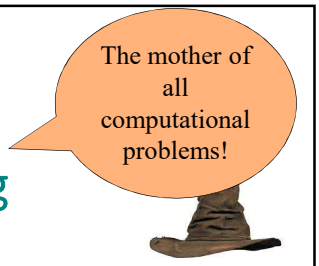
Sorting



6.006 Lecture 3

Image from pottermore.com

Sorting

ordering

alphabetizing

Why?

Easier to find things later!

Images from www.davie-fl.gov and enel460crawford.blogspot.com

Some uses of sorting:

- Problems that become easy once items are in sorted order
  - Searching: Is "Erik Demaine" in the database?
  - Statistics: Identify statistical outliers, or a median
- Non-obvious applications
  - Find closest pair
  - Data compression: sorting finds duplicates
  - Computer graphics: rendering scenes front to back

Today's plan

- Sorting definition
- Two sorting algorithms
  - Insertion Sort
  - Merge Sort
- Solving Recurrences

The problem of sorting

Input: array A[1…n] of numbers.

Output: permutation B[1…n] of A such that $B[1] \le B[2] \le \cdots \le B[n]$

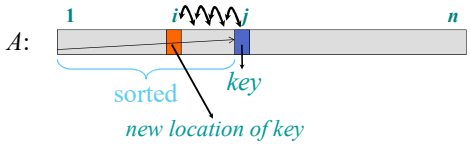
e.g. A = [7, 2, 5, 5, 9.6] → B = [2, 5, 5, 7, 9.6]

How can we do it efficiently ?

Insertion sort

INSERTION-SORT (A, n) ▷ $A[1 \,..\, n]$
for $j \leftarrow 2$ **to** n
  insert key $A[j]$ **into the (already sorted) sub-array** $A[1 \,..\, j\text{-}1]$**.**
  by pairwise key-swaps down to its sorted position

Illustration of iteration j



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

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Example of insertion sort

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Example of insertion sort

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2 4 8 9 3 6
2 3 4 8 9 6

Example of insertion sort

8 2 4 9 3 6
2 8 4 9 3 6
2 4 8 9 3 6
2 4 8 9 3 6
2 3 4 8 9 6
2 3 4 6 8 9 *done*

Running time:

- Upper bound $O(n^2)$
  - n times through loop
  - at most n swaps in each loop
- Algorithm will need $\Omega(n^2)$
  - E.g. Input in reverse sorted order n, n-1, n-2,…
    - i comparisons and swaps in loop i
    - Total $\Sigma i = \frac{n(n-1)}{2} = \theta(n^2)$

A nice property of insertion sort:

"in place sorting": A gets modified into B
without using additional memory!!

An improvement? Binary Insertion sort

BINARY-INSERTION-SORT (A, n) ▷ $A[1 \,..\, n]$
for $j \leftarrow 2$ **to** n
insert key $A[j]$ **into the (already sorted) sub-array** $A[1 \,..\, j\text{-}1]$.
Use binary search to find the right position

Binary search takes only $\Theta(\log n)$ time.
But, shifting elements after insertion still takes $\Theta(n)$ time!

Complexity: $\Theta(n \log n)$ comparisons
$\Theta(n^2)$ total time

Another approach: Divide and conquer

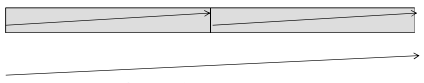
- Divide input into parts
- Conquer (solve) each part recursively
- Combine results to solve original

Run time:
$T(n) = divide\ time$
$+T(n_1) + T(n_2) + \ \ldots + T(n_k)$
$+\ combine\ time$

Meet Merge Sort

divide and conquer

MERGE-SORT $A[1 \,..\, n]$
1. If $n = 1$, done (nothing to sort).
2. Otherwise, recursively sort $A[\ 1 \,..\, n/2\ ]$ and $A[\ n/2{+}1 \,..\, n\ ]$.
3. "***Merge***" the two sorted sub-arrays.



Key subroutine: **MERGE**

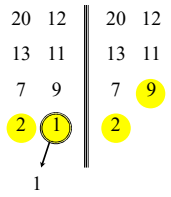
Merging two sorted arrays

20 12
13 11
7 9
2 1

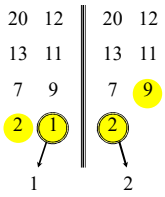
Merging two sorted arrays

20 12
13 11
7 9
2 1
1

Merging two sorted arrays



Merging two sorted arrays



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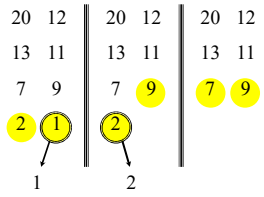

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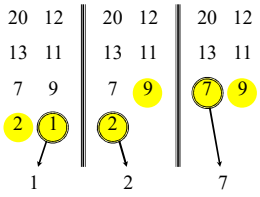

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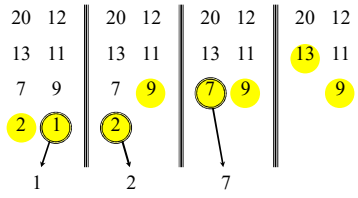

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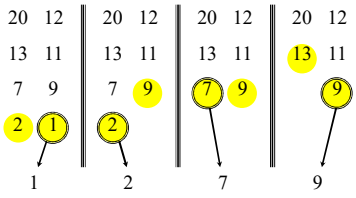
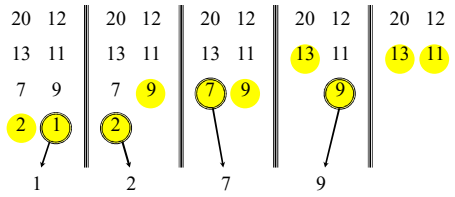


Merging two sorted arrays



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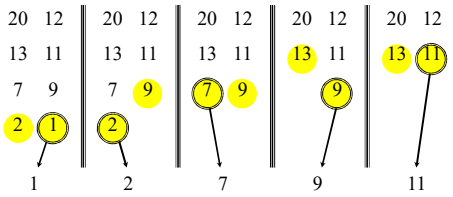

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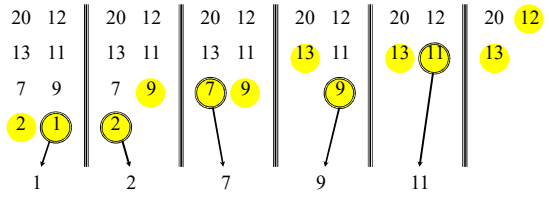

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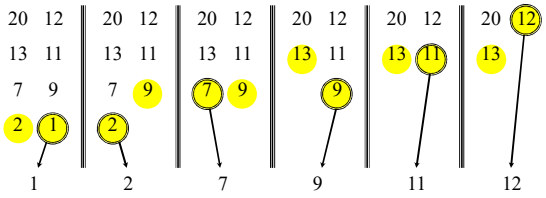

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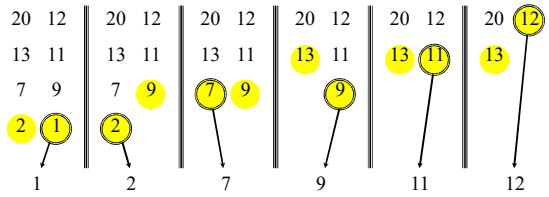


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

MERGE-SORT $A[1 \,..\, n]$

1. If $n = 1$, done. — $T(n)$ = $\Theta(1)$
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$. — $2T(n/2)$
3. *"Merge"* the two sorted lists — $\Theta(n)$

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

$T(n) = ?$

Recurrence solving

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree method

- Running time represented by sum of nodes
  - Describes where the work happens in the recursive call structure
- Will continually modify the tree in ways that
  - PRESERVE the sum
  - Make it progressively "simpler"
  - Note that "recursive calls" are only at leaves

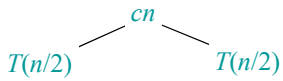
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$T(n)$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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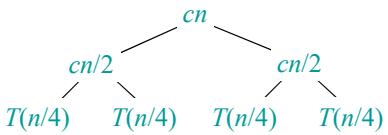

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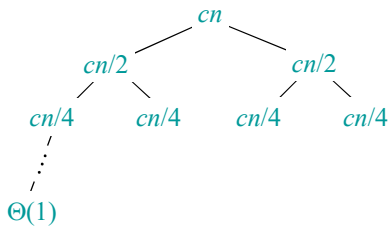

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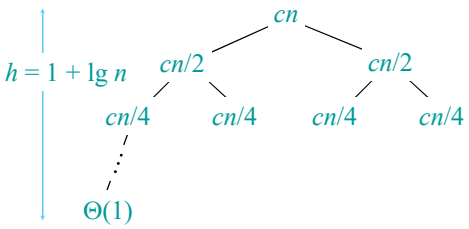

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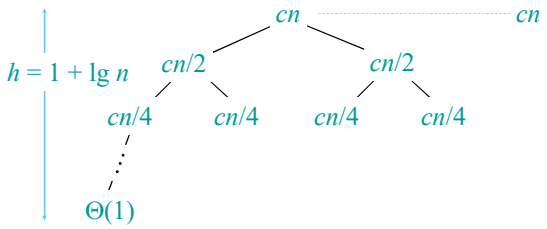

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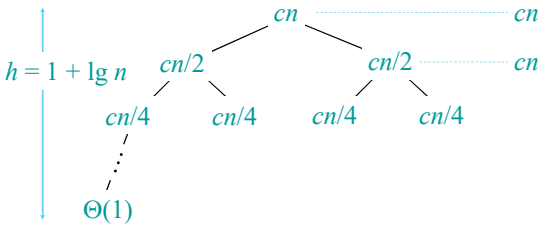

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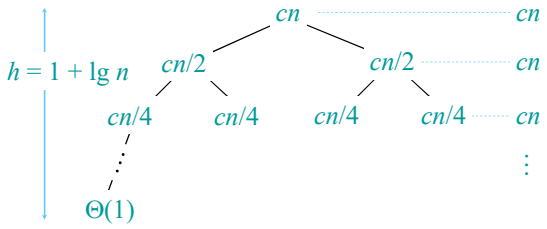

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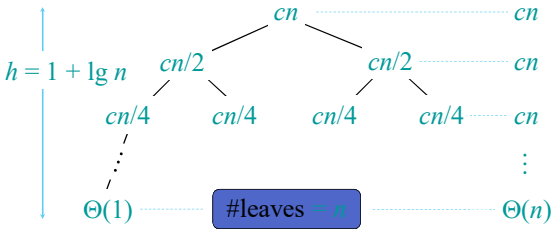

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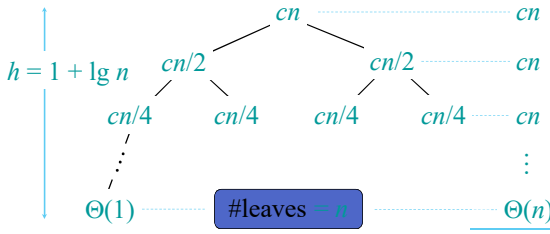
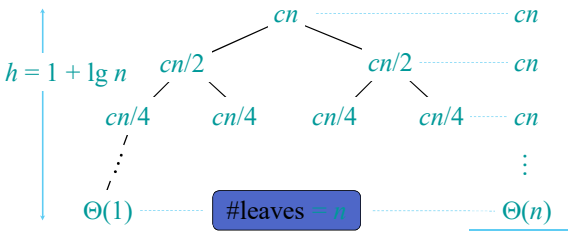


Total ?

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

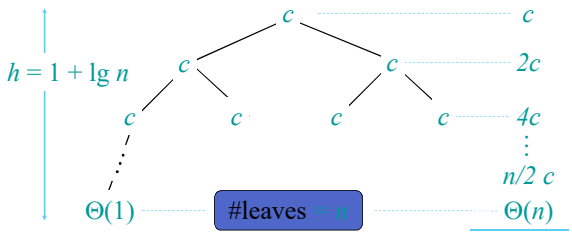


Total = $\Theta(n \lg n)$

Equal amount of work done at each level

Tree for different recurrence

Solve $T(n) = 2T(n/2) + c$, where $c > 0$ is constant.



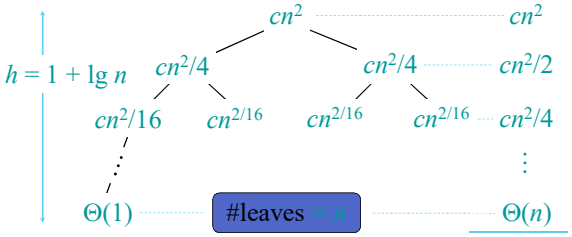
Note that $1 + ½ + ¼ + \ldots < 2$

Total = $\Theta(n)$

All the work done at the leaves

Tree for yet another recurrence

Solve $T(n) = 2T(n/2) + cn^2$, $c > 0$ is constant.



Note that $1 + ½ + ¼ + \ldots < 2$

Total = $\Theta(n^2)$

All the work done at the root

Fun fact:

- If level sums "same", total = $\theta(level\ sum \cdot number\ of\ levels)$
- If level sums geometrically decreasing, total = $\theta(top\ level) = \theta(root\ node)$
- If level sums geometrically increasing, total = $\theta(bottom\ level) = \theta(number\ of\ leaves)$

Master theorem

- A recipe for solving a typical form of divide and conquer recurrence
- Formalizes the "fun fact"
- A bit messy to get used to, use with care!

Master Theorem: T(n) = a T(n/b) + f(n)

- Some observations about recursion tree:
  - h = number of levels = $\log_b n = O(\log n)$
  - L = number of leaves = $a^h = a\text{^}(\log_b n) = n\text{^}(\log_b a)$
- Case 1: Geometrically increasing (e.g. $T(n) = 2T(n/2) + c$)

  If $f(n) = O(L^{1-\epsilon})$ =O($n\text{^}((\log_b a) - \epsilon)$)
  then $T(n) = \theta(L) = \theta(n\text{^}(\log_b a))$

Master Theorem: T(n) = a T(n/b) + f(n)

- Some observations about recursion tree:
  - h = number of levels = $\log_b n = O(\log n)$
  - L = number of leaves = $a^h = a\text{^}(\log_b n) = n\text{^}(\log_b a)$
- Case 2: Equal levels (e.g. *T(n) = 2T(n/2)+cn)*
  If $f(n) = \theta(L) = O(n\text{^}(\log_b a))$ then $T(n) = \theta(L \log n) = \theta(n\text{^}(\log_b a) \cdot \log n)$

Master Theorem: T(n) = a T(n/b) + f(n)

- Some observations about recursion tree:
  - h = number of levels = $\log_b n = O(\log n)$
  - L = number of leaves = $a^h = a\text{^}(\log_b n) = n\text{^}(\log_b a)$
- Case 3: Geometrically decreasing (e.g. *T(n) = 2T(n/2) + cn^2)*
  If $f(n) = O(L^{1+\epsilon}) = O(n\text{^}((\log_b a) + \epsilon))$ and $af\left(\frac{n}{b}\right) \leq (1-\delta)f(n)$ then $T(n) = \theta(f(n))$

Why does it work?

- See textbook for proof

Distinction to note:

Running time of algorithm $\theta(n^2)$

vs.

Time required by a problem $O(n \log n)$