$x \geq 0$

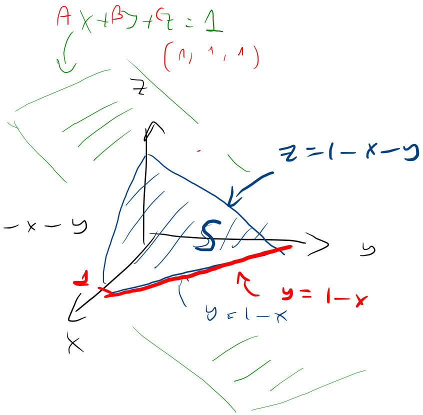
$y \geq 0$

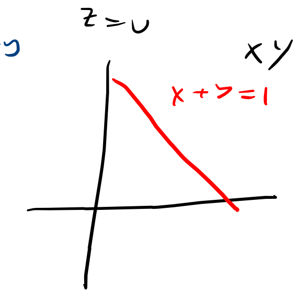
$z \geq 0$

Problema 1. Calcular las integrales:

(a) $\displaystyle\iint_S xyz\, dS$, donde S es la parte del plano $x+y+z = 1$ que está en el primer octante.

(b) $\displaystyle\iint_S xy\, dS$, donde S es la parte del cilindro $x^2+z^2 = 1$ que se encuentra en el primer cuadrante y esta acotada por el plano $y = x$.





(a) Parametrizar S $\qquad z = 1 - x - y$

$\varphi(x,y) = (x, y, 1-x-y)$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

$\varphi_x = (1, 0, -1)$

$\varphi_y = (0, 1, -1)$

$\vec{n} = (1, 1, 1) \qquad \|\vec{n}\| = \sqrt{3}$

$\vec{n} = \varphi_x \times \varphi_y$

$\varphi_y \times \varphi_x$

$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \\ 0 \leq z \leq 1-x-y \end{array} \right\}$

$\displaystyle\iint_S xyz\, dS = \int_0^1 \int_0^{1-x} xy(1-x-y) \cdot \|\vec{n}\|\, dy\, dx$

$\rightsquigarrow f(x,y,z)$

$f(\varphi(x,y))$

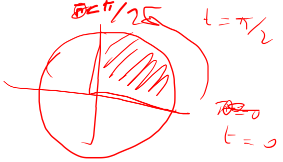
$\displaystyle = \sqrt{3} \int_0^1 \int_0^{1-x} xy - x^2y - xy^2\, dy\, dx$

$\displaystyle = \sqrt{3} \int_0^1 \left(x\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3} \right) \Bigg|_{y=0}^{y=1-x} dx$

$\displaystyle = \sqrt{3} \int_0^1 \frac{1}{2}(x-x^2)(1-x)^2 - x\frac{(1-x)^3}{3}\, dx$

$\displaystyle = \sqrt{3} \int_0^1 \frac{1}{2}x(1-x)^3 - \frac{1}{3}x(1-x)^3\, dx$

$\displaystyle = \frac{\sqrt{3}}{6} \int_0^1 x(1-x)^3\, dx = \frac{\sqrt{3}}{120}$



Problema 1. Calcular las integrales:

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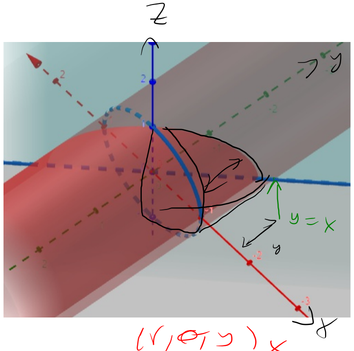
(b) $\displaystyle\iint_S xy\, dS$, donde S es la parte del cilindro $x^2+z^2 = 1$ que se encuentra en el primer ~~cuadrante y~~ esta acotada por el plano $y = x$.

$$\left.\begin{array}{l} x^2+z^2=1 \\ x=y \end{array}\right\rfloor$$

$$y^2 = 1 - z^2$$

$\sqrt{1-\sin^2 t} = \cos t$



(r,θ,y) x

$\varphi(\theta,y)$ (θ,y)

$$\varphi:\begin{cases} x = \cos t \\ y = y \\ z = \sin t \end{cases} \qquad \begin{array}{l} 0 \le y \le \sqrt{1-z^2} \\ 0 \le y \le \sin t \end{array}$$

$$S = \left\{ (y,t) \;\middle|\; \begin{array}{l} 0 \le y \le \cos t \\ 0 \le t \le \pi/2 \end{array} \right\}$$

$$\varphi(y,t) = (\cos t,\, y,\, \sin t)$$

$$\varphi_y = (0, 1, 0)$$

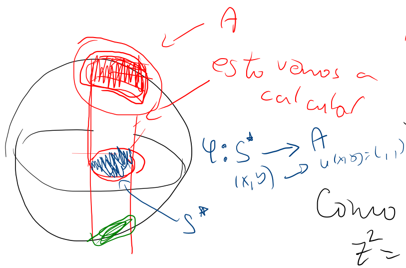
$$\varphi_t = (-\sin t, 0, \cos t)$$

$$\vec{n} = \varphi_y \times \varphi_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{vmatrix} = (\cos t, 0, \sin t)$$

$$\|\vec{n}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\iint_S xy\, dS = \int_0^{\pi/2}\int_0^{\cos t} y\cos t \, dy\, dt = \frac{1}{3}$$

Problema 2. Hallar el área la porción de la esfera $x^2+y^2+z^2=a^2$ incluida dentro del cilindro $x^2+y^2=ay$, con $a>0$.



$$\text{Área} = 2A$$

$$S^* = \{(x,y) \mid x^2+y^2 \le ay\}$$

Como $x^2+y^2+z^2=a^2$, entonces

$$z^2 = a^2-(x^2+y^2)$$

Restringiendonos a A se tiene

$$z = \sqrt{a^2-(x^2+y^2)}$$

Así $\phi(x,y) = (x, y, \sqrt{a^2-x^2-y^2})$

$$\phi_x = \left(1, 0, \frac{x}{\sqrt{a^2-x^2-y^2}}\right)$$

$$\phi_y = \left(0, 1, \frac{y}{\sqrt{a^2-x^2-y^2}}\right)$$

$$\vec{n} = \phi_x \times \phi_y = \left(\frac{-x}{\sqrt{a^2-x^2-y^2}}, \frac{-y}{\sqrt{a^2-x^2-y^2}}, 1\right) \leftarrow \left(\frac{-x}{\sqrt{\ }}, -\frac{y}{\sqrt{\ }}, 1\right)$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{x}{\sqrt{\ }} \\ 0 & 1 & \frac{y}{\sqrt{\ }} \end{vmatrix}$$

$$\|\vec{n}\| = \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

$$A = \iint_{S^*} dS = \iint_{S^*} \|\vec{n}\|\, dA = \iint_{\{x^2+y^2\le ay\}} \overbrace{\frac{a}{\sqrt{a^2-x^2-y^2}}}^{\|\vec{n}\|}\, dA$$

$r^2 \le ar\,\text{sen}\theta$

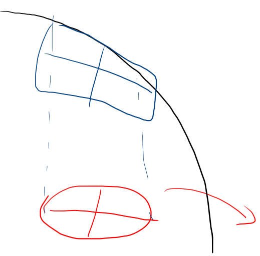
$r \le a\,\text{sen}\theta$

Cambio Coordenadas $\begin{cases} x(\theta) = r\cos\theta \\ y(\theta) = r\,\text{sen}\theta \end{cases}$

$$J = r$$

$$r^2 \le ar\,\text{sen}\theta$$

$$\boxed{\begin{array}{c} 0 \le r \le a\,\text{sen}\theta \\ 0 \le \theta \le \pi \end{array}}$$



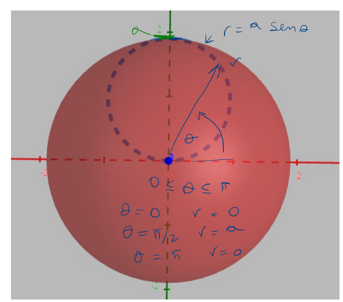
$$= 2\int_0^{\pi/2}\int_0^{a\,\text{sen}\theta} \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \left(-a\sqrt{a^2-r^2}\right)\Big|_{r=0}^{r=a\,\text{sen}\theta} d\theta$$

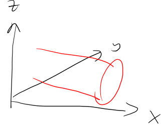
$$= -2a^2\int_0^{\pi/2} (\cos\theta - 1)\, d\theta$$

$$= -2a^2\left[\text{sen}\theta - \theta\right]\Big|_0^{\pi/2} = \frac{2\pi a^2 - 4a^2}{2}$$

Plano XY



$$\text{Área} = 2\pi a^2 - 4a^2$$

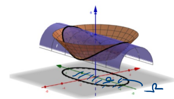


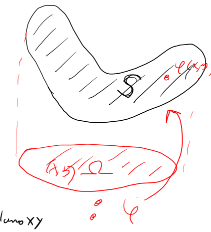
Problema 3. Sea, $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto z$ una función. Dado $a > 0$ definamos la superficie que sigue:

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ x^2 + z^2 = 2az\,,\ z \geq \sqrt{x^2+y^2} \right\}$$

Es decir, S es la parte de la superficie $x^2 + z^2 = 2az$, $(a > 0)$, recortada por la superficie $z = \sqrt{x^2+y^2}$. Calcular la integral $\displaystyle\iint_S z\,dS$.

$$\left.\begin{array}{l} x^2 + z^2 = 2az \\ z = \sqrt{x^2+y^2} \end{array}\right| \Longrightarrow 2x^2 + y^2 = 2a\sqrt{x^2+y^2}$$





Haciendo coordenadas polares $\quad x = r\cos\theta$, $y = r\,\mathrm{sen}\,\theta$

$$2r^2\cos^2\theta + r^2\,\mathrm{sen}^2\theta = 2a\,r$$

$$r^{\cancel{2}}\cos^2\theta + r^{\cancel{2}} = 2a\cancel{r}$$

$$r = \frac{2a}{1+\cos^2\theta} \qquad 0 \leq \theta \leq 2\pi$$

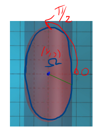
Dominio

Parametrización

$\varphi : \Omega \to S$

Luego $= \{(x,y) \in \mathbb{R}^2 \mid$

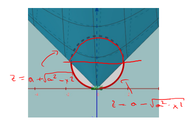
$$\Omega = \left\{ (r,\theta) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \leq r \leq \frac{2a}{1+\cos^2\theta} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$



Plano XY

$$x^2 + z^2 = 2az \longrightarrow x^2 + (z-a)^2 = a^2$$

$$z = a + \sqrt{a^2 - x^2}$$



Como el cono jamás intersecta a $z = a - \sqrt{a^2-x^2}$, entonces este caso no se analiza, luego

Parametrización

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = a + \sqrt{a^2-x^2}\}$$

Luego una parametrización de S es

$$\varphi(x,y) = (x, y, a + \sqrt{a^2-x^2})$$

$$\varphi_x = \left(1, 0, \frac{x}{\sqrt{a^2-x^2}}\right)$$

$$\varphi_y = (0, 1, 0)$$

$$\vec{n} = \varphi_x \times \varphi_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{x}{\sqrt{a^2-x^2}} \\ 0 & 1 & 0 \end{vmatrix} = \left(-\frac{x}{\sqrt{a^2-x^2}}, 0, 1\right)$$

$$\|\vec{n}\| = \sqrt{\frac{x^2}{a^2-x^2}+1} = \frac{a}{\sqrt{a^2-x^2}}$$

$f(\varphi(x,y))$

$$\iint_S z\,dS = \iint_\Omega \underbrace{\left(a + \sqrt{a^2-x^2}\right)}_{z} \underbrace{\frac{a}{\sqrt{a^2-x^2}}}_{\|\vec{n}\|}\,dA \qquad \iint f(\varphi(x,y))\,\|\vec{n}\|$$

$$= \iint_\Omega \left(\frac{a^2}{\sqrt{a^2-x^2}} + a\right) dA \qquad J = r \qquad \begin{array}{l} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \end{array}$$

Simetría

Coord Polares

$$= 4\int_0^{\pi/2}\int_0^{\frac{2a}{1+\cos^2\theta}} \left(\frac{a^2}{\sqrt{a^2 - r^2\cos^2\theta}} + a\right) r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{2a} \frac{a^2 r}{\sqrt{a^2 - r^2\cos^2\theta}} + ar \quad dr\,d\theta$$

$$= \frac{7\sqrt{2}}{2}a^3\pi$$

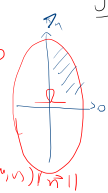
$\varphi(u,v)$

$(u,v) \in \Omega$

$0 \leq u \leq 1$

$2 \leq v \leq 3$

$$\int_0^1\int_2^3 f(\varphi(u,v))\|\vec{n}\|$$

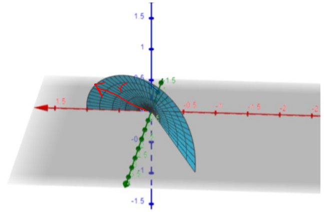


$$\varphi : \underset{\text{Dom}}{A}^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

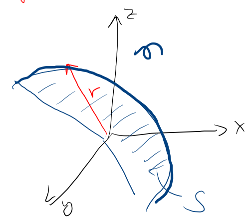
$$\varphi(A) = S$$

Problema 4. Sea γ la hélice $(\cos(\theta), \theta, \mathrm{sen}(\theta))$ con $0 \le \theta \le \pi$ y S la superficie que se forma al unir cada punto de γ con el origen, mediante un segmento de recta.

Si la densidad de masa es $\delta(x, y, z) = y$, entonces el momento de inercia respecto del eje y es:



$$d = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{1 + \theta^2}$$



■ **Momentos de inercia (segundos momentos) con respecto a los ejes coordenados:**

$I_x = \iint_S (y^2+z^2)\,\rho(x,y,z)\,\mathrm{dS}$, $\quad I_y = \iint_S (x^2+z^2)\,\rho(x,y,z)\,\mathrm{dS}\quad$ y $\quad I_z = \iint_S (x^2+y^2)\,\rho(x,y,z)\,\mathrm{dS}$

$$I_y = \iint_S (x^2+z^2)\, y \, dS$$

Parametrizar S

$x = r\cos\theta$
$y = r\theta$
$z = r\cos\theta$

$$\varphi(r,\theta) = (r\cos\theta, r\theta, r\,\mathrm{sen}\theta)$$

$0 \le r \le 1$ ← cuando $r = 1$ se recupera en $x = \cos t$, $y = \mathrm{sen}\, t$
$0 \le \theta \le \pi$

dominio de φ

cuando $r < 1$ se tienen radios "más cortos"

$\varphi : \Omega \to S \subseteq \mathbb{R}^3$
$\Omega \subseteq \mathbb{R}^2$

$$\varphi_r = (\cos\theta, \theta, \mathrm{sen}\theta)$$
$$\varphi_\theta = (-r\,\mathrm{sen}\theta, r, r\cos\theta)$$

$$\vec{n} = \varphi_r \times \varphi_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \theta & \mathrm{sen}\theta \\ -r\,\mathrm{sen}\theta & r & r\cos\theta \end{vmatrix} =$$

$$= \big(r(\theta\cos\theta - \mathrm{sen}\theta), -r, r(\cos\theta + \theta\,\mathrm{sen}\theta)\big)$$

$\varphi : \Omega \to S$

$$\|\vec{n}\|^2 = r^2\big((\theta\cos\theta - \mathrm{sen}\theta)^2 + (\cos\theta + \theta\,\mathrm{sen}\theta)^2 + 1\big)$$
$$= r^2\big(\theta^2\cos^2\theta + \mathrm{sen}^2\theta - \cancel{2\theta\cos\theta\,\mathrm{sen}\theta} + \cos^2\theta + \theta^2\mathrm{sen}^2\theta + \cancel{2\theta\cos\theta\,\mathrm{sen}\theta} + 1\big)$$

$$\iint_\Omega f(\varphi(u,v))\,\|\vec{n}\|\,dA$$

$$= r^2(2 + \theta^2) \Rightarrow \|\vec{n}\| = r\sqrt{2+\theta^2}$$

$$I_y = \int_0^\pi \int_0^1 \overbrace{(r^2\cos^2\theta + r^2\mathrm{sen}^2\theta)}^{x^2+z^2}\,\overbrace{r\theta}^{y}\,\overbrace{r\sqrt{2+\theta^2}}^{\|\vec{n}\|}\,dr\,d\theta$$
$$= \int_0^\pi \int_0^1 r^4\,\theta\sqrt{2+\theta^2}\,dr\,d\theta$$
$$= \int_0^1 r^4\,dr \int_0^\pi \theta\sqrt{2+\theta^2}\,d\theta \qquad u = \theta^2 \quad du = 2\theta\,d\theta$$
$$= \frac{1}{5}\cdot\frac{1}{2}\int_0^{\pi^2}\sqrt{2+u}\,du$$
$$= \frac{1}{10}\cdot\frac{2}{3}(2+u)^{3/2}\Big|_{u=0}^{u=\pi^2}$$
$$= \frac{1}{15}\Big[(2+\pi^2)^{3/2} - 2^{3/2}\Big] \quad //$$