2 (a) Prove the base case: $n=1$

A has the form $A = \begin{pmatrix} \alpha_{11} \\ a_{21} \end{pmatrix}$ where α_{11} is a scalar

don't forget $A = LU$ (factorization of A)

$$A = \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \quad L = \begin{pmatrix} L_{TL} & 0 \\ L_{BL} & L_{BR} \end{pmatrix} \quad U = \begin{pmatrix} U_{TL} & U_{TR} \\ 0 & U_{BR} \end{pmatrix}$$

$$A = \left(\begin{array}{c|c} L_{TL} U_{TL} & L_{TL} U_{TR} \\ L_{BL} U_{TL} & L_{BL} U_{TR} + L_{BR} U_{BR} \end{array}\right) = \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}$$

U is nonsingular since it can't have a zero on the diagonal

All principal leading submatrices are nonsingular, making $\alpha_{11} \neq 0$

Therefore $A = LU = \underbrace{\begin{pmatrix} 1 \\ a_{21}/\alpha_{11} \end{pmatrix}}_{L} \underbrace{\alpha_{11}}_{U}$

The factorization is unique since the first element of L must be 1

another interpretation of $A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$

2(b) <u>Inductive Step</u>: Assume that for $A_{00} \in \mathbb{R}^{n \times n}$ the bordered LU factorization computes $\check{L}_{00}$ and $\check{U}_{00}$

where $A_{00} + \Delta A_{00} = \check{L}_{00} \check{U}_{00}$ with $|\Delta A_{00}| \leq \gamma |\check{L}_{00}| |\check{U}_{00}|$

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$$

$a_{01} = L_{00} u_{01}$ ← according to algorithm

factoring in errors we can consider

$$\Delta A = \left(\begin{array}{c|c} \Delta A_{00} & \partial a_{01} \\ \hline \partial a_{10}^T & \partial \alpha_{11} \end{array}\right)$$

$$a_{01} + \partial a_{01} = L_{00} \underbrace{u_{01}}_{a_{01}} (1 + \underbrace{\epsilon_{01}}_{\partial a_{01}})$$

$a_{10}^T = l_{10}^T U_{00}$ ← according to the algorithm

$$a_{10}^T + \partial a_{10}^T = l_{10}^T \underbrace{U_{00}}_{a_{01}} (1 + \underbrace{\epsilon_{10}}_{\partial a_{01}})$$

$\alpha_{11} = u_{11}$ ← according to the algorithm

$$\alpha_{11} + \partial \alpha_{11} = (\alpha_{11} - a_{10}^T a_{01})(1 + \epsilon_{11})$$

$$= (\alpha_{11} - (a_{10}^T + \partial a_{10}^T)(a_{10} + \partial a_{10}))(1 + \epsilon_{11})$$

$$\Delta A_{00} = L_{00} U_{00} (1 + \epsilon_{00})$$

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$|\Delta A| \leq \gamma_n |\check{L}||\check{U}|$

2(b) $\left|\begin{pmatrix} \Delta A_{00} & \partial a_{01} \\ \partial a_{10}^T & \partial \alpha_{11} \end{pmatrix}\right| \leq \gamma_{n+1} \left|\begin{pmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{pmatrix}\right| \left|\begin{pmatrix} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{u}_{11} \end{pmatrix}\right|$

$|\Delta A_{00}| \leq \gamma_{n+1} |\check{L}_{00}||\check{U}_{00}|$

$|\partial a_{01}| \leq \gamma_{n+1} |\check{L}_{00}||\check{u}_{01}|$

This coincides with the instruction:

Solve $L_{00} u_{01} = a_{01}$ overwriting a_{01} with the result

$|\partial a_{10}^T| \leq \gamma_{n+1} |l_{10}^T||U_{00}|$

This coincides with the instruction:

Solve $l_{10}^T U_{00} = a_{10}^T$ overwriting a_{10}^T with the result

$|\partial \alpha_{11}| \leq \gamma_{n+1}(\underbrace{|l_{10}^T||\check{u}_{01}|}+|\check{u}_{11}|)$

Not a matrix-matrix multiplication, this can be omitted

$|\partial \alpha_{11}| \leq \gamma_{n+1} |\check{u}_{11}|$

This coincides with the instruction:

$\alpha_{11} := u_{11} = \alpha_{11} - a_{10}^T a_{01}$

$\check{L} = \begin{pmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{pmatrix}$ $\quad \check{U} = \begin{pmatrix} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{u}_{11} \end{pmatrix}$

$A + \Delta A = \check{L}\check{U}$

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$$\check{L}\check{U} = \left(\begin{array}{c|c} \check{L}_{00}\check{U}_{00} & \check{L}_{00}\check{u}_{01} \\ \check{l}_{10}^T\check{U}_{00} & \check{l}_{10}^T\check{u}_{01} + \check{u}_{11} \end{array}\right)$$

$$A + \Delta A = L_{00}U_{00} + \gamma_{n+1}\check{L}_{00}\check{U}_{00}$$

$$a_{01} + da_{01} = L_{00}u_{01} + \gamma_{n+1}\check{L}_{00}\check{u}_{01}$$

$$a_{10}^T + da_{10}^T = l_{10}^T U_{00} + \gamma_{n+1}\check{l}_{10}^T\check{U}_{00}$$

$$\alpha_{11} + d\alpha_{11} = l_{10}^T u_{01} + u_{11} + \gamma_{n+1}\check{l}_{10}^T\check{u}_{11}$$

This proves that $\check{L} = \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T\check{U}_{00} & 1 \end{array}\right)$

and $\check{U} = \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{u}_{11} \end{array}\right)$

are valid for backward error analysis of the LU factorization of A