it. This isn't right as the action isn't free, but we can make it free by multiplying with E_π first. Thus we define $D_\pi X = E_\pi \times_\pi X^{\wedge p}$.

Theorem 4.4. *Suppose n is the minimal cohomology of X. Then $H^{np}(D_\pi X) \cong (H^n(X)^{\otimes p})^\pi$.*

Proof. There is a fibration $X^{\wedge p} \to D_\pi X \to B_\pi$. Note that this fibration is not simple. Rather, the action of $\pi = \pi_1(B_\pi)$ on the fiber $X^{\wedge p}$ is exactly the permutation of coordinates. Considering the Serre spectral sequence, with coefficients in a local system, and noting that $\widetilde{H}^*(X^{\wedge p}) \cong \widetilde{H}^*(X)^{\otimes p}$ equivariantly (a fact which isn't true in every cohomology theory) gives the proof, as it gives a map from $H^0\left(B_\pi, \{\widetilde{H}^n(X)^{\otimes p}\}\right) \cong \left(\widetilde{H}^n(X)^{\otimes p}\right)^\pi$ □

We have a square of π-covering spaces

$$\begin{tikzcd} E_\pi \times X \arrow[r, "\Delta"] \arrow[d] & E_\pi \times X^{\wedge p} \arrow[d] \\ B_\pi \times X \arrow[r, "\Delta"] & D_\pi X \end{tikzcd}$$

Recall the norm map induced by covering maps. This gives us a commutative diagram on the cohomologies

$$\begin{tikzcd} H^*(E_\pi \times X) \arrow[d, shift left, "N"] & H^*(E_\pi \times X^{\wedge p}) \arrow[l, "\Delta^*"'] \arrow[d, shift left, "N"] \\ H^*(B_\pi \times X) \arrow[u, shift left] & H^*(D_\pi X) \arrow[l, "\Delta^*"'] \arrow[u, shift left] \end{tikzcd}$$

which gives us:

Proposition 4.5. *Suppose X has minimal cohomology $H^n(X)$. Let $\alpha \in H^{pn}(D_\pi X)$ and suppose it corresponds to a norm in $H^n(X)^{\otimes p}$. Then $\Delta^*\alpha = 0$.*

Now, for such an X and $\alpha \in H^n(X)$, we have that $\alpha^{\otimes p}$ is π-invariant and thus corresponds to $\lambda(\alpha) \in H^{np}(D_\pi X)$. We define $D(\alpha) = \Delta^*(\lambda(\alpha))$. Since this lies in $H^{pn}(B_{\pi+} \wedge X)$, and $H^*(B_\pi)$ has one component in each degree, we see that D corresponds to a bunch of cohomology elements of X. As this construction is natural and works for the universal K_n, we find that D actually defines cohomology operations for general spaces.

Proposition 4.6. $D(\alpha + \beta) = D(\alpha) + D(\beta)$

Proof. It suffices to prove this in the case of $e_n \otimes 1, 1 \otimes e_n$ of $K_n \times K_n$. In that case the space has minimal cohomology n and the equality follows from the fact $(\alpha + \beta)^{\otimes p} - \alpha^{\otimes p} - \beta^{\otimes p}$ is a norm □

Proposition 4.7. $D(\alpha \times \beta) = (-1)^{|\alpha||\beta|\binom{p}{2}} D(\alpha) \times D(\beta)$. *In particular* $D(\alpha\beta) = (-1)^{|\alpha||\beta|\binom{p}{2}} D(\alpha)D(\beta)$.

Proof. Wlog we may move to e_n, e_m. This should follow from the isomorphism $(K_n \times K_m)^p \cong K_n^p \times K_m^p$ (I believe the nontrivial content is Eilenberg-Zilber) □

Let β be the Bockstein. On the cochain level, we have $\delta e_n = p \cdot \beta e_n$, hence

$$\beta\lambda(e_n) = \frac{1}{p}\delta\lambda(e_n) = \frac{1}{p}\delta(e_n^{\otimes p}) = \sum_i (-1)^{ni} e_n^{\otimes i} \cdot \frac{1}{p}\delta e_n \cdot e_n^{\otimes(p-1-i)}$$

Which either vanishes or is a norm. In any case we find $\beta D = 0$. If we write this explicitly for $D(e_n) = \sum \alpha_k \otimes D_k(e_k)$, for $D_k : H^n \to H^{np-k}$, we find that

$$\sum_k \Big(\beta\alpha_k \wedge D_k(e_n) + (-1)^k \alpha_k \wedge \beta D_k(e_n)\Big) = 0$$

which means $\beta D_{2k}(e_n) + D_{2k-1}(e_n) = 0$ and $\beta D_{2k-1}(e_n) = 0$.

From standard theory of operations, $D_k e_n$ vanishes for $k > n(p-1)$ and is a constant a_n for $k = n(p-1)$.

Claim 4.8. $a_n = (-1)^{\binom{p}{2}\binom{n}{2}} a_1^n$.

Proof. this is true for $n = 0, 1$, and for $e_1 \in H^1(S^1)$ we have $D(e_n \times e_1) = (-1)^{n\binom{p}{2}} D(e_n) \times D(e_1)$. Looking at the bottom cohomology this means $a_{n+1} = (-1)^{n\binom{p}{2}} a_n a_1$ □

Claim 4.9. $a_1 = \left(\frac{p-1}{2}\right)!$ *for* p *odd and* 1 *for* $p = 2$.

Proof. This is very hard and involves doing everything by hand. (Turns out that Hatcher does this, pages 510-513) □

We can now define, for $p > 2$, $P^i e_n = (-1)^i a_n^{-1} D_{(n-2i)(p-1)} e_n \in H^{2i(p-1)}$. For $p = 2$ we define $\mathrm{Sq}^i e_n = D_{n-i}$. Then $P^0 = 1$, and then

$$\sum_{i+j=k} P^i e_n \times P^j e_m = (-1)^k a_n^{-1} a_m^{-1} \sum_{i+j=k} D_{(n-2i)(p-1)} e_n \times D_{(m-2j)(p-1)} e_m =$$
$$(-1)^{k+nm\binom{p}{2}} a_n^{-1} a_m^{-1} D_{(n+m-2k)(p-1)}(e_n \times e_m) = (-1)^{nm\binom{p}{2}} a_n^{-1} a_m^{-1} a_{m+n} P^k(e_n \times e_m) =$$
$$P^k(e_n \times e_m)$$

This calculation also works for Sq, giving the Cartan formula. For the last equality, the coefficient is $(-1)^{\binom{p}{2}\left(qr+\binom{q}{2}+\binom{r}{2}+\binom{q+r}{2}\right)} = 1$. Plugging the generator t of S^1 we find that $P^{k+1}(\alpha \times t) = P^k(\alpha) \times t$, i.e. the operations are stable. We note that the sub-algebra of $\mathbb{A}_p$ they generate has a Hopf algebra structure compatible with the Kunneth isomorphism.

Now, P^i kills $H^{<2i}$ ($H^{<i}$ for Sq) and acts by $u \to (-1)^i a_{2i}^{-1} u^p = u^p$ on H^{2i} (u^2 for Sq). Note, this only used $a_2 = -1$. Also, we have $\mathrm{Sq}^1 = \beta$.

This gives us all the basic facts about the Steenrod Powers and Squares. What remains is to show every stable operation comes from them (and β), to prove a splitting principle, and to find their Adem relations

4.2.2. *Steenrod Squares - another definition.* We want to define stable operations Sq^i satisfying $\mathrm{Sq}^0(e_i) = e_i, \mathrm{Sq}^i(e_i) = e_i^2, \mathrm{Sq}^{>i}(e_i) = 0$. Suppose we defined the action on e_{n-1} and we want to define it on e_n. For $i < n-1$, f is bijective, so $\mathrm{Sq}^i e_n = f^{-1}(\mathrm{Sq}^i e_{n-1})$. For $i = n-1$, we must have $f(\mathrm{Sq}^{n-1} e_n) = e_{n-1}^2$, so from injectivity of f we should only prove e_{n-1}^2 is transgressive, which is true as its only mid-image is $2e_{n-1}e_n = 0$ (by characteristic). For $i = n$ we should make sure that $f(e_n^2) = 0$. As f factors through $H^*(\Sigma K_{n-1})$ and cohomological products vanish in suspensions, we are done.

Let $\mathrm{Sq}(\alpha) = \sum \mathrm{Sq}^i \alpha$.

Theorem 4.10 (Cartan formula)**.** *$Sq(\alpha\beta) = Sq(\alpha)Sq(\beta)$. More generally, $Sq(\alpha \times \beta) = Sq(\alpha) \times Sq(\beta)$.*

Proof. It suffices to prove this on $e_n \times e_m \in K_n \wedge K_m$. There is a map $H^i(K_n \wedge K_m) \to H^i(\Sigma K_{n-1} \wedge K_m) \cong H^{i-1}(K_{n-1} \wedge K_m)$ that sends $e_n \times e_m \to e_{n-1} \times e_m$, and another one doing the other direction. Those kill $\gamma = \mathrm{Sq}(\alpha \times \beta) - \mathrm{Sq}(\alpha) \times \mathrm{Sq}(\beta)$ by induction. Let its k-th component be $\gamma_k = \sum \alpha \times \beta$ with $\{\alpha\}$ linearly independent and $\{\beta\}$ linearly independent. Then $\sum f(\alpha) \times \beta = \sum \alpha \times f(\beta) = 0$, meaning $f(\alpha) = f(\beta) = 0$. This implies $\deg \alpha \geq 2n, \deg \beta \geq 2m$, and as $\deg \gamma < 2(n+m)$, we get $\gamma = 0$ □

Since this was the only possible construction for stable operations with $\mathrm{Sq}^i(e_i) = e_i^2, \mathrm{Sq}^i(e_{<i}) = 0$, we find an equivalence of definitions. I don't know if this can be extended to $p > 2$ (the spectral sequence gets more complicated).

4.2.3. *Computation of* $\mathbb{A}_2$.

Theorem 4.11. *Suppose we have a fibration $F \to * \to B$ with $H^*(F; \mathbb{F}_p)$ the free skew-commutative polynomial algebra over transgressive a_i's. Then $H^*(B)$ is a free skew-commutative polynomial algebra over arbitrary $b_{i,k} \in \tau(a_i^{p^k})$ (with $k = 0$ or $|a_i|$ even) and their $\beta b_{i,k}$ (for $|a_i|$ even).*

Proof. We construct a formal spectral sequence E. The groups E_2^{ij} will be made of skew-commutative monomials in a_i, b_i. We define the differentials ∂_r by the skew-commutative action induced by $a_i^{p^k} \mapsto b_{i,k}$ and $a_i^{p^k-1} b_i \mapsto \beta b_{i,k}$ on $r = |a_i| p^k$ and $r = |a_i| p^k + 1$, respectively. Then we can inductively reduce E_r to symbols of degree $\geq r$ (and $> r$ in the b side), allowing ∂_{r+1} to be well-defined on it and to satisfy Leibnitz. Then we conclude that $E \Rightarrow 0$ is a spectral sequence.

We do, however, have to show that $\tau(a^{p^k-1}b) = \beta\tau(a^{p^k})$. It suffices to show this for a the generator of $H^*(K_{2r})$. It suffices to show $\beta \dots P^{pr} P^r e_{2r+1} \neq 0$, so that its killer must be $a^{p^k-1}b$ (this only proves the claim up to a constant $c(r,n)$, but this is enough). And indeed, the action of this operator on $e_2^r e_1 \in H^{2r+1}(K_1^{r+1})$ is nonzero.

There is a spectral sequence map $E \to E_\xi$, and as both sequences $\Rightarrow 0$, it is an isomorphism on the bottom row; Indeed, any minimal $c \in \widetilde{H}^*(B)$ not in the image or minimal polynomial relation $P(b_{i,k}) = 0$ would give a contradiction. $\square$

This implies that $H^*(K_n)$ is the free skew-commutative polynomial algebra, with basis being $e_1, \beta e_1$ for $n = 1$, $e_2, \beta e_2, P^1 \beta e_2, \beta P^1 \beta e_2, P^p P^1 \beta e_2, \dots$ for $n = 2$,

$$e_3, \beta e_3, P^1 \beta e_3, \beta P^1 \beta e_3, P^2 e_3, \beta P^2 e_3, \dots, P^{p+1} \beta P^1 \beta e_3, \dots$$

for $n = 3$, and so on. i.e., we consider $\beta^{n_0} P^{s_1} \beta^{n_1} \dots P^{s_k} \beta^{n_k}$, with $n = \pm 1$, so that $s_i \geq n_i + p \cdot s_{i+1}$ and $\mathrm{exc} < n$, where $\mathrm{exc} = \sum_{j \geq 1} (n_j + (s_j - p s_{j+1} - n_j)) = s_1 - (p-1) \sum_{j > 1} s_j$.

The case of Sq is simpler, with $H^*(K_n)$ being the polynomial algebra generated by $\mathrm{Sq}^I e_n$ with $\mathrm{exc} I < n$, where $\mathrm{Sq}^{i_1, \dots, i_k} = \mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_k}$.

Therefore $\mathbb{A}_2$ is a polynomial algebra in the proper Sq^I and $\mathbb{A}_p$ is a skew-symmetric polynomial algebra in the proper P^I, where the notion of properness is similar though slightly different.

4.3. **Splitting Principle.** Let $\mathcal{P} = K(\mathbb{Z}/p, 1)^{n+m}$. We know the action of $\mathbb{A}_p$ on $H^*(\mathcal{P})$, by $P(e) = e + e^p$ and $\beta t = e$. Consider the map $\mathbb{A}_p^q \to H^{2n+m+q}(\mathcal{P})$ given by action on $u = e_1 \dots e_n t_1 \dots t_m$. Its image consists of n-symmetric, m-skew-symmetric polynomials with monomial powers being e^{p^k} or t, and nontrivial powers of t's must be unique. The Splitting Principle is the statement that this map is injective. $\dim \mathbb{A}_p^q$ is the amount of partitions $(n_0, s_1, n_1, \dots, s_k, n_k)$ with $n = \pm 1$ and $s_i \geq p s_{i+1} + n_i$ and $\sum_i (2 s_i (p-1) + n_i) = q$. The dimension of the polynomial space we described is the amount of partitions of q into numbers of the form $2(p^k - 1)$ and $2(p^k - 1) + 1$, where numbers of the second form must be unique.

These amounts are equal, with bijection coming from reducing each s_i by p^{i-1}, and taking n_k, all together composing the first summand. As these spaces have the same dimension, it suffices to prove the map is surjective. We begin by eliminating the t's, this being done by first creating the non-t-part with P^i's (with the method shown later), then doing a composition of multiple copies of $P^{p^k} \dots P^p P^1 \beta$ with different k's. This allows us to lexicographically reduce the t-monomials, so Wlog there are only $e_1 \dots e_n$ now. For these, applying $P^{p^{k-1} i_k} P^{p^{k-2} i_{k-1}} \dots P^{i_1}$ has the maximal element with i_k p^k-powers, $(i_{k-1} - i_k)$ p^{k-1}-powers, and so on. This finishes the proof by lexicographic induction.

The case for $p = 2$ is easier: we act on $u = x_1 \dots x_n \in H^n(\mathcal{P})$, the combinatorial claim is that the amount of partitions of q into numbers of the form $2^k - 1$ is equal to the amount of partitions of q where every element is at least twice the previous. The last part of the proof is done by the $\mathrm{Sq}^{2^{k-1}i_k}\dots\mathrm{Sq}^{i_k}$ argument.

This allows us to check when a polynomial in Sq^n or P^n vanishes, just by checking its action on one element of a polynomial algebra. For instance, $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$ (both take $x_1\dots x_n \to \sum x_1^4 x_2^2 x_3\dots x_n$). In general the relations we get are

(1) $\mathrm{Sq}^a\mathrm{Sq}^b = \sum_c \binom{b-c-1}{a-2c}\mathrm{Sq}^{a+b-c}\mathrm{Sq}^c$ for $a<2b$
(2) $P^aP^b = \sum_t (-1)^{a+t}\binom{(p-1)(b-t)-1}{a-pt}P^{a+b-t}P^t$ for $a<pb$
(3) $P^a\beta P^b = \sum_i (-1)^{a+i}\left(\binom{(p-1)(b-i)}{a-pi}\beta P^{a+b-i}P^i + \binom{(p-1)(b-i)-1}{a-pi-1}P^{a+b-i}\beta P^i\right)$ for $a \le pb$

5. Adams' Spectral Sequence

5.1. **Motivation.** Suppose X is $N-1$-connected, and we want to find the p-components of its homotopy groups of indexes $N, N+1, \dots, N+n$. We can kill all cohomologies of X mod p in these dimensions, $X(1)\xrightarrow{\prod K(\mathbb{Z}_p, N+q_i-1)} X$. The spectral sequence of this map provides an epimorphism of a free (at least, in small dimensions) $\mathbb{A}_p$-module onto $H^*(X;\mathbb{Z}_p)$. The kernel is (approximately, in small dimensions) $H^*(X(1);\mathbb{Z}_p)$. We then make $X(2)$ and so on.

Note we can kill $\mathbb{A}_p$-, and not $\mathbb{Z}/p$-, generators. Counting the amount of cohomology generators over this process in a given degree $N+q$, we get an upper bound on the p-part of $\pi_{N+q}(X)$. This is the first page of the spectral sequence.

5.2. **Construction.** Let $X \in \mathrm{CW}$ with finite skeletons and $\mathbb{A} = \mathbb{A}_p$. Construct a free $\mathbb{A}$-resolution $\{B_i\}_{i\ge 0}$ of $H^*(X;\mathbb{Z}_p)$, and let $N \gg$. Let $X_0 = \Sigma^N X$. All cohomologies have coefficients in $\mathbb{Z}/p$. Denote $A \sim B$ for graded modules if they agree up to some large (but $\ll N$) degree.

We inductuvely define X_i so that $H^*(X_i) \sim \mathrm{im}\,(B_i \to B_{i-1})\,[N-i]$: Generators of $H^{N-i+q_j}(X_i) \cong \mathrm{im}\,(B_i \to B_{i-1})_{q_j}$ coming from generators of B_i induce a map $X_i \to \prod_j K_{N-i+q_j} = Y_i$ so that $H^*(Y_i) \sim B_i[N-i]$ with fiber X_{i+1}, and by the fibration $\Omega Y_i \to X_{i+1} \to X_i$ we see that

$$H^*(X_{i+1}) \sim \ker\,(B_i \to B_{i-1})\,[N-i-1] = \mathrm{im}\,(B_{i+1} \to B_i)\,[N-i-1].$$

Note that all X_i's are $N-1$-connected: this is because $\mathrm{im}\,(B_i \to B_{i-1})$ vanishes in degrees $<i$. We now have a filtered space $\ldots \subseteq X_2 \subseteq X_1 \subseteq X_0 = \Sigma^N X$, and we denote $X_{-j} = X_0$ for $j>0$. This filtered space provides a spectral sequence. Note - if N was replaced by $N+1$, each X_i could be replaced by ΣX_i. For $r>0, s,t \ge 0$, define

$$E_r^{s,t} = \mathrm{im}\left[\pi_{N+t-s}\left(X_s, X_{s+r}\right) \to \pi_{N+t-s}\left(X_{s+1-r}, X_{s+1}\right)\right]$$

and from the last statement, we see this is well defined if $N \gg r,s,t$. Since X_i are $N-1$-connected, $E^{s,t}=0$ for $t<s$. There is an obvious way to define $E_\infty^{s,t}$.

The differentials go by $\partial : E_r^{s,t} \to E_r^{s+r,t+r-1}$ (this is nonstandard). The actual map comes from the square

$$\begin{tikzcd}
\pi_{N+t-s}(X_s, X_{s+r}) \arrow[r, "\partial"] \arrow[d] & \pi_{N+t-s-1}(X_{s+r}, X_{s+2r}) \arrow[d] \\
\pi_{N+t-s}(X_{s+1-r}, X_{s+1}) \arrow[r, "\partial"] & \pi_{N+t-s-1}(X_{s+1}, X_{s+r+1})
\end{tikzcd}$$

It is clear that $\partial^2 = 0$, and now we state the Adams Theorem:

(1) $E_r' = E_{r+1}$
(2) $E_2^{s,t} = \mathrm{Ext}_{\mathbb{A}}^{s,t}\left(\widetilde{H}^*(X), \mathbb{Z}_p\right)$
(3) For $k>0$, there is a sequence $\ldots \subseteq B^{1,k+1} \subseteq B^{0,k} \subseteq \pi_k^S(X)$ such that $B^{s,t}/B^{s+1,t+1} \cong E_\infty^{s,t}$.
(4) $\bigcap_{t-s=k} B^{s,t}$ is the non-p-part of $\pi_k^S(X)_{\mathrm{Tor}}$.

Note, that if B_j are chosen well, the maps $\mathrm{Hom}_A(B_j, \mathbb{Z}_p) \to \mathrm{Hom}_{\mathbb{A}}(B_{j+1}, \mathbb{Z}_p)$ all vanish, hence $\mathrm{Ext}^k(\widetilde{H}^*(X), \mathbb{Z}_p) = \mathrm{Hom}_{\mathbb{A}}(B_k, \mathbb{Z}_p)$.

5.2.1. *Proof of Adams Theorem.* By definition, $E_1^{s,t} = \pi_{N+t-s}(X_s, X_{s+1}) \cong \pi_{N+t-s}(Y_s)$. The free $\mathbb{A}$-generators of $H^*(Y_s)$ correspond to the $\mathbb{Z}_p$-generators of $\bigoplus_t E_1^{s,t}$ (stably), i.e.

$$\bigoplus_t E_1^{s,t} \cong \mathrm{Hom}_{\mathbb{A}}\left(H^*(Y_s)[-(N-s)], \mathbb{Z}_p\right) \sim \mathrm{Hom}_{\mathbb{A}}(B_s, \mathbb{Z}_p).$$

This isomorphism is compatible with ∂, as seen by the commutative diagram

$$\begin{tikzcd}
\bigoplus_t \pi_{N+t-s}(X_s, X_{s+1}) \arrow[r] \arrow[dd, "\partial"] & \bigoplus_t \pi_{N+t-s}(Y_s) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{A}}\left(H^*(Y_s), \mathbb{Z}_p\right)[N-s] \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{A}}\left(B_s, \mathbb{Z}_p\right) \arrow[dd] \\
& \bigoplus_t \pi_{N+t-s-1}(\Omega Y_s) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{A}}\left(H^*(\Omega Y_s), \mathbb{Z}_p\right)[N-s-1] \arrow[d] & \\
\bigoplus_t \pi_{N+t-s-1}(X_{s+1}) \arrow[r] \arrow[d] & \bigoplus_t \pi_{N+t-s-1}(X_{s+1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{A}}\left(H^*(X_{s+1}), \mathbb{Z}_p\right)[N-s-1] \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{A}}\left(\mathrm{im}(B_{s+1} \to B_s), \mathbb{Z}_p\right) \arrow[d] \\
\bigoplus \pi_{N+t-s-1}(X_{s+1}, X_{s+2}) \arrow[r] & \bigoplus_t \pi_{N+t-s-1}(Y_{s+1}) \arrow[r] & \mathrm{Hom}_{\mathbb{A}}\left(H^*(Y_{s+1}), \mathbb{Z}_p\right)[N-s-1] \arrow[r] & \mathrm{Hom}_{\mathbb{A}}\left(B_{s+1}, \mathbb{Z}_p\right)
\end{tikzcd}$$

The left side says that

$$\begin{tikzcd}
\pi_n(X, F) \arrow[r] \arrow[d] & \pi_{n-1}(F) \\
\pi_n(B) \arrow[r] & \pi_{n-1}(\Omega B) \arrow[u]
\end{tikzcd}$$

commutes, this can be seen by the explicit maps on a spheroid; and the right side says that

$$\begin{tikzcd}
H^*(Y_s) \arrow[d] & H^*(\Omega Y_s)[-1] \arrow[l] & H^*(X_{s+1})[-1] \arrow[l] \arrow[d] \\
B_s[N-s] & & \mathrm{im}\left(B_{s+1}, B_s\right)[N-s] \arrow[ll]
\end{tikzcd}$$

commutes, which is true by the computation of $H^*(X_{s+1})$. All this implies (1) $\to$ (2) by the definition of Ext.

Let us prove (1). We will first define a map $\ker d_r^{s,t} \to E_{r+1}^{s,t}$. Let $n = N + t - s$. Any $\alpha \in \ker d_r^{s,t} \subseteq E_r^{s,t}$ is the image of $F \in \pi_n(X_s, X_{s+r}) \to \pi_n(X_{s+1-r}, X_{s+1})$, and we want it to get in $\mathrm{im}\left(\pi_n(X_s, X_{s+r+1}) \to \pi_n(X_{s-r}, X_{s+1})\right)$. It is clear which element we want, and to show it is an image we need to show that actually $\partial F \subseteq X_{s+r+1}$. We know that $\partial\alpha$ vanishes in $\pi_{n-1}(X_{s+1}, X_{s+r+1})$, and this suffices. This homomorphism is onto by reading this in reverse.

if α is in the kernel, it is generated by $F \in \pi_n(X_s, X_{s+r})$ which vanishes in $\pi_n(X_{s-r}, X_{s+1})$. This vanishing is expressed by $G \in \pi_{n+1}(X_{s-r}, X_s)$ with $\partial G = F \bmod X_{s+1}$. Thus $\alpha \in \mathrm{im}\, d_r^{s,t}$. Similarly, $\mathrm{im}\, d_r^{s,t}$ is in the kernel.

This proves (1).

We will assume, for now, that all $\pi_*^S(X)$ are finite. We will finish the general case later.

As the homotopies of Y_s are purely p, the non-p-part of $\pi_{N+q}(X_s)$ does not depend on s.

I claim that for $s \gg q$, the p-part of $\pi_{N+q}(X_s)$ vanishes: From $\mathcal{C}$-Hurewicz and UCT, the minimal homology and homotopy p-part are equal, and the minimal $\mathbb{Z}_p$-cohomology is them mod p. Suppose $H_{N+q}(X_s)$ has the minimal p-part and consider the homological spectral sequence of $\Omega Y_s \to X_{s+1} \to X_s$. Seeking the minimal homological p-part of X_{s+1}, we see it is either on a higher dimension than that of

X_s, or is on the same dimension and strictly smaller; This proves the claim. This argument used the finiteness of homotopies.

From here it follows that $\pi_{N+q}(X_s, X_{s'})$ are p-groups, that equal the p-part of $\pi_{N+q}(X_s)$ for $s' \gg$.

Now we finish the Adams Theorem. Denote $q = t - s$. By the last claims,

$$E_\infty^{s,t} = \mathrm{im}\,[\pi_{N+q}(X_s, X_{s+\infty}) \to \pi_{N+q}(X_{s-\infty+1}, X_{s+1})] =$$
$$\mathrm{im}\,[\pi_{N+q}(X_s) \to \pi_{N+q}(X_0) \to \pi_{N+q}(X_0, X_{s+1})] =$$
$$\frac{\mathrm{im}\,[\pi_{N+q}(X_s) \to \pi_{N+q}(X_0)]}{\mathrm{im}\,[\pi_{N+q}(X_{s+1}) \to \pi_{N+q}(X_0)]} = \frac{B^{s,t}}{B^{s+1,t+1}}$$

With $B^{s,t} = \mathrm{im}\,[\pi_{N+q}(X_s) \to \pi_{N+q}(X_0)]$. We got our filtration $B^{s,t} \subseteq B^{s-1,t-1}$ of $B^{0,q} = \pi_{N+q}(X_0) = \pi_q^S(X)$.

This, together with B^∞ being the p-part, proves (3) and (4) in the case X has finite stable homotopy groups.

5.2.2. *Functoriality.* Let $f : X \to X'$ be a continuous map. It induces maps between the $\mathbb{A}$-resolutions of $H^*(X), H^*(X')$ which is unique up to homotopy. This induces $Y_0 \to Y_0'$, which then induces $X_1 \to X_1'$ on the fiber, compatible with the resolution maps. Continuing this process we get $X_i \to X_i'$, which makes all parts of the theorem evidently functorial.

5.2.3. *Finishing the general case.* Suppose X is a CW complex, with finite skeletons, that may have infinite stable homotopy groups. They will still be finitely generated (say, by killing spaces). Now, extending the previous argument, we immediately see the non-p-part of $\pi_{N+q}(X_s)_{\mathrm{Tor}}$ is independent of s. Extending the $\pi_{N+q}(X_s, X_{s'})$ being p-groups that converge to $\pi_{N+q}(X_s)$ is harder. We make a proposition:

prop. $\bigcap_M \mathrm{im}\,[\pi_{N+q}(X_{s+M}) \to \pi_{N+q}(X_s)]$ is the non-p-part of $\pi_{N+q}(X_s)_{\mathrm{Tor}}$ (for $q \ll N$ and any s).

Wlog $s = 0$ as $\{X_{s+i}\}$ is the Adams filtration for X_s. Consider $h : \Sigma X \to \Sigma X$ coming from an order p^k map of and denote its cone X'. Denote $\pi_q^N(Y) = \pi_{N+q}(\Sigma^N Y)$. There is a map $\Sigma X \to X'$, and we get $\pi_q^N(X', \Sigma X) \cong \pi_q^N(X'/\Sigma X) \cong \pi_q^N(\Sigma^2 X) \cong \pi_{q-1}^N(\Sigma X)$ (The first iso comes from stable homotopy excision, as $q \ll N$). The map $\pi_{q-1}^N(\Sigma X) \cong \pi_q^N(X', \Sigma X) \xrightarrow{\partial} \pi_{q-1}^N(\Sigma X)$ is multiplication by p^k, as seen by taking a spheroid on the base ΣX, translating it up to a fiber of the other (mapped) ΣX, and then sort-of suspending it, via the proper cone at the top and a connection to p^k times the original spheroid at the bottom. Therefore, the groups $\pi_q^N(X')$, composed from the kernels and cokernels, are finite p-groups.

The map $\Sigma X \to X'$ induces $X_i \to X_i'$ (with $X_0 = \Sigma^N X, X_0' = \Sigma^{N-1} X'$). Now pick $\alpha \in \pi_q^N(\Sigma X)$ of order ∞ or p^n. We should prove that it is not an image from $\pi_{N+q}(X_M)$. We may choose $k \gg \alpha$ so that α is not divisible by p^k, thus its image in $\pi_q^N(X')$ is nonzero. X' has finite stable homotopies, thus $\pi_{N+q}(X_M')$ has no p-part for $M \gg$, thus the image of α does not come from $\pi_{N+q}(X_M')$, implying α does not come from $\pi_{N+q}(X_M)$. This proves the proposition.

Now we finish the proof of $(3), (4)$.

We would like to prove that $\bigcap E_M^{t,s} = \mathrm{im}\left[\pi_{N+q}(X_s) \to \pi_{N+q}(\Sigma^N X, X_{s+1})\right]$, and the rest of the proof will be identical. Thus, we want to show that if $\beta \in \pi_{N+q}(\Sigma^N X, X_{s+1})$ comes from arbitrary $\pi_{N+q}(X_s, X_M)$ then it comes from $\pi_{N+q}(X_s)$. And indeed, $\partial\beta \in \pi_{N+q}(X_{s+1})$ comes from arbitrary $\pi_{N+q}(X_M)$ and is thus torsion of order coprime to p. If $\partial\beta = 0$ then β comes from $X_s \to \Sigma^N X$ and we are done. Otherwise its order is a nontrivial number coprime to p. However, $\pi_{N+q}(\Sigma^N X, X_{s+1})$ cannot map to that part of $\pi_{N+q}(X_{s+1})$ because the boundary as by isomorphism on this part. This finishes the proof of Adams' Theorem.

5.3. **The Stable Homotopy Ring.** There is an obvious product $\circ$ in $\pi_*^S(S^0) = \bigoplus_q \pi_q^S(S^0)$. It is not hard to see it defines an associative graded skew-symmetric ring (for skew-symmetry, note that the map $S^k \wedge S^l \to S^l \wedge S^k$ has degree kl). Every $\pi_*^S(X)$ is a graded module over it, and there is a product map $\pi_*^S(X) \otimes \pi_*^S(Y) \to \pi_*^S(X \vee Y)$.

5.4. **Hopf Algebras.** If A is a Hopf k-algebra, then A-modules have a tensor product. $A \otimes A$ is a free A-module (with basis $1 \otimes a_i$ for a_i a k-basis), implying that a tensor of free modules is free. It also induces a product $\mathrm{Ext}_A^{a,b}(M,N) \otimes \mathrm{Ext}_A^{a',b'}(M',N') \to \mathrm{Ext}_A^{a+a',b+b'}(M \otimes M', N \otimes N')$ and we let $H^{*,*}(A) = \mathrm{Ext}_A^{*,*}(k,k)$; this is a bi-graded ring, and every $\mathrm{Ext}^{*,*}(M,N)$ is a module over it.

5.5. **Continuation of Adams' theorem.** Let $X', X'' \in \mathrm{CW}$ with finite skeletons, with Adams filtrations X'_i, X''_i. Then $X_n = \bigvee_{i+j=n} X'_i \wedge X''_j$ is an Adams filtration for $X' \vee X''$ (with $N = N' + N''$) by Kunneth.

Theorem 5.1. *There is an associative, skew-commutative product*

$$E_r^{t',s'}(X') \otimes E_r^{t'',s''}(X'') \to E^{t'+t'',s'+s''}(X' \wedge X'')$$

which satisfies Leibnitz, commutes with differentials, coincides with the Ext *product on page 2, and is adjoint to the π_*^S product on E_∞.*

Proof. the product of

$$\overline{\alpha'} \in \mathrm{im}\left(\pi_{N'+t'-s'}(X'^{s'}_{s'+r} \to X'^{s'-r+1}_{s'+1})\right), \overline{\alpha''} \in \mathrm{im}\left(\pi_{N''+t''-s''}(X''^{s''}_{s''+r} \to X''^{s''-r+1}_{s''+1})\right)$$

will be defined as follows: let α' be defined by $I^{N'+t'-s'} \to X'_{s'}$ with boundary in $X'_{s'+r}$, and the same for α''. Then we construct $\alpha = \left(I^{N'+t'-s'} \times I^{N''+t''-s''} \to X'_{s'} \wedge X''_{s''} \to X_s\right)$, and take its image in $\pi_{N+t-s}(X^{s-r+1}_{s+1})$ multiplied by $(-1)^{N'(t'-s')}$ (???). □

The operation is well defined, distributive and associative. Skew-commutativity comes from taking $(N'+t'-s')(N''+t''-s'')$ reflections, and then $N'N''$ more from the stabilization (??? verify the theorem)

The Adams spectral sequence, together with its multiplicative structure, allows for the computation of the stable homotopy groups π_i^S up to $i = 13$. For $i = 14$ there are differentials and things get more difficult.

6. K-Theory

From now on, X is a finite CW. Let $K(X) = K^0(X)$ be the Grothendieck group of complex vector bundles on X. there is a homomorphism $\mathrm{rk} : K(X) \to \mathbb{Z}$, and we let $\widetilde{K} = \ker \mathrm{rk}$. Every bundle is a summand of a free one, and actually $\widetilde{K}(X) = [X, \mathbb{C}\mathrm{G}_{\infty,\infty}]$. From the fibration $U(n) \to \mathbb{C}\mathrm{V}(\infty, n) \to \mathbb{C}\mathrm{G}_{\infty,n}$, we see $\Omega\left(\mathbb{C}\mathrm{G}_{\infty,\infty}\right) = U$, and denote this Grassmannian BU.

It follows $\widetilde{K}(S^r) = \pi_{r-1}(U)$, in particular, $\widetilde{K}(S^2) = \mathbb{Z}$. Consider the hopf bundle ζ on S^2, which is $\mathcal{O}(-1)$ under $S^2 \cong \mathbb{P}^1_{\mathbb{C}}$. Since $c_1(\zeta)$ is the generator of $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$, we find that $\zeta - 1$ has to generate $\widetilde{K}(S^2)$. Further, the exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ proves that $(\zeta - 1)^2 = 0$ in K-theory.

The definition $K^0(X) = [X, \mathbb{C}\mathrm{G}_{\infty,\infty}]$ immideately proves there is an exact sequence

$$\cdots \to K^0(\Sigma(X \cup CA)) \to K^0(\Sigma X) \to K^0(\Sigma A) \to K^0(X \cup CA) \to K^0(X) \to K^0(A)$$

so it makes sense to define $K^{-q}(X, A) = \widetilde{K}^0(\Sigma^q(X \cup CA))$.

Let X be a finite CW complex. We define the bott map $K^0(X) \oplus K^0(X) \to K^0(X \times S^2)$ by $(a, \beta) \mapsto \alpha \otimes 1 + \beta \otimes \zeta$. A fundamental result in K theory is that the bott map is an isomorphism.

Theorem 6.1. *The bott map is surjective.*

Proof. We begin with surjectivity. Bundles on $X \times S^2$ are pairs (E, u) of a bundle E on X and a continuous map $u : S^1 \to \operatorname{Aut} E$. We now approximate u by its Fourier series; if $S_N u = \sum_{n=-N}^{N} \hat{u}_k z^k$ are the partial sums, then $S_N u$ may not converge to u, but $\frac{1}{N}\sum_{k=1}^{N} S_k u$ does, and it does so uniformly (using the compactness of X). A sufficiently close approximation to u would correspond to an isomorphic bundle, and we may assume that u is a polynomial in z, including negative powers. Tensoring with ζ^N, which corresponds to multiplication by z^n, we get it is a true polynomial $u = \sum_{k=0}^{N} u_k z^k$.

Add a large trivial bundle of the right dimension to get that u is the matrix

$$\begin{pmatrix} \sum_{k\le N} u_k z^k & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I \end{pmatrix}$$

Since row and column operations are homotopically trivial, we are allowed to apply them. Use row operations on the identities on the diagonal to put other appropriate polynomials on the top row, and then use column operations to subtract each column from the column to its left (in the right order). Done properly, this leaves

$$\begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_N \\ -zI & I & 0 & \dots & 0 \\ 0 & -zI & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \end{pmatrix}$$

which we write as $u(z) = a + bz$. Now consider the vector space V at a specific fiber. We know that $\ker a \cap \ker b = 0$. The function mapping $[\lambda : \mu] \in \mathbb{CP}^1$ to $\det(\lambda a + \mu b)$ has finitely many roots, and for each root $(\lambda : \mu)$ we define $A_{[\lambda:\mu]} = \ker((\lambda a + \mu b)^\infty)$. Then $A = \bigoplus A_{[\lambda:\mu]}$ (if a or b is invertible this is the Jordan decomposition for $a^{-1}b$ or $b^{-1}a$; If not, we can do a base change to make it so). Note that, for $B_{[\lambda:\mu]} = aA_{[\lambda:\mu]} + bA_{[\lambda:\mu]}$ we have that $V = \bigoplus B_{[\lambda:\mu]}$.

If we group together the roots inside and outside of S^1, we reach get natural decompositions $V = V_+ \oplus V_- = V'_+ \oplus V'_-$ such that $\lambda a + \mu b : V_+ \cong V'_+$ for all $\frac{|\lambda|}{|\mu|} \ge 1$ and $\lambda a + \mu b : V_- \cong V'_-$ for all $\frac{|\lambda|}{|\mu|} \le 1$. These decompositions are natural across all of X, since roots moving continuously but avoiding S^1 cannot pass through it. Rewriting the condition in an equivalent way, $xI + a^{-1}b : V_+ \cong V_+$ for $|x| \ge 1$ and $xI + b^{-1}a : V_- \cong V_-$ for $|x| \ge 1$.

On our original bundle, this induces a factorization $(E, u(z)) \cong (E_+, I + a^{-1}bz) \oplus (E_-, I + b^{-1}az)$. Since $I + a^{-1}bz \cong I$ over E_+ and $I + b^{-1}az \cong b^{-1}az \cong z$ over E_-, we find that the original bundle indeed has the form $\alpha \otimes 1 + \beta \otimes \zeta$ □

Proposition 6.2. *The map $K^0(X) \to K^0(\Sigma^2 X)$ given by $\alpha \mapsto \alpha \wedge \zeta$ is surjective.*

Proof. This follows from the square

$$\begin{tikzcd} K^0(X) \arrow[r, "{(1,-1)}"] \arrow[d] & K^0(X) \oplus K^0(X) \arrow[r, "+"] \arrow[d] & K^0(X) \arrow[d, hook] \\ K^0(X \wedge S^2) \arrow[r] & K^0(X \times S^2) \arrow[r] & K^0(X) \oplus K^0(S^2) \end{tikzcd}$$

□

Proposition 6.3. $\pi_q(BU) \cong \pi_{q+2}(BU)$

Proof. For X a sphere, the previous proposition this gives surjectivity of $K^0(S^q) \to K^0(S^{q+2})$, or $\pi_q(BU) \to \pi_{q+2}(BU)$. Note that $\pi_1(BU) = \pi_0(U) = 0$ and $\pi_2(BU) = \pi_1(U) = \mathbb{Z}$, so it suffices to prove that high homotopies of BU (or U, or SU) contain a rank component.

However, $H^*(SU,\mathbb{Q}) \cong H^*(S^3 \times S^5 \times \dots,\mathbb{Q})$, so there is a map $SU \to K(\mathbb{Z},3) \times K(\mathbb{Z},5) \times \dots$ that is an isomorphism in rational cohomology, and viewing this as a fibration, the fiber has to have finite cohomology and thus finite homotopies, meaning that $SU \to K(\mathbb{Z},3) \times K(\mathbb{Z},5) \times \dots$ is also an isomorphism in rational homotopies. Thus, $\pi_{\mathrm{odd}}(SU)$ has rank 1, which finishes the proof (Note: this is a ganeric argument for spaces whose cohomology ring is free). □

Proposition 6.4. *The map $K^0(X) \to K^0(\Sigma^2 X)$ is an isomorphism.*

Proof. Let X be a finite CW complex and let $S^q \to X$ be any map. Up to homotopy, this can be assumed to be an embedding. Then we have

$$\begin{tikzcd} K^0(X \cup CS^q) \arrow[r] \arrow[d] & K^0(X) \arrow[r] \arrow[d] & K^0(S^q) \arrow[d] \\ K^0(\Sigma^2(X \cup CS^q)) \arrow[r] & K^0(\Sigma^2 X) \arrow[r] & K^0(S^{q+2}) \end{tikzcd}$$

which means that elements of the K-theory that vanish under the map can be lifted to the space X where we fill the sphere. We can fill a finite number of spheres until X becomes contractible, proving the result □

Proposition 6.5. *The bott map $K^0(X) \oplus K^0(X) \to K^0(X \times S^2)$ is an isomorphism.*

Proof. Recalling the diagram

$$\begin{tikzcd} K^0(X) \arrow[r, "{(1,-1)}"] \arrow[d, "\cong"] & K^0(X) \oplus K^0(X) \arrow[r, "+"] \arrow[d] & K^0(X) \arrow[d, hook] \\ K^0(X \wedge S^2) \arrow[r] & K^0(X \times S^2) \arrow[r] & K^0(X) \oplus K^0(S^2) \end{tikzcd}$$

we only need to prove that $K^0(X \wedge S^2) \to K^0(X \times S^2)$ is injective, i.e. $K^{-1}(X \times S^2) \to K^{-1}(X \vee S^2)$. But $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$, and any pair of bundles on ΣX and ΣY can be pulled to a bundle on $\Sigma(X \vee Y)$, whose restrictions to $\Sigma X, \Sigma Y$ is the original bundles using the fact there are no nontrivial bundles on S^1 □

From the periodicity we can trivially define K^n for all $n \in \mathbb{Z}$, and we have a long exact sequence.

Previously, we defined a multiplicative $\mathrm{ch} : K^0(X,A) \to H^{\mathrm{even}}(X,A;\mathbb{Q}) = \mathcal{H}^0(X,A)$. It now extends to a multiplicative map between the cohomology theories. It commutes with suspension isomorphisms, and it also commutes with their 2-periodicity, as $\mathrm{ch}(\zeta - 1)$ is the generator of $H^2(S^2;\mathbb{Q})$. We define $\mathrm{ch}_{\mathbb{Q}} : K \otimes \mathbb{Q} \to \mathcal{H}$. This is an isomorphism of cohomology theories, as it is for $i = 0, X = S^0, S^1$.

The Serre spectral sequence, when applied to K-theory, becomes $H^i(B, K^j(F)) \Rightarrow K^{i+j}(E)$ (note we have negative indices for K). When applying this to the trivial fibration $* \to X \to X$, and since $K^{\mathrm{odd}}(*) = 0$ and $K^{\mathrm{even}}(*) = \mathbb{Z}$, the spectral sequence has copies of the cohomology row of X. As $K \otimes \mathbb{Q} \cong \mathcal{H}$, Differentiation cannot reduce rank in the spectral sequence, thus all differentials are torsion.

Theorem 6.6. *$K^0(\mathbb{C}P^n) \cong \mathbb{Z}[x]/x^{n+1}$, generated by $\zeta - 1$.*

Proof. The spectral sequence is supported on even coordinates, so there are no differentials and the rank $n+1$ is correct. To prove independence of the given elements, note that $\mathrm{ch}(\zeta - 1)$ starts with $x + \dots$, and thus $\mathrm{ch}((\zeta-1)^k) = x^k + \dots$, and vanishes for $k = n+1$. It remains to prove there aren't additional elements $\sum r_i(\zeta-1)^i$for rational r_i. But the restriction to $\mathbb{C}\mathrm{P}^{n-1}$ only kills $(\zeta-1)^n$, so we may assume we have $r_i(\zeta-1)^n$. But his lifts to $K^0(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1}) \cong K^0(S^{2n})$, and its image in $H^{2n}(S^{2n},\mathbb{Q})$ is r_i times the generator. But the images have to be in $H^{2n}(S^{2n},\mathbb{Z})$ by Bott periodicity □

Proposition 6.7 (splitting principle)**.** *The map $(\mathbb{C}P^\infty)^n \to \mathrm{Gr}_{\mathbb{C}}(\infty, n)$ is injective in K-theory.*

Proof. It was injective for cohomology (being the injection of symmetric polynomials). As the K theory of the Grassmannian comes from the spectral sequence of $\mathrm{Gr}_{\mathbb{C}}$, which embeds in the differential-free sequence of $(\mathbb{C}\mathrm{P}^\infty)^n$, the claim is clear $\square$

6.1. **Adams Operations.** We will make use of the splitting principle. Consider the polynomials P_k that transform the elementary symmetric polynomials into sums of k-powers, and define $\psi_k(\xi) = P_k(\xi, \bigwedge^2 \xi, \bigwedge^3 \xi, \dots)$ for a vector bundle ξ. Note that $\dim \psi_k(\xi) = P_k(n, \binom{n}{2}, \dots) = k$.

Claim 6.8. $\psi_k(\alpha + \beta) = \psi_k(\alpha) + \psi_k(\beta)$

Proof. By the splitting principle it suffices to prove this for the K-theory elements $x_1 + \cdots + x_n, y_1 + \cdots + y_n$ on $(\mathbb{C}\mathrm{P}^\infty)^n \times (\mathbb{C}\mathrm{P}^\infty)^n$. The kth antisymmetric power of such a sum corresponds to the kth symmetric polynomial, so by definition of P_k, our required equality boils down to $\sum x_i^k + \sum y_i^k = \sum (x_i^k + y_i^k)$ $\square$

Here are more properties of the Adams operations, many of which follow from the splitting principle:

(1) $\psi^k(\xi \otimes \eta) = \psi^k(\xi)\psi^k(\eta)$, follows from $(\sum x_i^k)(\sum y_i^k) = \sum (x_i y_j)^k$
(2) $\psi^l \psi^k = \psi^{kl}$, follows from $\sum (x^k)^l = \sum x^{kl}$
(3) $\mathrm{ch}^q(\psi^k \xi) = k^q \,\mathrm{ch}^q \xi$, follows from $\sum (e^x)^k = \sum e^{kx}$
(4) $\psi^p \xi \equiv \xi^p \mod p$. I would like to say it follows from $\sum x^p \equiv (\sum x)^p \mod p$, but that requires injectivity modulo p. Instead just note that $P_p(x) \equiv x^p \mod p$.

We extend ψ by $\psi^0 \xi = \dim \xi$ and $\psi^{-k}\xi = \psi^k \overline{\xi}$. Since $\psi^k \overline{\xi} = \overline{\psi^k \xi}$, everything still holds.

Corollary 6.9 (Hopf number 1 theorem)**.** *Let $f \in \pi_{4n-1}(S^{2n})$ and consider the space X obtained by by adding the $4n$-cell. Let $a \in H^{2n}(X), b \in H^{4n}(X)$ be generators. If $a^2 = b$ then $n = 1, 2, 4$.*

Proof. Consider the short exact sequence

$$0 \to K^0(S^{4n}) \to K^0(X) \to K^0(S^{2n}) \to 0$$

and choose $\alpha, \beta \in K^0(X)$ corresponding to generators of the $K^0(S^{4n})$ and $K^0(S^{2n})$. Whose images under ch are $a + eb, b$, for some rational e. This implies $\alpha^2 = \beta$. The properties of ψ^k imply $\psi^k(\alpha) = k^n \alpha + \mu_k \beta$ and $\psi^k \beta = k^{2n} \beta$. Since $\psi^2 \alpha \equiv \beta \mod 2$, μ_2 must be odd, and since $\psi^2 \psi^3 \alpha = \psi^3 \psi^2 \alpha$ we find $2^n | 3^n - 1$, which only holds for $n = 1, 2, 4$ $\square$

6.2. **J-theory and Π-theory.**

Theorem 6.10. *If a bundle map is a homotopy equivalence on every fiber, then it has an inverse bundle map, with homotopy a bundle map.*

Proof. let the map be $f : X \to Y$ over B. First, the LES of homotopies shows f is a homotopy equivalence; With this assumption, we show it is a fiber homotopy equivalence. If we find a left-inverse fiber homotopy inverse g with $gf \sim_B 1$, then g is a homotopy equivalence and has a left-inverse fiber homotopy inverse $h \sim_B f$, and we would be done.

To find g, start with $gf \sim 1$, then using homotopy extension and $pg = pfg \sim p$ we may assume wlog g is over B. Now it suffices to show gf has a left fiber homotopy inverse, i.e. wlog $X = Y$ and $f \sim \mathrm{id}_X$ is a map over B. Let $h : X \times I \to X$ connect $f \sim \mathrm{id}_X$ then $ph_t : X \times I \to B$ is a homotopy $p \to p$, and lifts to a homotopy $k : X \times I \to X$ connecting $\mathrm{id}_X \to k_1$. I claim k_1, which is clearly a fiber map, is the answer; let $J : X \times I \to X$ be the composition of homotopies $k_1 f \sim f \sim \mathrm{id}_X$. Then pJ is homotopic via some $K : X \times I \times I \to B$, preserving endpoints, to the constant $p \to p$, and we can lift K to some $L : X \times I \times I \to X$ connecting J to some fiber homotopy over B. Then,

$$k_1 f = J_0 = L_{0,0} \sim_B L_{0,1} \sim_B L_{1,1} \sim_B L_{1,0} = J_1 = \mathrm{id}_X$$

$\square$

Consider the set of orientable sphere bundles on X. We define their dimension are the fiber dimension plus 1. We define the sum of E_1, E_2 as the pointwise join $E_1 *_X E_2$, and note this operation is commutative, associate, and adds the dimensions; In fact, if E_1, E_2 come from vector bundles, $E_1 *_X E_2$ comes from their direct sum. The corresponding Grothendieck group is denoted $\Pi(X)$.

Let J be the image of K in Π (taking the vectors of norm 1), and T be the kernel. Composition $U(n) \to \mathrm{Maps}(S^{2n-1}, S^{2n-1}) \xrightarrow{\Sigma} \mathrm{Maps}_*\left(S^{2n}, S^{2n}\right)$ makes for $U(n) \to \Omega^{2n} S^{2n}$ and the corresponding $\pi_k\left(U(n)\right) \to \pi_{k+2n}(S^{2n})$ defines J-homomorphism $\pi_k(U) \to \pi_k^S$, which is an interesting stable element for k odd. We define $\widetilde{J}, \widetilde{\Pi}$ in the obvious way.

Theorem 6.11. $\widetilde{J}(S^m) = \mathrm{im}\left(\pi_{m-1}(U) \to \pi_{m-1}^S\right)$

Proof. Vector bundles on the sphere are maps $S^{m-1} \to U$, and two maps are J-equivalent if the map is homotopic to constant in $\mathrm{Maps}\left(S^N, S^N\right)$ for $N \gg$ □

Theorem 6.12. $\widetilde{\Pi}(S^n) = \pi_{n-1}^S$ *for* $n > 1$*, zero otherwise.*

Proof. $\widetilde{\Pi}(S^n) = \varinjlim_N \pi_{n-1}\left(\mathrm{Maps}_{\deg 1}(S^N, S^N)\right)$. From the fibration $\left(\Omega^N S^N\right)_0 \to \mathrm{Maps}_{\deg 1}\left(S^N, S^N\right) \to S^N$, we are done □

As a corollary, $\widetilde{\Pi}(S^n)$ is finite. Thus, by using the exact sequence and gluing cells until triviality, $\widetilde{\Pi}(X)$ is finite for all finite CW complexes. In particular $\widetilde{J}$ is also finite.

$\widetilde{\Pi}$ makes for half a cohomology theory like K, but there is no periodicity this time.

6.3. **Adams Conjecture.** As $\widetilde{\Pi}(\Sigma X) = \varinjlim_N \left[X, \Omega^N S^N\right] = \widetilde{P}^0(X) = \widetilde{P}^1(\Sigma X)$, where P is the cohomotopy, we have $\widetilde{\Pi}^q = P^{q+1}$ for $q \le -1$. It is open for 0, wtf.

Let X be finite CW, $\alpha \in K(X)$, and $k \in \mathbb{Z}$. Then there is N so that $k^N(\psi^k\alpha - \alpha) \in T(X)$.

In other words, ψ^p preserves J away from p.

To prove Adam's conjecture for one-dimensional geometric vector bundles, we prove:

Theorem 6.13 (Adams-Dold)**.** *Let* ξ_1, ξ_2 *be vector bundles over* X *of the same dimension. Suppose there is a map between* $S(\xi_1), S(\xi_2)$ *of degree* k*. Then, under* J*, they are stably equivalent away from* k*. (the implication is from* $z \mapsto z^k$*.)*

Adding something to ξ_2 preserves the statement, so wlog $\xi_2 = n$. Let E be the bundle of ξ_1, and $f : E \to X \times S^{2n-1}$ a map of degree k on each fiber. Then it suffices to prove that for $N \gg$, there is a square

$$\begin{array}{ccc} k^N E & \xrightarrow{g} & X \times S^{2k^N n-1} \\ \Big\| & & \Big\downarrow \mathrm{id}_X \times h \\ k^N E & \xrightarrow{2^k f} & X \times S^{2k^N n-1} \end{array}$$

with $\deg h = k$, and g of degree 1 on each fiber. I will not write the proof, but if you read it, use the fact that addition of an H-space induces the addition of its homotopy (these are two monoid operations homomorphic wrt each other, also we should make explicit identifications with $\mathcal{G}(M, 0)$ via subtracting id), and that the reduced join is just $X * Y = \Sigma\left(X \wedge Y\right)$, and that suspension maps $[S^{N-1}, S^{N-1}] \to [S^N, S^N]$ create the stable homotopy groups in the colimit.

Now, there is a geometric theorem, the Transfer theorem of Becker and Gottlieb:

Let G be a compact Lie group acting on the compact smooth manifold M. Let $E \to B$ be a smooth fiber bundle with action of G preserving the base and fiber M. Note that $\Sigma^N\left(Y^+\right) = \left(Y \times S^n\right) / \left(Y \times *\right)$. Then, for $N \gg$, there is a transfer map $t : \Sigma^N X^+ \to \Sigma^N E^+$ so that the composition $\Sigma^N X^+ \to \Sigma^N E^+ \to \Sigma^N X^+$ maps all spheres to the same sphere, with maps of degree $\chi(M)$.

As a corollary, if $\chi(M)=1$, the composition is an isomorphism on cohomology. As a result, projection induces a $H^*(X)\subseteq H^*(E)$ a direct summand, for any coefficient group. Further, from the spectral sequence of $*\to X\to X$ of any cohomology theory k, it follows that $k^*(X)\subseteq k^*(E)$ a direct summand.

Given an n-sheeted cover $\pi:Y\to X$, there is a map $\pi_!:K(Y)\to K(X)$ coming from the direct sum over the inverse image. Clearly, it multiplies dimension by n, it commutes with ψ^k, and it preserves J-triviality.

Then the book finishes the proof in the general case, by showing that any $\xi\in K(X)$ can be pulled to a $K(E)$ as a direct summand (with a fibration $F\to E\to X$ s.t. $\chi(F)=1$), so that this pull is a pushforward of a line bundle

6.4. **Application to homotopy groups of spheres.** As $\widetilde{K}(S^{2n})=\mathbb{Z}$, on which ψ^k acts by k^n, we see that $\left|J(S^{2n})\right|$ divides $k^\infty(k^n-1)$ for all k, so $\left|J\left(S^{2n}\right)\right|\leq \gcd_k\left(k^\infty(k^n-1)\right)$.

There is also a lower bound to obtain:

Given $\lambda\in\pi_{2N+2k-1}(S^{2N})$, we can consider the space X_λ, obtained by gluing D^{2N+2k} to S^{2N}. It has cohomology generators a,b, and K-theory generators α,β, with $\operatorname{ch}\alpha=a+eb$, for some $e\in\mathbb{Q}$, defined in $\mathbb{Q}/\mathbb{Z}$. This is the e-invariant of λ. It satisfies $e(\lambda+\mu)=e(\lambda)+e(\mu)$ (like additivity of Hopf invariant: consider $X_{\lambda+\mu}\to X_{\lambda,\mu}\leftarrow X_\lambda,X_\mu$, and note that $X_{\lambda,\mu}$ has cohomology a,b_λ,b_μ and K-theory $\alpha,\beta_\lambda,\beta_\mu$, with maps being the obvious choice on these). It also satisfies $e(\Sigma^2\lambda)=e(\lambda)$ (I saw this from the splitting principle). Computing the e allows us to prove that an element of π^S_{2k-1} has high order.

Claim 6.14. *For $\alpha\in K(S^{2n})$ represented by a geometric ξ, $\dim\xi=N$, $T(\xi)$ is homotopy equivalent to X_λ, for $\lambda=J(\alpha)$.*

Proof. To see the cell structure of $T(\xi)$, note that by pulling to D^{2n}, we get a trivial $D^{2n}\times S^{2N}/D^{2n}\times *$ (an image of D^{2n+2N}), and that the image of its boundary on the base S^{2N} comes from the $\Phi:S^{2n-1}\to \operatorname{Map}_*\left(S^{2N},S^{2N}\right)$ describing the pointed sphere bundle $T(\xi)$.

Explicitly, the map $S^{2N+2n-1}\to S^{2N}$ comes from $S^{2N+2n-1}\to\left(S^{2n-1}\times S^{2N}\right)/\left(S^{2n-1}\times *\right)\xrightarrow{\Phi} S^{2N}$, where the first map comes from $S^{2N+2n-1}\cong\partial\left(D^{2n}\times D^{2N}\right)$.

The J-homomorphism map comes from $S^{2n-1}\to U\to[S^{2N},S^{2N}]_*$, which makes for a map in $\left[\left(S^{2n-1}\times S^{2N}\right)/\left(S^{2n-1}\times *\right),S^{2N}\right]_{*\times S^{2N}\xrightarrow{\mathrm{id}}S^{2N}}$, via Φ. These are, then, basically the same map. ∎

7. Riemann Roch

Let h be a multiplicative cohomology theory. Let $s_n\in\widetilde{h}^*(S^n)$ be the canonical generators over $h^*(*)$. It follows that multiplication by s_n is an isomorphism $\widetilde{h}^q(X)\to\widetilde{h}^{q+n}(\Sigma^nX)$.

A vector bundle ξ is h-orientable if there exists $u\in\widetilde{h}^n\left(T(\xi)\right)$, where $n=\dim_{\mathbb{R}}\xi$, whose restriction to every sphere is a generator. This is equivalent to having compatible u's on a covering of X (???)

Claim 7.1. *if ξ is h-orientable via u, and E' is the complement of the zero section, then $t_\xi:h^q(X)=h^q(E)\xrightarrow{u}h^{q+n}(E,E')$ is an isomorphism (the Thom Isomorphism, note that $(T_\xi,*)\cong(E,E')$).*

Proof. There are compatible CW structures, so by 5-lemma it suffices to prove it for $(S^m,*)$ (and $*$, which is trivial). In that case, the bundle is trivial, and we get the map $\widetilde{h}^q(S^m)\to\widetilde{h}^{q+n}(S^{m+n})$, coming from multiplication by s_n, which is an isomorphism.

Consider now a morphism between two multiplicative cohomology theories, $\tau:k^*\to h^*$. Let X be a space which is orientable with respect to both. Then, on X, $\tau t_k\neq t_h\tau$, but for $\mathcal{T}_\xi=t_h^{-1}(\tau(t_k(1)))$ we do have

$$\tau\left(t_k\left(\alpha\right)\right)=\tau\left(\alpha\cdot t_k\left(1_k\right)\right)=\tau\left(\alpha\right)\cdot\tau\left(t_k\left(1_k\right)\right)=\tau(\alpha)\cdot t_h(\mathcal{T}_\xi)=t_h(\tau(\alpha)\cdot\mathcal{T}_\xi)$$

meaning $t_h^{-1}\tau t_k(\alpha)=\tau(a)\cdot\mathcal{T}_\xi$. This is the Riemann Roch theorem. ∎

As an example, consider $\mathrm{Sq} : H^*(-;\mathbb{Z}_2) \to H^*(-;\mathbb{Z}_2)$. We can show that $\mathrm{Sq}(t(1)) = t(w)$, so $\mathcal{T} = w$ in this case. Indeed, $t^{-1}(\mathrm{Sq}(t(1)))$ is a characteristic class, so it suffices to check this on a product of hopf bundles, $\bigoplus \xi$. But as $T_{\bigoplus \zeta} = \bigwedge T_\zeta$ (smash product), with $t(\alpha) = \prod t_i(\alpha)$, it suffices to show for a single $\mathbb{RP}^\infty$. The Thom space of ζ on $\mathbb{RP}^n$ is $\mathbb{RP}^{n+1}$, by putting the $\mathbb{RP}^n$ on KAV HAMASHVE, with Thom iso by $t \in H^1(\mathbb{RP}^n;\mathbb{Z}/2)$. Thus, the Thom iso takes 1 to $\prod x_i$, which is taken to $\prod x_i(x_i+1)$ by Sq, which is taken to $\prod (x_i+1) = w$ by t^{-1}.

7.1. **The Riemann Roch for K-theory.** Let ξ be a complex vector bundle on X. Let $p : E \to X$ be its total space and E' the complement of the zero section. Then we can take $u(\xi) = 1 - p^*\bar{\xi} + p^* \bigwedge^2 \bar{\xi} - p^* \bigwedge^3 \bar{\xi} + \cdots \in K(E, E')$, where triviality along E' comes from the nonzero section there, giving a splitting $p^*\bar{\xi} = 1 \oplus \eta$ and telescoping sum.

I claim that $u(\xi)$ is an orientation for ξ in K-theory. Also, denoting $G(\sigma_1, \ldots, \sigma_n) = \prod \frac{1-e^{-x_i}}{x_i}$, (using the fact all complex bundles are $H^*(-;\mathbb{Q})$-orientable), I claim the Todd class $\mathcal{T}(\xi) = \left(t^\xi_{H^*(-;\mathbb{Q})}\right)^{-1} \mathrm{ch}\,(u(\xi))$ is equal to $G(c_1(\xi), \ldots c_n(\xi))$.

The last claim proves $u(\xi)$ is an orientation, as then its restriction to some S^n is $u([n])$ over a point, and $c_i(\xi_0) = 0$, so $\left(t^{\xi_0}_{H^*(-;\mathbb{Q})}\right)^{-1} \mathrm{ch}\,(u(\xi_0)) = 1$, thus $\mathrm{ch}\,(u(\xi_0)) = t^{\xi_0}_{H^*(-;\mathbb{Q})}(1)$ is the generator of $H^{2n}(S^{2n};\mathbb{Z})$, thus $\mathrm{ch}\, u(\xi_0)$ is the generator for K.

Now, to prove the second claim, note that $\mathcal{T}(\xi)$ is a characteristic class for ξ, so by splitting we reduce to $X = (\mathbb{CP}^\infty)^n$ with ζ^n. As $u(\xi_1 \oplus \xi_2) = u(\xi_1) \times u(\xi_2) \in K(E_1, E_1') \otimes K(E_2, E_2')$ we see that $\mathcal{T}$ is additive (???), so it suffices to assume $n = 1$. Now $T_\zeta = \mathbb{CP}^{n+1}$ (geometrically, this comes from taking some $p \in \mathbb{CP}^{n+1} - \mathbb{CP}^n$ and drawing all complex lines (i.e. S^2) through it, each intersecting $\mathbb{CP}^n$ is a point) (???). It turns out (???) that the lift of ζ really is $1 - \bar{\zeta}$. Applying chern gives $1 - e^{-x}$, and t^{-1} divides by x. In conclusion:

For an n-dimensional complex vector bundle ξ over X, there exists a natural Thom isomorphsim $t^\xi_K : K(X) \to \widetilde{K}(T_\xi)$ so that $\mathrm{ch}(t^\xi_K \alpha) = t^\xi_H(\mathrm{ch}\,\alpha \cdot \mathcal{T}(\xi))$, where $\mathcal{T}(\xi) = G(c_1(\xi), \ldots, c_n(\xi))$.

Recall that elements of π^S_{2n} had an e-invariant, $e(\lambda) \in \mathbb{Q}/\mathbb{Z}$, determined by $\mathrm{ch}\,\widetilde{\alpha} = a + eb$ on X_λ. For $\lambda = J(\xi)$ we have $X_\lambda = T_\xi$, and we can pick $\widetilde{\alpha} = u(\xi) \in \widetilde{K}(T_\xi)$, for its restriction to S^{2N} is the generator by definition. Then by RR, $\mathrm{ch}\, u(\xi) = t^\xi_{H^*(-;\mathbb{Q})}(\mathcal{T}(\xi)) = t^\xi(G(c_1(\xi), \ldots))$. The only nonzero Chern is $c_n(\xi) \in H^{2n}(S^{2n})$, and as $\mathrm{ch}\,\xi = \dim \xi + s_{2n}$, with $s_{2n} \in H^{2n}(S^{2n};\mathbb{Z})$ the generator (by some $(\zeta - 1)^{\otimes n}$ argument (???)), and

$$\mathrm{ch}_n = \frac{1}{n!} P_n(c_1, \ldots, c_n)$$

where $P_n(\sigma_1, \ldots, \sigma_n) = \sum x_i^n$, we can see (by $x_j = e^{\frac{j\tau i}{n}}$) that $P_n(0, \ldots, 0, (-1)^{\frac{n(n-1)}{2}}) = n$, thus $c_n(\xi) = \pm(n-1)! s_n$. And so, $\mathrm{ch}\,\widetilde{\alpha} = t^\xi(1 \pm \mu(n-1)! s_{2n}) = a \pm \mu(n-1)! b$, where μ is the coefficient of c_n in G. Thus $e(\lambda) = \pm(n-1)!\mu$, but we can calculate μ. This gives a lower bound on the image of the J-homomorphism.

8. Spin

Consider $\bigwedge^* \mathbb{C}^n$. It has a Hermitian product, $(\bigwedge u_i, \bigwedge v_i) = \det((u_i, v_j))$, and we let $\phi_v = F_v + F_v^*$ where F_v is the operation $(-) \wedge v$. As $F_v^2 = 0, F_v F_v^* + F_v^* F_v = \|v\|^2$ we have $\phi_v^2 = \|v\|^2$. We thus defined an $\mathbb{R}$-homomorphism $\phi : \mathbb{C}^n \to \mathrm{End}_{\mathrm{Herm}}(\bigwedge^* \mathbb{C}^n)$ that maps unit vectors to unitary matrices. Note that $\phi_v \phi_w + \phi_w \phi_v = (w, v) + (v, w)$ (check on a basis, using $\mathbb{R}$-linearity). Thus $\phi_v \phi_w \phi_v^{-1} = \phi_{\frac{(u,v)+(v,u)}{\|v\|^2} v - w}$, v acts as the (real) reflection across the vector $\vec{v}$.

Now, consider the collection of invertible unitary endomorphisms x of $\bigwedge^* \mathbb{C}^n$ that preserve the set $\{\bigwedge^{\mathrm{even}} \mathbb{C}^n, \bigwedge^{\mathrm{odd}} \mathbb{C}^n\}$ and such that conjugation by x preserves the image of ϕ. This is $\mathrm{Pin}^{\mathbb{C}}_{2n}$. Clearly, ϕ_v is in it. Let $\mathrm{Spin}^{\mathbb{C}}_{2n}$ be the index 2 subgroup preserving both $\bigwedge^{\mathrm{even}} \mathbb{C}^n, \bigwedge^{\mathrm{odd}} \mathbb{C}^n$. Define $\tau : \mathrm{Pin}^{\mathbb{C}}_{2n} \to O_{2n}$

by the action on the ϕ_v, negated on $\mathrm{Pin} - \mathrm{Spin}$. Then $\tau(\phi_v)$ is the reflection over the hyperplane orthogonal to v. As reflections generate O_{2n}, τ is surjective. Its kernel is multiplication by unit complex numbers, this can be proven by induction on n. Thus, $\mathrm{Pin}^{\mathbb{C}}$ is generated by ϕ_v and S^1, and we have given its unitary representation in $\bigwedge^* \mathbb{C}^n$. Its subgroup $\mathrm{Spin}^{\mathbb{C}}$ has representations in $\bigwedge^{\text{even}} \mathbb{C}^n, \bigwedge^{\text{odd}} \mathbb{C}^n$.

Those, I claim, are isomorphic as representations of $\mathrm{Spin}^{\mathbb{C}}_{2n-2}$. Indeed, applying ϕ_v with v orthogonal to $\mathbb{C}^{n-1}$ induces the isomorphism.

There are analogous groups $\mathrm{Pin}_n(\mathbb{R}), \mathrm{Spin}_n(\mathbb{R})$ of endomorphisms of $\bigwedge^* \mathbb{R}^n$. In this case, $\mathrm{Spin}_n(\mathbb{R})$ is a two-covering of SO_n, which is nontrivial on $n > 1$ (as seen by an explicit path $1 = \phi_v^2 \sim \phi_{-v}\phi_v = -1$). There is an embedding $\mathrm{Pin}_{2n}(\mathbb{R}) \to \mathrm{Pin}_n(\mathbb{C})$ (as every element is just $k^\times$ and some ϕ_v's). This allows us to see that the fibration $S^1 \to \mathrm{Spin}_n(\mathbb{C}) \to SO_{2n}$ is nontrivial (and it is simple, with orientation coming from the group structure of S^1), and its spectral sequence then gives

$$H^1(\mathrm{Spin}_n(\mathbb{C})) = \mathbb{Z}, H^2(\mathrm{Spin}_n(\mathbb{C})) = 0$$

The data of an oriented bundle is equivalent to a family of compatible transition maps $\phi_{ij} : U_i \cap U_j \to SO_n$, up to the right notion of equivalence. Thus, a ¡u¿Complex Spinor Structure¡/u¿ on an even-dimensional orientable bundle is the data of compatible lifts of these homomorphisms to $\mathrm{Spin}_n(\mathbb{C})$. A bundle with this structure is called a Spinor Bundle.

Recall, there are $BU_n = \mathbb{C}G_{\infty,n}, BO_n = \mathbb{R}G_{\infty,n}, BSO_n = \mathbb{R}G^+_{\infty,n}, B\pi = K(\pi, 1)$ (for π discrete). Maps to these define cohomology theories. What about $B\,\mathrm{Spin}_n(\mathbb{C})$? We have a fibration $BS^1 \to B\,\mathrm{Spin}_n(\mathbb{C}) \to BSO_{2n}$, i.e. $\mathbb{C}\mathrm{P}^\infty \to B\,\mathrm{Spin}_n(\mathbb{C}) \to \mathbb{R}G^+_{\infty,n}$. Taking the Serre spectral sequence, while using the sequences of $G \to EG \to BG$ for $G = \mathrm{Spin}, SO$ we see that $H^3(\mathbb{R}G^+_{\infty,n}) \cong \mathbb{Z}/2$ generated by $w_3^{\mathbb{Z}}$.

Theorem 8.1. *A $2n$-dimensional orientable bundle $\xi : X \to BSO_{2n}$ has a spinor structure iff $\xi^*(w_3^{\mathbb{Z}}) = 0$, or equivalently, the Stiefel Whitney $\xi^*(w_2) \in H^2(X; \mathbb{Z}/2)$ is integral. The collection of possible Spinor structures corresponds to the collection of integral lifts for $\xi^*(w_2)$.*

Proof. As $K(\mathbb{Z}, 2)$ only has π_2, the only obstruction to a lift can be $\xi^* w_3^{\mathbb{Z}}$, and the only other possibility is that there is no obstruction at all, i.e. the fibration is trivial, and this is not the case. Now, let $\xi : X \to BSO_{2n}$ satisfy $\xi^*(w_3^{\mathbb{Z}}) = 0$, or equivalently, $\xi^*(w_2)$ is integral. Now, there is an $\widetilde{w}_2 \in H^2(B\,\mathrm{Spin}^{\mathbb{C}}_n; \mathbb{Z})$ which covers the pullback of the Stiefel Whitney $w_2 \in H^2(B\,\mathrm{Spin}^{\mathbb{C}}_n; \mathbb{Z}/2)$, and we can classify the lifts s of f by $s^*\widetilde{w}_2 \in H^2(X; \mathbb{Z})$ which cover w_2. Given two lifts s, t, we can consider $c_{st} \in H^2(X; \mathbb{Z})$; It will suffice to show that $c_{st} = s^*\widetilde{w}_2 - t^*\widetilde{w}_2$. ∎