Robot Motion Planning:
An Efficient Algorithm for the Minkowski Sum

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The Minkowski difference of two convex polygons A and B is defined as

$$A \ominus B \doteq \{\mathbf{a} - \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B\}.$$

Note that this can be expressed as the Minkowski sum as

$$A \ominus B = A \oplus (-B)$$

where $-B = \{-\mathbf{b} \,|\mathbf{b} \in B\}$. For polygon cases, we can only consider the vertices of A and B in computing the Minkowski operations. Here we introduce an efficient algorithm for computing the Minkowski sum. See [1] for more details.

Pseudocode:

Inputs: A convex polygon A with vertices $\mathbf{a}_1, \cdots, \mathbf{a}_n$, and a convex polygon B with vertices $\mathbf{b}_1, \cdots, \mathbf{b}_m$. The lists of vertices in all the polygons are in counterclockwise order, with $\mathbf{a}_1$ and $\mathbf{b}_1$ being the vertices with smallest y-coordinate (and smallest x-coordinate in case of ties). The pseudocode below is modified from [1].

Output: Minkowski Sum $Q = A \oplus B$.

function Q = Minkowski_Sum(A, B)
  $i \doteq 1$; $j \doteq 1$
  $\mathbf{a}_{n+1} \doteq \mathbf{a}_1$; $\mathbf{a}_{n+2} \doteq \mathbf{a}_2$; $\mathbf{b}_{m+1} \doteq \mathbf{b}_1$; $\mathbf{b}_{m+2} \doteq \mathbf{b}_2$
  $Q \doteq \emptyset$
  while $size(Q, 2) < n + m$ **do**:
    Add $\mathbf{a}_i + \mathbf{b}_j$ and put it in Q.
    if $angle(\overrightarrow{\mathbf{a}_i\mathbf{a}_{i+1}}) < angle(\overrightarrow{\mathbf{b}_j\mathbf{b}_{j+1}})$ **then**:
      $i \leftarrow i + 1$
    else if $angle(\overrightarrow{\mathbf{a}_i\mathbf{a}_{i+1}}) > angle(\overrightarrow{\mathbf{b}_j\mathbf{b}_{j+1}})$ **then**:
      $j \leftarrow j + 1$
    else
      $i \leftarrow i + 1$; $j \leftarrow j + 1$
    end if
    Make sure that $i \leq n + 1$ and $j \leq m + 1$.
  end while
  Remove possible redundant points (due to collinear edges).

References

[1] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf. *Computational Geometry*. Springer, 2nd edition, 2000.