$u(x,t) = X(x)\,T(t)$

$u_{xx}(x,t) = X''(x)\,T(t)$

$u_{tt}(x,t) = X(x)\,T''(t)$

Problema 1. Hallar la solución $u = u(x,t)$ de:

$$\begin{cases} u_t - (1+2t)u_{xx} = 0 & 0 < x < \pi, \quad t > 0, \\ u(x,0) = 3\sin(\frac{5x}{2}) & 0 < x < \pi, \\ u(0,t) = 0 & t > 0, \\ u_x(\pi,t) = 0 & t > 0. \end{cases}$$

1 Separación Variables

$u = u(x,t) = X(x)\,T(t)$

$u_{xx} = X''(x)T(t) \qquad u_x = X'(x)T(t)$

$u_t = X(x)T'(t)$

$XT' - (1+2t)X''T = 0$

$$\frac{X''}{X} = \frac{T'}{(1+2t)T} = -\lambda$$

$\Longrightarrow$ Desacoplado

$$\left.\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = X'(\pi) = 0 \end{array}\right\}$$

$m^2 + \lambda = 0$

$m = \pm\sqrt{\lambda}\, i$

$X(x) = C e^{\pm\sqrt{\lambda}\, i x}$

$= C_1 \operatorname{sen}(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)$

$X'(x) = C_1\sqrt{\lambda}\cos(\sqrt{\lambda}x) - \sqrt{\lambda}C_2\operatorname{sen}(\sqrt{\lambda}x)$

$X(0) = C_2 = 0$

$X'(\pi) = C_1\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$

$\sqrt{\lambda}\,\pi = \frac{\pi}{2} + n\pi \qquad n \in \mathbb{N} \cup \{0\}$

$\lambda_n = \left(\frac{2n+1}{2}\right)^2 \qquad n = 0, 1, 2, \ldots$

$X_n(x) = C_n \operatorname{sen}\left(\frac{2n+1}{2}x\right)$

Por otro lado

$T' + \lambda(1+2t)T = 0 \quad / \quad T(0) = 3\operatorname{sen}(\frac{5}{2}x)$

$\frac{dT(t)}{dt} + \lambda_n(1+2t)T(t) = 0$

$\frac{dT(t)}{T(t)} = -\lambda_n(1+2t)\,dt \quad / \int$

$\ln(T(t)) = -\lambda_n(t+t^2)$

$T_n(t) = \exp\left(-\left(\frac{2n+1}{2}\right)^2(t+t^2)\right)$

Aplicando series de Fourier

$$u(x,t) = \sum_{n=0}^{\infty} C_n X_n(x) T_n(x)$$

$$u(x,t) = \sum_{n=0}^{\infty} C_n \operatorname{sen}\left(\frac{2n+1}{2}x\right) e^{-\left(\frac{2n+1}{2}\right)^2(t+t^2)}$$

$$u(x,0) = \sum_{n=0}^{\infty} C_n \operatorname{sen}\left(\frac{2n+1}{2}x\right) = 3\operatorname{sen}\left(\frac{5}{2}x\right)$$

$\underline{n=2|}\quad C_2\operatorname{sen}\left(\frac{5}{2}x\right) = 3\operatorname{sen}\left(\frac{5}{2}x\right) \Longrightarrow C_2 = 3$

$\underline{n\neq 2|}\quad C_n\operatorname{sen}\left(\frac{2n+1}{2}x\right) = 0 \quad \forall x$

$\Longrightarrow C_n = 0.$

Luego

$$u(x,t) = 3\operatorname{sen}\left(\frac{5}{2}x\right) e^{-\left(\frac{5}{2}\right)^2(t+t^2)}$$

$\frac{X''}{X} = -\lambda$

$X'' + \lambda X = 0$

$X(t)$

$aX'' + bX + c$

$am^2 + bm + c = 0$

$X = e^{-mt}$

$X = \sum C_i e^{-m_i t}$

$\lambda < 0 \to m > 0$

$X(x) = C_1 e^{-\sqrt{\lambda}x} + C_2 e^{\sqrt{\lambda}x}$

$X(0) = C_1 + C_2 = 0$

$C_1 = -C_2$

$X'(x) = -\sqrt{\lambda}C_1 e^{-\sqrt{\lambda}x} + C_2\sqrt{\lambda}e^{\sqrt{\lambda}x}$

$X'(\pi) = -\sqrt{\lambda}C_1 e^{-\sqrt{\lambda}\pi} + C_2\sqrt{\lambda}e^{\sqrt{\lambda}\pi} = 0$

$= C_2\sqrt{\lambda}e^{-\sqrt{\lambda}\pi} + C_2\sqrt{\lambda}e^{\sqrt{\lambda}\pi} = 0$

$C_2\underbrace{\sqrt{\lambda}}_{>0}\big(\underbrace{e^{-\sqrt{\lambda}\pi} + e^{\sqrt{\lambda}\pi}}_{>0}\big) = 0$

$\to C_2 = 0 = C_1$

$u = \sum X_n T_n$

$u = XT$

X_n

T_n

Problema 2. Resuelva

$$\begin{cases} u_{tt} = x^2 u_{xx} + 3x u_x + u & 1 < x < e^e, \quad t > 0, \\ u(1,t) = u(e^e, t) = 0 & t > 0, \\ u(x,0) = \frac{1}{x^2} & 1 < x < e^e, \\ u_t(x,0) = 0 & 1 < x < e^e. \end{cases}$$

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$$u(x,t) = X(x)T(t) \qquad u_{xx}(x,t) = X''(x)T(t)$$

$$u_{tt}(x,t) = X(x)T''(t)$$

$$u_x(x,t) = X'(x)T(t)$$

$$u_{tt} = x^2 u_{xx} + 3x u_x + u$$

$$X(x)T''(t) = x^2 X''(x)T(t) + 3x X'(x)T(t) + X(x)T(t)$$

$$X(x)T''(t) = \left(x^2 X''(x) + 3x X'(x) + X(x)\right) T(t)$$

$$\frac{X(x)}{x^2 X''(x) + 3x X'(x) + X(x)} = \frac{T(t)}{T''(t)} = -\frac{1}{\lambda}$$

Sistema desacoplado

$$x^2 X''(x) + 3x X'(x) + X(x) = -\lambda X(x)$$

$$x^2 X''(x) + 3x X'(x) + (1+\lambda) X(x) = 0$$

$$X(1) = X(e^e) = 0$$

$X'' + 3X' + (\lambda+1)X = 0 \qquad x^2 X'' + 3x X' + (\lambda+1) X = 0$

$x^2 X''(x) + 3x X'(x) + (\lambda+1) X(x) = 0$

Hacemos $x = e^u \qquad \frac{dx}{du} = e^u$

$\frac{dX}{dx} = \frac{dX}{du} \cdot \frac{du}{dx} = \frac{dX}{du} \cdot e^{-u} = \frac{dX}{du} \cdot \frac{1}{x}$

$X''(u) + 3X'(u) + (\lambda+1) X(u) = 0$

$m^2 + 3m + (\lambda+1) = 0$

$m = \frac{-3 \pm \sqrt{9 - 4(\lambda+1)}}{2}$

$x = e^u$

$u = \ln x$

$9 - 4\lambda - 4 = 5 - 4\lambda \qquad X(u) = e^{-mu} \qquad X(x) = e^{-m \ln x}$

Si $5 - 4\lambda > 0$

$X(x) = e^{-mu} = C_1 e^{-\frac{-3+\sqrt{5-4\lambda}}{2} \ln x} + C_2 e^{-\frac{-3-\sqrt{5-4\lambda}}{2} \ln x}$

$X(1) = C_1 + C_2 = 0 \to C_1 = -C_2$

$X(e^e) = C_1 e^{-\frac{3+\sqrt{5-4\lambda}}{2} e} - C_1 e^{-\frac{-3-\sqrt{5-4\lambda}}{2} e} = 0$

$= C_1 \Big(\underbrace{e^{-\frac{-3+\sqrt{5-4\lambda}}{2} e} - e^{-\frac{-3-\sqrt{5-4\lambda}}{2} e}}_{>0} \Big) = 0$

$\Rightarrow C_1 = C_2 = 0$

Si $5 - 4\lambda = 0 \to \lambda = 5/4$

$\to m = -3/2 \qquad X(u) = C_1 e^{mu} + C_2 u e^{mu}$

$X(u) = C_1 e^{3/2 u} + C_2 u e^{3/2 u}$

$X(x) = C_1 x^{3/2} + C_2 \ln x \cdot x^{3/2}$

$X(1) = C_1 + 0 = 0 \to C_1 = 0$

$X(e^e) = C_2 e \cdot e^{\frac{3}{2} e} = 0 \to C_2 = 0 \to C_1 = C_2 = 0$

Si $5 - 4\lambda < 0$

$m = -\frac{3}{2} \pm i \frac{\sqrt{4\lambda - 5}}{2}$

$X(u) = e^{-mu} = e^{\left(-\frac{3}{2} \pm i \frac{\sqrt{4\lambda-5}}{2}\right) u}$

$= e^{-\frac{3}{2} u} \left(C_1 \cos\left(\frac{\sqrt{4\lambda-5}}{2} u\right) + C_2 \operatorname{sen}\left(\frac{\sqrt{4\lambda-5}}{2} u\right)\right)$

$X(x) = x^{3/2} \left(C_1 \cos\left(\frac{\sqrt{4\lambda-5}}{2} \ln x\right) + C_2 \operatorname{sen}\left(\frac{\sqrt{4\lambda-5}}{2} \ln x\right)\right)$

$X(1) = C_1 \cdot 1 = 0 \to C_1 = 0$

$X(e^e) = e^{\frac{3}{2} e} \cdot C_2 \operatorname{sen}\left(\frac{\sqrt{4\lambda-5}}{2} e\right) = 0$

$\operatorname{Sen}\left(\frac{\sqrt{4\lambda-5}}{2} e\right) = 0$

$\frac{\sqrt{4\lambda-5}}{2} e = n\pi \qquad n \in \mathbb{N}, \; n \geq 0$

$\lambda_n = \frac{1}{4}\left(\left(\frac{2n\pi}{e}\right)^2 + 5\right)$

$X_n(x) = x^{-3/2} C_n \operatorname{sen}\left(\frac{2n\pi}{e} \ln x\right)$

$T_n(t)$

Problemas de Sturm Liouville más conocidos

Problema	$\{\lambda_n\}_{n=1}^{\infty}$	$\{u_n\}_{n=1}^{\infty}$
$y'' + \lambda y = 0$; $y(0) = y(L) = 0$	$\left\{\left(\frac{n\pi}{L}\right)^2\right\}_{n=1}^{\infty}$	$\left\{\operatorname{sen}\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$
$y'' + \lambda y = 0$; $y'(0) = y'(L) = 0$	$\left\{\left(\frac{(n-1)\pi}{L}\right)^2\right\}_{n=1}^{\infty}$	$\left\{\cos\left(\frac{(n-1)\pi x}{L}\right)\right\}_{n=1}^{\infty}$
$y'' + \lambda y = 0$; $y(0) = y'(L) = 0$	$\left\{\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2\right\}_{n=1}^{\infty}$	$\left\{\operatorname{sen}\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)\right\}_{n=1}^{\infty}$
$y'' + \lambda y = 0$; $y'(0) = y(L) = 0$	$\left\{\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2\right\}_{n=1}^{\infty}$	$\left\{\cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)\right\}_{n=1}^{\infty}$
$y'' + \lambda y = 0$; $y(0) = y(L)$; $y'(0) = y'(L)$	$\left\{\left(\frac{2(n-1)\pi}{L}\right)^2\right\}_{n=1}^{\infty}$	$\left\{\operatorname{sen}\left(\frac{2(n-1)\pi}{L}\right), \cos\left(\frac{2(n-1)\pi}{L}\right)\right\}_{n=1}^{\infty}$

Problema 3. Hallar la solución de:

$$\begin{cases} u_{tt} - 2u_{xx} = 0 & x \in [0,1], \quad t > 0, \\ u(0,t) = u(1,t) & t > 0, \\ u_x(0,t) = u_x(1,t) & t > 0, \\ u(x,0) = \cos(2\pi x) & x \in [0,1], \\ u_t(x,0) = 2\cos(6\pi x) & x \in [0,1]. \end{cases}$$

$u(x,t) = X(x)T(t)$

$u_x = X'T \qquad u_t = XT'$

$u_{xx} = X''T \qquad u_{tt} = XT''$

$u_{tt} - 2u_{xx} = 0$

$XT'' - 2X''T = 0$

$\frac{X}{X''} = \frac{2T}{T''} = -\frac{1}{\lambda}$

Desacoplado

Sturm - Liouville

$$\begin{aligned} &X'' + \lambda X = 0 \\ &X(0) = X(1) \\ &X'(0) = X'(1) \end{aligned} \implies \begin{aligned} &\lambda_n = (2(n-1)\pi)^2 \\ &X_n = A_n \operatorname{sen}(2n\pi) + B_n \cos(2n\pi x) \end{aligned}$$

Usando la parte temporal

$$T_n'' + 2\lambda_n T_n = 0 \implies T_n(t) = \sum_{n=0}^{\infty} C_n \operatorname{sen}(2\sqrt{2}\, n\pi) + D_n \cos(2\sqrt{2}\, n\pi)$$

$$u(x,t) = \sum_{n=1}^{\infty} X_n T_n = \sum_{n=1}^{\infty} \left(A_n \operatorname{sen}(2n\pi x) + B_n \cos(2n\pi x)\right)\left(C_n \operatorname{sen}(2\sqrt{2}\, n\pi t) + D_n \cos(2\sqrt{2}\, n\pi t)\right)$$

Por las condiciones iniciales

$$u(x,0) = \sum_{n=1}^{\infty} D_n A_n \operatorname{sen}(2\pi n x) + D_n B_n \cos(2n\pi x) = \cos(2\pi x)$$

$\implies A_n D_n = 0 \ \forall n \qquad B_n D_n = 0 \ \forall n \neq 1$

$B_1 D_1 = 1$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(A_n \operatorname{sen}(2\pi n x) + B_n \cos(2n\pi x)\right)\left(C_n 2\sqrt{2}\pi n \cos(2\sqrt{2}\, n\pi t) - D_n 2\sqrt{2}\, n\pi \operatorname{sen}(2\sqrt{2}\, n\pi t)\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(A_n \operatorname{sen}(2\pi n x) + B_n \cos(2n\pi x)\right) C_n 2\sqrt{2}\pi n = 2\cos(6\pi x)$$

$A_n C_n = 0 \ \forall n \qquad B_n C_n = 0 \quad \forall n \neq 3$

$B_3 C_3 = \frac{1}{3\sqrt{2}\pi}$

$$u(x,t) = \cos(2\pi x)\cos(2\sqrt{2}\pi t) + \frac{1}{3\sqrt{2}\pi}\cos(6\pi x)\cos(6\sqrt{2}\pi t).$$

Problema 4. Si se sabe que

$$\begin{cases} X'' - 3X' + 2(1+\lambda)X = 0 \\ X(0) = X(1) = 0 \end{cases} \implies \begin{cases} \lambda_n = \frac{1}{8} + \frac{n^2\pi^2}{2} \\ X_n(x) = e^{3x/2}\sin(n\pi x), \quad n \geq 1. \end{cases}$$

Resuelva

$$\begin{cases} u_{xx} - 3u_x = 2u_{tt} - 2u & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0 & 0 < x < 1, \\ \cancel{u(x,0) = e^{3x/5}} & 0 < x < 1 \end{cases}$$

$u_t(x,0) = e^{3x/2}$

Separación variables

$u(x,t) = X(x)\,T(t)$

$u_{xx} = X''T \qquad u_{tt} = XT''$

$u_x = X'T \qquad u_t = XT'$

$u_{xx} - 3u_x = 2u_{tt} - 2u$

$X''T - 3X'T = 2XT'' - 2XT$

$(X'' - 3X' + 2X)\,T = 2XT''$

$$\frac{X'' - 3X' + 2X}{2X} = \frac{T''}{T} = -\lambda$$

Sistema desacoplado

$X'' - 3X' + 2X(1+\lambda) = 0$

$X(0) = X(1) = 0$

Problema 4. Si se sabe que

$$\begin{cases} X'' - 3X' + 2(1+\lambda)X = 0 \\ X(0) = X(1) = 0 \end{cases} \implies \begin{cases} \lambda_n = \frac{1}{8} + \frac{n^2\pi^2}{2} \\ X_n(x) = e^{3x/2}\sin(n\pi x), \quad n \geq 1. \end{cases}$$

Resuelva

$$\begin{cases} u_{xx} - 3u_x = 2u_{tt} - 2u & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0 & 0 < x < 1, \\ \cancel{u_t(x,0) = e^{3x/5}} & 0 < x < 1 \end{cases}$$

$u_t(x,0) = e^{3x/2}$

$$\begin{cases} X'' - 3X' + 2(1+\lambda)X = 0 \\ X(0) = X(1) = 0 \end{cases} \qquad \lambda_n = \frac{1}{8} + \frac{n^2\pi^2}{2}$$

$$X_n(x) = e^{3x/2}\,\mathrm{sen}(n\pi x)$$

Analizando la parte temporal

$$T'' + \lambda T = 0$$

$$m^2 + \lambda = 0$$

$$m = \pm\sqrt{\lambda}\,i$$

$$\longrightarrow T(t) = e^{\pm m t} = e^{\pm\sqrt{\lambda}\,i t}$$

$$T(t) = A_n \cos(\sqrt{\lambda}\,t) + B_n\,\mathrm{sen}(\sqrt{\lambda}\,t)$$

$$T_n(t) = A_n \cos(\sqrt{\lambda_n}\,t) + B_n\,\mathrm{sen}(\sqrt{\lambda_n}\,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{3x}{2}}\,\mathrm{sen}(n\pi x)\left(A_n\cos(\sqrt{\lambda_n}\,t) + B_n\,\mathrm{sen}(\sqrt{\lambda_n}\,t)\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} e^{\frac{3x}{2}}\,\mathrm{sen}(n\pi x)\left(-A_n\sqrt{\lambda_n}\,\mathrm{sen}(\sqrt{\lambda_n}\,t) + B_n\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\,t)\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} \underbrace{e^{\frac{3x}{2}}}_{>0}\,\underbrace{\mathrm{sen}(n\pi x)}_{\text{Base ortonormal series Fourier}}\,A_n = 0$$

$$\implies A_n = 0 \;\forall n.$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n\, e^{\frac{3x}{2}}\,\mathrm{sen}(n\pi x)\cos(\sqrt{\lambda_n}\,t) = e^{\frac{3x}{2}}$$

$$\implies \sum_{n=1}^{\infty} B_n\,\mathrm{sen}(n\pi x)\cos(\sqrt{\lambda_n}\,t) = 1$$

Aplicamos Series de Fourier para hallar B_n