a)

$$\iiint_\Omega e^{(x^2+y^2+z^2)^{3/2}} (x^2+y^2)\, dz\,dy\,dx$$

$r^2 \operatorname{sen}^2\varphi$

Problema 1. Sobre la región $\Omega \subset \mathbb{R}^3$ se distribuye masa siendo $\rho(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$ la densidad volumétrica (de masa). Para Ω definida como

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 / \quad 9 < x^2+y^2+z^2 < 16, \quad x^2+y^2-z^2 < 1, \quad x^2+y^2-z^2 \geq 0, \quad z \geq 0\}.$$

Exprese la(s) integral(s) iterada(s) para el cálculo de las siguientes cantidades

(a) Momento de inercia con respecto al eje z

(b) Masa de Ω utilizando coordenadas cilíndricas según el orden de integración $drdzd\theta$

$x^2+y^2+z^2$

$x^2+y^2-z^2=0$

$x^2+y^2=z^2$

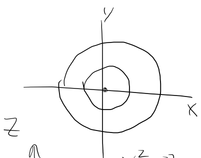
$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 9 < x^2+y^2+z^2 < 16 \\ z^2 \leq x^2+y^2 < z^2+1 \\ z \geq 0 \end{array} \right\}$$

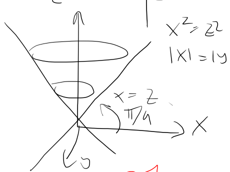
coord. esféricas

$x = r\cos\theta \operatorname{sen}\varphi$

$y = r\operatorname{sen}\theta \operatorname{sen}\varphi$

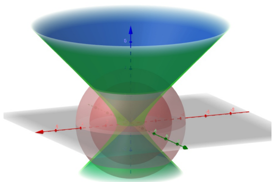
$z = r\cos\varphi$

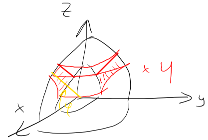


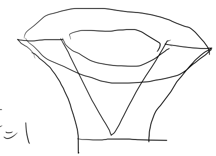


$x^2 = z^2$

$|x| = |y|$



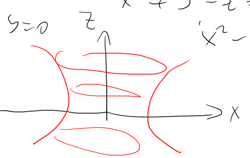




$x^2+y^2-z^2=1$

$y=0$

$x^2-z^2=1$



Buscamos intersecciones

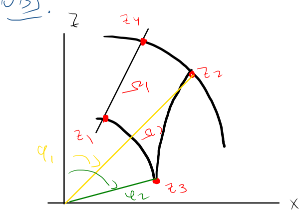
(1) $x^2+y^2+z^2=9$, $x^2+y^2-z^2=0$

(2) $x^2+y^2+z^2=16$, $x^2+y^2-z^2=1$

(3) $x^2+y^2+z^2=9$, $x^2+y^2-z^2=1$

(4) $x^2+y^2+z^2=16$, $x^2+y^2-z^2=0$

OBS.



$z_1 = \sqrt{9/2}$ $\quad z_3 = \sqrt{8/2}$

$z_2 = \sqrt{15/2}$ $\quad z_4 = \sqrt{16/2}$

Coordenadas esféricas

$x = r\cos\theta \operatorname{sen}\varphi$

$y = r\operatorname{sen}\theta \operatorname{sen}\varphi$

$z = r\cos\varphi$

$|J| = r^2 \operatorname{sen}\varphi$

Determinamos φ

$z = z_3 \quad r = 3$

$3\cos\varphi = \sqrt{8/2}$

$\varphi_2 = \arccos(2/3)$

$z = z_2 \quad r = 4$

$4\cos\varphi = \sqrt{15/2}$

$\varphi_1 = \arccos\left(\frac{1}{4}\sqrt{15/2}\right)$

$$\Omega_1 = \left\{ (r,\theta,\varphi) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 3 \leq r \leq 4 \\ 0 \leq \theta \leq 2\pi \\ \frac{\pi}{4} \leq \varphi \leq \varphi_1 \end{array} \right\}$$

$$\Omega_2 = \left\{ (r,\theta,\varphi) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 3 \leq r \leq \sqrt{\frac{1}{-\cos(2\varphi)}} \\ 0 \leq \theta \leq 2\pi \\ \varphi_1 \leq \varphi \leq \varphi_2 \end{array} \right\}$$

$$I_z = \int_{\pi/4}^{\varphi_1}\int_0^{2\pi}\int_3^4 e^{r^3} \underbrace{r^2\operatorname{sen}^2\varphi}_{(x^2+y^2)} \underbrace{r^2\operatorname{sen}\varphi}_{|J|}\, dr\,d\theta\,d\varphi$$

$$+ \int_{\varphi_1}^{\varphi_2}\int_0^{2\pi}\int_3^{\sqrt{\frac{1}{-\cos(2\varphi)}}} e^{r^3} \underbrace{r^2\operatorname{sen}^2\varphi}_{(x^2+y^2)} \underbrace{r^2\operatorname{sen}\varphi}_{|J|}\, dr\,d\theta\,d\varphi$$

$$I_z = \iiint_\Omega (x^2+y^2)\,\rho(x,y,z)\,dV$$

Determinamos r

En $\Omega_1 \rightarrow 3 \leq r \leq 4$

En $\Omega_2 \rightarrow$ Por la figura $r \geq 3$

Usamos coord. esféricas en $x^2+y^2-z^2=1$,

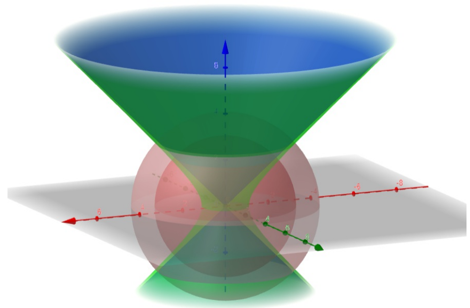
$r^2\operatorname{sen}^2\varphi - r^2\cos^2\varphi = 1$

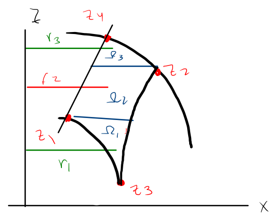
$-r^2\cos(2\varphi) = 1$

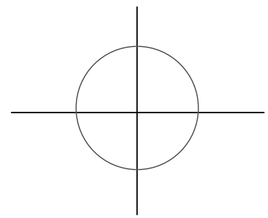
$r = \sqrt{\frac{1}{-\cos(2\varphi)}}$

Luego

$3 \leq r \leq \sqrt{\frac{1}{-\cos(2\varphi)}}$







Problema 1. Sobre la región $\Omega \subset \mathbb{R}^3$ se distribuye masa siendo $\rho(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$ la densidad volumétrica (de masa). Para Ω definida como

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Exprese la(s) integral(s) iterada(s) para el cálculo de las siguientes cantidades

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(b) Masa de Ω utilizando coordenadas cilíndricas según el orden de integración $drdzd\theta$

b) Coord. cilíndricas

$x = r\cos\theta$

$y = r\,\text{sen}\theta$ $\qquad |J| = r$

$z = z$

Determinar z

En Ω_1 $\quad z_3 \leq z \leq z_1$

En Ω_2 $\quad z_1 \leq z \leq z_2$

En Ω_3 $\quad z_2 \leq z \leq z_4$

$$\Omega_1 = \left\{(r,\theta,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \sqrt{9-z^2} \leq r \leq \sqrt{1+z^2} \\ z_3 \leq z \leq z_1 \\ 0 \leq \theta \leq 2\pi \end{array}\right\}$$

$$\Omega_2 = \left\{(r,\theta,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z \leq r \leq \sqrt{1+z^2} \\ z_1 \leq z \leq z_2 \\ 0 \leq \theta \leq 2\pi \end{array}\right\}$$

$$\Omega_3 = \left\{(r,\theta,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z \leq r \leq \sqrt{16-z^2} \\ z_2 \leq z \leq z_4 \\ 0 \leq \theta \leq 2\pi \end{array}\right\}$$

$$\rho(r,\theta,z) = e^{(r^2+z^2)^{3/2}}$$

$$M = \int_0^{2\pi}\int_{z_3}^{z_1}\int_{\sqrt{9-z^2}}^{\sqrt{1+z^2}} e^{(r^2+z^2)^{3/2}}\, r\, dr\,dz\,d\theta$$

$$+ \int_0^{2\pi}\int_{z_1}^{z_2}\int_{\sqrt{9-z^2}}^{\sqrt{1+z^2}} e^{(r^2+z^2)^{3/2}}\, r\, dr\,dz\,d\theta$$

$$+ \int_0^{2\pi}\int_{z_1}^{z_2}\int_{z}^{\sqrt{16-z^2}} e^{(r^2+z^2)^{3/2}}\, r\, dr\,dz\,d\theta$$

Determinar r

En Ω_1:

$x^2+y^2+z^2 \geq 9$

$x^2+y^2-z^2 \leq 1$

$r^2+z^2 \geq 9$

$r^2-z^2 \leq 1$

$\sqrt{9-z^2} \leq r \leq \sqrt{1+z^2}$

En Ω_2:

$x^2+y^2-z^2 \geq 0$

$x^2+y^2-z^2 \leq 1$

$r^2 \geq z^2$

$r^2 \leq 1+z^2$

$z \leq r \leq \sqrt{1+z^2}$

En Ω_3:

$x^2+y^2-z^2 \geq 0$

$x^2+y^2+z^2 \leq 16$

$r^2-z^2 \geq 0$

$r^2+z^2 \leq 16$

$z \leq r \leq \sqrt{16-z^2}$

$X = \rho \cos\theta \operatorname{sen} 4\theta$

$y = \rho \operatorname{sen}\theta \operatorname{sen} 4\theta$

$X^2 + y^2 = \rho^2 \operatorname{sen}^2 4\theta$

$r^2(\theta) = \rho^2 \operatorname{sen}(4\theta)$

$r^2(\theta) = x^2 + y^2$ lemniscata

Coord polares $r(\theta) = \alpha \operatorname{sen}(\beta\theta)$

$X^2 + y^2 = \rho^2 \operatorname{sen}^2 4\theta$

Si consideramos ρ fijo

$x = r\cos\theta \quad y = r \operatorname{sen}\theta$

se tiene que

Problema 2. Dada la función $T : [0, 2\pi] \times \mathbb{R}_0^+ \to \mathbb{R}^2$ definida por

$$T(\theta, \rho) = (\rho \cos\theta \operatorname{sen} 4\theta, \rho \operatorname{sen}\theta \operatorname{sen} 4\theta) = (x, y).$$

(a) Considere el rectángulo $R = [0, \pi/4] \times [0, 2]$. Dibuje la imagen de $T(R) = \Omega$.

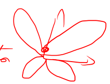
(b) Calcular la masa del cilindro S con base Ω y altura dada por la superficie $z = \sqrt{4 - x^2 - y^2}$ con densidad proporcional a la distancia de cada punto del sólido al plano xy.

Sugerencia: Para obtener la imagen del conjunto R bajo la transformación T es posible utilizar coordenadas polares.

a) $x^2 + y^2 = \rho^2 \operatorname{sen}^2(4\theta)$

Recuerdo de polares

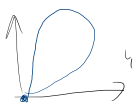
$r(\theta) = \alpha \operatorname{sen}(\beta\theta) \leftarrow$ lemniscata



Para $\rho > 0$ fijo y $x = r\cos\theta \quad y = r\operatorname{sen}\theta$
se tiene que

$$r(\theta) = \rho \underbrace{\operatorname{sen}(4\theta)}_{\text{lemniscata}}$$

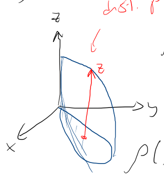
Gráfica



$\rho\cos\theta$

$\rho\operatorname{sen}\theta$

b)



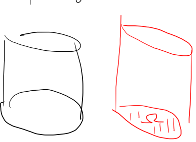
dist. pto. plano xy

z depende de $(x,y) \in \Omega$

$\rho(x,y,z) = K\, z(x,y) \qquad (x,y) \in \Omega$

$= K\sqrt{4 - x^2 - y^2}$

$\rho(\rho, \theta, 4) = K\sqrt{4 - \rho^2 \operatorname{sen}^2(4\theta)}$



$$\begin{matrix} \rho(-\operatorname{sen}\theta \operatorname{sen} 4\theta + 4\cos\theta\cos 4\theta) \\ \rho(\cos\theta \operatorname{sen} 4\theta + 4 \operatorname{sen}\theta \cos 4\theta) \end{matrix}\Bigg],$$

$X = \rho\cos\theta \operatorname{sen}(4\theta)$

$y = \rho \operatorname{sen}\theta \operatorname{sen}(4\theta)$

$z = z$

$$|J| = \begin{vmatrix} \cos\theta \operatorname{sen}(4\theta) & \rho(-\operatorname{sen}\theta \operatorname{sen}4\theta + 4\cos\theta\cos 4\theta) & 0 \\ \operatorname{sen}\theta \operatorname{sen}(4\theta) & \rho(\cos\theta \operatorname{sen}4\theta + 4\operatorname{sen}\theta\cos 4\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \operatorname{sen}^2 4\theta$$

en valor absoluto

$$M = \int_0^{\pi/4}\int_0^2\int_0^{\sqrt{4-\rho^2\operatorname{sen}^2(4\theta)}} K z\, \rho \operatorname{sen}^2(4\theta)\, dz\, d\rho\, d\theta$$

$= \frac{K}{2}\int_0^{\pi/4}\int_0^2 \rho\operatorname{sen}^2(4\theta)\left(4 - \rho^2\operatorname{sen}^2(4\theta)\right) d\rho\, d\theta$

$= \frac{K}{2}\int_0^{\pi/4}\int_0^2 4\operatorname{sen}^2(4\theta)\rho - \rho^3 \operatorname{sen}^4(4\theta)\, d\rho\, d\theta$

$= \frac{K}{2}\int_0^{\pi/4} 4\operatorname{sen}^2(4\theta) \left.\frac{\rho^2}{2}\right|_0^2 - \operatorname{sen}^4(4\theta) \left.\frac{\rho^4}{4}\right|_0^2 d\theta$

$= \frac{K}{2}\int_0^{\pi/4} 8\operatorname{sen}^2(4\theta) - 4\operatorname{sen}^4(4\theta)\, d\theta$

$= 2K\int_0^{\pi/4} 2\operatorname{sen}^2(4\theta) - \operatorname{sen}^4(4\theta)\, d\theta$

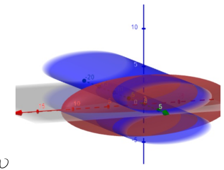
$= 2K\left(1 - \frac{3\pi}{32}\right)$

$$\bar{y} = \frac{M_y}{M} \qquad M_y = \iiint y\rho(\ )\,dV \qquad M = \iiint \rho(\ )\,dV$$

Problema 3. Determine la componente $\bar{y}$ del centroide del sólido definido por

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 / \ \frac{x^2}{9} + (y-1)^2 + z^2 \le 16, \quad z^2 + \frac{x^2}{9} \ge 4|z|\right\}$$

$z > 0$



$$\frac{x^2}{9} + z^2 + (y-1)^2 \le 16$$

$$\frac{x^2}{9} + (z+2)^2 \ge 4 \iff \frac{x^2}{9} + z^2 \ge 4z \qquad z < 0$$

$$\frac{x^2}{9} + (z-2)^2 \ge 4 \iff \frac{x^2}{9} + z^2 \ge -4z$$

$$|J| = \begin{vmatrix} 3\cos\theta & -3r\sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & r\cos\theta & 0 \end{vmatrix} = 3r$$

$x = 3r\cos\theta$

$y = y$

$z = r\sin\theta$

$r^2 \ge 4r\sin\theta$

$r \ge 4\sin\theta$

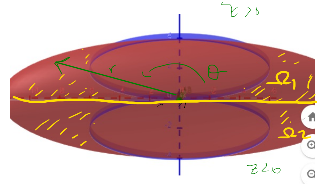
$r^2 + (y-1)^2 \le 16$

$r \le \sqrt{16-(y-1)^2}$

$-3 \le y \le 5$

$4\sin\theta \le r \le \sqrt{16-(y-1)^2}$

$0 \le \theta \le \pi$



$\leftarrow \Omega = \Omega_1 \cup \Omega_2$

$\int_\Omega f = \int_{\Omega_1} f + \int_{\Omega_2} f$

$$\frac{x^2}{9} + (y-1)^2 + z^2 = 16$$

$$\frac{x^2}{9} + z^2 = 4z$$

$$(y-1)^2 + 4z = 16$$

$$z = 4 - \frac{(y-1)^2}{4}$$

$$z = -\left(4 - \frac{(y-1)^2}{4}\right)$$

$$\frac{(y-1)^2}{4} - 4 \le z \le 4 - \frac{(y-1)^2}{4}$$

$16-(y-1)^2 \ge 16\sin^2\theta$

$(y-1)^2 \le 16(1-\sin^2\theta)$

$-4\sqrt{1-\sin^2\theta} \le y-1 \le 4\sqrt{1-\sin^2\theta}$

$1-4\cos\theta \le y \le 1+4\cos\theta$ (*)

$0 \le \theta \le \pi$

$z \le 0$

$|z| = |r\sin\theta| = -r\sin\theta$

$-4\sin\theta \le r \le \sqrt{16-(y-1)^2}$

$16-(y-1)^2 \ge 16\sin\theta$

$\Rightarrow$ lo mismo que (*)

$\pi \le \theta \le 2\pi$

$$\Omega_1 = \begin{cases} 0 \le \theta \le \pi \\ 1-4\cos\theta \le y \le 1+4\cos\theta \\ 4\sin\theta \le r \le \sqrt{16-(y-1)^2} \end{cases} \qquad \Omega_2 = \begin{cases} \pi \le \theta \le 2\pi \\ 1-4\cos\theta \le y \le 1+4\cos\theta \\ -4\sin\theta \le r \le \sqrt{16-(y-1)^2} \end{cases}$$

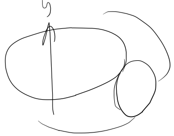
$$M = \iiint_{\Omega_1} 3r\,dr\,dy\,d\theta + \iiint_{\Omega_2} 3r\,dr\,dy\,d\theta$$

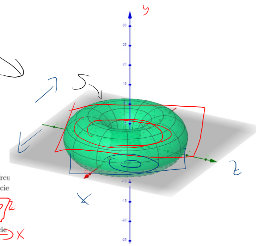
$$= \int_0^\pi \int_{1-4\cos\theta}^{1+4\cos\theta} \int_{4\sin\theta}^{\sqrt{16-(y-1)^2}} 3r\,dr\,dy\,d\theta + \int_\pi^{2\pi} \int_{1-4\cos\theta}^{1+4\cos\theta} \int_{4\sin\theta}^{\sqrt{16-(y-1)^2}} 3r\,dr\,dy\,d\theta$$

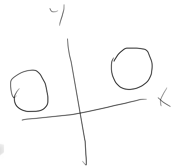
$$M_y = \iiint_{\Omega_1} y\,3r\,dr\,dy\,d\theta + \iiint_{\Omega_2} y\,3r\,dr\,dy\,d\theta$$

$$= \int_0^\pi \int_{1-4\cos\theta}^{1+4\cos\theta} \int_{4\sin\theta}^{\sqrt{16-(y-1)^2}} 3ry\,dr\,dy\,d\theta + \int_\pi^{2\pi} \int_{1-4\cos\theta}^{1+4\cos\theta} \int_{4\sin\theta}^{\sqrt{16-(y-1)^2}} 3r\,dr\,dy\,d\theta$$

$$\bar{y} = \frac{M_y}{M}$$

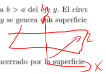






Problema 4. El centro de un círculo de radio a se ubica a una distancia $b > a$ del eje y. El círcu
y el eje están en el mismo plano. Cuando el círculo gira alrededor del eje y se genera una superficie
cuya ecuación es

$$x^2 + z^2 = (b + \sqrt{a^2 - y^2})^2.$$



Usando integrales múltiples determine la masa del sólido Ω el cual está encerrado por la superficie
considere que la densidad en cada punto de Ω es

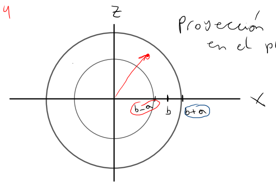
$$\rho(x, y, z) = 2|y|\sqrt{x^2 + z^2}.$$

$y = 3 \quad y = 4$

Plano XZ ($y = 0$)

$x^2 + z^2 - (b + |a|)^2 \leq |b + |a||$

$(b-a)^2 \leq x^2 + z^2 \leq (b+a)^2$



Proyección de S en el plano XZ

Usamos coord cilíndricas

$x = r\cos\theta$
$z = r\sin\theta \qquad |J| = r$
$y = y$

Luego determinamos los límites de Ω donde Ω es el volumen encerrado en S

Despejamos y

$$x^2 + z^2 = (b + \sqrt{a^2 - y^2})^2$$

$$y^2 = a^2 - \left(\sqrt{x^2+z^2} - b\right)^2$$

Usamos coor cilíndricas

$$y^2 = a^2 - (r-b)^2$$

Luego

$$-\sqrt{a^2 - (r-b)^2} \leq y \leq \sqrt{a^2 - (r-b)^2}$$

Finalmente

$$\Omega = \left\{ (r, \theta, y) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} (b-a) \leq r \leq (b+a) \\ -\sqrt{a^2-(r-b)^2} \leq y \leq \sqrt{a^2-(r-b)^2} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$M = \int_0^{2\pi} \int_{(b-a)}^{(b+a)} \int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} \underbrace{|y|\, r}_{\rho(y,r,\theta)} \cdot \overset{\text{Jacobiano}}{r} \, dy\, dr\, d\theta$$

simetría plano YZ

$$= 2 \int_0^{\pi} \int_{(b-a)}^{(b+a)} \int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} |y|\, r^2 \, dy\, dr\, d\theta$$

simetría plano XY

$$= 4 \int_0^{\pi} \int_{(b-a)}^{(b+a)} \int_0^{\sqrt{a^2-(r-b)^2}} y\, r^2 \, dy\, dr\, d\theta$$

$$= 4 \int_0^{\pi} \int_{(b-a)}^{(b+a)} (a^2 - (r-b)^2) r^2 \, dr\, d\theta$$

$$= \frac{16}{15}(a^5 + 5a^3b^2)$$

Problema 5. Considere la curva C intersección de las superficies:

$$S_1: \; z - x^2 - 4y^2 = 0 \qquad S_2: z = 2x+3, \quad x \geq 1.$$

a Encuentre una parametrización para C.
b Encuentre el vector velocidad junto con su módulo.
c Indique si la parametrización es simple, suave y regular.

a) Intersectamos S_1 con S_2

$$z = x^2 + 4y^2 \qquad \Rightarrow \qquad x^2 + 4y^2 = 2x+3$$
$$z = 2x+3$$
$$x^2 - 2x + 4y^2 = 3$$
$$(x-1)^2 + 4y^2 = 4$$
$$\frac{(x-1)^2}{4} + y^2 = 1$$

Luego

$$x = 1 + 2\cos t$$
$$y = \operatorname{sen} t$$

Reemplazando en z

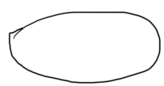
$$z(t) = 1 + 4\cos t + 4\cos^2 t + 4\operatorname{sen}^2 t = 5 + 4\cos t$$

$$r(t) = (1 + 2\cos t, \operatorname{sen} t, 5 + 4\cos t) \qquad t \in [0, 2\pi]$$

b) $$V(t) = r'(t) = (-2\operatorname{sen} t, \cos t, -4\operatorname{sen} t)$$

$$\|v(t)\|^2 = V_x^2 + V_y^2 + V_z^2$$
$$= 4\operatorname{sen}^2 t + \cos^2 t + 16\operatorname{sen}^2 t$$
$$= 19\operatorname{sen}^2 t + 1 \qquad /\sqrt{}$$

$$\|v(t)\| = \sqrt{19\operatorname{sen}^2 t + 1}$$

c) Simple ✓ Es una elipse 

Suave ✓ $1 + 2\cos t, \operatorname{sen} t, 5 + 4\cos t \in C^1 [0, 2\pi)$

Regular ✓ $r'(t) = (0,0,0) \Leftrightarrow V(t) = (0,0,0)$

$$\Leftrightarrow \quad -2\operatorname{sen} t = 0 \longrightarrow t = 0, \pi, 2\pi$$
$$\cos t = 0 \longrightarrow t = \pi/2, 3\pi/2$$
$$-4\operatorname{sen} t = 0 \longrightarrow t = 0, \pi, 2\pi$$

no tienen t en común

$$\Rightarrow \quad V(t) \neq (0,0,0) \quad \forall t \in [0, 2\pi]$$