Newsvendor Model Structure

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1 The Structure

Let C_u and C_o denote the *underage cost* and the *overage cost*, respectively. Suppose the *profit* is given by, when the *order quantity* is Q and the *demand* is x,

$$\text{Profit} = C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0) \tag{1}$$

where the first and second terms represent the *gain from sales* and the *cost of leftover inventory*, respectively.

Let $\mathbf{D}$ be a continuous random variable of the demand, whose distribution function and probability density function are $F(x)$ and $f(x)$, respectively, for $x \in [0, \infty)$. The *expected profit*, $\pi(Q)$, is

$$\pi(Q) = \int_0^\infty (C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0))f(x)\mathrm{d}x \tag{2}$$

Proposition 1. The expected profit is rewritten as follows:

$$\pi(Q) = \int_0^Q (C_u(1 - F(x)) - C_oF(x))\mathrm{d}x \tag{3}$$

Corollary 2. The optimal order quantity Q^*, which maximizes the expected profit, satisfies

$$F(Q^*) = \frac{C_u}{C_o + C_u} \quad (\textit{critical raito}) \tag{4}$$

Proof. Because $F(x)$ is a nondecreasing function, Eq. (3) is maximized when $C_u(1-F(Q^*))-C_oF(Q^*) = 0$. □

Proof of Proposition 1.

$$\begin{aligned}
\pi(Q) &= \int_0^Q (C_ux - C_o(Q - x))f(x)\mathrm{d}x + \int_Q^\infty C_uQf(x)\mathrm{d}x && (5)\\
&= (C_u + C_o)\int_0^Q xf(x)\mathrm{d}x - C_oQ\int_0^Q f(x)\mathrm{d}x + C_uQ\int_Q^\infty f(x)\mathrm{d}x && (6)\\
&= (C_u + C_o)\int_0^Q xf(x)\mathrm{d}x - C_oQF(Q) + C_uQ(1 - F(Q)) && (7)
\end{aligned}$$

By integration by parts,

$$\int_0^Q xf(x)\mathrm{d}x = [xF(x)]_0^Q - \int_0^Q F(x)\mathrm{d}x = QF(Q) - \int_0^Q F(x)\mathrm{d}x \tag{8}$$

Hence, the expected profit is written as

$$\begin{align}
\pi(Q) &= (C_u + C_o)\left(QF(Q) - \int_0^Q F(x)\mathrm{d}x\right) - C_oQF(Q) + C_uQ(1 - F(Q)) \tag{9}\\
&= C_uQ - (C_u + C_o)\int_0^Q F(x)\mathrm{d}x \tag{10}\\
&= \int_0^Q C_u\mathrm{d}x - (C_u + C_o)\int_0^Q F(x))\mathrm{d}x \tag{11}\\
&= \int_0^Q (C_u(1 - F(x)) - C_oF(x))\mathrm{d}x \tag{12}
\end{align}$$

□

2 Problems

The following fact is used throughout the following subsections:

$$v = \min(v, w) + \max(v - w, 0) \tag{13}$$

2.1 Newsvendor Model

Suppose sales price p, wholesale price c, and salvage value s, where $p > c > s$. The profit is given as, with order quantity Q and demand x,

$$\text{Profit} = p \cdot \min(x, Q) + s \cdot \max(Q - x, 0) - cQ \tag{14}$$

where each term represents, in order, the revenue from sales, the revenue from leftover inventory, and the purchase cost, respectively. By using Eq. (13) with $v = Q$ and $w = x$, the expected profit is rewritten as

$$\begin{align}
\text{Profit} &= p \cdot \min(x, Q) + s \cdot \max(Q - x, 0) - c(\min(x, Q) + \max(Q - x, 0)) \tag{15}\\
&= (p - c) \cdot \min(x, Q) - (c - s) \cdot \max(Q - x, 0) \tag{16}
\end{align}$$

Hence,

$$\begin{align}
C_u &= p - c \tag{17}\\
C_o &= c - s \tag{18}
\end{align}$$

(Note that in Eq. (16) it is clear that the first and second terms represent the gain from sales and the cost of leftover inventory, respectively, as described in Sec. 1.)

2.2 Quick Response with Reactive Capacity

Suppose sales price p, initial wholesale price c, and salvage value s. In addition, the second order can be made with premium wholesale price $c'(> c)$. The profit is given as, with initial order quantity Q_0 and demand over the entire season x,

$$\text{Profit} = px + s \cdot \max(Q_0 - x, 0) - cQ_0 - c' \cdot \max(0, x - Q_0) \tag{19}$$

where each term represents, in order, the gain from sales, the gain from leftover inventory, the initial purchase cost, and the purchase cost of the second order, respectively. (Note that it is assumed that all demand is fulfilled regardless of the initial order quantity.) By using Eq. (13),

$$\begin{aligned}\text{Profit} &= p(\min(x, Q_0) + \max(0, x - Q_0)) + s \cdot \max(Q_0 - x, 0) \\ &\quad - c(\min(x, Q_0) + \max(Q_0 - x, 0)) - c' \cdot \max(0, x - Q_0) &(20)\\ &= (p - c) \cdot \min(x, Q_0) + (p - c') \cdot \max(0, x - Q_0) - (c - s) \cdot \max(Q_0 - x, 0) &(21)\end{aligned}$$

Now, each term in Eq. (21) shows the sales gain from the initial order, the sales gain from the second order, and the cost of leftover inventory, respectively. Again,

$$\begin{aligned}\text{Profit} &= (p - c' + c' - c) \cdot \min(x, Q_0) + (p - c') \cdot \max(0, x - Q_0) - (c - s) \cdot \max(Q_0 - x, 0) &(22)\\ &= (c' - c) \cdot \min(x, Q_0) - (c - s) \cdot \max(Q_0 - x, 0) + (p - c')(\min(x, Q_0) + \max(0, x - Q_0)) &(23)\\ &= (c' - c) \cdot \min(x, Q_0) - (c - s) \cdot \max(Q_0 - x, 0) + (p - c')x &(24)\end{aligned}$$

Even though there is an additional term, $(p - c')x$, the structure is the same with that in Eq. (1) because $\int_0^\infty (p - c')xf(x)\mathrm{d}x = (p - c')\mu$ is a constant. Hence,

$$C_u = c' - c \tag{25}$$

$$C_o = c - s \tag{26}$$

2.3 Order-up-to Model

Let l, h, and b be the lead time in periods, the holding cost of a unit for a period, and the back-order penalty cost of a unit, respectively. When the order-up-to level is S and the demand over $(l + 1)$ periods is x, the profit of the period is

$$\begin{aligned}\text{Profit} &= R - b \cdot \max(x - S, 0) - h \cdot \max(S - x, 0) &(27)\\ &= b \cdot \min(x, S) - h \cdot \max(S - x, 0) + R - bx \qquad \text{by Eq. (13)} &(28)\end{aligned}$$

where R is a fixed revenue (by assumption). Hence,

$$C_u = b \tag{29}$$
$$C_o = h \tag{30}$$

2.4 Protection Level

Let r_h and r_l be the regular fare and the discount fare, respectively, for a room. Let K be the capacity. When the high-fare protection level is Q and the high-fare demand is x,

$$\begin{aligned} \text{Revenue} &= r_h \cdot \min(x, Q) + r_l(K - Q) && (31)\\ &= r_h \cdot \min(x, Q) + r_l(K - \min(x, Q) - \max(Q - x, 0)) && (32)\\ &= (r_h - r_l) \cdot \min(x, Q) - r_l \cdot \max(Q - x, 0) + r_l K && (33) \end{aligned}$$

Hence,

$$C_u = r_h - r_l \tag{34}$$
$$C_o = r_l \tag{35}$$

2.5 Overbooking

Let r and K be the fare for a room and the capacity, respectively. A no-show customer is allowed to cancel with penalty p. The cost of a bumped customer is c. Let Y be the number of additional reservations beyond capacity that the hotel is willing to accept. Assume that there are plenty of customers and there will be always $(K+Y)$ customer reservations. When there are x no-show customers,

$$\text{Revenue} = r(K + Y - x) + px - (c + r) \cdot \max(Y - x, 0) \tag{36}$$

where each term represents, in order, the gain from show-up customers, the gain from no-show customers, and the loss from bumped customers, respectively. It is rearranged as

$$\begin{aligned} \text{Revenue} &= rY - (c + r) \cdot \max(Y - x, 0) + rK + (p - r)x && (37)\\ &= r(\min(x, Y) + \max(Y - x, 0)) - (c + r) \cdot \max(Y - x, 0) + rK + (p - r)x && (38)\\ &= r \cdot \min(x, Y) - c \cdot \max(Y - x, 0) + rK + (p - r)x && (39) \end{aligned}$$

Hence,

$$C_u = r \tag{40}$$
$$C_o = c \tag{41}$$

3 The Structure in Discrete Case

Suppose $\mathbf{D}$ is a discrete random variable of the demand with distribution function $F(x)$ and probability mass function p_x, for $x \in \{0, 1, 2, \ldots\}$. Similarly to the continuous case, the expected profit is

$$\pi(Q) = \sum_{0}^{\infty} (C_u \cdot \min(x, Q) - C_o \cdot \max(Q - x, 0)) \cdot p_x \tag{42}$$

$$= \sum_{x=0}^{Q} (C_u x - C_o(Q - x)) \cdot p_x + \sum_{x=Q+1}^{\infty} C_u Q \cdot p_x \tag{43}$$

$$= (C_u + C_o) \sum_{x=0}^{Q} x p_x - C_o Q \sum_{x=0}^{Q} p_x + C_u Q \sum_{x=Q+1}^{\infty} p_x \tag{44}$$

$$= (C_u + C_o) \left(Q F(Q) - \sum_{x=0}^{Q-1} F(x) \right) - C_o Q F(Q) + C_u Q (1 - F(Q)) \tag{45}$$

$$= C_u Q - (C_u + C_o) \sum_{x=0}^{Q-1} F(x) \tag{46}$$

$$= \sum_{x=0}^{Q-1} C_u - \sum_{x=0}^{Q-1} (C_u + C_o) F(x) \tag{47}$$

$$= \sum_{x=0}^{Q-1} (C_u(1 - F(x)) - C_o F(x)) \tag{48}$$

Hence, the optimal order quantity Q^* is the minimum quantity x that satisfies $C_u(1 - F(x)) < C_o F(x)$, i.e.,

$$Q^* = \min \left\{ x \;\middle|\; F(x) > \frac{C_u}{C_o + C_u} \right\} \tag{49}$$

Notice that, when $F(Q) < C_u/(C_o + C_u) < F(Q + 1)$, $Q^* = Q + 1$, which justifies the *round-up rule*.