On the Kruskal Count Trick and its Variations

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In this paper, we explain a wonderful trick, commonly known as Kruskal's Count and analyse variations of the same. The trick is ostensibly complicated, but upon closer inspection gives way to a simple recurring process that lies at the center of the magic. Understanding the mathematical fundamentals behind it, however, takes a bit more effort, and relies on the principles of statistics and linear algebra. It is a trick that can be performed with or without a deck of cards at a moment's notice without any preparation. Behold the magic.

THE EFFECT

Here is what the audience sees: The performer holds a deck of cards in his hand, wrapped with a few rubber bands to prevent disaster. The performer claims this trick is one of great mind-reading mystique, requiring him to predict the audience member's card after much shuffling has been done to it. The deck is tossed to a random audience member who shuffles the deck however he or she pleases. While that process is occuring, the magician asks another random audience member to be a part of the demonstration and come onstage. The performer explains to the victim, "I want you to select your favorite number in your mind between 1 and 10. The way this trick works is that you begin on the nth card of the deck as I display them, n being your favorite number, and from there on you move over how many ever spots is the value of that original nth card. For example, if it is an Ace, move over 1, if it is a nine, move over 9, and if it is a face card, move over 5. Repeat this process until you reach the end of the deck and run out of a next move." The performer then claims to the audience that this increases the confusion and difficulty in predicting the audience member's final card. The performer then begins to deal out the cards and the audience member tracks his card throughout. Upon dealing out the whole deck, the performer appears to read the audience member's mind and says "Oh, ________, you're so predictable, I would bet $50 that your final card is ______," and places a $50 bill on the selected card.

If the cards were shuffled randomly by an audience member and the victim could have chosen any card he wanted, how did the performer guess the correct card?

THE SECRET

The secret lies in a small, but important, omitted detail. As the audience member chooses a particular card between 1 and 10 to track, the performer also chooses to track any card from the first row. The performer either has the option to choose a random card between 1-10 or always start by tracking from the first card. This allows the performer to also have a "path" through the deck of cards, which, as we will later show, couples with the audience member's path at some point, thereby converging to the same final card. It is rather intuitive that the more cards there are in the deck, the longer the path will go on for, which means the greater the probability that the two paths converge at any given point (remember that once the paths couple, they remain together for the rest of the journey till the end). Likewise, it is equally intuitive that the smaller the average card values, the smaller each step will be, again meaning a longer path time and greater likelihood of convergence. These natural findings are somewhat indeterminate, so in attempting to quantify them, we encounter the math question at the crux of the trick: What is the probability that the magician's path converges with the audience member's path before the deck ends? In posing such a question, we are essentially asking, what is the success rate of Kruskal's Count and can any variations improve it?

THE MATH

In order to understand the math behind this trick we must grasp the underlying probability that it requires. We will be working with a Geometric Distribution because we are looking for k-1 failures until the first success, k. Think of this analogous to flipping a coin. If we are interested in the probability of the first heads occuring on the 6th flip, we must have 5 failures followed by 1 success, i.e. $5^{0.5} * 1^{0.5}$. Likewise, each card that the two paths do not converge is defined as a failure, and so the probability that coupling first happens on the k^{th} card is: $p_k = (1-p) \cdot p^{k-1}$, where p is defined as the probability of failure

We are also interested in finding the expected value, which is given by $E(x) = \sum_{k=1}^{\infty} k \cdot p(1-p)^{k-1}$. Here we work with p defined as the probability of success for the simplicity of the proof. This is because we are looking for the probability of success on a k^{th} given trial: $p(1-p)^{k-1}$ over the entire distribution of possible k values. If we expand the notation we get:

$$E(x) = p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 \dots$$

Multiply the entire expansion by (1-p) to get:

$$(1-p)E(x) = p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + 4p(1-p)^4 \dots$$

Subtract the two series to get: $p \cdot E(x) = p + p(1-p) + p(1-p)^2 \ldots$

This is a convergent Geometric Series so use the definition to find its sum:
$\frac{1}{1-(1-p)} \Rightarrow E(x) = \frac{1}{p}$

We are using p as the probability of failure so in our case, $E(x) = \frac{1}{1-p}$. If $x = \frac{1}{1-p}$, then we can solve for $p = \frac{x-1}{x}$. Knowing $p_k = (1-p) \cdot p^{k-1}$, by substitution this becomes $p_k = \frac{1}{x} \cdot (\frac{x-1}{x})^{k-1}$.

Also, for Kruskal's Count, the average card value is x= 70/13. This is because each number (Ace through King) has a 1/13 probability of appearing (4/52), and the sum of their values is 70 when we assign face cards a value of 5.

Having established the aforementioned, we are interested in when the paths of the magician and spectator converge, i.e. looking at coupling time, t, which is represented by variable k in our formulations. For example, P[t=1] is the probability that coupling happens on the first card. Using the formula $p_k = \frac{1}{x} \cdot (\frac{x-1}{x})^{k-1}$, this becomes $p_1^2 = \frac{1}{x^2}$.

We know that Kruskal's Count fails when coupling happens after the deck is over. That is to say that the paths don't converge within the deck and so the magician mis predicts the spectator's card. We can represent this probability by the following:

$$P[t > N] = P[t > N | t = 1]P[t = 1] + P[t > N | t \neq 1]P[t \neq 1]$$

$$= 0 \cdot \frac{1}{x^2} + P[t > N | t \neq 1] \cdot 1 - \frac{1}{x^2}$$

$$= P[t > N - 1] \cdot \frac{x^2-1}{x^2}$$

We can perform this same operation for $P[t > N - 1]$ and continue this all the way till $P[t > N - N]$ to get $P[t > N] = [\frac{x^2-1}{x^2}]^N$. Note that $P[t > 0]$ is 1.

After deriving this formula for the probability of failure, we subtract it from one to find the probability of success. $P(success) = 1 - [\frac{x^2-1}{x^2}]^N$, where $x = \frac{70}{13}$ and $n = 52$ gives us P(success) = 83.88%.

Two important properties to understand the aforementioned method is firstly that: $P[t > N|t = 1] = 0$. This is because we give t as 1 so for any t before N smaller than N it is impossible for it to be larger than N, or in other words has a 0% probability. The second property to understand is the memoryless property of Markov Chains. This just essentially means that the next state is not dependent on its history but only the present state. Mathematically it looks as such:

$$P[t = k|t \geq 1] = P[t = k - 1]$$

$$= \frac{P[t=k \cap t \geq 1]}{P[t \geq 1]} = \frac{P[t=k]}{P[t \geq 1]} = \frac{p(1-p)^{k-1}}{1-P(t \leq 1)} = p(1-p)^{k-l-1}$$

$$P(t \leq l) = P(t = 0) + P(t = 1) + ... + P(t = l)$$

$$\Rightarrow p + p(1-p) + p(1-p)^2 + ... + p(1-p)^l$$

$$\Rightarrow 1 - (1-p)^l$$

COMPUTER SIMULATION

As much as we would love to, we naturally cannot afford to repeat the experiment thousands of times to discover an average rate of success. Thankfully, however, we can take the aid of computer technology and write code that will simulate this experiment over and over for us.

The code was written using the Python language due to its particularly strong libraries. In our case the 'random' library was used (Code given in the appendix).

Upon running the simulation for a million iterations, we found a rate of success around ~ 84.1%. Interestingly, however, if the magician always chooses the first card as his starting card the probability goes up to ~ 85.3%. This logically follows as it guarantees the longest possible path length throughout the deck of cards, thereby slightly raising the possibility of coupling. When it was ran again with two decks i.e. 104 cards in all, the probability becomes ~97.8%

Then we consider the case where the value of each of the cards is the number of letters in the name of the card, insead of the number written on it. This would mean that an Ace holds a value of 3, Five holds a value of 4, and King holds a value of 4 so on and so forth. Running the simulation with this variation (for the one deck and the magician choosing a random card) we get a probability of ~95.3%. Likewise, for the same setup but with the magician now choosing the first card we get ~95.9%. Finally, with two decks and the same setup we get ~99.8%.

VARIATIONS

It was previously shown that for the mathematical formula for the standard version of the trick, the probability of success turns out to be 83.88%.

Now, we can use the same formula to see the probabilities for all the variations mentioned above and observe how well correlated the math and simulation findings are.

So, for the standard version of the trick with 104 cards we apply the formula $P(success) = 1 - [\frac{x^2-1}{x^2}]^N$, for x=70/13 and N=104

We get

$$P(success) = 1 - [\frac{[\frac{70}{13}]^2-1}{[\frac{70}{13}]^2}]^{104}$$

P(success)= 97.40%

We now consider the case where we use the names of the cards as our key. Here, our average card value ‘x’ changes. First, we find x

$$x = \frac{4(3+3+5+4+4+3+5+5+4+3+4+5+4)}{4(13)} = 4$$

And as usual N=52

Putting this new x in our equation,

$$P(success) = 1 - [\frac{4^2-1}{4^2}]^{52}$$

P(success) = 96.51%

With the same setup, but with two decks we have N=104

Then we get

$$P(success) = 1 - [\frac{4^2-1}{4^2}]^{104}$$

P(success) = 99.87%

Thus, we see that the simulation results and the math work out to be quite close to each other.

APPENDIX

The simulation code:

```
import random

NO_OF_CARDS = 52
Deck =[]
Ctr_victim = 0 #the counter of the victim
Ctr_mag = 0 #the counter of the magician
match = 0
Run_count = 0

#initialising the deck for the spelled out card variation
#words = [3, 3, 5, 4, 4, 3, 5, 5, 4, 3, 4, 5, 4]
#Deck = [words[i%13] for i in range (NO_OF_CARDS)]

#initialising the deck for the original trick with face cards as 5
Deck =[(i % 13) + 1 for i in range (NO_OF_CARDS)]
for i in range (NO_OF_CARDS):
    if ((i % 13) + 1) > 10:
        Deck[i] = 5

#takes the given deck and its size as parameters and returns the
shuffled deck
def shuffle_deck(Deck, i):
    while i > 0:
        j = (random.randint (0, 1000000)) % i
        temp = Deck[i]
        Deck[i] = Deck[j]
        Deck[j] = temp
        i = i - 1
    return Deck


for k in range (1000000):
    shuffle_deck(Deck, NO_OF_CARDS-1)
```

```
    Ctr_victim = random.randint (0, 1000000) % 10
    while Ctr_victim + Deck[Ctr_victim] < NO_OF_CARDS:
        Ctr_victim = Ctr_victim + Deck[Ctr_victim]

    #the case where the magician chooses a random number
    Ctr_mag = random.randint (0, 1000000) % 10

    #the case where the magician chooses the first card
    #Ctr_mag = 0
    while Ctr_mag + Deck[Ctr_mag] < NO_OF_CARDS:
        Ctr_mag = Ctr_mag + Deck[Ctr_mag]

    if Ctr_mag == Ctr_victim:
        match = match + 1
    Run_count = Run_count + 1

print (100 *(match / Run_count))
```

REFERENCES

http://www.singingbanana.com/Kruskal.pdf