The truth state is a motorcycle model in the 2D plane

$$\boldsymbol{x}_t = \begin{bmatrix} r_{xi} \\ r_{yi} \\ v_{xb} \\ \psi \\ \phi \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{1}$$

The corresponding dynamics are

$$\dot{\boldsymbol{x}}_t = \begin{bmatrix} v_{xb}\cos\psi \\ v_{xb}\sin\psi \\ a \\ \frac{v_{xb}}{L}\tan\phi \\ \xi \\ -\frac{1}{\tau_a}\boldsymbol{b}_a + \boldsymbol{w}_a \\ -\frac{1}{\tau_g}\boldsymbol{b}_g + \boldsymbol{w}_g \\ \boldsymbol{0} \end{bmatrix} \tag{2}$$

The continuous measurements consist of accelerometer and gyro measurements, where the y component includes the lateral accelerations felt during a turn and the z component includes gravity.

$$\tilde{\boldsymbol{a}}^b = \begin{bmatrix} a & \frac{v_{xb}^2}{L}\tan\phi & -g \end{bmatrix}^T + \boldsymbol{b}_a + \boldsymbol{n}_a \tag{3}$$

$$\tilde{\boldsymbol{\omega}}^b_{b/i} = \begin{bmatrix} 0 & 0 & \frac{v_{xb}}{L}\tan\phi \end{bmatrix}^T + \boldsymbol{b}_g + \boldsymbol{n}_g \tag{4}$$

The design states model is a conventional inertial navigation system augmented with the position of the circuit

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{5}$$

The dynamics are defined as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{v}^i_{b/i} \\ T^i_b\left(\tilde{\boldsymbol{a}}^b - \boldsymbol{b}_a - \boldsymbol{n}_a\right) + \boldsymbol{g}^i \\ \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}^b_{b/i} - \boldsymbol{b}_g - \boldsymbol{n}_g \end{bmatrix} \otimes \boldsymbol{q}^b_i \\ -\frac{1}{\tau_a}\boldsymbol{b}_a + \boldsymbol{w}_a \\ -\frac{1}{\tau_g}\boldsymbol{b}_g + \boldsymbol{w}_g \\ \boldsymbol{0} \end{bmatrix} \tag{6}$$

The method for propagating the navigation states is

$$\dot{\hat{\boldsymbol{x}}} = \begin{bmatrix} \hat{\boldsymbol{v}}^i_{b/i} \\ \hat{T}^i_b\left(\hat{\boldsymbol{q}}^b_i\right)\left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a\right) + \boldsymbol{g}^i \\ \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}^b_{b/i} - \hat{\boldsymbol{b}}_g \end{bmatrix} \otimes \hat{\boldsymbol{q}}^b_i \\ -\frac{1}{\tau_a}\hat{\boldsymbol{b}}_a \\ -\frac{1}{\tau_g}\hat{\boldsymbol{b}}_g \\ \boldsymbol{0} \end{bmatrix} \tag{7}$$

The relationship between the navigation state, error state, and the true navigation state is the following

$$\begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{r}}^i_{b/i} + \delta\boldsymbol{r}^i_{b/i} \\ \hat{\boldsymbol{v}}^i_{b/i} + \delta\boldsymbol{v}^i_{b/i} \\ \begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}^b_b}{2} \end{bmatrix} \otimes \hat{q}^b_i \\ \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a \\ \hat{\boldsymbol{b}}_g + \delta\boldsymbol{b}_g \\ \hat{\boldsymbol{r}}^i_{c/i} + \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{8}$$

or equivalently for attitude

$$T^b_i = \left[I_{3\times3} - \left(\delta\boldsymbol{\theta}^b_b\right)\times\right]\hat{T}^b_i \tag{9}$$

The error injection mapping is the following

$$\begin{bmatrix} \hat{\boldsymbol{r}}^i_{b/i} \\ \hat{\boldsymbol{v}}^i_{b/i} \\ \hat{q}^b_i \\ \hat{\boldsymbol{b}}_a \\ \hat{\boldsymbol{b}}_g \\ \hat{\boldsymbol{r}}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} - \delta\boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} - \delta\boldsymbol{v}^i_{b/i} \\ \begin{bmatrix} 1 \\ -\frac{\delta\boldsymbol{\theta}^b_b}{2} \end{bmatrix} \otimes q^b_i \\ \boldsymbol{b}_a - \delta\boldsymbol{b}_a \\ \boldsymbol{b}_g - \delta\boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} - \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{10}$$

The error calculation mapping is the following, where it is assumed that the quaternion contains the scalar in the first element of a 4 element vector

$$\begin{bmatrix} \delta\boldsymbol{r}^i_{b/i} \\ \delta\boldsymbol{v}^i_{b/i} \\ \delta\boldsymbol{\theta}^b_b \\ \delta\boldsymbol{b}_a \\ \delta\boldsymbol{b}_g \\ \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} - \hat{\boldsymbol{r}}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} - \hat{\boldsymbol{v}}^i_{b/i} \\ 2\begin{bmatrix} 0_{3\times1} & I_{3\times3} \end{bmatrix} q^b_i \otimes \left(\hat{q}^b_i\right)^* \\ \boldsymbol{b}_a - \hat{\boldsymbol{b}}_a \\ \boldsymbol{b}_g - \hat{\boldsymbol{b}}_g \\ \boldsymbol{r}^i_{c/i} - \hat{\boldsymbol{r}}^i_{c/i} \end{bmatrix} \tag{11}$$

Finally, the mapping between truth and design states is

$$\begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_{xi} \\ r_{yi} \\ 0 \end{bmatrix} \\ \begin{bmatrix} v_{xb}\cos\psi \\ v_{xb}\sin\psi \\ 0 \end{bmatrix} \\ \begin{bmatrix} \cos\left(\frac{\psi}{2}\right) \\ 0 \\ 0 \\ \sin\left(\frac{\psi}{2}\right) \end{bmatrix} \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{12}$$

To linearize the dynamics, we must use perturbation techniques, since the partial derivatives w.r.t. attitude parameters are difficult to obtain. The two difficult linearizations are related to the $\dot{\boldsymbol{v}}^i_{b/i}$ equation and the $\dot{q}^b_i$ equation. Expanding the former from Eqn. 6 yields

$$\dot{\hat{\boldsymbol{v}}}^i_{b/i} + \delta\dot{\boldsymbol{v}}^i_{b/i} = \left(\left[I_{3\times3} - \left(\delta\boldsymbol{\theta}^b_b\right)\times\right]\hat{T}^b_i\right)\left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a - \boldsymbol{n}_a\right) + \boldsymbol{g}^i \tag{13}$$

and defining the nominal differential equation to be

$$\dot{\hat{\boldsymbol{v}}}^i_{b/i} = \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a \right) + \boldsymbol{g}^i \tag{14}$$

Subtracting Eqns. 13 and 14 yields

$$\begin{aligned}
\delta \dot{\boldsymbol{v}}^i_{b/i} &= \left(\left[I_{3\times3} - \left(\delta\boldsymbol{\theta}^b_b \right) \times \right] \hat{T}^b_i \right) \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a - \boldsymbol{n}_a \right) + \boldsymbol{g}^i - \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a \right) - \boldsymbol{g}^i \\
&= \left(\hat{T}^b_i - \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \right) \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a - \boldsymbol{n}_a \right) + \boldsymbol{g}^i - \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a \right) - \boldsymbol{g}^i \\
&= \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a - \boldsymbol{n}_a \right) - \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a - \boldsymbol{n}_a \right) + \boldsymbol{g}^i - \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a \right) - \boldsymbol{g}^i \\
&= \hat{T}^b_i \tilde{\boldsymbol{a}}^b - \hat{T}^b_i \hat{\boldsymbol{b}}_a + \hat{T}^b_i \delta\boldsymbol{b}_a - \hat{T}^b_i \boldsymbol{n}_a - \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \tilde{\boldsymbol{a}}^b + \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \hat{\boldsymbol{b}}_a - \cancelto{0}{\left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \delta\boldsymbol{b}_a} + \cancelto{0}{\left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \boldsymbol{n}_a} + \boldsymbol{g}^i - \hat{T}^b_i \tilde{\boldsymbol{a}}^b + \hat{T}^b_i \hat{\boldsymbol{b}} \\
&= \hat{T}^b_i \delta\boldsymbol{b}_a - \hat{T}^b_i \boldsymbol{n}_a - \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \tilde{\boldsymbol{a}}^b + \left(\delta\boldsymbol{\theta}^b_b \right) \times \hat{T}^b_i \hat{\boldsymbol{b}}_a \\
&= \hat{T}^b_i \delta\boldsymbol{b}_a - \hat{T}^b_i \boldsymbol{n}_a + \left(\hat{T}^b_i \tilde{\boldsymbol{a}}^b \right) \times \delta\boldsymbol{\theta}^b_b - \left(\hat{T}^b_i \hat{\boldsymbol{b}}_a \right) \times \delta\boldsymbol{\theta}^b_b \\
&= \left[\left\{ \hat{T}^b_i \left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a \right) \right\} \times \right] \delta\boldsymbol{\theta}^b_b + \hat{T}^b_i \delta\boldsymbol{b}_a - \hat{T}^b_i \boldsymbol{n}_a
\end{aligned}$$

From Markley and Crassidis, Fundamentals of Spacecraft Attitude Determination and Control, 2014, pages 37-48, 71, 76-77, 127-128, 239-246, 257-260, 267, the attitude error dynamics are

$$\delta\dot{\boldsymbol{\theta}}^b = \left(\boldsymbol{\omega} - \hat{\boldsymbol{\omega}} \right) - \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{\theta}^b \tag{22}$$

where

$$\hat{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}}^b_{b/i} - \hat{\boldsymbol{b}}_g \tag{23}$$

Substitution yields

$$\delta\dot{\boldsymbol{\theta}}^b = \left(\boldsymbol{\omega} - \tilde{\boldsymbol{\omega}}^b_{b/i} + \hat{\boldsymbol{b}}_g \right) - \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{\theta}^b \tag{24}$$

Recall that measured gyro is

$$\tilde{\boldsymbol{\omega}}^b_{b/i} = \boldsymbol{\omega} + \boldsymbol{b}_g + \boldsymbol{n}_g \tag{25}$$

Substitution yields

$$\delta\dot{\boldsymbol{\theta}}^b = \left(\boldsymbol{\omega} - \boldsymbol{\omega} - \boldsymbol{b}_g - \boldsymbol{n}_g + \hat{\boldsymbol{b}}_g \right) - \hat{\boldsymbol{\omega}} \times \delta\boldsymbol{\theta}^b \tag{26}$$

Simplification and rearranging yields

$$\delta\dot{\boldsymbol{\theta}}^b = -\hat{\boldsymbol{\omega}} \times \delta\boldsymbol{\theta}^b - \delta\boldsymbol{b}_g - \boldsymbol{n}_g \tag{27}$$

1 Measurement model linearization

Begin with the design model of a simple GPS-based position measurement

$$\tilde{\boldsymbol{z}}_{gps} = \boldsymbol{h}_{gps}\left(\boldsymbol{x}, t \right) + G_{gps} \nu_{gps} \tag{28}$$

where the nonlinear measurement function is

$$\boldsymbol{h}_{gps}\left(\boldsymbol{x}, t \right) = \boldsymbol{r}^i_{b/i} \tag{29}$$

$$G_{gps} = I_{3\times3} \tag{30}$$

Step 1: Define the perturbations

$$\boldsymbol{r}^i_{b/i} = \hat{\boldsymbol{r}}^i_{b/i} + \delta\boldsymbol{r}^i_{b/i} \tag{31}$$

$$\tilde{\boldsymbol{z}}_{gps} = \hat{\tilde{\boldsymbol{z}}}_{gps} + \delta\tilde{\boldsymbol{z}}_{gps} \tag{32}$$

Step 2: Not needed
Step 3: Define the "nominal"

$$\hat{\tilde{\boldsymbol{z}}}_{gps} = \boldsymbol{h}_{gps}\left(\hat{\boldsymbol{x}}, t\right) = \hat{\boldsymbol{r}}^i_{b/i} \tag{33}$$

Step 4: Substitute

$$\hat{\tilde{\boldsymbol{z}}}_{gps} + \delta\tilde{\boldsymbol{z}}_{gps} = \hat{\boldsymbol{r}}^i_{b/i} + \delta\boldsymbol{r}^i_{b/i} + \nu_{gps} \tag{34}$$

Step 5: Cancel the nominal

$$\delta\tilde{\boldsymbol{z}}_{gps} = \delta\boldsymbol{r}^i_{b/i} + \nu_{gps} \tag{35}$$

Step 6: Discard second and higher order terms, not needed
Determine the H matrix

$$\delta\tilde{\boldsymbol{z}}_{gps} = \left[I_{3\times3} 0_{3\times15}\right]\delta\boldsymbol{x} + \nu_{gps} \tag{36}$$

Now apply the approach to the more complicated phase difference measurement. The design model of the measurement is

$$\tilde{\boldsymbol{z}}_{pd} = k\left(d_2 - d_1\right) + \nu \tag{37}$$

where

$$d_1 = \left\|r^i_{c/i} - r^i_{b/i} - T^i_b r^b_{1/b}\right\| = \sqrt{\boldsymbol{\rho}_1^T\boldsymbol{\rho}_1} = \left(\boldsymbol{\rho}_1^T\boldsymbol{\rho}_1\right)^{1/2} \tag{38}$$

$$d_2 = \left\|r^i_{c/i} - r^i_{b/i} - T^i_b r^b_{2/b}\right\| = \sqrt{\boldsymbol{\rho}_2^T\boldsymbol{\rho}_2} = \left(\boldsymbol{\rho}_2^T\boldsymbol{\rho}_2\right)^{1/2} \tag{39}$$

Step 1: Define perturbations

$$\tilde{\boldsymbol{z}}_{pd} = \hat{\tilde{\boldsymbol{z}}}_{pd} + \delta\tilde{\boldsymbol{z}}_{pd} \tag{40}$$

$$r^i_{c/i} = \hat{r}^i_{c/i} + \delta r^i_{c/i} \tag{41}$$

$$r^i_{b/i} = \hat{r}^i_{b/i} + \delta r^i_{b/i} \tag{42}$$

$$T^b_i = \left[I_{3\times3} - \left(\delta\boldsymbol{\theta}^b_b\right)\times\right]\hat{T}^b_i \tag{43}$$

$$T^i_b = \hat{T}^i_b\left[I_{3\times3} + \left(\delta\boldsymbol{\theta}^b_b\right)\times\right] \tag{44}$$

$$d_2 = \hat{d}_2 + \delta d_2 \tag{45}$$

$$d_1 = \hat{d}_1 + \delta d_1 \tag{46}$$

Playing around

$$\hat{\tilde{\boldsymbol{z}}}_{pd} + \delta\tilde{\boldsymbol{z}}_{pd} = k\left(\hat{d}_2 + \delta d_2 - \hat{d}_1 - \delta d_1\right) + \nu \tag{47}$$

Subtract the nominal

$$\delta\tilde{\boldsymbol{z}}_{pd} = k\left(\delta d_2 - \delta d_1\right) + \nu \tag{48}$$

Note that

$$\delta d_1 \approx \frac{\partial d_1}{\partial\boldsymbol{\rho}_1}\delta\boldsymbol{\rho}_1 \tag{49}$$

$$\delta d_2 \approx \frac{\partial d_2}{\partial\boldsymbol{\rho}_2}\delta\boldsymbol{\rho}_2 \tag{50}$$

where

$$\frac{\partial d_1}{\partial\boldsymbol{\rho}_1} = \frac{1}{2}\left(\boldsymbol{\rho}_1^T\boldsymbol{\rho}_1\right)^{-1/2}2\boldsymbol{\rho}_1^T = \left(\boldsymbol{\rho}_1^T\boldsymbol{\rho}_1\right)^{-1/2}\boldsymbol{\rho}_1^T = \frac{\boldsymbol{\rho}_1^T}{\|\boldsymbol{\rho}_1\|} = \boldsymbol{u}_1^T \tag{51}$$

$$\frac{\partial d_2}{\partial\boldsymbol{\rho}_2} = \frac{1}{2}\left(\boldsymbol{\rho}_2^T\boldsymbol{\rho}_2\right)^{-1/2}2\boldsymbol{\rho}_2^T = \left(\boldsymbol{\rho}_2^T\boldsymbol{\rho}_2\right)^{-1/2}\boldsymbol{\rho}_2^T = \frac{\boldsymbol{\rho}_2^T}{\|\boldsymbol{\rho}_2\|} = \boldsymbol{u}_2^T \tag{52}$$

The last step is to determine the perturbations in $\delta\boldsymbol{\rho}_i$. Expanding $\boldsymbol{\rho}_1$ about the estimated quantities yields

$$\begin{aligned}\hat{\rho}_1 + \delta\rho_1 &= \hat{r}^i_{c/i} + \delta r^i_{c/i} - \hat{r}^i_{b/i} - \delta r^i_{b/i} - \hat{T}^i_b\left[I_{3\times3} + \left(\delta\boldsymbol{\theta}^b_b\right)\times\right]r^b_{1/b} && (53)\\ &= \hat{r}^i_{c/i} + \delta r^i_{c/i} - \hat{r}^i_{b/i} - \delta r^i_{b/i} - \hat{T}^i_b r^b_{1/b} - \hat{T}^i_b\left(\delta\boldsymbol{\theta}^b_b\right)\times r^b_{1/b} && (54)\end{aligned}$$

where

$$\hat{\rho}_1 = \hat{r}^i_{c/i} - \hat{r}^i_{b/i} - \hat{T}^i_b r^b_{1/b} \tag{55}$$

Subtracting the nominal yields

$$\delta\rho_1 = \delta r^i_{c/i} - \delta r^i_{b/i} - \hat{T}^i_b \left(\delta\boldsymbol{\theta}^b_b\right) \times r^b_{1/b} \tag{56}$$

which is equivalent to

$$\delta\rho_1 = \delta r^i_{c/i} - \delta r^i_{b/i} + \hat{T}^i_b \left[\left(r^b_{1/b}\right)\times\right] \delta\boldsymbol{\theta}^b_b \tag{57}$$

Following the same process, we obtain

$$\delta\rho_2 = \delta r^i_{c/i} - \delta r^i_{b/i} + \hat{T}^i_b \left[\left(r^b_{2/b}\right)\times\right] \delta\boldsymbol{\theta}^b_b \tag{58}$$

Working our way back as follows

$$\delta\tilde{\boldsymbol{z}}_{pd} = k\left(\frac{\partial d_2}{\partial \boldsymbol{\rho}_2}\delta\boldsymbol{\rho}_2 - \frac{\partial d_1}{\partial \boldsymbol{\rho}_1}\delta\boldsymbol{\rho}_1\right) + \nu \tag{59}$$

$$\delta\tilde{\boldsymbol{z}}_{pd} = k\left(\boldsymbol{u}_2^T\left\{\delta r^i_{c/i} - \delta r^i_{b/i} + \hat{T}^i_b \left[\left(r^b_{2/b}\right)\times\right] \delta\boldsymbol{\theta}^b_b\right\} - \boldsymbol{u}_1^T\left\{\delta r^i_{c/i} - \delta r^i_{b/i} + \hat{T}^i_b \left[\left(r^b_{1/b}\right)\times\right] \delta\boldsymbol{\theta}^b_b\right\}\right) + \nu \tag{60}$$

Distribution yields

$$\begin{aligned}
\delta\tilde{\boldsymbol{z}}_{pd} &= k\left(\boldsymbol{u}_2^T - \boldsymbol{u}_1^T\right)\delta r^i_{c/i} && (61)\\
&+ k\left(\boldsymbol{u}_1^T - \boldsymbol{u}_2^T\right)\delta r^i_{b/i} && (62)\\
&+ k\left(\boldsymbol{u}_2^T\hat{T}^i_b\left[\left(r^b_{2/b}\right)\times\right] - \boldsymbol{u}_1^T\hat{T}^i_b\left[\left(r^b_{1/b}\right)\times\right]\right)\delta\boldsymbol{\theta}^b_b && (63)\\
&+ \nu && (64)
\end{aligned}$$

The resulting H matrix is

$$H_{pd} = \left[\begin{array}{cccccc} k\left(\boldsymbol{u}_1^T - \boldsymbol{u}_2^T\right) & 0_{1\times 3} & k\left(\boldsymbol{u}_2^T\hat{T}^i_b\left[\left(r^b_{2/b}\right)\times\right] - \boldsymbol{u}_1^T\hat{T}^i_b\left[\left(r^b_{1/b}\right)\times\right]\right) & 0_{1\times 3} & 0_{1\times 3} & k\left(\boldsymbol{u}_2^T - \boldsymbol{u}_1^T\right)\end{array}\right] \tag{65}$$

where

$$\boldsymbol{u}_1 = \frac{\boldsymbol{\rho}_1}{\|\boldsymbol{\rho}_1\|} = \frac{\hat{r}^i_{c/i} - \hat{r}^i_{b/i} - \hat{T}^i_b r^b_{1/b}}{\left\|\hat{r}^i_{c/i} - \hat{r}^i_{b/i} - \hat{T}^i_b r^b_{1/b}\right\|} \tag{66}$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{\rho}_2}{\|\boldsymbol{\rho}_2\|} = \frac{\hat{r}^i_{c/i} - \hat{r}^i_{b/i} - \hat{T}^i_b r^b_{2/b}}{\left\|\hat{r}^i_{c/i} - \hat{r}^i_{b/i} - \hat{T}^i_b r^b_{2/b}\right\|} \tag{67}$$

Double check to make sure H is correct

$$\delta\tilde{\boldsymbol{z}}_{pd} = H_{pd}\begin{bmatrix}\delta\boldsymbol{r}^i_{b/i}\\ \delta\boldsymbol{v}^i_{b/i}\\ \delta\boldsymbol{\theta}^b_b\\ \delta\boldsymbol{b}_a\\ \delta\boldsymbol{b}_g\\ \delta\boldsymbol{r}^i_{c/i}\end{bmatrix} + \nu \tag{68}$$

$$\delta\tilde{\boldsymbol{z}}_{pd} = \left[\begin{array}{cccccc} k\left(\boldsymbol{u}_1^T - \boldsymbol{u}_2^T\right) & 0_{1\times 3} & k\left(\boldsymbol{u}_2^T\hat{T}^i_b\left[\left(r^b_{2/b}\right)\times\right] - \boldsymbol{u}_1^T\hat{T}^i_b\left[\left(r^b_{1/b}\right)\times\right]\right) & 0_{1\times 3} & 0_{1\times 3} & k\left(\boldsymbol{u}_2^T - \boldsymbol{u}_1^T\right)\end{array}\right]\begin{bmatrix}\delta\boldsymbol{r}^i_{b/i}\\ \delta\boldsymbol{v}^i_{b/i}\\ \delta\boldsymbol{\theta}^b_b\\ \delta\boldsymbol{b}_a\\ \delta\boldsymbol{b}_g\\ \delta\boldsymbol{r}^i_{c/i}\end{bmatrix} + \nu \tag{69}$$

which yields what we had before

$$\delta\tilde{\boldsymbol{z}}_{pd} = k\left(\boldsymbol{u}_1^T - \boldsymbol{u}_2^T\right)\delta\boldsymbol{r}^i_{b/i} + k\left(\boldsymbol{u}_2^T\hat{T}^i_b\left[\left(r^b_{2/b}\right)\times\right] - \boldsymbol{u}_1^T\hat{T}^i_b\left[\left(r^b_{1/b}\right)\times\right]\right)\delta\boldsymbol{\theta}^b_b + k\left(\boldsymbol{u}_2^T - \boldsymbol{u}_1^T\right)\delta\boldsymbol{r}^i_{c/i} + \nu \tag{70}$$