

Appendix A

Summary of Basic Formulae

Summary

Two sets of tables are provided for reference. The first records the definition, and the first two moments of the most common probability distributions used in this volume. The second records the basic elements of standard Bayesian inference processes for a number of special cases. In particular, it records the appropriate likelihood function, the sufficient statistics, the conjugate prior and corresponding posterior and predictive distributions, the reference prior and corresponding reference posterior and predictive distributions.

A.1 PROBABILITY DISTRIBUTIONS

The first section of this Appendix consists of a set of tables which record the notation, parameter range, variable range, definition, and first two moments of the probability distributions (discrete and continuous, univariate and multivariate) used in this volume.

*Univariate Discrete Distributions***Br**($x \mid \theta$) *Bernoulli* (p. 115)

$$\begin{array}{ll}
0 < \theta < 1 & x = 0, 1 \\
p(x) = \theta^x (1 - \theta)^{1-x} & \\
E[x] = \theta & V[x] = \theta(1 - \theta)
\end{array}$$

Bi($x \mid \theta, n$) *Binomial* (p. 115)

$$\begin{array}{ll}
0 < \theta < 1, n = 1, 2, \dots & x = 0, 1, \dots, n \\
p(x) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} & \\
E[x] = n\theta & V[x] = n\theta(1 - \theta)
\end{array}$$

Bb($x \mid \alpha, \beta, n$) *Binomial-Beta* (p. 117)

$$\begin{array}{ll}
\alpha > 0, \beta > 0, n = 1, 2, \dots & x = 0, 1, \dots, n \\
p(x) = c \binom{n}{x} \Gamma(\alpha + x) \Gamma(\beta + n - x) & c = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + \beta + n)} \\
E[x] = n \frac{\alpha}{\alpha + \beta} & V[x] = \frac{n\alpha\beta}{(\alpha + \beta)^2} \frac{(\alpha + \beta + n)}{(\alpha + \beta + 1)}
\end{array}$$

Hy($x \mid N, M, n$) *Hypergeometric* (p. 115)

$$\begin{array}{ll}
N = 1, 2, \dots & x = a, a + 1, \dots, b \\
M = 1, 2, \dots & a = \max(0, n - M) \\
n = 1, \dots, N + M & b = \min(n, N) \\
p(x) = c \binom{N}{x} \binom{M}{n-x} & c = \binom{N+M}{n}^{-1} \\
E[x] = n \frac{N}{N+M} & V[x] = \frac{nNM}{(N+M)^2} \frac{N+M-n}{N+M-1}
\end{array}$$

Univariate Discrete Distributions (continued)

Nb($x \mid \theta, r$) *Negative-Binomial* (p. 116)

$$0 < \theta < 1, r = 1, 2, \dots \qquad x = 0, 1, 2, \dots$$

$$p(x) = c \binom{r+x-1}{r-1} (1-\theta)^x \qquad c = \theta^r$$

$$E[x] = r\theta \qquad V[x] = r \frac{1-\theta}{\theta^2}$$

Nbb($x \mid \alpha, \beta, r$) *Negative-Binomial-Beta* (p. 118)

$$\alpha > 0, \beta > 0, r = 1, 2, \dots \qquad x = 0, 1, 2, \dots$$

$$p(x) = c \binom{r+x-1}{r-1} \frac{\Gamma(\beta+x)}{\Gamma(\alpha+\beta+r+x)} \qquad c = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+r)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$E[x] = \frac{r\beta}{\alpha-1} \qquad V[x] = \frac{r\beta}{(\alpha-1)} \left[\frac{\alpha+\beta+r-1}{(\alpha-2)} + \frac{r\beta}{(\alpha-1)(\alpha-2)} \right]$$

Pn($x \mid \lambda$) *Poisson* (p. 116)

$$\lambda > 0 \qquad x = 0, 1, 2, \dots$$

$$p(x) = c \frac{\lambda^x}{x!} \qquad c = e^{-\lambda}$$

$$E[x] = \lambda \qquad V[x] = \lambda$$

Pg($x \mid \alpha, \beta, \nu$) *Poisson-Gamma* (p. 119)

$$\alpha > 0, \beta > 0, \nu > 0 \qquad x = 0, 1, 2, \dots$$

$$p(x) = c \frac{\Gamma(\alpha+x)}{x!} \frac{\nu^x}{(\beta+\nu)^{\alpha+x}} \qquad c = \frac{\beta^\alpha}{\Gamma(\alpha)}$$

$$E[x] = \nu \frac{\alpha}{\beta} \qquad V[x] = \frac{\nu\alpha}{\beta} \left[1 + \frac{\nu}{\beta} \right]$$

*Univariate Continuous Distributions***Be**($x \mid \alpha, \beta$) *Beta* (p. 116)

$\alpha > 0, \beta > 0$

$0 < x < 1$

$p(x) = c x^{\alpha-1} (1-x)^{\beta-1}$

$c = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}$

$E[x] = \frac{\alpha}{\alpha + \beta}$

$V[x] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$

Un($x \mid a, b$) *Uniform* (p. 117)

$b > a$

$a < x < b$

$p(x) = c$

$c = (b - a)^{-1}$

$E[x] = \frac{1}{2}(a + b)$

$V[x] = \frac{1}{12}(b - a)^2$

Ga($x \mid \alpha, \beta$) *Gamma* (p. 118)

$\alpha > 0, \beta > 0$

$x > 0$

$p(x) = c x^{\alpha-1} e^{-\beta x}$

$c = \frac{\beta^\alpha}{\Gamma(\alpha)}$

$E[x] = \alpha\beta^{-1}$

$V[x] = \alpha\beta^{-2}$

Ex($x \mid \theta$) *Exponential* (p. 118)

$\theta > 0$

$x > 0$

$p(x) = c e^{-\theta x}$

$c = \theta$

$E[x] = 1/\theta$

$V[x] = 1/\theta^2$

Gg($x \mid \alpha, \beta, n$) *Gamma-Gamma* (p. 120)

$\alpha > 0, \beta > 0, n > 0$

$x > 0$

$p(x) = c \frac{x^{n-1}}{(\beta + x)^{\alpha+n}}$

$c = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + n)}{\Gamma(n)}$

$E[x] = n \frac{\beta}{\alpha - 1}$

$V[x] = \frac{\beta^2(n^2 + n(\alpha - 1))}{(\alpha - 1)^2(\alpha - 2)}$

Univariate Continuous Distributions (continued)

 $\chi^2(x \mid \nu) = \chi_\nu^2$ *Chi-squared (p. 120)*

$\nu > 0$	$x > 0$
$p(x) = c x^{(\nu/2)-1} e^{-x/2}$	$c = \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)}$
$E[x] = \nu$	$V[x] = 2\nu$

 $\chi^2(x \mid \nu, \lambda)$ *Non-central Chi-squared (p. 121)*

$\nu > 0, \lambda > 0$	$x > 0$
$p(x) = \sum_{i=0}^{\infty} \text{Pn} \left(i \mid \frac{\lambda}{2} \right) \chi^2(x \mid \nu + 2i)$	
$E[x] = \nu + \lambda$	$V[x] = 2(\nu + 2\lambda)$

 $\text{Ig}(x \mid \alpha, \beta)$ *Inverted-Gamma (p. 119)*

$\alpha > 0, \beta > 0$	$x > 0$
$p(x) = c x^{-(\alpha+1)} e^{-\beta/x}$	$c = \frac{\beta^\alpha}{\Gamma(\alpha)}$
$E[x] = \frac{\beta}{\alpha - 1}$	$V[x] = \frac{\beta^2}{(\alpha - 1)^2(\alpha - 2)}$

 $\chi^{-1}(x \mid \nu)$ *Inverted-Chi-squared (p. 119)*

$\nu > 0$	$x > 0$
$p(x) = c x^{-(\nu/2+1)} e^{-1/2x^2}$	$c = \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)}$
$E[x] = \frac{1}{\nu - 2}$	$V[x] = \frac{2}{(\nu - 2)^2(\nu - 4)}$

 $\text{Ga}^{-1/2}(x \mid \alpha, \beta)$ *Square-root Inverted-Gamma (p. 119)*

$\alpha > 0, \beta > 0$	$x > 0$
$p(x) = c x^{-(2\alpha+1)} e^{-\beta/x^2}$	$c = \frac{2\beta^\alpha}{\Gamma(\alpha)}$
$E[x] = \frac{\sqrt{\beta}\Gamma(\alpha - 1/2)}{\Gamma(\alpha)}$	$V[x] = \frac{\beta}{\alpha - 1} - E[x]^2$

*Univariate Continuous Distributions (continued)***Pa**($x \mid \alpha, \beta$) *Pareto* (p. 120)

$$\begin{array}{ll}
\alpha > 0, \beta > 0 & \beta \leq x < +\infty \\
p(x) = c x^{-(\alpha+1)} & c = \alpha\beta^\alpha \\
E[x] = \frac{\beta\alpha}{\alpha-1}, \quad \alpha > 1 & V[x] = \frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2
\end{array}$$

Ip($x \mid \alpha, \beta$) *Inverted-Pareto* (p. 120)

$$\begin{array}{ll}
\alpha > 0, \quad \beta > 0 & 0 < x < \beta^{-1} \\
p(x) = c x^{\alpha-1} & c = \alpha\beta^\alpha \\
E[x] = \beta^{-1}\alpha(\alpha+1)^{-1} & V[x] = \beta^{-2}\alpha(\alpha+1)^{-2}(\alpha+2)^{-1}
\end{array}$$

N($x \mid \mu, \lambda$) *Normal* (p. 121)

$$\begin{array}{ll}
-\infty < \mu < +\infty, \lambda > 0 & -\infty < x < +\infty \\
p(x) = c \exp\left\{-\frac{1}{2}\lambda(x-\mu)^2\right\} & c = \lambda^{1/2}(2\pi)^{-1/2} \\
E[x] = \mu & V[x] = \lambda^{-1}
\end{array}$$

St($x \mid \mu, \lambda, \alpha$) *Student t* (p. 122)

$$\begin{array}{ll}
-\infty < \mu < +\infty, \lambda > 0, \alpha > 0 & -\infty < x < +\infty \\
p(x) = c [1 + \alpha^{-1}\lambda(x-\mu)^2]^{-(\alpha+1)/2} & c = \frac{\Gamma(\frac{1}{2}(\alpha+1))}{\Gamma(\frac{1}{2}\alpha)} \left(\frac{\lambda}{\alpha\pi}\right)^{1/2} \\
E[x] = \mu & V[x] = \lambda^{-1}\alpha(\alpha-2)^{-1}
\end{array}$$

F($x \mid \alpha, \beta$) = **F** _{α, β} *Snedecor F* (p. 123)

$$\begin{array}{ll}
\alpha > 0, \beta > 0 & x > 0 \\
p(x) = c \frac{x^{\alpha/2-1}}{(\beta + \alpha x)^{(\alpha+\beta)/2}} & c = \frac{\Gamma(\frac{1}{2}(\alpha+\beta)) \alpha^{\alpha/2} \beta^{\beta/2}}{\Gamma(\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)} \\
E[x] = \frac{\beta}{\beta-2}, \quad \beta > 2 & V[x] = \frac{2\beta^2(\alpha+\beta-2)}{\alpha(\beta-2)^2(\beta-4)}, \quad \beta > 4
\end{array}$$

Univariate Continuous Distributions (continued)

Lo($x \mid \alpha, \beta$) *Logistic* (p. 122)

$$-\infty < \alpha < +\infty, \beta > 0 \qquad -\infty < x < +\infty$$

$$p(x) = \beta^{-1} \exp \{ -\beta^{-1}(x - \alpha) \} [1 + \exp \{ -\beta^{-1}(x - \alpha) \}]^{-2}$$

$$E[x] = \alpha \qquad V[x] = \beta^2 \pi^2 / 3$$

Multivariate Discrete Distributions

Mu_k($\mathbf{x} \mid \boldsymbol{\theta}, n$) *Multinomial* (p. 133)

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \qquad \mathbf{x} = (x_1, \dots, x_k)$$

$$0 < \theta_i < 1, \quad \sum_{\ell=1}^k \theta_\ell \leq 1 \qquad \sum_{\ell=1}^k x_\ell \leq n$$

$$n = 1, 2, \dots \qquad x_i = 0, 1, 2, \dots$$

$$p(\mathbf{x}) = \frac{n!}{\prod_{\ell=1}^{k+1} x_\ell!} \prod_{\ell=1}^k \theta_\ell^{x_\ell}, \quad \theta_{k+1} = 1 - \sum_{\ell=1}^k \theta_\ell, \quad x_{k+1} = n - \sum_{\ell=1}^k x_\ell$$

$$E[x_i] = n\theta_i \qquad V[x_i] = n\theta_i(1 - \theta_i) \qquad C[x_i, x_j] = -n\theta_i\theta_j$$

Md_k($\mathbf{x} \mid \boldsymbol{\theta}, n$) *Multinomial-Dirichlet* (p. 135)

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k+1}) \qquad \mathbf{x} = (x_1, \dots, x_k)$$

$$\alpha_i > 0 \qquad x_i = 0, 1, 2, \dots$$

$$n = 1, 2, \dots \qquad \sum_{\ell=1}^n x_\ell \leq n$$

$$p(\mathbf{x}) = c \prod_{\ell=1}^{k+1} \frac{\alpha_\ell^{x_\ell}}{x_\ell!} \qquad c = \frac{n!}{\left(\sum_{\ell=1}^{k+1} \alpha_\ell \right)^{[n]}}$$

$$\alpha^{[s]} = \prod_{\ell=1}^s (\alpha + \ell - 1) \qquad x_{k+1} = n - \sum_{\ell=1}^k x_\ell$$

$$E[x_i] = np_i \qquad V[x_i] = \frac{n + \sum_{\ell=1}^{k+1} \alpha_\ell}{1 + \sum_{\ell=1}^{k+1} \alpha_\ell} np_i(1 - p_i)$$

$$p_i = \frac{\alpha_i}{\sum_{\ell=1}^{k+1} \alpha_\ell} \qquad C[x_i, x_j] = -\frac{n + \sum_{\ell=1}^{k+1} \alpha_\ell}{1 + \sum_{\ell=1}^{k+1} \alpha_\ell} np_i p_j$$

*Multivariate Continuous Distributions***Di_k($\mathbf{x} \mid \boldsymbol{\alpha}$)** *Dirichlet (p. 134)*

$$\begin{aligned}
\boldsymbol{\alpha} &= (\alpha_1, \dots, \alpha_{k+1}) & \mathbf{x} &= (x_1, \dots, x_k) \\
\alpha_i &> 0 & 0 < x_i < 1, \quad \sum_{\ell=1}^k x_\ell &\leq 1 \\
p(\mathbf{x}) &= c \left(1 - \sum_{\ell=1}^k x_\ell\right)^{\alpha_{k+1}-1} \prod_{\ell=1}^k x_\ell^{\alpha_\ell-1} & c &= \frac{\Gamma(\sum_{\ell=1}^{k+1} \alpha_\ell)}{\prod_{\ell=1}^{k+1} \Gamma(\alpha_\ell)} \\
E[x_i] &= \frac{\alpha_i}{\sum_{\ell=1}^{k+1} \alpha_\ell} & V[x_i] &= \frac{E[x_i](1 - E[x_i])}{1 + \sum_{\ell=1}^{k+1} \alpha_\ell} & C[x_i, x_j] &= \frac{-E[x_i]E[x_j]}{1 + \sum_{\ell=1}^{k+1} \alpha_\ell}
\end{aligned}$$

Ng($x, y \mid \mu, \lambda, \alpha, \beta$) *Normal-Gamma (p. 136)*

$$\begin{aligned}
\mu &\in \Re, \lambda > 0, \alpha > 0, \beta > 0, & x &\in \Re, y > 0 \\
p(x, y) &= N(x \mid \mu, \lambda y) \text{Ga}(y \mid \alpha, \beta) \\
E[x] &= \mu & E[y] &= \alpha\beta^{-1} & V[x] &= \beta\lambda^{-1}(\alpha - 1)^{-1} & V[y] &= \alpha\beta^{-2} \\
p(x) &= \text{St}(x \mid \mu, \alpha\beta^{-1}\lambda, 2\alpha)
\end{aligned}$$

N_k($\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\lambda}$) *Multivariate Normal (p. 136)*

$$\begin{aligned}
\boldsymbol{\mu} &= (\mu_1, \dots, \mu_k) \in \Re^k & \mathbf{x} &= (x_1, \dots, x_k) \in \Re^k \\
\boldsymbol{\lambda} &\text{ symmetric positive-definite} \\
p(\mathbf{x}) &= c \exp \left\{ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\lambda}(\mathbf{x} - \boldsymbol{\mu}) \right\} & c &= |\boldsymbol{\lambda}|^{1/2} (2\pi)^{-k/2} \\
E[\mathbf{x}] &= \boldsymbol{\mu} & V[\mathbf{x}] &= \boldsymbol{\lambda}^{-1}
\end{aligned}$$

Pa₂($x, y \mid \alpha, \beta_0, \beta_1$) *Bilateral Pareto (p. 141)*

$$\begin{aligned}
(\beta_0, \beta_1) &\in \Re^2, \beta_0 < \beta_1, \alpha > 0 & (x, y) &\in \Re^2, x < \beta_0, y > \beta_1 \\
p(x, y) &= c (y - x)^{-(\alpha+2)} & c &= \alpha(\alpha + 1)(\beta_1 - \beta_0)^\alpha \\
E[x] &= \frac{\alpha\beta_0 - \beta_1}{\alpha - 1} & E[y] &= \frac{\alpha\beta_1 - \beta_0}{\alpha - 1} & V[x] = V[y] &= \frac{\alpha(\beta_1 - \beta_0)^2}{(\alpha - 1)^2(\alpha - 2)}
\end{aligned}$$

Multivariate Continuous Distributions (continued)

Ng_k($\mathbf{x}, y \mid \boldsymbol{\mu}, \boldsymbol{\lambda}, \alpha, \beta$) *Multivariate Normal-Gamma (p. 140)*

$$\begin{aligned}
-\infty < \mu_i < +\infty, \alpha > 0, \beta > 0 & (\mathbf{x}, y) = (x_1, \dots, x_k, y) \\
\boldsymbol{\lambda} \text{ symmetric positive-definite} & -\infty < x_i < \infty, y > 0 \\
p(\mathbf{x}, y) = N_k(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\lambda} y) \text{Ga}(y \mid \alpha, \beta) & \\
E[\mathbf{x}, y] = (\boldsymbol{\mu}, \alpha\beta^{-1}), \quad V[\mathbf{x}] = (\alpha - 1)^{-1}\beta\boldsymbol{\lambda}^{-1}, \quad V[y] = \alpha\beta^{-2} & \\
p(\mathbf{x}) = \text{St}_k(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\lambda}\alpha\beta^{-1}, 2\alpha) & p(y) = \text{Ga}(y \mid \alpha, \beta)
\end{aligned}$$

Nw_k($\mathbf{x}, \mathbf{y} \mid \boldsymbol{\mu}, \boldsymbol{\lambda}, \alpha, \beta$) *Multivariate Normal-Wishart (p. 140)*

$$\begin{aligned}
-\infty < \mu_i < +\infty, \lambda > 0 & \mathbf{x} = (x_1, \dots, x_k) \\
2\alpha > k - 1 & -\infty < x_i < +\infty \\
\boldsymbol{\beta} \text{ symmetric non-singular} & \mathbf{y} \text{ symmetric positive-definite} \\
p(\mathbf{x}, \mathbf{y}) = N_k(\mathbf{x} \mid \boldsymbol{\mu}, \lambda \mathbf{y}) \text{Wi}_k(\mathbf{y} \mid \alpha, \boldsymbol{\beta}) & \\
E[\mathbf{x}, \mathbf{y}] = \{\boldsymbol{\mu}, \alpha\boldsymbol{\beta}^{-1}\} & V[\mathbf{x}] = (\alpha - 1)^{-1}\boldsymbol{\beta}\boldsymbol{\lambda}^{-1} \\
p(\mathbf{x}) = \text{St}_k(\mathbf{x} \mid \boldsymbol{\mu}, \lambda\alpha\boldsymbol{\beta}^{-1}, 2\alpha) & p(\mathbf{y}) = \text{Wi}_k(\mathbf{y} \mid \alpha, \boldsymbol{\beta})
\end{aligned}$$

St_k($\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\lambda}, \alpha$) *Multivariate Student (p. 139)*

$$\begin{aligned}
-\infty < \mu_i < +\infty, \alpha > 0 & \mathbf{x} = (x_1, \dots, x_k) \\
\boldsymbol{\lambda} \text{ symmetric positive-definite} & -\infty < x_i < +\infty \\
p(\mathbf{x}) = c \left[1 + \frac{1}{\alpha} (\mathbf{x} - \boldsymbol{\mu})^t \boldsymbol{\lambda} (\mathbf{x} - \boldsymbol{\mu}) \right]^{-(\alpha+k)/2} & c = \frac{\Gamma(\frac{1}{2}(\alpha + k))}{\Gamma(\frac{1}{2}\alpha)(\alpha\pi)^{k/2}} |\boldsymbol{\lambda}|^{1/2} \\
E[\mathbf{x}] = \boldsymbol{\mu}, \quad V[\mathbf{x}] = \boldsymbol{\lambda}^{-1}(\alpha - 2)^{-1}\alpha &
\end{aligned}$$

Wi_k($\mathbf{x} \mid \alpha, \boldsymbol{\beta}$) *Wishart (p. 138)*

$$\begin{aligned}
2\alpha > k - 1 & \mathbf{x} \text{ symmetric positive-definite} \\
\boldsymbol{\beta} \text{ symmetric non-singular} & \\
p(\mathbf{x}) = c |\mathbf{x}|^{\alpha-(k+1)/2} \exp\{-\text{tr}(\boldsymbol{\beta}\mathbf{x})\} & c = \frac{\pi^{-k(k-1)/4} |\boldsymbol{\beta}|^\alpha}{\prod_{\ell=1}^k \Gamma(\frac{1}{2}(2\alpha + 1 - \ell))} \\
E[\mathbf{x}] = \alpha\boldsymbol{\beta}^{-1}, \quad E[\mathbf{x}^{-1}] = (\alpha - \frac{k+1}{2})^{-1}\boldsymbol{\beta} &
\end{aligned}$$

A.2 INFERENCE PROCESSES

The second section of this Appendix records the basic elements of the Bayesian learning processes for many commonly used statistical models.

For each of these models, we provide, in separate sections of the table, the following: the sufficient statistic and its sampling distribution; the conjugate family, the conjugate prior predictives for a single observable and for the sufficient statistic, the conjugate posterior and the conjugate posterior predictive for a single observable.

When clearly defined, we also provide, in a final section, the reference prior and the corresponding reference posterior and posterior predictive for a single observable. In the case of uniparameter models this can always be done. We recall, however, from Section 5.2.4 that, in multiparameter problems, the reference prior is only defined *relative to an ordered parametrisation*. In the univariate normal model (Example 5.17), the reference prior for (μ, λ) happens to be the same as that for (λ, μ) , namely $\pi(\mu, \lambda) = \pi(\lambda, \mu) \propto \lambda^{-1}$, and we provide the corresponding reference posteriors for μ and λ , together with the reference predictive distribution for a future observation.

In the multinomial, multivariate normal and linear regression models, however, there are very many different reference priors, corresponding to different inference problems, and specified by different ordered parametrisations. These are not reproduced in this Appendix.

Bernoulli model

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad x_i \in \{0, 1\}$$

$$p(x_i | \theta) = \text{Br}(x_i | \theta), \quad 0 < \theta < 1$$

$$t(\mathbf{z}) = r = \sum_{i=1}^n x_i$$

$$p(r | \theta) = \text{Bi}(r | \theta, n)$$

$$p(\theta) = \text{Be}(\theta | \alpha, \beta)$$

$$p(x) = \text{Bb}(r | \alpha, \beta, 1)$$

$$p(r) = \text{Bb}(r | \alpha, \beta, n)$$

$$p(\theta | \mathbf{z}) = \text{Be}(\theta | \alpha + r, \beta + n - r)$$

$$p(x | \mathbf{z}) = \text{Bb}(x | \alpha + r, \beta + n - r, 1)$$

$$\pi(\theta) = \text{Be}(\theta | \tfrac{1}{2}, \tfrac{1}{2})$$

$$\pi(\theta | \mathbf{z}) = \text{Be}(\theta | \tfrac{1}{2} + r, \tfrac{1}{2} + n - r)$$

$$\pi(x | \mathbf{z}) = \text{Bb}(x | \tfrac{1}{2} + r, \tfrac{1}{2} + n - r, 1)$$

Poisson Model

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad x_i = 0, 1, 2, \dots$$

$$p(x_i | \lambda) = \text{Pn}(x_i | \lambda), \quad \lambda \geq 0$$

$$t(\mathbf{z}) = r = \sum_{i=1}^n x_i$$

$$p(r | \lambda) = \text{Pn}(r | n\lambda)$$

$$p(\lambda) = \text{Ga}(\lambda | \alpha, \beta)$$

$$p(x) = \text{Pg}(x | \alpha, \beta, 1)$$

$$p(r) = \text{Pg}(x | \alpha, \beta, n)$$

$$p(\lambda | \mathbf{z}) = \text{Ga}(\lambda | \alpha + r, \beta + n)$$

$$p(x | \mathbf{z}) = \text{Pg}(x | \alpha + r, \beta + n, 1)$$

$$\pi(\lambda) \propto \lambda^{-1/2}$$

$$\pi(\lambda | \mathbf{z}) = \text{Ga}(\lambda | r + \tfrac{1}{2}, n)$$

$$\pi(x | \mathbf{z}) = \text{Pg}(x | r + \tfrac{1}{2}, n, 1)$$

Negative-Binomial model

$$\mathbf{z} = (x_1, \dots, x_n), \quad x_i = 0, 1, 2, \dots$$

$$p(x_i | \theta) = \text{Nb}(x_i | \theta, r), \quad 0 < \theta < 1$$

$$t(\mathbf{z}) = s = \sum_{i=1}^n x_i$$

$$p(s | \theta) = \text{Nb}(s | \theta, nr)$$

$$p(\theta) = \text{Be}(\theta | \alpha, \beta)$$

$$p(x) = \text{Nbb}(x | \alpha, \beta, r)$$

$$p(s) = \text{Nbb}(s | \alpha, \beta, nr)$$

$$p(\theta | \mathbf{z}) = \text{Be}(\theta | \alpha + nr, \beta + s)$$

$$p(x | \mathbf{z}) = \text{Nbb}(x | \alpha + nr, \beta + s, r)$$

$$\pi(\theta) \propto \theta^{-1}(1 - \theta)^{-1/2}$$

$$\pi(\theta | \mathbf{z}) = \text{Be}(\theta | nr, s + \tfrac{1}{2})$$

$$\pi(x | \mathbf{z}) = \text{Nbb}(x | nr, s + \tfrac{1}{2}, r)$$

Exponential Model

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad 0 < x_i < \infty$$

$$p(x_i | \theta) = \text{Ex}(x_i | \theta), \quad \theta > 0$$

$$t(\mathbf{z}) = t = \sum_{i=1}^n x_i$$

$$p(t | \theta) = \text{Ga}(t | n, \theta)$$

$$p(\theta) = \text{Ga}(\theta | \alpha, \beta)$$

$$p(x) = \text{Gg}(x | \alpha, \beta, 1)$$

$$p(t) = \text{Gg}(t | \alpha, \beta, n)$$

$$p(\theta | \mathbf{z}) = \text{Ga}(\theta | \alpha + n, \beta + t)$$

$$p(x | \mathbf{z}) = \text{Gg}(x | \alpha + n, \beta + t, 1)$$

$$\pi(\theta) \propto \theta^{-1}$$

$$\pi(\theta | \mathbf{z}) = \text{Ga}(\theta | n, t)$$

$$\pi(x | \mathbf{z}) = \text{Gg}(x | n, t, 1)$$

Uniform Model

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad 0 < x_i < \theta$$

$$p(x_i | \theta) = \text{Un}(x_i | 0, \theta), \quad \theta > 0$$

$$t(\mathbf{z}) = t = \max\{x_1, \dots, x_n\}$$

$$p(t | \theta) = \text{Ip}(t | n, \theta^{-1})$$

$$p(\theta) = \text{Pa}(\theta | \alpha, \beta)$$

$$p(x) = \frac{\alpha}{\alpha+1} \text{Un}(x | 0, \beta), \text{ if } x \leq \beta, \quad \frac{1}{\alpha+1} \text{Pa}(x | \alpha, \beta), \text{ if } x > \beta$$

$$p(t) = \frac{\alpha}{\alpha+n} \text{Ip}(t | n, \beta^{-1}), \text{ if } t \leq \beta, \quad \frac{n}{\alpha+n} \text{Pa}(t | \alpha, \beta), \text{ if } t > \beta$$

$$p(\theta | \mathbf{z}) = \text{Pa}(\theta | \alpha + n, \beta_n), \quad \beta_n = \max\{\beta, t\}$$

$$p(x | \mathbf{z}) = \frac{\alpha+n}{\alpha+n+1} \text{Un}(x | 0, \beta_n), \text{ if } x \leq \beta_n, \quad \frac{1}{\alpha+n+1} \text{Pa}(x | \alpha, \beta_n), \text{ if } x > \beta_n$$

$$\pi(\theta) \propto \theta^{-1}$$

$$\pi(\theta | \mathbf{z}) = \text{Pa}(\theta | n, t)$$

$$\pi(x | \mathbf{z}) = \frac{n}{n+1} \text{Un}(x | 0, t), \text{ if } x \leq t, \quad \frac{1}{n+1} \text{Pa}(x | n, t), \text{ if } x > t$$

Normal Model (known precision λ)

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad -\infty < x_i < \infty$$

$$p(x_i | \mu, \lambda) = \mathbf{N}(x_i | \mu, \lambda), \quad -\infty < \mu < \infty$$

$$t(\mathbf{z}) = \bar{x} = n^{-1} \sum_{i=1}^n x_i$$

$$p(\bar{x} | \mu, \lambda) = \mathbf{N}(\bar{x} | \mu, n\lambda)$$

$$p(\mu) = \mathbf{N}(\mu | \mu_0, \lambda_0)$$

$$p(x) = \mathbf{N}(x | \mu_0, \lambda \lambda_0 (\lambda_0 + \lambda)^{-1})$$

$$p(\bar{x}) = \mathbf{N}(\bar{x} | \mu_0, n \lambda \lambda_0 \lambda_n^{-1}), \quad \lambda_n = \lambda_0 + n\lambda,$$

$$p(\mu | \mathbf{z}) = \mathbf{N}(\mu | \mu_n, \lambda_n), \quad \mu_n = \lambda_n^{-1} (\lambda_0 \mu_0 + n \lambda \bar{x})$$

$$p(x | \mathbf{z}) = \mathbf{N}(x | \mu_n, \lambda \lambda_n (\lambda_n + \lambda)^{-1})$$

$$\pi(\mu) = \text{constant}$$

$$\pi(\mu | \mathbf{z}) = \mathbf{N}(\mu | \bar{x}, n\lambda)$$

$$\pi(x | \mathbf{z}) = \mathbf{N}(x | \bar{x}, \lambda n(n+1)^{-1})$$

Normal Model (known mean μ)

$$\mathbf{z} = \{x_1, \dots, x_n\}, \quad -\infty < x_i < \infty$$

$$p(x_i | \mu, \lambda) = \mathbf{N}(x_i | \mu, \lambda), \quad \lambda > 0$$

$$t(\mathbf{z}) = t = \sum_{i=1}^n (x_i - \mu)^2$$

$$p(t | \mu, \lambda) = \mathbf{Ga}(t | \tfrac{1}{2}n, \tfrac{1}{2}\lambda), \quad p(\lambda t) = \chi^2(\lambda t | n)$$

$$p(\lambda) = \mathbf{Ga}(\lambda | \alpha, \beta)$$

$$p(x) = \mathbf{St}(x | \mu, \alpha \beta^{-1}, 2\alpha)$$

$$p(t) = \mathbf{Gg}(x | \alpha, 2\beta, \tfrac{1}{2}n)$$

$$p(\lambda | \mathbf{z}) = \mathbf{Ga}(\lambda | \alpha + \tfrac{1}{2}n, \beta + \tfrac{1}{2}t)$$

$$p(x | \mathbf{z}) = \mathbf{St}(x | \mu, (\alpha + \tfrac{1}{2}n)(\beta + \tfrac{1}{2}t)^{-1}, 2\alpha + n)$$

$$\pi(\lambda) \propto \lambda^{-1}$$

$$\pi(\lambda | \mathbf{z}) = \mathbf{Ga}(\lambda | \tfrac{1}{2}n, \tfrac{1}{2}t)$$

$$\pi(x | \mathbf{z}) = \mathbf{St}(x | \mu, nt^{-1}, n)$$

Normal Model (both parameters unknown)

$$\begin{aligned} \mathbf{z} &= \{x_1, \dots, x_n\}, & -\infty < x_i < \infty \\ p(x_i | \mu, \lambda) &= \mathbf{N}(x_i | \mu, \lambda), & -\infty < \mu < \infty, \quad \lambda > 0 \end{aligned}$$

$$\begin{aligned} t(\mathbf{z}) &= (\bar{x}, s), & n\bar{x} = \sum_{i=1}^n x_i, & \quad ns^2 = \sum_{i=1}^n (x_i - \bar{x})^2 \\ p(\bar{x} | \mu, \lambda) &= \mathbf{N}(\bar{x} | \mu, n\lambda) \\ p(ns^2 | \mu, \lambda) &= \mathbf{Ga}(ns^2 | \tfrac{1}{2}(n-1), \tfrac{1}{2}\lambda), & \quad p(\lambda ns^2) = \chi^2(\lambda ns^2 | n-1) \end{aligned}$$

$$\begin{aligned} p(\mu, \lambda) &= \mathbf{Ng}(\mu, \lambda | \mu_0, n_0, \alpha, \beta) = \mathbf{N}(\mu | \mu_0, n_0\lambda) \mathbf{Ga}(\lambda | \alpha, \beta) \\ p(\mu) &= \mathbf{St}(\mu | \mu_0, n_0\alpha\beta^{-1}, 2\alpha) \\ p(\lambda) &= \mathbf{Ga}(\lambda | \alpha, \beta) \\ p(x) &= \mathbf{St}(x | \mu_0, n_0(n_0+1)^{-1}\alpha\beta^{-1}, 2\alpha) \\ p(\bar{x}) &= \mathbf{St}(\bar{x} | \mu_0, n_0n(n_0+n)^{-1}\alpha\beta^{-1}, 2\alpha) \\ p(ns^2) &= \mathbf{Gg}(ns^2 | \alpha, 2\beta, \tfrac{1}{2}(n-1)) \\ p(\mu | \mathbf{z}) &= \mathbf{St}(\mu | \mu_n, (n+n_0)(\alpha + \tfrac{1}{2}n)\beta_n^{-1}, 2\alpha + n), \\ &\quad \mu_n = (n_0 + n)^{-1}(n_0\mu_0 + n\bar{x}), \\ &\quad \beta_n = \beta + \tfrac{1}{2}ns^2 + \tfrac{1}{2}(n_0 + n)^{-1}n_0n(\mu_0 - \bar{x})^2 \\ p(\lambda | \mathbf{z}) &= \mathbf{Ga}(\lambda | \alpha + \tfrac{1}{2}n, \beta_n) \\ p(x | \mathbf{z}) &= \mathbf{St}(x | \mu_n, (n+n_0)(n+n_0+1)^{-1}(\alpha + \tfrac{1}{2}n)\beta_n^{-1}, 2\alpha + n) \end{aligned}$$

$$\begin{aligned} \pi(\mu, \lambda) &= \pi(\lambda, \mu) \propto \lambda^{-1}, & \quad n > 1 \\ \pi(\mu | \mathbf{z}) &= \mathbf{St}(\mu | \bar{x}, (n-1)s^{-2}, n-1) \\ \pi(\lambda | \mathbf{z}) &= \mathbf{Ga}(\lambda | \tfrac{1}{2}(n-1), \tfrac{1}{2}ns^2) \\ \pi(x | \mathbf{z}) &= \mathbf{St}(x | \bar{x}, (n-1)(n+1)^{-1}s^{-2}, n-1) \end{aligned}$$

Multinomial Model

$$\mathbf{z} = \{r_1, \dots, r_k, n\}, \quad r_i = 0, 1, 2, \dots, \quad \sum_{\ell=1}^k r_\ell \leq n$$

$$p(\mathbf{z} | \boldsymbol{\theta}) = \text{Mu}_k(\mathbf{z} | \boldsymbol{\theta}, n), \quad 0 < \theta_i < 1, \quad \sum_{i=1}^k \theta_i \leq 1$$

$$t(\mathbf{z}) = (\mathbf{r}, n), \quad \mathbf{r} = (r_1, \dots, r_k)$$

$$p(\mathbf{r} | \boldsymbol{\theta}) = \text{Mu}_k(\mathbf{r} | \boldsymbol{\theta}, n)$$

$$p(\boldsymbol{\theta}) = \text{Di}_k(\boldsymbol{\theta} | \boldsymbol{\alpha}), \quad \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{k+1}\}$$

$$p(\mathbf{r}) = \text{Md}_k(\mathbf{r} | \boldsymbol{\alpha}, n)$$

$$p(\boldsymbol{\theta} | \mathbf{z}) = \text{Di}_k\left(\boldsymbol{\theta} | \alpha_1 + r_1, \dots, \alpha_k + r_k, \alpha_{k+1} + n - \sum_{\ell=1}^k r_\ell\right)$$

$$p(\mathbf{x} | \mathbf{z}) = \text{Md}_k\left(\mathbf{x} | \alpha_1 + r_1, \dots, \alpha_k + r_k, \alpha_{k+1} + n - \sum_{\ell=1}^k r_\ell, n\right)$$

Multivariate Normal Model

$$\mathbf{z} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}, \quad \mathbf{x}_i \in \mathbb{R}^k$$

$$p(\mathbf{x}_i | \boldsymbol{\mu}, \boldsymbol{\lambda}) = \text{N}_k(\mathbf{x}_i | \boldsymbol{\mu}, \boldsymbol{\lambda}), \quad \boldsymbol{\mu} \in \mathbb{R}^k, \quad \boldsymbol{\lambda} \text{ } k \times k \text{ positive-definite}$$

$$t(\mathbf{z}) = (\bar{\mathbf{x}}, \mathbf{S}), \quad \bar{\mathbf{x}} = n^{-1} \sum_{i=1}^n \mathbf{x}_i, \quad \mathbf{S} = \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^t$$

$$p(\bar{\mathbf{x}} | \boldsymbol{\mu}, \boldsymbol{\lambda}) = \text{N}_k(\bar{\mathbf{x}} | \boldsymbol{\mu}, n\boldsymbol{\lambda})$$

$$p(\mathbf{S} | \boldsymbol{\lambda}) = \text{Wi}_k\left(\mathbf{S} | \frac{1}{2}(n-1), \frac{1}{2}\boldsymbol{\lambda}\right)$$

$$p(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \text{Nw}_k(\boldsymbol{\mu}, \boldsymbol{\lambda} | \boldsymbol{\mu}_0, n_0, \alpha, \boldsymbol{\beta}) = \text{N}_k(\boldsymbol{\mu} | \boldsymbol{\mu}_0, n_0\boldsymbol{\lambda}) \text{Wi}_k(\boldsymbol{\lambda} | \alpha, \boldsymbol{\beta})$$

$$p(\mathbf{x}) = \text{St}_k(\mathbf{x} | \boldsymbol{\mu}_0, (n_0 + 1)^{-1}n_0(\alpha - \frac{1}{2}(k-1))\boldsymbol{\beta}^{-1}, 2\alpha - k + 1)$$

$$p(\boldsymbol{\mu} | \mathbf{z}) = \text{St}_k(\boldsymbol{\mu} | \boldsymbol{\mu}_n, (n + n_0)\alpha_n\boldsymbol{\beta}_n^{-1}, 2\alpha_n),$$

$$\boldsymbol{\mu}_n = (n_0 + n)^{-1}(n_0\boldsymbol{\mu}_0 + n\bar{\mathbf{x}}),$$

$$\boldsymbol{\beta}_n = \boldsymbol{\beta} + \frac{1}{2}\mathbf{S} + \frac{1}{2}(n + n_0)^{-1}nn_0(\boldsymbol{\mu}_0 - \bar{\mathbf{x}})(\boldsymbol{\mu}_0 - \bar{\mathbf{x}})^t$$

$$p(\boldsymbol{\lambda} | \mathbf{z}) = \text{Wi}_k(\boldsymbol{\lambda} | \alpha + \frac{1}{2}n, \boldsymbol{\beta}_n)$$

$$p(\mathbf{x} | \mathbf{z}) = \text{St}_k(\mathbf{x} | \boldsymbol{\mu}_n, (n_0 + n + 1)^{-1}(n_0 + n)\alpha_n\boldsymbol{\beta}_n^{-1}, 2\alpha_n),$$

$$\alpha_n = \alpha + \frac{1}{2}n - \frac{1}{2}(k-1)$$

Linear Regression

$$\mathbf{z} = (\mathbf{y}, \mathbf{X}), \quad \mathbf{y} = (y_1, \dots, y_n)^t \in \mathbb{R}^n, \quad \mathbf{x}_i = (x_{i1}, \dots, x_{ik}) \in \mathbb{R}^k, \quad \mathbf{X} = (x_{ij})$$

$$p(\mathbf{y} | \mathbf{X}, \boldsymbol{\theta}, \lambda) = N_n(\mathbf{y} | \mathbf{X}\boldsymbol{\theta}, \lambda \mathbf{I}_n), \quad \boldsymbol{\theta} \in \mathbb{R}^k, \quad \lambda > 0$$

$$t(\mathbf{z}) = (\mathbf{X}^t \mathbf{X}, \mathbf{X}^t \mathbf{y})$$

$$p(\boldsymbol{\theta}, \lambda) = \text{Ng}(\boldsymbol{\theta}, \lambda | \boldsymbol{\theta}_0, \mathbf{n}_0, \alpha, \beta) = N_k(\boldsymbol{\theta} | \boldsymbol{\theta}_0, \mathbf{n}_0 \lambda) \text{Ga}(\lambda | \alpha, \beta)$$

$$p(\boldsymbol{\theta}) = \text{St}_k(\boldsymbol{\theta} | \boldsymbol{\theta}_0, \mathbf{n}_0 \alpha \beta^{-1}, 2\alpha)$$

$$p(\lambda) = \text{Ga}(\lambda | \alpha, \beta)$$

$$p(\mathbf{y} | \mathbf{x}) = \text{St}(y | \mathbf{x}\boldsymbol{\theta}_0, f(\mathbf{x})\alpha\beta^{-1}, 2\alpha)$$

$$f(\mathbf{x}) = 1 - \mathbf{x}(\mathbf{x}^t \mathbf{x} + \mathbf{n}_0)^{-1} \mathbf{x}^t,$$

$$p(\boldsymbol{\theta} | \mathbf{z}) = \text{St}_k(\boldsymbol{\theta} | \boldsymbol{\theta}_n, (\mathbf{n}_0 + \mathbf{X}^t \mathbf{X})(\alpha + \frac{1}{2}n)\beta_n^{-1}, 2\alpha + n),$$

$$\boldsymbol{\theta}_n = (\mathbf{n}_0 + \mathbf{X}^t \mathbf{X})^{-1}(\mathbf{n}_0 \boldsymbol{\theta}_0 + \mathbf{X}^t \mathbf{y}),$$

$$\beta_n = \beta + \frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}_n)^t \mathbf{y} + \frac{1}{2}(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_n)^t \mathbf{n}_0 \boldsymbol{\theta}_0$$

$$p(\lambda | \mathbf{z}) = \text{Ga}(\lambda | \alpha + \frac{1}{2}n, \beta_n)$$

$$p(\mathbf{y} | \mathbf{x}, \mathbf{z}) = \text{St}(y | \mathbf{x}\boldsymbol{\theta}_n, f_n(\mathbf{x})(\alpha + \frac{1}{2}n)\beta_n^{-1}, 2\alpha + n),$$

$$f_n(\mathbf{x}) = 1 - \mathbf{x}(\mathbf{x}^t \mathbf{x} + \mathbf{n}_0 + \mathbf{X}^t \mathbf{X})^{-1} \mathbf{x}^t$$

$$\pi(\boldsymbol{\theta}, \lambda) = \pi(\lambda, \boldsymbol{\theta}) \propto \lambda^{-(k+1)/2} \quad (\text{for all reorderings of the } \theta_j)$$

$$\pi(\boldsymbol{\theta} | \mathbf{z}) = \text{St}_k(\boldsymbol{\theta} | \hat{\boldsymbol{\theta}}_n, \frac{1}{2} \mathbf{X}^t \mathbf{X}(n - k)\hat{\beta}_n^{-1}, n - k),$$

$$\hat{\boldsymbol{\theta}}_n = (\mathbf{X}^t \mathbf{X})^{-1} \mathbf{X}^t \mathbf{y},$$

$$\hat{\beta}_n = \frac{1}{2}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\theta}}_n)^t \mathbf{y}$$

$$\pi(\lambda | \mathbf{z}) = \text{Ga}(\lambda | \frac{1}{2}(n - k), \hat{\beta}_n)$$

$$\pi(\mathbf{y} | \mathbf{x}, \mathbf{z}) = \text{St}(y | \mathbf{x}\hat{\boldsymbol{\theta}}_n, \frac{1}{2}f_n(\mathbf{x})(n - k)\hat{\beta}_n^{-1}, n - k),$$

$$f_n(\mathbf{x}) = 1 - \mathbf{x}(\mathbf{x}^t \mathbf{x} + \mathbf{X}^t \mathbf{X})^{-1} \mathbf{x}^t$$
