6.006	Lecture 7	Sept. 29, 2016

TODAY: Linear-Time Sorting

- comparison model
- lower bounds:
  - searching: $\Omega(\lg n)$
  - sorting: $\Omega(n \lg n)$
- $O(n)$ sorting algorithms (for small integers)
  - counting sort
  - radix sort

theorem
proof
counterexample

Lower bounds: claim

- searching among n preprocessed items requires $\Omega(\lg n)$ time
  $\Rightarrow$ binary search, AVL tree search optimal
- sorting n items requires $\Omega(n \lg n)$
  $\Rightarrow$ mergesort, heap sort, AVL sort optimal

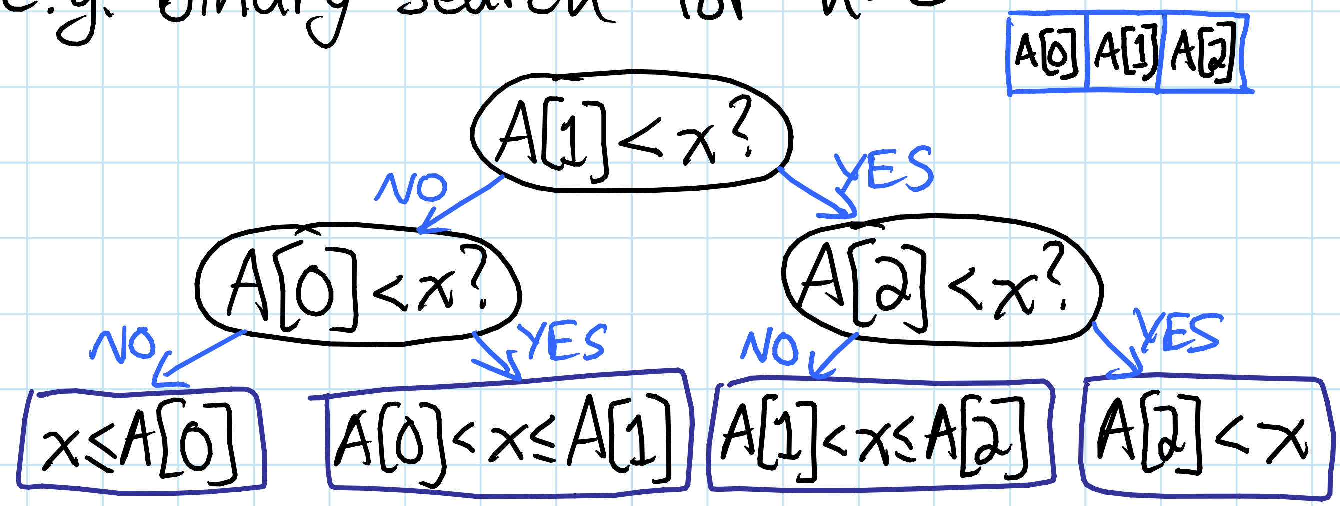
... in the comparison model

Comparison model of computation:

- input items are black boxes (ADTs)
- only support comparisons ($<$, $>$, $\leq$, etc.)
- time cost = # comparisons

Decision tree: any comparison algorithm can be viewed/specified as a tree of all possible comparison outcomes & resulting output, for a particular n:

- e.g. binary search for $n=3$:



- internal node = binary decision
- leaf = output (algorithm is done)
- root-to-leaf path = algorithm execution
- path length (depth) = running time
- height of tree = worst-case running time

In fact, binary decision tree model is more powerful than comparison model, and lower bounds extend to it.

Search lower bound:

- # leaves $\geq$ # possible answers
  $\geq n$ (at least 1 per $A[i]$)
- decision tree is binary

$\Rightarrow$ height $\geq \lg n$

Sorting lower bound:

- leaf specifies answer as permutation:
  $A[3] \leq A[1] \leq A[9] \leq \cdots$
- all $n!$ are possible answers

$\Rightarrow$ # leaves $\geq n!$

$\Rightarrow$ height $\geq \lg n!$

$$= \lg(1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n)$$
$$= \lg 1 + \lg 2 + \cdots + \lg(n-1) + \lg n$$
$$= \sum_{i=1}^{n} \lg i$$
$$\geq \sum_{i=n/2}^{n} \lg i$$
$$\geq \sum_{i=n/2}^{n} \lg \frac{n}{2} \quad \rightarrow = \lg n - 1$$
$$= \frac{n}{2} \lg n - \frac{n}{2} \quad = \boxed{\Omega(n \lg n)}$$

- in fact $\lg n! = n \lg n - O(n)$ via:

Sterling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\Rightarrow \lg n! \sim n \lg n - \underbrace{(\lg e)n + \tfrac{1}{2}\lg n + \tfrac{1}{2}\lg(2\pi)}_{O(n)}$$

Linear-time sorting:

if n keys are integers $\in \{0, 1, \ldots, k-1\}$, ↗ fitting in a word
can do more than compare them
⇒ lower bounds don't apply
– if $k = n^{O(1)}$, can sort in $O(n)$ time

OPEN: $O(n)$ time possible for all k?

Counting sort:

– L = array of k empty lists $\}\ O(k)$
linked or Python lists
– for j in range(n): $\}\ O(n)$
L[key(A[j])].append(A[j]) $\}\ O(1)$
↖ random access using integer key
– output = []
– for i in range(k):
output.extend(L[i])
$\Big\}\ O\left(\sum_i (1 + |L[i]|)\right) = O(k+n)$

Time: $\Theta(n+k)$ (also $\Theta(n+k)$ space)

Intuition: count key occurrences using RAM [L2]
output <count> copies of each key in order
... but item is more than just a key

CLRS has cooler implementation of counting sort with counters, no lists ~ but time bound is the same

Radix sort: [Herman Hollerith 1887 punchcards!] 80 cols.

- imagine each integer in base b
- $\Rightarrow d = \log_b k$ digits $\in \{0, 1, \dots, b-1\}$
- sort by least significant digit → can extract in $O(1)$ time $(x // b^i) \% b$
- ... → all n items
- sort by most significant digit
  - ↳ sort must be stable: preserve relative order of items with the same key
  - $\Rightarrow$ don't mess up previous sorting

e.g.

		⇒		⇒		⇒	
	3 2 9		7 2 0		7 2 0		3 2 9
	4 5 7		3 5 5		3 2 9		3 5 5
	6 5 7		4 3 6		4 3 6		4 3 6
	8 3 9		4 5 7		8 3 9		4 5 7
	4 3 6		6 5 7		3 5 5		6 5 7
	7 2 0		3 2 9		4 5 7		7 2 0
	3 5 5		8 3 9		6 5 7		8 3 9
	sort		sorted		sorted		sorted

- use counting sort for digit sort
- $\Rightarrow \Theta(n+b)$ per digit
- $\Rightarrow \Theta((n+b)d) = \Theta((n+b)\log_b k)$ total time
- minimized when $b = n$

$$\boxed{\begin{aligned} &\Rightarrow \Theta(n \log_n k) \\ &= O(nc) \quad \text{if } k \le n^c \end{aligned}}$$

running time of radix sort