The randomized level set method and an associated reaction-diffusion equation to model wildland fire propagation

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Abstract Front propagation can be studied by two alternative approaches: the level set method and the reaction-diffusion equation. When a front propagates in a random environment it gets a random character and these two approaches can indeed be considered complementary and reconciled. In fact, if the level set contour is randomized accordingly to the probability density function of the front particle displacement, the resulting averaged process emerges to be governed by an evolution equation of the reaction-diffusion type. This approach turns out to be useful to simulate random effects in wildland fire propagation as those due to turbulent heat convection and fire spotting phenomena.

1 Introduction

Wildland fire propagation is a complex multi-scale, as well as a multi-physics and multi-discipline process, strongly influenced by the atmospheric wind. Wildland fire is fed by the fuel on the ground and displaced, beside meteorological and orographical factors, also by the hot air that pre-heats the fuel and aids the fire propagation. Heat transfer is turbulent due to the Atmospheric Boundary Layer and the fire-induced flow. Moreover, fire generates firebrands that when land on the ground are

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further sources of fire. Both turbulence and jump-length of firebrands are random processes that affect the fireline propagation.

Fire propagation has been mainly modelled in the literature by using reaction-diffusion type equations, see e.g. [1, 5], and the level set method, see e.g. [3, 4]. Here, an approach based on the level set method is proposed to model the global random effects on fire front propagation due to turbulence and fire spotting. Actually, a reaction-diffusion equation associated to the level-set method is derived.

2 Model Formulation

Let $\Gamma(t)$ be the fire line contour then, in a two dimensional domain, it can be represented as an isoline of an auxiliary function $\gamma(\boldsymbol{x},t)$, i.e. $\Gamma(t) = \{\boldsymbol{x},t : \gamma(\boldsymbol{x},t) = \gamma_0 = \mathit{constant}\}$. The evolution equation of the isoline γ_0 is given by

$$\frac{D\gamma}{Dt} = \frac{\partial \gamma}{\partial t} + \frac{d\boldsymbol{x}}{dt} \cdot \nabla \gamma = \frac{D\gamma_0}{Dt} = 0\,. \tag{1}$$

Let the motion of the surface points be directed towards the normal direction then

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{V}(\boldsymbol{x},t) = \mathscr{V}(\boldsymbol{x},t)\,\widehat{\boldsymbol{n}}\,, \quad \widehat{\boldsymbol{n}} = -\frac{\nabla\gamma}{||\nabla\gamma||}\,, \tag{2}$$

and (1) becomes

$$\frac{\partial \gamma}{\partial t} = \mathscr{V}(\boldsymbol{x},t)\,||\nabla\gamma||\,, \tag{3}$$

which is the *ordinary* level set equation. Let $\varphi(\gamma(\boldsymbol{x},t))$ be an indicator function such that

$$\varphi(\gamma(\boldsymbol{x},t)) = \begin{cases} 1\,,\ \gamma(\boldsymbol{x},t) > \gamma_0\,,\ \boldsymbol{x} \in \Omega(t)\,,\ \text{burned area}\,, \\ \\ 0\,,\ \gamma(\boldsymbol{x},t) \le \gamma_0\,,\ \boldsymbol{x} \notin \Omega(t)\,,\ \text{unburned area}\,. \end{cases} \tag{4}$$

The boundary of $\Omega(t)$ is $\Gamma(t)$, that is the front line contour of the wildland fire. Quantity $\mathscr{V}(\boldsymbol{x},t)$ is identified with the so-called Rate Of Spread (ROS) [3, 4]. Several determinations of the ROS have been proposed, see e.g. [2, 3, 9]. The present formulation holds for any determination of the ROS.

Let the burning fireline be embodied by a large number of *active* flame holders. Let the motion of each *active* flame holder belonging to the fireline be random due to turbulence and fire spotting effects. For any realization indexed by ω, the random trajectory of each *active* flame holder is stated to be $\boldsymbol{X}^\omega(t,\bar{\boldsymbol{x}}_0) = \bar{\boldsymbol{x}}_{ROS}(t,\bar{\boldsymbol{x}}_0) + \chi^\omega + \xi^\omega$, where χ and ξ are two random noises that reproduce the randomness of turbulence and fire spotting. The deterministic component $\bar{\boldsymbol{x}}_{ROS}$ corresponds to the motion obtained by literature determination of the ROS [2, 3, 9]. The trajectory of a single *active* flame holder is marked out by the one-particle density function $f^\omega(\boldsymbol{x};t) = \delta(\boldsymbol{x} - \boldsymbol{X}^\omega(t,\bar{\boldsymbol{x}}_0))$, where $\delta(\boldsymbol{x})$ is the Dirac-delta function. The random trajectory $\boldsymbol{X}(t,\bar{\boldsymbol{x}}_0)$ has the same fixed initial condition $\boldsymbol{X}^\omega(0,\bar{\boldsymbol{x}}_0) = \bar{\boldsymbol{x}}_{ROS}(0,\bar{\boldsymbol{x}}_0) = \bar{\boldsymbol{x}}_0$

in all realizations. Let $\gamma(\bar{\boldsymbol{x}}_0, 0)$ be the initial fixed fireline contour, the evolution in time of the fireline according to the ω-realization of the trajectories of the *active* flame holders follows to be

$$\gamma^{\omega}(\boldsymbol{x}(t)) = \int_{\Gamma_0} \gamma(\bar{\boldsymbol{x}}_0, 0)\, \delta(\boldsymbol{x} - \boldsymbol{X}^{\omega}(t, \bar{\boldsymbol{x}}_0))\, d\bar{\boldsymbol{x}}_0\,, \tag{5}$$

where $\Gamma_0 = \{\boldsymbol{x} : \gamma(\bar{\boldsymbol{x}}, 0) = \gamma_0\}$.

Denoting by $\langle \cdot \rangle$ the ensemble average, the average trajectory $\langle \boldsymbol{X}(t; \bar{\boldsymbol{x}}_0)\rangle = \bar{\boldsymbol{x}}(t, \bar{\boldsymbol{x}}_0)$ is driven by the deterministic velocity field $d\bar{\boldsymbol{x}}/dt = \boldsymbol{V}(\bar{\boldsymbol{x}}, t)$. Then, trajectory $\bar{\boldsymbol{x}}(t, \bar{\boldsymbol{x}}_0)$ emerges to be time-reversible and the Jacobian of the transformation follows to be $J = d\bar{\boldsymbol{x}}_0/d\bar{\boldsymbol{x}} \neq 0$. When the fireline length $\mathscr{L}(t)$ grows, the number $\mathscr{N}(t)$ of the *active* flame holders composing the fireline grows as well. Then the growing ratio of the fireline, i.e. $\mathscr{L}(t)/\mathscr{L}(0)$, and that of the number of the *active* flame holders, i.e. $\mathscr{N}(t)/\mathscr{N}(0)$, are equal. Hence, to each *active* flame holder it can be associated an *action length d* stated as $d = \mathscr{L}(t)/\mathscr{N}(t) = \mathscr{L}(0)/\mathscr{N}(0) = constant$. As a consequence of this reasoning, a condition of incompressibility type follows: $J = 1$. Finally, by time inversion and ensemble averaging, from (5) the effective fire front contour emerges to be in terms of the indicator function $\varphi(\boldsymbol{x}, t)$ as follows

$$\begin{aligned}\langle \varphi^{\omega}(\boldsymbol{x}(t))\rangle &= \langle \int_{R^2} \varphi(\bar{\boldsymbol{x}}, t)\, \delta(\boldsymbol{x} - \boldsymbol{X}^{\omega}(t, \bar{\boldsymbol{x}}))\, d\bar{\boldsymbol{x}}\rangle = \int_{R^2} \varphi(\bar{\boldsymbol{x}}, t)\, \langle \delta(\boldsymbol{x} - \boldsymbol{X}^{\omega}(t, \bar{\boldsymbol{x}}))\rangle\, d\bar{\boldsymbol{x}} \\ &= \int_{R^2} \varphi(\bar{\boldsymbol{x}}, t)\, f(\boldsymbol{x}; t|\bar{\boldsymbol{x}})\, d\bar{\boldsymbol{x}} = \varphi_e(\boldsymbol{x}, t)\,, \end{aligned} \tag{6}$$

where $f(\boldsymbol{x}; t|\bar{\boldsymbol{x}}) = \langle \delta(\boldsymbol{x} - \boldsymbol{X}^{\omega}(t, \bar{\boldsymbol{x}}))\rangle$ is the probability density function (PDF) of the distribution of the particles of the fireline contour around the average front location $\bar{\boldsymbol{x}}$ and the definition of $\varphi(\bar{\boldsymbol{x}}, t)$ stated in (4) has been used.

Field variable $\varphi_e(\boldsymbol{x}, t)$ is computed from formula (6) where indicator function $\varphi(\bar{\boldsymbol{x}}, t)$ follows from solving the level set equation driven by an average front velocity.

By applying the Reynolds transport theorem to (6), the evolution equation of the effective fire front $\varphi_e(\boldsymbol{x}, t)$ is [6]

$$\frac{\partial \varphi_e}{\partial t} = \int_{\Omega(t)} \frac{\partial f}{\partial t}\, d\bar{\boldsymbol{x}} + \int_{\Omega(t)} \nabla_{\bar{\boldsymbol{x}}} \cdot [\boldsymbol{V}(\bar{\boldsymbol{x}}, t)\, f(\boldsymbol{x}; t|\bar{\boldsymbol{x}})]\, d\bar{\boldsymbol{x}}\,, \tag{7}$$

that is the reaction-diffusion equation associated to the level set equation (3).

Since the effective fireline contour $\varphi_e(\boldsymbol{x}, t)$ is a smooth function continuously ranging from 0 to 1, a criterion to mark burned points have to be stated. Here points $\boldsymbol{x}$ such that $\varphi_e(\boldsymbol{x}, t) > 0.5$ are marked as burned and the effective burned area emerges to be $\Omega_e(t) = \{\boldsymbol{x}, t : \varphi_e(\boldsymbol{x}, t) > 0.5\}$. However, beside this criterion, a further criterion associated to an ignition delay due to the pre-heating action of the hot air or to the landing of firebrands is introduced. Hence, in the proposed modelling approach, an unburned point $\boldsymbol{x}$ will be marked as burned when one of these two criteria is met.

This ignition delay, due to a certain *heating-before-burning mechanism*, can be depicted as an accumulation in time of heat [7], i.e.

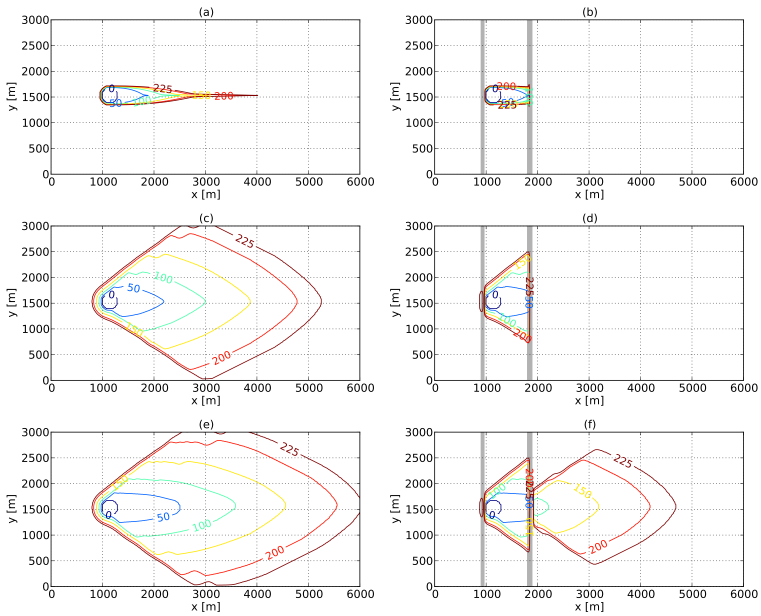

Fig. 1 Time evolution of the firefront in absence (on the left) and presence (on the right) of two fire-break zones (grey stripes). The results are obtained by adopting the level set method (top row), by the present modelling approach when only turbulence is taken into account (middle raw), and when both turbulence and fire spotting are considered (bottom raw). The labels on the contour lines represent the propagation time (expressed in minutes). Following [8], turbulence has been parameterized with a Gaussian PDF and fire spotting with a stationary log-normal distribution for jump-length of embers with mean stated equal to $\mu = 1.32 I_f^{0.26} U_t^{0.11} - 0.02$ and standard deviation $s = 4.95 I_f^{-0.01} U_t^{-0.02} - 3.48$, where U_t is the modulus of the mean wind, assumed constant both in value ($6.70\,\mathrm{m s^{-1}}$) and direction (x-axis), and $I_f = I + I_t$ where $I = 10\,000\,\mathrm{kW m^{-1}}$ is the fire intensity and $I_t = 0.015\,\mathrm{kW m^{-1}}$ is the tree torching intensity. Other simulation parameters are: $V_{ROS} = I/(H w_0)$ where $H = 22\,000\,\mathrm{kJ kg^{-1}}$ is the fuel low heat of combustion and $w_0 = 2.243\,\mathrm{kg m^{-2}}$ is the oven-dry mass of fuel, $\mathscr{D} = 0.04\,\mathrm{m^2 s^{-1}}$, $\tau_h = 600\,\mathrm{s}$, $\tau_f = 60\,\mathrm{s}$ and the width of fire-breaks is 60 m in the windward sector and 90 m in the leeward sector.

$$\psi(\boldsymbol{x},t) = \int_0^t \varphi_e(\boldsymbol{x},\eta)\,\frac{d\eta}{\tau}\,, \tag{8}$$

where $\psi(\boldsymbol{x},0) = 0$ corresponds to the unburned initial condition and τ is a characteristic ignition delay. Since the fuel can burn because of two pathways, i.e. hot-air heating and firebrand landing, the resistance analogy suggests that τ can be approximatively computed as resistances acting in parallel, i.e.

$$\frac{1}{\tau} = \frac{1}{\tau_h} + \frac{1}{\tau_f} = \frac{\tau_f + \tau_h}{\tau_h \tau_f}\,, \tag{9}$$

where τ_h and τ_f are the ignition delays due to hot air and firebrands, respectively.

The amount of heat is proportional to the increasing of the fuel temperature $T(\boldsymbol{x},t)$, then

$$\psi(\boldsymbol{x},t) \propto \frac{T(\boldsymbol{x},t) - T(\boldsymbol{x},0)}{T_{ign} - T(\boldsymbol{x},0)}\,, \quad T(\boldsymbol{x},t) \leq T_{ign}\,, \tag{10}$$

where T_{ign} is the ignition temperature. Finally, when $\psi(\boldsymbol{x},t) = 1$ the ignition temperature is assumed to be reached, so that a new ignition occurs in $(\boldsymbol{x},t)$ and, with reference to (6), the modelled fire goes on by setting $\varphi(\boldsymbol{x},t) = 1$.

3 Discussion and Conclusions

The present analysis constitutes a proof-of-concept and it needs to be subjected to a future validation. Hence, numerical results showed in Figs. 1-5 are understood as explorative exercises to investigate the potentialities of the approach. From comparison of the level set method against the proposed model when only turbulence and when both turbulence and fire spotting are taken into account, it emerges the suitability of the proposed approach to simulate a fire that overcomes a fire-break zone, in contrast to the level set method. Moreover, it emerges also that the inclusion of turbulence allows for simulating fire flank and backing fire and the inclusion of hot air preheating and ember landing enhances the frontline propagation. This richness of model behaviours supports the proposed formulation as a promising approach to simulate the complex phenomenology of real wildland fire propagation.

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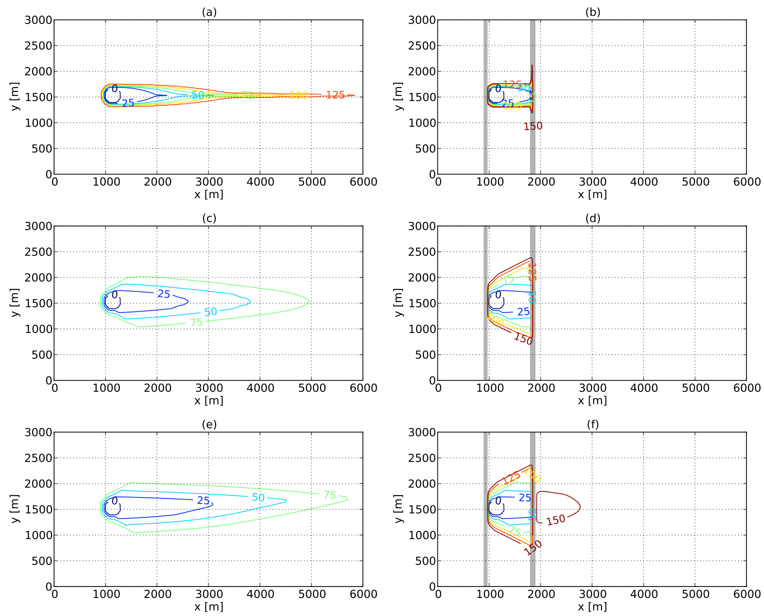

Fig. 2 The same as in Fig. 1 but when $U_{\rm t} = 6.70\,{\rm m\,s^{-1}}$ and $I = 30\,000\,{\rm kW\,m^{-1}}$.

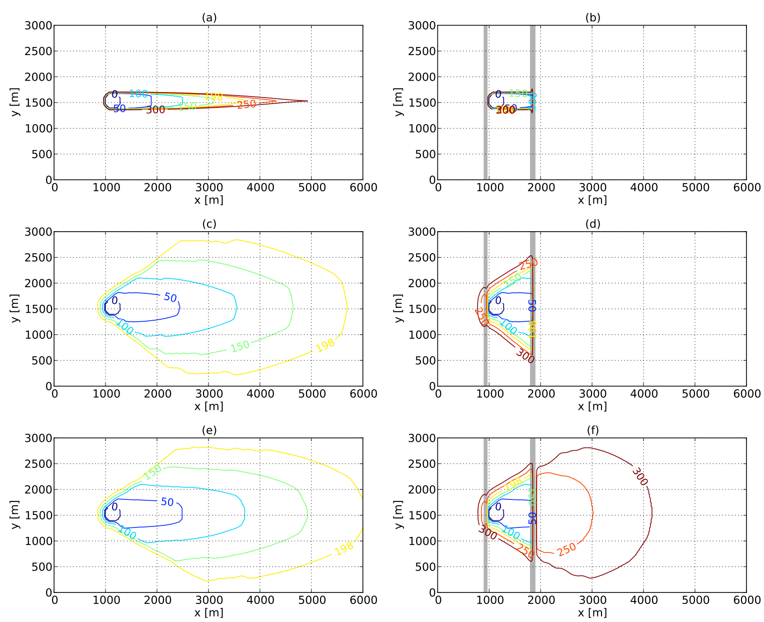

Fig. 3 The same as in Fig. 1 but when $U_{\rm t} = 17.88\,{\rm m\,s^{-1}}$ and $I = 10\,000\,kW\,{\rm m^{-1}}$.

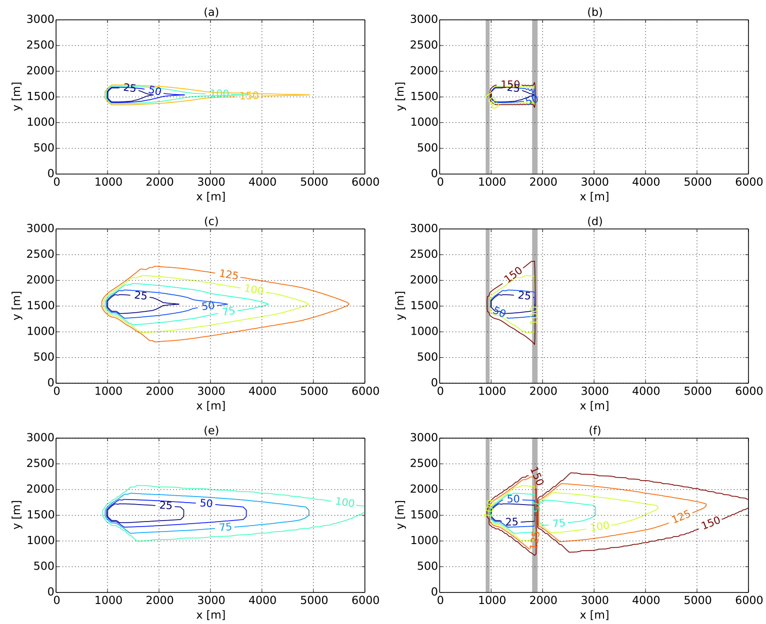

Fig. 4 The same as in Fig. 1 but when $U_t = 17.88\,\mathrm{m\,s^{-1}}$ and $I = 20\,000\,kW\,\mathrm{m}^{-1}$.

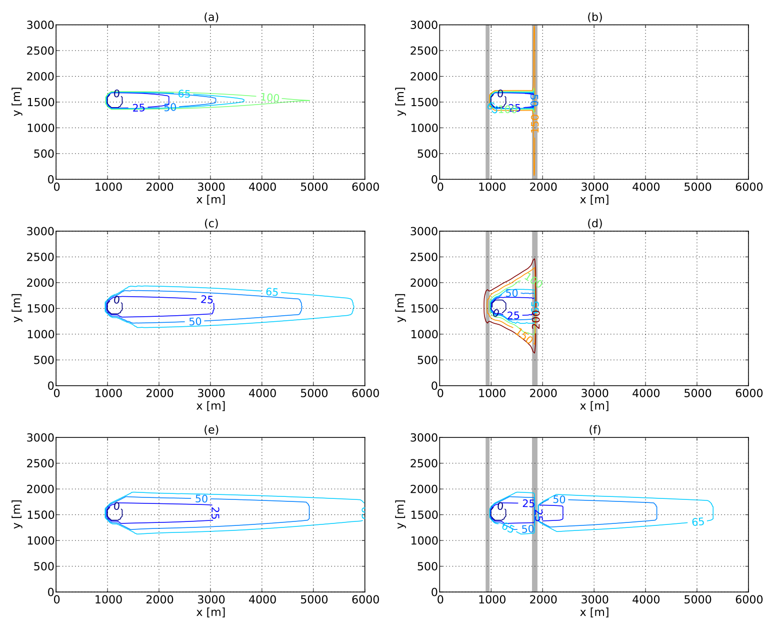

Fig. 5 The same as in Fig. 1 but when $U_t = 17.88\,\mathrm{m\,s^{-1}}$ and $I = 30\,000\,kW\,\mathrm{m}^{-1}$.