$$\iint_W \overset{(u,v)}{f\,|J_T|\,du\,dv} = \iint_S \overset{(x,y)}{f\,dx\,dy}$$

$$\underbrace{\iint dx\,dy}_{\text{Area}}$$

$x = v$
$y = -v = -x$
$y = -x$

Problema 1. Considerar la transformación T definida por las ecuaciones

$$x = u + v, \qquad y = u^2 - v.$$

a. Determine el jacobiano de la transformación.
b. Un triángulo W en el plano uv tiene vértices en $(0,0)$, $(2,0)$, $(0,2)$. Representar mediante un gráfico, la región de la imagen $T(W) = S$ en el plano xy.
c. Verifique el teorema del cambio de variables calculando el área de S directamente y con la transformación antes definida.
d. Calcular la integral

$$\iint_S (y - x + 1)\, dA$$

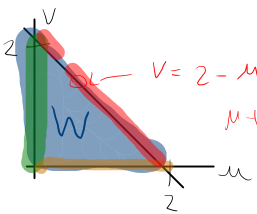
a) $T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u+v \\ u^2 - v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$J_T = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J_T = \begin{vmatrix} 1 & 1 \\ 2u & -1 \end{vmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \to \left(\frac{\partial x}{\partial u} \;\; \frac{\partial x}{\partial v}\right.$$

$$= -1 - 2u$$

b)



$v = 2 - u$

$u + v = 2$

$T(W) = \{ \;? \le x \le ?, \;\; ? \le y \le ? \}$

$$W = \left\{ (u,v) \in \mathbb{R}^2 : \begin{array}{l} 0 \le u \le 2 \\ 0 \le v \le 2 - u \end{array} \right\}$$

¿Qué pasa en los bordes de W?

$u = 0 \quad 0 \le v \le 2 \quad \leftarrow$ eje v

$T\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x = v \\ y = -v \end{array}$

$\Rightarrow \boxed{y = -x} \; \ell_1$

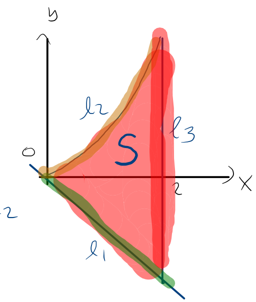
$0 \le u \le 2 \quad v = 0 \quad \leftarrow$ eje u

$T\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ u^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x = u \\ y = u^2 \end{array}$

$\Rightarrow \boxed{y = x^2} \; \ell_2$

$u + v = 2 \Rightarrow \boxed{x = 2} \; \ell_3$

$$W = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le x \le 2 \\ -x \le y \le x^2 \end{array} \right\}$$



c) Calculamos con u y v

$$\int_0^2 \int_0^{2-u} |J_T| \cdot dv\,du$$

$-1 - 2u \ge 0$
$-\frac{1}{2} \ge u$

$-1 - 2u \le 0$
$-(1 + 2u) \le 0$
$2u + 1 \ge 0$

$$\int_0^2 \int_0^{2-u} 2u + 1 \; dv\,du$$

$$= \int_0^2 (2u+1)(2-u)\,du$$

$$= \int_0^2 4u - 2u^2 + 2 - u \; du$$

$$= \int_0^2 -2u^2 + 3u + 2 \; du$$

$$= -\frac{2u^3}{3} + \frac{3}{2}u^2 + 2u \Big|_0^2$$

$$= \frac{14}{3}$$

$$\int_0^2 \int_{-x}^{x^2} dy\,dx$$

$x^2-(-x)$

Calculamos con respecto a x, y

$$\int_0^2 \int_{-x}^{x^2} dy\,dx = \int_0^2 x^2+x \, dx$$
$$= \frac{x^3}{3}+\frac{x^2}{2}\Big|_0^2$$
$$= \frac{14}{3}$$

d

$$\iint_S y-x+1 \, dy\,dx =$$

Valor abs del jacobiano

$$= \iint_D (u^2-v-u-v+1)(2u+1)\,d\ \ u$$
$$= \int_0^2 \int_0^{2-u} ((u^2-u+1)-2v)(2u+1)\,dv\,du$$
$$= \int_0^2 \left[\int_0^{2-u} (u^2-u+1)(2u+1)\,dv - (2u+1)\int_0^{2-u} 2v\,dv\right] du$$
$$= \int_0^2 (u^2-u+1)(2u+1)\,v\Big|_{v=0}^{v=2-u} - (2u+1)v^2\Big|_{v=0}^{v=2-u} du$$
$$= \int_0^2 (u^2-u+1)(2u+1)(2-u) - (2u+1)(2-u)^2\,du$$
$$= \int_0^2 (2u+1)(2-u)(u^2-u+1-(2-u))\,du$$
$$= \int_0^2 (2u+1)(2-u)(u^2-1)\,du$$
$$= -\frac{2u^5}{5}+\frac{3}{4}u^4+\frac{4}{3}u^3-\frac{3}{2}u^2-2u\Big|_0^2$$
$$= -\frac{2}{15}$$

$v\,d$

Problema 2. Calcule la integral

$$\iint_R \frac{x}{(x^2+y^2)^2}\,dA$$

donde $R=\{(x,y)\in\mathbb{R}^2,\ x+y\geq 2,\quad x^2+y^2\leq 2y\}$.

$y \geq 2-x$

$R = \{(x,y)\in\mathbb{R}^2 \mid x+y\geq 2,\ x^2+y^2\leq 2y\}$

$\iff x^2+y^2-2y\leq 0 \quad /+1$

$x^2+y^2-2y+1\leq 1$

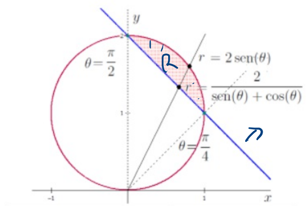
$x^2+(y-1)^2\leq 1$

$x = r\cos\theta$

$y = r\,\text{sen}\,\theta$

$R=\{(r,\theta)\in\mathbb{R}^2 \mid r(\cos\theta+\text{sen}\,\theta)\geq 2,\ r^2\leq 2\,\text{sen}\,\theta\}$

$=\left\{(r,\theta)\in\mathbb{R}^2 \mid r\geq \frac{2}{\cos\theta+\text{sen}\,\theta},\ r^2\leq 2\,\text{sen}\,\theta\right\}$



Luego

Jacobiano

$$\iint_R \frac{x}{(x^2+y^2)^2}\,dy\,dx = \int_{\pi/4}^{\pi/2}\int_{\frac{2}{\text{sen}\,\theta+\cos\theta}}^{2\,\text{sen}\,\theta} \frac{r\cos\theta}{r^4}\cdot r\,dr\,d\theta$$

$$\iint_R \frac{x}{(x^2+y^2)^2}\,dy\,dx = \int_{\pi/4}^{\pi/2}\int_{\frac{2}{\text{sen}\,\theta+\cos\theta}}^{2\,\text{sen}\,\theta} \frac{r\cos\theta}{r^4}\cdot r\,dr\,d\theta$$

$\hookrightarrow \frac{1}{r^2}$

$$= \int_{\pi/4}^{\pi/2} \cos\theta\left(-\frac{1}{r}\right)\Bigg|_{r=\frac{2}{\text{sen}\,\theta+\cos\theta}}^{r=2\,\text{sen}\,\theta} dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta\left(\frac{\text{sen}\,\theta+\cos\theta}{2} - \frac{1}{2\,\text{sen}\,\theta}\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \text{sen}\,\theta\cos\theta + \cos^2\theta - \frac{\cos\theta}{\text{sen}\,\theta}\,d\theta$$

$\int \underbrace{\text{sen}}_{u}\,\underbrace{\cos}_{du}$

$\int u\,du$

$\frac{u^2}{2}$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \text{sen}\,\theta\cos\theta + \left(\frac{1-\cos(2\theta)}{2}\right) - \frac{\cos\theta}{\text{sen}\,\theta}\,d\theta$$

regla coseno [illegible] 2

$$= \frac{1}{2}\left(\frac{1}{2}\text{sen}^2\theta + \frac{\theta}{2} - \frac{\text{sen}(2\theta)}{4} - \ln(\text{sen}\,\theta)\right)\Bigg|_{\theta=\pi/4}^{\theta=\pi/2}$$

$$= \frac{\pi}{16} - \frac{1}{4}\ln 2$$

$R = R_1 \cup R_2$

Problema 3. Dada la siguiente integral

$$I = \underbrace{\int_{-3}^{1}\int_{-\sqrt{1-y}}^{y+1} f(x,y)\,dx dy}_{R_1} + \underbrace{\int_{1}^{2}\int_{1-\sqrt{2y-y^2}}^{1+\sqrt{2y-y^2}} f(x,y)\,dx dy}_{R_2}.$$

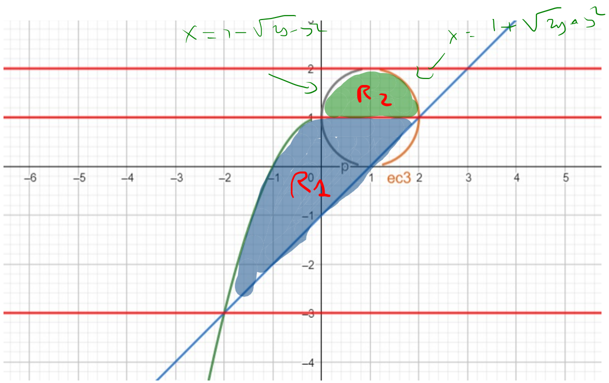
Cambie el orden de integración.

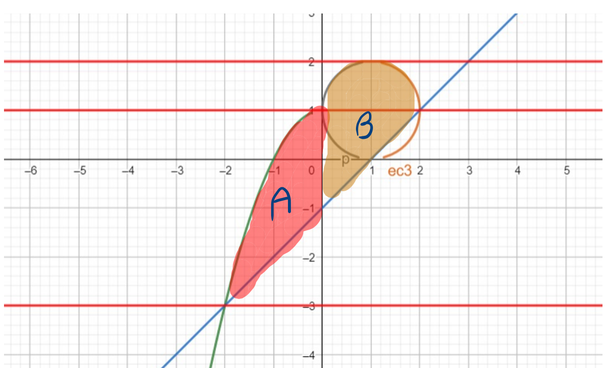
Región a integrar $= R$

$R = R_1 \cup R_2$

$$R_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -3 \leq y \leq 1 \\ -\sqrt{1-y} \leq x \leq y+1 \end{array} \right\}$$

$$R_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 1 \leq y \leq 2 \\ 1-\sqrt{2y-y^2} \leq x \leq 1+\sqrt{2y-y^2} \end{array} \right\}$$





$R = A \cup B$

$$A = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -2 \leq x \leq 0 \\ x-1 \leq y \leq 1-x^2 \end{array} \right\}$$

$x = -\sqrt{1-y}$

$x^2 = 1-y$

$y = 1-x^2$

$$B = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \leq x \leq 2 \\ x-1 \leq y \leq 1+\sqrt{2x-x^2} \end{array} \right\}$$

$x = 1 \pm \sqrt{2y - y^2}$

$2y - y^2 = (1-x)^2$

$y^2 - 2y + 1 = 1 - (1-x)^2$

$(y-1)^2 = \cancel{1} - (\cancel{1} - 2x + x^2)$

$y = 1 + \sqrt{2x - x^2}$

$$\int_{-2}^{0}\int_{x-1}^{1-x^2} f\,dy dx + \int_{0}^{2}\int_{x-1}^{1+\sqrt{2x-x^2}} f\,dy dx$$

Problema 4. Sea D la región del plano delimitada por las curvas $xy = 1$, $y = 2x$, $x = 2$. Determine el valor de

$$\iint_D min\{x, y\}\, dx dy.$$

$y = \frac{1}{x}$ $\quad$ $y = 2x$ $\quad$ $x = 2$

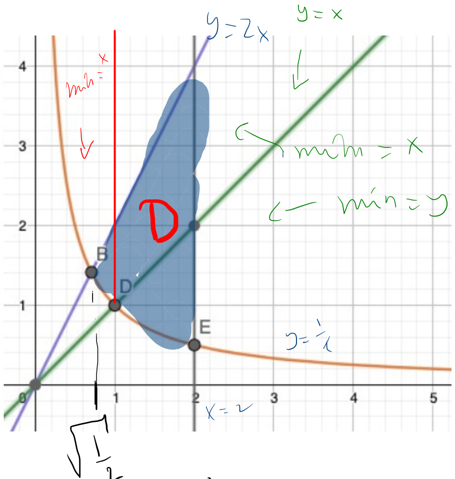
Entre $\sqrt{\frac{1}{2}} \leq x \leq 1$

$\frac{1}{x} \leq y \leq 2x$ y también $y \geq x$

Entre $1 \leq x \leq 2$

$x \geq y$ para $\frac{1}{x} \leq y \leq x$

$y \geq x$ para $x \leq y \leq 2x$



Luego

$$\iint_D \text{mín}\{x, y\}\, dx\, dy = \overbrace{\int_{\sqrt{\frac{1}{2}}}^{1} \int_{1/x}^{2x} x\, dy\, dx}^{I_1} + \overbrace{\int_1^2 \int_{1/x}^{x} y\, dy\, dx}^{I_2} + \underbrace{\int_1^2 \int_x^{2x} x\, dy\, dx}_{I_3}$$

$D_1 = \{$

$D_2 :$

$D_3 :$

$D_1 = \sqrt{\frac{1}{2}} \leq x \leq 1$

$1 \leq x \leq 2$

$1 \leq x \leq 2$

$$I_1 = \int_{\sqrt{\frac{1}{2}}}^{1} \int_{1/x}^{2x} x \, dy \, dx = \int_{\sqrt{1/2}}^{1} x\left(2x - 1/x\right) dx$$

$$= \int_{\sqrt{1/2}}^{1} 2x^2 - 1 \, dx = \left(\frac{2}{3}x^3 - x\right)\Big|_{\sqrt{1/2}}^{1} = \frac{1}{3}\left(\sqrt{2} - 1\right)$$

$$I_2 = \int_1^2 \int_{1/x}^{x} y \, dy \, dx = \int_1^2 \frac{y^2}{2}\Big|_{y=1/x}^{y=x} dx$$

$$= \int_1^2 \frac{x^2}{2} - \frac{1}{2x^2} \, dx$$

$$= \left(\frac{x^3}{6} + \frac{1}{2x}\right)\Big|_2^1 \approx 0{,}92$$

$$I_3 = \int_1^2 \int_x^{2x} x \, dy \, dx = \int_1^2 x(2x - x) \, dx$$

$$= \int_1^2 x^2 dx$$

$$= \frac{x^3}{3}\Big|_1^2 = \frac{7}{3}$$

$\int_0^x I(t) \cdot 1 \cdot dt$

$I(t) = \int_0^t f(u)\,du$

$I'(t) = f(t)$

$I(t)$

Problema 5. Demuestre que

$$\int_0^x \boxed{\int_0^t f(u)\,du}\,dt = \int_0^x (x-u) f(u)\,du.$$

Sea $I(t) = \int_0^t f(u)\,du$, luego

$$\int_0^x \underbrace{I(t)}_{u} \cdot \underbrace{1\,dt}_{dv} = t\,I(t)\Big|_0^x - \int_0^x t\,\overbrace{f(t)}^{I'(t)}\,dt$$

$du = I'(t) = f(t)\,dt$

$v = t$

$$= x\,I(x) - \int_0^x t\,f(t)\,dt$$

$\int_0^x s\,f(s)\,ds$

$\int_0^x w\,f(w)\,dw$

$\int_0^x \alpha\,f(\alpha)\,d\alpha$

$$= x\int_0^x f(u)\,du - \int_0^x t\,f(t)\,dt$$

da lo mismo usar t ó u

$$= x\int_0^x f(u)\,du - \int_0^x u\,f(u)\,du$$

$$= \int_0^x (x-u)\,f(u)\,du.$$