Strongly Connected Components and Minimum Spanning Trees
CSCI 700 - Algorithms I

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Last Time

- Depth-First Search
- Breadth-First Search

Today

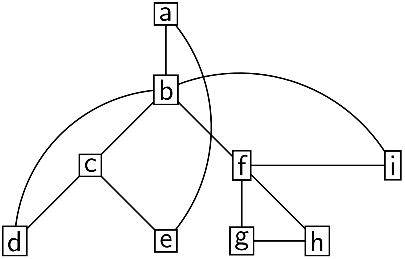
- Strongly Connected Components
- Minimum Spanning Trees: Prim's Algorithm

Ordering Of Nodes using DFS

We can track the **discovery time** $d[v]$ and **finish time** $f[v]$ of each node.

- Sorting by discovery time gives a **preorder traversal**
- Sorting by finish time gives a **postorder traversal**
- Is a node u is a DFS ancestor of a node v?
  - $d[u] < d[v] < f[v] < f[u]$
- Is a node u to the left is a node v?
  - $d[u] < f[u] < d[v] < f[v]$

Ordering Of Nodes using DFS



Strongly Connected Components

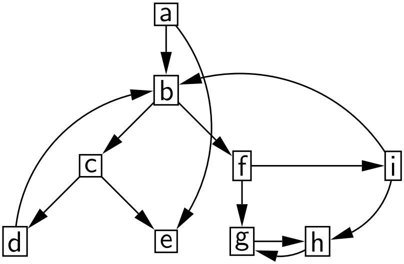
A Graph G is **strongly connected** if for all $u, v \in V$, u and v are mutually reachable.

We can decompose any Graph into a set of **Strongly Connected Components** – SCCs.

Every node in a cycle appears in the same SCC.

SCCs

How can we identify SCCs?



Identifying SCCs

Test reachability from every node to every other node.

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How can we do better?

- Run a DFS – keep track of $f[v]$.
- Reverse the Graph.
- Run a DFS on the reversed Graph – traversing in order of **decreasing** $f[v]$.
- The DFS Tree contains only the strongly connected components with internal edges included.

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$O(3(n+e)) = O(n+e)$

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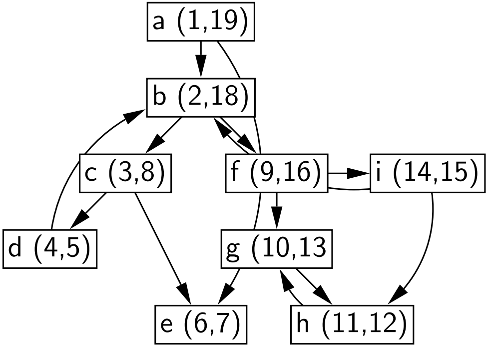
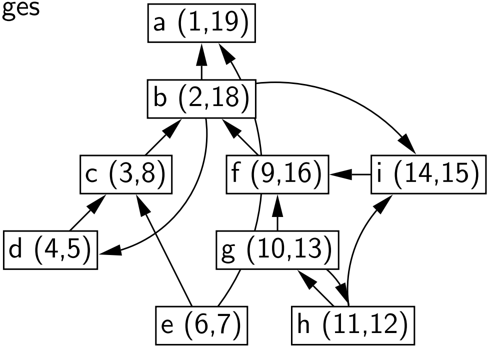


Identifying SCCs

Reverse the edges



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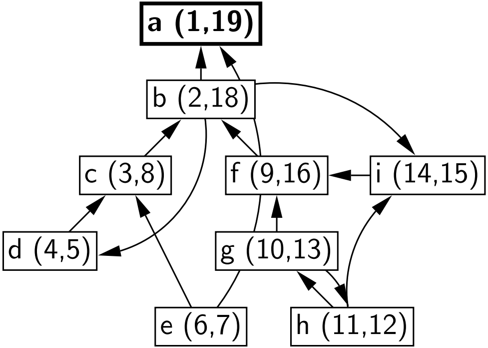

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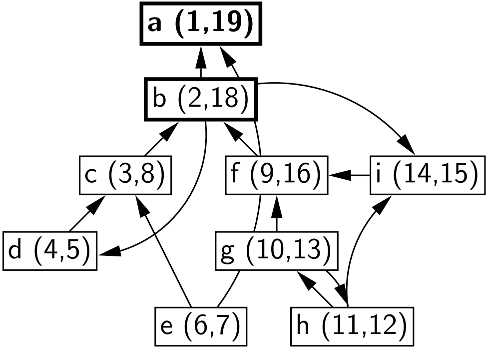

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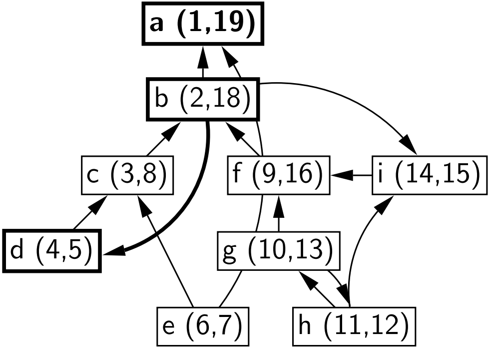

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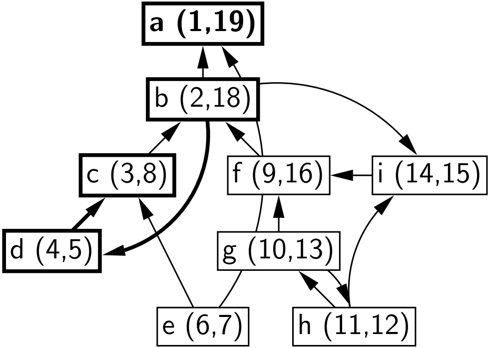

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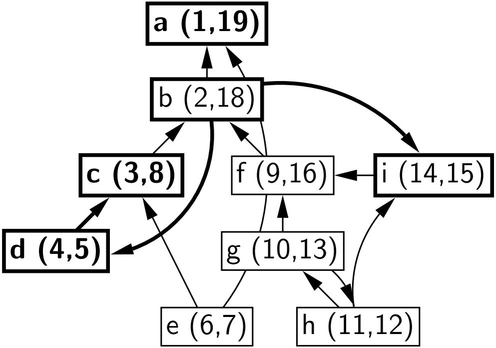

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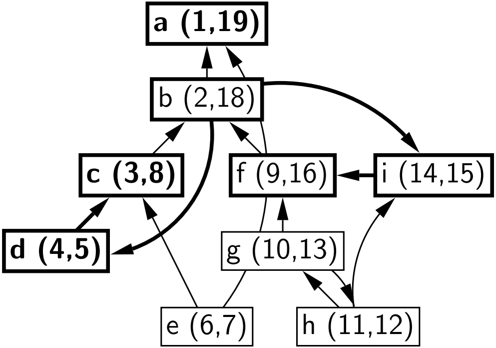

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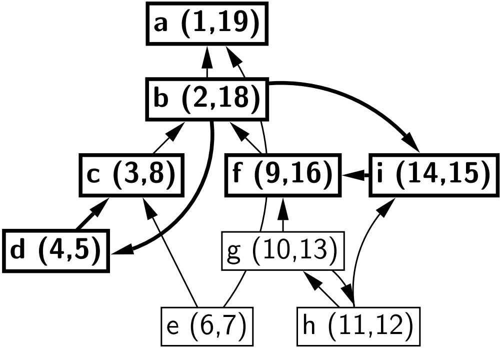

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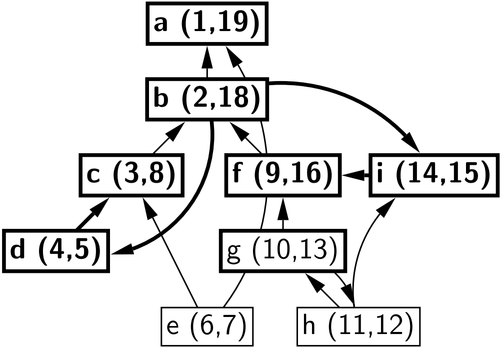

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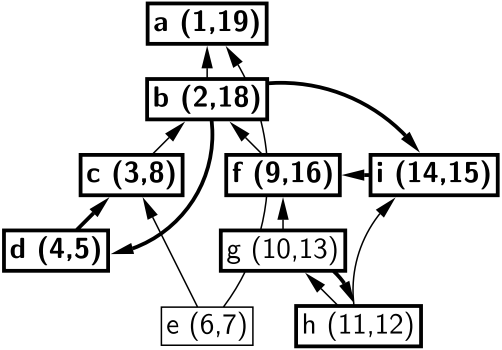

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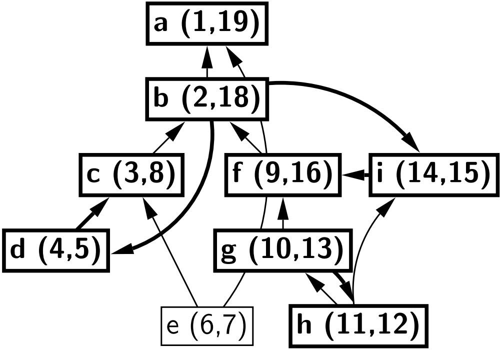

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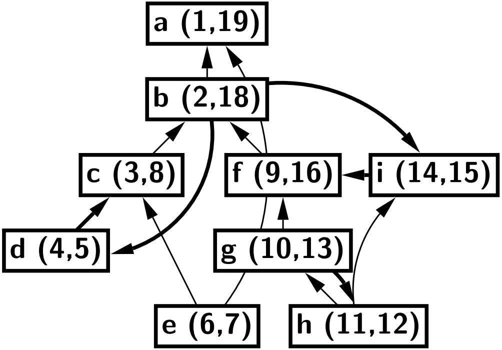

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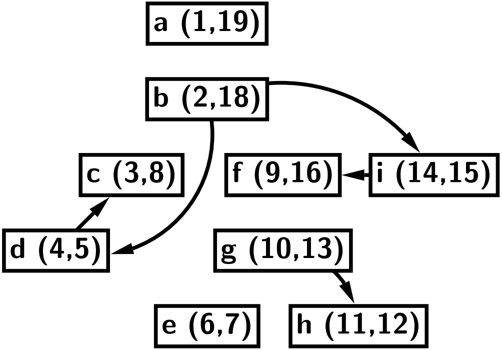


SCC Proof

Prove that the SCC procedure identifies all strongly connected components Must show

- If u and v are mutually reachable then they are in the same DFS tree generated by the second DFS.
- If u and v are in the same DFS tree, then they are mutually reachable.

SCC Proof

1. If u and v are mutually reachable in G, then they are mutually reachable in G_R.

If they are mutually reachable in G_R then they will appear in the same SCC.

Case 1) The DFS finds u first. Since u is reachable from v in G, then v is reachable from u in G_R. If v is reachable from u in G_R, the DFS subtree from u will include v.

Case 2) The reverse. The DFS finds v first. Since v is reachable from u in G, then u is reachable from v in G_R. If u is reachable from v in G_R, the DFS subtree from v will include u.

SCC Proof

2. If u and v are in the same DFS tree in G_R, then they are mutually reachable.

- Let x be the root of the DFS tree containing u and v.
- Want to show: x and u are mutually reachable, and x and v are mutually reachable. (Only one is necessary)
- $f[v] < f[x]$, otherwise we would have started the DFS with v.
- Since x can reach v in the reverse graph G_R, v can reach x in the original graph G.

- Partition the **first** DFS into four regions when the active node is x.
  - Black nodes – completed.
  - Grey nodes – ancestors of x.
  - White nodes reachable from x.
  - White nodes not reachable from x.

SCC Proof

- $f[greynodes] > f[x]$
- $f[nodesnotreachablefromx] > f[x]$.

- Recall $f[v] < f[x]$, thus, v is either black or in reachable from x.
- v cannot be black, because v can reach x in G.
- Thus, v must be below x, therefore reachable from x in G.
- Since x and v are mutually reachable in G, they are mutually reachable in G_R.

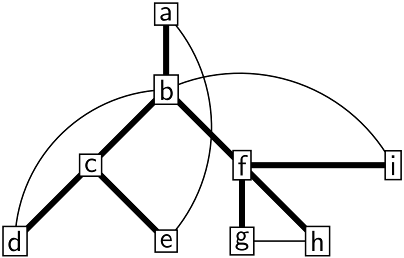
- The same logic holds for u.
- Thus, u and v are mutually reachable in G_R if they are in the same DFS Tree.

Definition of a Spanning Tree

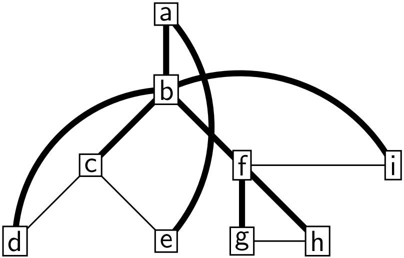
A **spanning tree** T of a graph G is a subgraph that contains

- Every vertex $v \in V$
- Edges $e \in E$ such at T is a tree – acyclic.

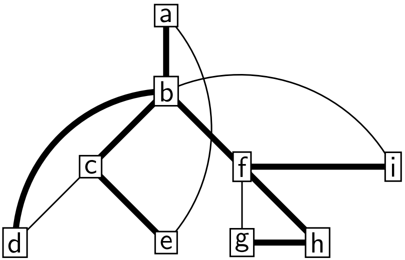
Example of Spanning Trees - DFS



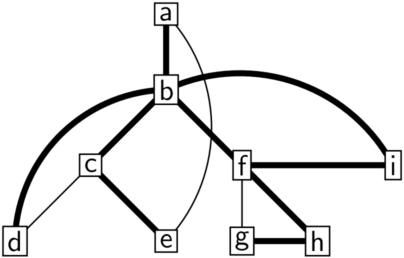
Example of Spanning Trees - BFS



Example of Spanning Trees - Other



Not a Spanning Tree



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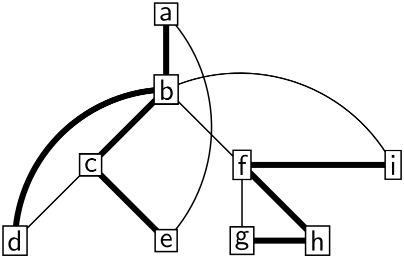
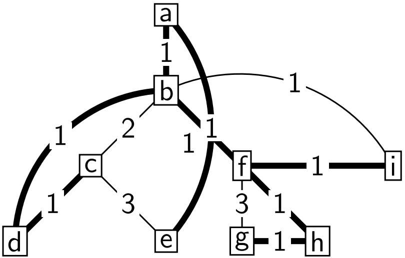


Minimum Spanning Tree

If the edges $e \in E$, have weights, a **minimum spanning tree** is a spanning tree with minimal cost.



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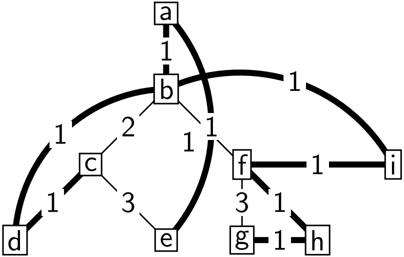

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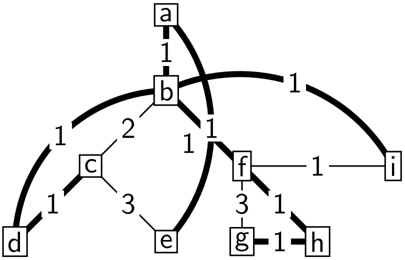
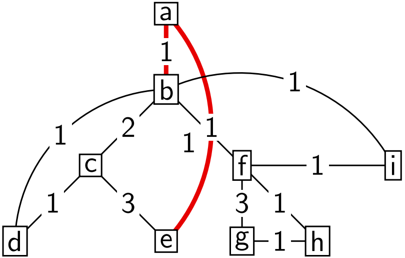


Proposition: Start at any node. Include the lowest cost "fringe" edge
Claim: This greedy choice works.

Prim's Algorithm

Example of Prim's Algorithm



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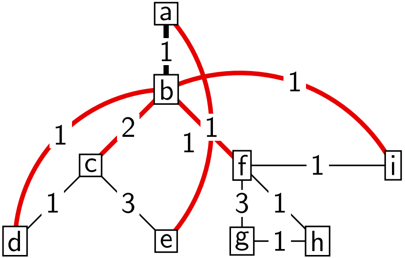

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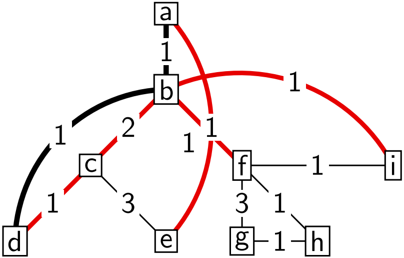

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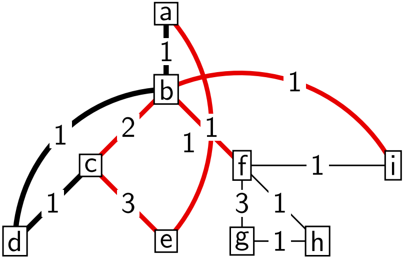

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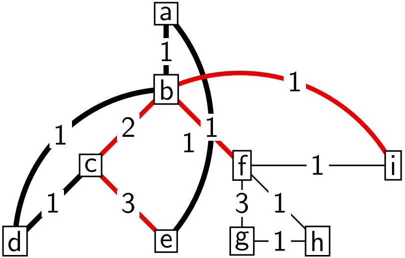

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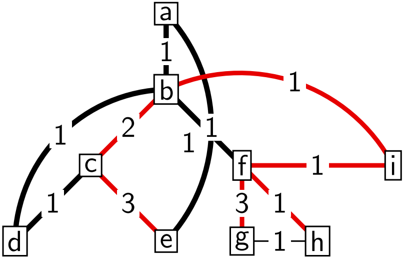

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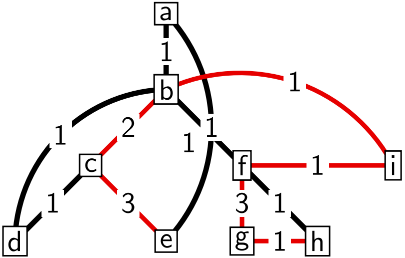

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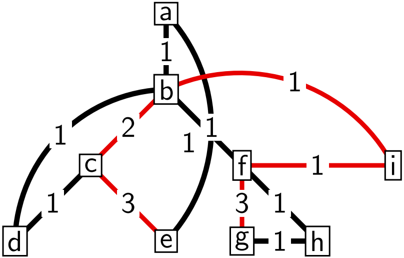

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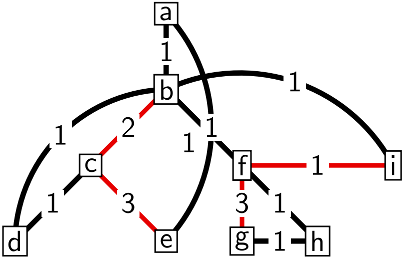

Prim's Algorithm

MST-Prim(G,w,r)

for $u \in V[G]$ **do**
    $key[u] = \infty$; $\pi[u] \leftarrow \emptyset$
end for
$key[r] \leftarrow 0$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$ **do**
    $u \leftarrow ExtractMin(Q)$
    for $v \in Adjacent(u)$ **do**
        if $v \in Q$ **and** $w(u,v) < key[v]$ **then**
            $key[v] = w(u,v)$; $\pi[u] \leftarrow u$
        end if
    end for
end while

Proof of Generic MST Algorithm

Greedy Strategy for a Generic MST growth.

Manage a set of edges A, such that prior to each iteration of the algorithm, A is the subset of a minimum spanning tree.

- At each step, identify an edge (u, v) that can be added to A without violating the invariant.
- We call (u, v) a **safe edge**.

How do we identify **safe edges**?

Identifying safe edges

Safe edges

Let G be a connected, undirected graph with weight functions w defined on E. Let A be a subset of E that is included in a MST. Let $(S, V - S)$ be any *cut* of G that *respects* A. Let (u, v) be a *light edge crossing* $(S, V - S)$. Then (u, v) is safe for A.

- A **cut** $(S, V - S)$ of an undirected graph G, is a partition of V.
- An edge **crosses** a cut if one of the end points are in S, and the other is in V.
- A cut **respects** a set of edges A if no edges in A cross the cut.
- A edge that crosses the cut is a **light edge**, if its weight is minimal crossing the cut.

Safe edges proof

- Let T be a MST that includes A, but doesn't include light edge (u, v). (If it does, we're done.)
- We will construct another MST T' that includes $A \bigcup (u, v)$ using the cut-and-paste technique. Showing that (u, v) is a **safe edge** for A.

- Assume the edge (u, v) forms a cycle with the edges on the unique path p from u to v in T.
- Thus, there is some edge (x, y) on the path p that also crosses the cut.
  - (x, y) is not in A because the cut respects A.
- Since, (x, y) is on the unique path removing (x, y) separates T into two components.
- Adding (u, v) connects these two components, forming a new spanning tree T'.

Safe edges proof

- Now we need to show that T' is Minimal.
- Since (u, v) is a light edge crossing the cut $(S, V - S)$, and (x, y) also crosses the cut, $w(u, v) \leq w(x, y)$.
- Thus:
  - $w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$
- Since T was a minimum spanning tree, so is T'.
- Since T' is a minimum spanning tree containing (u, v), (u, v) is safe for A.

Prim's algorithm

In Prim's algorithm, we identify the cut $(S, V - S)$, by starting at one node, and building up A by including light edges, and expanding A.

Next time:

- Another way to identify safe edges and build up A – Kruskal's Algorithm

Bye

- Next time
  - Another Minimum Spanning Tree Algorithm: Kruskal's
  - Shortest Paths: Bellman-Ford.