Newsvendor Model Structure

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1 The Structure

Let C_u and C_o denote the *underage cost* and the *overage cost*, respectively. Suppose the *profit* is given by, when the *order quantity* is Q and the *demand* is x,

$$\text{Profit} = C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0) \tag{1}$$

where the first and second terms represent the *gain from sales* and the *cost of leftover inventory*, respectively.

Let $\mathbf{D}$ be a continuous random variable of the demand, whose distribution function and probability density function are $F(x)$ and $f(x)$, respectively, for $x \in [0, \infty)$. The *expected profit*, $\pi(Q)$, is

$$\pi(Q) = \int_0^\infty (C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0)) f(x) \mathrm{d}x \tag{2}$$

Proposition 1. The expected profit is rewritten as follows:

$$\pi(Q) = \int_0^Q (C_u(1 - F(x)) - C_o F(x)) \mathrm{d}x \tag{3}$$

Corollary 2. The optimal order quantity Q^*, which maximizes the expected profit, satisfies

$$F(Q^*) = \frac{C_u}{C_o + C_u} \tag{4}$$

Proof. Because $F(x)$ is a nondecreasing function, Eq. (3) is maximized when

$$C_u(1 - F(Q^*)) - C_o F(Q^*) = 0 \tag{5}$$

□

Proof of Proposition 1.

$$\pi(Q) = \int_0^Q (C_u x - C_o(Q - x)) f(x) \mathrm{d}x + \int_Q^\infty C_u Q f(x) \mathrm{d}x \tag{6}$$

$$= (C_u + C_o) \int_0^Q x f(x) \mathrm{d}x - C_o Q \int_0^Q f(x) \mathrm{d}x + C_u Q \int_Q^\infty f(x) \mathrm{d}x \tag{7}$$

$$= (C_u + C_o) \int_0^Q x f(x) \mathrm{d}x - C_o Q F(Q) + C_u Q(1 - F(Q)) \tag{8}$$

By integration by parts,

$$\int_0^Q x f(x) \mathrm{d}x = [x F(x)]_0^Q - \int_0^Q F(x) \mathrm{d}x = Q F(Q) - \int_0^Q F(x) \mathrm{d}x \tag{9}$$

Hence, the expected profit can be written as

$$\pi(Q) = (C_u + C_o)\left(Q F(Q) - \int_0^Q F(x) \mathrm{d}x\right) - C_o Q F(Q) + C_u Q(1 - F(Q)) \tag{10}$$

$$= C_u Q - (C_u + C_o) \int_0^Q F(x) \mathrm{d}x \tag{11}$$

$$= \int_0^Q C_u \mathrm{d}x - (C_u + C_o) \int_0^Q F(x)) \mathrm{d}x \tag{12}$$

$$= \int_0^Q (C_u(1 - F(x)) - C_o F(x)) \mathrm{d}x \tag{13}$$

□

2 Newsvendor Problems

Suppose *sales price* p, *wholesale price* c, and *salvage value* s, where $p > c > s$. The profit is given as, with order quantity Q and demand x,

$$\text{Profit} = p \cdot \min(x, Q) + s \cdot \max(Q - x, 0) - cQ \tag{14}$$

where each term represents, in order, the revenue from sales, the revenue from leftover inventory, and the purchase cost, respectively. Since

$$Q = \min(x, Q) + \max(Q - x, 0) \tag{15}$$

the expected profit is rewritten as

$$\text{Profit} = p \cdot \min(x, Q) + s \cdot \max(Q - x, 0) - c(\min(x, Q) + \max(Q - x, 0)) \tag{16}$$

$$= (p - c) \cdot \min(x, Q) - (c - s) \cdot \max(Q - x, 0) \tag{17}$$

Hence,

$$C_u = p - c \tag{18}$$

$$C_o = c - s \tag{19}$$

(Note that in Eq. (17) it is clear that two terms represent the gain from sales and the cost of leftover inventory, respectively, as described in Sec. 1.)

3 Quick Response with Reactive Capacity

Suppose sales price p, *initial wholesale price* c, and salvage value s. In addition, the *second order* can be made with *premium wholesale price* $c'(> c)$. The profit is given as, with *initial order quantity* Q_0 and *demand over the entire season* x,

$$\text{Profit} = px + s \cdot \max(Q_0 - x, 0) - cQ_0 - c' \cdot \max(0, x - Q_0) \tag{20}$$

where each term represents, in order, the revenue from sales, the revenue from leftover inventory, the initial purchase cost, and the purchase cost from the second order, respectively. (Note that it is assumed that all demand is fulfilled regardless of the initial order quantity.) By using $x = \min(x, Q_0) + \max(0, x - Q_0)$ and Eq. (15),

$$\begin{aligned}\text{Profit} &= p(\min(x, Q_0) + \max(0, x - Q_0)) + s \cdot \max(Q_0 - x, 0) \\ &\quad - c(\min(x, Q_0) + \max(Q_0 - x, 0)) - c' \cdot \max(0, x - Q_0) \qquad (21) \\ &= (p - c) \cdot \min(x, Q_0) + (p - c') \cdot \max(0, x - Q_0) - (c - s) \cdot \max(Q_0 - x, 0) \qquad (22)\end{aligned}$$

Now, each term in Eq. (22) shows the sales gain from the initial order, the sales gain from the second order, and the cost of leftover inventory. Again,

$$\begin{aligned}\text{Profit} &= (p - c' + c' - c) \cdot \min(x, Q_0) + (p - c') \cdot \max(0, x - Q_0) - (c - s) \cdot \max(Q_0 - x, 0) \qquad (23) \\ &= (c' - c) \cdot \min(x, Q_0) - (c - s) \cdot \max(Q_0 - x, 0) + (p - c')(\min(x, Q_0) + \max(0, x - Q_0)) \qquad (24) \\ &= (c' - c) \cdot \min(x, Q_0) - (c - s) \cdot \max(Q_0 - x, 0) + (p - c')x \qquad (25)\end{aligned}$$

Even though there is an additional term, $(p - c')x$, it is in the same structure as in Eq. (1) is because $\int_0^\infty (p - c')xf(x)\mathrm{d}x = (p - c')\mu$ is a constant. Hence,

$$C_u = c' - c \tag{26}$$

$$C_o = c - s \tag{27}$$

4 The Structure in Discrete Case

Suppose $\mathbf{D}$ is a discrete random variable of the demand with distribution function $F(x)$ and probability mass function p_x for $x \in \{0, 1, 2, \ldots\}$. Similarly to the continuous case, the expected profit is

$$\pi(Q) = \sum_{0}^{\infty}(C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0)) \cdot p_x \tag{28}$$

$$= \sum_{x=0}^{Q}(C_u x - C_o(Q - x)) \cdot p_x + \sum_{x=Q+1}^{\infty} C_u Q \cdot p_x \tag{29}$$

$$= (C_u + C_o)\sum_{x=0}^{Q} x p_x - C_o Q \sum_{x=0}^{Q} p_x + C_u Q \sum_{x=Q+1}^{\infty} p_x \tag{30}$$

$$= (C_u + C_o)\sum_{x=0}^{Q} x p_x - C_o Q F(Q) + C_u Q(1 - F(Q)) \tag{31}$$

$$= (C_u + C_o)\left(QF(Q) - \sum_{x=0}^{Q-1} F(x)\right) - C_o Q F(Q) + C_u Q(1 - F(Q)) \tag{32}$$

$$= C_u Q - (C_u + C_o)\sum_{x=0}^{Q-1} F(x) \tag{33}$$

$$= \sum_{x=0}^{Q-1} C_u - \sum_{x=0}^{Q-1}(C_u + C_o)F(x) \tag{34}$$

$$= \sum_{x=0}^{Q-1}(C_u(1 - F(x)) - C_o F(x)) \tag{35}$$

Hence, the optimal order quantity Q^* is the minimum quantity x that satisfies $C_u(1 - F(x)) < C_o F(x)$, i.e.,

$$Q^* = \min\left\{x \;\middle|\; F(x) > \frac{C_u}{C_o + C_u}\right\} \tag{36}$$

Notice that, if $F(Q) < C_u/(C_o + C_u) < F(Q + 1)$, $Q^* = Q + 1$, which justifies the *round-up rule.*