Universitat Politécnica de Catalunya - Universitat de Barcelona
Faculty of Mathematics & Statistics

Master's Degree in Statistics and Operations Research
Master's Thesis

Probabilistic and Statistical Methods for Power Laws in Complex Systems: Applications to Finance

Iker Caballero Bragagnini

Supervised by
Àlvaro Corral - Centre de Recerca Matemàtica (Complex Systems Group)
Luis Ortiz - Universitat de Barcelona (RiskCenter UB)

29th May 2024

This master thesis is dedicated to all my family, who always trusted me, and to the one who has supported me always, Jinni.

Thanks to Dr. Álvaro Corral, Dr. Luis Ortiz, and Dr. Isabel Serra for their help and guidance during the elaboration of this thesis, and to Dr. Elisa Alòs for making me delve into mathematical finance and econophysics.

Abstract

Keywords: Complex Systems, Power Law, Finance, Extreme Value Theory, Stable Processes, Lévy, Mandelbrot

The application of statistical methodologies and probabilistic results of power laws (PLs) could prove useful for financial modelling and risk quantification. In this project, we delve into the connections between financial markets, PLs, extreme value theory (EVT), and stochastic processes such as α-stable processes. We provide empirical applications of statistical methods based on PLs and EVT results using real high-frequency data for stock indexes in the Japanese (Nikkei 225) and United Kingdom (FTSE 100) markets, where we reproduce statistical analyses from the references but also discuss interesting patterns found and their implications for financial practice. Moreover, we discuss the relevance of different probabilistic forces, distributions, and processes found in the econophysics literature and the different methodologies considered to estimate relevant elements for our analysis.

List of Acronyms

CV .... Coefficient of Variation

EVT .... Extreme Value Theory

CLT .... Central Limit Theorem

GCLT .... Generalized Central Limit Theorem

GEV .... Generalized Extreme Value

GPD .... Generalized Pareto Distribution

iid .... Identical and Independently Distributed

IQR .... Interquartile Range

K-S .... Kolmogorov-Smirnov

MDA .... Maximum Domain of Attraction

ML .... Maximum Likelihood

PL .... Power law

OLS .... Ordinary Least Squares

List of Notations

$\mathbb{R}$ . . . . Set of real numbers

$\mathbb{D}$ . . . . Skorokhod Space

$\sim$ . . . . Distributed as

$\stackrel{D}{=}$ . . . . Equality in distribution

$\xrightarrow{D}$ . . . . Convergence in distribution

$(X_i)_{i\in\mathcal{A}}$ . . . . Sequence of variables X_i indexed by i in set $\mathcal{A}$

Table of Contents

List of Figures

List of Tables

1 | Introduction

In the realm of economics and finance, an uncommon synergy has emerged between mathematicians, scientists, and economists, igniting academic and practical interest in deciphering and modelling dynamics of markets and economic agents. This interdisciplinary intrigue has given rise to unorthodox domains within economics, notably econophysics, which wants to unveil the hidden order within the apparent chaos of financial systems by using theories and tools from scientific fields such as mathematics, physics, engineering, and many others. Most of the works in the literature focus especially on financial markets, and all of these ideas stem from the same principle: financial markets can be understood and modelled as complex systems.

One of the most common mathematical relationships found in complex systems are power laws (PLs). These offer relevant insights into the underlying mechanics of complex systems and have important implications for models that try to reproduce them. Therefore, the application of statistical methodologies and probabilistic results of PLs could prove useful for financial modelling and risk quantification. Indeed, the econophysics literature jointly with the financial risk management literature have given special emphasis to this topic, being explored nowadays by different academic institutions such as Oxford University (Johnson, Jefferies, & Hui, 2003) and Santa Fe Institute (Farmer & Lo, 1999). Relevant authors such as Mandelbrot, Sornette, and Gabaix developed different hypotheses and models having PLs and their special properties as core elements.

In this project, we delve into the connections between complex systems, PLs, extreme value theory (EVT), and stochastic processes such as α-stable processes, due to the relevance each of these has for finance. The main goal is to justify the usage of PLs by viewing financial markets as complex systems and through probabilistic results stemming from EVT and to showcase that the statistical analysis of PLs provides a promising toolset for analyzing and modelling financial data. Moreover, we provide empirical applications of some of the results presented using high-frequency data, where we will reproduce statistical analyses and discuss interesting patterns found and implications for financial practice.

This thesis is structured in the following way. The second chapter is concerned with the relationship between financial markets and complex systems, first giving some context about financial markets and then continuing to compare the properties of markets and complex systems. The next chapter lays out the necessary theoretical results about PLs, EVT, risk, and stable distributions and processes, as all these elements are interrelated and allow justifying the usage of PLs for statistical analysis of financial data. Moreover, we also discuss the relevance of different PLs, distributions, and processes found in econophysics and the different methodologies considered to estimate relevant elements for our analysis. In the fourth chapter, we delve into the empirical applications proposed using real high-frequency data for stock indexes in the Japanese (Nikkei 225) and United Kingdom (FTSE 100) markets, where we use the statistical methodologies explained and discuss the different results concerning the econophysics and finance literature and their implications. Finally, we end this thesis by summarizing the different results and insights and proposing different ideas following this line of research.

2 | Complex Systems and Finance

2.1 Financial Markets and Complex Systems

The United States Department of Treasury defines financial markets as any place or system that provides buyers and sellers the means to trade financial instruments, facilitating the interaction between those who need capital with those who have capital to invest and allowing participants to transfer risk and promote commerce (Office of the Controller of the Currency, 2019). Financial markets are one of the main objects of study in economics and are often studied from the microeconomic and the macroeconomic point of view, hence leading to an analytical approach where one decomposes these "systems" in components (particular investors' behaviour, the behaviour of banks in a whole country, how asset prices change with consumption, etc.) hoping that a detailed understanding of each component leads to understanding the functioning of the whole (Sornette, 2003). This leads to making restricting assumptions for a formal and axiomatic treatment but not met in reality (Stiglitz, 2004) and creates models that might not address features that can have important implications in different domains (Farmer & Geanakoplos, 2008).

Therefore, even though standard economic theory understands financial markets as systems, mainstream contemporary models might not reflect some important aspects of financial markets' complexity, such as interrelations between various non-trivially related aspects, agents with different or unpredictable behaviours, non-linear interactions or relationships, and others (Kuhlmann, 2014). There is a need for a systemic and multidisciplinary approach that leads to better outcomes when it comes to understanding and usefulness in real life. Due to these facts and the interest of physicists and mathematicians in social sciences, new non-orthodox multidisciplinary fields such as econophysics were born and have gained interest among scientists and economists. Even though there is no official definition, econophysics can be defined as the usage of theories and methods from physics for economics and finance.

In this discipline, statistical, physical, and mathematical models are used to study different problems in finance. One of the most important features is that it advocates that financial markets should be thought of as complex systems so that one can apply tools from statistics, data science, and complexity science in a justified manner. When understood as complex systems, markets might not need to be in equilibrium, there might not be perfect rationality of participants and situations are not always well-defined (in contrast to common models in financial economics). The economy and the markets are more like an "ecology of beliefs, organizing principles and behaviours" (Arthur, 2021). This shift allows us to analyze and study the different aspects of financial markets that are not commonly studied in standard economic disciplines, by using different tools and ideas that potentially lead to more realistic or at least credible results.

Understanding markets like this serves as the basis that justifies the aim of this thesis: exploring some probabilistic results and statistical tools for PLs for financial modelling in different areas. Nevertheless, it remains to justify why financial markets can, indeed, be compared with complex systems. Hence, in the next section, we delve into the shared features of complex systems and financial markets so that we ultimately see that both are similar in this regard and hence the view we take is reasonable.

2.2 Features of Financial Markets

We argued that financial markets share various features with complex systems, so it is reasonable to view these markets as complex systems and, consequently, apply tools used for complex systems modeling. This section aims to expose some of these shared characteristics and put the spotlight on a crucial feature that motivates this project: the presence of PLs in finance. Even though a lengthy discussion could shed more light on the complete features, we just mention and briefly explain some of the most prominent features for information and basic understanding. We refer to Sornette (2003) and Kuhlmann (2014) for a more detailed comparison and analysis of the features of complex systems in financial markets. Some of the most important features that allow the categorization of financial markets as complex systems are the following:

Interconnectedness: It refers to the presence of multiple connections between multiple parts of the system that lead to interdependence, which may occur between individual elements, subsystems, or even other systems. A great example of interconnectedness might be the financial crisis of 2008, where a bubble in the real estate market in some countries led to the collapse of the global financial system (including other markets as well).

Self-organization: Self-organization represents the spontaneous emergence of order in natural and physical systems (Kauffman, 1993), but it emerges economic systems as well (Witt, 1997). A clear example of self-organizing behaviour in financial markets is a bank run, where investors organize themselves to get the money out of the banks (herding behaviour).

Chaotic Behaviour: This behavior consists on the tiniest differences on the micro level being amplified to an effect of any size due to the non-linear interaction between the system's elements (Kuhlmann, 2014). Many examples of chaotic behaviour can be found in financial markets, such as the crash of October 1987 (Sornette, 2003) or the recent "meme stocks" rally, which generated a shock in American markets (CNN, 2022).

Robust Behaviour: A robust behavior means that the system is insensitive to variations in the micro level in terms of coordination of the interacting parts that compose the system, also due to the non-linearity of interactions. Examples of this behaviour are volatility clustering or PL decay of price variations (henceforth, returns): they are statistical patterns found in financial data in which the initial conditions' variations seem to be dissolved into collective statistics so that perturbations in the initial conditions (inside a reasonable range) do not affect these patterns that emerge in financial markets (Kuhlmann, 2014).

Scale Invariance of Fluctuations: This means that there is no concrete size or scale for the fluctuations of market prices of different stocks. For example, relative changes in stock prices maintain their distribution independently of time and magnitude scale (Mantegna & Stanley, 2000; Kuhlmann, 2014). Hence, large and small changes are distributed identically and the tails of the distribution are fatter than the tails of a normal distribution, which implies that the probability distribution for these large changes in price behaves like a PL (the tails have a PL function form).

Clusterization: In financial markets, large changes tend to be followed by other large changes, so that clusters are formed. This is a statistical phenomenon that can be easily observed in volatility time series for stock returns in different stock markets.

Openness: Openness is a characteristic of a system that is coupled to the environment or other systems so that there are endogenous effects (caused by elements of the same system) and exogenous effects (from other systems connected to it), and this openness also manifests in financial markets (Sornette, Deschâtres, Gilbert, & Ageon, 2004). This openness can be better understood, for example, by looking at the relationship between climate change and financial and economic decisions, which has gained recent attention due to its potential economic effects.

Feedback loops: These are mechanisms by which change in a variable results in either an amplification or a dampening of that change, and financial markets are full of these because there is usage of past information to make decisions that alter the future. In this regard, a clear feedback loop can be a bubble caused by the increase in share prices, which leads to investors buying and increasing the prices again (positive feedback).

Some of these characteristics are part of the list of statistical patterns found in financial data which are called "stylized facts" (Cont, 2001), such as volatility clustering, but others purely pertain to fields in which complex systems show up. Even though there is no widely accepted theory, academics and researchers in the area of econophysics (and us) support the idea that some of the features explained above could partly justify the "stylized fact" we focus on in this thesis: the presence of heavy and fat tails in the distribution of financial returns.

It is widely observed that, in reality, the probability distribution of returns has heavier or, in some cases, fatter tails than that of normal distributions (Cont, 2001; Embrechts, Kluppelberg, & Mikosch, 2013). Because this is pervasive in different assets (stocks, derivatives, currencies, etc.) and markets such as London, New York, Tokyo, and more (Sornette, 2003), academics and practitioners have found it worth studying this phenomenon and discussing its implications in different areas of finance such as risk management or asset pricing. However, different theories and justifications based on statistical and economic patterns have emerged for explaining heavy and fat tails, econophysics is the most relevant area of research that has provided theories based on features stemming from the complexity of financial markets to justify similar phenomena.

One of the most important contributions from econophysics for explaining and modelling the properties of the tails of financial returns' distributions has been to model them through PLs, which was first proposed by Mandelbrot (1963) through α-stable processes (which have PL tail behaviour) and is commonly studied nowadays. This idea is also deeply related to EVT, as this theory focuses on modelling extreme values (tails) through related distributions (like the generalized Pareto) and is commonly used for financial risk management.

Moreover, it is not just in the tails of a distribution where we can find PLs. Like in many physical complex systems, many PL relationships and behaviours also arise in economics and finance (Gabaix, 2009), and they are empirically observed in different aspects of financial markets. Hence, the interest of some authors to involve PLs in their theories and the relevance of PLs in complex systems motivates our focus on applying statistical and probabilistic methodologies for PLs in finance, giving some examples of how PLs can be used for modelling using real data.

All in all, financial markets seem to behave and share typical features of complex systems in natural sciences, so it is sensible to understand them as complex systems henceforth, complying with the view of econophysics (Kuhlmann, 2014). Additionally, we covered why PLs are relevant for complex systems such as (but not limited to) economic and financial systems and how they are related to other mathematical theories such as EVT, which motivates us to apply both PLs and EVT to study financial returns, the central topic of this project.

3 | Power Laws, EVT, and Finance

3.1 Power Laws

We have referred several times to PLs when explaining our view of financial markets as complex systems and the various common features, but we have not yet given a formal definition nor explained important characteristics related to them. In this section, we proceed to do so and discuss relevant properties and details about PLs, first starting with PL distributions. Then, we continue with a discussion on their appearance and relevance in finance and we close by exposing interesting results from the econophysics literature that motivate some of our empirical applications.

3.1.1 Power Law Distributions

Definition 3.1.1. Given a continuous real variable X, we say X follows a power law (PL) distribution, denoted as $PL(\alpha)$ for $\alpha > 0$, if its density function has the form

$$f(x) = Cx^{-\alpha-1} \tag{3.1}$$

where $\alpha > 0$ is the power law exponent and C is a constant.

Few real-world distributions follow a PL over their entire range, particularly for small values of the variable (Newman, 2005), so the distribution deviates from the PL form below some minimum value x_{min} at which the PL behaviour is obeyed. Hence, the range of the variable is $x \geq x_{min}$ and its PL distribution is denoted as $PL(\alpha, x_{min})$.

One can prove that the value of the constant C must be $C = \alpha x_{min}^{\alpha}$ by a normalization argument (see Appendix A.1) so that we obtain the following result:

Theorem 3.1.2. The probability density function of a continuous real variable X that follows a $PL(\alpha, x_{min})$ for $\alpha > 0$ is

$$f(x) = \frac{\alpha}{x_{min}^{-\alpha}} x^{-\alpha-1} = \frac{\alpha}{x_{min}} \left(\frac{x}{x_{min}}\right)^{-\alpha-1} \quad \text{for} \quad x \geq x_{min} \tag{3.2}$$

From this theorem, the cumulative distribution function $F(x)$ can be obtained by direct integration, which results in

$$F(x) = \begin{cases} 1 - \left(\frac{x}{x_{min}}\right)^{-\alpha}, & \text{for } x \geq x_{min} \\ 0, & \text{for } x < x_{min} \end{cases} \tag{3.3}$$

where $\alpha > 0$. Two important properties of this kind of distribution are the scale invariance and the divergence of high-order moments, stated in the next theorems (the proof for each one is given in Appendix A.2 and A.3).

Theorem 3.1.3. Given the density function $f(x)$ of a PL distribution $PL(\alpha, x_{min})$ for $\alpha > 0$ and a function g, then

$$f(bx) = g(b)f(x) \quad \text{where} \quad g(b) = \frac{f(b)}{f(1)} \tag{3.4}$$

for any $b > 0$. We say that the PL distribution is scale-invariant or that it is a free-scale distribution.

Theorem 3.1.4. Given a random variable X that follows a PL distribution $PL(\alpha, x_{min})$ for $\alpha > 0$, then

$$E(X^k) = \begin{cases} \frac{\alpha}{\alpha-k} x_{min}^k, & \text{for } k < \alpha \\ \infty, & \text{for } k \geq \alpha \end{cases} \tag{3.5}$$

so that the moments of the PL distribution do not exist for $k \geq \alpha$.

Finally, there is an important feature that will allow us to detect PL behaviour and represent it graphically. Given a PL function $f(x) = Cx^{-\alpha-1}$, taking logarithms on both sides of the equality yields

$$\ln[f(x)] = \ln(C) - (\alpha + 1)\ln(x) \tag{3.6}$$

so that the logarithm of $f(x)$ is a linear function with intercept $\ln(C)$ and slope $-(\alpha + 1)$ and the representation on a plot would be a straight line. The same linearity can be found when computing the logarithm of $1 - F(x)$ for the $F(x)$

defined in (3.3). For all values $x \geq x_{min}$,

$$\ln[1 - F(x)] = \alpha \ln(x_{min}) - \alpha \ln(x) = C - \alpha \ln(x) \tag{3.7}$$

where $C = \alpha \ln(x_{min})$. As we will discuss, the PL distribution is normally used for characterizing the behaviour of the tails of a distribution, which means that, by plotting the logarithm of x on the logarithm of $1 - F(x)$ (the right tail of the distribution) we should observe a linear behaviour if the tail is PL distributed, even though the converse does not hold (Newman, 2005; Deluca & Corral, 2013). This is why these kinds of plots are known as tail plots when used for analyzing the tail of a distribution, and we will use them throughout the empirical applications to detect possible PL behaviour on our data and represent it graphically.

3.1.2 Power Laws in Finance

In the first chapter, we remarked on the idea that PLs emerge in financial markets and that they can be applied for financial modelling. We now talk about where these PLs can appear and some of the mechanisms that generate them, while we also comment on some of the areas in Finance where those are applied.

Apart from the PL behavior of the tails in financial returns' distributions, there are other instances in which one can observe PL behaviour. For example, one can find a PL relationship in the autocorrelation of absolute returns (which creates clustered volatility and is related to the so-called Hurst exponent and long-range dependency) in the trading volume of different stocks, in firm sizes, in the transaction order volume, in the distribution of income and wealth, and many more (Farmer & Geanakoplos, 2008; Lux & Alfarano, 2016). This makes PL functions and distributions relevant for modelling certain variables or aspects of financial markets. Indeed, PLs have been used for modelling in different areas, such as asset pricing (Mandelbrot, 1963), risk management (Embrechts, Resnick, & Samorodnitsky, 1999; Embrechts et al., 2013; McNeil, Embrechts, & Frey, 2015), behavioural finance (Lux & Alfarano, 2016), volatility modelling (LeBaron, 2001; Jusselin & Rosenbaum, 2018), price impact modelling and market microstructure (Gabaix, Gopikrishnan, Plerou, & Stanley, 2006; Jusselin & Rosenbaum, 2018), using different data such as returns with different frequencies, trading volumes, the number of transactions, etc.

As Farmer and Geanakoplos (2004) argue, the requirements to generate PLs are not well-developed and the same phenomenon can be explained from different layers of detail. However, there is extensive literature on the mechanisms that can generate these PLs in economics and finance, which throws some light on how these emerge and how are related to the type of features of complex systems that we presented in the previous section. Aside from the tail behaviour of distributions, some mechanisms that can create PLs are self-similar processes (which will be discussed in the following sections), multiplicative processes (such as the log-normal distribution), the presence of hierarchies, deterministic dynamics and critical points, and dimensional constraints common in physics (Farmer & Geanakoplos, 2008).

As stated, we especially focus on the tails of the distribution of returns because it is one of the biggest contributions of econophysics that involve PLs and because financial returns are the main variable of interest in many finance areas such as risk management, asset pricing, and others. Specifically, we will use high-frequency data for this study because it is the kind of data in which PL tail behaviour has been observed in the econophysics literature, so we can apply statistical tools developed for the PL distribution and for others related to EVT to analyze the behaviour of returns, compare results with those of the literature, and explain the implications of the findings.

3.1.3 Power Laws from Econophysics

As argued, there are different areas and instances in which one observes PL behaviour, and the econophysics literature contains interesting findings regarding the presence of PL behaviour in asset returns. For this project, we focus on two PLs related to returns: the inverse cubic law for the tails of the returns distribution and the PL proposed by Müller et al. (1990), which relates the frequency of the returns with their mean absolute value or "volatility".

The inverse cubic PL distribution for stock price fluctuations or stock returns is one of the most relevant and empirically documented PL distributions in econophysics (Gopikrishnan, Plerou, Nunes Amaral, Meyer, & Stanley, 1999; Mantegna & Stanley, 2000; Plerou, Gabaix, & Stanley, 2001; Gabaix, Gopikrishnan, Plerou, & Stanley, 2003; Gabaix, 2009). This law states that

$$\overline{F}(x) \propto x^{-\alpha} \quad \text{with} \quad \alpha \approx 3 \tag{3.8}$$

where $\overline{F}(x) = P(|r_t| > x)$ and X represent the random variable for the stock returns r_t, defined as logarithmic returns $r_t = \log(P_t) - \log(P_{t-1})$ over a time interval Δt. Empirical estimates based on a first model by Lévy yielded an exponent $\alpha \in (0, 2)$, but the literature converged to the value of $\alpha \approx 3$ (Lux & Alfarano, 2016). This inverse cubic PL holds for various firm sizes and seems to hold internationally (Gabaix, 2009), implying that the tails are fatter than that of a normal distribution. This law has been estimated using high-frequency data (from 15 seconds to months) using different lengths of time and an adequate number of years for the frequency chosen (for seconds, one could use a year or less, while for months one needs more years).

The other interesting PL distribution, the one proposed by Müller et al. (1990), relates the average change in logarithmic prices (logarithmic returns), denoted by $\overline{|r_t|}$ with the frequency or time interval, denoted by Δt. This PL is

$$\overline{|r_t|} = C(\Delta t)^{1/\lambda} \tag{3.9}$$

where $1/\lambda$ is the PL exponent and C is a scaling constant. Müller et al. (1990) state that $|\hat{r}_t|$ was generated by an α-stable process, then one can relate λ to the characteristic exponent α of a stable characteristic function (see Section 3.4). Given that $|\hat{r}_t|$ can be interpreted as the volatility, especially by authors in option theory (Müller et al., 1990), then this would mean that the volatility is proportional to some power of the time interval or frequency. The pattern does not only support the pervasive presence of PL behaviour in financial markets but also has important implications, as this scaling law can be used as a reference of the average returns that an investor can obtain in a given time horizon or frequency and this has applications for areas such as risk management and volatility modelling.

In this thesis, the empirical applications explore the presence of all the previous PLs in real stock index data. We look specifically for these results because they are a clear opportunity to demonstrate the applications of statistical methods for PLs with real financial data, justified through the vision of complex systems and probabilistic results. However, we are not extremely concerned about carrying out a complete analysis nor obtaining very precise results: we just illustrate statistical analyses that are common in the literature and have sound statistical rationale given the theory.

Each PL is interesting because we will also analyze complementary statistical aspects relevant to financial practice. For the first PL, the inverse cubic PL, two forces play an important role in shaping the behaviour of the tails of the returns' distributions, the central limit theorem, and the summation of PL-behaved tails, which will be explained in later sections. For the second law, the scaling law proposed by Müller et al. (1990), we carry on complementary statistical analysis to assess whether we could discard (or not) the hypothesis of returns following an α-stable process, as this would explain partly the presence of scaling laws like the one proposed.

3.2 Extreme Value Theory

Apart from modelling tails as PLs, what has been a common practice in mainstream financial risk management in recent years is to consider a branch of statistics that is concerned with extreme observations and behaviour: extreme value theory (EVT). It serves as a theoretical basis for understanding PL tail behaviour and is intimately related to PL distributions (Alfarano & Lux, 2010), so one can also use the theory and tools of EVT to infer and obtain insights from the data we analyze, especially for risk management applications. This adds value to our applications as we do not just obtain a PL fit, but we also compare it to commonly used distributions for extremes (like the GPD, introduced next) for risk modelling and measuring.

Hence, in this section we delve into heavy-tailed and fat-tailed distributions (which justify PL tails), defining the GEV and GPD distributions, and finally explaining the relevance of EVT in finance.

3.2.1 Heavy-tailed Distributions

The one "stylized fact" or empirical pattern we focus on is that the distributions of stock returns have heavier or fatter tails than the normal distribution (which is a common assumption for financial models). Different references use the terms "fat" and "heavy" to state it, but in probability and statistics, there is a difference between both types of tails. Heavy-tailed distributions are a superset of fat-tailed ones, and this last kind has a straight relation to PLs. To make this distinction clear and discuss the implications of each type of distribution, we introduce the class of heavy-tailed distributions and its principal subclasses, defined and presented based on Embrechts, Kluppelberg & Mickosch (2013).

Definition 3.2.1. The class of heavy-tailed distributions, denoted as $\mathcal{K}$, is defined as

$$\mathcal{K} \equiv \{F \text{ on } (0,\infty) : M(t) = \infty, \quad \forall t > 0\} \tag{3.10}$$

where $M(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} dF(x)$ is the moment generating function of a random variable X, which has distribution function F defined on $(0,\infty)$.

Therefore, heavy-tailed distributions are distributions where the moment generating is infinite (diverges). Note, however, that subclasses can have some finite moments without necessarily having a finite $M(t)$ for all $t > 0$, as the divergence of this function does not mean there are no finite moments. This is the case for common heavy-tailed distributions such as the log-normal, the t-Student, and the PL distribution, which are used in the finance literature and have only some finite moments (there is a functional form for just some values k for $E(X^k)$).

This class of distributions contains other useful subclasses (Embrechts et al., 2013), but the most relevant for us is the class of regularly varying distributions or fat-tailed distributions $\mathcal{R}$, one being a subset of the other. We refer to Appendix C.1 for further results and other subclasses.

Definition 3.2.2. A positive Lebesgue measurable function L on $(0, \infty)$ is slowly varying at ∞, denoted at $L \in \mathcal{R}_0$ if

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1, \quad \forall t > 0 \tag{3.11}$$

Definition 3.2.3. A positive Lebesgue measurable function h on $(0, \infty)$ is regularly varying at ∞ of index $\alpha \in \mathbb{R}$, denoted at $h \in \mathcal{R}_\alpha$ if

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\alpha, \quad \forall t > 0 \tag{3.12}$$

Through these definitions, we can see that a regularly varying function h can be expressed as $h(x) = L(x)x^\alpha$, which will be important for understanding fat-tailed distributions.

Definition 3.2.4. The class of regularly varying distributions, also known as fat-tailed distributions, denoted as $\mathcal{R}$, is defined as

$$\mathcal{R} \equiv \{F \;\text{ on }\; (0, \infty) : \overline{F} \in \mathcal{R}_{-\alpha} \;\text{ for some }\; \alpha \geq 0\} \tag{3.13}$$

where F is a distribution function, so that $\mathcal{R}_{-\alpha}$ is the class of regularly varying functions at ∞ with index $-\alpha$.

Given the definition of $\mathcal{R}$, we can see that it is the class of distributions for which their tails have the form $\overline{F}(x) \sim L(x)x^{-\alpha}$ for $x \to \infty$, where $\sim$ is understood as the asymptotic equivalence of functions. This means that fat-tailed distributions decay like a PL function perturbed by some function $L \in \mathcal{R}_0$, which is more common and realistic than the "purest" PL case (in which $\overline{F}(x) \sim x^{-\alpha}$ for $x \to \infty$ as its density function implies) because of the possible presence of deviations from the exact dependency (Voitalov, van der Hoorn, van der Hofstad, & Krioukov, 2019). Consequently, regularly varying tails reflect a PL-like behaviour, and we we will refer this like a PL behaviour from now on.

From this brief discussion, one can conclude two important ideas:

1. Fat-tailed distributions (regularly varying distributions $\mathcal{R}$) are a subclass of

heavy-tailed distributions $\mathcal{K}$, so both types are different even though they are used interchangeably in the finance literature.

2. Fat-tailed distributions include more distributions apart from the pure PL distribution we defined.

EVT is concerned with studying the behaviour of extreme values, and one of the most important classes of distributions that help study them is the class of heavy-tailed distributions $\mathcal{K}$, which stems from the most important theorem of EVT: the Fisher-Tippet-Gnedenko theorem. We stated in the following theorem, extracted from Embrechts, Kluppelberg & Mickosch (2013).

Theorem 3.2.5. Let $(X_n)_{n\geq 0}$ be a sequence of iid random variables. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate distribution function H such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{D} H \tag{3.14}$$

then H belongs to the type of one of the following distribution functions:

$$\text{Fréchet:} \quad \Phi_\alpha(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ \exp\{-x^{-\alpha}\}, & \text{for } x > 0 \end{cases}, \quad \alpha > 0 \tag{3.15}$$

$$\text{Weibull:} \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{for } x \leq 0 \\ 1, & \text{for } x > 0 \end{cases}, \quad \alpha > 0 \tag{3.16}$$

$$\text{Gumbel:} \quad \Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathbb{R} \tag{3.17}$$

This theorem is the most important result from EVT because it states that the limiting behaviour of extreme events is governed by one of three possible classes of distributions, called extreme value distributions. One justification for the emergence of PL behaviour in the tails of different distributions of financial returns (included in the heavy-tailed class) lies in the fact that extreme events are governed by the Fréchet distribution (Farmer & Geanakoplos, 2008; Alfarano & Lux, 2010), as the PL distribution pertains to the maximum domain of attraction (MDA) of this family (this statement is further explained and justified in Appendix C.2).

The justification of modelling the tails of returns' distributions as PLs and not through other distributions in the MDA of the Fréchet distribution, however, is justified through another relevant result from EVT: the Pickans-Balkema-de Haan theorem, which needs from the generalized Pareto distribution.

3.2.2 Generalized Pareto Distribution

We have defined and established relationships between heavy tails, fat tails, and PLs, but we did not justify why exactly PLs could be used to model tails (and not other distributions). Hence, we introduce EVT results that allow justify this. The main distribution of interest that generalizes the PL distribution and is derived from the Generalized Extreme Value (GEV) distribution (a generalization from the distributions in the Fisher-Tippet-Gnedenko theorem) is the Generalized Pareto Distribution (GPD).

A detailed derivation of the GPD from the GEV distribution and other results are presented in Appendix C.2, but we just mention here that, due to the characterizations of the maximum domain of attraction of the Fréchet distribution and the GEV distribution, we can obtain an equivalence result that relates the distribution of excesses over a high threshold to a distribution function F pertaining to the maximum domain of attraction of the GEV distribution. This result complements the ones we present next, but are not entirely relevant for understanding the usage of the GPD for modelling tails.

Definition 3.2.6. The Generalized Pareto Distribution for a centered random variable, denoted as $G_{\xi,\beta}$ for $\beta > 0$, is defined by

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \xi \frac{x}{\beta}\right)^{-\xi^{-1}}, & \text{if } \xi \neq 0 \\ 1 - e^{-x/\beta}, & \text{if } \xi = 0 \end{cases} \tag{3.18}$$

where

$$\begin{cases} x \in [0, \infty), & \text{if } \xi \geq 0 \\ x \in [0, -\beta/\xi), & \text{if } \xi < 0 \end{cases} \tag{3.19}$$

The shape parameter ξ determines the shape of the $G_{\xi,\beta}$ by linking it to the shape parameter of the extreme value distributions (the Fréchet, the Weibull, and the Gumbel distributions). This is fundamental for studying the tails of different distributions and quantifying tail risk (Embrechts et al., 2013) because we can relate the tails of a distribution with the GPD. Nevertheless, we first need to define the excess distribution function and the mean excess distribution function and then present the theorem that links tails, excess distributions, and the GPD: the Pickands-Balkema-de Haan theorem.

Definition 3.2.7. Let X be a random variable with distribution function F and right endpoint $x_F = \sup\{x \in \mathbb{R} : F(x) < 1\}$. For a fixed $u < x_F$

$$F_u(x) = P(X - u \leq x | X > u), \quad x \geq 0 \tag{3.20}$$

is the excess distribution function of X over the threshold u. The function

$$e(u) = E(X - u | X > u) = \frac{1}{\overline{F}(u)} \int_u^{x_F} \overline{F}(x) dx, \quad 0 < u < x_F \tag{3.21}$$

is called the mean excess function of X.

The excess distribution function can be interpreted as the conditional probability that X exceeds a higher threshold $x + u$ given that it has exceeded the threshold u, so that it relates to high values of X over a threshold and, consequently, to the extremes of the distribution (the tail values). When this high threshold tends to infinity, the limit of F_u turns out to be the GPD, as the Pickands-Balkema-de Haan theorem shows (extracted from Embrechts, Kluppelberg & Mickosch, 2013).

Theorem 3.2.8. For every $\xi \in \mathbb{R}$ and $u \in \mathbb{R}$, there exists a function $\beta(u)$ such that

$$\lim_{u \to \infty} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0 \tag{3.22}$$

The excess distribution function tends to the GPD whenever we increase the threshold u (whenever we move further to the tail of the distribution), so we can use the GPD to analyze the tail behaviour of distributions. As we mentioned, ξ

determines the shape of the GPD, and hence the shape of the tail:

$$\begin{cases} F \text{ is fat-tailed,} & \text{for } \xi > 0 \\ F \text{ is exponentially-tailed,} & \text{for } \xi = 0 \\ F \text{ is light-tailed,} & \text{for } \xi < 0 \end{cases} \tag{3.23}$$

The GPD includes more cases apart from the case $\xi > 0$, but this corresponds to the fat-tailed distributions such as the PL case, which hence justifying its use for modelling tails. At first, one could obtain different results or inferences and a potentially different fit using a GPD than just a PL distribution (as it includes more parameters for adjusting to the data). Furthermore, this can also affect risk modelling and measuring. Hence, we will compare PL and GPD distribution results and also their effect on risk measuring using risk measures such as the "Value-at-Risk" (VaR) and the "Expected Shortfall" (ES), which are defined in the next section.

3.2.3 PL and GPD

Both PL and the GPD distributions are intimately related, and we stated that one is a special case of the other. We now delve into the relationship of both distributions to understand why the PL is a special case of the GPD.

The PL distribution is also called Pareto distribution, given the name of the early researcher who documented it, the economist Vilfredo Pareto. Depending on the name given to certain other distributions, this creates some confusion in the literature, but it is effectively synonymous with our PL distribution (Newman, 2005). Accordingly, in this thesis, the Pareto distribution will be understood as a PL distribution with density as in (3.2), and we can establish a formal relation between both PL and GPD through the next theorem, which presents the necessary equivalences in the parameters.

Theorem 3.2.9. Given a centered random variable X, suppose $X \sim PL(\alpha, x_{min})$ for $\alpha > 0$ and define $Y = X - x_{min}$. Then, $Y \sim G_{\xi,\beta}$ for $\xi > 0$ when $\alpha = \xi^{-1}$ and $x_{min} = \beta/\xi$ hold. Moreover, given a centered random variable Y such that

$Y \sim G_{\xi,\beta}$ for $\xi > 0$ and applying the same relations for parameters and variables, then $X \sim PL(\alpha, x_{min})$.

The formal proof of the theorem appears in Appendix A.4. Fixing the parameters adequately, one can see that the PL distribution is just an instance of the GPD given the relationship between α and ξ and the appropriate fixing of the minimum value x_{min} (which determines the support). This justifies applying statistical methodologies for the GPD to cases where a PL behaviour emerges (when $\alpha > 0$), but also in the cases where we do not exactly know if there is a pure PL behaviour. As we mentioned, the generality of the cases covered by the GPD could lead us to obtain other results and even fit the data better, and because of its usage in risk management, it is interesting to use both distributions.

Apart from this direct relationship, one can also establish this same relationship through Fréchet's MDA we mentioned earlier, which is shown in Appendix C.2. However, we restrict ourselves to this relevant relationship as it will be used in our applications.

3.3 Risk

One of the most important concepts in the finance discipline is risk, which can represent both negative outcomes and opportunities or potential for gain. The modern financial industry exists due to the presence of economic risk and the opportunities that companies took advantage of (McNeil et al., 2015).

Because risk is related to uncertainty and randomness, financial practitioners use probability theory and statistics to model and measure risk. The most used approach is to model risk through the distribution of losses (negative returns) for a given asset. Both PLs and distributions from EVT have proved to be extremely useful for modelling and measuring risk related to its tails, called tail risks. Therefore, we now briefly discuss the notion of tail risk and present the measures of risk that will be used in our empirical applications and their relation to PLs.

3.3.1 Tail Risk

There exist different types of risk, but the most relevant ones for the goals of this thesis are those related to the tail of the distribution of returns, also called tail risks. The tail risk stems from the mischance of suffering losses due to the occurrence of extreme events and given that tails of distribution represent extreme occurrences, PLs and EVT have proved to be helpful tools for modelling tail behaviour and measuring tail risk in financial data.

The foundational paper that proposed the usage of EVT distributions for risk management was Embrechts et al. (1999), in which the authors argue that EVT yields methods for quantifying extreme events in a statistically optimal way and hence they offer an interesting set of techniques for modelling risk in different sectors such as insurance or credit risk management. Most of the developed methodologies for quantification of tail risks are based on EVT (McNeil et al., 2015), but given the relationship between the GPD and the PL discussed earlier, we could also mix results from econophysics with those from EVT for risk management.

In our empirical application, we try to model the tail behaviour in the distribution of returns for different frequencies and stock indices using PL fits and GPD fits, so that we can compare how they fit the data and their differences when measuring risk with each model. Therefore, we now need to define the measures of risk that will be used for the tail.

3.3.2 Measures of risk

A risk measure associates a financial position with loss L with a real number that measures the riskiness of L, and they are relevant for determining capital against unexpected future losses, fixing margin requirements for investors, limiting the risk that a business unit can take, and many others (McNeil et al., 2015). There are different approaches to risk measurement, but the one we will use in our empirical applications is measuring based on a loss distribution (the opposite of the distribution of returns), as we use the most used risk measures: the Value-at-Risk (VaR) and the Expected Shortfall (ES). In our empirical applications, we will compute the VaR and the ES for our tail fits to get an insight about how different fits affect risk measurement in practice, highlighting the relevance of using PL or GPD. We now present the definition of these measures and their expression for the GPD case, extracted from McNeil, Embrechts and Frey (2015).

Definition 3.3.1. Given some confidence level $\alpha \in (0, 1)$, the Value-at-Risk (VaR) of a portfolio with loss L at the confidence level α is

$$VaR_\alpha = VaR_\alpha(L) = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\} \tag{3.24}$$

Definition 3.3.2. Given some confidence level $\alpha \in (0, 1)$, the Expected Shortfall (ES) of a portfolio with loss L at the confidence level α is

$$ES_\alpha = ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du \tag{3.25}$$

In this case, L represents the negative logarithmic returns $\ln(P_t/P_{t-1})$ (where P_t are the prices at moment t), which are called losses. Assuming that for a high threshold u, $L \sim G_{\xi,\beta}$, then we can obtain the following results:

Theorem 3.3.3. Assuming $x \geq u$, for $\alpha > F(u)$ the VaR is

$$VaR_\alpha = u + \frac{\beta}{\xi}\left(\left(\frac{1-\alpha}{\overline{F}(u)}\right)^{-\xi} - 1\right) \tag{3.26}$$

Furthermore, assuming that $\xi < 1$, the associated ES is

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du = \frac{VaR_\alpha}{1-\xi} + \frac{\beta - \xi u}{1-\xi} \tag{3.27}$$

Due to the PL or Pareto distribution being an instance of the GPD, we could use the transformations and the PL distribution density specified earlier to obtain similar equations to compute the VaR and the ES directly.

Nevertheless, these values are also commonly obtained through simulation, such that we use the tail model to simulate n observations (being n a large enough number) and then we get the value corresponding to the percentile $(1-\alpha) \times 100\%$ as an estimate for the VaR and the mean for all observations above the VaR as an estimate for the ES. In our empirical applications, we obtain the VaR and the ES of the PL and the EVT fit through simulations using a very high number of simulated observations for increased precision.

3.4 Stochastic Processes, Stability and Mandelbrot

In previous sections, we argued that our choice of studying the tails of distributions through PLs and EVT is motivated by the complex system features of financial markets, the academic research brought by econophysics, and the relationship between PLs and EVT. However, it is also motivated by Mandelbrot (1963), as he was the first to propose PL tail behaviour in the distribution of returns through the usage of stable processes for modelling the time series of financial returns. This contributed to a relevant (yet old) debate in finance about the process that returns follow (given how they fit the data) and also pointed out how the complexity of financial markets played a role in how returns behave in later publications (Mandelbrot, 1983). The usual approach is to describe the behaviour of prices through a Brownian motion, which makes returns normally distributed, but Mandelbrot proposed an alternative process, the α-stable process, because of the "peakedness" of the distribution and the tail behaviour.

We now give a detailed discussion of the ideas proposed by Mandelbrot regarding the behaviour of returns and their relationship to stability through the notion of stable distributions. These are not just in the heart of Mandelbrot's financial theories, but are vital for presenting the Generalized Central Limit Theorem (GCLT), playing an important role in our empirical applications and some results of the econophysics literature. We first define stable distributions and processes and link them with the GCLT, and then we finally discuss Mandelbrot's hypothesis and their relation to PLs, heavy tails, and complex systems.

3.4.1 Stable Distributions & the GCLT

The following definitions and theorems are extracted from Embrechts et al. (2013). A stable distribution can be defined and characterized as follows:

Definition 3.4.1. A random variable is said to be stable if it satisfies

$$c_1X_1 + c_2X_2 \stackrel{D}{=} b(c_1, c_2)X + a(c_1, c_2) \tag{3.28}$$

for X and iid X_1, X_2, all non-negative numbers c_1,c_2 and appropriate real numbers $b(c_1, c_2) > 0$ and $a(c_1, c_2) \in \mathbb{R}$.

Theorem 3.4.2. A stable distribution has a characteristic function

$$\phi_X(t) = E(e^{itX}) = \exp\{i\gamma t - c|t|^\alpha(1 - i\beta\text{sign}(t)z(t, \alpha)\}, \quad t \in \mathbb{R} \tag{3.29}$$

where γ is a real constant, $c > 0$, $\alpha \in (0, 2]$, $\beta \in [-1, 1]$ and

$$z(t, \alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}), & \text{if } \alpha \neq 1 \\ \frac{2}{\pi}\ln|t|, & \text{if } \alpha = 1 \end{cases} \tag{3.30}$$

The parameter α of the characteristic function in (3.30) is the characteristic exponent. This type of distribution is also known as an α-stable distribution given the importance of this parameter, and its distribution function is denoted as G_α. There are special cases of this type of distribution: for $\alpha = 1/2$ and $\beta = 1$ the distribution reduces to a Lévy distribution; for $\alpha = 1$ and $\beta = 0$ the distribution reduces to a Cauchy distribution; and for $\alpha = 2$ the distribution is a Gaussian or normal distribution.

The latter case is interesting, as the normal is the limit distribution for sums of iid variables when the second moment is finite (so that $E(X^2) < \infty$), as stated by the classical central limit theorem (CLT). It turns out that stable distributions allow us to generalize this theorem to the generalized central limit theorem (GCLT) by using α-stable distributions.

Theorem 3.4.3. Let $S_n = X_1 + X_2 + ... + X_n$ be a sum of a sequence of n iid stable random variables. If there exist constants $a_n \in \mathbb{R}$ and $b_n > 0$ such that

$$b_n^{-1}(S_n - a_n) \xrightarrow{D} G_\alpha \tag{3.31}$$

for some $\alpha \in (0, 2]$, then, for $L \in \mathcal{R}_0$, and denoting the standard normal distribution function by Φ, then

$$\begin{cases} (\sigma n^{1/2})^{-1}(S_n - \mu n) \xrightarrow{D} \Phi, & \text{if } E(X^2) < \infty \\ (n^{1/\alpha}L(n))^{-1}(S_n - \mu n) \xrightarrow{D} G_\alpha, & \text{if } E(X^2) = \infty \text{ and } \alpha = 2 \\ (n^{1/\alpha}L(n))^{-1}S_n \xrightarrow{D} G_\alpha, & \text{if } E(X^2) = \infty \text{ and } \alpha < 2 \end{cases} \tag{3.32}$$

This theorem has important implications: it states that the sum of iid random variables can converge to an α-stable distribution if it does not have finite variance (on the class of heavy-tailed distributions) but can also converge to a Gaussian distribution if it has a finite second moment, which would be the classical CLT. Therefore, sums of iid random variables do not necessarily converge to the normal distribution but can converge to other classes of distributions, and this is one of the implications of Mandelbrot's stable Paretian hypothesis for the distribution of returns (Mandelbrot, 1963).

For $\alpha < 2$, the tails exhibit PL behaviour with PL exponent α, which is why they also call these distributions "Paretian stable". Hence,

$$X \sim G_\alpha \text{ for } \alpha \in (0, 2) \Rightarrow P(|X| > x) \sim x^{-\alpha} \tag{3.33}$$

and this means that we can use PLs to study the tails of this type of distributions whenever $\alpha \in (0, 2)$, as tails are regularly varying. We will use regularly varying and PL-behaved tails interchangeably from now on. If $\alpha = 2$, then tails would be exponential because the distribution would be normal, so the tails should not be PL-behaved. However, we must notice that the econophysics literature (Plerou et al., 2001; Gabaix et al., 2003; Gabaix, 2009) defend the idea that the tails of the distribution of returns are PL-behaved with $\alpha = 3$, which is not considered in this theorem and motivates analyzing the tails of these distributions.

We defined the α-stable distribution and established the relationship between this distribution, the GCLT, and PLs (due to its tails). Now, we define what an α-stable motion is (Embrechts et al., 2013):

Definition 3.4.4. A stochastic process $(X_t)_{t\in[0,1]}$ with sample paths in $\mathbb{D}[0,1]$ is said to be an α-stable motion if the following properties hold:

- It starts with $X_0 = 0$
- It has independent, stationary increments.
- For every $t \in [0,1]$, X_t has an α-stable distribution with fixed parameters $\beta \in [-1,1]$ and $\gamma = 0$ in the characteristic function.

The α-stable motions are also called α-stable processes (as we call them from now on), but we must highlight that the specialized literature makes a distinction between both as more general processes (like those without independent increments) occur (Embrechts et al., 2013).

An important lemma which allows obtaining the distribution of the α-stable motions at different points $t \in [0,1]$ is the following, extracted from Embrechts, Kluppelberg & Mickosch (2013):

Lemma 3.4.5. For an α-stable motion $(X_t)_{t\in[0,1]}$, we have

$$X_t - X_s = (t-s)^{1/\alpha} X_1, \quad 0 \leq s < t \leq 1 \tag{3.34}$$

This lemma is also explanatory for some results of econophysics that will be used in this thesis, as it motivates the PL relationship presented in (3.9). Müller et al. (1990) argue that the exponent $1/\lambda$ in the equation could be identified as the characteristic exponent of an α-stable distribution if Δx is generated by a random process with stable distributions (so that $\alpha = \lambda$ if returns follow an α-stable motion). Even though the authors do not explain why is this the case, it is motivated by this lemma, as it relates the increments to the time interval (just like the law).

However, they do not assume in any case a distribution for Δx, and they use different statistical analyses to try to test for the stability of the observed distribution of returns. Hence, the work from Müller et al. (1990) also provides some analyses that we can use for discussing the possible stability of the process of returns.

The exponent $1/\alpha$ is relevant in financial statistics for analyzing financial time series, as it is related to the Hurst exponent H, which determines the degree of long-term memory behaviour in a process (autocorrelation between points decays slower than an exponential decay) and is used for different applications in finance to model jumps and other behaviours.

3.4.2 Mandelbrot and Finance

Mandelbrot defended that financial markets behave like complex systems (Mandelbrot & Hudson, 2010) and that the presence of heavy tails in the distribution of observed stock returns induced to think that the normal distribution is not adequate for modelling, even though it is common for modelling the logarithmic returns for stocks (Fama, 1963; Mandelbrot, 1963). All of these led him to develop the stable Paretian hypothesis, where stock returns are distributed like an α-stable distribution instead of a normal, caused by financial returns following not a Gaussian random walk but an α-stable stochastic process with $\alpha \in (1, 2)$. This, of course, would cast doubts on the validity of models that use Gaussian processes to model returns, so it is a highly relevant proposal that had an important impact on mainstream finance and is fundamental for econophysics.

Although Mandelbrot's proposals were fundamental to the econophysics literature at first, academics in the area widely accepted the hypothesis presented in many empirical studies, in which the tail index is $\alpha \approx 3 > 2$ (Plerou et al., 2001; Gabaix et al., 2003, 2006), discarding then the possibility that high-frequency returns follow an α-stable process like Mandelbrot defined (as $\alpha \in (1, 2)$). Therefore, to give some insights related to the differences between econophysics and Mandelbrot's stable Paretian hypothesis, the estimation of α for the tail of returns using different frequencies will also serve to discuss the possibility of an α-stable process being the process generating the returns in our data jointly with statistical analyses used in Müller et al. (1990) to test for the stability property it must satisfy.

3.5 Estimation Methodologies

Now that the theory for PLs and EVT has been presented, we discuss different procedures for which we can obtain estimates of parameters like the exponent α and other relevant quantities such as the threshold for the PL and the GPD distribution.

3.5.1 Estimators for the Exponent

The principal estimators used in practice and the ones we will use are the linear regression estimator and the ML estimator, even though other methods exist for both types of distributions (Embrechts et al., 2013; Deluca & Corral, 2013)), such as the Hill estimator (which is used to approximate the tail of a PL but also used for GPD) or Bayesian estimation methods. Anyway, we restrict to these two as both methods are extensively used in the literature and the references used, and are specifically useful for our applications.

Linear Regression Estimator

Given the linear relationship between $\ln[f(x)]$ and $\ln(x)$ for a given PL $f(x) = Cx^{-\alpha}$, a straightforward method to obtain an estimate for both the logarithm of the constant and the α exponent is fitting a linear regression through the usual OLS method.

$$\ln[f(x)] = \ln(C) - \alpha \ln(x) \tag{3.35}$$

However, the presence of dependency between different $f(x)$ points creates problems when using OLS estimators and this procedure is only valid for untruncated PLs (as in the case we presented), as for truncated PLs the results are not optimal (Deluca & Corral, 2013). The authors and Clauset et al. (2009) also state that these methods usually have some problems that can cause biased results, so other estimation methodologies such as ML are preferable.

Maximum Likelihood Estimators

The maximum likelihood (ML) estimators are obtained the usual way: maximizing the likelihood or log-likelihood function. For the sake of brevity, we do not show the complete derivation but just the estimators. By maximizing the log-likelihood function, we obtain the following ML estimator for the PL case:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n\ln(\hat{x}_{min})} \quad \text{for} \quad x_i > x_{min}, \quad i = 1, 2, ..., n \tag{3.36}$$

$$\hat{\sigma}_\alpha = \frac{n\hat{\alpha}}{(n-1)\sqrt{(n-2)}} \quad \text{for} \quad x_i > x_{min}, \quad i = 1, 2, ..., n \tag{3.37}$$

where $\hat{x}_{min} = \min x_i$ is the ML estimator of the threshold point. Corral and Deluca (2013) mention that there are comparative studies that conclude that the ML estimators for the PL distribution outperform other methods when comparing them through the variance and bias (so that these yield the lowest).

In the case of the GPD, the ML estimators are derived and solved numerically, as it is a restricted optimization problem due to the domain of X depending on ξ

$$\begin{aligned} \max \ \ln\left(L(\xi, \beta)\right) = -n\ln(\beta) - (\xi^{-1} + 1)\sum_{i=1}^{n}\left(1 + \frac{\xi}{\beta}X_i\right) \\ s.t. \quad x_i \in [0, \infty) \qquad \text{if } \xi \geq 0 \\ x_i \in [0, -\beta/\xi] \quad \text{if } \xi < 0 \end{aligned} \tag{3.38}$$

For both the PL and the GPD, however, the most important difficulty is to fix a threshold such that the PL or GPD behaviour is respected above this threshold. Consequently, we need to use estimation methodologies such as the following for this purpose.

3.5.2 Estimators for Thresholds

For getting estimates of the thresholds we will use Clauset et al.'s method (Clauset, Shalizi, & Newman, 2009) for determining the threshold of both the PL and the GPD. We chose this method because it is used in the literature when fixing a threshold for PL distributions, even though there are some drawbacks when using this method (Deluca & Corral, 2013) which will be discussed later. Of course, other methods have a more rigorous theoretical justification, such as the threshold algorithm based on the coefficient of variation (CV) plot (Castillo, Serra, Padilla,

& Moriña, 2019), and there are other methods used specifically for the GPD in practice, such as the mean excess plot (called ME-plot).

However, given that our focus is PLs and the empirical applications try to exemplify the usage of PLs for statistical financial analysis, we restrict ourselves to this common method and adapt it to both the PL distribution and the GPD.

Clauset et al.'s method

The method developed by the authors consists of two parts: the first one is selecting the threshold or minimum value x_{min} for a PL distribution which yields the minimum Kolmogorov-Smirnov (K-S) statistic, and the second part is assessing the validity of the fit through a p-value for all the possible values of the threshold we consider. The statistic is defined as the supremum of the distance between the empirical cumulative distribution function and the theoretical cumulative distribution function we want to fit (the largest distance), which depends on our selection of x_{min} (as the goal is to fit the tails of the distribution):

$$D_{x_{min}} = \sup_{x \geq x_{min}} |\hat{F}_{x_{min}}(x) - F_{x_{min}}(x)| \tag{3.39}$$

where

$$\hat{F}_{x_{min}}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}} \quad \text{for } X_i, x \geq x_{min} \tag{3.40}$$

and $\mathbf{1}_{\{arg\}}$ is an indicator function that equals 1 whenever the argument is true and 0 if not. Then, this method yields an x_{min} such that $D_{x_{min}}$ is the smallest distance between both distributions. Once we obtain an x_{min}, then the second part of the method is to obtain a p-value corresponding to this fit and discard it if it is too low (according to the desired significance level) (Deluca & Corral, 2013). The authors generate samples from the original data and create synthetic power laws for $x \geq x_{min}$ so that we can apply the first part of the methodology to each sample and obtain a distribution of x_{min} for the p-value.

Although this method is acknowledged in the PL literature, Corral and Deluca

(2013) highlight that the authors do not justify it and it has some problems, being the most noticeable one the K-S test sometimes rejects fits even if the data truly comes from a synthetic PL. This is probably due to the high sensitivity of the K-S test to deviations from the hypothesized distribution, especially in the tails (Mason & Schuenemeyer, 1983). Therefore, the p-values obtained are of no practical use when assessing whether the chosen fit seems valid or not, and we should recur to other ways such as visual inspection through tail plots and complementary distribution function plots to assess the goodness of fit of the estimated models.

Our method

Consequently, in this thesis, we consider the first part of the methodology but not the second, assessing the goodness of fit of the distribution by comparing the sample data and the fit for both the PL and the GPD using tail plots and complementary distribution function plots.

For the PL case, we use the R package *poweRlaw* created by some of the authors of Clauset et al. (2009), which includes a function to directly estimate x_{min} through this method given a vector of possible thresholds. For the GPD case, no package directly allows us to estimate the threshold u in this way, so we constructed the algorithm using the *evir* package functions for estimating the GPD. In this case, we try all possible values of the threshold and estimate the GPD, so that we can compare it to the data and obtain the K-S statistic value for each threshold and finally choose the lowest one which allows obtaining a good fit for the sample.

To determine the vector of possible thresholds, we look at a range of values for which we detect a linear behaviour starts through tail plots. Therefore, the vector of possible values is the same for both PL and GPD. Nevertheless, note that the best threshold for each case might not be the same even if the data is the same. This is due to the GPD being more flexible than the PL distribution, so the thresholds could fit the data better at lower thresholds compared to those that we would need to use for the PL to obtain a good fit. This makes K-S statistic values not comparable to each other.

Keep in mind, however, that the purpose of these applications is to illustrate the applications of PLs to financial practice so that we are not specifically concerned with being extremely accurate when choosing a fit or choosing the best methodology for carrying on this exercise. Instead, we want to show a possible method for using PLs to study tail behaviour and use sound statistical analysis for discussing findings related to the econophysics literature and practical uses for financial practice.

4 | Applications to Finance

4.1 Applications and Data

The theoretical justification and the motivations for using PLs to analyze financial data in this project have been explained. Moreover, we have reviewed the relevant mathematical and statistical theory necessary to develop the empirical applications in this thesis. We now focus on applying all of these tools to real financial data so that we show their potential for financial analysis and modelling. We expose how PL methods can be used in finance and we also delve into statistical and probabilistic aspects which are very interesting from the point of view of a modeller or practitioner through the use of EVT and stable processes.

The analysis is divided into two different (but related) sections: the analysis of the PL and extreme behaviour of financial returns, and the analysis of scaling laws in these returns. Before delving into them, we present the data that will be used for the applications and we establish which are the main goals of each part.

4.1.1 Data for the Applications

Due to the various empirical patterns and hypotheses being derived from high-frequency data, our interest resides in this type of data. Therefore, we use high-frequency data of prices from the Nikkei 225 stock index (representing the Japanese stock market) and the FTSE 100 stock index (representing the UK stock market) to check that different results can hold in other markets, as many results from the literature are based on American stock market data. The data has been extracted from Tickstory, a public free database that obtains its from the Dukascopy Bank database (a Swiss banking group that provides data services). Given that we will focus on working with returns, we chose to work with logarithmic returns, thus computing them from the prices as $r_t = \ln(P_t) - \ln(P_{t-1})$, where P_t is the closing price at time t.

We use frequencies of 10 seconds, 1 minute, 5 minutes, 15 minutes, 30 minutes, 1 hour, 4 hours, 12 hours and 1 day. The usage of these frequencies is also justified by the limited availability of public high-frequency data, which constrains the options for research purposes. For the analysis of the tail behaviour, we just use frequencies from 10 seconds to 1 hour, as the number of observations in frequencies of 4 hours, 12 hours, and 1 day would be low (even though one can apply the same techniques used), but we use all the frequencies for the analyses in the second part (for the scaling law), given that it is of primary importance to have a diverse range of frequencies (the number of observations for each frequency not being too relevant in this case).

The chosen period of study is from 14 July 2023 to 13 October 2023, as it is one of the most recent periods with available data at the time of the elaboration of this thesis and because it is also interesting from the historical point of view, as many important events such as the Israel-Palestine conflict and the Russia-Ukraine war were going on, which creates shocks in the financial markets analyzed and likely creates extreme observations (captured by the tails of the distribution).

4.1.2 Goals of the Applications

First Part: PL and Extreme Behaviour

In the first part, we use high-frequency data ranging from 10 seconds to 1 hour to fit PL distributions and GPDs for both the positive and the negative tails while we also compute estimates for VaR and ES risk measures for both. The main goal for this part is to estimate the tail exponent of the PL and the GPD to discuss the results obtained and relate them to the results obtained in the econophysics literature and the possible process of returns (relating to the discussion about Mandelbrot's proposal). When fitting a GPD distribution, the analysis becomes richer in two senses:

- Given that the PL and the GPD are related, we can discuss the differences and the results of both models regarding the hypothesis of Mandelbrot and the empirical pattern from econophysics.
- Because the GPD is commonly used for risk measuring, we can compare the results that both models yield in terms of the VaR and the ES measures

and their implications.

Moreover, we also take a look at two forces that come into play: the convergence of the sum of returns to a normal distribution due to the CLT, and the persistence of regularly varying tails when summing independent variables with regularly varying tails for a given α (Gabaix, 2009). Gabaix (2009) states the last force differently, saying that "the sum of independent PLs is a distribution with a PL tail", so that tails should be regularly varying and, for a better comprehension, we will generalize the statement using the sum of variables with regularly varying tails instead.

Considering a lower frequency for log-returns is equivalent to sum the corresponding log-returns with a higher frequency. Assume t_0, t_1 and t_2 are three equispaced times and define $\Delta_2 t = t_2 - t_0$ and $\Delta_1 = t_1 - t_0$. Because times are equispaced, then $\Delta_2 t = 2\Delta_1 t$. Let $P(t_i)$ be the price of the asset for $i = 0, 1, 2$, then we can define:

$$r_1(t) = \log[P(t + \Delta_1 t)] - \log[P(t)] \tag{4.1}$$

$$r_2(t) = \log[P(t + \Delta_2 t)] - \log[P(t)] \tag{4.2}$$

From this equations, we can obtain the following equality:

$$r_2(t_0) = \log[P(t_2)] - \log[P(t_0)] = \log[P(t_0 + \Delta_2 t)] - \log[P(t_0)] \tag{4.3}$$

By summing and subtracting the same quantity, we obtain:

$$\begin{aligned} r_2(t_0) &= \log[P(t_0 + 2\Delta_1 t)] - \log[P(t_0)] \\ &= \log[P(t_0 + 2\Delta_1 t)] - \log[P(t_0 + \Delta_1 t)] + \log[P(t_0 + \Delta_1 t)] \\ &- \log[P(t_0)] = r_1(t_0 + \Delta_1 t) + r_1(t_0) \end{aligned} \tag{4.4}$$

Hence, log-returns of lower frequencies are the sum of log-returns of higher frequencies, and, by the GCLT, the distribution of these returns will converge to an α-stable or a Gaussian distribution when one increases the frequency Δt. If the second moment exists, the CLT applies, but if it does not exist, then, depending on α, it will tend to an α-stable distribution G_α.

The second force states that, if tails are regularly varying, then the increase of the frequency Δt should not change the behaviour of the tails given that the summation of independent variables with regularly varying tails has a regularly varying tail as well.

This result can be formulated in the following lemma, extracted from Embrechts, Kluppelberg & Mickosch (2013):

Lemma 4.1.1. Suppose F_1, F_2 are two distribution functions such that $1-F_i(x) = x^{-\alpha}L_i(x)$ for $\alpha \geq 0$ and $L_i \in \mathcal{R}_0$,$i = 1, 2$, then the convolution $G = F_1 * F_2$ (sum of independent random variables) has a regularly varying tail such that:

$$1 - G(x) \sim x^{-\alpha}(L_1(x) + L_2(x)), \quad x \to \infty \tag{4.5}$$

Gabaix (2009) states that it is the CLT that occurs because of his findings on the tails of the distribution of returns being PL-behaved with $\alpha \approx 3$, as the second moment would be finite. Therefore, the CLT would make the central part of the distribution more and more Gaussian, while tails would remain regularly varying, which makes PLs relevant for studying longer-horizon returns (Gabaix, 2009). This would be consistent with the stylized fact of fat-tailed distributions, so it is worth checking for our data.

We can analyze both forces by comparing the tail data with both PL and exponential tail behaviours (through tail plots) and by comparing the complete distribution of the data (through histograms) for each frequency. Moreover, because the GPD also contemplates the exponential behaviour case (where $\xi = 0$), we can also look at the results of the fits for the tails and discuss the results.

Second Part: Scaling Laws

For the second part, we expand the range of frequencies considered and use high-frequency data ranging from 10 seconds to 1 day to estimate the mean of absolute log-returns $\overline{|x_t|}$ (and other related statistics as well) and the size of the interval Δt for each frequency and index. The goals now are to assess whether this PL holds for our data and discuss what the estimated exponent $1/\lambda$ can tell us about the underlying process, diving also into the implications for financial practice.

We look at the scaling law that data could exhibit through Müller et al. (1990) PL relationship, which relates Δt and the mean of absolute log-returns $\overline{|x_t|}$. The authors found that it holds for some assets such as currencies and commodities, even if the returns do not follow an α-stable process. Given that the exponent $1/\lambda$ of the proposed PL could be identified as $1/\alpha$ if returns follow an α-stable process, then we can estimate this PL to check if it holds and also if the values of α stay in the $(0, 2)$ range for returns to possibly follow an α-stable process. If this PL holds and returns do not follow this latter process, then the scaling law is phenomenological (not due to the stability of the process). Additionally, we also carry on the same analyses that Müller et al. (1990) did for arguing whether returns respected stability or not, which is related to the discussion of Mandelbrot's hypothesis.

4.2 PL and Extreme Behaviour in Returns

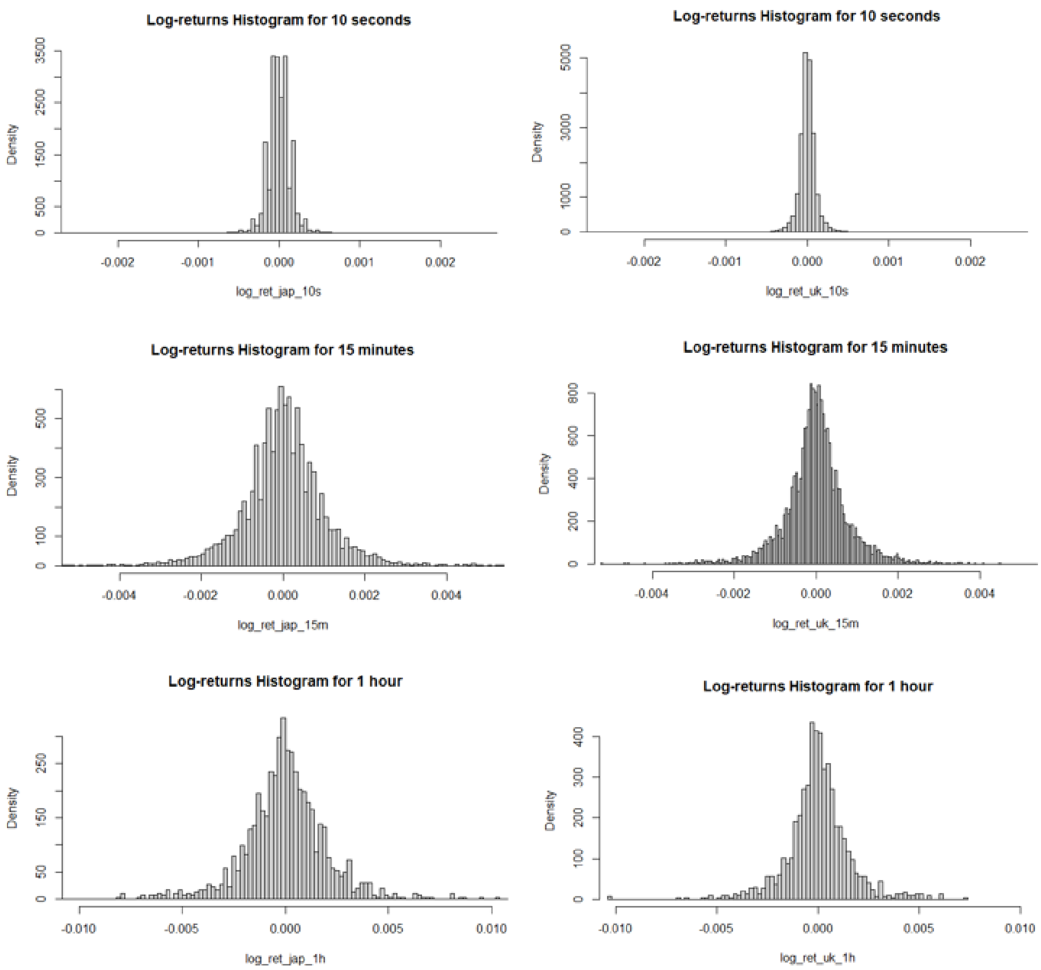
4.2.1 Exploratory Analysis

After computing the returns from the closing prices for each index and each frequency, we start by doing a brief exploratory analysis of returns for different frequencies, where we take a closer look at how returns are distributed and their tail behaviour. First, we look at the distribution of returns for the different frequencies and both indices, shown in Figure 4.1.

Looking at the histograms for some frequencies, we can see that the distributions of returns seem to have "peaked" central bulk, as the central bulk has a notorious peak that is different from the bell-shaped bulk of the normal distribution, even though the decrease of frequency makes returns more bell-shaped. Moreover, tails seem heavier than those from a normal distribution, looking like they could be PL-behaved. Hence, we can observe that the behaviour of both the positive and the negative tails gives a hint that the distribution of the log-returns has a more extreme behaviour in the tails than that implied by a normal distribution.

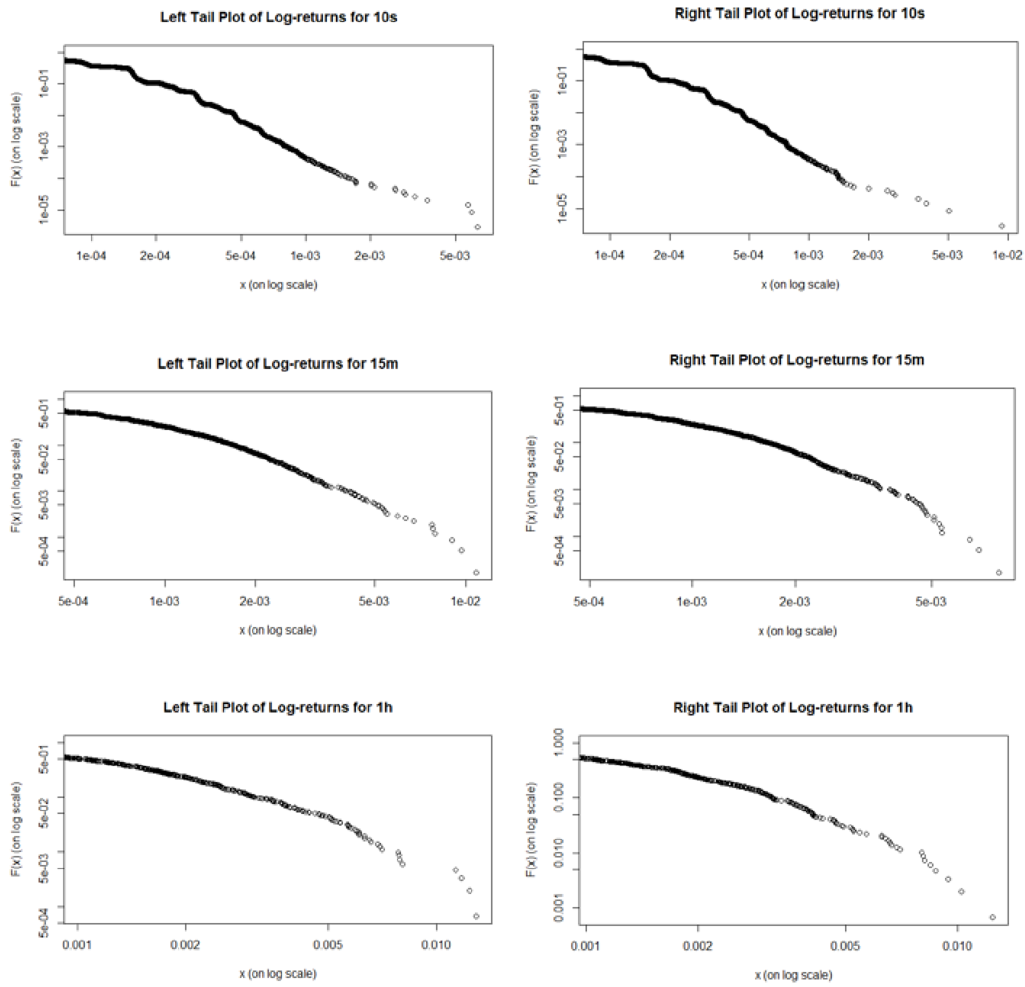
Additionally, we can use tail plots to visualize how the tails behave more specifically. A linear behaviour would indicate a PL behaviour, while a curved one

Figure 4.1: Histograms of log-returns for the Nikkei 225 and the FTSE 100



would be more typical of other behaviours such as exponential or normal tails. These are shown in Figure 4.2 and Figure 4.3.

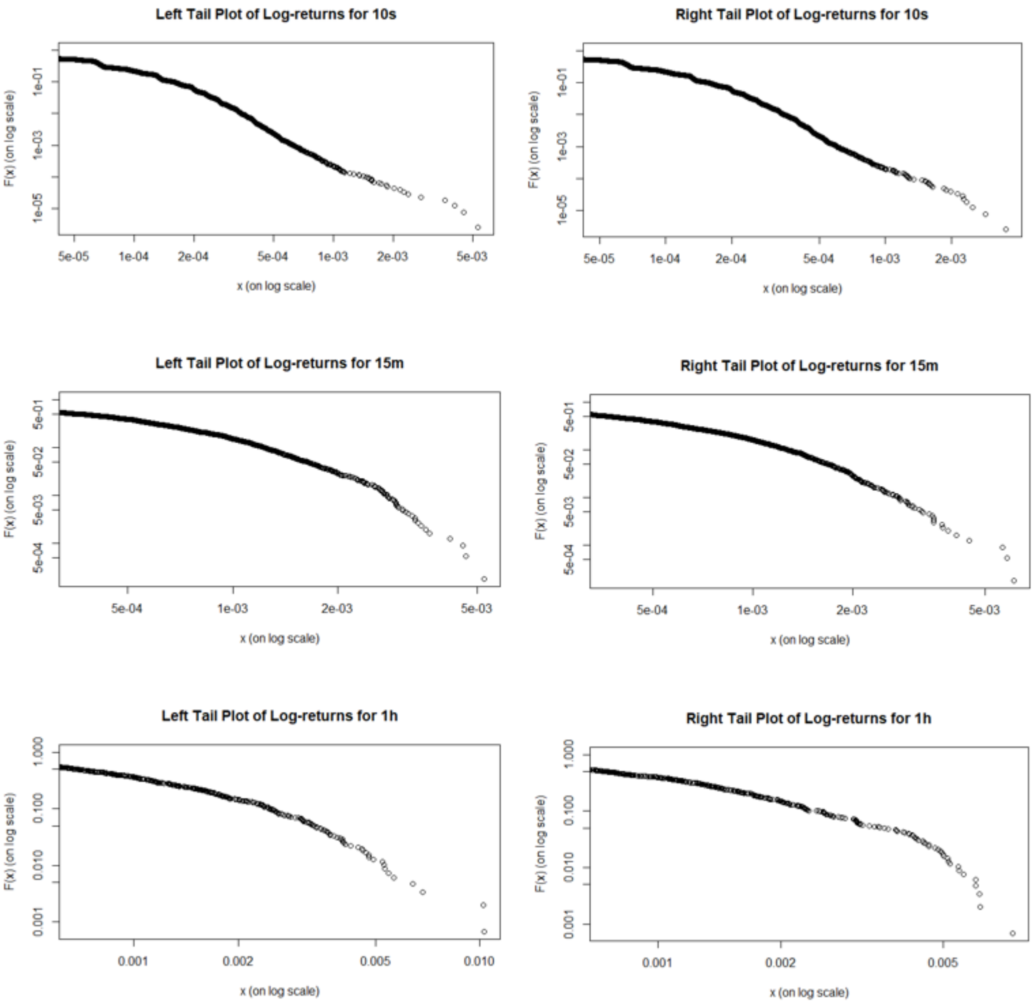
Figure 4.2: Negative (left) and Positive (right) tail plots for the Nikkei 225 log-returns



These tail plots show that for very high frequencies such as 10 seconds, the tail seems linear, indicating a possible PL behaviour, but it changes to a more curved one as we decrease the frequency, indicating that an exponential fit might be better.

Hence, this preliminary analysis allows us to spot the effect of the two probabilistic forces of the CLT and the sum of regularly varying tails firsthand, while also showing that distributions might not seem to be stable for the different frequen-

Figure 4.3: Negative (left) and Positive (right) tail plots for the FTSE 100 log-returns



cies, as the distribution seems to change depending on the frequency. We could see that there is "peakedness" in the distribution of returns for very high frequencies, leading to a distribution different from normal, but as we decrease the frequency, the distribution is more resemblant to a normal because the central bulk tends to a bell shape (possibly caused by the CLT) even though tails still seem to be heavy (because of the possible effect of the sum of regularly varying tails). Hence, even though the "peakedness" of the central bulk may resemble to that of an α-stable distribution, the fact that distributions could change depending on the frequency indicates stability would not be respected for the returns in our data.

Moreover, we can see that the tails of the distribution for both indexes could follow a PL (because of the linear behaviour when using logarithmic axes) at high frequencies, but as we decrease the frequency, the linear behaviour turns slightly into a more curved one, which might make exponential tails from normal distribution a better fit.

More images and graphics regarding the distribution and the tail behaviour of both indices for all frequencies considered are available in Appendix B.1.

4.2.2 Results

In the preliminary exploratory analysis, we found that distributions of returns for different frequencies and both indices seem to change, and we hypothesized that the CLT and the sum of random variables with regularly varying tails could partially explain these observations. The tails for some frequencies do not seem to be normal or exponential, but more like a PL, and the central bulk of the distribution for some frequencies does not seem bell-shaped but more peaked, as with some α-stable distributions. Consequently, we now provide the statistical analyses mentioned earlier for both the positive and negative tails of the distribution for different frequencies and indices.

The numerical results for the PL fit of the positive and the negative tails for frequencies ranging from 10 seconds to 1 hour with the Nikkei 225 index data are presented in Table 4.1 and Table 4.2, while similar results for the FTSE 100 index are shown in Table 4.3 and Table 4. We estimated the threshold through Clauset et al.'s (2009) method, as presented in Section 3.5.

Table 4.1: PL fit results for the positive tail of the Nikkei 225 index returns

	10s	**1m**	**5m**
x_{min}	8×10^{-4}	8.5×10^{-4}	1.35×10^{-3}
$\hat{\alpha}$	3.310 ± 0.280	3.724 ± 0.155	3.405 ± 0.184
K-S	0.046	0.040	0.037
$N_{x_{min}}$	140	574	334
n	179113	41041	8669
	15m	**30m**	**1h**
x_{min}	1.95×10^{-3}	1.7×10^{-3}	2.9×10^{-3}
$\hat{\alpha}$	3.190 ± 0.227	2.454 ± 0.151	2.829 ± 0.281
K-S	0.044	0.052	0.059
$N_{x_{min}}$	197	263	101
n	2931	1472	745

Table 4.2: PL fit results for the negative tail of the Nikkei 225 index returns

	10s	**1m**	**5m**
x_{min}	7×10^{-4}	1×10^{-3}	1.35×10^{-3}
$\hat{\alpha}$	3.483 ± 0.197	3.374 ± 0.178	3.205 ± 0.195
K-S	0.050	0.042	0.035
$N_{x_{min}}$	313	360	270
n	178414	40633	8546
	15m	**30m**	**1h**
x_{min}	1.85×10^{-3}	2.15×10^{-3}	2×10^{-3}
$\hat{\alpha}$	2.793 ± 0.181	2.641 ± 0.199	2.004 ± 0.151
K-S	0.032	0.035	0.058
$N_{x_{min}}$	237	176	176
n	2870	1475	761

The tables show that for high frequencies (when Δt is small) the exponent is high, being most of the times a little bit higher than $\alpha \approx 3$, but then it goes to a value of

Table 4.3: PL fit results for the positive tail of the FTSE 100 index returns

	10s	**1m**	**5m**
x_{min}	6×10^{-4}	6×10^{-4}	9.5×10^{-4}
$\hat{\alpha}$	3.040 ± 0.218	3.742 ± 0.131	3.320 ± 0.159
K-S	0.037	0.025	0.032
$N_{x_{min}}$	195	811	438
n	192650	40431	8476

	15m	**30m**	**1h**
x_{min}	1.85×10^{-3}	2.85×10^{-3}	1.65×10^{-3}
$\hat{\alpha}$	3.728 ± 0.359	4.264 ± 0.603	2.105 ± 0.170
K-S	0.045	0.066	0.071
$N_{x_{min}}$	108	50	153
n	2816	1410	737

Table 4.4: PL fit results for the negative tail of the FTSE 100 index returns

	10s	**1m**	**5m**
x_{min}	5.5×10^{-4}	6×10^{-4}	1.3×10^{-3}
$\hat{\alpha}$	3.016 ± 0.178	3.825 ± 0.137	4.364 ± 0.337
K-S	0.030	0.020	0.032
$N_{x_{min}}$	288	780	168
n	192140	39958	8274

	15m	**30m**	**1h**
x_{min}	1.2×10^{-3}	8.5×10^{-4}	3×10^{-3}
$\hat{\alpha}$	2.757 ± 0.159	1.823 ± 0.086	3.344 ± 0.468
K-S	0.046	0.067	0.050
$N_{x_{min}}$	299	449	51
n	2836	1466	745

$\alpha \approx 2$ when we decrease the frequency (when Δt gets larger). For the Nikkei 225, the values of the exponents exhibit this evolution of the exponent in a clearer way in both the positive and the negative tails, while there are some "ups and downs" throughout when looking at both tails (it is not as smooth as with the Nikkei 225 case). This could be caused by the threshold selected through our methodology, as the estimated thresholds imply a low number of observations above this threshold (as demonstrated by the error). Many studies in econophysics used much more data than we do as they have access to private and better data sources for high-frequency data (Plerou et al., 2001; Gabaix et al., 2003, 2006), which are normally available for many EE.UU. securities, so that they fit the PL for the tails with a higher number of observations and possibly obtain more accurate results.

However, we highlight that we used a public database for high-frequency data and we also highlight that authors study the distribution of returns aggregating many assets (Plerou et al., 2001; Gabaix et al., 2003) to study the general distribution of returns and they use a considerably longer horizon than we do. Despite this, we consider our chosen time horizon and the number of observations to be relatively adequate taking into account access constraints to better data and the usage of just two assets (both indices).

There are two important takeaways from these patterns in the estimated models for the tails of both indexes. By looking at the fits for the different frequencies, it seems that the empirical patterns observed in the econophysics literature (Plerou et al., 2001; Gabaix et al., 2003; Gabaix, 2009) in which $\alpha \approx 3$ is respected in our data most of the time. However, this value goes to a lower value of $\alpha \approx 2$ when the frequency decreases. Even though we do not know the process that returns follow, the fit of the PLs shows that the tail exponent $\alpha \notin (0, 2)$, which would be contrary to the hypothesis that returns follow an α-stable process, at least for this data. Additionally, looking at histograms in Figure 4.1 and Appendix B.1, we can see that the central part of the distributions turns more bell-shaped as we decrease the frequency. Hence, these patterns could exemplify the combination of a Gaussian central bulk and PL tails explained by Gabaix (2009) possibly caused by the effect of both the CLT and the summation of regularly varying tails.

Still, even if the results give preference to the empirical observed $\alpha \approx 3$, there are also some patterns from econophysics literature that we do not observe in our data. Plerou et al. (2001) present evidence supporting that $\alpha \approx 3$ for a long range of frequencies from minutes to months when computing the average αs for 1000 American stocks. Additionally, the series of mean αs for both the positive and the negative tail show an upward trend, meaning that as Δt increases, so does the exponent. However, the evidence found in this exercise does not show an upward trend, as values become lower when the frequency decreases. We can observe how the exponential tail behaviour of a normal distribution seems to fit better the data as we decrease the frequency in Figures 4.4 and 4.5 (more figures available in Appendix B.1):

Therefore, we notice that Gabaix's (2009) prediction of a PL tail behaviour with $\alpha \approx 3$ even for lower frequencies does not seem to hold for our data. These results indicate that the effect of the CLT could be more powerful than that of the summation of regularly varying tails and make the distributions converge to a normal distribution. Hence, we might need a longer time horizon and more data to observe or reproduce the same results found in the relevant econophysics literature (Gopikrishnan et al., 1999; Plerou et al., 2001; Gabaix et al., 2003, 2006).

Now, we can compare the results obtained for the PL fit with the ones for the GPD fit. We present the results of the GPD fit for the Nikkei 225 in Tables 4.5 and 4.6 and the FTSE 100 in Tables 4.7 and 4.8. In these tables, we only display the error for the ξ estimate, as the error for the β is not as relevant for our purposes.

When looking at the results using the GPD, we can see how the estimations for ξ for the Nikkei 225 tails are greater in magnitude than those for the FTSE 100. Despite using a similar method to estimate the thresholds, the thresholds for the GPD are lower than for the PL distribution for most of the frequencies. One can explain such results by understanding the nature of the GPD distribution: as the GPD is a generalization of the PL distribution and contains more parameters, it is more flexible and can adjust better to the data from a lower threshold. For the PL, we needed to identify a clear linear behaviour in the tail plot, but for the GPD we can allow for some curvature given its parametrization, making possible lower

Figure 4.4: Comparison of data to PL and normal tail behaviour for the positive tail of Nikkei 225 (left) and FTSE 100 (right) returns

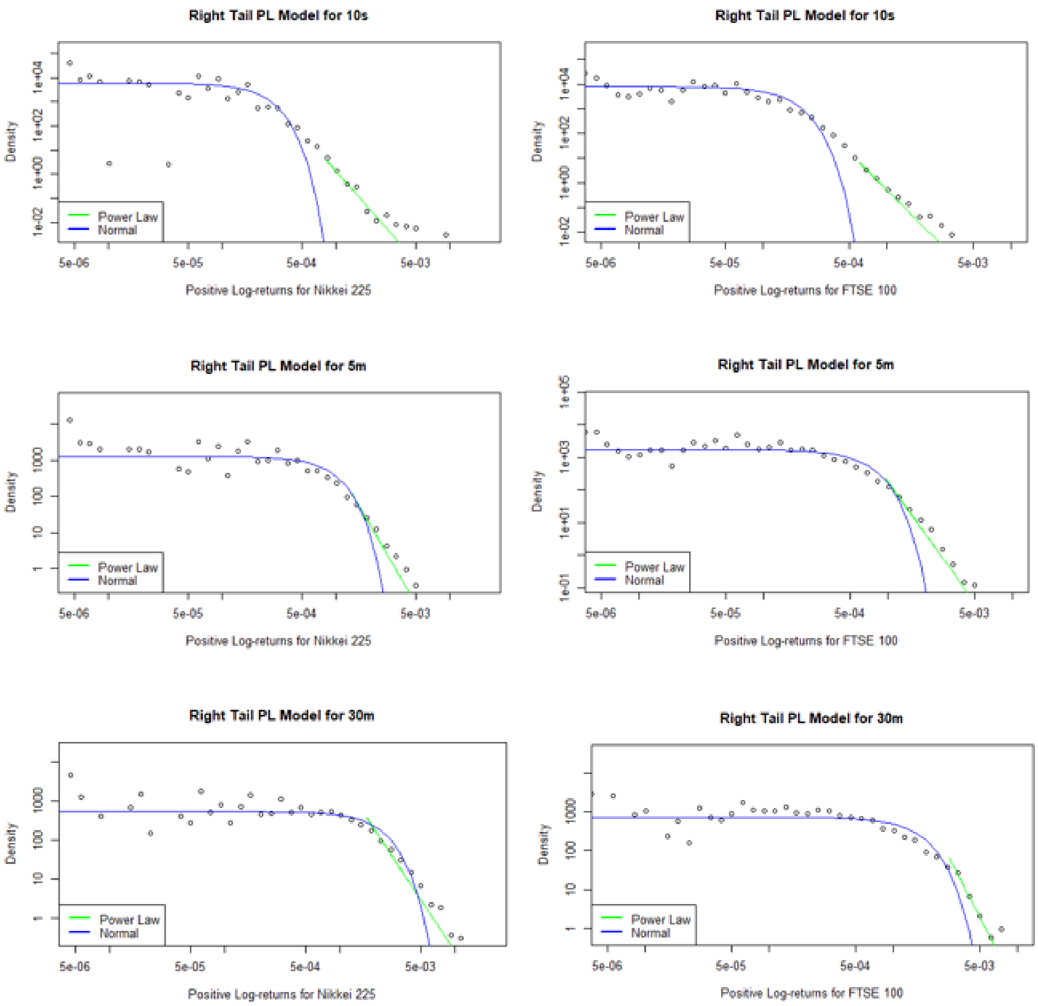


Figure 4.5: Comparison of data to PL and normal tail behaviour for the negative tail of Nikkei 225 (left) and FTSE 100 (right) returns

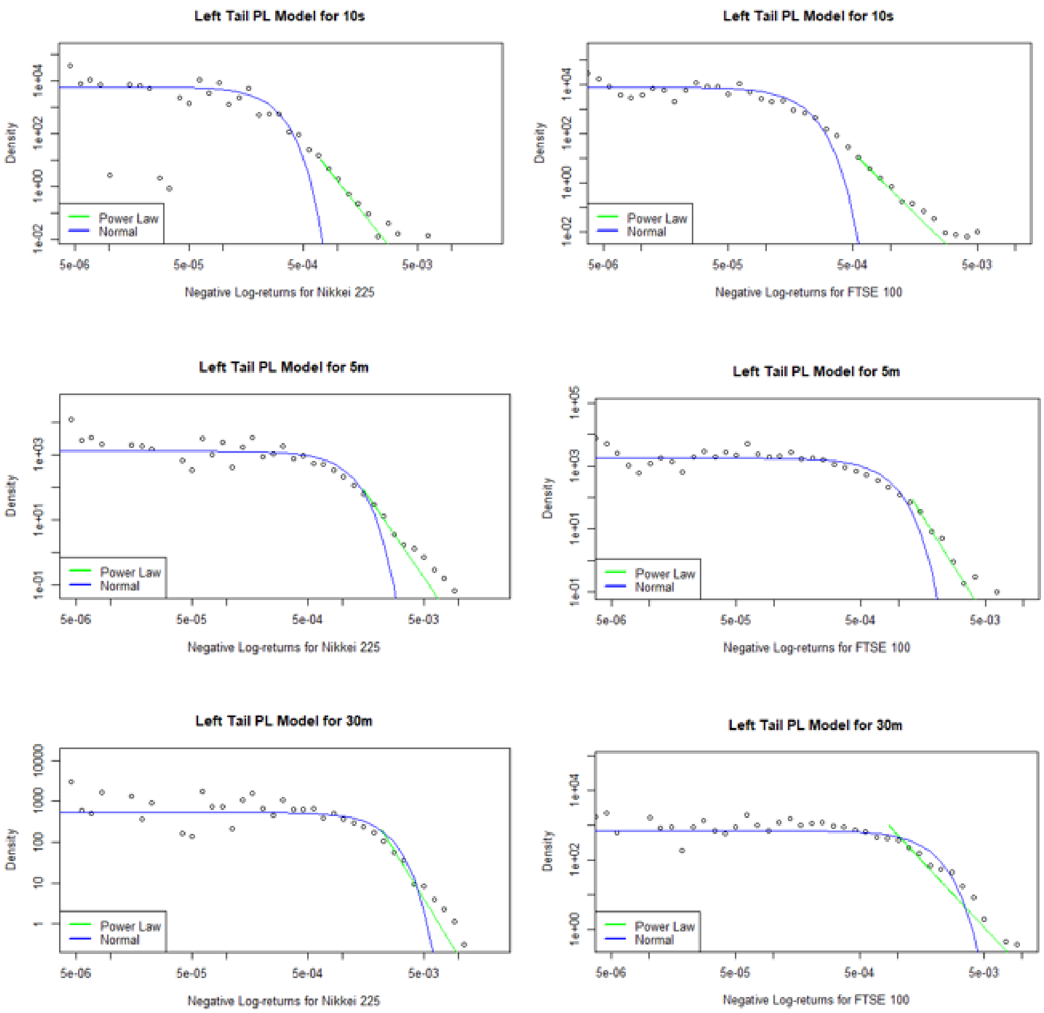


Table 4.5: GPD fit results for the positive tail of the Nikkei 225 index returns

	10s	**1m**	**5m**
u	5.5×10^{-4}	7.5×10^{-4}	1.1×10^{-3}
$\hat{\xi}$	0.306 ± 0.035	0.274 ± 0.034	0.152 ± 0.037
$\hat{\beta}$	1.26×10^{-4}	1.92×10^{-4}	4.4×10^{-4}
K-S	0.168	0.090	0.067
N_u	787	940	568
n	179113	41041	8669

	15m	**30m**	**1h**
u	9.5×10^{-4}	2.25×10^{-3}	2.15×10^{-3}
$\hat{\xi}$	0.078 ± 0.028	0.155 ± 0.126	0.069 ± 0.062
$\hat{\beta}$	7.11×10^{-4}	1.01×10^{-3}	1.46×10^{-3}
K-S	0.051	0.048	0.041
N_u	764	142	156
n	2931	1472	745

Table 4.6: GPD fit results for the negative tail of the Nikkei 225 index returns

	10s	**1m**	**5m**
u	6×10^{-4}	5.5×10^{-4}	6.5×10^{-4}
$\hat{\xi}$	0.556 ± 0.050	0.256 ± 0.020	0.168 ± 0.020
$\hat{\beta}$	9.88×10^{-5}	1.88×10^{-4}	4.04×10^{-4}
K-S	0.173	0.086	0.081
N_u	672	2375	1707
n	178414	40633	8546

	15m	**30m**	**1h**
u	1.1×10^{-3}	1.6×10^{-3}	5.5×10^{-4}
$\hat{\xi}$	0.171 ± 0.035	0.209 ± 0.054	0.083 ± 0.034
$\hat{\beta}$	7.16×10^{-4}	9.21×10^{-4}	1.36×10^{-3}
K-S	0.051	0.039	0.036
N_u	614	295	511
n	2870	1475	761

Table 4.7: GPD fit results for the positive tail of the FTSE 100 index returns

	10s	**1m**	**5m**
u	2×10^{-4}	5×10^{-4}	5×10^{-4}
$\hat{\xi}$	0.186 ± 0.008	0.194 ± 0.023	0.116 ± 0.021
$\hat{\beta}$	7.14×10^{-5}	1.6×10^{-4}	3.11×10^{-4}
K-S	0.122	0.112	0.047
N_u	11048	1439	1592
n	192650	40431	8476

	15m	**30m**	**1h**
u	5×10^{-4}	6×10^{-4}	1.15×10^{-3}
$\hat{\xi}$	0.075 ± 0.024	0.093 ± 0.027	0.089 ± 0.080
$\hat{\beta}$	5.34×10^{-4}	7.42×10^{-4}	1.04×10^{-3}
K-S	0.031	0.027	0.026
N_u	1085	631	249
n	2816	1410	737

Table 4.8: GPD fit results for the negative tail of the FTSE 100 index returns

	10s	**1m**	**5m**
u	2×10^{-4}	5×10^{-4}	1.2×10^{-3}
$\hat{\xi}$	0.203 ± 0.009	0.201 ± 0.024	0.236 ± 0.063
$\hat{\beta}$	7.08×10^{-5}	1.54×10^{-4}	2.83×10^{-4}
K-S	0.122	0.094	0.037
N_u	10934	1411	228
n	192140	39958	8274

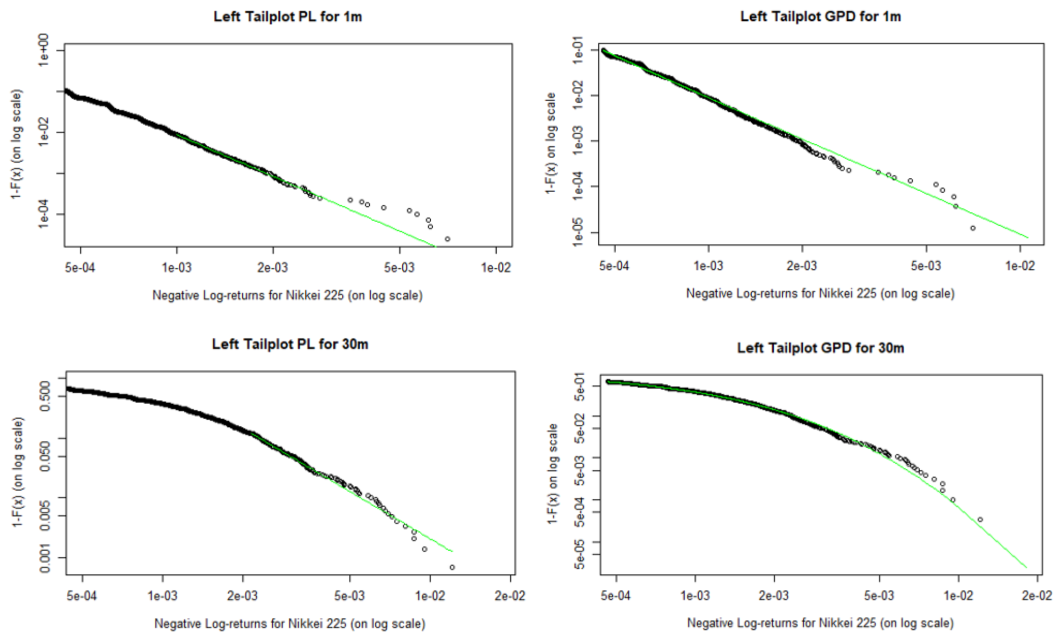
	15m	**30m**	**1h**
u	5×10^{-4}	6×10^{-4}	5.5×10^{-4}
$\hat{\xi}$	0.093 ± 0.025	0.088 ± 0.032	0.137 ± 0.042
$\hat{\beta}$	5.01×10^{-4}	7.22×10^{-3}	9.22×10^{-4}
K-S	0.026	0.031	0.028
N_u	1094	612	434
n	2836	1466	745

thresholds.

Recalling that for $\xi = 0$ the GPD is the exponential distribution, an important pattern inferred from the results is that the shape parameters go to zero as we decrease the frequency of the data, which is the same pattern observed for the PL fits and would be consistent with the effect of the CLT that we were discussing through Figures 4.4 and 4.5.

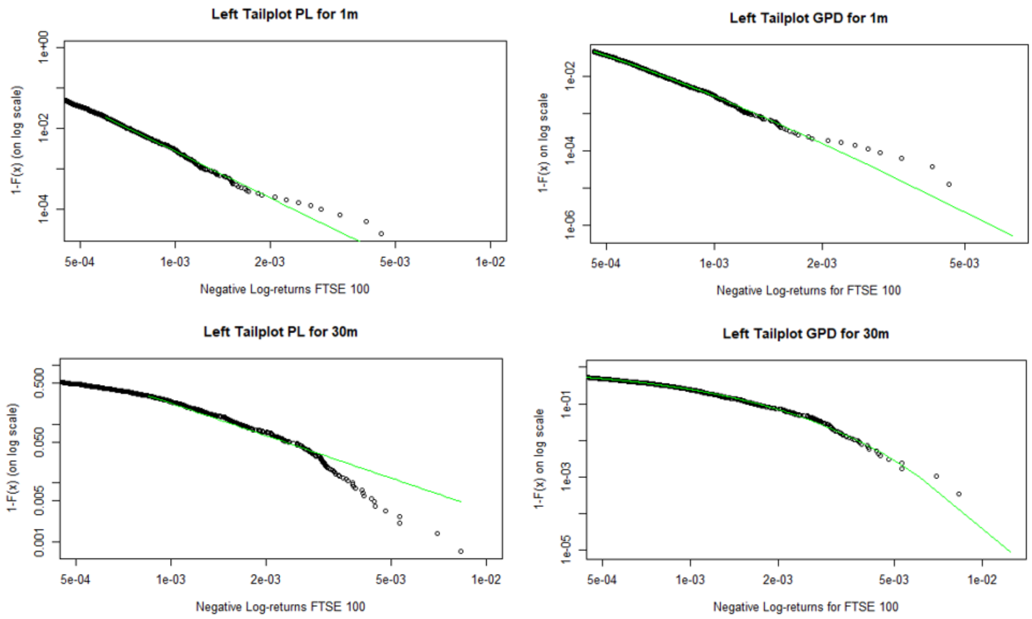
In Figures 4.6 to 4.9 we compare the fit of the PL and the GPD distributions for the negative tail of the log-returns using frequencies of 1 minute and 30 minutes through the complementary distribution function $1 - F(x)$ and with tail plots, which is more visual for how well does each one fit the data. We chose to use the negative tail because this is the relevant tail for risk management purposes (the losses), but we can do the same plots and comparisons for the positive tails. Other frequencies and the positive tails are available in Appendix B.1.

Figure 4.6: Tail plots for the negative tails PL and GPD fit for 1 and 30-minute frequencies for Nikkei 225 log-returns



The best PL fits have higher thresholds than the best GPD fits, as observed by the numerical results in the tables and the tail plots, which was expected because of

Figure 4.7: Tail plots for the negative tails PL and GPD fit for 1 and 30-minute frequencies for FTSE 100 log-returns



the flexibility of the GPD.

Looking at the figures, the GPD might visually fit the data better (even though the thresholds are not the same as with the PL), despite being cases in which the fit curve over or underestimates the tail. The PL also over or underestimates the tail in some instances, but sometimes the fit is relatively further from the tail data than the GPD, such as the one using 30 minutes for the FTSE 100, which would make the GPD a better fit for some frequencies.

Finally, we compare both distributions in terms of implications for finance and risk measurement. Now that we have two different fits for the tails of the distribution of returns of both indices and all frequencies, we can analyze the implications of using one or another to model the tails of the distribution of returns. For risk management, the relevant tail is the negative one, as we want to manage the risk of negative returns (losses), so we just look at the left tail. The principal interest resides in looking at how the measures of tail risk are affected by each model, and how would this translate in practice. Tables 4.9 to 4.12 show the VaR and the ES

Figure 4.8: Complementary distribution function plots for the negative tails PL and GPD fit for 1 and 30-minute frequencies for Nikkei 225 log-returns

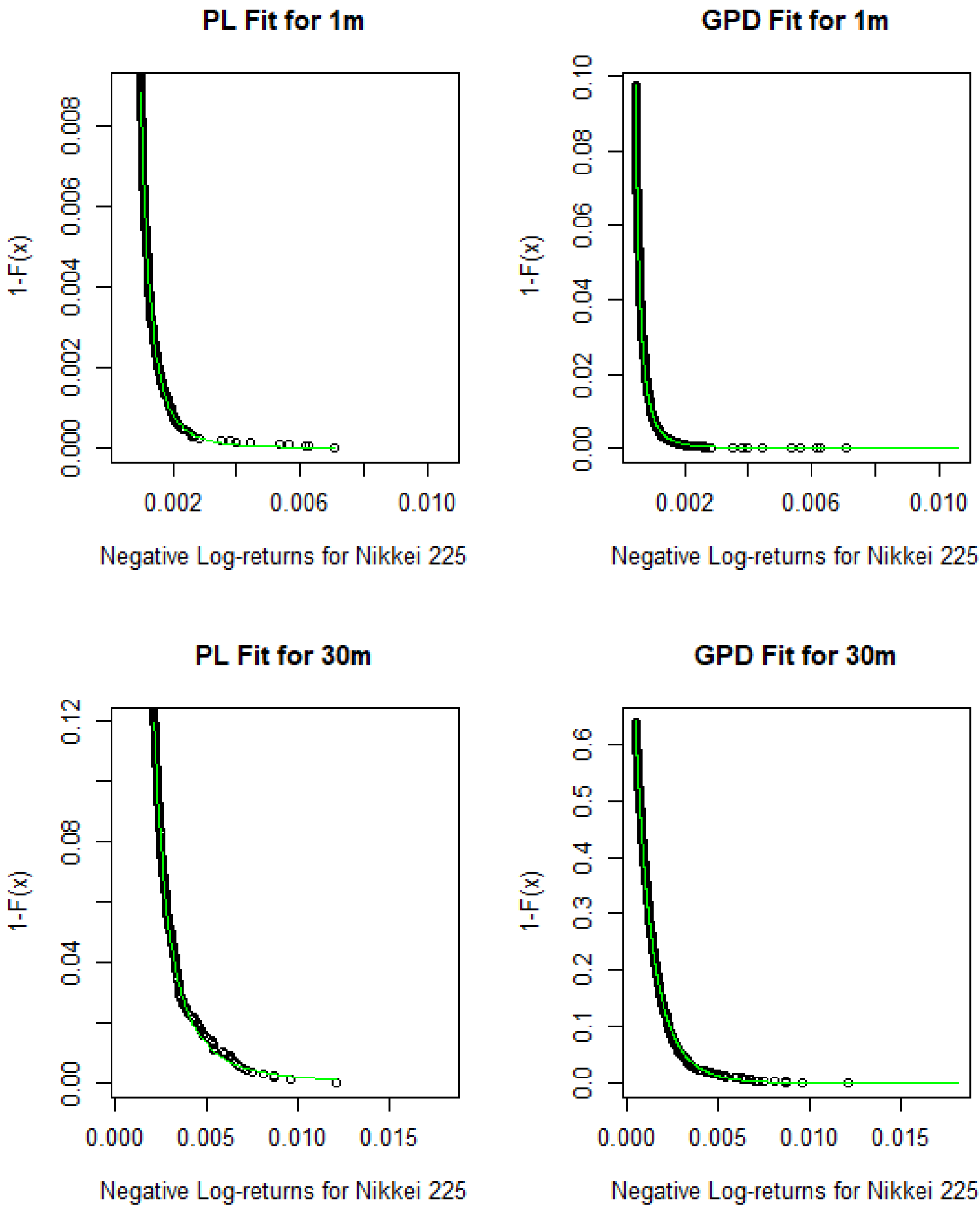
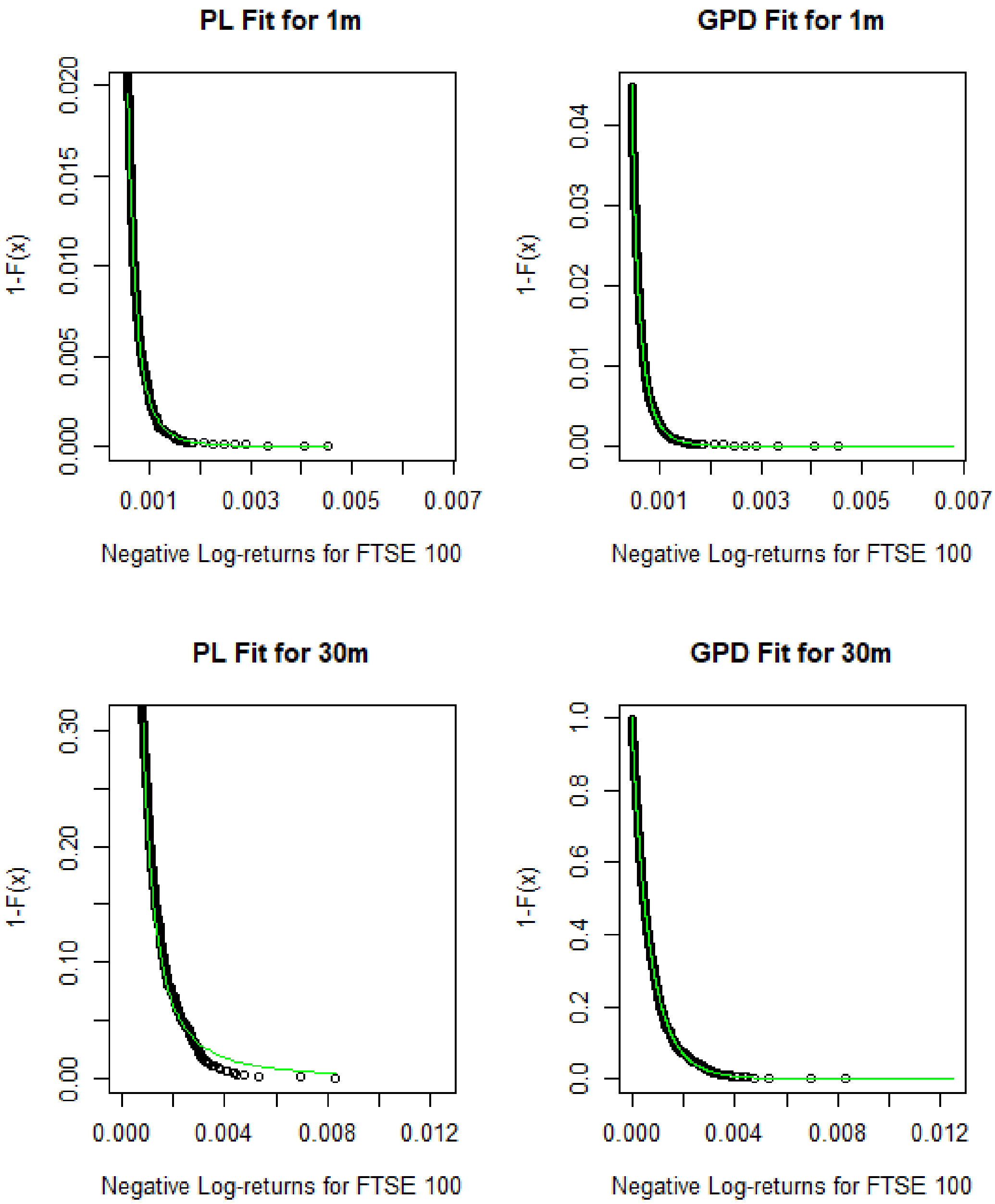


Figure 4.9: Complementary distribution function plots for the negative tails PL and GPD fit for 1 and 30-minute frequencies for FTSE 100 log-returns



estimations for both indexes and each frequency and distribution used.

Table 4.9: VAR for the Nikkei 225 negative tail

	10s	**1m**	**5m**	**15m**	**30m**	**1h**
PL	0.002578	0.003901	0.006457	0.0091298	0.012451	0.020269
GPD	0.002131	0.001657	0.002812	0.005018	0.007127	0.007648

Table 4.10: ES for the Nikkei 225 negative tail

	10s	**1m**	**5m**	**15m**	**30m**	**1h**
PL	0.003316	0.005853	0.009091	0.013616	0.021423	0.037552
GPD	0.004587	0.002458	0.003914	0.006546	0.010078	0.009699

Table 4.11: VAR for the FTSE 100 negative tail

	10s	**1m**	**5m**	**15m**	**30m**	**1h**
PL	0.002478	0.001994	0.003798	0.006047	0.010826	0.012018
GPD	0.000420	0.001163	0.002359	0.002880	0.004104	0.005924

Table 4.12: ES for the FTSE 100 negative tail

	10s	**1m**	**5m**	**15m**	**30m**	**1h**
PL	0.003332	0.002828	0.004835	0.009075	0.026386	0.016660
GPD	0.000544	0.001806	0.003659	0.003751	0.005340	0.008035

As we can see, the PL distributions imply greater VaR and ES, while the GPD fits would result in lower values for both measures. This can be explained due to the tail exponent α being lower than ξ^{-1} in most of the cases (making the density flatter and hence increasing the fatness of the tail).

For credit risk management, where the VaR and the ES are used for determining the economic and regulatory capital, a bank and other financial institutions such as insurance carriers would prefer to use a GPD for measuring tail risk, as this would make the institution allocate fewer assets to risk capital and obtain higher returns from other investments. However, this changes when looking at asset managers, traders, and other practitioners not obliged to keep capital like banks. In this case, the major issue would be to effectively hedge against very bad outcomes and "black swans" (Taleb, 2017), so by following the econophysics literature results and ours, a better option would be to model the tail risk through a PL distribution. This will make a more conservative assessment of the possible risk an investor is facing when dealing with some assets, or in our case, with stock indexes.

4.3 Scaling Laws in Returns

4.3.1 Exploratory Analysis

For this application, we want to know whether the PL relationship proposed by Müller et al. (1990) is also found in our data for the Nikkei 225 and the FTSE 100 stock indexes and discuss the implications of the results for the underlying process of returns and financial practice. To be consistent, we use the same methodology used in their original paper, which is ultimately based on the linearity of the logarithm of the PL function. We compute different statistics for absolute price changes (log-returns) for each frequency, creating also the time intervals of the frequencies (expressed in seconds), to carry on the analysis. Before starting with the statistical procedure, however, we first carry on a brief exploratory analysis of the data.

The proposed PL establishes a scaling law between the mean of absolute log-returns and the time intervals. Hence, we can first explore the relationship between both variables for both the Nikkei 225 and the FTSE 100 and observe whether this PL seems to hold. The mean absolute returns for each frequency and index are shown in Table 4.13:

We plot the data in Figure 4.8, where we observe that there seems to be a relationship of the form $(\Delta t)^{1/\lambda}$ for both indexes when looking at the linear plots and a linear relation of the form $(1/\lambda)\log(\Delta t)$ when looking at the log-log plots.

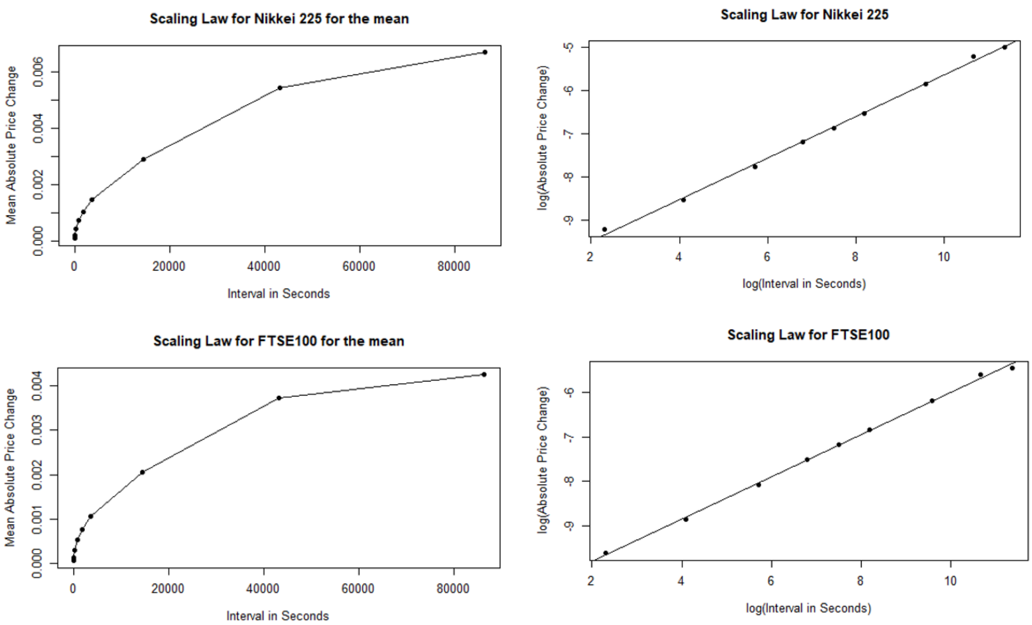
Apart from using the mean, the authors use other statistics such as the standard deviation, the interquartile range (IQR), the skewness, and the kurtosis. The last

Table 4.13: Mean of absolute log-returns and time intervals considered for Nikkei 225 and FTSE 100

	10s	**1m**	**5m**	**15m**	**30m**
Nikkei 225	9.96×10^{-5}	1.95×10^{-4}	4.27×10^{-4}	7.47×10^{-4}	1.03×10^{-3}
FTSE 100	6.65×10^{-5}	1.42×10^{-4}	3.10×10^{-4}	5.41×10^{-4}	7.57×10^{-4}
Seconds	10	60	300	900	1800

	1h	**4h**	**12h**	**1d**
Nikkei 225	1.46×10^{-3}	2.90×10^{-3}	5.44×10^{-3}	6.69×10^{-3}
FTSE 100	1.07×10^{-3}	2.05×10^{-3}	3.72×10^{-3}	4.24×10^{-3}
Seconds	3600	14400	43200	86400

Figure 4.10: Linear (left) and log-log (right) plots for the mean of absolute log-returns vs time intervals for Nikkei 225 and FTSE 100



two statistics are reserved for another part of this analysis, but we can mention now that, like the authors, we have found also a PL pattern for both the standard deviation $(\overline{|r_t|^2})^{1/2}$ and the interquartile range of absolute log-returns. These results will be key for analyzing scaling laws present in the time series of returns. More figures and tables that describe the different data are available in Appendix B.2.

4.3.2 Results

As results in the different tables in the exploratory analysis and Appendix B.2 show, the mean, the standard deviation, and the IQR seem to follow the proposed PL: the linear plots show a PL behaviour, and the log-log plots confirm this by showing a clear linear behaviour. Therefore, we can proceed just like Muller et al. (1990) and estimate the exponent of the PL. We recall that the authors stated that if the underlying process for the log-returns is an α-stable process, then, the exponent $1/\lambda$ can be identified as the $1/\alpha$ exponent of Lemma 3.4.5. However, we do not make any assumptions about the generating process, so we can just estimate the exponent to conclude whether it could be possible that returns were generated by this type of process or not (like in the previous application).

If the generating process is not an α-stable process, then the presence of this scaling law will not be a consequence of the stability of an α-stable process, but it would be a phenomenological scaling law (Müller et al., 1990).

The method we use to estimate the exponent in the PL is the ordinary least squares regression method, which is justified by the linear PL behaviour when considering the logarithms. If the PL behaviour holds, then by taking logarithms on both sides of the relationship we should see a linear behaviour on a plot with logarithmic axes. And, if the results of the graphical analysis are consistent with this linear PL behaviour, then one can use the linear regression estimation method to obtain the PL exponent and the constant of the relation, as we are treating with an untruncated PL. We point out that this method has a problem: $\overline{|r_t|}$ values for different intervals Δt are not independent, as the larger time interval is a sum of smaller intervals so that the regression is applied to dependent observations. Despite this problem, we justify the usage of this method as an approximation because the multiplicative factor between neighboring time intervals is at least 2 (so that the dependency is not too important), just as Müller et al. (1990).

As the scaling law seems to hold for the indexes, we use the OLS regression fit for the data to obtain the value of the parameters of the PL, shown in Table 4.13.

Table 4.14: Parameters of the PL law for the mean of absolute log-returns for Nikkei 225 and FTSE 100

	Nikkei 225	**FTSE 100**
$\ln(C)$	-10.434 ± 0.066	-10.750 ± 0.057
$1/\hat{\lambda}$	0.479 ± 0.008	0.474 ± 0.007
$\hat{\lambda}$	2.086	2.109

The results for the mean of absolute log-returns PL show that the exponent $1/\lambda$ is close to 0.475 for both indexes, indicating that the generating process should not be an α-stable process because $\alpha > 2$ in our sample. Despite that, the scaling law between the mean absolute log-returns and the frequencies or time intervals seems to hold. If we look at $(\overline{|r_t|^2})^{1/2}$ and at the IQR OLS fit in Tables 4.14 and 4.15, we obtain the following results:

Table 4.15: Parameters of the PL law for the standard deviation of absolute log-returns for Nikkei 225 and FTSE 100

	Nikkei 225	**FTSE 100**
$\ln(C)$	-10.265 ± 0.040	-10.520 ± 0.041
$1/\hat{\lambda}$	0.455 ± 0.005	0.449 ± 0.005
$\hat{\lambda}$	2.198	2.227

These results show that the exponent $1/\lambda$ has changed, and can be different depending on the statistic we are using. These results contrast in some aspects with those found by Müller et al. (1990). The authors obtained values near $1/\lambda \approx 0.59$ for the mean of absolute log-returns, implying that $\lambda \approx 1.66$, when analyzing foreign exchange time series. Due to this result, which would not be contrary to the α-stable hypothesis, they decided to compare the estimations for the exponent of

Table 4.16: Parameters of the PL law for the IQR of absolute log-returns for Nikkei 225 and FTSE 100

	Nikkei 225	**FTSE 100**
$\ln(C)$	-10.308 ± 0.151	-10.606 ± 0.098
$1/\hat{\lambda}$	0.473 ± 0.019	0.461 ± 0.012
$\hat{\lambda}$	2.114	2.169

the scaling law with the standard deviation and the IQR, as they should be statistically the same given that there would be only one generating process and a unique characteristic exponent α if the process is stable. Additionally, they also looked at the cumulative distribution functions for the different frequencies, which must stay the same if data is generated by an α-stable process.

For their data, the scaling law for the standard deviation yielded $1/\lambda \approx 0.52$ and for the IQR $1/\lambda \approx 0.7$, which shows a change in the exponent and hence in α, leading to the conclusion that the process must not be stable (if so, the characteristic exponent would not change). For our data, the exponent changes between different distributions but are not as clear as with the authors' case, because differences for our estimations are quite small and there are overlaps between the shown intervals. Consequently, an interesting result of our analysis is that even though it depends on the assets and the market we are looking at, the PLs seem to hold and, for both indexes studied, the PL relationships using different statistics that proxy volatility or risks such as the mean, the standard deviation or the IQR of absolute log-return, seem to behave more or less equally, having little differences among exponents (and, hence, among the scaling law that each statistic follows).

The clearest way of eliminating doubts about the possible stability of the underlying process is to compare the distributions for different time frequencies. The authors found that distributions changed depending on the frequency, so that the underlying process cannot be stable and hence they confirmed the PLs found are phenomenological. One can start by analyzing them by using the estimated statistics for the mean, the variance, the skewness, and the kurtosis for some frequencies. We consider frequencies of 5 minutes, 4 hours, and 1 day for each index so that we can compare quite different frequencies. Results are shown in Tables 4.15 and 4.16.

Table 4.17: Moments for some frequencies of the distribution of the Nikkei 225 log-returns

	Mean	**Variance**	**Skewness**	**Kurtosis**
$\Delta t = 5m$	4.267×10^{-4}	2.109×10^{-7}	3.709	31.395
$\Delta t = 4h$	2.901×10^{-3}	7.247×10^{-6}	1.729	3.737
$\Delta t = 1d$	6.691×10^{-3}	3.735×10^{-5}	1.222	1.129

Table 4.18: Moments for some frequencies of the distribution of the FTSE100 log-returns

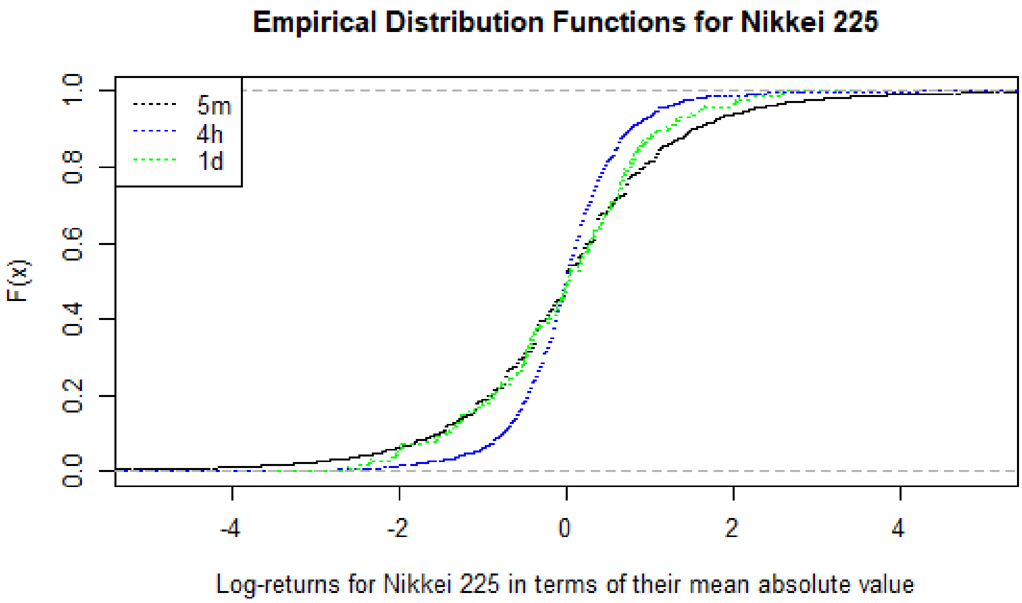
	Mean	**Variance**	**Skewness**	**Kurtosis**
$\Delta t = 5m$	3.101×10^{-4}	1.137×10^{-7}	2.825	15.954
$\Delta t = 4h$	2.051×10^{-3}	4.009×10^{-6}	1.950	5.362
$\Delta t = 1d$	4.244×10^{-3}	1.937×10^{-5}	1.987	4.812

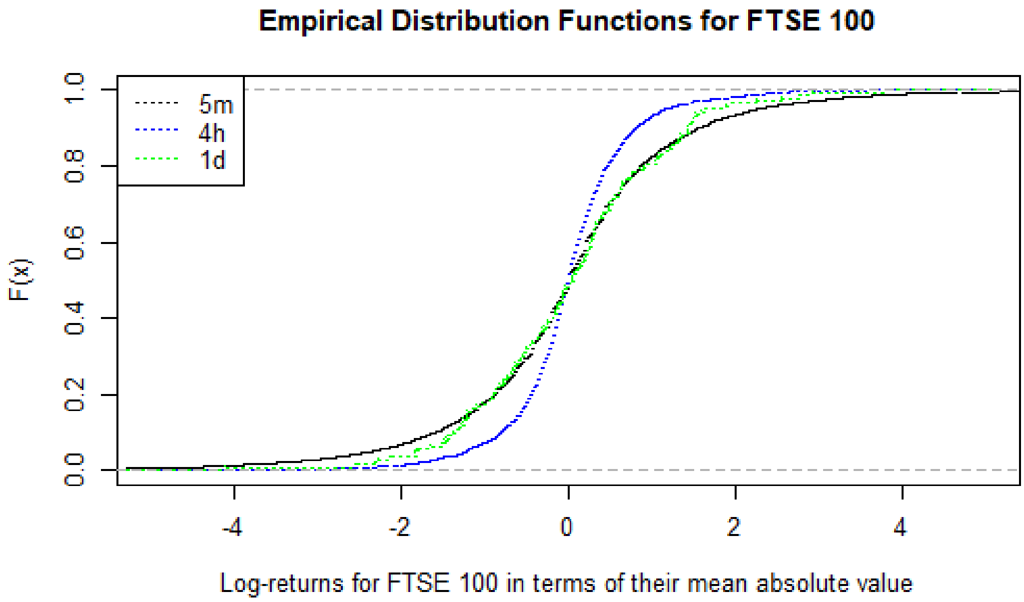
The results appearing in these tables show that the distribution changes depending on the frequency, but these statistics are subject to a scaling effect, as higher frequencies are aggregated when considering lower frequencies. Therefore, we can do the same as Müller et al. (1990) and graphically display the cumulative distributions for each frequency considered in terms of their mean absolute value, so that the scaling is compensated to make distributions comparable. We show the different distributions for both the Nikkei 225 and the FTSE 100 in Figure 4.9.

The images show that the distribution changes depending on the frequency, as some cumulative distributions have fatter tails than others. Because of the change observed, then the underlying process of returns should not be stable (as stability would make distributions similar), at least for our sample. Due to this reason, the evidence does not support identifying the exponent $1/\lambda$ with that exponent $1/\alpha$ from Lemma 3.4.5.

We can also conclude that the PL proposed by Müller et al. (1990) holds for our data even though the underlying process of returns is not stable, and hence does

Figure 4.11: Cumulative distributions for some frequencies of Nikkei 225 and FTSE 100





not stem from its properties. This, instead, is a phenomenological scaling law for log-returns, and this is why there is interest in why this scaling law holds. Indeed, it does just not hold for the mean of absolute of log-returns, but also for their standard deviation and IQR. The authors did not give explanations about why this phenomenon occurs, they just mentioned different causes of the instability such as different information flows or autoregressive conditional heteroskedasticity (ARCH) effects (Müller et al., 1990).

Even though no theoretical explanation has been found, an interesting economic interpretation is that this law holds due to a mixture of risk profiles of investors trading at different time horizons (Guillaume et al., 1997). Because one can interpret $|\overline{r_t}|$ as a measure of volatility for a time horizon Δt (while $(\overline{|r_t|^2})^{1/2}$ and IQR can also be used as measures of risk), then this measure indicates the maximum return an investor should expect on average for a given Δt. Therefore, we can use this PL to characterize the different expected returns for the different time horizons, which can be useful for different financial models and can be also used to have insights into the volatility of the asset (in our case, for the index).

5 | Conclusions & Discussion

In the first section, we discussed why interpreting financial markets as complex systems can be reasonable and useful. Comparing the different features of financial markets with those of complex systems, we concluded that many of them are shared and, hence, by using statistical methods and results from non-orthodox areas of economics and science such as econophysics, we could obtain broader insights and interesting empirical results that can be applied to financial modelling and analysis. Indeed, we concentrated on one crucial aspect which is the focal point of this thesis: the emergence of PLs in these complex systems and financial markets and their utility. These relationships seemed to appear commonly in financial data, and many results from the econophysics literature supported the idea that their usage and analysis could be worthy for financial practice.

One of the most important applications of PLs in finance is their usage for modelling the tails of the distribution of returns so that practitioners can model and quantify tail risk. To justify their usage, we used EVT results as a theoretical and probabilistic basis for which one can understand how PL behaviour and extreme observations are related. We also established a relationship between the PL distribution and one of the most used distributions for quantifying tail risk in finance, the GPD, establishing that the first is a particular instance of the second.

We presented different results and hypotheses from econophysics relevant to finance literature and practice, concerning the process that returns follow and their tail behaviour. For the discussion on the process of returns, we highlighted the importance of Mandelbrot's stable Paretian hypothesis and their relationship with PLs, and we explained different PLs proposed in the econophysics literature that will be used as a basis for our empirical applications, namely the inverse cubic law of returns (proposed by many authors in the field) and the scaling law for financial returns proposed by Müller et al. (1990).

In the first empirical application, we used high-frequency data (10 seconds to 1 hour) to fit PLs and GPDs to the tails of financial returns, estimating tail exponents and computing Value at Risk (VaR) and Expected Shortfall (ES). We discussed these results in the context of Mandelbrot's proposal and the results obtained in the econophysics literature. We also considered the effects of two probabilistic forces, the central limit theorem and the persistence of regularly varying tails, on the observed behaviour of the tails. The results of the application showed that the exponent for the tails of the distribution for each frequency changes so that the hypothesis of $\alpha \approx 3$ for different intraday frequencies found in econophysics does not hold. We instead observe that they go to $\alpha \approx 2$ and that the tails of a normal distribution fit the tail data better as we decrease the frequency of the data, illustrating the effect of the CLT. This has also implications for the stable Paretian hypothesis, because, even if we do not know whether returns come from an α-stable process, the estimated tail exponents are not compatible with the hypothesis, and the effect of the CLT shows how a normal might also fit the data well enough, even though the effect of the regularly varying tail persistence is still visible.

Additionally, when we compare the GPD fits with the PL distribution fits, we observe the same effect of the CLT and the persistence of regularly varying tails, even if the exponents do vary (recalling the relationship between α coming from a PL distribution and ξ coming from the GPD). Through graphical means, we observe that the flexibility of the GPD makes it fit the data better than the PL, even though both models overestimate and underestimate the observed values in the tail sometimes.

When measuring the tail risk through the fitted distributions, we can see that the GPD yields lower VaR and ES values, given that the tails are thinner in our results. Therefore, we argued that practical implications of this would be that financial institutions such as insurers or banks would prefer to model tails with a GPD distribution because of their obligation to allocate assets to risk capital depending on tail risk measures, while other practitioners such as traders, risk managers, or asset managers might prefer a PL distribution if they are conservative enough and want to protect themselves from highly unlikely events.

In the second application, we extend the analysis to a broader range of frequencies (10 seconds to 1 day) to examine scaling laws, specifically the relationship between log-return intervals and their mean absolute values, standard deviations, and IQRs, to determine if returns do respect the scaling law and discuss whether returns could follow an α-stable process depending on the estimated value of the exponent $1/\lambda$.

The statistical analysis carried out consisted of an estimation of this PL exponent $1/\lambda$ for the mean, the standard deviation, and the IQR of the absolute log-returns for the different time horizons or frequencies, and a posterior analysis of the distributions for the different time series for various frequencies to assess the stability of the underlying distributions of returns. Results show that the scaling law also holds for our data and that there is no evidence (with our data) of the scaling law deriving from the properties of an α-stable process, which is mentioned by the authors of the original paper. Therefore, this PL relationship is an empirical pattern that does not seem to have a clear explanation, even though Guillaume et al. (1997) provide the idea of this PL representing a mix of risk profiles of investors in financial markets, which can be used for financial modelling of these markets.

All in all, from the theory developed and the results provided through the empirical applications, the usage of statistical and probabilistic methodologies related to PLs seems to be very insightful and useful for the financial data of different markets and assets, as illustrated by our examples. The idea of using them is backed principally by the interpretation of markets as complex systems but is also justified by EVT probabilistic results. The econophysics literature and Mandelbrot's work are based on this view and provide relevant results for financial practice and theoretical finance discussions. Moreover, the application of different methodologies and results related allowed us to observe the effect of probabilistic forces like the CLT and the sum of regularly varying tails to the behavior of the tails of high-frequency returns' distributions and the presence of a scaling law among the different frequencies. Both could be used in financial modelling, specifically for risk management.

6 | Further Research

In this last part of the thesis, we briefly develop some ideas for further research derived from the connections and applications presented in this thesis and our interest in combining complex systems and financial markets, which we think can contribute to the econophysics literature.

6.1 Multivariate PL Distributions for Stock Market Data

Even though we used univariate statistical analyses applied to the time series of returns for the Nikkei 225 and the FTSE 100 stock market indices, we could model the tail risk using multivariate methods, so that we consider both assets at the same time.

Specifically, given the importance of PL relationships and PL tails in the econophysics literature (Gopikrishnan et al., 1999; Plerou et al., 2001; Gabaix et al., 2003, 2006; Gabaix, 2009), we propose modelling the tails of the joint distribution of returns for two or more assets using multivariate distributions such as the multivariate Pareto distribution and the multivariate GPD. Then, we can compare the different implications of using one type of model or another for the tails when it comes to risk measuring in financial practice.

6.2 A Market Risk Model for Extreme Climate Events

This thesis explored how financial markets could be understood as complex systems given their shared characteristics. One of these characteristics, openness, has received more attention this last decade due to the effect of human activity on climate change, and many regulators have tackled the issue of how to measure the risk that the openness and interconnections present in financial markets pose (Basel Committee on Banking Supervision, 2021).

One of the most interesting challenges is measuring physical climate change risk, given that extreme climate or weather events can have a great impact. Therefore, one intriguing idea would be to design a market risk model that allows the application of EVT techniques to quantify physical climate change risk and take it into account for a given portfolio of financial assets.

Relevant references that motivate this idea are Caby (2019), which explores EVT applied to dynamical systems with specific applications to climate, and McNeil et al. (2015), which presents different frameworks for risk quantification that can be used to design our model.

We must specify that the author of this same thesis has been researching how to take into account this type of climate risk in models using stochastic differential equations at the RiskCenter research department of the University of Barcelona so that this idea would be a natural continuation of his interests.

References

Alfarano, S., & Lux, T. (2010, September). Extreme value theory as a theoretical background for power law behavior. *Kiel Working Papers*, *2*(1), 31–58.

Arthur, W. B. (2021). Foundations of complexity economics. *Nature Reviews Physics*, *3*(2), 136 to 145. doi: 10.1038/s42254-020-00273-3

Basel Committee on Banking Supervision. (2021). *Climate-related risk drivers and their transmission channels*. Bank for International Settlements.

Caby, T. (2019). *Extreme value theory for dynamical systems, with applications in climate and neuroscience* (Unpublished doctoral dissertation).

Castillo, d., Joan, Serra, I., Padilla, M., & Moriña, D. (2019). Fitting tails by the empirical residual coefficient of variation: The ercv package. *The R Journal*, *11*(2), 56. doi: 10.32614/rj-2019-044

Clauset, A., Shalizi, C. R., & Newman, M. E. (2009). Power-law distributions in empirical data. *SIAM Review*, *51*(4), 661â703. doi: 10.1137/070710111

CNN. (2022, May). *Melvin capital to shut after heavy losses on meme stocks, market slump.* Retrieved from `https://edition.cnn.com/2022/05/19/investing/melvin-capital-hedge-fund-closes/index.html`

Cont, R. (2001, Feb). Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance*, *1*(2), 223â236. doi: 10.1088/1469-7688/1/2/304

Deluca, A., & Corral, A. (2013). Fitting and goodness-of-fit test of non-truncated and truncated power-law distributions. *Acta Geophysica*, *61*(6), 1351–1394. doi: 10.2478/s11600-013-0154-9

Embrechts, P., Kluppelberg, C., & Mikosch, T. (2013). *Modelling extremal events for insurance and finance*. Springer Berlin / Heidelberg.

Embrechts, P., Resnick, S. I., & Samorodnitsky, G. (1999). Extreme value theory as a risk management tool. *North American Actuarial Journal*, *3*(2), 30–41. doi: 10.1080/10920277.1999.10595797

Fama, E. F. (1963). Mandelbrot and the stable paretian hypothesis. *The Journal of Business*, *36*(4), 420. doi: 10.1086/294633

Farmer, J. D., & Geanakoplos, J. (2008). *Power laws in economics and elsewhere*. Retrieved from `https://www.inet.ox.ac.uk/publications/power-laws-in-economics-and-elsewhere/`

Farmer, J. D., & Lo, A. (1999). Retrieved from `https://www.santafe.edu/research/results/working-papers/frontiers-of-finance-evolution-and-efficient-marke`

Gabaix, X. (2009). Power laws in economics and finance. *Annual Review of Economics*, *1*(1), 255-294. Retrieved from `https://doi.org/10.1146/annurev.economics.050708.142940` doi: 10.1146/annurev.economics.050708.142940

Gabaix, X., Gopikrishnan, P., Plerou, V., & Stanley, H. E. (2003). A theory of power-law distributions in financial market fluctuations. *Nature*, *423*(6937), 267–270. doi: 10.1038/nature01624

Gabaix, X., Gopikrishnan, P., Plerou, V., & Stanley, H. E. (2006). Institutional investors and stock market volatility. *The Quarterly Journal of Economics*, *121*(2), 461–504. doi: 10.1162/qjec.2006.121.2.461

Gopikrishnan, P., Plerou, V., Nunes Amaral, L. A., Meyer, M., & Stanley, H. E. (1999). Scaling of the distribution of fluctuations of financial market indices. *Physical Review E*, *60*(5), 5305–5316. doi: 10.1103/physreve.60.5305

Guillaume, D. M., Dacorogna, M. M., Davé, R. R., Müller, U. A., Olsen, R. B., & Pictet, O. V. (1997). From the birdâs eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics*, *1*(2), 95â129. doi: 10.1007/s007800050018

Johnson, N. F., Jefferies, P., & Hui, P. M. (2003, 07). Financial markets as complex systems. In *Financial Market Complexity.* Oxford University Press. doi: 10.1093/acprof:oso/9780198526650.003.0001

Jusselin, P., & Rosenbaum, M. (2018). No-arbitrage implies power-law market impact and rough volatility. *SSRN Electronic Journal*. doi: 10.2139/ssrn.3180582

Kauffman, S. A. (1993, Jun). Self-organization and adaptation in complex systems. *The Origins of Order*, 173â236. doi: 10.1093/oso/9780195079517.003.0005

Kuhlmann, M. (2014). Explaining financial markets in terms of complex systems. *Philosophy of Science*, *81*(5), 1117-1130. Retrieved 2023-07-13, from `http://www.jstor.org/stable/10.1086/677699`

LeBaron, B. D. (2001). Stochastic volatility as a simple generator of financial power-laws and long memory. *SSRN Electronic Journal*. doi: 10.2139/ssrn.278427

Lux, T., & Alfarano, S. (2016). Financial power laws: Empirical evidence, models, and mechanisms. *Chaos, Solitons amp;amp; Fractals*, *88*, 3–18. doi: 10.1016/j.chaos.2016.01.020

Mandelbrot, B. (1963). The variation of certain speculative prices. *The Journal of Business*, *36*(4), 394. doi: 10.1086/294632

Mandelbrot, B. (1983). *The fractal geometry of nature*. Freeman.

Mandelbrot, B., & Hudson, R. L. (2010). *The (mis)behaviour of markets: A fractal view of risk, ruin, and reward*. Profile Books.

Mantegna, R. N., & Stanley, H. E. (2000). *An introduction to econophysics: Correlations and complexity in finance*. Cambridge University Press.

Mason, D. M., & Schuenemeyer, J. H. (1983, Sep). A modified kolmogorov-smirnov test sensitive to tail alternatives. *The Annals of Statistics*, *11*(3). doi: 10.1214/aos/1176346259

McNeil, A. J., Embrechts, P., & Frey, R. (2015). *Quantitative risk management: Concepts, techniques and tools*. Princeton University Press.

Müller, U. A., Dacorogna, M. M., Olsen, R. B., Pictet, O. V., Schwarz, M., & Morgenegg, C. (1990). Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis. *Journal of Banking & Finance*, *14*(6), 1189–1208. doi: 10.1016/0378-4266(90)90009-q

Newman, M. (2005). Power laws, pareto distributions and zipf's law. *Contemporary Physics*, *46*(5), 323–351. doi: 10.1080/00107510500052444

Office of the Controller of the Currency. (2019, Apr). Retrieved from `https://www.occ.treas.gov/topics/supervision-and-examination/capital-markets/financial-markets/index-financial-markets.html`

Plerou, P., V. Gopikrishnan, Gabaix, A. L., X., & Stanley, H. (2001). Price fluctuations, market activity and trading volume. *Quantitative Finance*, *1*(2), 262â269. doi: 10.1088/1469-7688/1/2/308

Sornette, D. (2003). *Why stock markets crash: Critical events in complex financial systems*. Princeton University Press.

Sornette, D., Deschâtres, F., Gilbert, T., & Ageon, Y. (2004, Nov). Endogenous versus exogenous shocks in complex networks: An empirical test using book sale rankings. *Physical Review Letters*, *93*(22). doi: 10.1103/physrevlett.93.228701

Stiglitz, J. E. (2004). Information and the change in the paradigm in economics. *New Frontiers in Economics*, 27–67. doi: 10.1017/cbo9780511754357.004

Taleb, N. N. (2017). *The black swan: The impact of the highly improbable*. Taylor and Francis.

Voitalov, I., van der Hoorn, P., van der Hofstad, R., & Krioukov, D. (2019). Scale-free networks well done. *Physical Review Research*, *1*(3). doi: 10.1103/physrevresearch.1.033034

Witt, U. (1997, Oct). Self-organization and economics: What is new? *Structural Change and Economic Dynamics*, *8*(4), 489â507. doi: 10.1016/s0954-349x(97)00022-2

A | Proofs & Notes

A.1 Proof of Theorem 3.1.2

Proof. We want to prove Theorem 3.1.2 regarding the exact functional form of the density function of a $PL(\alpha, x_{min})$ distribution for $\alpha > 0$. The integral from x_{min} to ∞ (the range for the values of X) of the density function $f(x)$ must be equal to 1. Assume $\alpha > 0$. Then, we integrate to obtain the following result that must hold for any density function:

$$\int_{x_{min}}^{\infty} f(x)dx = C\int_{x_{min}}^{\infty} x^{-\alpha-1}dx = -\frac{C}{\alpha}[x^{-\alpha}]_{x_{min}}^{\infty} = 1 \tag{A.1}$$

By isolating C, we obtain:

$$-\frac{C}{\alpha}[x^{-\alpha}]_{x_{min}}^{\infty} = 1 \quad \Rightarrow \quad C = \frac{\alpha}{x_{min}^{-\alpha}} \tag{A.2}$$

So, substituting the value of C, the density function is

$$f(x) = \frac{\alpha}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha-1} \tag{A.3}$$

which is the same as (3.2). Hence, the proof is complete. ■

A.2 Proof of Theorem 3.1.3

Proof. We want to prove that, given the density f of a $PL(\alpha, x_{min})$ distribution, $f(bx) = g(b)f(x)$ for any b where $g(b) = f(b)/f(1)$. Let $f(x) = \frac{\alpha}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha-1}$ and $g(b) = f(b)/f(1)$. Then

$$g(b) = \frac{f(b)}{f(1)} = \frac{\frac{\alpha}{x_{min}}\left(\frac{b}{x_{min}}\right)^{-\alpha-1}}{\frac{\alpha}{x_{min}}\left(\frac{1}{x_{min}}\right)^{-\alpha-1}} = b^{-\alpha-1} \tag{A.4}$$

and hence

$$g(b)f(x) = b^{-\alpha-1}\frac{\alpha}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha-1} = \frac{\alpha}{x_{min}}\left(\frac{bx}{x_{min}}\right)^{-\alpha-1} \tag{A.5}$$

However,

$$f(bx) = \frac{\alpha}{x_{min}}\left(\frac{bx}{x_{min}}\right)^{-\alpha-1} = g(b)f(x) \tag{A.6}$$

Therefore, we conclude that if f is the density function of a $PL(\alpha, x_{min})$, then $f(bx) = g(b)f(x)$ where $g(b) = f(b)/f(1)$ and the proof is complete. ■

A.3 Proof of Theorem 3.1.4

Proof. We want to prove Theorem 3.1.4, which can be done through direct computation of the kth moment of a $PL(\alpha, x_{min})$ distribution. Assuming X follows a PL distribution like the one specified, then

$$\begin{aligned} E(X^k) = \int_{x_{min}}^{\infty} x^k f(x)dx &= \frac{\alpha}{x_{min}^{-\alpha}}\int_{x_{min}}^{\infty} x^{k-\alpha-1}dx \\ &= \frac{\alpha}{x_{min}^{-\alpha}}\left[\frac{x^{k-\alpha}}{k-\alpha}\right]_{x_{min}}^{\infty} \end{aligned} \tag{A.7}$$

As we can see, the evaluation of the second factor will depend on the exponent $k-\alpha$. If $k > \alpha$, then

$$\begin{aligned} E(X^k) &= \frac{\alpha}{x_{min}^{-\alpha}}\left[\lim_{x\to\infty}\frac{x^{k-\alpha}}{k-\alpha} - \frac{x_{min}^{k-\alpha}}{k-\alpha}\right] \\ &= \frac{\alpha}{x_{min}^{-\alpha}}\left[\infty - \frac{x_{min}^{k-\alpha}}{k-\alpha}\right] = \infty \end{aligned} \tag{A.8}$$

so that higher moments than k do not exist (diverge). Instead, if $k < \alpha$

$$\begin{aligned} E(X^k) &= \frac{\alpha}{x_{min}^{-\alpha}}\left[\lim_{x\to\infty}\frac{x^{k-\alpha}}{k-\alpha} - \frac{x_{min}^{k-\alpha}}{k-\alpha}\right] \\ &= \frac{\alpha}{x_{min}^{-\alpha}}\left[0 - \frac{x_{min}^{k-\alpha}}{k-\alpha}\right] = \frac{\alpha}{\alpha-k}x_{min}^{k} \end{aligned} \tag{A.9}$$

so the moments lower or equal to k exist. Therefore, we conclude that

$$E(X^k) = \begin{cases} \frac{\alpha}{\alpha-k} x_{min}^k, & \text{for } k < \alpha \\ \infty, & \text{for } k \geq \alpha \end{cases} \tag{A.10}$$

as intended. Consequently, the proof is complete. ■

A.4 Proof of Theorem 3.2.9

Proof. For the first implication, suppose $X \sim PL(\alpha, x_{min})$ and apply a change of variable $Y = X - x_{min}$. Fix $\alpha = \xi^{-1}$ and $x_{min} = \beta/\xi$. We substitute the values in the $PL(\alpha, x_{min})$ distribution function to obtain

$$F(y) = \begin{cases} 1 - \left(\frac{y+x_{min}}{x_{min}}\right)^{-\alpha}, & \text{for } y + x_{min} \geq x_{min} \\ 0, & \text{for } y + x_{min} < x_{min} \end{cases} \tag{A.11}$$

$$\Rightarrow F(y) = G_{\xi,\beta}(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-\xi^{-1}}, & \text{for } y \geq 0 \\ 0, & \text{for } y < 0 \end{cases} \tag{A.12}$$

Hence, $Y \sim G_{\xi,\beta}$ for $\xi > 0$. It is now left to prove the second implication. Starting from the random variable Y found above, apply the same change of variable $Y = X - \beta/\xi$ and fix $\xi = \alpha^{-1}$ and $\beta = \xi x_{min}$ as before. We substitute the values in the $G_{\xi,\beta}$ distribution function to obtain

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \xi \frac{x-\beta/\xi}{\beta}\right)^{-1/\xi}, & \text{for } x - \beta/\xi \geq 0 \\ 0, & \text{for } x - \beta/\xi < 0 \end{cases} \tag{A.13}$$

$$\Rightarrow G_{\xi,\beta}(x) = F(x) = \begin{cases} 1 - \left(\frac{x}{x_{min}}\right)^{-\alpha}, & \text{for } x \geq x_{min} \\ 0, & \text{for } x < x_{min} \end{cases} \tag{A.14}$$

Hence, $X \sim PL(\alpha, x_{min})$ with $\alpha > 0$ as desired, so the proof is complete. ■

B | Additional Figures & Tables

B.1 PL and Extreme Behaviour in Returns

Figure B.1: Histograms of log-returns for the Nikkei 225

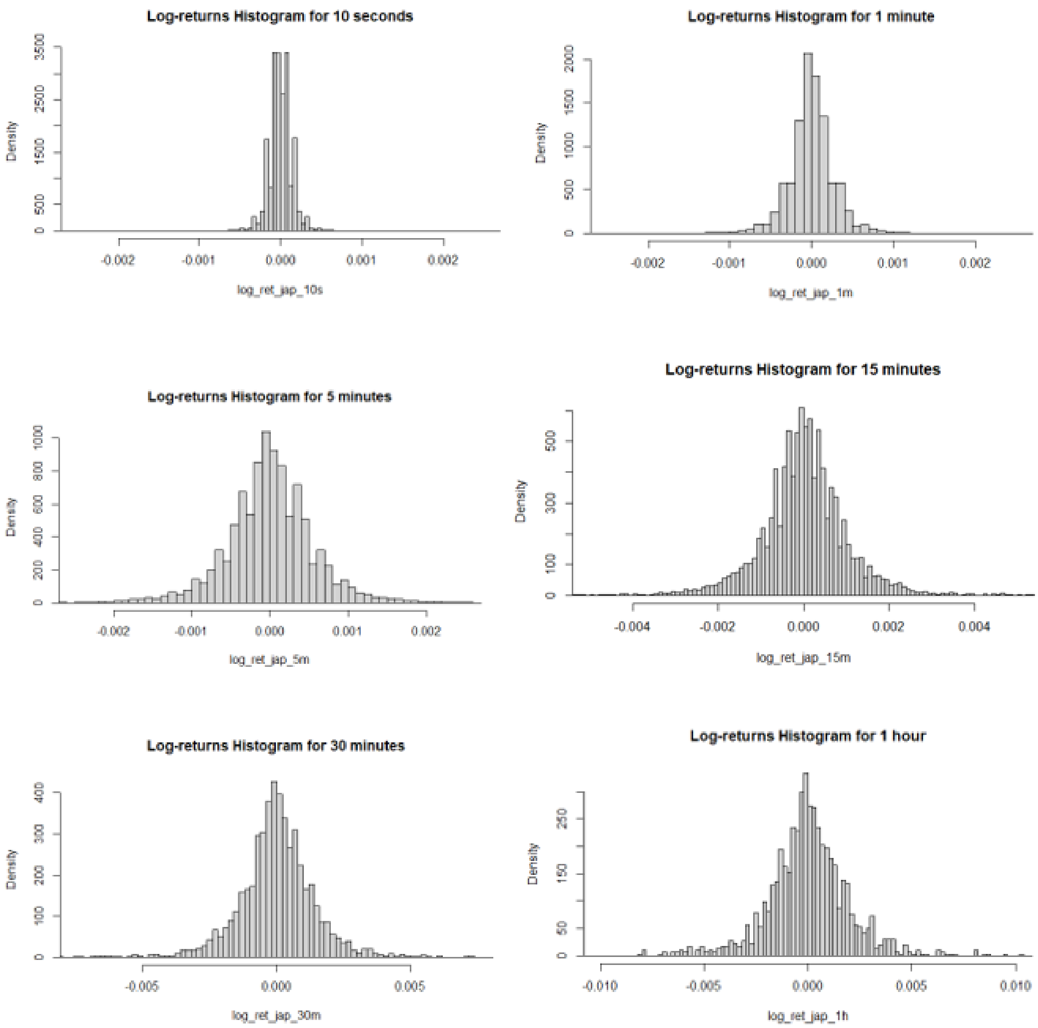


Figure B.2: Histograms of log-returns for the FTSE 100

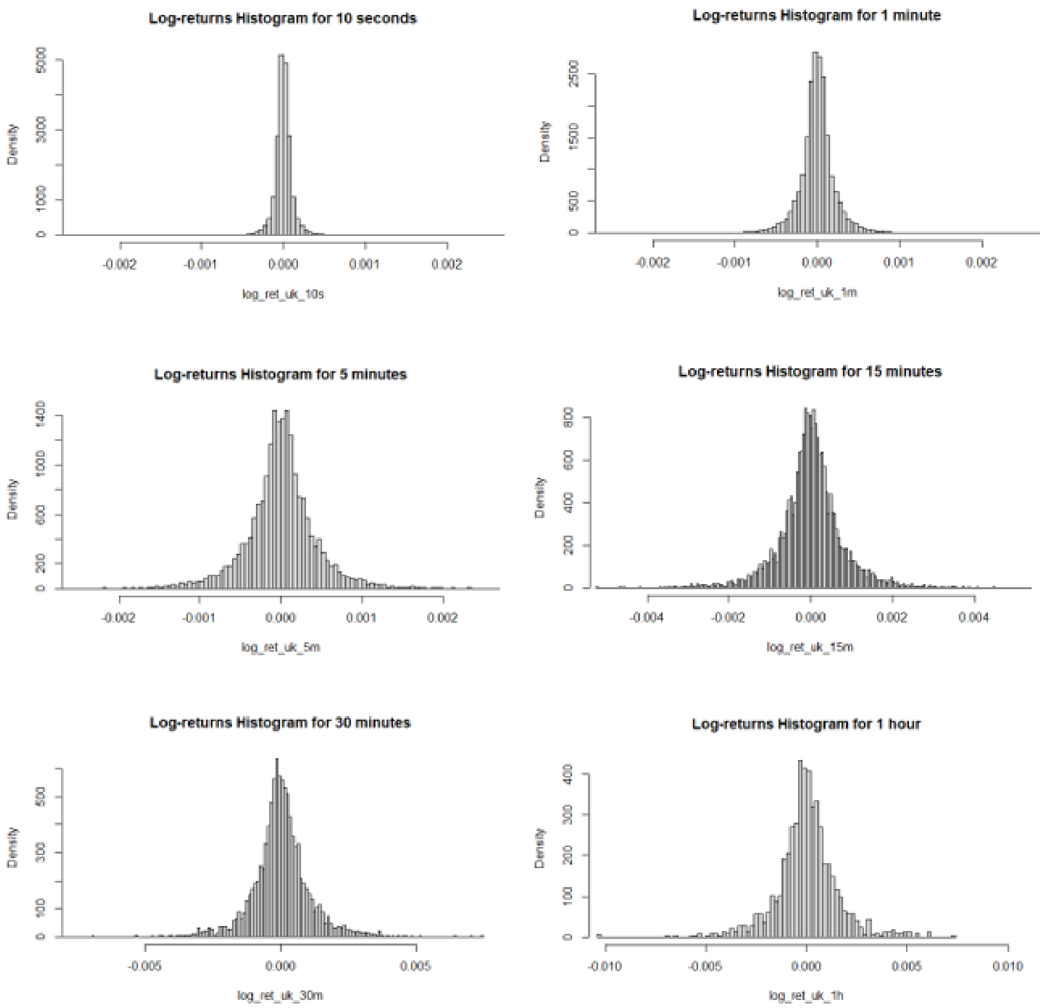


Figure B.3: Histograms of positive tails for the Nikkei 225

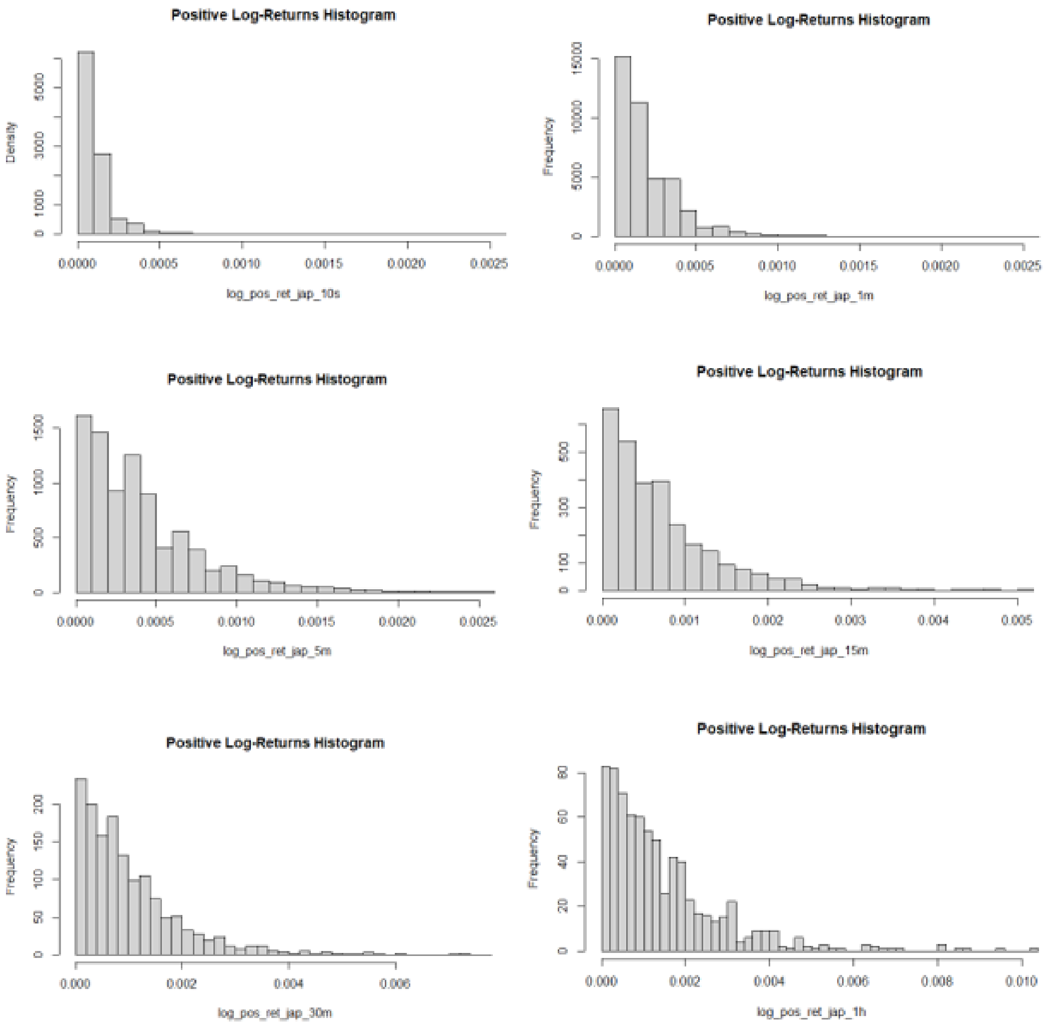


Figure B.4: Histograms of positive tails for the FTSE 100

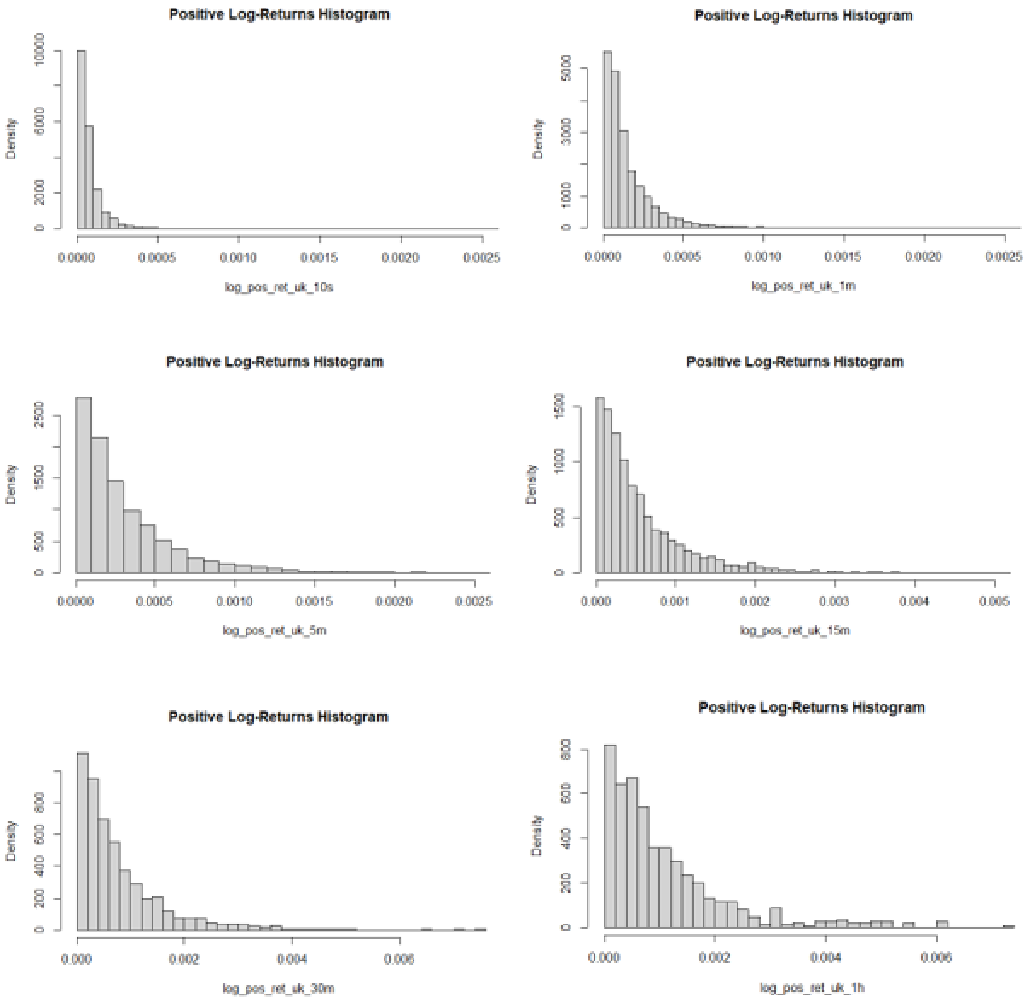


Figure B.5: Histograms of negative tails for the Nikkei 225

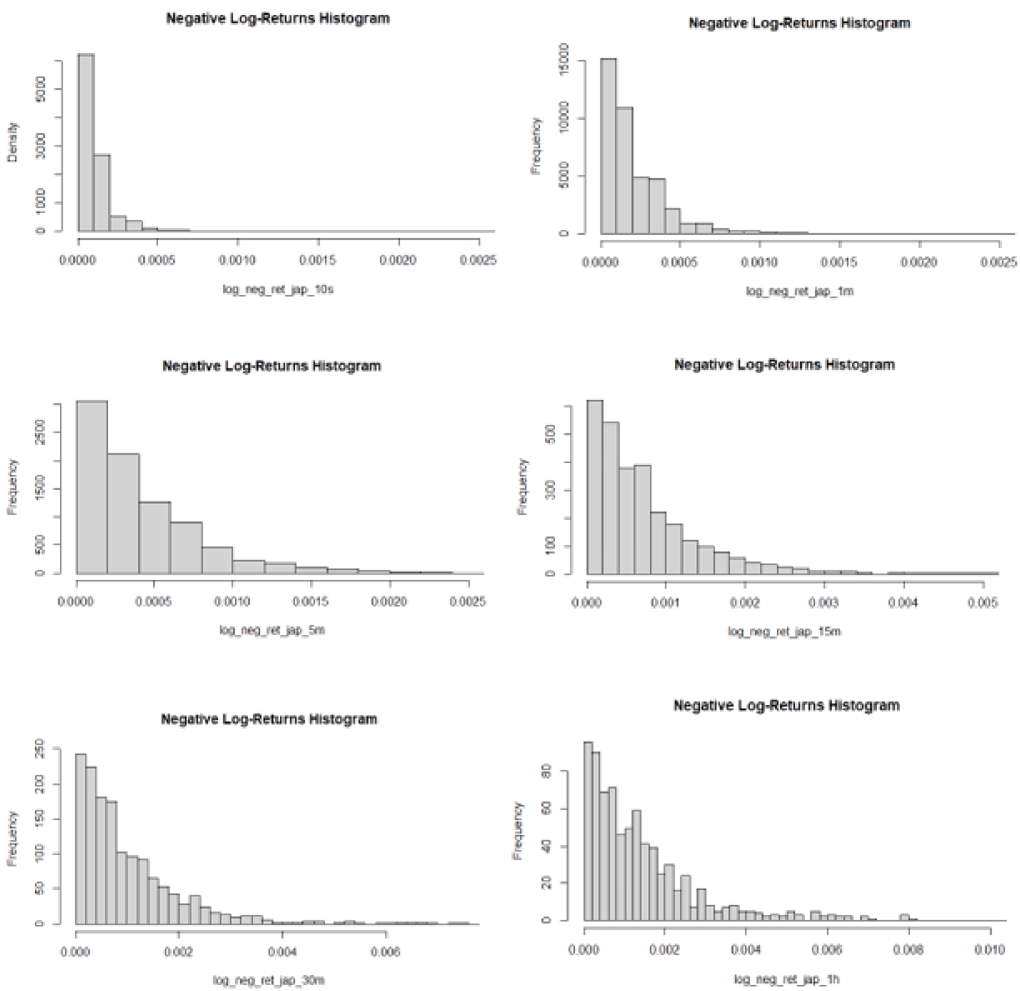


Figure B.6: Histograms of negative tails for the FTSE 100

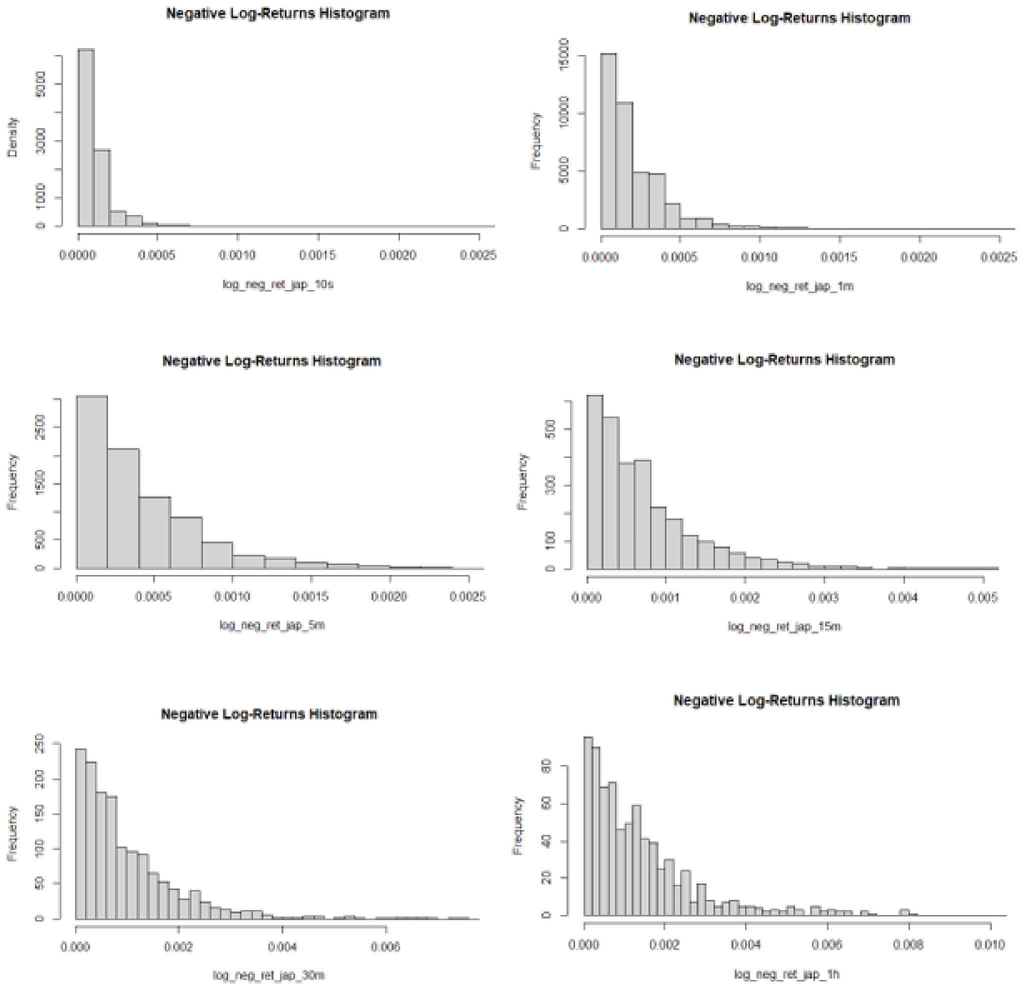


Figure B.7: Negative tail plots for the Nikkei 225 log-returns

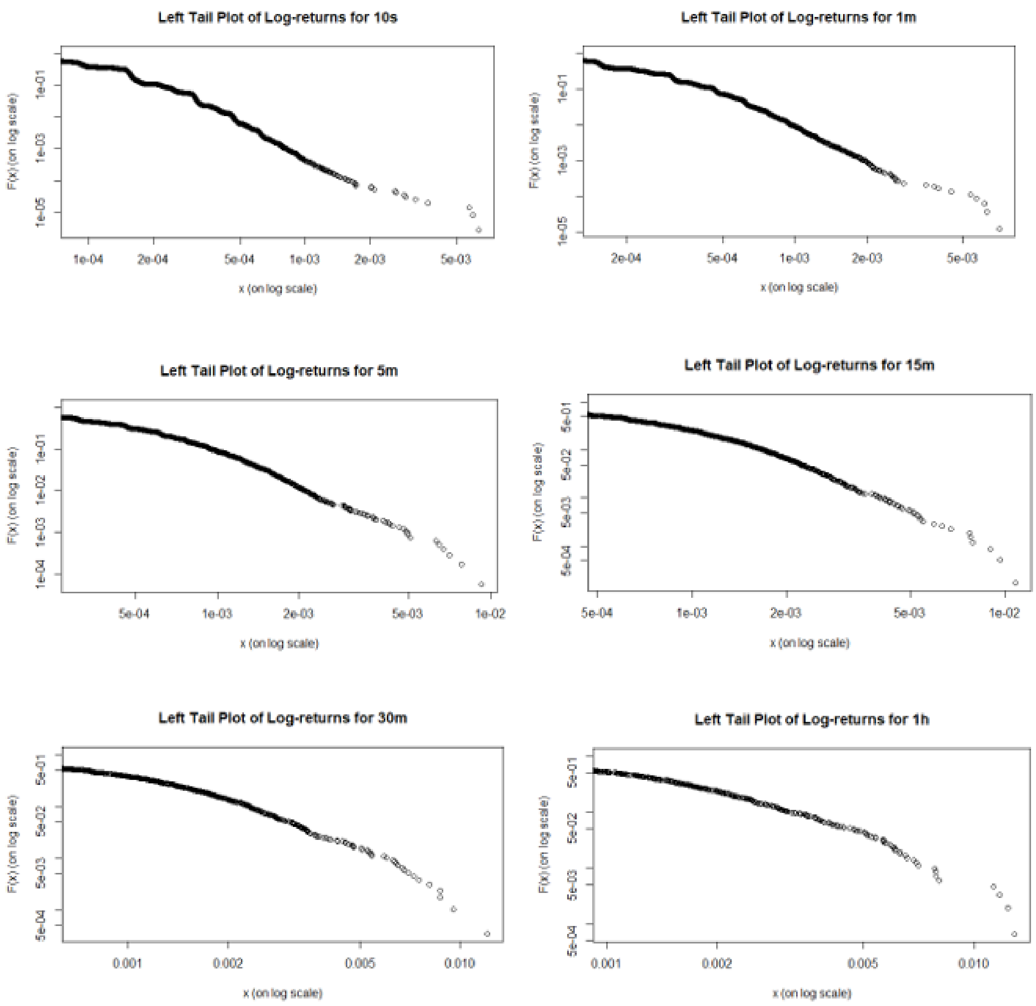


Figure B.8: Positive tail plots for the Nikkei 225 log-returns

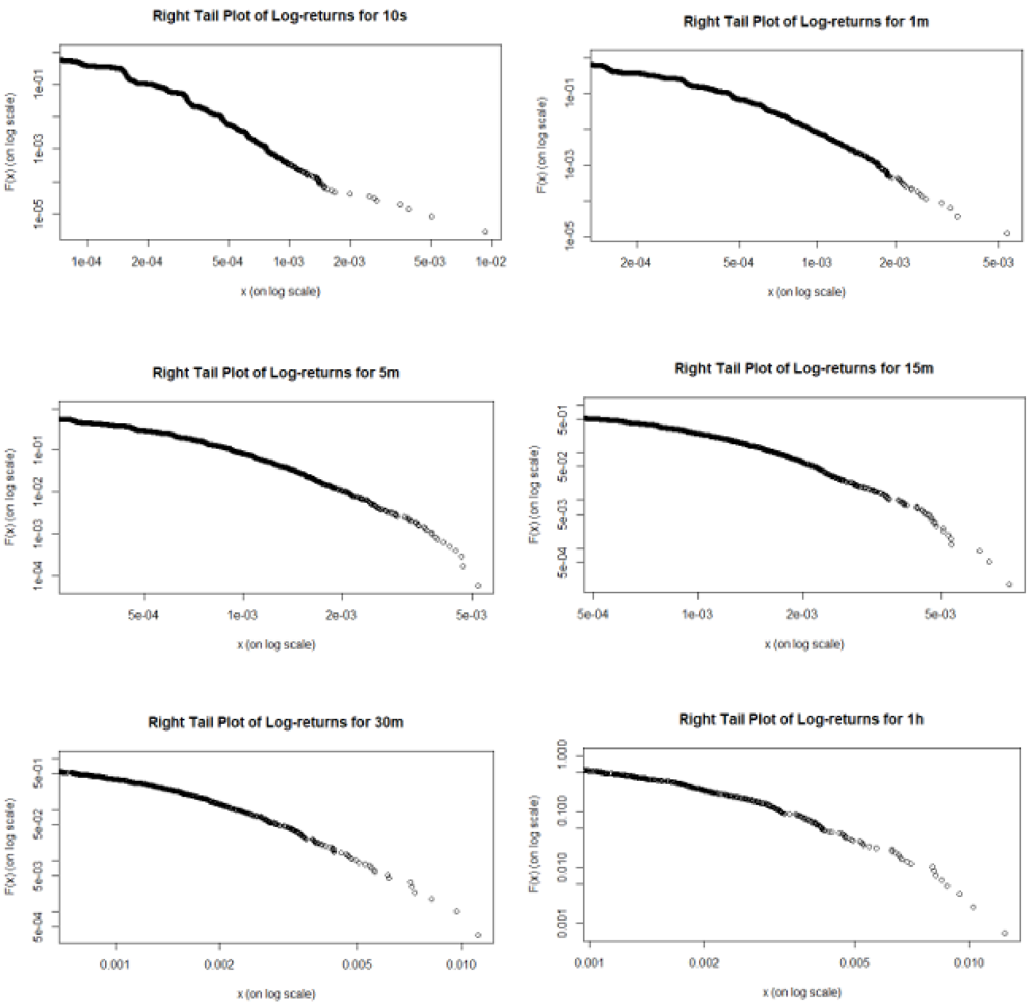


Figure B.9: Negative tail plots for the FTSE 100 log-returns

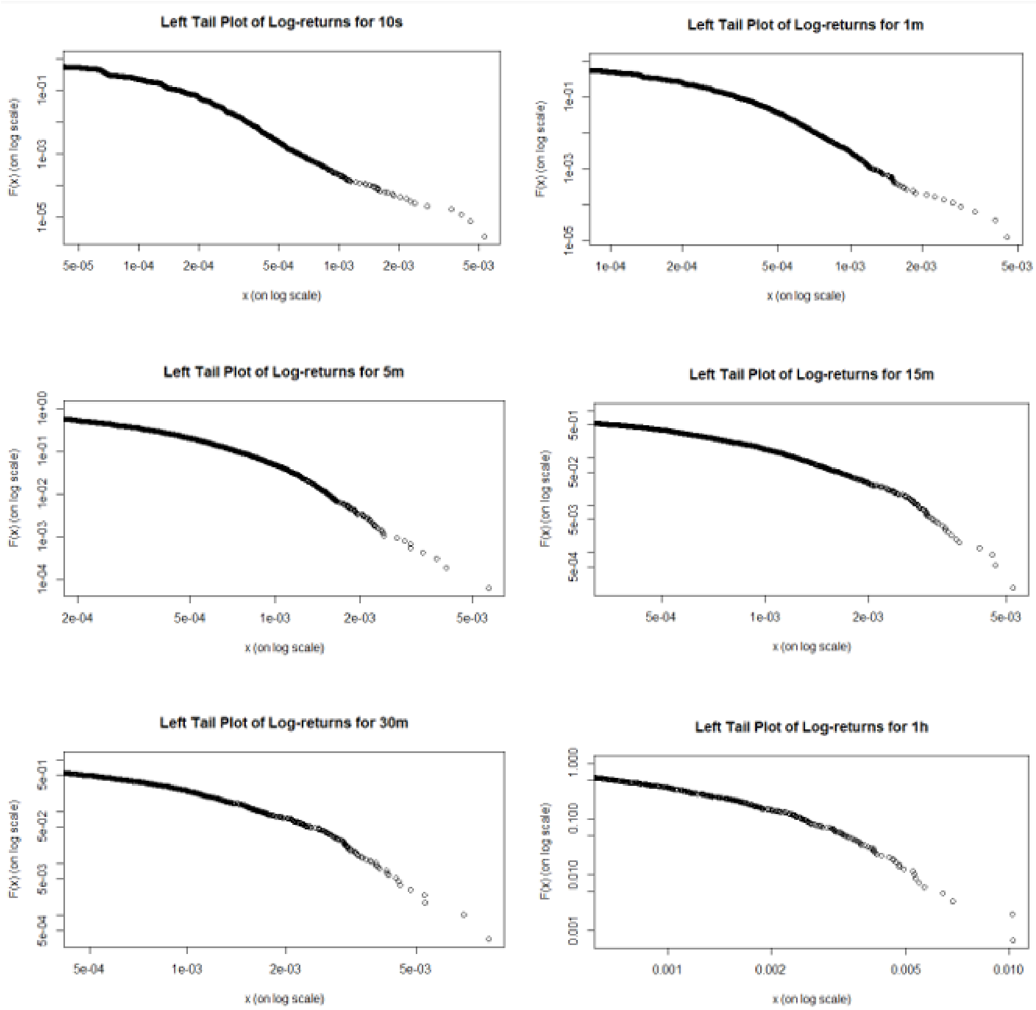


Figure B.10: Positive tail plots for the FTSE 100 log-returns

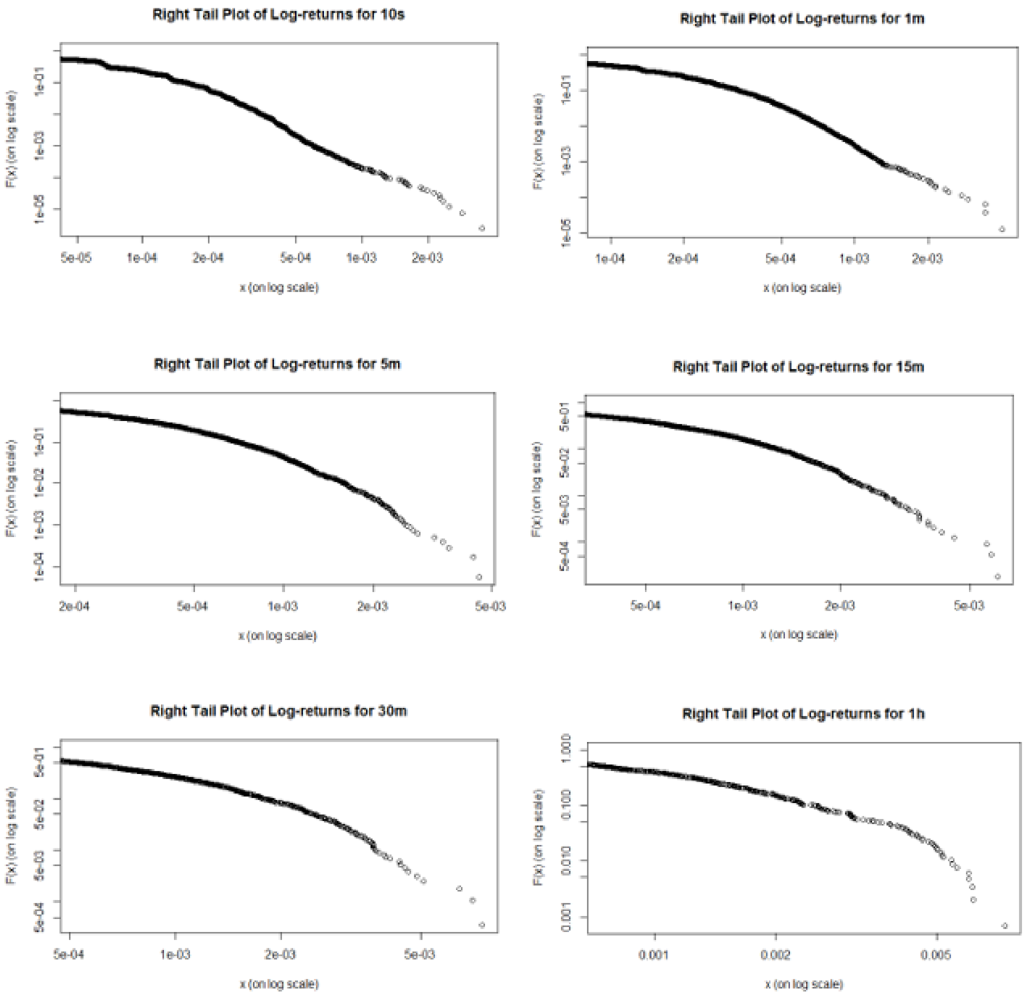


Figure B.11: QQ-plots for the Nikkei 225 log-returns

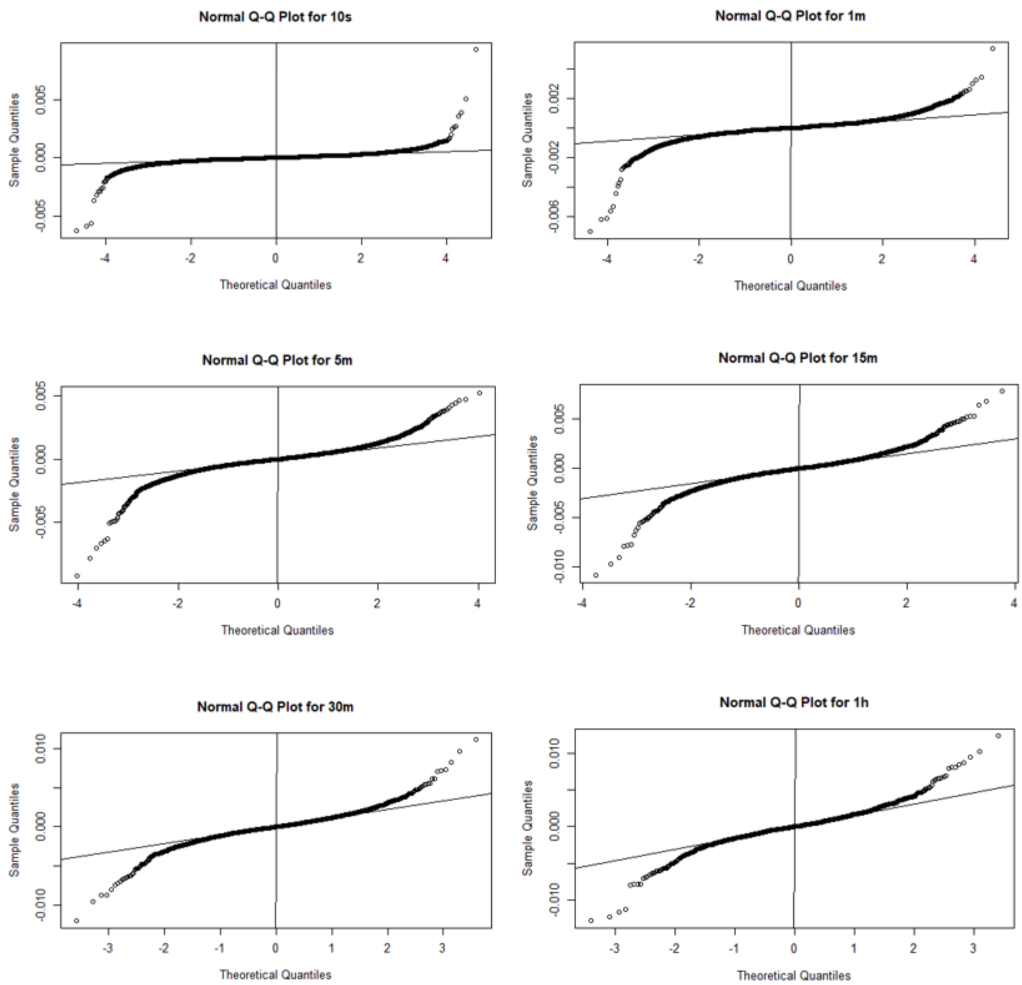


Figure B.12: QQ-plots for the FTSE 100 log-returns

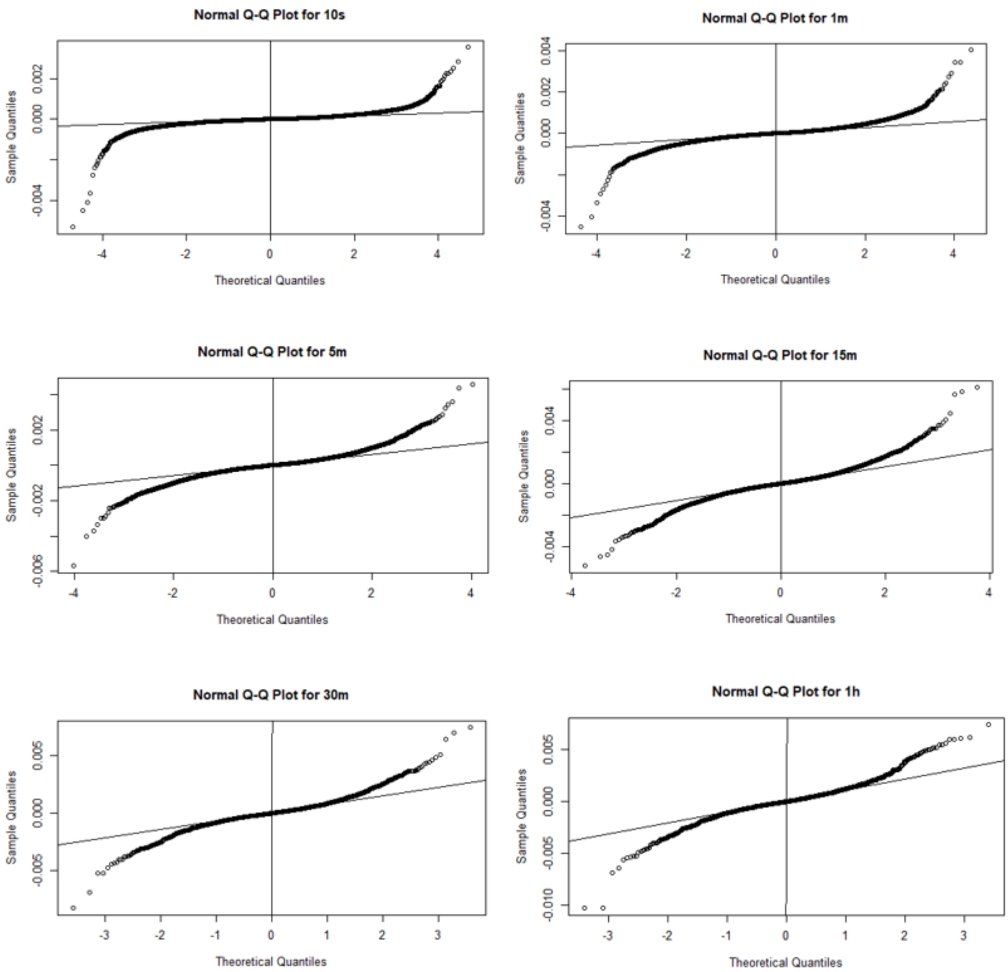


Figure B.13: Comparison of data to PL and normal tail behaviour for the negative tail of Nikkei 225 returns

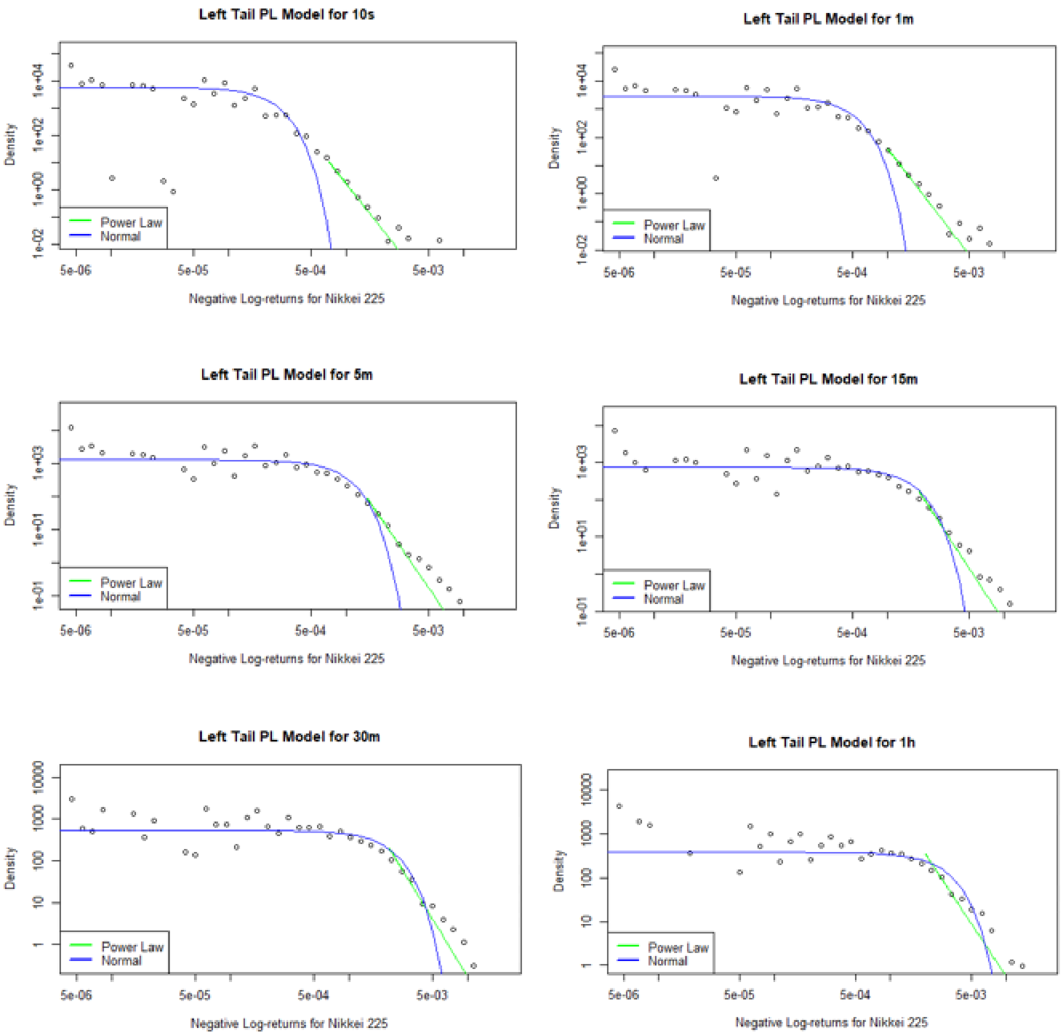


Figure B.14: Comparison of data to PL and normal tail behaviour for the positive tail of Nikkei 225 returns

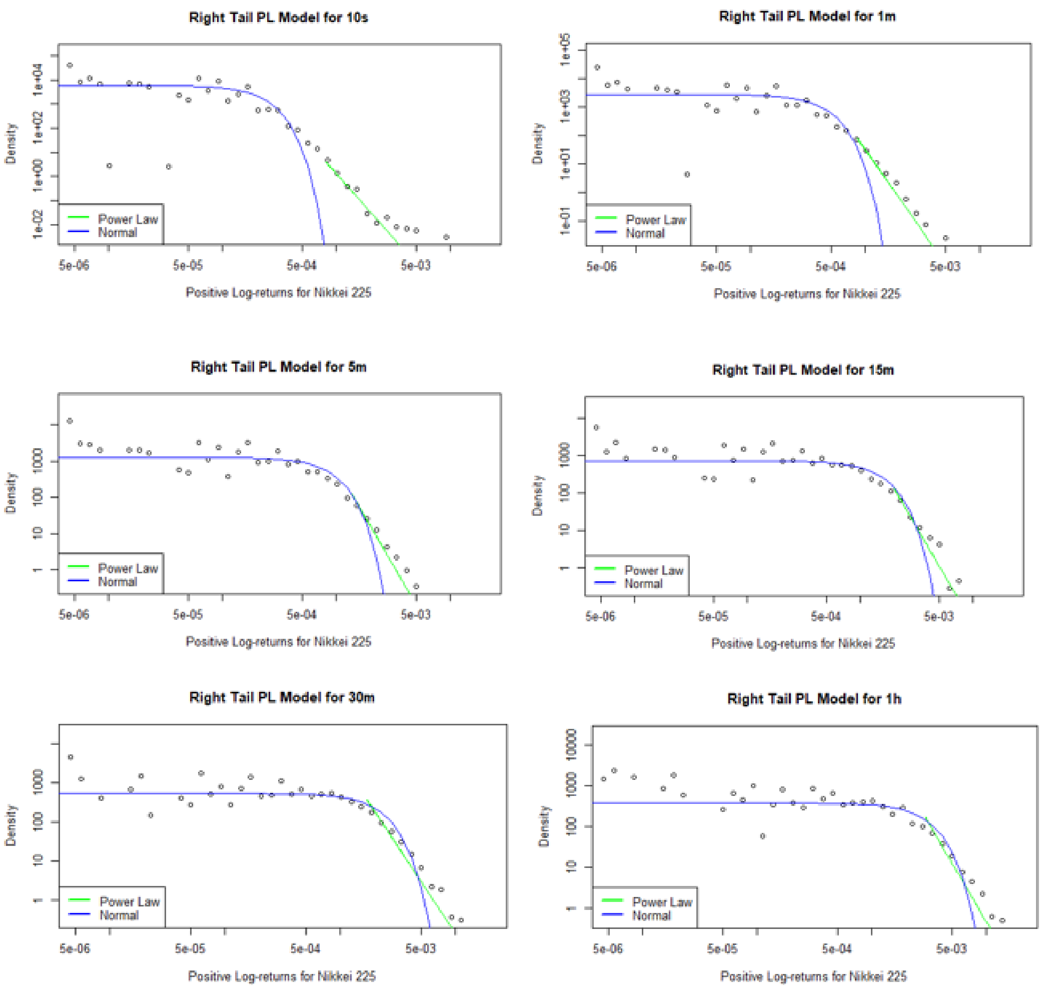


Figure B.15: Comparison of data to PL and normal tail behaviour for the negative tail of FTSE 100 returns

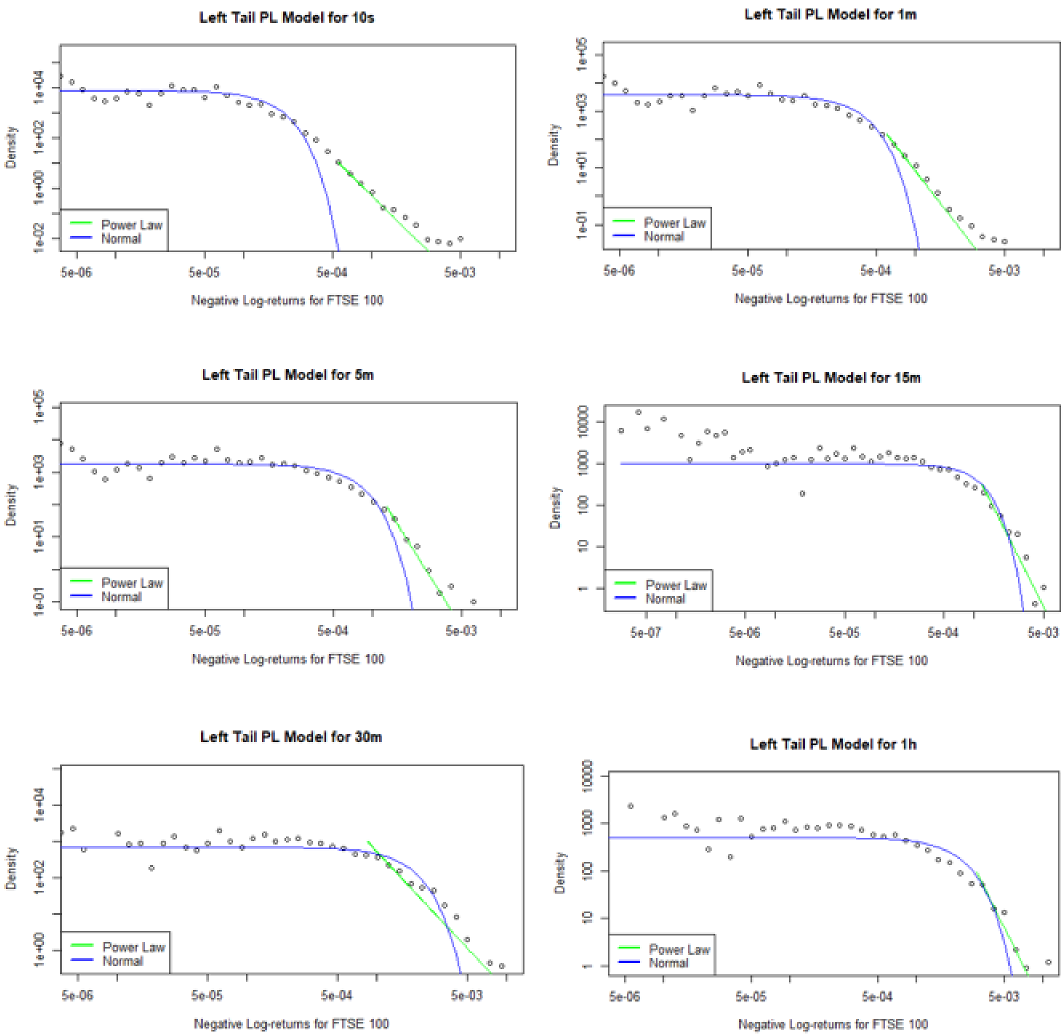


Figure B.16: Comparison of data to PL and normal tail behaviour for the positive tail of FTSE 100 returns

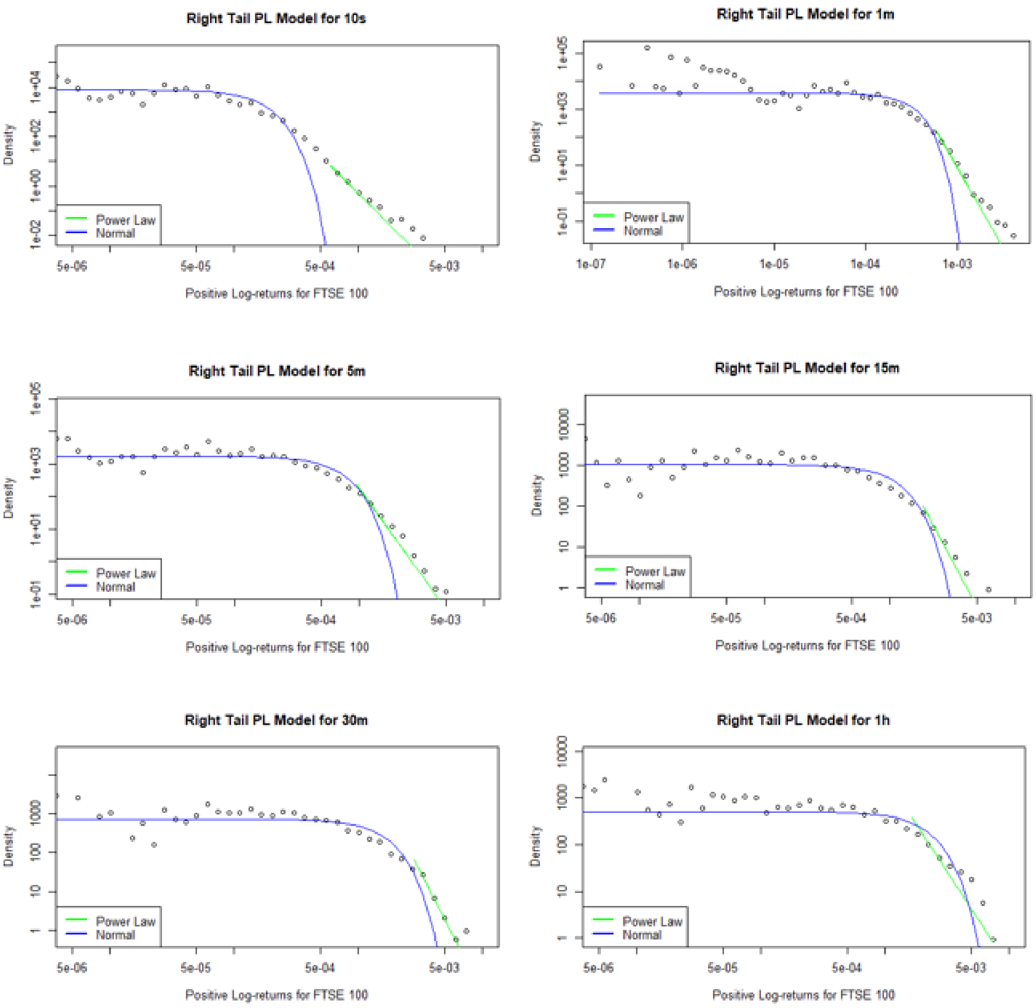


Figure B.17: Tail plots for the positive tails PL and GPD fit for 10-second to 5-minute frequencies for Nikkei 225 log-returns

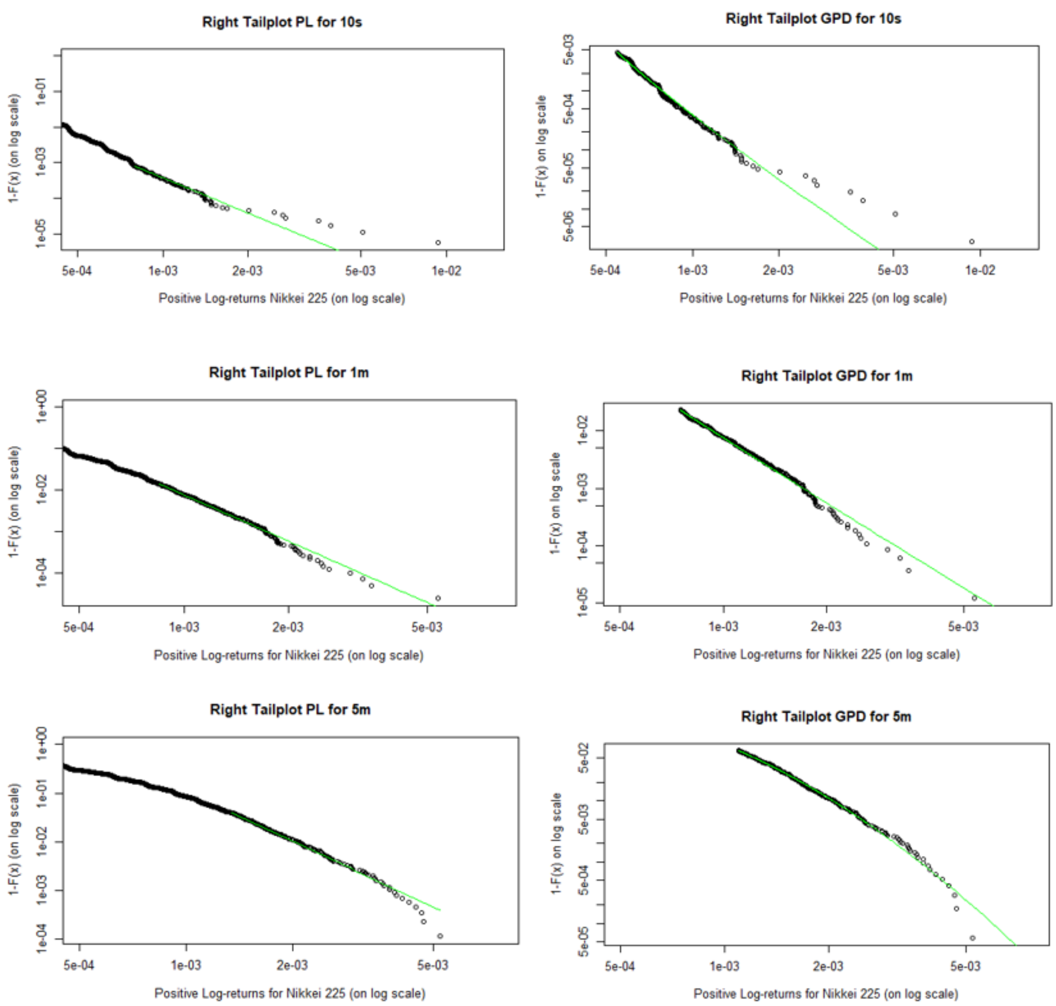


Figure B.18: Tail plots for the positive tails PL and GPD fit for 15-minute to 1-hour frequencies for Nikkei 225 log-returns

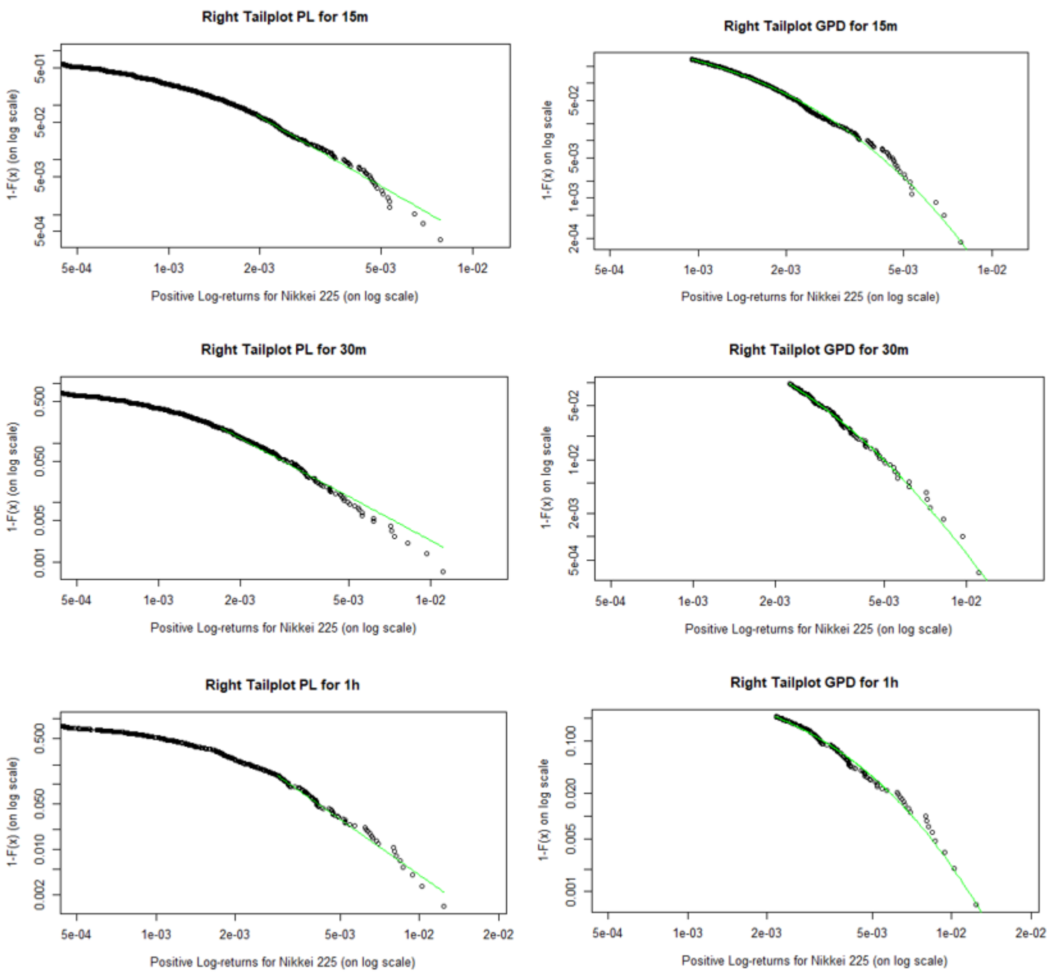


Figure B.19: Tail plots for the negative tails PL and GPD fit for 10-second to 5-minute frequencies for Nikkei 225 log-returns

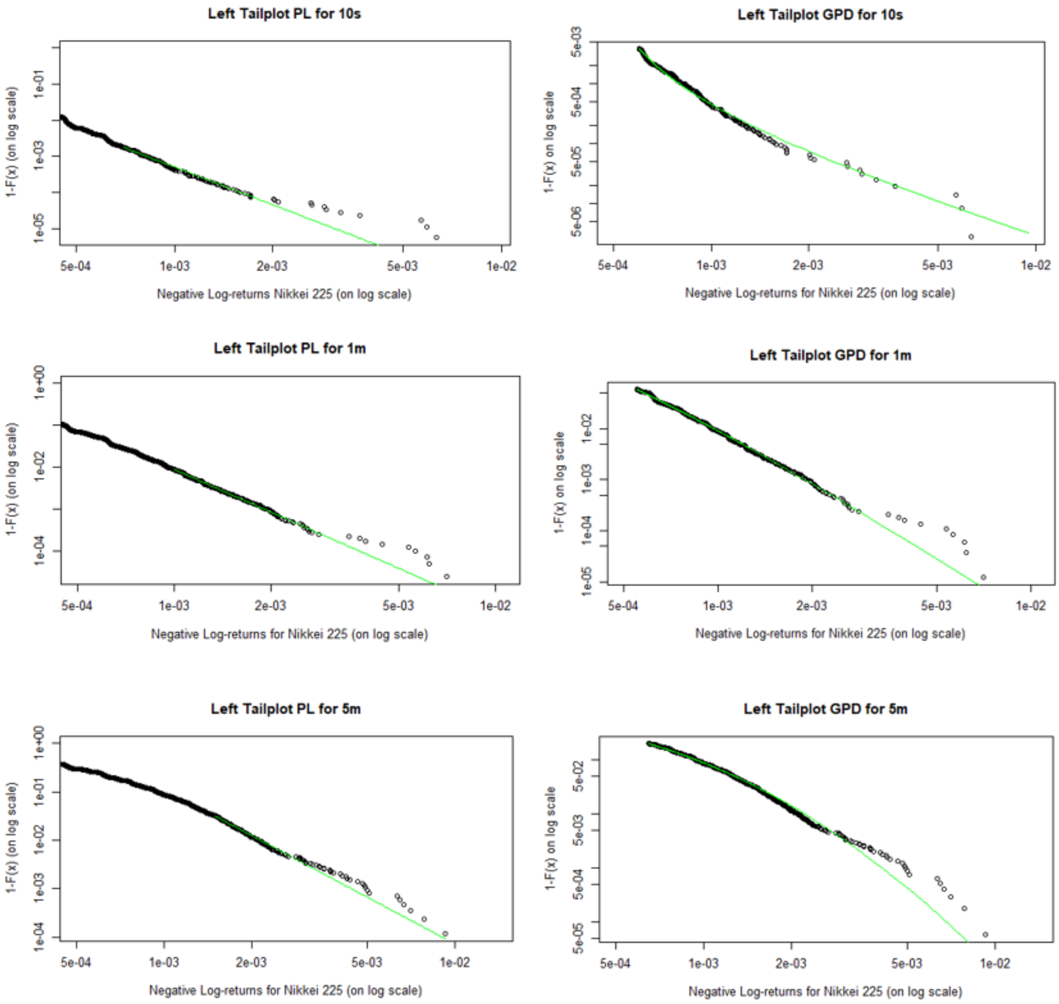


Figure B.20: Tail plots for the negative tails PL and GPD fit for 15-minute to 1-hour frequencies for Nikkei 225 log-returns

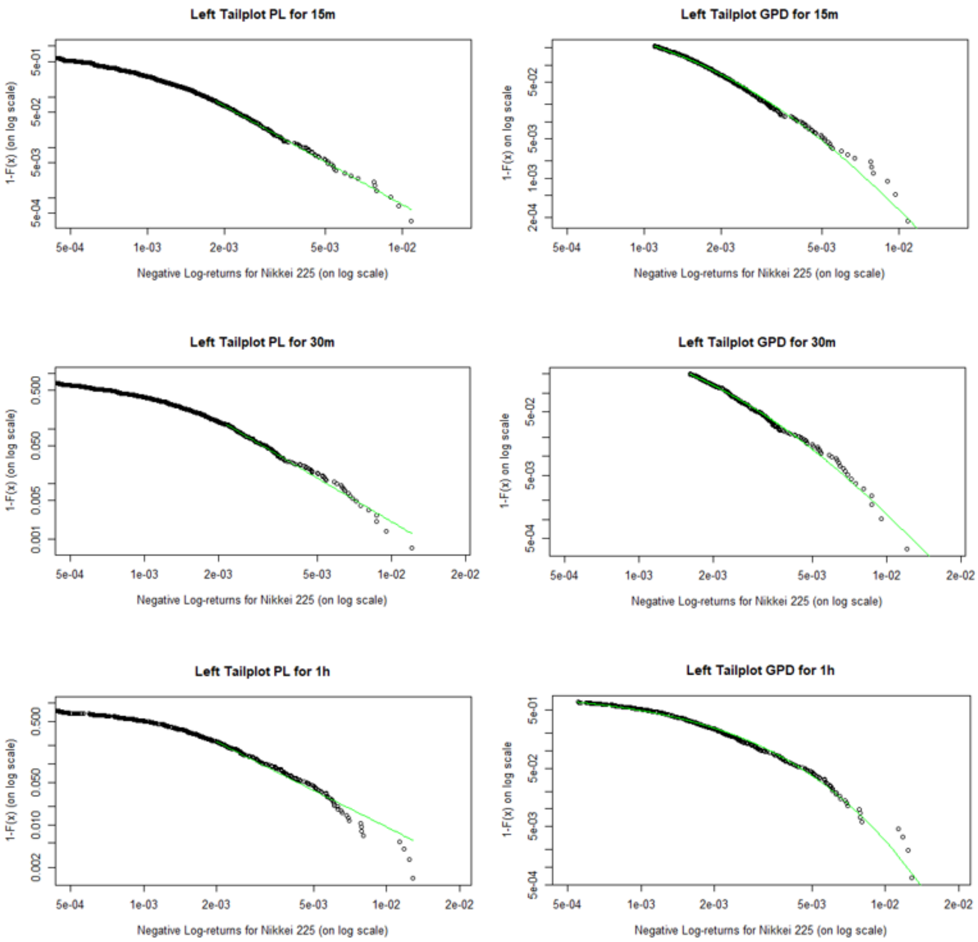


Figure B.21: Tail plots for the positive tails PL and GPD fit for 10-second to 5-minute frequencies for FTSE 100 log-returns

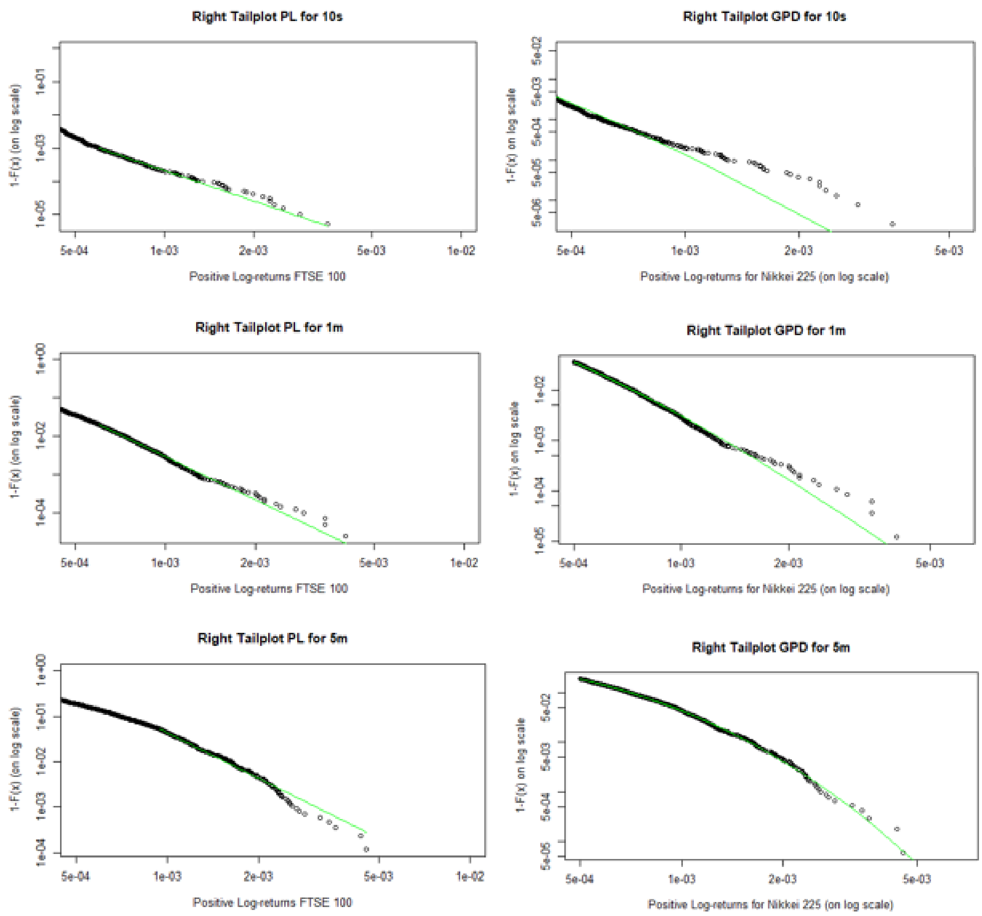


Figure B.22: Tail plots for the positive tails PL and GPD fit for 15-minute to 1-hour frequencies for FTSE 100 log-returns

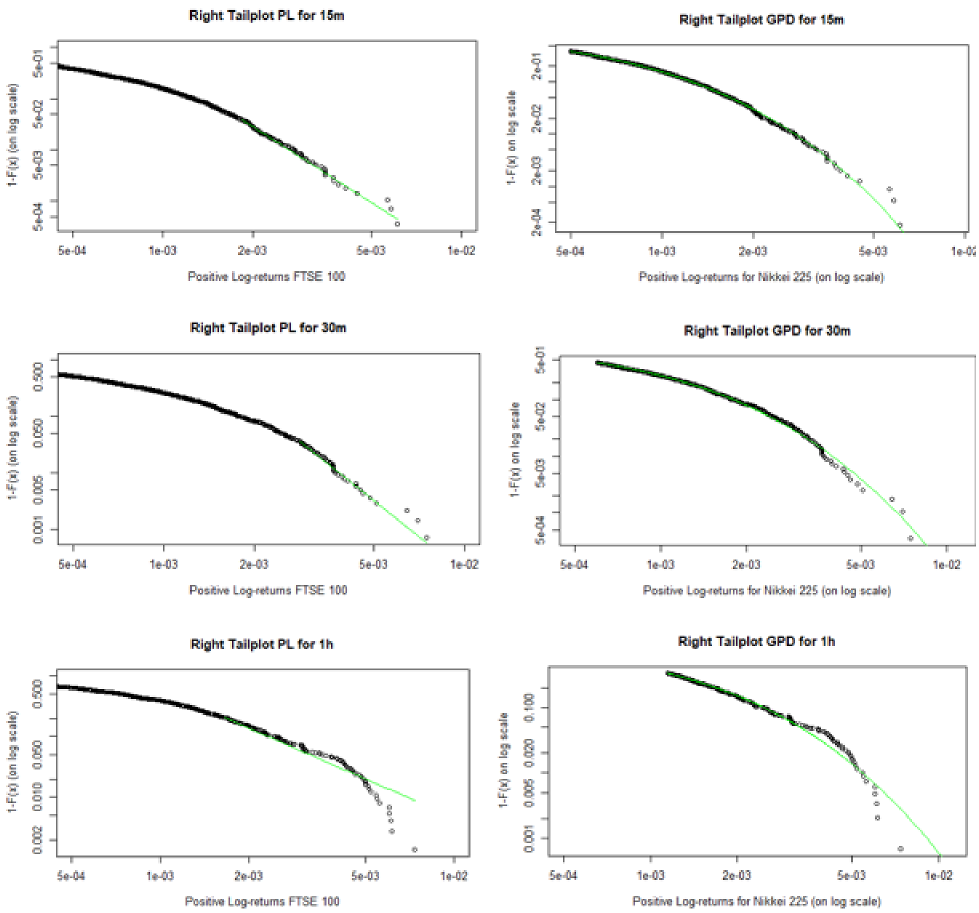


Figure B.23: Tail plots for the negative tails PL and GPD fit for 10-second to 5-minute frequencies for FTSE 100 log-returns

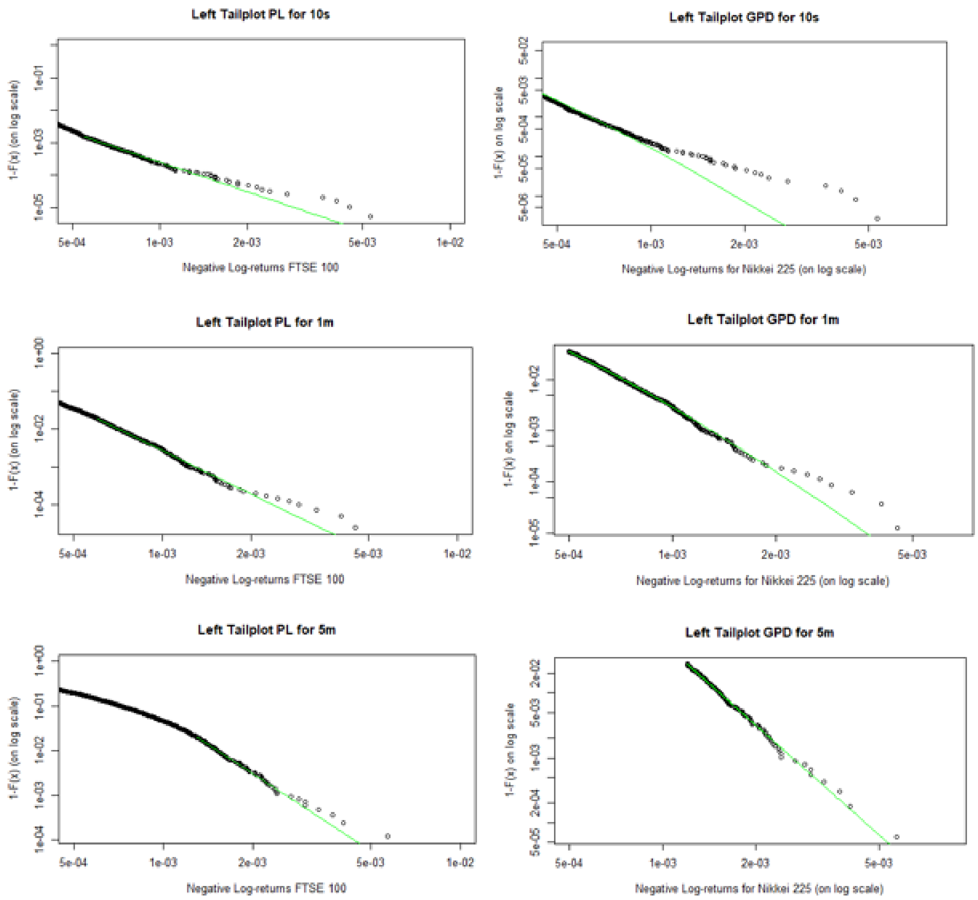


Figure B.24: Tail plots for the negative tails PL and GPD fit for 15-minute to 1-hour frequencies for FTSE 100 log-returns

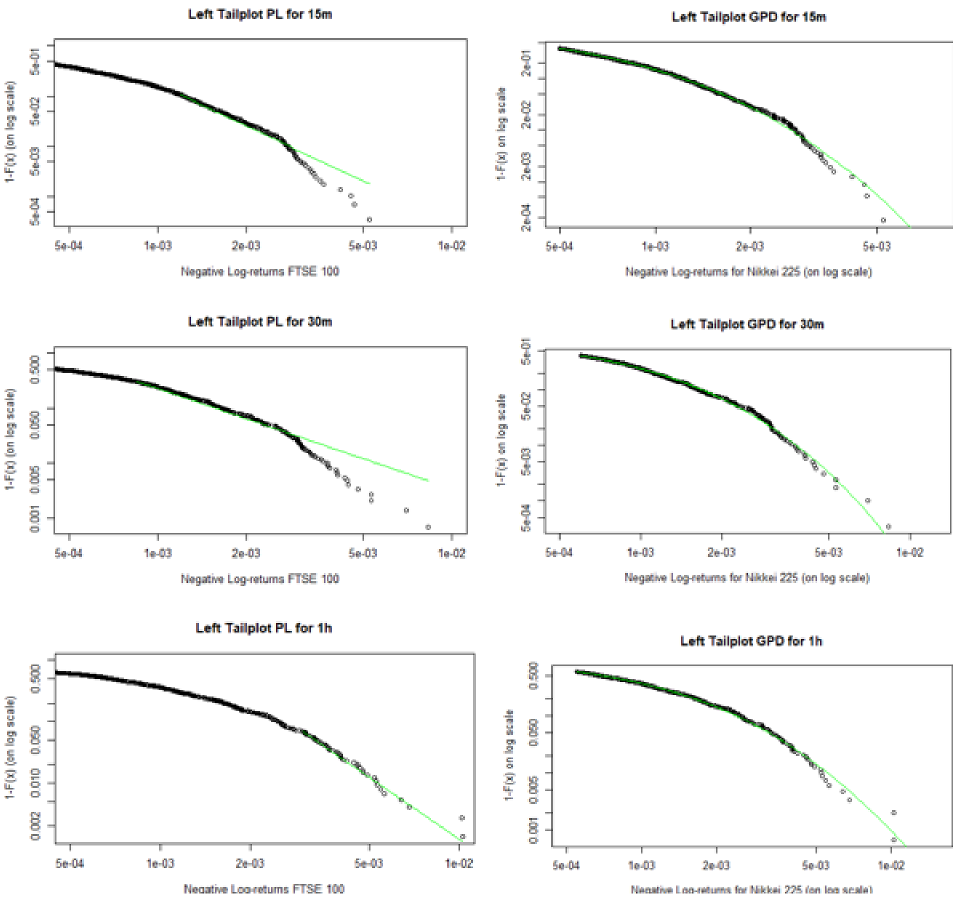


Figure B.25: Complementary distribution function plots for the negative tails PL and GPD fit for 10-second to 5-minute frequencies for Nikkei 225 log-returns

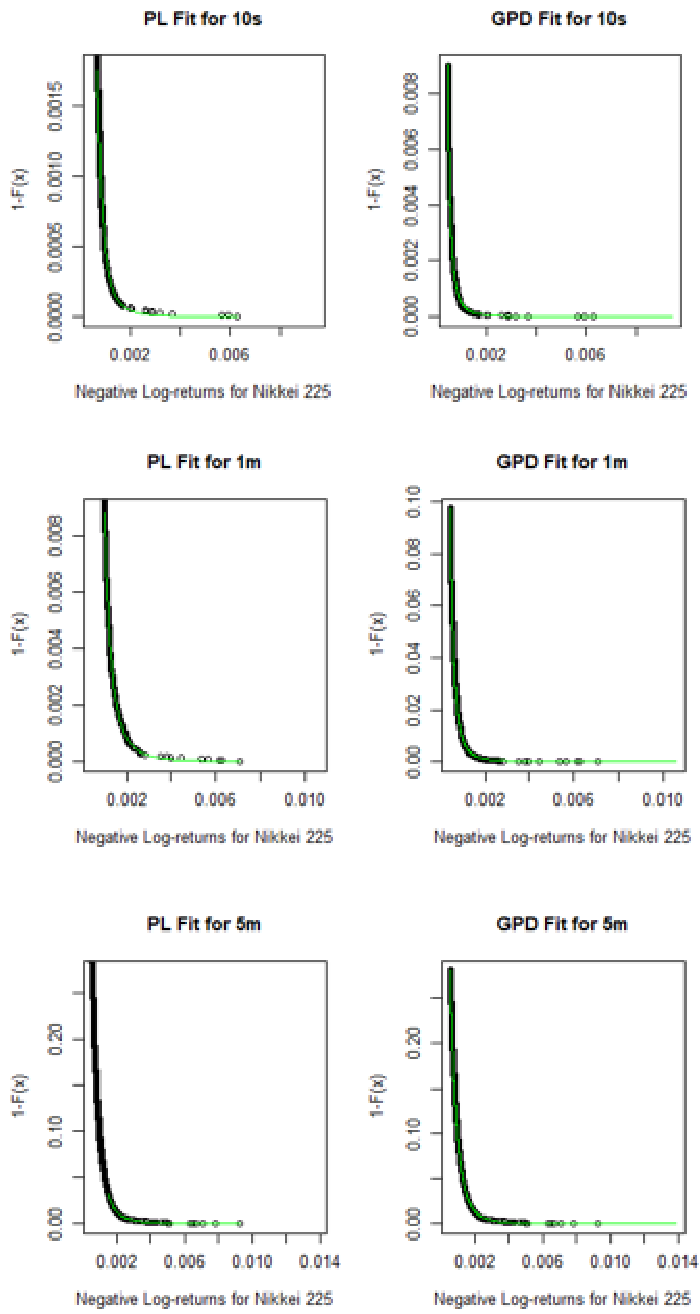


Figure B.26: Complementary distribution function plots for the negative tails PL and GPD fit for 15-minute to 1-hour frequencies for Nikkei 225 log-returns

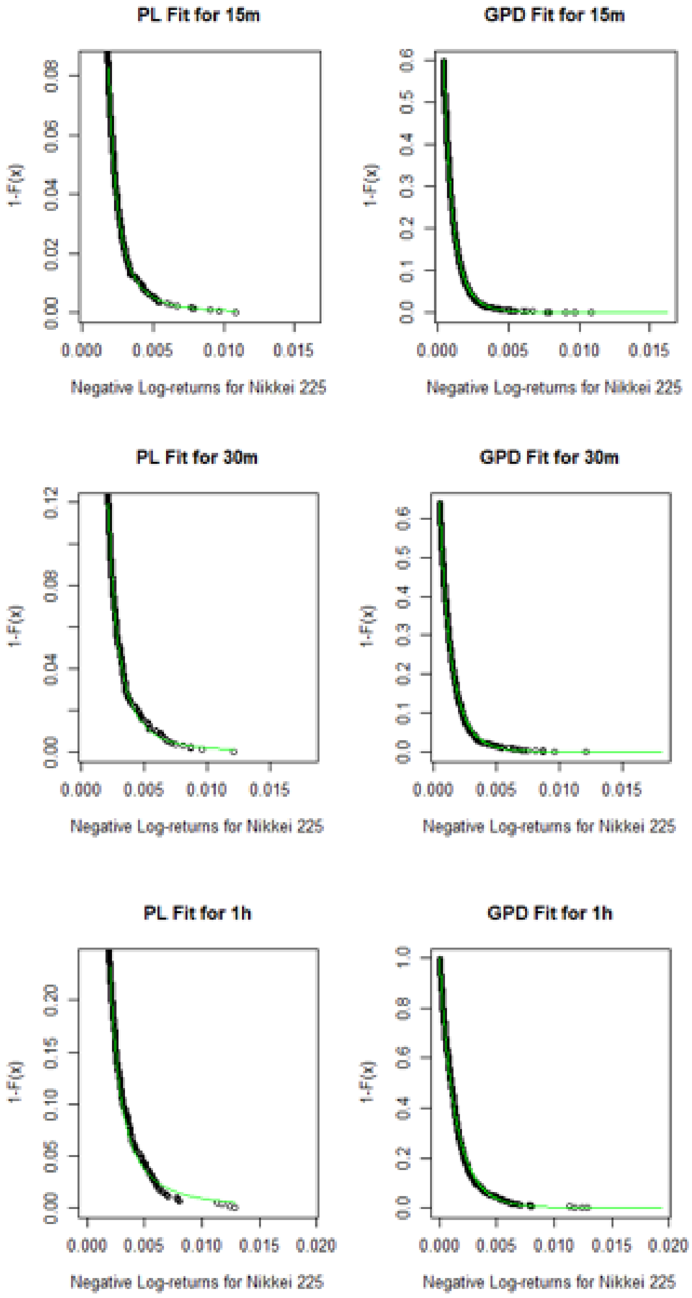


Figure B.27: Complementary distribution function plots for the positive tails PL and GPD fit for 10-second to 5-minute frequencies for Nikkei 225 log-returns

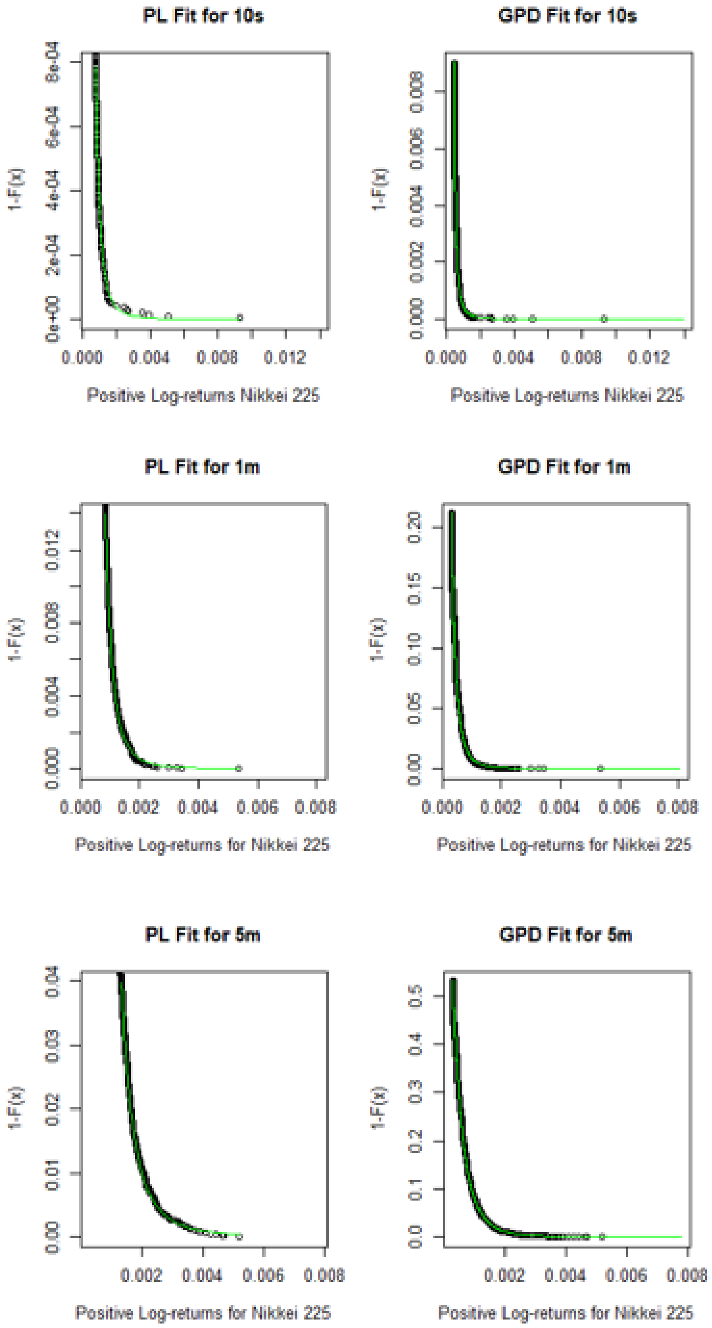


Figure B.28: Complementary distribution function plots for the positive tails PL and GPD fit for 15-minute to 1-hour frequencies for Nikkei 225 log-returns

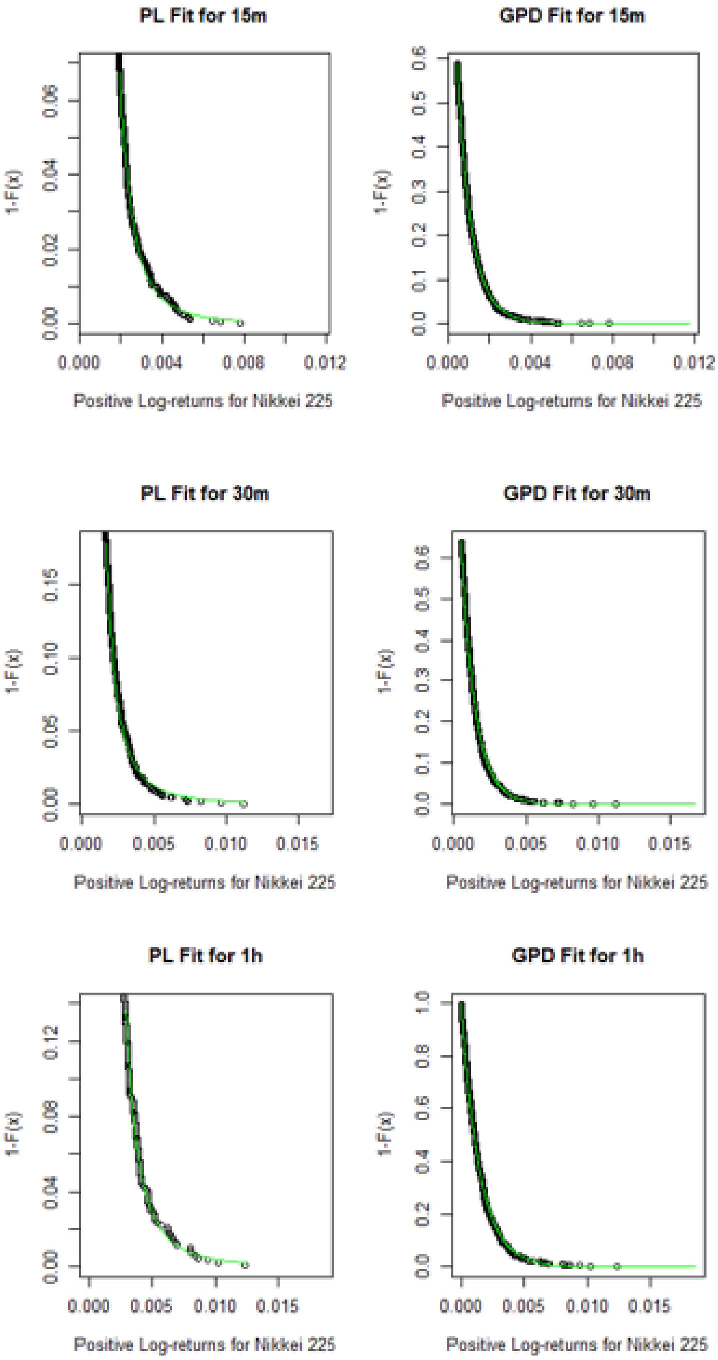


Figure B.29: Complementary distribution function plots for the negative tails PL and GPD fit for 10-second to 5-minute frequencies for FTSE 100 log-returns

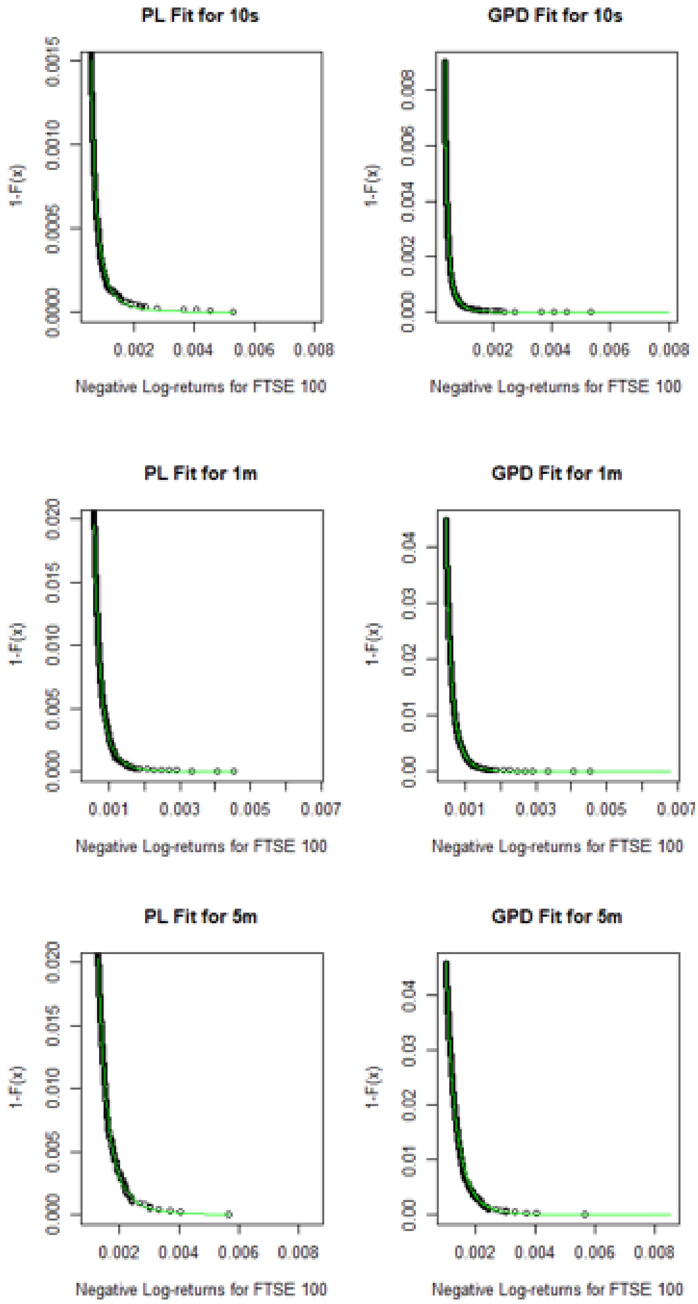


Figure B.30: Complementary distribution function plots for the negative tails PL and GPD fit for 15-minute to 1-hour frequencies for FTSE 100 log-returns

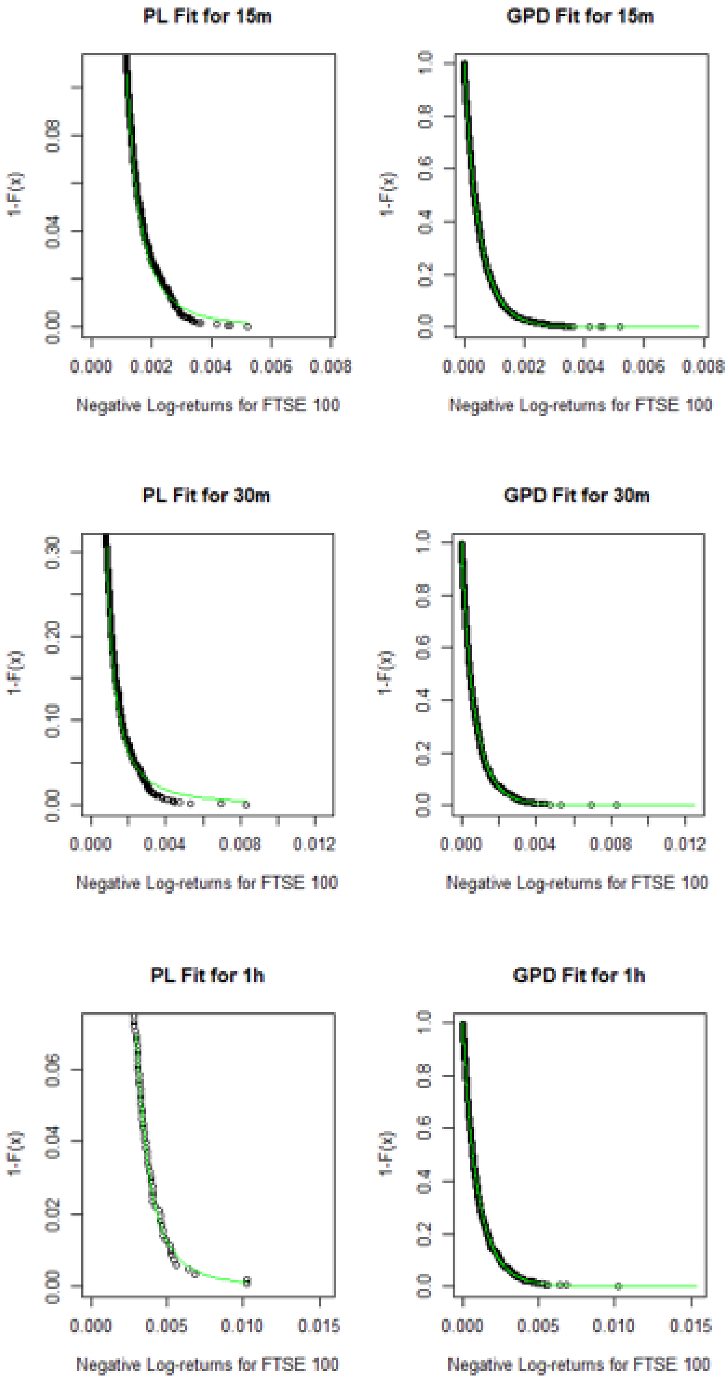


Figure B.31: Complementary distribution function plots for the positive tails PL and GPD fit for 10-second to 5-minute frequencies for FTSE 100 log-returns

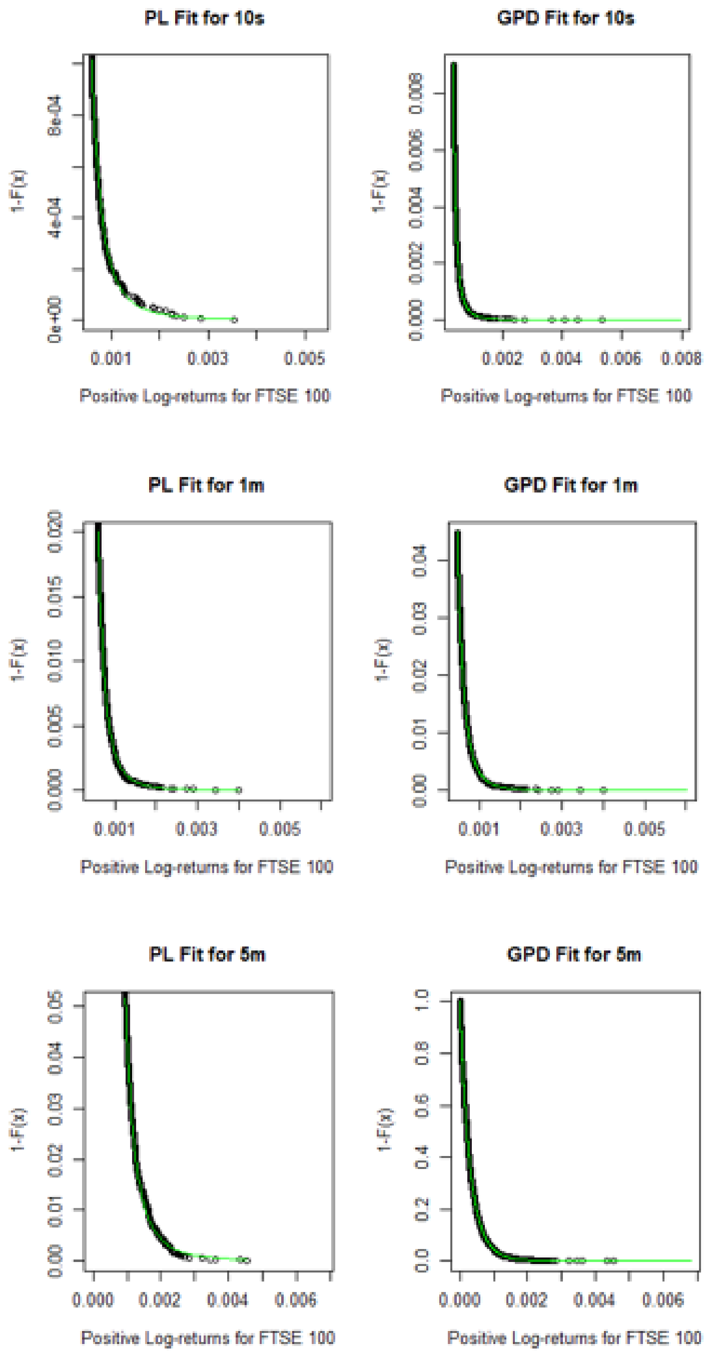
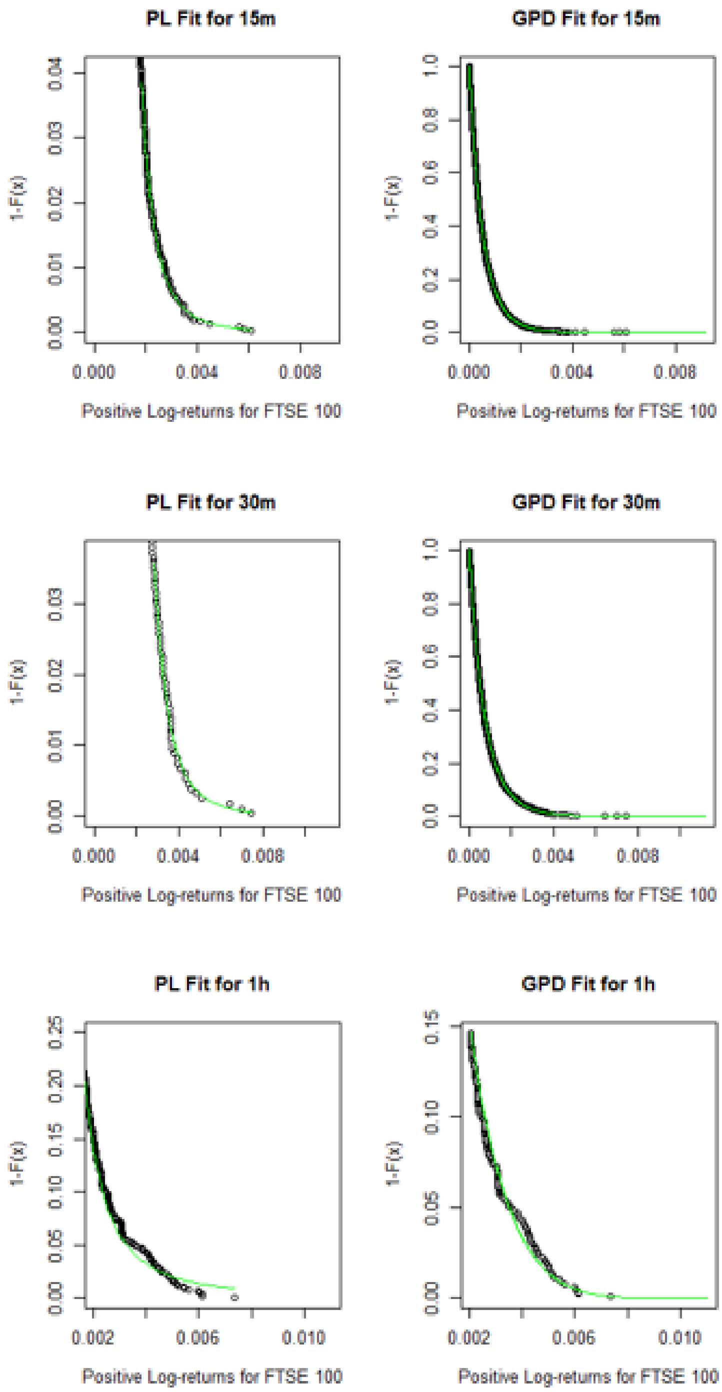


Figure B.32: Complementary distribution function plots for the positive tails PL and GPD fit for 15-minute to 1-hour frequencies for FTSE 100 log-returns



B.2 Scaling Laws in Returns

Figure B.33: Log-log plots for the standard deviation of absolute log-returns vs time intervals for Nikkei 225 (left) and FTSE 100 (right)

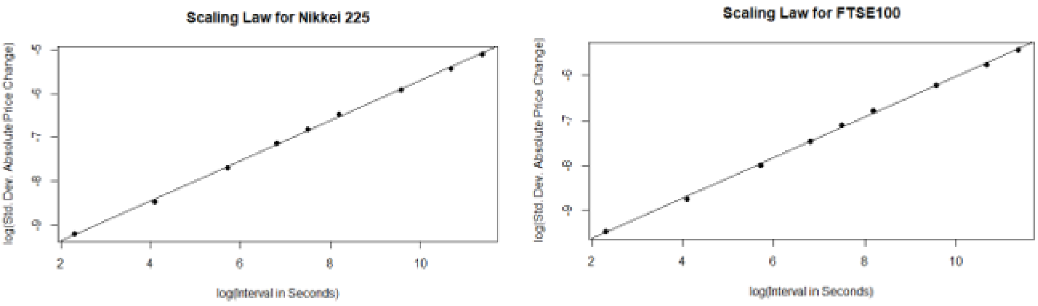


Figure B.34: Log-log plots for the IQR of absolute log-returns vs time intervals for Nikkei 225 (left) and FTSE 100 (right)

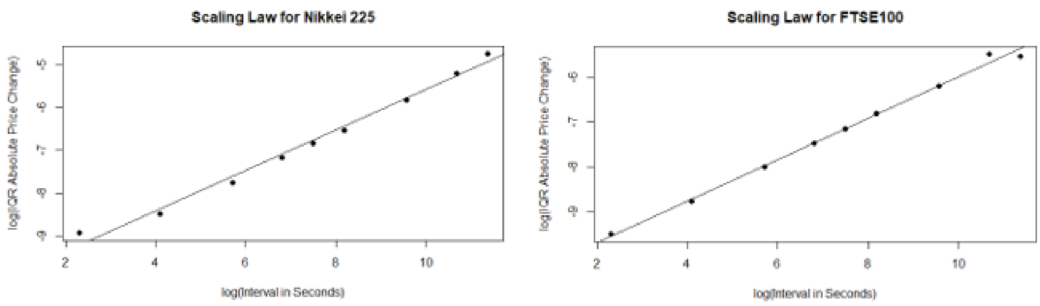
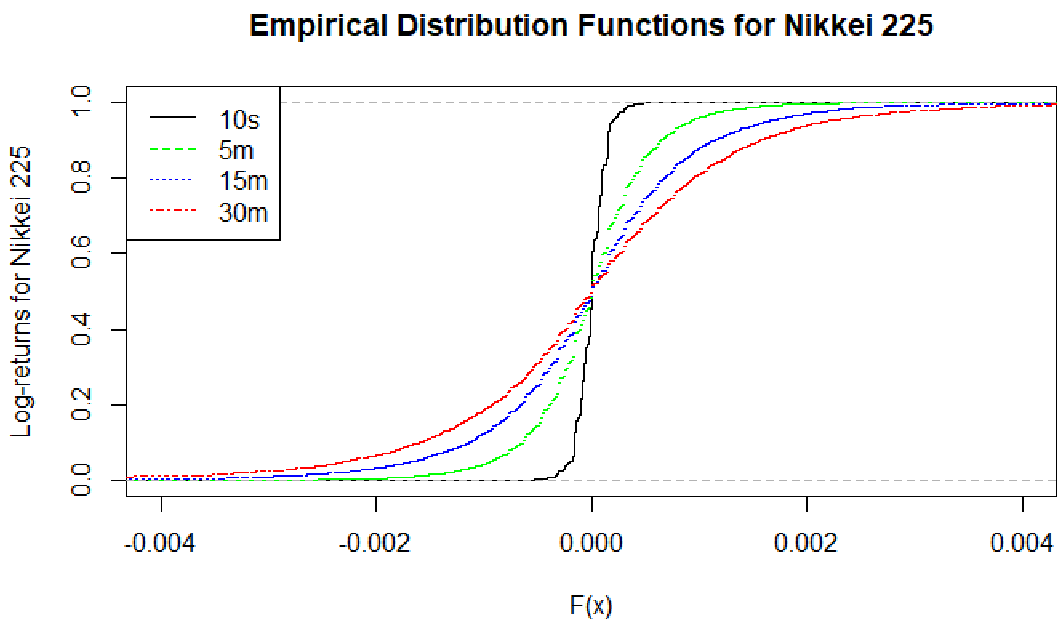


Figure B.35: Alternative cumulative distributions for some frequencies of Nikkei 225 and FTSE 100



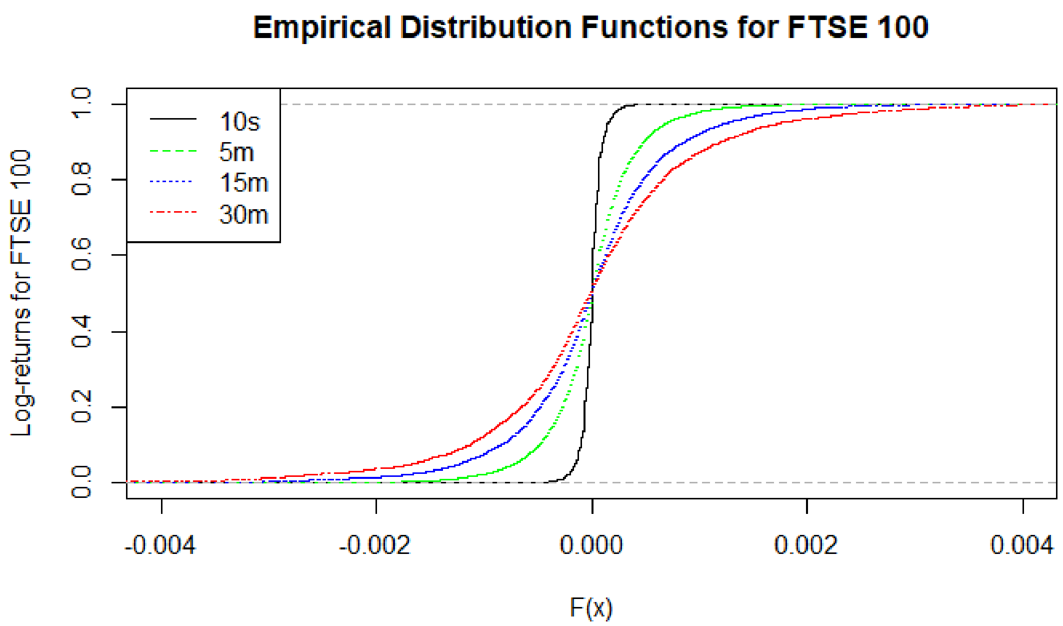


Table B.1: Standard deviation of absolute log-returns and time intervals considered for Nikkei 225 and FTSE 100

	10s	**1m**	**5m**	**15m**	**30m**
Nikkei 225	1.01×10^{-4}	2.10×10^{-4}	4.60×10^{-4}	8.03×10^{-4}	1.10×10^{-2}
FTSE 100	7.82×10^{-5}	1.60×10^{-4}	3.37×10^{-4}	5.74×10^{-4}	8.19×10^{-4}
Seconds	10	60	300	900	1800

	1h	**4h**	**12h**	**1d**
Nikkei 225	1.54×10^{-3}	2.70×10^{-3}	4.36×10^{-3}	6.11×10^{-3}
FTSE 100	1.14×10^{-3}	2.00×10^{-3}	3.15×10^{-3}	4.40×10^{-3}
Seconds	3600	14400	43200	86400

Table B.2: IQR of absolute log-returns and time intervals considered for Nikkei 225 and FTSE 100

	10s	**1m**	**5m**	**15m**	**30m**
Nikkei 225	1.33×10^{-4}	2.07×10^{-4}	4.30×10^{-4}	7.64×10^{-4}	1.07×10^{-3}
FTSE 100	7.49×10^{-5}	1.56×10^{-4}	3.33×10^{-4}	5.65×10^{-4}	7.85×10^{-4}
Seconds	10	60	300	900	1800

	1h	**4h**	**12h**	**1d**
Nikkei 225	1.45×10^{-3}	2.98×10^{-3}	5.52×10^{-3}	8.78×10^{-3}
FTSE 100	1.10×10^{-3}	2.05×10^{-3}	4.19×10^{-3}	3.9×10^{-3}
Seconds	3600	14400	43200	86400

C | Additional Results

C.1 Heavy-tailed Distributions

We mentioned earlier that the class of heavy-tailed distributions included important subclasses that, even if they are not entirely relevant for our subject, are useful for understanding the different types of distributions depending on its tails. Hence, we start by defining the related subclasses $\mathcal{L}$ (long-tailed distributions), $\mathcal{S}$ (subexponential distributions) and $\mathcal{D}$ (dominatedly varying distributions). We already defined heavy-tailed distributions and fat-tailed distributions in Chapter 3. As before, we get the definitions from Embrechts, Kluppelberg & Mickosch (2013).

Definition C.1.1. Let $\overline{F}(x) = 1 - F(x)$. The class of long-tailed distributions, denoted as $\mathcal{L}$, is defined as

$$\mathcal{L} \equiv \{F \text{ on } (0,\infty) : \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1, \quad \forall y > 0\} \tag{C.1}$$

Definition C.1.2. Let $\overline{F}(x) = 1 - F(x)$ and $\overline{F^{n*}}(x) = 1 - F^{n*}(x) = 1 - P(\sum_{i=1}^{n} X_i \leq x)$ for a sequence of random variables $X, X_1, ..., X_n$. The class of subexponential distributions, denoted as $\mathcal{S}$, is defined as

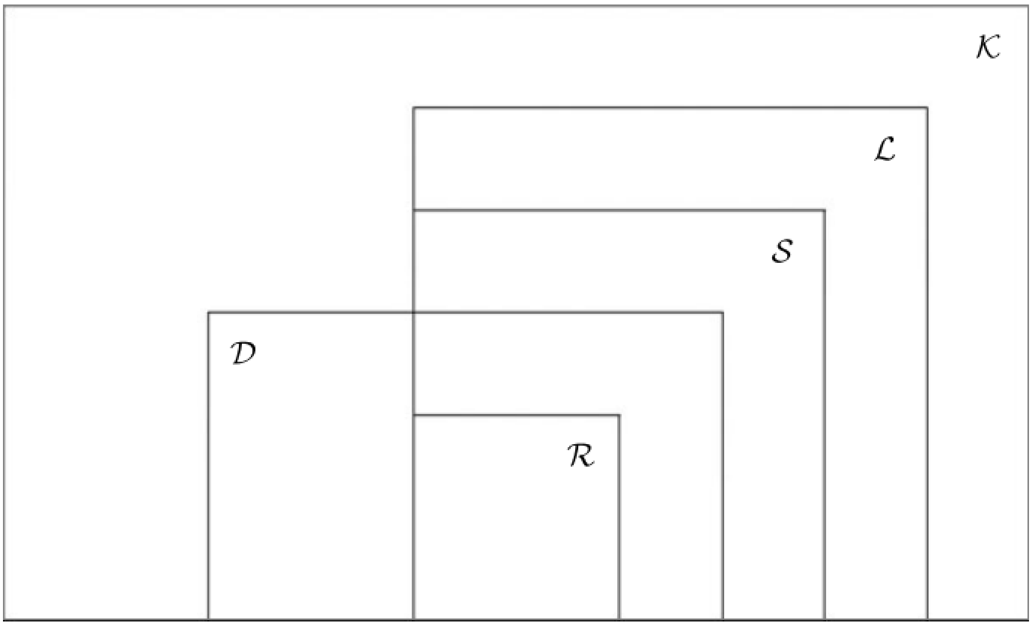
$$\mathcal{S} \equiv \{F \text{ on } (0,\infty) : \lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n, \quad \forall n \geq 2\} \tag{C.2}$$

Definition C.1.3. Let $\overline{F}(x) = 1 - F(x)$. The class of dominatedly varying distributions, denoted as $\mathcal{D}$, is defined as

$$\mathcal{D} \equiv \{F \text{ on } (0,\infty) : \lim_{x\to\infty} \frac{\overline{F}(x/2)}{\overline{F}(x)} < \infty\} \tag{C.3}$$

The following diagram, extracted from Embrechts, Kluppelberg & Mickosch (2013), perfectly illustrates the relationships between the different subclasses of heavy-

Figure C.1: Subclasses of heavy-tailed distributions (Embrechts et al., 2013)



tailed distributions.

In the finance literature, different academics and authors might use the name of the different classes to refer others, such as the $\mathcal{K} = \mathcal{R}$ or $\mathcal{S} = \mathcal{D}$, but we obey the definitions from the references because they give a clear framework to work with from a mathematical point of view.

From Figure C.1, we can see that the following relations hold:

$$\begin{aligned} &(a) \quad \mathcal{R} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{K} \\ &(b) \quad \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \\ &(c) \quad \mathcal{D} \not\subset \mathcal{S} \quad \text{and} \quad \mathcal{S} \not\subset \mathcal{D} \end{aligned} \tag{C.4}$$

An important result related to regular variation distributions or fat-tailed distributions $\mathcal{R}$ is that $F \in \mathcal{L} \ \Leftrightarrow \ \ln(\overline{F}) \in \mathcal{R}_0$. Hence, a distribution function pertains to the class of long-tailed distributions if and only if the logarithm of its tail is a slowly varying function.

C.2 GEV and GPD

In this section we layout some fundamental results of EVT related to the derivation of the GPD but that are not directly related to our goals in the thesis, just for an increased analytical understanding of the different results we use.

The most important theorem of EVT is the Fisher-Tippett extreme value theorem, which was shown in Chapter 3. The distributions of the theorem are called the (standard) extreme value distributions. One can generalize these distributions into one general distribution considering a parametric family of distributions $(H_\xi)_{\xi\in\mathbb{R}}$, known as the family of Generalized Extreme Value (GEV) distributions.

Definition C.2.1. The generalized extreme value (GEV) distribution, denoted by H_ξ, is defined as

$$H_\xi(x) = \begin{cases} \exp\{-(1+\xi x)^{1/\xi}\}, & \text{if } \xi \neq 0 \\ \exp\{-\exp\{-x\}\}, & \text{if } \xi = 0 \end{cases} \tag{C.5}$$

where $1 + \xi x > 0$, so that its support is

$$\begin{cases} x > \xi^{-1}, & \text{if } \xi > 0 \\ x < \xi^{-1}, & \text{if } \xi < 0 \\ x \in \mathbb{R}, & \text{if } \xi = 0 \end{cases} \tag{C.6}$$

This distribution includes the Fréchet for $\xi = \alpha^{-1} > 0$, the Gumbel for $\xi = 0$ and the Weibull for $\xi = -\alpha^{-1} < 0$, so that it compacts the extreme value distributions into one single distribution. One can introduce a related location-scale family $H_{\xi;\mu,\beta}$ (also referred as GEV distributions) by replacing x by $(x-\mu)/\beta$ for $\mu \in \mathbb{R}$ and $\beta > 0$, with the support being adjusted accordingly.

Using the notion of maximum domains of attraction, we can prove that the GEV distribution leads naturally to the GPD for values over high thresholds. The definitions are extracted from the same reference as before.

Definition C.2.2. A random variable X or its distribution function F belongs to the maximum domain of attraction of the extreme value distribution H if there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{D} H \tag{C.7}$$

holds, where M_n are normalized maxima. We write $X \in MDA(H)$ or $F \in MDA(H)$ to denote this belonging.

Because the extreme value distributions are continuous on $\mathbb{R}$, the condition above is equivalent to

$$\lim_{x\to\infty} P(M_n \leq c_n x + d_n) = \lim_{x\to\infty} F^n(c_n x + d_n) = H(x), \quad x \in \mathbb{R} \tag{C.8}$$

which is based on the distribution function of the maximum.

An important result that allows characterizing $MDA(H)$ is the following:

Theorem C.2.3. The distribution function F belongs to $MDA(H)$ with morning constants $c_n > 0$ and $d_n \in \mathbb{R}$ if and only if

$$\lim_{x\to\infty} n\overline{F}(c_n x + d_n) = -\ln[H(x)], \quad x \in \mathbb{R} \tag{C.9}$$

When $H(x) = 0$, then the limit is interpreted as ∞.

Through this theorem, we can now characterize the domain of attraction of the Fréchet, which englobes PL and regularly varying distributions.

Theorem C.2.4. The distribution function F belongs to the maximum domain of attraction of Φ_α (the Fréchet distribution) with $\alpha > 0$, if and only if $\overline{F}(x) = x^{-\alpha}L(x)$ for some slowly varying function L. If $F \in MDA(\Phi_\alpha)$, then

$$c_n^{-1}M_n \xrightarrow{D} \Phi_\alpha \tag{C.10}$$

where $c_n = \inf\{x \in \mathbb{R} : 1/\overline{F}(x) \geq n\}$.

This result implies that $F \in MDA(\Phi_\alpha) \Leftrightarrow \overline{F} \in \mathcal{R}_\alpha$. Therefore, any distribution function with regularly varying tails, such as the PL distribution or regularly varying distributions, pertain to $MDA(\Phi_\alpha)$. Moreover, if a variable follows an α-stable distribution, then it must also pertain to $MDA(\Phi_\alpha)$, as it has regularly varying tails. Therefore, PLs, EVT and the Paretian stable hypothesis of Mandelbrot can be related through the notion of regularly varying tails and the maximum domain of attraction of the Fréchet distribution.

Theorem C.2.4 and results related to the characterization of other maximum domains of attraction (for the Gumbel and the Weibull) help proving the following result, which relates $MDA(H_\xi)$ with the GPD. An sketch of the proof using these results appears in Embrechts, Kluppelberg & Mickosch (2013), but we just state the theorem without proof for brevity.

Theorem C.2.5. For $\xi \in \mathbb{R}$, the following assertions are equivalent:

(a) $F \in MDA(H_\xi)$

(b) There exist a positive, measurable function $a()$ such that for $1 + \xi x > 0$ and denoting the right endpoint of the distribution F as x_F,

$$\lim_{u \to x_F} \frac{\overline{F}(u + xa(u))}{\overline{F}(u)} = \begin{cases} (1+\xi x)^{-1/\xi}, & \text{for } \xi \neq 0 \\ \exp\{-x\}, & \text{for } \xi = 0 \end{cases} \tag{C.11}$$

Part (b) of Theorem C.2.5 relates the limit with a distribution which we defined as the GPD in Chapter 3. Like the GPD, this result can be interpreted as the following limit when $x > 0$:

$$\lim_{u \to x_F} P\left(\frac{X-u}{a(u)} > x | X > u\right) = \begin{cases} (1+\xi x)^{-1/\xi}, & \text{for } \xi \neq 0 \\ \exp\{-x\}, & \text{for } \xi = 0 \end{cases} \tag{C.12}$$

Hence, we have shown the different results which motivates the definition of the GPD from the GEV distribution through the usage of the notion of the maximum domain of attraction.

D | R Code

The R code is available at: https://github.com/ikercb2000/TFM_UPC.git