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Exam 2

1(a) Bordered algorithm for computing the cholesky factorization of a symmetric positive definite matrix

Step 1: partition $A \Rightarrow \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$ and $L \Rightarrow \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$

where $A = LL^T$, which means the following:

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)\left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right)^T = \left(\begin{array}{c|c} L_{00}L_{00}^T & L_{00}l_{10} \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

This shows that $L_{00}L_{00}^T$ is the Cholesky Factorization of A_{00}

Step 3 →	$a_{10}^T = l_{10}^T L_{00}^T$ ↓ $a_{10}^T L_{00}^{-T} = l_{10}^T$	and	$\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2$ ← Step 4
		a_{01} is irrelevant	$\lambda_{11}^2 = \alpha_{11} - l_{10}^T l_{10}$ ↓
	$A_{00} = L_{00}L_{00}^T$		$\lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

Step 2: $\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \Rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right)$ Partition until you reach A_{mm}

Step 3: $a_{10}^T = a_{10} L_{00}^{-T}$

Step 4: $\alpha_{11} = \alpha_{11} - a_{10} a_{10}^T$

Step 5: Do cholesky factorization of A_{11} (aka α_{11})

This will progress to step 3: a_{21}, Step 4: A_{22} ... until you reach A_{mm}

1(b) <u>Theorem</u>: Given a SPD matrix A, there exists a lower triangular matrix L such that $A = LL^T$. If the diagonal elements of L are restricted to be positive, L is unique

Prove the above theorem by showing that the bordered Cholesky factorization algorithm is well-defined for a matrix A that is SPD

Start with Proof by induction

<u>Base case, $n=1$</u>. Clearly true. For a 1×1 matrix, $A = \alpha_{11}$. By definition, a matrix is SPD iff its eigenvalues are positive. The cholesky factor $\lambda_{11} = \sqrt{\alpha_{11}}$ is unique since it's positive, making L unique

<u>inductive step, assume $n=k$ is true, test $n=k+1$</u>

Let $A \in \mathbb{R}^{(k+1)\times(k+1)}$ be SPD. Partition

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \qquad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

e_1 = standard basis vector

1. $\lambda_{11} = \sqrt{\alpha_{11}}$, α_{11} is well defined since $0 < e_1 A e_1 = \alpha_{11}$
2. $l_{10}^T = a_{10}^T / \lambda_{11}$ l_{10}^T is unique just like λ_{11}, and well defined
3. $\alpha_{11} - l_{10}^T l_{10} = \lambda_{11}^2$, subsequently due to 2 (also shown in answer to 1(a))

L is unique, and the desired cholesky factorization of A

Proved by induction