THE WARING PROBLEM

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1. Schnirelmann density

Definition 1.1. For $A \subseteq \mathbb{Z}^{>0}$ we define $A_k = |A \cap \{1, 2, \cdot, k\}|$ and $\rho(A) = \inf_{k \geq 1} \frac{A_k}{k}$. This is the Schnirelmann density of A.

Definition 1.2. For $A, B \subseteq \mathbb{Z}^{>0}$, we define $A + B = A \cup B \cup \{a + b \mid a \in A, b \in B\}$.

Theorem 1.3. *If $\rho(A) > 0$ then there is h such that $hA = \mathbb{Z}^{>0}$.*

Proof. This follows from the following two claims.

Claim 1.4. If $\rho(A) + \rho(B) \geq 1$ then $A + B = \mathbb{Z}^{>0}$.

Claim 1.5. For all A, B, $1 - \rho(A + B) \leq (1 - \rho(A))(1 - \rho(B))$.

The first claim follows from pigeonhole principle. For the second claim, let $n \geq 1$, and express $\{1, 2, \ldots, n\} \setminus A$ as a disjoint union $\bigcup_r I_r$ where each I_r has the form $\{a_r + 1, a_r + 2, \ldots, a_r + d_r\}$ such that $a_r \in A$ (To do this we need to assume $1 \in A$, but the claim is trivial in the case $1 \notin A$). Since $B_{d_r} \geq \rho(B) d_r$ we have $|I_r \cap \{a_r + b : b \in B\}| \geq \rho(B)|I_r|$. The claim follows. $\square$

2. Counting solutions

Lemma 2.1. *Let n be a nonzero integer. Then the amount of integer solutions to $x_1 y_1 + x_2 y_2 = n$ where $|x_i| \leq X$, $|y_i| \leq Y$ is $O(XY \sum_{d|n} \frac{1}{d})$.*

Proof. It suffices to show that the amount is $O(XY)$ under the assumption $\gcd(x_1, x_2) = 1$. Let us fix such x_1, x_2. Given one solution (y_1, y_2), all the others are obtained by $(y_1 + t x_2, y_2 - t x_1)$. Since $|t| \leq \frac{2Y}{\max(|x_1|, |x_2|)}$, the total amount of solutions is bounded by

$$\sum_{\substack{|x_1|, |x_2| \leq X \\ \gcd(x_1, x_2) = 1}} \frac{4Y}{\max(|x_1|, |x_2|)} \leq 8Y \sum_{\substack{|x_1| \leq |x_2| \leq X \\ x_2 \neq 0}} \frac{1}{|x_2|} = 8Y \sum_{0 < |x_2| \leq X} \frac{2|x_2| + 1}{|x_2|} \leq 48XY$$

$\square$

Lemma 2.2. *The amount of integer solutions to $x_1 y_1 + x_2 y_2 = x_3 y_3 + x_4 y_4$ with $|x_i| \leq X, |y_i| \leq Y$ and also $\sum x_i^2 \neq 0, \sum y_i^2 \neq 0$ is $O((XY)^3)$.*

Proof. First we bound the amount of solutions with both sides nonzero:

$$\sum_{0 < |n| \leq 2XY} \left(XY \sum_{d|n} \frac{1}{d} \right)^2 = 2(XY)^2 \sum_{1 \leq n \leq 2XY} \sum_{d_1, d_2 | n} \frac{1}{d_1 d_2} \leq 2(XY)^2 \sum_{d_1, d_2} \frac{2XY}{\sqrt{d_1 d_2}} \cdot \frac{1}{d_1 d_2}$$
$$= 4(XY)^3 \zeta \left(\frac{3}{2} \right)^2.$$

Then, if any of the products $x_i y_i$ is nonzero, we can rearrange the terms and get a form of the equation when both sides are nonzero. The amount of solutions when all $x_i y_i$ are zero is $O(X^3 Y + X Y^3)$. $\square$

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3. Fourier

Definition 3.1. $r_{k,g}(m)$ is the amount of solutions in $\mathbb{Z}^{\geq 0}$ to the equation $m = x_1^k + \cdots + x_g^k$. We also define $g(k) = \inf(\{g \mid \forall m, r_{k,g}(m) > 0\})$ and $G(k) = \inf(\{g \mid \exists N, \forall m > N, r_{k,g}(m) > 0\})$ (the infimum of the empty set is ∞).

Proposition 3.2. *If there is g such that $r_{k,g}(N) = O(N^{g/k-1})$ then $g(k) < \infty$.*

Proof. Let $A = \{m : r_{k,g}(m) > 0\}$. Then

$$(N/k)^{g/k} \leq \sum_{m \leq N} r_{k,g}(m) \ll \sum_{\substack{m \leq N \\ r_{k,g}(m) > 0}} N^{g/k-1} = N^{g/k-1} A_N$$

implying that $\rho(A) > 0$, so $hA = \mathbb{Z}^{>0}$ for some h. ◻

Proposition 3.3. *We have $r_{k,g}(N) \leq \int_0^1 \left| \sum_{x=0}^{[m^{\frac{1}{k}}]} e^{2\pi i x^k t} \right|^g dt$.*

Proof. Note that $r_{k,g}(N)$ is the Nth Fourier coefficient of $\left(\sum_{x=0}^{[m^{\frac{1}{k}}]} e^{2\pi i x^k t} \right)^g$. The proposition then follows from the general inequality $|\hat{f}(n)| \leq \int_0^1 |f(t)| dt$. ◻

Proposition 3.4. *Let $k \geq 2$ be an integer and let $g = 8^{k-1}$. Let P be a large integer and $f(x) = \sum_{j=0}^k c_j x^j \in \mathbb{Z}[x]$ a polynomial of degree k such that $c_j = O(P^{k-j})$. Also, let S be a set of integers that are $O(P)$. Then*

$$\int_0^1 \left| \sum_{x \in S} e^{2\pi i f(x) t} \right|^g dt = O(P^{g-k})$$

Proof. Induction on k. For $k = 2$, say $f(x) = ax^2 + bx + c$ with $a \neq 0$, the integrand is $\left| \sum_{x \in S} e^{2\pi i f(x) t} \right|^8$ whose 0th Fourier coefficient is exactly the amount of solutions to

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) = f(y_1) + f(y_2) + f(y_3) + f(y_4)$$

with $x_i, y_i \in S$. This equation is equivalent to $\sum_{i=1}^4 (x_i - y_i)(ax_i + ay_i + b)$. Since $x_i - y_i$ and $ax_i + ay_i + b$ determine x_i, y_i and are $O(P)$, the amount of solutions when not all $x_i = y_i$ and not all $ax_i + ay_i + b = 0$ is $O(P^6) = O(P^{g-k})$. Solutions where all $x_i = y_i$ or all $x_i + y_i + a$ only add $O(P^4)$.

We move to the induction step. We will use the inequalities $|x+y|^n \ll x^n + y^n$ and $\left| \sum_{j=0}^{P-1} x_j \right|^n \leq P^{n-1} \sum_{j=0}^{P-1} |x_j|^n$.

$$\int_0^1 \left|\sum_{x\in S} e^{2\pi i f(x)t}\right|^g dt = \int_0^1 \left(\left(\left|\sum_{x\in S} e^{2\pi i f(x)t}\right|^2\right)^{g/8}\right)^4 dt$$

$$= \int_0^1 \left(\left(P+1+\sum_{0<|h|\le P}\sum_{x,x+h\in S} e^{2\pi i h\Delta_h f(x)t}\right)^{g/8}\right)^4 dt$$

$$\ll \int_0^1 \left(P^{g/8} + P^{g/8-1}\sum_{0<|h|\le P}\left|\sum_{x,x+h\in S} e^{2\pi i h\Delta_h f(x)t}\right|^{g/8}\right)^4 dt$$

$$\ll P^{g/2} + P^{g/2-4}\int_0^1 \left(\sum_{0<|h|\le P}\left|\sum_{x,x+h\in S} e^{2\pi i h\Delta_h f(x)t}\right|^{g/8}\right)^4 dt$$

Consider the Fourier coefficients of $\left|\sum_{x,x+h\in S} e^{2\pi i h\Delta_h f(x)t}\right|^{g/8}$. We write it as $\sum_n A_h(n)e^{2\pi i nht}$, where the sum is over $|n| = O(P^{k-1})$. We use the induction hypothesis on $k-1, g/8, \Delta_h f(x), S\cap(S-h)$ to deduce that

$$\int_0^1 \left|\sum_{x,x+h\in S} e^{2\pi i h\Delta_h f(x)t}\right|^{g/8} dt = \int_0^1 \left|\sum_{x,x+h\in S} e^{2\pi i \Delta_h f(x)u}\right|^{g/8} du = O(P^{g/8-(k-1)})$$

Thus, the same bound holds for the Fourier coefficients $A_h(n)$. Finally, we have

$$\int_0^1 \left|\sum_{x\in S} e^{2\pi i f(x)t}\right|^g dt \ll P^{g/2} + P^{g/2-4}\int_0^1 \left(\sum_{0<|h|\le P}\sum_{n=O(P^{k-1})} A_h(n)e^{2\pi i nht}\right)^4 dt$$

$$= P^{g/2} + P^{g/2-4} \sum_{\substack{0\neq h_i=O(P)\\ n_i=O(P^{k-1})\\ h_1n_1+h_2n_2+h_3n_3+h_4n_4=0}} A_{h_1}(n_1)A_{h_2}(n_2)A_{h_3}(n_3)A_{h_4}(n_4)$$

We know that the amount of solutions to $h_1n_1+h_2n_2+h_3n_3+h_4n_4 = 0$ with $h_i = O(P)$, $n_i = O(P^{k-1})$ and $\sum h_i^2 \neq 0, \sum n_i^2 \neq 0$ is $O(P^{3k})$. The sum that appeared in the equation considers also solutions where $n_1 = n_2 = n_3 = n_4$, but there are only $P^4 = O(P^{3k})$ such solutions. It follows that

$$\int_0^1 \left|\sum_{x\in S} e^{2\pi i f(x)t}\right|^g dt \ll P^{g/2-4}\cdot P^{3k}\cdot (P^{g/8-(k-1)})^4 = P^{g-k}$$

□

Corollary 3.5. *For every $k \geq 1$ we have $g(k) < \infty$.*