Local Volatilities and Parametric Methods for Option Pricing

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Local Volatilities and Parametric Methods for Option Pricing

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ABSTRACT

In this project, we discuss theoretical and practical aspects of the local volatility model while applying parametric methods to obtain the implied volatility surface through the model's calibration and create an option pricing algorithm, to give some sense of how these models are used in real financial practice. We show how the calibrated SVI model reproduces the implied volatility surface accurately, how there are practical problems for option pricing algorithms with local volatilities grid and the SVI model, and how using the surface SVI we can create a simpler algorithm that uses local volatilities to price European options and exotic options such as binary options without no static arbitrage opportunities.

KEYWORDS:

Local Volatility, Dupire, Calibration, Parametric Methods, Implied Volatility, SVI, SSVI, Gatheral

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1. Introduction

Since the Black-Scholes model was proposed by Black and Scholes (1973) approximately 50 years ago, it has been the most used model in both the literature and the financial industry because of its simplicity and its pricing equations. However, it is well-known that the empirical evidence about volatility of underlying asset prices in real financial markets does not behave like the implied in the model. Moreover, the analysis of the implied volatility has shown that it has some properties which differ significantly from the assumptions made by Black and Scholes.

Hence, academics and practitioners have tried to construct models that can accommodate these empirical properties, in order to achieve a model that can price options more consistently and that is easy to use for professional traders and risk managers. One of the most basic models which reproduce some of the empirical properties of implied volatility is the local volatility model, which replicates the market prices through the marginal distributions of the underlying asset price. In this project, we aim to discuss theoretical and practical aspects of the local volatility model while applying parametric methods to obtain the implied volatility surface through the model's calibration and create an option pricing algorithm, to give some sense of how these models are used in real financial practice.

In the first part, we introduce some basic knowledge of volatility modeling so the reader can be familiar with some of the terminology and the building blocks of option pricing, such as the Black-Scholes model and its alternatives and the implied volatility surface. Then, we proceed to present and discuss the local volatility model, delving into important results such as Gyöngy's lemma and Dupire's formula, and we explain and discuss the calibration of the model and our results.

Then, we will discuss some of the issues regarding the construction of an option pricing algorithm using the grid of local volatilities obtained through Dupire's formula to motivate the use of parametric methods such as the SVI and the surface SVI, which are presented and applied for implied volatility surface calibration and for obtaining option prices through Monte Carlo methods. Moreover, we adapt the pricing algorithm for binary options, a special kind of exotic options, to illustrate the usefulness of the combination of parametric methods with local volatilities.

2. Option Pricing Basics

A model's validity is not derived from its accuracy when describing historical dynamics nor the realism of its assumptions, but if it can price derivatives consistently (not allowing for arbitrage opportunities) and make the least assumptions about future conditions (Bergomi, 2016). The main goal of a model, then, is to be able to decompose the different risks coming from different contributions in the carry profits and losses (P&L) of a derivative position and secure a consistent valuation of these risks that allows to control them individually. The Black-Scholes model allows to price simple options easily and consistently, but real market data about volatility shows how its constant volatility assumption is not met and hinders its ability to price and hedge other kind of options. Therefore, extensions and alternative models have been developed by academics and practitioners, aiming to overcome the limitations of the Black-Scholes model.

In this section, we will take a closer look at the Black-Scholes model, its implications for volatility and the implied volatility surface, and some of the alternative models that have been proposed, including the local volatility model, the main focus of this project.

2.1. The Black-Scholes Model

The Black-Scholes model, developed by Black and Scholes (1973) is an option pricing model that estimates the fair value of an option given some certain conditions (Glantz and Kissell, 2014). Specifically, the model assumes a diffusion process for asset prices with constant volatility

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2.1}$$

where r is the risk-free interest rate, S_t is W_t is a geometric Brownian motion, or equivalently, by assuming the option price satisfies the following partial differential equation

$$\frac{\partial P_t}{\partial t} + \frac{\partial P_t}{\partial S} r S_t + \frac{1}{2}\frac{\partial^2 P_t}{\partial F^2}\sigma^2 S_t^2 = rP_t \tag{2.2}$$

where P_t is the Black-Scholes price of a European option at time t, the model allows us to obtain the price of European options through simple pricing equations

$$C_{BS} = S_0 N(d_1) - Ke^{-rT}N(d_2) \qquad P_{BS} = Ke^{-rT}N(-d_2) - S_0 N(-d_1) \tag{2.3}$$

$$for\ \ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + rT}{\sigma\sqrt{T}}\ \ and\ \ d_2 = d_1 - \frac{\sigma\sqrt{T}}{2}$$

where K is the strike price, T is the option's maturity and $N(\cdot)$ is a normal probability distribution function. Even though the literature on this model has proven it is not consistent with asset prices' dynamics and makes unrealistic assumptions, the pricing equations derived from the model are used in daily financial practice, as they allow for decomposing of the P&L of derivative positions such as a short-hedged option position.

However, the one parameter in the formulae which cannot be observed in the market is volatility, as it refers to the volatility that asset prices will have during the life of the option. Even though it can be estimated with the historical volatility, in financial practice, traders usually work with what is called implied volatilities (Hull, 2021). Thus, the next section is dedicated to this crucial concept in option pricing.

2.2. The Implied Volatility and Its Surface

The implied volatility is the volatility parameter σ_{BS} that allows obtaining the market price of an option through the Black-Scholes model pricing equations. Moreover, Alòs and García (2021) show how the implied volatility can be understood as a weighted mean of the future values of the volatility process, so it can also be interpreted as an estimate of the option's volatility until maturity is reached. Considering a European call option with strike K and maturity T, we can see that, based on the previous definitions, its market price V_t would equal its Black-Scholes price for the adequate volatility

$$V_t = C_{BS}(K, T-t, S_t, \sigma_{BS}) \tag{2.4}$$

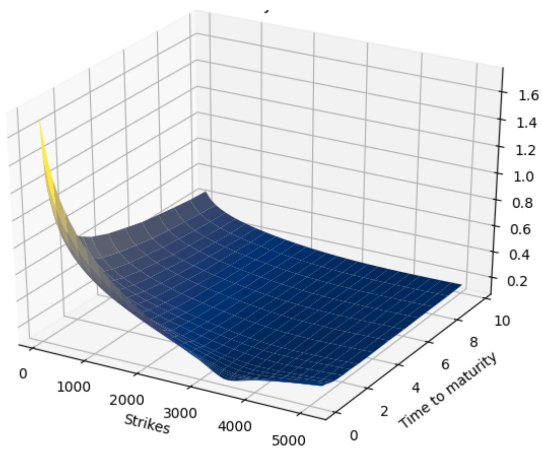
so the implied volatility σ_{BS} can be defined as follows:

$$\sigma_{BS} = C_{BS}^{-1}(K, T-t, S_t, V_t) \tag{2.5}$$

Hence, under the Black-Scholes model, we can obtain the implied volatility of a vanilla option just from its market price. As there is no analytical expression for C_{BS}^{-1}, numerical methods are usually applied in financial practice to obtain the implied volatility.

Nevertheless, empirical evidence using real market data shows how the implied volatility is not the same for all strikes and maturities (Dumas et al., 1998), which is a basic assumption of the Black-Scholes model. In contrast, the implied volatility of a European option depends on its strike and time to maturity, which means we can obtain a set of implied volatilities for different values of K and T, which conforms the so-called implied volatility surface. Moreover, an empirical analysis, as explained by Alòs and García (2021), shows how the implied volatility tends to exhibit a U-shaped pattern across different strikes for a given time to maturity. We can call this phenomenon the “smile effect” if this pattern is symmetric, while we use the term “skew effect” when it is not. These smiles and skews change with time to maturity, being very pronounced for short maturities and flattening as the time to maturity increases, and the empirical skews seem to follow a power law as a function of time to maturity. These properties can be spotted in Figure 1, which is the implied volatility surface using real market data provided by Caixa Bank.

Figure 1. *Implied volatility surface from real market data*



Source: *Own elaboration with real market data provided by Caixa Bank*

Other properties of volatility have been highlighted by Alòs and García (2021), such as volatility roughness (Gatheral et al., 2018), the presence of jumps (Eraker et al., 2003; Todorov and Tauchen, 2008), clustering (Mandelbrot, 1963) and long-memory properties (Harvey, 1993). Thus, the Black-Scholes model is not able to reproduce volatility dynamics correctly. Moreover, there are other empirical properties apart from being non-constant that the model cannot reproduce, which motivates the use of alternative models. From a practitioner’s point of view, it is necessary to use a model that can accurately reproduce the properties and behaviour of the implied volatility surface due to its effect on pricing financial derivatives and impact on the P&L of derivative positions. And this is the reason why many have shifted their focus to a kind of models that can help for this task, such as stochastic volatility models, the local volatility model and others.

2.3. Alternative Models

Bergomi (2016) shows how directly modelling the implied volatility dynamics (by assuming a diffusion process for S_t and another for the implied volatility I_t) is impractical given that the dynamics of the underlying determine the dynamics of the implied volatility. Additionally, Bergomi (2016) defends that more sophisticated models would allow to precisely characterize the P&L and the conditions when it vanishes, which makes it easier to spot the contribution of each effect rather than grouping them by omitting market microstructure-related parameters, as simpler models do. Consequently, many researchers and practitioners have tried to construct models which allow specifying implied volatility dynamics in a less restricted way.

The simplest one of all the alternative models that can be used in option pricing is the local volatility model, which is considered an extension of the Black-Scholes model and assumes that volatility is a deterministic function depending on the moment t and the underlying asset price S_t. However, this is a very constrained model because of the dependence on the market data calibration (Bergomi, 2016) and they have dynamic inconsistencies with the data (Alòs and García, 2021), so other kind of models such as the stochastic volatility models were constructed. These models assume a diffusion process for volatility (apart from the one of prices), which allows to work with simple expressions and reproduce better the dynamics of volatility. But, again, these do not reproduce all of these properties, so more models have been developed to account for them, such as long-memory volatility models (Comte and Renault, 1998), volatility models based on fractional brownian motions to reproduce roughness (Alòs et al., 2007), models based on jumps (Medveved and Scaillet, 2006), and many others.

In the following section, however, we will delve into the simplest one, the local volatility model. We aim to discuss about some important theoretical results and practical aspects that might be interesting to practitioners, while we also exemplify the calibration procedure by calibrating the model ourselves.

3. The Local Volatility Model

The local volatility model is known as the simplest stochastic volatility and most commonly used model in option pricing (Bergomi, 2016). The model was first introduced by Dupire (1994) and Derman and Kani (1994), and it is based on the spot volatility σ_t being a deterministic function $\sigma(t, S_t)$ of the moment t and the price of the underlying asset at that moment S_t, so that vanilla prices can be replicated, an outcome that relies on Gyöngy's lemma. It is normally understood as an extension of the Black-Scholes model and, according to Gatheral (2006), it is likely that it was thought to represent some kind of average rather than reproducing volatility behaviour, which can be shown through Dupire's most famous result: Dupire's formula.

This section aims to present relevant and practical aspects of the local volatility model, Gyöngy's lemma, the backbone of the model, and Dupire's formula and its extensions.

3.1. Gyöngy's lemma

The Gyöngy's lemma is a mathematical result introduced by Gyöngy (1986) to mimic one-dimensional marginal distributions of processes with an Itô differential. Given the importance of Gyöngy's lemma for the model, it is natural to introduce this result before the model itself, as it is very helpful to understand its origins and why does it work. We do formalize the lemma without taking into account α_t, which is a special case but does not affect the generality of the lemma:

Gyöngy's lemma. *Given a stochastic differential equation of the form*

$$dX_t = \alpha_t dt + \beta_t dW_t \tag{3.1}$$

where W_t is a standard Brownian motion and β_t is some adapated stochastic process, with initial condition $X_T = x$, there exists another stochastic differential equation of the form

$$dY_t = a_t(t, S_t)dt + b_t(t, S_t)dW_t \tag{3.2}$$

where $a_t(t, S_t)$ and $b_t(t, S_t)$ are deterministic functions, with initial condition $Y_T = x$, that admits a weak solution such that

$$\Pr(X_t \in [x, x + dx]) = \Pr(Y_t \in [x, x + dx]) \tag{3.3}$$

for all t and x.

We do not include a formal proof of this lemma on this project given that it is out of scope and its interest lies in its application rather than in its justification. However, for interested readers, a formal proof of the multidimensional version of Gyöngy's lemma is presented by Alòs and García (2021).

The lemma shows how one can mimic the process for underlying asset prices through a stochastic differential equation that yields the same marginal distributions of prices given the same initial condition, which in the option pricing context would be the vanilla option price being equal to the option's payoff at maturity T. This allows to replicate the underlying asset prices and obtain the observed option prices for options depending on the final underlying price S_T. Nevertheless, it also means that a stochastic differential equation like (3.2) would not allow to obtain the prices for path-depending options, as the diffusion cannot replicate the joint distribution of prices.

Starting from this result, then we can see how volatility σ_t could be modelled by a deterministic function $\sigma(t, S_t)$ to replicate vanilla prices, hence justifying the existence of the local volatility model, which will be explained in the following section.

3.2. The Model

The local volatility model is a model for underlying asset prices, whose volatility is a deterministic function of the form $\sigma(t, S_t)$ of the moment t and the price of the underlying S_t. Based on the original model, we ca mathematically define the model by the following stochastic differential equation

$$dS_t = rdt + \sigma(t, S_t)dW_t \tag{3.4}$$

where W_t is a Brownian motion and $\sigma(t, S_t)$ is the deterministic function of volatility, also called the local volatility function. The stochastic differential equation associated with the model is identical to (2.2), except for the local volatility function substituting the Black-Scholes volatility parameter

$$\frac{\partial P_t}{\partial t} + \frac{\partial P_t}{\partial S} r S_t + \frac{1}{2} \frac{\partial^2 P_t}{\partial F^2} \sigma^2(t, S_t) S_t^2 = r P_t \tag{3.5}$$

where P_t is the option price.

This model has three important characteristics about the behaviour of volatilities that reflect some practical problems. First, the model is fully determined by the calibration on the market skew or smile (Bergomi, 2016), as the calibration is done through vanilla options and, hence, determine the implied volatilities and the behaviour of volatility. Second, Alòs and García (2021) show that the short-time limit of the skew slope for the implied volatility of the local volatility model is

$$\lim_{T \to t} \frac{\partial I_t}{\partial k}(k_t^*) = \frac{1}{2} \frac{\partial \sigma(t, S_t)}{\partial X} \tag{3.6}$$

where I_t is the implied volatility in time t, X_t is the log-asset price, k is the log-strike and k_t^* is the at-the-money log-strike in time t. This means that, for short maturities, the at-the-money local volatility is approximately double the implied volatility, which is consistent with the one-half rule presented by Derman et al. (1996), and that the model cannot reproduce the short-time skew. Finally, Alòs and García (2021) mention that, because there is a need of a singularity at S_t and cannot be satisfied for every pair (t, S_t), the model cannot reproduce the surface for every t and needs to be calibrated in a daily basis. Hence, we

Another problem from the practitioner's point of view, implied by Gyöngy's lemma, might be to find the specific function $\sigma(t, S_t)$ such that the marginal distributions of the prices are equal, as not all functions work. Yet, this is not a problem given that one of the simplest ways to obtain an expression for this deterministic function is through Dupire's formula, which is discussed in the next section.

3.3. Dupire's Formula and Extensions

Dupire's formula (Dupire, 1994) is an equation which relates the local variance with the derivatives of the option price (with respect to time and the strike price). Through this formula, a practitioner can obtain the local volatility for a given strike price K and maturity T from the vanilla option prices. The equation for an European call option can be mathematically expressed as follows:

$$\sigma^2(K, T) = \frac{\frac{\partial C}{\partial T} - -rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}} \tag{3.7}$$

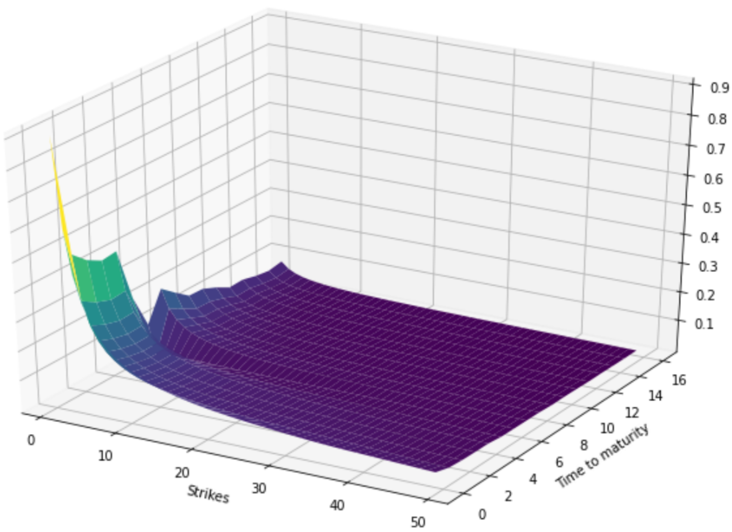
In words of Gatheral (2006) this formula can be understood as a definition of local volatility because it does not depend on the process that determines the behaviour of volatility. Moreover, Bergomi (2016) presents proofs which show how that the formula yields arbitrage-free results as long as the vanilla prices are free of arbitrage as well, which supports the practicality of this result in financial industry.

However, Alòs and García (2021) highlight that the computation of local volatilities through this formula is numerically unstable, as the denominator can be very small for at-the-money options. Consequently, other formula should be used to obtain local volatilities in financial practice. One alternative way is to use an equivalent formula in terms of implied volatilities, which solve the numerical problem posed by the original formula. By using the Black-Scholes total implied variance $\omega(y,T)$ and the log-strike y, it is possible to obtain

$$\sigma^2(K,T) = \frac{\frac{\partial \omega}{\partial T}}{\left(\frac{y}{2\omega}\frac{\partial \omega}{\partial y} - 1\right)^2 + \frac{1}{2}\frac{\partial^2 \omega}{\partial y^2} - \frac{1}{4}\left(-\frac{1}{4} + \frac{1}{\omega}\right)\left(\frac{\partial \omega}{\partial y}\right)^2} \tag{3.8}$$

where $\omega = I^2(y,T)T$, I is the Black-Scholes implied volatility, $y = \ln(K/F_t)$ and $F_t = S_0 e^{r(T-t)}$. To exemplify the usage of this formula, we have constructed an algorithm to obtain the local volatilities for each strike price and maturity. The results are shown in Figure's 2 local volatility surface.

Figures 2. *Local volatility surface from the data*



Source: *Own elaboration with real market data provided by Caixa Bank*

Yet, there are further practical problems with the local volatility's expressions found above. A main problem of these equation is that they show how asset prices and local volatilities move in opposite ways, while the empirical evidence shows a positive relation between prices and implied volatilities (Alòs and García, 2021). Hagan et al. (2002) claim that this fact can cause problems when using the model for delta and vega hedging strategies, which emphasizes the practical problems of the model. Furthermore, Bergomi (2016) accentuates another practical problem such as the discrete nature of market prices, which requires a discretization and interpolation for non-market traded strikes and maturities (taking care of not creating arbitrage opportunities).

Due to these kinds of problems, many methods have been developed, such as interpolation techniques as Kahale interpolation (Kahale, 2004) or the Tikhonov regularization (Egger and Engl, 2005). Moreover, if few prices are known, then the Dupire formula becomes less effective (Turinici, 2008), and other methods such as relative-entropy minimization methods (Avellaneda et al., 1997), non-parametric methods (Boddurtha and Jermakyan, 1999) or parametric methods such as the SVI (Gatheral, 2004). In fact, in the following section we will discuss the usage of parametric methods in order to calibrate the implied volatility surface and to price options using local volatilities.

4. Option Pricing with Local Volatilities

One common method in financial practice for option pricing using local volatilities is to run Monte Carlo simulations using the model specification and the Dupire formula. However, it is easier said than done, as the local volatility function depends on t and S_t, so it changes depending on the moment and the price at that moment. This complicates the construction of an algorithm to price European option, as it requires some adjustments and constraint for consistency and avoid arbitrage. Hence, we apply parametric methods such as the SVI and the surface SVI in order to calibrate the implied volatility surface and construct an algorithm to price European options and exotic options such as binary options.

In this section we discuss the problems arising from using the local volatility grid obtained through Dupire's formula and two widely-used parametric methodologies: the SVI and the surface SVI. Then, we present and discuss a simple SVI calibration and the algorithms constructed to price both European and exotic options.

4.1. The Problem with the Local Volatility Grid

As mentioned, one of the most common methods in financial industry is the Monte Carlo method, applied to the simulation of the underlying asset prices in order to obtain the options' price using the discounted risk-neutral expectation of its payoff

$$P_t = e^{-rT} E^*[h(S_T)] \tag{4.1}$$

where P_t is the price of the option, E^* is the risk-neutral expectation (the expected value using the risk-neutral probabilities) and h is the payoff function that depends on the final asset price S_T. In the context of the local volatility model, the diffusion process of the underlying prices would be the one represented by (3.4), so by simulating the path of prices multiple times, we can compute the expectation and obtain the price of the option.

Hence, we first need to obtain the local volatilities. To do so, one of the most straightforward methods is to use Dupire's formula in terms of the implied volatilities through a discretization of the derivatives of ω with respect to time and the log strike (due to the market prices being discrete)

$$\frac{\partial \omega}{\partial t} \approx \frac{\omega(t+\Delta t, y) - \omega(t, y)}{\Delta t} \tag{4.2}$$

$$\frac{\partial \omega}{\partial y} \approx \frac{\omega(t,y+\Delta y)-\omega(t,y)}{\Delta y} \tag{4.3}$$

$$\frac{\partial^2 \omega}{\partial y^2} \approx \frac{\omega(t,y+\Delta y)+2\omega(t,y)-(t,y-\Delta y)}{\Delta y} \tag{4.4}$$

As a result, we will obtain local volatilities for each value of ω and y, and because these depend on K and T, is equivalent to obtain a grid of the local volatilities for each K and T traded in the market.

Nevertheless, the biggest difference with a standard Monte Carlo simulation using Black-Scholes, is that, due to the local volatility function depending on t and S_t, the diffusion process term $\sigma(t, S_t)$ changes for every time step, so the algorithm should change the value of the term for each moment t. However, we do not have available the local volatility for each t, just the volatilities for each one of the maturities traded in the market, so we would need to use methods such as interpolation to obtain the volatility between maturities. But this is also posing a precision problem, as depending on the time steps, we might need to be very precise with the interpolation or the method applied to obtain the local volatility for t.

A similar problem occurs with the strike prices. Because we model randomness through a continuous Brownian motion, the price of the underlying is a continuous random value that can take a large number of values, but we do not have available all the strike prices, just the ones traded in the market. Hence, there is a need of using interpolation or other kind of methods in order to obtain the corresponding local volatility for S_t.

As one might deduce, applying Monte Carlo simulations turns a highly complex task if one wants to use the local volatility grid obtained. Despite of that, by using parametric approaches such as the SVI and the surface SVI, we can construct a simpler algorithm that allows to work better with local volatilities and overcome some of the problems. We discuss these approaches and exemplify their use in the following sections.

4.2. The SVI and its Calibration

The stochastic volatility inspired model or SVI model is a parametric model originally developed at Merrill Lynch in 1999 and published years after by Gatheral (2004). The model can be mathematically expressed as

$$\hat{\sigma}_{BS}^2(k) = a + b\left[\rho(k-m) + \sqrt{(k-m)^2\sigma^2}\right] \tag{4.5}$$

where $\hat{\sigma}_{BS}$ is the Black-Scholes implied volatility, k is the log-strike, and the other coefficients depend on maturity T. The model is very attractive for practitioners because the parametrization allows to interpret the effect of each parameter: a increases implied volatility; b affects the smile wings' slope positively; ρ affects the skew positively; m laterally shifts the volatility smile; and σ affects the at-the-money curvature negatively.

One of the most popular features of the model is that, through some restrictions and conditions, one can make the SVI model avoid arbitrage opportunities. The model can be fit for all maturities simultaneously given that there are no arbitrage opportunities between maturities. However, one can also obtain the parameters for each maturity and, adding further restrictions, one can even use it as an extrapolation formula.

To exemplify the usage of the model to obtain the implied volatility surface, we carry on a raw calibration of the model using real market data from Caixa Bank. This dataset contains both the implied volatilities for a European call option, represented by the surface in Figure 1, and the forward prices for each maturity. The goal is to obtain a set of parameters that allows to reproduce the same market volatility surface for all strikes and maturities.

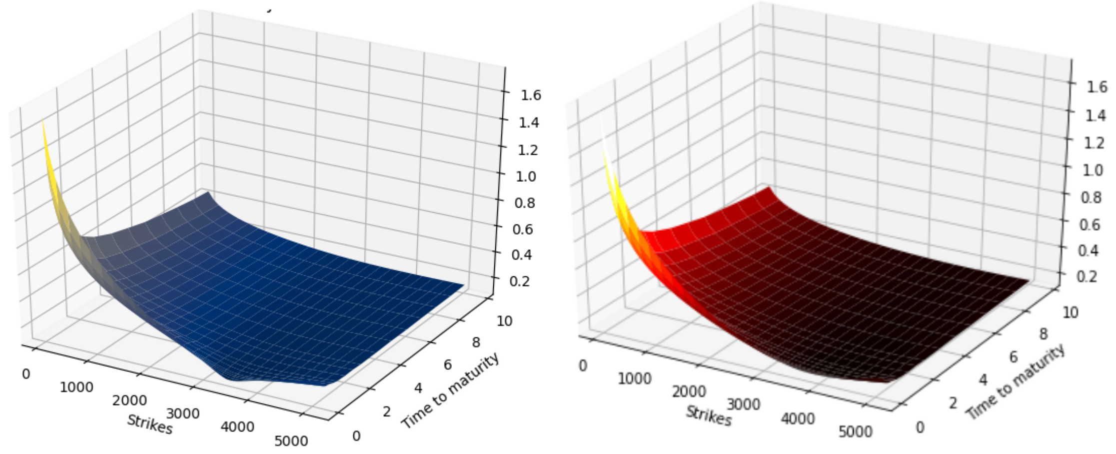
To proceed, we fit the SVI model for each maturity available, hence solving a minimization problem of the sum of squared errors

$$\left(\hat{a}, \hat{b}, \hat{\rho}, \hat{m}, \hat{\sigma}\right)_T = \underset{a,b,\rho,m,\sigma}{\operatorname{argmin}} \sum_{K_T} \left(\sigma_{K,T}^{mkt} - \sigma_{K,T}^{SVI}(a, b, \rho, m, \sigma)\right)^2 \tag{4.6}$$

where $\sigma_{K,T}^{mkt}$ is the market implied volatility and σ^{SVI} is the SVI implied volatility for a given strike K and maturity T. The SVI implied volatility is obtained through (4.5) and the fit in our Python code is done using the "*curve_fit*" function from the "*Scipy*", yielding a set of parameters for each maturity. The reader can find the calibrated parameters' values in Table B1 and other interesting figures in the Appendix, as we do not aim to create an extense discussion on the SVI model, but to give an introduction and a practical insight on how this model can be used in financial practice.

By comparing the resulting surface with Caixa Bank's professionally-calibrated implied volatility surface through Figures 2 and 3, we can see that our results are very similar, indicating the accuracy of our fit and the usefulness of this approach.

Figures 3 (left) & 4 (right). *Implied volatility surface from the data (left) and implied SVI volatility using calibrated parameters (right)*



Source: *Own elaboration with real market data provided by Caixa Bank*

Due to the SVI parameters depending on the maturity but not on the strike, then we could obtain the implied volatility for any k, and use Dupire's formula in terms of implied volatilities to obtain the associated local volatility. Nevertheless, there are still practical problems when using this method. A first problem is that applying this method in the Monte Carlo simulation algorithm creates convergence problems in our algorithm, and the second one is that we did not use any restrictions or constraints to avoid the creation of arbitrage opportunities with our calibration, while it is know that SVI smiles are arbitrageable (Gatheral and Jacquier, 2012). By using time-dependent parameters, we might be creating temporal arbitrage opportunities such as calendar or butterfly arbitrage opportunities, so the prices would not coincide with market prices.

This is why Gatheral and Jacquier (2012) presented one closed-form representation of the implied volatility surface that allows to eliminate static arbitrage: the surface SVI or SSVI model. The next section is dedicated to discuss the most relevant aspects of the SSVI model and its implementation through Monte Carlo simulations for option pricing avoiding static arbitrage opportunities.

4.3. The Surface SVI and Monte Carlo Simulations

The surface SVI is an extension of the natural SVI parametrization (which is the implied variance in the Heston model (Heston, 1993) in the limit $T \to \infty$ and a generalization of (4.6)) presented by Gatheral and Jacquier (2011). Using the authors' notation, we can define the model mathematically as

$$\sigma^2(k,\theta_t) = \frac{\theta_t}{2}\left[1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)+\rho)^2 + (1-\rho^2)^2}\right] \tag{4.7}$$

where $\theta_t = \sigma_{BS}^2(0,t)t$ is the at-the-money implied total variance and φ is a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$ such that the limit of the product $\theta_t\varphi(\theta_t)$ when $t \to 0$ exists in $\mathbb{R}$. Through the formula, we can obtain the implied volatilities for the whole surface, so there is no need of obtaining different sets of parameters for each maturity as with the SVI parametrization.

The most important characteristics of the model is that it yields a volatility surface in terms of at-the-money variance time rather than common calendar time, while it also ensures the absence of butterfly arbitrage and calendar arbitrage through specific conditions for the parameters. The conditions for the absence of static arbitrage proposed by Gatheral and Jacquier (2012) are the following:

$$\frac{\partial\theta_t}{\partial t} \geq 0 \tag{4.8}$$

$$0 \leq \frac{\partial\theta_t\varphi(\theta_t)}{\partial\theta} \leq \frac{1}{\rho^2}\left(1-\sqrt{1-\rho^2}\right)\varphi(\theta_t) \quad for \quad \forall\theta_t > 0 \tag{4.9}$$

$$\theta_t\varphi(\theta_t) \leq \min\left(\frac{4}{1+|\rho|}, 2\sqrt{\frac{\theta_t}{1+|\rho|}}\right) \quad for \quad \forall\theta_t > 0 \tag{4.10}$$

Hence, the parametrization is very advantageous for practitioners, as it offers a large class of closed-form volatility surfaces without static arbitrage opportunities.

Now, we can use this model to price options through a Monte Carlo simulation using local volatilities and be sure that there will be no static arbitrage opportunities. We have built an algorithm that prices options using local volatilities based on the SSVI implied volatilities, available on the link in the Appendix. In this algorithm we first define a functional form for $\varphi(\theta_t)$ that satisfies the above-mentioned requirements in order to obtain the implied volatilities. We used a Heston-like surface, defined as

$$\varphi(\theta_t) \equiv \frac{1}{\lambda\theta_t}\left[1 - \frac{1-e^{-\lambda\theta_t}}{\lambda\theta_t}\right] \tag{4.11}$$

where $\lambda \geq (1+|\rho|)/4$ to satisfy the non-arbitrage conditions. Function (4.11) is consistent with the skew of the Heston model's implied variance, even though other functional forms might be suitable, such as the power-law function, which yields a volatility surface more consistent with the term structure observed from empirical evidence.

Once we obtained the implied volatility with the given parameters and for a given t and k, we can obtain the corresponding local volatility using Dupire's formula in terms of implied volatilities and we can use this local volatility in the Monte Carlo simulation for simulating the underlying asset price given the local volatility for each t and S_t with

$$\sigma_t = \sqrt{\sigma^2(t, S_t)} \tag{4.12}$$

$$S_t = S_0 \exp\left(-\frac{1}{2}\sigma_t^2 \Delta t + \sqrt{\Delta t}\sigma_t z_t\right) \tag{4.13}$$

where Δt is the time variation for each time step and z_t is normally distributed random variable. With the risk-neutral valuation result (4.1), we can obtain the price of the desired European option.

Therefore, the algorithm allows to obtain option prices through local volatilities given a vector of parameters' values, which could be the calibrated for the whole surface using the SSVI model or others which satisfy conditions (4.8) to (4.10). The algorithm is far simpler than other algorithms that used different methods and techniques (apart from the parametric ones presented) to use the local volatilities in the Monte Carlo simulations and can be extended to price exotic options, highlighting its practical value.

In the following section, we present a similar algorithm that uses local volatilities and the SSVI parametrization in order to obtain the option prices of digital or binary options.

4.4. The Binary Option and its Valuation

A binary option is an exotic option with a discontinuous payoff, normally being cash-or-nothing (receiving a predetermined quantity of money or nothing) or asset-or-nothing (receiving the asset or nothing) binary options (Hull, 2021). This kind of exotic options is commonly used in academic or theoretical settings, given that fraud surrounding these options is usual (FBI, 2017) and regulators have banned them in many jurisdictions.

However, because binary options are non-path dependent options and they only depend on the final price of the underlying asset, we can use local volatilities to price them. To exemplify the potential of local volatilities and parametric methods together, we have constructed a similar algorithm as the previous one that prices binary options. This one is also available on the link attached in the Appendix. Throughout the algorithm, we assume the goal is to price a binary option with a payoff equal to $Q\$$ if $S_T \geq K$ and 0$ otherwise,

$$Binary\ option\ payoff = \begin{cases} Q\$\ \ if\ S_T \geq K \\ 0\$\ \ if\ S_T < K \end{cases} \tag{4.14}$$

where Q is the predetermined money quantity. Hence, there is the need of using the final underlying asset price for each simulation in order to obtain the expected value of the payoff. the Consequently, we can use an identical algorithm as the previous one (using SSVI parametrization and local volatilities) but modifying the payoff function and the option price computation.

In order to adjust the algorithm for digital options payoff function, we defined logical conditions such that, for each path simulated, the we store S_T and determine the payoff for the jth simulation through logical conditions according to (4.14). However, the functioning is the same as before: the implied volatilities are obtained through the SSVI model and the algorithm uses these values to obtain the local volatility through Dupire's formula in terms of implied volatilities. Then, the expected value is computed and the final option price is obtained. Different binary option prices are simulated for different maturities and strike prices with initial parameters $S_0 = 2800$, $n = 100{,}000$ (the number of simulated paths), $\sigma = 0.2, \gamma = 0.8, \rho = -0.7$, $m = 1500$ (the number of time steps) and $Q = 100$. The results are shown in Table 1.

Table 1. *Simulated prices for binary options for different strikes and maturities*

k	t = 1	t = 2	t = 3	t = 4	t = 5
1000	99.99	99.61	97.98	95.81	93.79
1250	99.86	97.76	94.03	90.55	85.9
1500	98.6	93	87.05	81.55	77.54
2000	87.07	75.96	68.34	63.58	60.29
2250	74.42	64.87	58.91	54.32	51.58
2500	60.26	54.71	50.04	46.14	44.52
2800	44.72	41.56	39.87	38.92	37.74
3100	30.17	32.23	32.44	30.98	31.03
3300	22.27	26.66	28.03	26.82	27.94

Source: *Own elaboration with real market data provided by Caixa Bank*

All in all, we can conclude that the algorithm that uses both SSVI parametrization and local volatilities allows to simply obtain European and exotic option prices without static arbitrage opportunities, showing its value for practitioners, as parameters can be calibrated for a whole surface to obtain the same observed market prices.

5. Final Conclusions

Option pricing models are constructed in order to price options consistently and have as few as possible assumptions about the future, which explains why the Black-Scholes model is used in financial practice and allows practitioners to work with its implied volatility, even though is not empirically accurate. However, real market data shows how the model is not able to reproduce some properties of implied volatilities, and hence this motivates the creation of alternative models such as the local volatility or the stochastic volatility models, which reproduce some properties of the implied volatility surface and allow practitioners to price options more consistently.

The simplest alternative model is the local volatility, which allows to obtain option prices through a mimicking diffusion process for underlying prices which uses a deterministic function of t and S_t as the volatility parameter, called the local volatility function. The model relies on Gyöngy's lemma and its very popular among practitioners because of Dupire's formula and its extension, which stablish a link between local volatilities, option market prices and implied volatilities.

When trying to apply Monte Carlo methods with the local volatility model and Dupire's formula, nevertheless, one can observe that the construction of an algorithm turns out to be complex because of the model's nature. Consequently, we use parametric approaches in order to obtain implied volatilities for different strikes and maturities and simplify the local volatilities option pricing algorithm. We consider two models: the SVI and the surface SVI. The first one is a widely-used parametric model which allows to obtain implied volatilities for any strike given a certain maturity, and its easily interpretable for a practitioner. Through our calibration of the SVI model, we showed how the model can accurately reproduce the implied volatility surface obtained by Caixa Bank professionals. Despite that, the usage of this model can create static arbitrage opportunities, so the prices obtained may differ from real market prices even though we use local volatilities and the calibrated SVI.

Therefore, we base our option pricing algorithm on the surface SVI model, which ensures static arbitrage-free implied volatilities given some certain conditions. Through this model, one can calibrate the whole volatility surface and build a Monte Carlo simulation algorithm that allows to obtain prices through local volatilities for both European and exotic options. Our algorithm uses the SSVI model assuming a Heston-like smooth function that, given certain parameters that respect the non-arbitrage conditions, yields implied volatilities for any strike and, through Dupire's formula in terms of implied volatilities, we can obtain the local volatilities that will be used in the diffusion process of the underlying prices. The algorithm results to be simpler and more intuitive than using the local volatility grid computed using Dupire's formula, highlighting its practicality for financial applications.

Finally, we adapt the previous algorithm based on the SSVI and local volatilities to price exotic options such as binary options, which are a kind of all-or-nothing options normally used in academic and theoretical settings. For developing the algorithm, we just adjust the payoff function for the final underlying asset price for each simulated path, and we obtain the price by using the same mechanics as before. Therefore, the developed algorithm allows to price both European and exotic options through SSVI parametrization and local volatilities in a simpler and static arbitrage-free way which can be very useful in financial practice.

References

Alòs, E. and García, D. (2021). Malliavin calculus in finance theory and practice. Boca Raton Chapman & Hall/Crc Press.

Alòs, E., Leon, J. and Vives, J. (2007). On the short-time behaviour of the implied volatility for jump-diffusion models with stochastic volatility. Finance and Stochastics, 11, 571-589.

Avellaneda, M., Friedman, C., Holmes, R. and Samperi, D. (1997). Calibrating volatility surfaces via relative-entropy minimization. Applied Mathematical Finance, 4(1), pp.37–64. doi:10.1080/135048697334827.

Bergomi, L. (2016). Stochastic volatility modeling. Boca Raton ; London ; New York: Crc Press, Cop.

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, [online] 81(3), pp. 637–654. doi:10.1086/260062.

Bodurtha, J. and Jermakyan, M. (1999). Nonparametric estimation of an implied volatility surface. The Journal of Computational Finance, 2(4), pp.29–60. doi:10.21314/jcf.1999.034.

Comte, F. and Renault (1998) E. Long memory in continuous-time stochastic volatility modelsMathematical Finance, Vol. 8, No. 4, pp. 291–323.

Derman, E. and Kani, I. (1994) Riding on a Smile. Risk, 7, 32-39.

Derman, E., Kani, I. and Zou, J.Z. (1996). The Local Volatility Surface: Unlocking the Information in Index Option Prices. Financial Analysts Journal, 52(4), pp.25–36. doi:10.2469/faj.v52.n4.2008.

Dumas, B., Fleming, J. and Whaley, R.E. (1998). Implied Volatility Functions: Empirical Tests. The Journal of Finance, 53(6), pp.2059–2106. doi:10.1111/0022-1082.00083.

Dupire, B. (1994) Pricing with a Smile. Risk, 7, 18-20.

Egger, H. and Engl, H.W. (2005). Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates. Inverse Problems, 21(3), pp.1027–1045. doi:10.1088/0266-5611/21/3/014.

Eraker, B., Johannes, M. and Polson, N. (2003). The Impact of Jumps in Volatility and Returns. The Journal of Finance, 58(3), pp.1269–1300. doi:10.1111/1540-6261.00566.

Federal Bureau of Investigation. (n.d.). Binary Options Fraud. [online] Available at: https://www.fbi.gov/news/stories/binary-options-fraud.

Gatheral, J. (2004). A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives.

Gatheral, J. (2006). The volatility surface: a practitioner's guide. Hoboken, N.J.: John Wiley & Sons.

Gatheral, J. and Jacquier, A. (2011). Convergence of Heston to SVI. Quantitative Finance, 11(8), pp.1129–1132. doi:10.1080/14697688.2010.550931.

Gatheral, J. and Jacquier, A. (2012). Arbitrage-free SVI Volatility Surfaces. *SSRN Electronic Journal*. doi:10.2139/ssrn.2033323.

Gatheral, J., Jaisson, T. and Rosenbaum, M. (2018). Volatility is rough. Quantitative Finance, 18(6), pp.933–949. doi:10.1080/14697688.2017.1393551.

Glantz, Morton and Kissell, Robert, (2013), Multi-Asset Risk Modeling, 1 ed., Elsevier, https://EconPapers.repec.org/RePEc:eee:monogr:9780124016903.

Hagan, P.S., Kumar, D., Lesniewski, A.S. and Woodward, D.E. (2002) Managing smile risk, Wilmott Magazine, m, pp. 84–108.

Harvey, A.C. (1993) Long Memory in Stochastic Volatility. Working Paper 10, School of Economics, London.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies, 6(2), 327- 343.

Hull, J.C. (2021). Options, futures, and other derivatives. Harlow (Gran Bretaña): Pearson.

Kahale, Nabil. (2003). An Arbitrage-free Interpolation of Volatilities. Risk. 17.

Mandelbrot, B. (1967). The Variation of Some Other Speculative Prices. The Journal of Business, 40(4), p.393. doi:10.1086/295006.

Medvedev, A. and Scaillet, O. (2006). Approximation and Calibration of Short-Term Implied Volatilities Under Jump-Diffusion Stochastic Volatility. Review of Financial Studies, 20(2), pp.427–459. doi:10.1093/rfs/hhl013.

Todorov, V. and Tauchen, G.E. (2008). Volatility Jumps. SSRN Electronic Journal. doi:10.2139/ssrn.1188509.

Turinici, G. (2008) "Local Volatility Calibration Using An Adjoint Proxy," Review of Economic and Business Studies, issue 2, pp. 93-105

Appendix A. Link for the Python Code Used

In order to the take a closer look to the Python code used to do the calibration and the option pricing algorithms, we make the code available through the GitHub link below:

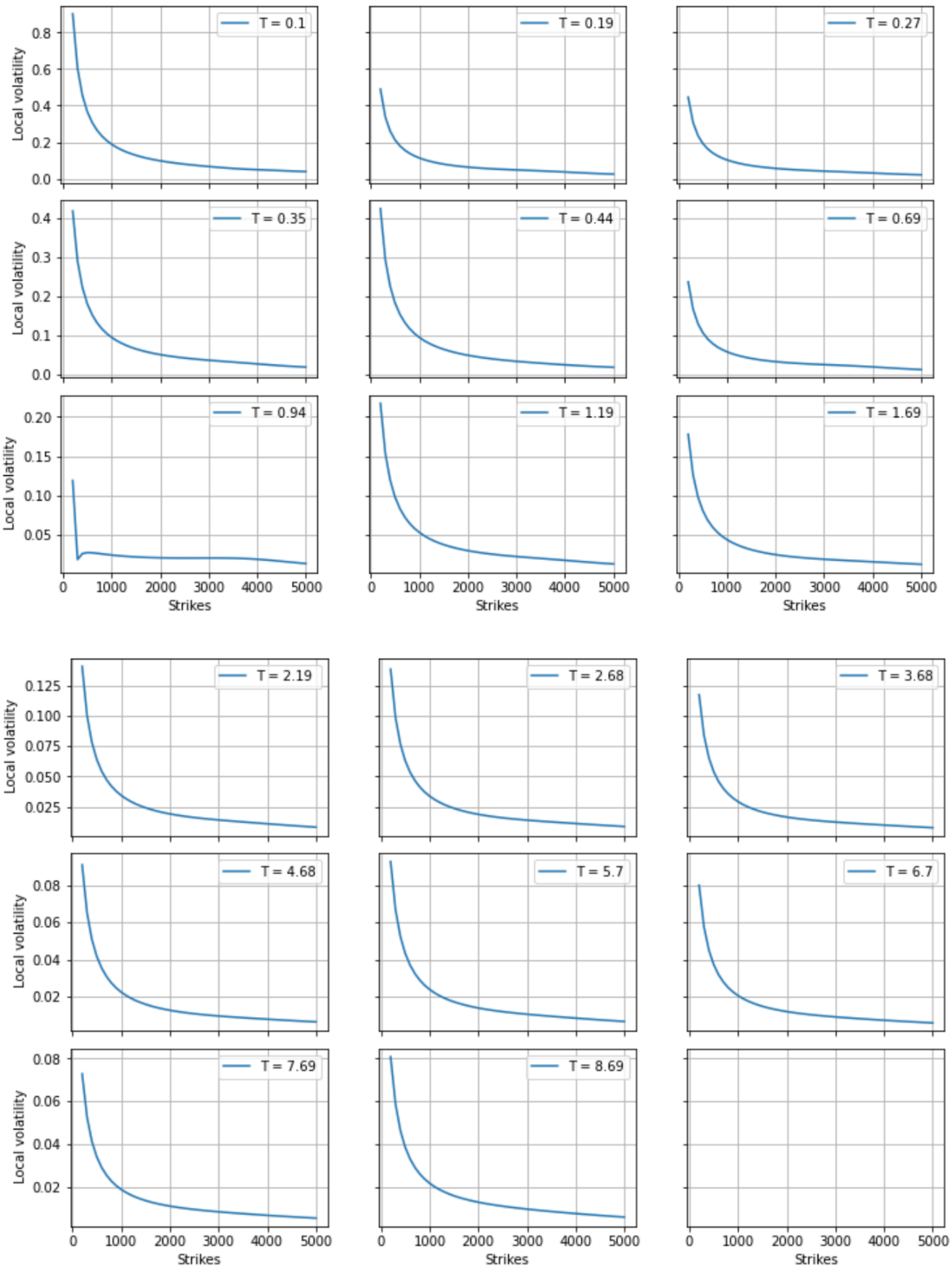
https://github.com/ikercb2000/OptionPricing/blob/main/Local_volatility_project_AOPM_definitivo.ipynb

Appendix B. Other Figures and Tables Used

Table B1. *Calibrated SVI parameters for each time to maturity*

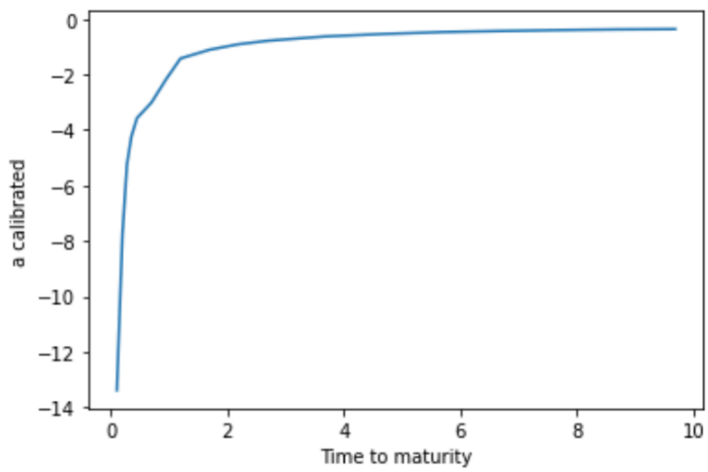
Maturity	a	b	m	rho	Sigma
0.0986301	-13.3853	874.288	12.2119	0.998062	0.246476
0.194521	-7.76304	594.646	11.7796	0.99814	0.214718
0.271233	-5.25642	551.778	11.482	0.99851	0.175239
0.347945	-4.23831	449.594	11.3592	0.99846	0.170724
0.443836	-3.57348	378.859	11.2698	0.998407	0.168086
0.693151	-3.01404	375.43	11.1914	0.998601	0.152767
0.942466	-2.17838	316.335	11.0533	0.998728	0.137707
1.19178	-1.424	199.417	10.8392	0.99856	0.134815
1.69041	-1.10925	152.246	10.7747	0.998471	0.134066
2.18904	-0.910767	118.469	10.6773	0.998314	0.135382
2.68767	-0.785218	106.973	10.6656	0.998371	0.131887
3.68493	-0.629259	99.6881	10.6131	0.998556	0.121392
4.68219	-0.539994	74.4561	10.5951	0.998316	0.130174
5.69863	-0.471532	73.943	10.5805	0.998511	0.122369
6.69589	-0.432947	64.9517	10.5979	0.998448	0.12612
7.69315	-0.401857	54.9844	10.6169	0.99831	0.133213
8.69041	-0.377635	44.3312	10.6407	0.99805	0.145185
9.68767	-0.365699	42.9126	10.694	0.99808	0.147288

Figure B1. *Sections of the local volatility surface for each maturity*



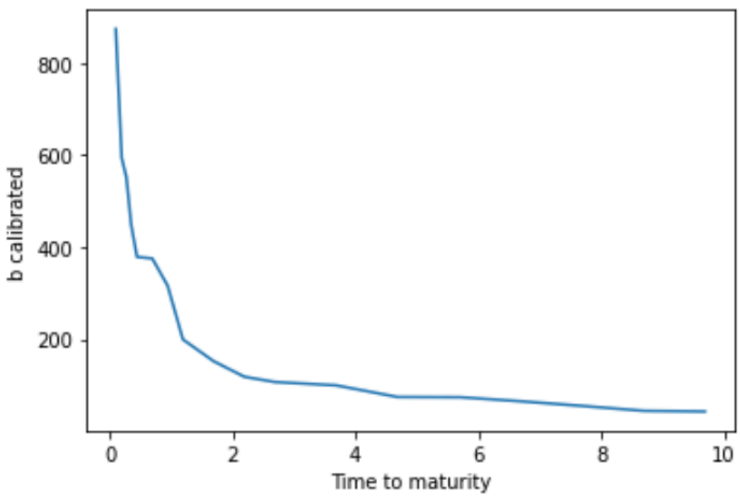
Source: *Own elaboration with real market data provided by Caixa Bank*

Figure B2. *Parameter a behaviour from the SVI calibration*



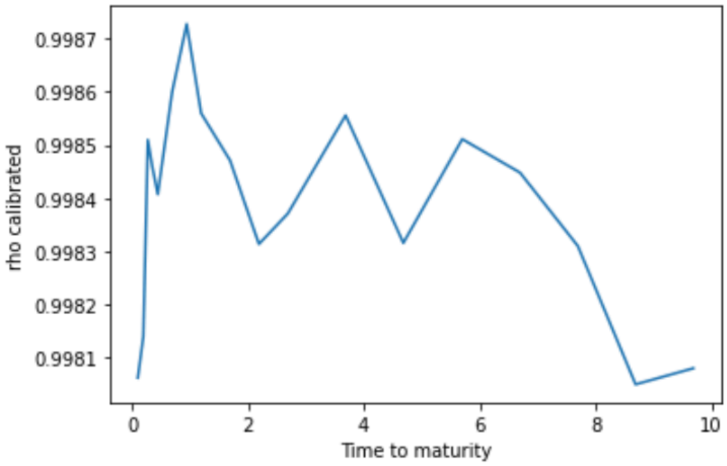
Source: *Own elaboration with real market data provided by Caixa Bank*

Figure B3. *Parameter b behaviour from the SVI calibration*



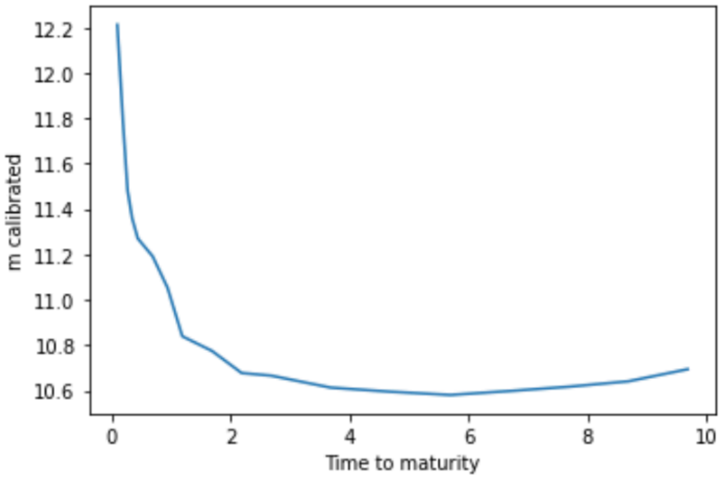
Source: *Own elaboration with real market data provided by Caixa Bank*

Figure B4. *Parameter ρ behaviour from the SVI calibration*



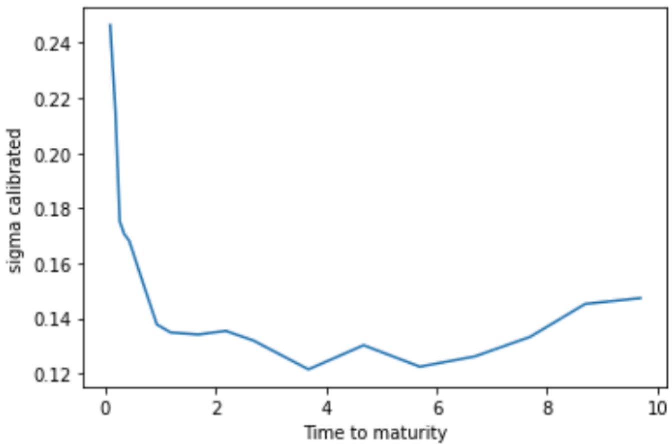
Source: *Own elaboration with real market data provided by Caixa Bank*

Figure B5. *Parameter m behaviour from the SVI calibration*



Source: *Own elaboration with real market data provided by Caixa Bank*

Figure B6. *Parameter σ behaviour from the SVI calibration*



Source: *Own elaboration with real market data provided by Caixa Bank*