The truth state is a motorcycle model in the 2D plane

$$\boldsymbol{x}_t = \begin{bmatrix} r_{xi} \\ r_{yi} \\ v_{xb} \\ \psi \\ \phi \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{1}$$

The corresponding dynamics are

$$\dot{\boldsymbol{x}}_t = \begin{bmatrix} v_{xb}\cos\psi \\ v_{xb}\sin\psi \\ a \\ \frac{v_{xb}}{L}\tan\phi \\ \xi \\ -\frac{1}{\tau_a}\boldsymbol{b}_a + \boldsymbol{w}_a \\ -\frac{1}{\tau_g}\boldsymbol{b}_g + \boldsymbol{w}_g \\ \boldsymbol{0} \end{bmatrix} \tag{2}$$

The continuous measurements consist of accelerometer and gyro measurements, where the y component includes the lateral accelerations felt during a turn and the z component includes gravity.

$$\tilde{\boldsymbol{a}}^b = \begin{bmatrix} a & \frac{v_{xb}^2}{L}\tan\phi & -g \end{bmatrix}^T + \boldsymbol{b}_a + \boldsymbol{n}_a \tag{3}$$

$$\tilde{\boldsymbol{\omega}}^b_{b/i} = \begin{bmatrix} 0 & 0 & \frac{v_{xb}}{L}\tan\phi \end{bmatrix}^T + \boldsymbol{b}_g + \boldsymbol{n}_g \tag{4}$$

The design states model is a conventional inertial navigation system augmented with the position of the circuit

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{5}$$

The dynamics are defined as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{v}^i_{b/i} \\ T^i_b\left(\tilde{\boldsymbol{a}}^b - \boldsymbol{b}_a - \boldsymbol{n}_a\right) + \boldsymbol{g}^i \\ \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}^b_{b/i} - \boldsymbol{b}_g - \boldsymbol{n}_g \end{bmatrix} \otimes \boldsymbol{q}^b_i \\ -\frac{1}{\tau_a}\boldsymbol{b}_a + \boldsymbol{w}_a \\ -\frac{1}{\tau_g}\boldsymbol{b}_g + \boldsymbol{w}_g \\ \boldsymbol{0} \end{bmatrix} \tag{6}$$

The method for propagating the navigation states is

$$\dot{\hat{\boldsymbol{x}}} = \begin{bmatrix} \hat{\boldsymbol{v}}^i_{b/i} \\ \hat{T}^i_b\left(\hat{\boldsymbol{q}}^b_i\right)\left(\tilde{\boldsymbol{a}}^b - \hat{\boldsymbol{b}}_a\right) + \boldsymbol{g}^i \\ \frac{1}{2}\begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}^b_{b/i} - \hat{\boldsymbol{b}}_g \end{bmatrix} \otimes \boldsymbol{q}^b_i \\ -\frac{1}{\tau_a}\hat{\boldsymbol{b}}_a \\ -\frac{1}{\tau_g}\hat{\boldsymbol{b}}_g \\ \boldsymbol{0} \end{bmatrix} \tag{7}$$

The relationship between the navigation state, error state, and the true navigation state is the following

$$\begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{r}}^i_{b/i} + \delta\boldsymbol{r}^i_{b/i} \\ \hat{\boldsymbol{v}}^i_{b/i} + \delta\boldsymbol{v}^i_{b/i} \\ \begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}^b_b}{2} \end{bmatrix} \otimes \hat{q}^b_i \\ \hat{\boldsymbol{b}}_a + \delta\boldsymbol{b}_a \\ \hat{\boldsymbol{b}}_g + \delta\boldsymbol{b}_g \\ \hat{\boldsymbol{r}}^i_{c/i} + \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{8}$$

The error injection mapping is the following

$$\begin{bmatrix} \hat{\boldsymbol{r}}^i_{b/i} \\ \hat{\boldsymbol{v}}^i_{b/i} \\ \hat{q}^b_i \\ \hat{\boldsymbol{b}}_a \\ \hat{\boldsymbol{b}}_g \\ \hat{\boldsymbol{r}}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} - \delta\boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} - \delta\boldsymbol{v}^i_{b/i} \\ \begin{bmatrix} 1 \\ -\frac{\delta\boldsymbol{\theta}^b_b}{2} \end{bmatrix} \otimes q^b_i \\ \boldsymbol{b}_a - \delta\boldsymbol{b}_a \\ \boldsymbol{b}_g - \delta\boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} - \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{9}$$

The error calculation mapping is the following, where it is assumed that the quaternion contains the scalar in the first element of a 4 element vector

$$\begin{bmatrix} \delta\boldsymbol{r}^i_{b/i} \\ \delta\boldsymbol{v}^i_{b/i} \\ \delta\boldsymbol{\theta}^b_b \\ \delta\boldsymbol{b}_a \\ \delta\boldsymbol{b}_g \\ \delta\boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}^i_{b/i} - \hat{\boldsymbol{r}}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} - \hat{\boldsymbol{v}}^i_{b/i} \\ 2\begin{bmatrix} 0_{3\times1} & I_{3\times3} \end{bmatrix} q^b_i \otimes \left(\hat{q}^b_i\right)^* \\ \boldsymbol{b}_a - \hat{\boldsymbol{b}}_a \\ \boldsymbol{b}_g - \hat{\boldsymbol{b}}_g \\ \boldsymbol{r}^i_{c/i} - \hat{\boldsymbol{r}}^i_{c/i} \end{bmatrix} \tag{10}$$

Finally, the mapping between truth and design states is

$$\begin{bmatrix} \boldsymbol{r}^i_{b/i} \\ \boldsymbol{v}^i_{b/i} \\ q^b_i \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_{xi} \\ r_{yi} \\ 0 \end{bmatrix} \\ \begin{bmatrix} v_{xb}\cos\psi \\ v_{xb}\sin\psi \\ 0 \end{bmatrix} \\ \begin{bmatrix} \cos\left(\frac{\psi}{2}\right) \\ 0 \\ 0 \\ \sin\left(\frac{\psi}{2}\right) \end{bmatrix} \\ \boldsymbol{b}_a \\ \boldsymbol{b}_g \\ \boldsymbol{r}^i_{c/i} \end{bmatrix} \tag{11}$$