$\mathrm{Aut}(F_n)$ IS GENERATED BY NIELSEN TRANSFORMATIONS

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We refer to elements of a free group over a finite set of generators by words. Elements which are either generators or their inverses are called letters. Given a word u, with reduced presentation $u = u_1 u_2 \dots u_m$ for u_i letters, we define the length of u as $|u| = m$ and we define the following notation for prefixes and suffixes of u:

(1) $\mathrm{head}_+(u) = u_1 \dots u_{\lceil \frac{m}{2} \rceil}$
(2) $\mathrm{head}_-(u) = u_1 \dots u_{\lfloor \frac{m}{2} \rfloor}$
(3) $\mathrm{tail}_+(u) = u_{\lfloor \frac{m}{2} \rfloor} \dots u_m$
(4) $\mathrm{tail}_-(u) = u_{\lceil \frac{m}{2} \rceil} \dots u_m$.

When m is known to be even, we omit the subscript and write simply $\mathrm{head}(u), \mathrm{tail}(u)$. Let $u, v \in F_n$, and assume that $|u| \geq |v|$. If $|uv| \leq |u|$, we say that uv is a right augmentation of u by v. If $|vu| \leq |u|$, we say that vu is a left augmentation of u by v. An augmentation can never increase the length of a word; if it leaves the length unchanged, we say that the augmentation is conservative.

The following lemma is straightforward.

Lemma 1. *If uv is a right conservative augmentation of u by v, then* $\mathrm{head}_+(uv) = \mathrm{head}_+(u)$*, $|v|$ is even, and* $\mathrm{tail}(v)$ *is a suffix of* $\mathrm{tail}_-(uv)$*. Similarly, if vu is a left conservative augmentation of u by v, then* $\mathrm{tail}_+(vu) = \mathrm{tail}_+(u)$*, $|v|$ is even, and* $\mathrm{head}(v)$ *is a prefix of* $\mathrm{head}_-(vu)$*.*

Given a finite subset $S \subseteq F_n$, we let F_S be the free group formally generated by the elements of S. Given $s \in S^{\pm 1}$, we write $\hat{s} \in F_S$ for the letter corresponding to s. There is an obvious homomorphism $\pi_S : F_S \to F_n$ sending each $\hat{s}$ to s. We denote general elements of F_S with a tilde on top to distinguish them from elements of F_n. For $s \in S$, we define a set $\mathrm{Aug}_S(s) \subseteq F_S$ with the following inductive definition:

(1) $\hat{s} \in \mathrm{Aug}_S(s)$
(2) If $\widetilde{s} \in \mathrm{Aug}_S(s)$ and $t \in (S \setminus \{s\})^{\pm 1}$ is an element such that $\pi_S(\widetilde{s})t$ (or $t\pi_S(\widetilde{s})$) is a right (left) augmentation of $\pi_S(\widetilde{s})$, then $\widetilde{s}\hat{t}$ (or $\hat{t}\widetilde{s}$) is in $\mathrm{Aug}_S(s)$.

We also define $\mathrm{Aug}_S = \bigcup_{s \in S} \mathrm{Aug}_S(s)$. We say that S is reducible if there exist $s \in S$ and $\widetilde{s} \in \mathrm{Aug}_S(s)$ such that $|\pi_S(\widetilde{s})| < |s|$. Otherwise, we say that S is irreducible.

We say that an element $\widetilde{u} \in F_S$ with reduced representation $\widetilde{u} = \hat{u}_1 \hat{u}_2 \dots \hat{u}_k \in F_S$ is small if $|u_i u_{i+1} \dots u_j| \leq \max(|u_i|, |u_{i+1}|, \dots, |u_j|)$ for every $1 \leq i \leq j \leq k$. Clearly, a subword of a small element is small.

Theorem 2. *Let $S \subseteq F_n$ be an irreducible subset. Then the elements of $\mathrm{Aug}_S^{\pm 1}$ are small.*

Proof. It suffices to prove this for an elements of Aug_S. Let $\widetilde{u} \in \mathrm{Aug}_S(s)$, with the reduced representation $\widetilde{u} = \hat{v}_1 \dots \hat{v}_k \hat{s} \hat{w}_1 \dots \hat{w}_l$. Denote $m = |s|$. From Lemma 1, we find that the left augmentations via v_i and the right augmentations via w_j do not interact; therefore, any subword $\widetilde{u}'$ of $\widetilde{u}$ that contains $\hat{s}$ belongs to $\mathrm{Aug}_S(s)$ and satisfies $|\pi_S(\widetilde{u}')| = m$. It remains to check the subwords that contain only v's or only w's. Without loss of generality, we consider the subword $\hat{v}_i \hat{v}_{i+1} \dots \hat{v}_j$.

For each r, the word $v_r v_{r+1} \dots v_k s$ is a conservative left augmentation of $v_{r+1} \dots v_k s$ by v_r. Therefore $|v_r|$ is even, these two words have the same length m, and they coincide except for their first $\frac{|v_r|}{2}$ letters. Inductively, we find that $v_i v_{i+1} \dots v_k s$ and $v_{j+1} v_{j+2} \dots v_k s$ are two words of length m

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that coincide except for their first $\max(\frac{|v_i|}{2}, \frac{|v_{i+1}|}{2}, \dots, \frac{|v_j|}{2})$ letters. This implies that $|v_i v_{i+1} \dots v_j| \leq \max(|v_i|, |v_{i+1}|, \dots, |v_j|)$, as required. □

Theorem 3. *Let $S \subseteq F_n$ be an irreducible subset. Let $\widetilde{u} \in F_S$ be a nontrivial small element. Let $\hat{s}$ be a letter that appears in the word $\widetilde{u}$, such that $|s| = \max(\pi_S(\hat{u}_1), \dots, \pi_S(\hat{u}_k))$. Then $\hat{s}$ appears exactly once in $\widetilde{u}$, its inverse letter does not appear in $\widetilde{u}$, and we have $\widetilde{u} \in \mathrm{Aug}_S(s)$ or $\widetilde{u} \in \mathrm{Aug}_S(s^{-1})^{-1}$ (depending on whether s or s^{-1} belongs to S). Further, $\pi_S(\widetilde{u})$ does not have a right or left augmentation by s, and it can only have a right or left augmentation by s^{-1} if it ends or begins with $\hat{s}$, respectively.*

Proof. Let $m = |s|$. Suppose that i is the first index such that u_i is equal to one of s, s^{-1}. By replacing s with s^{-1} if necessary, we may assume that $u_i = s$. Let $i \leq j \leq k$ be the largest index such that $u_{i+1}, \dots, u_j$ are all different from s and from s^{-1}.

Every subword of $\hat{u}_1 \hat{u}_2 \dots \hat{u}_j$ that contains $\hat{u}_i = \hat{s}$ needs to project to a word of length at most m, by smallness. It follows inductively, using the irreducibility of S, that those projections all belong to $\mathrm{Aug}_S(s)$ and have length exactly m.

It remains to show that $j = k$, and also to prove inexistence of augmentations by s or s^{-1}.

However, if $j < k$, then $u_i \dots u_j$ has a right augmentation by $u_{j+1} \in \{s, s^{-1}\}$. Since $\mathrm{tail}_+(u_1 \dots u_j) = \mathrm{tail}_+(u_i \dots u_j)$, one of them has a right augmentation by $s' \in \{s, s^{-1}\}$ if and only if the other one does. Therefore, all the remaining parts of the theorem would follow if we prove that $u_i \dots u_j$ has no right augmentations by s, and that it can only have right augmentations by s^{-1} if $j = i$. (the case of left augmentations by $s^{\pm 1}$ follows by symmetry).

Suppose that $u_i \dots u_j$ has a right augmentation by $s' = s^{\pm 1}$. The element $(u_i \dots u_j)s'$ is a product of two words of length exactly m, which has length at most m. By Lemma 1 we have $\mathrm{head}_+(u_i \dots u_j) = \mathrm{head}_+(u_i) = \mathrm{head}_+(s)$, and it follows that $u_i \dots u_j s' = \mathrm{head}_+(s)\,\mathrm{tail}_+(s')$.

If $s' = s$ and $|s|$ is odd, then the last letter of $\mathrm{head}_+(s)$ and the first letter of $\mathrm{tail}_+(s)$ are both equal to the middle letter of s; thus, the product $\mathrm{head}_+(s)\,\mathrm{tail}_+(s)$ has length $m+1$, which is a contradiction.

If $s' = s$ and $|s|$ is even, we get $u_i \dots u_j s = \mathrm{head}_+(s)\,\mathrm{tail}_+(s) = s$, and thus $u_i \dots u_j = e$, which is a contradiction.

Finally, we have the case where $s' = s^{-1}$ and $j > i$. We get $s u_{i+1} \dots u_j s^{-1} = \mathrm{head}_+(s)\,\mathrm{tail}_+(s^{-1}) = e$, so $u_{i+1} u_{i+2} \dots u_j = e$. The element $\widetilde{u}' = \hat{u}_{i+1} \dots \hat{u}_j \in F_S$ is nontrivial since $j > i$, and it is small as a subword of $\widetilde{u}$. Note that $\widetilde{u}'$ lies in the subgroup $F_{S \setminus \{s\}} \subseteq F_S$. Applying the above arguement inductively for the smaller irreducible set $S \setminus \{s\}$, we find that $\widetilde{u}'$ belongs to $\mathrm{Aug}^{\pm 1}_{S \setminus \{s\}} \subseteq \mathrm{Aug}^{\pm 1}_S$. In particular, it cannot belong to the kernel of π_S, a contradiction. This concludes the proof. □

Corollary 4. *Let $S \subseteq F_n$ be an irreducible subset. Then the nontrivial small elements of F_S are precisely the elements of $\mathrm{Aug}^{\pm 1}_S$.*

Corollary 5. *Let $S \subseteq F_n$ be an irreducible subset. Let $\widetilde{u} \in F_S$ be a nontrivial small element with reduced presentation $\widetilde{u} = \hat{u}_1 \hat{u}_2 \dots \hat{u}_k$. Then $|\pi_S(\widetilde{u})| = \max(|u_1|, |u_2|, \dots, |u_k|)$.*

Corollary 6. *Let $S \subseteq F_n$ be an irreducible subset. Let $\widetilde{u} \in F_S$ be a small element, with reduced representation $\widetilde{u} = \hat{u}_1 \hat{u}_2 \dots \hat{u}_k$. Then $\mathrm{head}_+(u_1)$ is a prefix of $\mathrm{head}_+(\pi_S(\widetilde{u}))$, and $\mathrm{tail}_+(u_k)$ is a suffix of $\mathrm{tail}_+(\pi_S(\widetilde{u}))$.*

Proof. This follows from Theorem 3 and from Lemma 1. □

Lemma 7. *Let $S \subseteq F_n$ be an irreducible subset. Suppose that $\widetilde{u}, \widetilde{v} \in F_S$ are small, and also that $|\pi_S(\widetilde{u}\widetilde{v})| \leq \max(|\pi_S(\widetilde{u})|, |\pi_S(\widetilde{v})|)$. Then $\widetilde{u}\widetilde{v}$ is small.*

Proof. Without loss of generality $|\pi_S(\widetilde{u})| \geq |\pi_S(\widetilde{v})|$. We may assume that the last letter of $\widetilde{u}$ and the first letter of $\widetilde{v}$ are not inverses of each other. We prove this by induction on $|\pi_S(\widetilde{v})|$, with the case $|\pi_S(\widetilde{v})| = 0$ being trivial.

Let $\widetilde{v} = \hat{v}_1 \hat{v}_2 \dots \hat{v}_k$ be the reduced representation of $\widetilde{v}$. From $|\pi_S(\widetilde{u})\pi_S(\widetilde{v})| \leq |\pi_S(\widetilde{u})|$ and $|\pi_S(\widetilde{v})| \leq |\pi_S(\widetilde{u})|$ we find that $\mathrm{head}_+(\pi_S(\widetilde{v}))^{-1}$ is a suffix of $\mathrm{tail}_+(\pi_S(\widetilde{u}))$. By Corollary 6, we know that $\mathrm{head}_+(v_1)$

is a prefix of $\mathrm{head}_+(\pi_S(\widetilde{v}))$. It follows that $\mathrm{head}_+(v_1)^{-1}$ is a suffix of $\mathrm{tail}_+(\pi_S(\widetilde{u}))$. Therefore, $\pi_S(\widetilde{u}) \cdot v_1$ is a right augmentation of $\pi_S(\widetilde{u})$ by v_1.

By Theorem 3 there is $s \in S$ such that $\widetilde{u} \in \mathrm{Aug}_S(s)^{\pm 1}$. Since $\widetilde{u}$ does not end with $\hat{v}_1^{-1}$, the second part of Theorem 3 implies that v_1 must be distinct from s and from s^{-1}. Therefore, $\widetilde{u}\hat{v}_1 \in \mathrm{Aug}_S(s)$. We can then proceed by induction for the pair $\widetilde{u}\hat{v}_1$ and $\hat{v}_1^{-1}\widetilde{v}$, where the first one is small from Theorem 2, and the second one is small as a subword of $\widetilde{v}$. This concludes the proof. □

Theorem 8. *Let $S \subseteq F_n$ be an independent irreducible subset. Let $\widetilde{u} \in F_S$, and denote $m = |\pi_S(\widetilde{u})|$. Then there are elements $\widetilde{s}_1, \widetilde{s}_2, \dots, \widetilde{s}_k \in \mathrm{Aug}_S^{\pm 1}$ with $k \leq m$, such that $\widetilde{u} = \widetilde{s}_1 \widetilde{s}_2 \dots \widetilde{s}_k$.*

Proof. The set $\mathrm{Aug}_S^{\pm 1}$ contains the generators of F_S, so there is some representation $\widetilde{u} = \widetilde{s}_1 \widetilde{s}_2 \dots \widetilde{s}_k$ with each $\widetilde{s}_i \in \mathrm{Aug}_S^{\pm 1}$. Let us choose a representation with the minimal possible k, and assume for the sake of contradiction that $k > m$.

Consider the sequence of reduced words $\pi_S(\widetilde{s}_1), \pi_S(\widetilde{s}_2), \dots, \pi_S(\widetilde{s}_k) \in F_n$. We can define a partial pairing of the letters appearing in them, such that each letter y is paired to a letter y^{-1} that cancels it when the product is being reduced, except for m letters that remain unpaired. Note that there might be more than one way to reduce the product, so there may be more than one way to define this pairing.

Since $k > m$, it follows that there must be some $\pi_S(\widetilde{s}_i)$ all of whose letters are paired. It then follows that there exists some $\pi_S(\widetilde{s}_i)$ such that all of its letters are paired to letters in its two immediate neighbors $\pi_S(\widetilde{s}_{i-1}), \pi_S(\widetilde{s}_{i+1})$ (or just one neighbor, for $i = 1$ or $i = k$). One of the neighbors has to cancel at least half of the letters of $\pi_S(\widetilde{s}_i)$. Without loss of generality this is $\pi_S(\widetilde{s}_{i-1})$, canceling them from the left. This implies $|\pi_S(\widetilde{s}_{i-1})\pi_S(\widetilde{s}_i)| \leq |\pi_S(\widetilde{s}_{i-1})|$. By Lemma 7 this implies $\widetilde{s}_{i-1}\widetilde{s}_i \in \mathrm{Aug}_S$, contradicting the minimality of k.

□

Theorem 9. *Let $S \subseteq F_n$ be an irreducible subset. Assume that $x \in \langle S \rangle$ for x a letter of F_n. Then $x \in S^{\pm 1}$.*

Proof. Let $\widetilde{u} \in F_S$ be an element in the preimage of x under π_S. By Theorem 8, it follows that $\widetilde{u} \in \mathrm{Aug}_S^{\pm 1}$. Let $s \in S$ be such that $\widetilde{u} \in \mathrm{Aug}_S(s)^{\pm 1}$. Since $|s| = |\widetilde{u}| = |x| = 1$, it follows that s is a letter of F_n. However, letters do not have any nontrivial augmentations. Thus $x = s^{\pm 1} \in S^{\pm 1}$. □