Problema 1. Encuentre la divergencia y el rotacional de los siguientes campos vectoriales:

a) $\vec{F}(x,y,z) = (\cos(yz) - x, \cos(xz) - y, \cos(xy) - z)$

b) $\vec{F}(x,y,z) = (y^2 z, e^{xyz}, x^2 z)$

c) $\vec{F}(x,y,z) = (xz - e^{2x}\cos(z), -yz, e^{2x}(\sin(y) + 2\sin(z)))$

$$F = (f_1, f_2, f_3)$$

$$\text{Rot}(F) = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{Div}(F) = \nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

c) $F(x,y,z) = (f_1, f_2, f_3)$

$f_1 = \cos(yz) - x$

$f_2 = \cos(xz) - y$

$f_3 = \cos(xy) - z$

$$\frac{\partial f_1}{\partial x} = -1 \qquad \frac{\partial f_1}{\partial y} = -z\,\text{sen}(yz) \qquad \frac{\partial f_1}{\partial z} = -y\,\text{sen}(yz)$$

$$\frac{\partial f_2}{\partial y} = -1 \qquad \frac{\partial f_2}{\partial x} = -z\,\text{sen}(xz) \qquad \frac{\partial f_2}{\partial z} = -x\,\text{sen}(xz)$$

$$\frac{\partial f_3}{\partial z} = -1 \qquad \frac{\partial f_3}{\partial x} = -y\,\text{sen}(xy) \qquad \frac{\partial f_3}{\partial y} = -x\,\text{sen}(xy)$$

$$\text{div}(F) = -3$$

$$\text{Rot}(F) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\imath} + \left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$= \left(-x\,\text{sen}(xy) + x\,\text{sen}(xz),\ y\,\text{sen}(xy) - y\,\text{sen}(yz),\ -z\,\text{sen}(xz) + z\,\text{sen}(yz)\right)$$

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$$F(x,y,z) = (f_1, f_2, f_3)$$

b)

$$f_1 = y^2 z$$
$$f_2 = e^{xyz}$$
$$f_3 = x^2 z$$

$$\frac{\partial f_1}{\partial x} = 0 \qquad \frac{\partial f_1}{\partial y} = 2yz \qquad \frac{\partial f_1}{\partial z} = y^2$$

$$\frac{\partial f_2}{\partial x} = yze^{xyz} \qquad \frac{\partial f_2}{\partial y} = xze^{xyz} \qquad \frac{\partial f_2}{\partial z} = xye^{xyz}$$

$$\frac{\partial f_3}{\partial x} = 2xz \qquad \frac{\partial f_3}{\partial y} = 0 \qquad \frac{\partial f_3}{\partial z} = x^2$$

$$\operatorname{div}(F) = x^2 + xze^{xyz}$$

$$\operatorname{rot}(F) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \left(-xye^{xyz},\ y^2 - 2xz,\ yze^{xyz} - 2yz\right)$$

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$$F(x,y,z) = (f_1, f_2, f_3)$$

$$f_1 = xz - e^{2x}\cos z$$

$$f_2 = -yz$$

$$f_3 = e^{2x}(\operatorname{sen} y + 2\operatorname{sen} z)$$

$$\frac{\partial f_1}{\partial x} = z - 2e^{2x}\cos z \qquad \frac{\partial f_1}{\partial y} = 0 \qquad \frac{\partial f_1}{\partial z} = x + e^{2x}\operatorname{sen} z$$

$$\frac{\partial f_2}{\partial x} = 0 \qquad \frac{\partial f_2}{\partial y} = -z \qquad \frac{\partial f_2}{\partial z} = -y$$

$$\frac{\partial f_3}{\partial x} = 2e^{2x}(\operatorname{sen} y + 2\operatorname{sen} z) \qquad \frac{\partial f_3}{\partial y} = e^{2x}\cos y \qquad \frac{\partial f_3}{\partial z} = 2e^{2x}\cos z$$

$$\operatorname{div}(F) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = z - 2e^{2x}\cos z - z + 2e^{2x}\cos z = 0$$

$$\operatorname{ROT}(F) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \left(e^{2x}\cos y + y,\ x - e^{2x}(\operatorname{sen} y + 3\operatorname{sen} z),\ 0\right)$$

Problema 2. Sea $\vec{F}(x,y) = (u(x,y), -v(x,y))$ un campo vectorial incompresible e irrotacional de clase C^2.

(a) Muestre que las funciones u, v satisfacen las **ecuaciones de Cauchy-Riemann.**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b) Muestre que u, v son funciones harmónicas, es decir

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

a) Incompresible $\Rightarrow$ $\operatorname{div}(F) = 0$

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial u}{\partial x} + \left(-\frac{\partial v}{\partial y}\right) = 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Irrotacional

$$\operatorname{Rot}(F) = \vec{0}$$

Note que $F: \mathbb{R}^2 \to \mathbb{R}^2$, pero es posible extender este campo a $\mathbb{R}^3$ y aplicar la fórmula

$$\operatorname{Rot}(F) = \nabla \times F$$

(Recuerde que el producto cruz solo tiene sentido en $\mathbb{R}^3$)

Luego

$$\tilde{F}(x,y,z) = (u(x,y), -v(x,y), 0)$$

$$\operatorname{Rot}(\tilde{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ u & -v & 0 \end{vmatrix} = \hat{\imath}\left(0 - \cancel{\frac{\partial v}{\partial z}}^{\,0}\right) - \hat{\jmath}\left(0 - \cancel{\frac{\partial u}{\partial z}}^{\,0}\right) + \hat{k}\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \left(0, 0, -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (0, 0, 0)$$

Luego

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Problema 2. Sea $\vec{F}(x,y) = (u(x,y), -v(x,y))$ un campo vectorial incompresible e irrotacional de clase C^2.

(a) Muestre que las funciones u, v satisfacen las **ecuaciones de Cauchy-Riemann.**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(b) Muestre que u, v son funciones harmónicas, es decir

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

b) Derivando $\partial u/\partial x$ con respecto a x y $\partial u/\partial y$ con respecto a y en la parte a) se tiene que

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial y\, \partial x} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial x\, \partial y}$$

Como $v \in C^2$ entonces

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Haciendo lo mismo para $\partial v/\partial x$, $\partial v/\partial y$ se sigue

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial u}{\partial y\, \partial x} \qquad \frac{\partial^2 v}{\partial y^2} = \frac{\partial u}{\partial x\, \partial y}$$

Como $u \in C^2$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Problema 3. Muestre las siguientes identidades

a) Si f, g son funciones de clase C^2, entonces $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.

b) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

c) $\nabla \cdot (f\nabla f) = ||\nabla f||^2 + f\nabla^2 f$.

donde $\nabla^2 = \nabla \cdot \nabla$

a) Recordemos la propiedad

Divergencia

producto punto

$$\nabla \cdot (ab) = \nabla a \cdot b + b(\nabla \cdot a)$$

a, b funciones

$$\nabla \cdot (\alpha f + \beta) = \alpha \nabla \cdot f + \beta$$

$\alpha, \beta \in \mathbb{R}$

$$\nabla \cdot (\varphi \mathbf{F}) = (\nabla \varphi) \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F}).$$

Gradiente

$$\nabla(fg) = f\nabla g + g\nabla f$$

producto de funciones

Luego

$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (f\nabla g + g\nabla f)$$

$$= \nabla \cdot (f\nabla g) + \nabla \cdot (g\nabla f)$$

$$= \nabla f \cdot \nabla g + f(\nabla \cdot \nabla g) + \nabla g \cdot \nabla f + g(\nabla \cdot \nabla f)$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \nabla g$$

Problema 3. Muestre las siguientes identidades

a) Si f, g son funciones de clase C^2, entonces $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.

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c) $\nabla \cdot (f\nabla f) = ||\nabla f||^2 + f\nabla^2 f$.

donde $\nabla^2 = \nabla \cdot \nabla$

b)

$$\nabla \cdot (f\nabla g - g\nabla f) = \nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f)$$

$$= \cancel{\nabla f \cdot \nabla g} + f\nabla \cdot \nabla g - (\cancel{\nabla g \cdot \nabla f} + g\nabla \cdot \nabla f)$$

$$= f\nabla^2 g - g\nabla^2 f$$

$$\nabla \cdot (\varphi \mathbf{F}) = (\nabla \varphi) \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F}).$$

c)

$$\nabla \cdot (f\nabla f) = \nabla f \cdot \nabla f + f\nabla \cdot \nabla f$$

$$= ||\nabla f||^2 + f\nabla^2 f$$

Problema 4. Sea $\vec{r} = (x, y, z)$ y supongamos que r denota $\|\vec{r}\|$. Verifique las siguientes identidades.

a) $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$

b) $\nabla \times (r^n \vec{r}) = \vec{0}$

a) $\nabla \cdot (r^n \vec{r}) = \nabla \cdot \Big(\underbrace{r(x,y,z)}_{\in \mathbb{R}}{}^n \, \underbrace{\vec{r}(x,y,z)}_{\in \mathbb{R}^n} \Big)$

$$\nabla \cdot (\varphi \mathbf{F}) = (\nabla \varphi) \cdot \mathbf{F} + \varphi (\nabla \cdot \mathbf{F}).$$

$= \underbrace{(\nabla (r^n))}_{\text{regla de la cadena}} \cdot \vec{r} + r^n \underbrace{(\nabla \cdot \vec{r})}$

$\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$= (n r^{n-1} \nabla r) \cdot \vec{r} + 3 r^n$

Notar que

$r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$

$\nabla r = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right) = \frac{1}{r}(x, y, z) = \frac{1}{r}\vec{r}$

Luego

$= n \frac{r^{n-1}}{r} \vec{r} \cdot \vec{r} + 3 r^n$

$\vec{r} \cdot \vec{r} = \|\vec{r}\|^2 = r^2$

$= n r^{n-\cancel{2}} \cancel{r^2} + 3 r^n = (n+3) r^n$

Problema 4. Sea $\vec{r} = (x, y, z)$ y supongamos que r denota $||\vec{r}||$. Verifique las siguientes identidades.

a) $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$
b) $\nabla \times (r^n \vec{r}) = \vec{0}$

Solución:

a) $\nabla \cdot (r^n \vec{r}) = r^n \nabla \vec{r} + \vec{r} \cdot \nabla r^n = 3r^n + n\vec{r} \cdot r^{n-2} \vec{r} = (n+3) r^n$.
b) $\nabla \times (r^n \vec{r}) = \vec{0} = r^n \nabla \times \vec{r} + \nabla r^n \times \vec{r} = \vec{0} + n r^{n-2} \vec{r} \times \vec{r} = \vec{0}$.

$$\nabla \times (\varphi \mathbf{F}) = \nabla \varphi \times \mathbf{F} + \varphi \nabla \times \mathbf{F}$$

b) $\nabla \times (r^n \vec{r}) = \nabla (r^n) \times \vec{r} + r^n \nabla \times \vec{r}$

por a)

$= n \dfrac{r^{n-1}}{r} \vec{r} \times \vec{r} + r^n \nabla \times \vec{r}$

$= \underbrace{n r^{n-2} \vec{r} \times \vec{r}}_{\text{"}0} + r^n \nabla \times \vec{r}$

$\vec{V} \times \vec{V} = 0$

$= r^n \nabla \times \vec{r}$

Note que $\mathrm{rot}(\vec{r}) = \vec{0}$ pues

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial x & \partial y & \partial z \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, -\frac{\partial z}{\partial x} + \frac{\partial x}{\partial z}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right)$$

$= (0, 0, 0)$

Luego se tiene lo pedido.