STRUCTURE THEORY FOR LOCAL FIELDS

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1. Notation

Let F be a field. We write $\mathrm{ch}(F)$ for its characteristic, $\overline{F}$ for an algebraic closure, $G_{E/F}$ for the Galois group of E/F, and $\chi_{\alpha,F}$ for the minimal polynomial of α over F.

If F is a non-Archimedean local field, we write $v_F : F \to \mathbb{Z} \cup \{\infty\}$ for its normalized valuation, $\mathcal{O}_F$ for its ring of integers $\{\alpha \in F : v_F(\alpha) \geq 0\}$, $\mathfrak{m}_F$ for the maximal ideal $\{\alpha \in \mathcal{O}_F : v_F(\alpha) > 0\}$, and k_F for the residue field $\mathcal{O}_F/\mathfrak{m}_F\mathcal{O}_F$. If E/F is an extension of local fields, we write $f_{E/F}$ for the inertia degree $[k_E : k_F]$ and $e_{E/F}$ for the ramification degree $[v_E(E^\times) : v_E(F^\times)]$. Recall that $[E : F] = f_{E/F}e_{E/F}$ and that $f_{L/F} = f_{L/E}f_{E/F}$, $e_{L/F} = e_{L/E}e_{E/F}$.

2. Unramified Extensions

Definition 2.1. Let F be a non-Archimedean local field. A finite extension E/F is called unramified if $e_{E/F} = 1$, or equivalently $[E : F] = f_{E/F}$. A general algebraic extension Ω/F is called unramified if every finite subextension E/F is unramified. An extension which is not unramified is called ramified.

The extension $\mathbb{C}/\mathbb{R}$ is considered ramified.

Proposition 2.2. *Let E/F be a finite extension of non-Archimedean local fields. Then it is unramified if and only if there is $\alpha \in \mathcal{O}_E$ such that $E = F(\alpha)$ and the minimal polynomial $\chi_{\alpha,F} \in \mathcal{O}_F[X]$ has irreducible reduction $\overline{\chi_{\alpha,F}} \in k_F[X]$. When this holds, we also have $k_{F(\alpha)} = k_F(\overline{\alpha})$.*

Note that the coefficients of $\chi_{\alpha,F}$ are integral because $\alpha \in \mathcal{O}_E$.

Proof. For any $\alpha \in \mathcal{O}_E$ with reduction $\overline{\alpha} \in k_E$ we have $\overline{\chi_{\alpha,F}}(\overline{\alpha}) = 0$ and thus $\chi_{\overline{\alpha},k_F} \mid \overline{f_{\alpha,F}}$. It follows that $\overline{\chi_{\alpha,F}}$ is irreducible if and only if $[k_F(\overline{\alpha}) : k_F] = [F(\alpha) : F]$. Clearly $k_F(\overline{\alpha}) \subseteq k_{F(\alpha)}$, so the last condition can only hold if the inertia degree of $F(\alpha)/F$ is equal to the degree $[F(\alpha) : F]$. To conclude, $\overline{\chi_{\alpha,F}}$ is irreducible if and only if $F(\alpha)/F$ is unramified, and when this holds we also have $k_{F(\alpha)} = k_F(\overline{\alpha})$.

This immediately proves the "if" direction. For the "only if" direction, suppose E/F is unramified. Let $\overline{\alpha}$ be a generator of the finite field k_E over k_F, represented by $\alpha \in \mathcal{O}_E$. Since $F(\alpha)/F$ is unramified (as a subextension of E/F), it follows that $\overline{\chi_{\alpha,F}}$ is irreducible. Further, we have

$$f_{F(\alpha)/F} = [k_{F(\alpha)} : k_F] = [k_F(\overline{\alpha}) : k_F] = f_{E/F}$$

and it follows that $E = F(\alpha)$. □

Remark 2.3. By Hensel's Lemma, a factorization of $\overline{f} \in k_F[X]$ into two coprime factors lifts into such a factorization for $f \in \mathcal{O}_F[X]$. Thus, $\overline{\chi_{\alpha,F}}$ is irreducible if and only if it is squarefree.

Proposition 2.4. *Let F be a local field.*

1. *(transitivity) If $L/E/F$ is a tower of finite extensions, then L/F is unramified if and only if L/E and E/F are unramified.*
2. *(base change) Let E/F and L/F be finite extensions over F. If E/F is unramified, then EL/L is also unramified.*

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Proof. In the Archimedean case there is nothing to prove since unramified extensions are trivial. In the non-Archimedean case, transitivity follows from $e_{L/F} = e_{L/E}e_{E/F}$. For base change, let $\alpha \in \mathcal{O}_E$ be as in Proposition 2.2. Then $EL = L(\alpha)$ and it is required to show that $\overline{\chi_{\alpha,L}}$ is irreducible. Since $\overline{\chi_{\alpha,F}}$ is irreducible as a polynomial over k_F, it is also squarefree as a polynomial over k_F, and thus squarefree as a polynomial over the extension k_L. Since $\overline{\chi_{\alpha,L}} \mid \overline{\chi_{\alpha,F}}$, it follows that $\overline{\chi_{\alpha,L}}$ is squarefree. By the remark following Proposition 2.2 we are done. ☐

Corollary 2.5. *The compositum of unramified extensions of F is unramified over F. In particular, if an extension is unramified, so is its Galois closure.*

It follows that there is a maximal unramified extension F^{ur}/F, which is unique up to isomorphism.

Proposition 2.6. *Let F be a non-Archimedean local field and let n be a positive integer coprime to $\mathrm{ch}(k_F)$. Then for any $u \in \mathcal{O}_F^\times$ and $u' \in \overline{F}$ an nth root of u, the extension $F(u')/F$ is unramified.*

Proof. Consider $\overline{\chi_{u',F}} \in k_F[X]$. This polynomial divides $X^n - \overline{u}$, which is a squarefree as its gcd with the derivative nX^{n-1} is equal to 1. Note that $n \neq 0$ in k_F. ☐

Theorem 2.7. *Let F be a non-Archimedean local field and let $q = |k_F|$. Then for every positive integer n, there exists a unique unramified extension of degree n over F, which is $F(\zeta)/F$ where ζ is a primitive $(q^n - 1)$-th primitive root of unity. In particular, F^{ur} is generated over F by all such roots of unity.*

Proof. Suppose ζ is a primitive $(q^n - 1)$-th root of unity. By Proposition 2.6 for $u = 1$, it follows that $F(\zeta)/F$ is unramified. We also have $k_F(\overline{\zeta}) = k_{F(\zeta)}$, so $[F(\zeta) : F] = [k_{F(\zeta)} : k_F] = [k_F(\overline{\zeta}) : k_F] = n$.

It remains to show that any unramified extension E/F of degree n has this form. The extension k_E/k_F also has degree n, so it is generated by an element $\lambda \in k_E$ of multiplicative order $q^n - 1$. Consider the polynomial $X^{q^n-1} - 1 \in F[X]$. Since its reduction is squarefree and has λ as a root, Hensel's lemma implies there is $\zeta \in \mathcal{O}_E$ such that $\zeta^{q^n-1} = 1$ and $\overline{\zeta} = \lambda$. Since $k_{F(\zeta)} = k_F(\overline{\zeta}) = k_E$, we have $E = F(\zeta)$ as E/F is unramified. ☐

Proposition 2.8. *Let F be a non-Archimedan local field. Let $p = \mathrm{ch}(k_F)$ and $q = |k_F|$. Any finite unramified extension E/F is Galois, and has a unique automorphism $\mathrm{Frob}_{E/F} \in G_{E/F}$ satisfying $\mathrm{Frob}_{E/F}(\omega) = \omega^q$ for every $\omega \in E$ a root of unity whose order is coprime to p. Further, $\mathrm{Frob}_{E/F}$ generates $G_{E/F}$, and for a tower $L/E/F$ of finite unramified extensions we have $\mathrm{Frob}_{L/F}|_E = \mathrm{Frob}_{E/F}$.*

Proof. Let E/F be an unramified extension of degree n. by Theorem 2.7 there is a primitive $(q^n - 1)$-th root of unity $\zeta \in E$ such that $E = F(\zeta)$. This already shows E/F is Galois.

Suppose that $\omega, \omega' \in E$ are two distinct roots of unity whose orders are coprime to p. Let m be a positive integer coprime to p such that $\omega^m = \omega'^m = 1$. Since $(X - \omega)(X - \omega') \mid X^m - 1$ over F, we have $(X - \overline{\omega})(X - \overline{\omega'}) \mid X^m - 1$ over k_F. As $X^m - 1$ is squarefree over k_F, we have $\overline{\omega} \neq \overline{\omega'}$. The collection of roots of unity in F whose order is coprime to p is a group, and we have just shown this group has an embedding into $k_E^\times$. Since $k_E^\times$ is cyclic of order $q^n - 1$, every root of unity in F whose order is coprime to p must be a power of ζ. Thus, we just need to show there is an automorphism sending ζ to ζ^q. It will be unique since $E = F(\zeta)$.

Since $k_F(\overline{\zeta}) = k_E$, and since $x \mapsto x^q$ is an automorphism of k_E/k_F, it follows that $\overline{\zeta}^q$ is a root of $\overline{\chi_{\zeta,F}}$. By Hensel's Lemma there is $\zeta' \in E$ such that $\chi_{\zeta,F}(\zeta') = 0$ and $\overline{\zeta'} = \overline{\zeta}^q$. The first part implies $(\zeta')^{q^n-1} = 1$, so there exists $q' \in \mathbb{Z}$ such that $\zeta' = \zeta^{q'}$. Then, the second part implies $\overline{\zeta}^q = \overline{\zeta}^{q'}$. By the discussion in the previous paragraph we must have $\zeta^q = \zeta^{q'}$ in E. Overall, this proved $\chi_{\zeta,F}(\zeta^q) = 0$, so there exists a unique automorphism $\mathrm{Frob}_{E/F} \in G_{E/F}$ sending ζ to ζ^q. By the primitivity of ζ, it follows that $\mathrm{Frob}_{E/F}$ has order n, and thus generates $G_{E/F}$. The property $\mathrm{Frob}_{L/F}|_E = \mathrm{Frob}_{E/F}$ is trivial. ☐

Since the Frobenius automorphisms are compatible, they combine to an automorphism $\mathrm{Frob}_F \in G_{F^{\mathrm{ur}}/F}$.

Corollary 2.9. *Let F be a non-Archimedean local field. Then $G_{F^{\mathrm{ur}}/F} \cong \hat{\mathbb{Z}}$, topologically generated by the Frobenius element.*

3. Tamely Ramified Extensions

Definition 3.1. Let F be a non-Archimedean local field. A finite extension E/F is called tamely ramified if $\mathrm{ch}(k_F) \nmid e_{E/F}$. A general algebraic extension Ω/F is called tamely ramified if every finite subextension E/F is tamely ramified. An extension which is not tamely ramified is called wildly ramified.

The extension $\mathbb{C}/\mathbb{R}$ is considered tamely ramified.

Lemma 3.2. *Let E/F be a tamely ramified extension of non-Archimedean local fields. Let M/F be its maximal unramified subextension. Denote $e = e_{E/F} = e_{E/M}$. Then there are uniformizers π_E of E and π_M of M such that $\pi_E^e = \pi_M$.*

Proof. Let $\pi_E \in E$ be a uniformizer. For every uniformizer π_M of M we have $\pi_E^e\pi_M^{-1} \in \mathcal{O}_E^\times$. Since $f_{E/M} = 1$, there is $\lambda \in \mathcal{O}_M^\times$ such that $\pi_E^e\pi_M^{-1} \equiv \lambda \mod \mathfrak{m}_E$. By changing our initial choice of π_M to $\lambda\pi_M$, we may assume without loss of generality that $\pi_E\pi_M^{-1} \equiv 1 \mod \mathcal{O}_E^\times$. Denote $u = \pi_E^e\pi_M^{-1}$. The polynomial $X^e - 1 \in k_E[X]$ is squarefree since $\gcd(X^e-1, eX^{e-1}) = 1$, using the fact that $e \neq 0$ in k_E, which follows from tame ramification. Thus, Hensel's Lemma implies that the root 1 of $X^e - 1 \in k_E[X]$ lifts to a root $v \in E$ for $X^e - u$. Therefore, we have $\pi_M = (v^{-1}\pi_E)^e$, proving the lemma. ☐

Lemma 3.3. *Let F be a local field.*

1. *(transitivity) If $L/E/F$ is a tower of finite extensions over F, then L/F is tamely ramified if and only if L/E and E/F are tamely ramified.*
2. *(base change) Let E/F and L/F be finite extensions over F. If E/F is tamely ramified, then EL/L is also tamely ramified.*

Proof. In the Archimedean case there is nothing to prove since all extensions are tamely ramified. In the non-Archimedean case, transitivity follows from $e_{L/F} = e_{L/E}e_{E/F}$. For base change, let M/F be the maximal unramified extension contained in E/F and let $e = e_{E/F} = e_{E/M}$. By Lemma 3.2 there are uniformizers π_E of E and π_M of M such that $\pi_E^e = \pi_M$. The extension ML/L is unramified by Proposition 2.4, so using transitivity of tame ramification it remains to show $EL = ML(\pi_E)$ is tamely ramified over ML. This will follow from Proposition 3.5. ☐

Corollary 3.4. *The compositum of tamely ramified extensions over F is tamely ramified over F. In particular, if an extension is tamely ramified, so is its Galois closure.*

It follows that there is a maximal tamely ramified extension F^{tame}/F, which is unique up to isomorphism.

Proposition 3.5. *Let F be a non-Archimedean local field and let n be a positive integer coprime to $\mathrm{ch}(k_F)$. Then for any $a \in F$ and $\alpha \in \overline{F}$ an nth root of a, the extension $F(\alpha)/F$ is tamely ramified.*

Proof. We prove this by induction on n. For $n = 1$ the claim is trivial. We thus assume the claim holds for all local fields F and all $n' < n$. Write $a = u\pi_F^k$ for $u \in \mathcal{O}_F^\times$ and π_F a uniformizer of F. Let $u' \in \overline{F}$ be an nth root of u. By Proposition 2.6, the extension $F(u')/F$ is unramified. By the transitivity of tame ramification, it suffices to show $F(\alpha, u')/F(u')$ is tamely ramified. Since π_F is a uniformizer of $F(u')$, and since $F(\alpha, u')$ is generated over $F(u')$ by the element $\alpha(u')^{-1}$ which is an nth root of π_F^k, it suffices to prove the original proposition in the case $a = \pi_F^k$.

Let $d = \gcd(n, k)$, and let us separate to several cases.

1. If $d = 1$ there are $s, t \in \mathbb{Z}$ such that $sn + tk = 1$. Therefore $\pi = \pi_F^s\alpha^t \in F(\alpha)$ satisfies $\pi^n = \pi_F$, implying that $e_{F(\alpha)/F} = n$. We started by assuming n is coprime to $\mathrm{ch}(k_F)$, so the extension is tamely ramified in this case.

(2) If $d = n$, then $\alpha\pi_F^{-\frac{k}{n}}$ is an nth root of unity. Thus $F(\alpha) = F(\alpha\pi_F^{-\frac{k}{n}})$ is an unramified extension of F by Proposition 2.6.
(3) If $1 < d < n$, the induction hypothesis for $d, \frac{n}{d} < n$ implies that $F(\alpha)/F(\alpha^{\frac{n}{d}})$ and $F(\alpha^{\frac{n}{d}})/F$ are tamely ramified. Thus, $F(\alpha)/F$ is tamely ramified.

□

Theorem 3.6. *Let F be a non-Archimedean local field and let π_F be a uniformizer of F. Then F^{tame} is generated over F by all roots of unity $\zeta \in \overline{F}$ whose order is coprime to $\mathrm{ch}(k_F)$ and all roots of π_F with order coprime to $\mathrm{ch}(k_F)$.*

Proof. By Proposition 3.5, it follows that nth roots of unity and nth roots of π_F, for n coprime to $\mathrm{ch}(k_F)$, generate tamely ramified extensions of F. It remains to show that all finite tamely ramified extensions E/F are contained in extensions generated by elements of this form.

Let M be the maximal unramfied subextension of E/F, and denote $e = e_{E/F} = e_{E/M}$, which is coprime to $\mathrm{ch}\, k_F$. By Lemma 3.2 there are uniforizers π_E of E and π_M of M such that $\pi_E^e = \pi_M$. Since M/F is unramified, there is $u \in \mathcal{O}_M^\times$ such that $\pi_M = u\pi_F$. Let $u' \in \overline{F}$ be an eth root of u. Since $M(u')/F$ is a finite unramified extension, Theorem 2.7 implies there is a root of unity $\zeta \in \overline{F}$ whose order is coprime to $\mathrm{ch}(k_F)$ such that $M(u') = F(\zeta)$. Let $\pi' = u'^{-1}\pi_E$, which is an eth root of π_F. Then we have $E \subseteq F(\zeta, \pi')$, finishing the proof. □

Theorem 3.7. *Let F be a non-Archimedean local field with $p = \mathrm{ch}(k_F)$ and $q = |k_F|$. Then $G_F^{\text{tame}} \cong \left(\prod_{r\neq p} \mathbb{Z}_r\right) \rtimes_q \hat{\mathbb{Z}}$, where the notation $\rtimes_q$ means that the generator $1 \in \hat{\mathbb{Z}}$ acts on $\prod_{r\neq p} \mathbb{Z}_r$ by multiplication with q. The projection to $G_{F^{\text{ur}}/F}$ is compatible with the projection to $\hat{\mathbb{Z}}$.*

Proof. The extension $F^{\text{ur}}(\pi_F^{\frac{1}{n}})/F^{\text{ur}}$ has Galois group $\mathbb{Z}/n$ for $p \nmid n$. This shows $G_{F^{\text{tame}}/F^{\text{ur}}}$ is isomorphic to the procyclic group $\prod_{r\neq p} \mathbb{Z}_r$, providing the exact sequence

$$0 \to \prod_{r\neq p} \mathbb{Z}_r \to G_{F^{\text{tame}}/F} \to \hat{\mathbb{Z}} \to 0$$

The sequence induces a semidirect product structure because any homomorphism to $\hat{\mathbb{Z}}$ has a section. To see the conjugation action of the Frobenius, Let $\tau \in G_{F^{\text{tame}}/F^{\text{ur}}}$ and let $\sigma \in G_{F^{\text{tame}}/F}$ be any lift of $\mathrm{Frob}_F \in G_{F^{\text{ur}}/F}$. Let π' be an nth root of π_F for $p \nmid n$. There are nth root of unity $\zeta, \omega \in F^{\text{ur}}$ such that $\sigma^{-1}\pi' = \omega\pi'$ and $\tau\pi' = \zeta\pi'$. It follows that

$$(\sigma\tau\sigma^{-1})(\pi') = \sigma\tau(\omega\pi') = \sigma(\omega\zeta\pi') = (\omega^q\zeta^q)(\omega^{-q}\pi') = \tau^q(\pi')$$

which proves the theorem, as elements of the form π' generate F^{tame} over F^{ur}. □

4. Wildly Ramified Extensions

Proposition 4.1. *Let E/F be a finite extension of non-Archimedean local fields, and let p be their residue characteristic. Let T/F be the maximal tamely ramified subextension of E/F. Then $[E : F]$ is a p-power.*

Proof. Let $\widetilde{E}$ denote the Galois closure of E/F and let $\widetilde{T}/F$ denote the maximal tamely ramified subextension of $\widetilde{E}/F$. Let P be a p-Sylow subgroup of $G_{\widetilde{E}/\widetilde{T}}$. Since $\widetilde{E}^P/\widetilde{T}$ is an extension whose order is coprime to p, it has to be tamely ramified, so $G_{\widetilde{E}/\widetilde{T}} = P$ is a p-group by maximality of $\widetilde{T}$. The extension $\widetilde{T} \cap E/T$ is tamely ramified, so $\widetilde{T} \cap E = T$ by maximality of T. This implies

$$[E : T] = [E : E \cap \widetilde{T}] \mid [\widetilde{E} : \widetilde{T}]$$

is a p-power. □

Corollary 4.2. *Let F be a non-Archimedean local field. Let $p = \mathrm{ch}(k_F)$ and $q = |k_F|$. Then the absolute Galois group of F fits into an exact sequence*

$$0 \to P \to G_{\overline{F}/F} \to \left(\prod_{r \neq p} \mathbb{Z}_r \right) \rtimes_q \hat{\mathbb{Z}} \to 0$$

where P is a pro-p group, and the notation $\rtimes_q$ means that the generator $1 \in \hat{\mathbb{Z}}$ acts on $\prod_{r \neq p} \mathbb{Z}_r$ by multiplication with q.

Corollary 4.3. *Let F be a local field. Then the absolute Galois group $G_{\overline{F}/F}$ is solvable.*