GROUP EXTENSIONS

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1. First Cohomology

Let H, G be groups and let $\varphi\colon G \to \mathrm{Aut}(H)$ be a group homomorphism. Given $h \in H, g \in G$ we denote $h^g = \varphi(g)(h)$ and $h^{-g} = \varphi(g)(h^{-1})$. Let $Z^1(G, H)$ denote the set of maps $s\colon G \to H$ satisfying

$$s(gh) = s(g)s(h)^g.$$

Such a map is called a 1-cocycle. There is an equivalence relation on $Z^1(G, H)$ given by

$$s_1 \sim s_2 \iff \exists x \in H \text{ such that } s_1(g) = x^{-1}s_2(g)x^g \text{ for all } g \in G$$

and the set of equivalence classes is denoted $H^1(G, H)$. In the case where H is abelian, $Z^1(G, H)$ is an abelian group under pointwise multiplication, and there is a subgroup of $Z^1(G, H)$ consisting of the elements of the form

$$g \mapsto x^{-1}x^g$$

for some fixed $x \in H$. This subgroup is called the group of 1-coboundaries, denoted $B^1(G, H)$, and in this case we have that $H^1(G, H) = Z^1(G, H)/B^1(G, H)$. Thus, when H is abelian, $H^1(G, H)$ is an abelian group.

Theorem 1. *Let H, G be groups and let $\varphi\colon G \to \mathrm{Aut}(H)$ be a group homomorphism. Then there is a bijection between the set of 1-cocycles $s\colon G \to H$ and the set of sections of the semidirect product $H \rtimes G \to G$, given by*

$$s \mapsto (g \mapsto (s(g), g)).$$

Two 1-cocycles are equivalent if and only if the corresponding sections are conjugate by an element of $H \rtimes \{e\}$. If H is abelian, the bijection is an isomorphism of abelian groups, where the abelian group structure on the set of sections is defined by

$$(g \mapsto (s_1(g), g)) \cdot (g \mapsto (s_2(g), g)) = (g \mapsto (s_1(g)s_2(g), g)).$$

Proof. This is straightforward to verify. □

2. Second Cohomology

A group extension $0 \to A \to E \to G \to 0$ gives rise to a homomorphism $\varphi\colon G \to \mathrm{Out}(A)$ by choosing a section $s\colon G \to E$ of the projection map and defining

$$\varphi(g) = \text{the class of the automorphism } (a \mapsto s(g)as(g)^{-1}).$$

In the case A is abelian, this induces a homomorphism $\varphi\colon G \to \mathrm{Aut}(A)$, i.e. a G-module structure on A. We denote the action of $g \in G$ on $a \in A$ by a^g.

Let A be a G-module, and let us study the set of extensions E of G by A that induce the given G-module structure on A (up to isomorphism of extensions). We define the sum of two such extensions E, E' as

$$0 \to A \to (E \times_G E')/\Delta_A \to G \to 0$$

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where Δ_A is the anti-diagonal $\{(a, a^{-1}) \mid a \in A\} \leq A \times A$, which is a normal subgroup of $E \times_G E'$ since both extensions induce the same G-module structure on A. This is clearly associative and commutative. There is an identity element, given by the split extension

$$0 \to A \to A \rtimes G \to G \to 0,$$

and an inverse operation, given by

$$0 \to A \xrightarrow{i} E \to G \to 0 \implies 0 \to A \xrightarrow{-i} E \to G \to 0.$$

where the splitting $G \to E \times_G E$ is given by $g \mapsto (s(g), s(g))$ for any section $s\colon G \to E$. Thus, the set of extensions of G by A inducing the given G-module structure on A forms an abelian group.

Given a G-module A, we let $Z^2(G, A)$ denote the set of 2-cocycles of G with values in A, which are maps $f\colon G \times G \to A$ satisfying

$$f(g_1g_2, g_3) + f(g_1, g_2) = f(g_1, g_2g_3) + f(g_2, g_3)^{g_1}.$$

These form an abelian group under pointwise addition. There is a subgroup $B^2(G, A) \leq Z^2(G, A)$ consisting of the 2-coboundaries, which are the maps of the form

$$f(g_1, g_2) = \phi(g_1g_2) - \phi(g_1) - \phi(g_2)^{g_1}$$

for some fixed map $\phi\colon G \to A$. We define the second cohomology group $H^2(G, A) = Z^2(G, A)/B^2(G, A)$.

Theorem 2. *The abelian group of extensions of G by A that induce the given G-module structure on A is isomorphic to $H^2(G, A)$, with the isomorphism that sends an extension E to the 2-cocycle*

$$f(g_1, g_2) = \hat{g}_1\hat{g}_2\widehat{g_1g_2}^{-1}$$

where $(g \mapsto \hat{g})\colon G \to E$ is any set-theoretic section of the projection map.

Proof. f is a 2-cocycle because

$$\begin{aligned}
&f(g_1g_2, g_3) + f(g_1, g_2) - f(g_1, g_2g_3) - f(g_2, g_3)^{g_1} \\
&= \widehat{g_1g_2}\hat{g}_3\widehat{g_1g_2g_3}^{-1} + \hat{g}_1\hat{g}_2\widehat{g_1g_2}^{-1} - \hat{g}_1\widehat{g_2g_3}\widehat{g_1g_2g_3}^{-1} - \left(\hat{g}_2\hat{g}_3\widehat{g_2g_3}^{-1}\right)^{g_1} \\
&= \left(\hat{g}_1\hat{g}_2\widehat{g_1g_2}^{-1}\right) \cdot \left(\widehat{g_1g_2}\hat{g}_3\widehat{g_1g_2g_3}^{-1}\right) - \left(\hat{g}_2\hat{g}_3\widehat{g_2g_3}^{-1}\right)^{g_1} \cdot \left(\hat{g}_1\widehat{g_2g_3}\widehat{g_1g_2g_3}^{-1}\right) \\
&= \hat{g}_1\hat{g}_2\hat{g}_3\widehat{g_1g_2g_3}^{-1} - \hat{g}_1\hat{g}_2\hat{g}_3\widehat{g_1g_2g_3}^{-1} \\
&= 0.
\end{aligned}$$

Changing the section to $g \mapsto \phi(g)\hat{g}$, with $\phi\colon G \to A$, alters f by a 2-coboundary:

$$f'(g, h) - f(g, h) = \phi(g)\hat{g}\phi(h)\hat{h}\left(\phi(gh)\widehat{gh}\right)^{-1} - \hat{g}\hat{h}\widehat{gh}^{-1} = \phi(g) - \phi(gh) + \phi(h)^g.$$

Thus, the map from extensions to $H^2(G, A)$ is well-defined. To verify it is a homomorphism, let E, E' be two extensions, denote sections for both of them by $g \mapsto \hat{g}$, and let f, f' be the corresponding 2-cocycles. Then a section of the sum extension $E \times_G E'/\Delta_A$ is given by

$$g \mapsto (\hat{g}, \hat{g})\Delta_A$$

and the corresponding 2-cocycle is

$$\begin{aligned}
f''(g, h) &= (\hat{g}, \hat{g})\Delta_A \cdot (\hat{h}, \hat{h})\Delta_A \cdot (\widehat{gh}, \widehat{gh})^{-1}\Delta_A \\
&= (\hat{g}\hat{h}\widehat{gh}^{-1}, \hat{g}\hat{h}\widehat{gh}^{-1})\Delta_A \\
&= f(g, h) + f'(g, h).
\end{aligned}$$

Thus, the map is a group homomorphism. If the extension E corresponds to the trivial element of $H^2(G, A)$, it follows that there is a section $g \mapsto \hat{g}$ which is a group homomorphism, so the extension is split. Thus, the map is injective. Finally, given any 2-cocycle $f \in Z^2(G, A)$, we select formal symbols $\hat{g}$ for each $g \in G$, and define the group E by the presentation

$$E_f = \langle A, \hat{g}\ (g \in G) \mid a_1 a_2 = a_1 + a_2,\ \hat{g} a \hat{g}^{-1} = a^g,\ \hat{g}_1 \hat{g}_2 = f(g_1, g_2)\widehat{g_1 g_2}\rangle.$$

It is clear that there is an exact sequence $0 \to A \to E_f \to G \to 0$; exactness at A is a little less obvious, but follows from the 2-cocycle condition on f. It is then clear that E_f induces the given G-module structure on A, and that the 2-cocycle corresponding to the set-theoretic section $g \mapsto \hat{g}$ is f. □

3. Third Cohomology

Given a group N, we have a natural action of $\mathrm{Out}(N)$ on $Z(N)$. We define a map $\Phi : \mathrm{Out}(N)^3 \to Z(N)$ as follows: Choose a set-theoretic section $\mathrm{Out}(N) \to \mathrm{Aut}(N)$, denoted $\alpha \mapsto \hat{\alpha}$, and a set-theoretic section $s : \mathrm{Inn}(N) \to N$. Denote $n_{\alpha,\beta} = s(\hat{\alpha}\hat{\beta}\widehat{\alpha\beta}^{-1})$. Then for $\alpha, \beta, \gamma \in \mathrm{Out}(N)$, we let

$$\Phi(\alpha, \beta, \gamma) = (n_{\alpha,\beta} \cdot n_{\alpha\beta,\gamma}) \cdot \left(n_{\beta,\gamma}^{\hat{\alpha}} \cdot n_{\alpha,\beta\gamma}\right)^{-1}.$$

It is easy to verify that $\Phi(\alpha, \beta, \gamma)$ vanishes in $\mathrm{Inn}(N)$, so it lies in $Z(N)$.

Theorem 3. *Φ satisfies a 3-cocycle condition*

$$\Phi(\alpha, \beta, \gamma) - \Phi(\alpha, \beta, \gamma\delta) + \Phi(\alpha, \beta\gamma, \delta) - \Phi(\alpha\beta, \gamma, \delta) + \Phi(\beta, \gamma, \delta)^{\alpha} = 0.$$

Further, Φ is dependent on the choice of sections only up to a 3-coboundary. Thus, Φ defines a canonical cohomology class in $H^3(\mathrm{Out}(N), Z(N))$.

Proof. We begin by checking the 3-cocycle condition:

$$\begin{aligned}
&- \Phi(\alpha, \beta, \gamma\delta) + \Phi(\alpha, \beta, \gamma) + \Phi(\beta, \gamma, \delta)^{\alpha} \\
&= \left(n_{\alpha,\beta} \cdot n_{\alpha\beta,\gamma\delta} \cdot n_{\alpha,\beta\gamma\delta}^{-1} \cdot n_{\beta,\gamma\delta}^{-\hat{\alpha}}\right)^{-1} \cdot \left(n_{\alpha,\beta} \cdot n_{\alpha\beta,\gamma} \cdot n_{\alpha,\beta\gamma}^{-1} \cdot n_{\beta,\gamma}^{-\hat{\alpha}}\right) \cdot \left(n_{\beta,\gamma} \cdot n_{\beta\gamma,\delta} \cdot n_{\beta,\gamma\delta}^{-1} \cdot n_{\gamma,\delta}^{-\hat{\beta}}\right)^{\alpha} \\
&= n_{\beta,\gamma\delta}^{\hat{\alpha}} \cdot n_{\alpha,\beta\gamma\delta} \cdot n_{\alpha\beta,\gamma\delta}^{-1} \cdot n_{\alpha\beta,\gamma} \cdot n_{\alpha,\beta\gamma}^{-1} \cdot n_{\beta\gamma,\delta}^{\hat{\alpha}} \cdot n_{\beta,\gamma\delta}^{-\hat{\alpha}} \cdot n_{\gamma,\delta}^{-\hat{\alpha}\hat{\beta}} \\
&= \left(n_{\alpha,\beta\gamma\delta} \cdot n_{\alpha\beta,\gamma\delta}^{-1} \cdot n_{\alpha\beta,\gamma} \cdot (n_{\alpha,\beta\gamma}^{-1} \cdot n_{\beta\gamma,\delta}^{\hat{\alpha}})\right)^{n_{\beta,\gamma\delta}^{\hat{\alpha}}} \cdot \left(n_{\gamma,\delta}^{-\widehat{\alpha\beta}}\right)^{n_{\alpha,\beta}} \\
&= \left(n_{\alpha,\beta\gamma\delta} \cdot n_{\alpha\beta,\gamma\delta}^{-1} \cdot n_{\alpha\beta,\gamma} \cdot (n_{\alpha\beta\gamma,\delta} \cdot n_{\alpha,\beta\gamma\delta}^{-1})\right)^{n_{\beta,\gamma\delta}^{\hat{\alpha}}} \cdot \left(n_{\gamma,\delta}^{-\widehat{\alpha\beta}}\right)^{n_{\alpha,\beta}} - \Phi(\alpha, \beta\gamma, \delta) \\
&= \left(n_{\alpha\beta,\gamma\delta}^{-1} \cdot (n_{\alpha\beta,\gamma} \cdot n_{\alpha\beta\gamma,\delta})\right)^{n_{\beta,\gamma\delta}^{\hat{\alpha}} \cdot n_{\alpha,\beta\gamma\delta}} \cdot \left(n_{\gamma,\delta}^{-\widehat{\alpha\beta}}\right)^{n_{\alpha,\beta}} - \Phi(\alpha, \beta\gamma, \delta) \\
&= \left(n_{\alpha\beta,\gamma\delta}^{-1} \cdot (n_{\gamma,\delta}^{\widehat{\alpha\beta}} \cdot n_{\alpha\beta,\gamma\delta})\right)^{n_{\beta,\gamma\delta}^{\hat{\alpha}} \cdot n_{\alpha,\beta\gamma\delta}} \cdot \left(n_{\gamma,\delta}^{-\widehat{\alpha\beta}}\right)^{n_{\alpha,\beta}} - \Phi(\alpha, \beta\gamma, \delta) + \Phi(\alpha\beta, \gamma\delta) \\
&= (n_{\gamma,\delta}^{\widehat{\alpha\beta}})^{n_{\beta,\gamma\delta}^{\hat{\alpha}} \cdot n_{\alpha,\beta\gamma\delta} \cdot n_{\alpha\beta,\gamma\delta}^{-1}} \cdot (n_{\gamma,\delta}^{-\widehat{\alpha\beta}})^{n_{\alpha,\beta}} - \Phi(\alpha, \beta\gamma, \delta) + \Phi(\alpha\beta, \gamma\delta) \\
&= (n_{\gamma,\delta}^{\widehat{\alpha\beta}})^{\Phi(\alpha,\beta,\gamma\delta)^{-1} \cdot n_{\alpha,\beta}} \cdot (n_{\gamma,\delta}^{-\widehat{\alpha\beta}})^{n_{\alpha,\beta}} - \Phi(\alpha, \beta\gamma, \delta) + \Phi(\alpha\beta, \gamma\delta) = -\Phi(\alpha, \beta\gamma, \delta) + \Phi(\alpha\beta, \gamma\delta).
\end{aligned}$$

It is clear that changing the section s into $\phi(h)s(h)$, for some map $\phi\colon \mathrm{Inn}(N) \to Z(N)$, alters Φ by the 3-coboundary

$$(\alpha, \beta, \gamma) \mapsto \phi(n_{\alpha,\beta}) - \phi(n_{\alpha,\beta\gamma}) + \phi(n_{\alpha\beta,\gamma}) - \phi(n_{\beta,\gamma})^{\alpha}.$$

It remains to show that altering the section $\mathrm{Out}(N) \to \mathrm{Aut}(N)$ into $\alpha \mapsto \psi(\alpha)\hat{\alpha}$, for some map $\psi\colon \mathrm{Out}(N) \to \mathrm{Inn}(N)$, changes Φ by a 3-coboundary as well. We may assume that ψ is actually a map $\psi\colon \mathrm{Out}(N) \to N$. Consider

$$\begin{aligned}
n'_{\alpha,\beta} &= s\left((\psi(\alpha)\hat{\alpha})\left(\psi(\beta)\hat{\beta}\right)\left(\psi(\alpha\beta)\widehat{\alpha\beta}\right)^{-1}\right) \\
&= s\left(\left(\psi(\alpha) \cdot \psi(\beta)^{\hat{\alpha}} \cdot \psi(\alpha\beta)^{-n_{\alpha,\beta}}\right) \cdot (\hat{\alpha}\hat{\beta}\widehat{\alpha\beta}^{-1})\right) = z_{\alpha,\beta} \cdot m_{\alpha,\beta} \cdot n_{\alpha,\beta}
\end{aligned}$$

where $m_{\alpha,\beta} = \psi(\alpha) \cdot \psi(\beta)^{\hat{\alpha}} \cdot \psi(\alpha\beta)^{-n_{\alpha,\beta}} \in N$ and $z_{\alpha,\beta} \in Z(N)$. It is clear that $z_{\alpha,\beta}$ only contributes a 3-coboundary to Φ, so we may ignore it. Thus, we have

$$\begin{aligned}
\Phi'(\alpha,\beta,\gamma) &= (m_{\alpha,\beta} \cdot n_{\alpha,\beta} \cdot m_{\alpha\beta,\gamma} \cdot n_{\alpha\beta,\gamma}) \cdot \left(m_{\beta,\gamma}^{\psi(\alpha)\hat{\alpha}} \cdot n_{\beta,\gamma}^{\psi(\alpha)\hat{\alpha}} \cdot m_{\alpha,\beta\gamma} \cdot n_{\alpha,\beta\gamma} \right)^{-1} \\
&= m_{\alpha,\beta} \cdot m_{\alpha\beta,\gamma}^{n_{\alpha,\beta}} \cdot \Phi(\alpha,\beta,\gamma) \cdot \left(\psi(\alpha) \cdot m_{\beta,\gamma}^{\hat{\alpha}} \cdot n_{\beta,\gamma}^{\hat{\alpha}} \cdot \psi(\alpha)^{-1} \cdot m_{\alpha,\beta\gamma} \cdot n_{\beta,\gamma}^{-\hat{\alpha}} \right)^{-1},
\end{aligned}$$

and we compute

$$\begin{aligned}
&\Phi'(\alpha,\beta,\gamma) - \Phi(\alpha,\beta,\gamma) \\
&= m_{\alpha,\beta} \cdot m_{\alpha\beta,\gamma}^{n_{\alpha,\beta}} \cdot n_{\beta,\gamma}^{\hat{\alpha}} \cdot m_{\alpha,\beta\gamma}^{-1} \cdot \psi(\alpha) \cdot n_{\beta,\gamma}^{-\hat{\alpha}} \cdot m_{\beta,\gamma}^{-\hat{\alpha}} \cdot \psi(\alpha)^{-1} \\
&= \left(\psi(\alpha) \cdot \psi(\beta)^{\hat{\alpha}} \cdot \psi(\alpha\beta)^{-n_{\alpha,\beta}} \right) \cdot \left(\psi(\alpha\beta) \cdot \psi(\gamma)^{\widehat{\alpha\beta}} \cdot \psi(\alpha\beta\gamma)^{-n_{\alpha\beta,\gamma}} \right)^{n_{\alpha,\beta}} \cdot \\
&\cdot n_{\beta,\gamma}^{\hat{\alpha}} \cdot \left(\psi(\alpha) \cdot \psi(\beta\gamma)^{\widehat{\alpha}} \cdot \psi(\alpha\beta\gamma)^{-n_{\alpha,\beta\gamma}} \right)^{-1} \cdot \psi(\alpha) \cdot n_{\beta,\gamma}^{-\hat{\alpha}} \cdot \left(\psi(\beta) \cdot \psi(\gamma)^{\hat{\beta}} \cdot \psi(\beta\gamma)^{-n_{\beta,\gamma}} \right)^{-\hat{\alpha}} \cdot \psi(\alpha)^{-1} \\
&= \left(\psi(\alpha) \cdot \psi(\beta)^{\hat{\alpha}} \cdot \psi(\gamma)^{n_{\alpha,\beta} \cdot \widehat{\alpha\beta}} \right) \cdot \left(\psi(\alpha\beta\gamma)^{-n_{\alpha,\beta} \cdot n_{\alpha\beta,\gamma}} \cdot \psi(\alpha\beta\gamma)^{n_{\beta,\gamma}^{\hat{\alpha}} \cdot n_{\alpha.\beta\gamma}} \right) \cdot n_{\beta,\gamma}^{\hat{\alpha}} \cdot \psi(\beta\gamma)^{-\hat{\alpha}} \cdot \\
&\cdot n_{\beta,\gamma}^{-\hat{\alpha}} \cdot \psi(\beta\gamma)^{\hat{\alpha} \cdot n_{\beta,\gamma}} \cdot \left(\psi(\gamma)^{-\hat{\alpha}\hat{\beta}} \cdot \psi(\beta)^{-\hat{\alpha}} \cdot \psi(\alpha)^{-1} \right) \\
&= n_{\beta,\gamma}^{\hat{\alpha}} \cdot \psi(\beta\gamma)^{-\hat{\alpha}} \cdot n_{\beta,\gamma}^{-\hat{\alpha}} \cdot \psi(\beta\gamma)^{\hat{\alpha} \cdot n_{\beta,\gamma}} = 0.
\end{aligned}$$

□

Theorem 4. *Let $f : G \to \mathrm{Out}(N)$ be a group homomorphism. Then there exists an extension $0 \to N \to E \to G \to 0$ inducing the homomorphism f if and only if $f^*(\Phi) \in H^3(G, Z(N))$ vanishes.*

Proof. Suppose that an extension

$$0 \to N \to E \to G \to 0$$

inducing f exists. There is a map $f' \colon E \to \mathrm{Aut}(N)$ given by conjugation, which has to be compatible with f, i.e. the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "f'"] & G \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathrm{Inn}(N) \arrow[r] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 0.
\end{tikzcd}$$

Let $g \mapsto \hat{g}$ be a set-theoretic section of $E/\ker(f') \to G/\ker(f)$. Lift it to set-theoretic sections of $E \to G$ and of $\mathrm{Aut}(N) \to \mathrm{Out}(N)$. Then we have $f'(\hat{g}) = \widehat{f(g)}$ for all $g \in G$. Let s be a section of $N \to \mathrm{Inn}(N)$, and let $z : N \to Z(N)$ be the map $z(n) = s(\hat{f}(n))n^{-1}$. Note that

$$n_{f(g),f(h)} = s(f'(\hat{g}) \cdot f'(\hat{h}) \cdot f'(\widehat{gh})^{-1}) = s(f'(\hat{g} \cdot \hat{h} \cdot \widehat{gh}^{-1})) = \left(\hat{g}\hat{h}\widehat{gh}^{-1} \right) \cdot z(\hat{g}\hat{h}\widehat{gh}^{-1})$$

It follows that

$$\begin{aligned}
f^*\Phi(g,h,k) &= n_{f(g),f(h)} \cdot n_{f(gh),f(k)} \cdot \left(n_{f(h),f(k)}^{\widehat{f(g)}} \cdot n_{f(g),f(hk)} \right)^{-1} \\
&= z(\hat{g}\hat{h}\widehat{gh}^{-1}) + z(\widehat{gh}\hat{k}\widehat{ghk}^{-1}) - z(\hat{h}\hat{k}\widehat{hk}^{-1})^g - z(\hat{g}\widehat{hk}\widehat{ghk}^{-1})
\end{aligned}$$

which is a 3-coboundary. Conversely, suppose that $f^*\Phi$ vanishes in $H^3(G, Z(N))$, i.e. there is a map $\phi \colon G \times G \to Z(N)$ such that

$$f^*\Phi(g,h,k) = \phi(g,h) + \phi(gh,k) - \phi(h,k)^g - \phi(g,hk).$$

Let $\alpha \mapsto \hat{\alpha}$ be a set-theoretic section of $\mathrm{Aut}(N) \to \mathrm{Out}(N)$. Let E be the group defined by the presentation

$$E = \langle N, \hat{g}\ (g \in G) \mid\ \hat{g} n \hat{g}^{-1} = n^{\widehat{f(g)}},\ \phi(g,h)\hat{g}\hat{h}\widehat{gh}^{-1} = s\left(\widehat{f(g)}\,\widehat{f(h)}\,\widehat{f(gh)}^{-1}\right)\rangle.$$

It follows that there is an exact sequence $0 \to N \to E \to G \to 0$. Exactness at N is nontrivial, but follows from $f^*\Phi = \delta\phi$. It is clear that the extension induces the homomorphism f. $\square$