

# A solution to Haagerup's problem on normal weights

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## Main result

In my master's thesis, I solved a 50-year-old problem posed by Haagerup in his master's thesis and obtained the following theorem!

### Theorem ([Cho25], arXiv:2501.16832)

*Let  $A$  be a  $C^*$ -algebra, and  $F^*$  be a weakly\* closed convex hereditary subset of  $A^{*+}$ . Then, for any  $\omega \in A^{*+} \setminus F^*$ , there exists  $a \in A^+$  such that*

$$\omega(a) > 1 \quad \text{and} \quad \omega'(a) \leq 1 \quad \text{for all } \omega' \in F^*.$$

The original proof has been simplified thanks to N. Ozawa. The positivity condition  $a \geq 0$  is the non-trivial point. Applying the idea used to prove the above theorem, I could simplify the proof of the following.

### Theorem ([Haa75])

*For a subadditive weight  $\varphi$  on  $M$ , the followings are equivalent:*

- ▶  $\varphi$  is  $\sigma$ -lower semi-continuous.
- ▶  $\varphi$  is given by the pointwise supremum of normal positive linear functionals.

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## $C^*$ -algebras and von Neumann algebras

We will always denote a  $C^*$ -algebra and a vN algebra by  $A$  and  $M$  respectively.

A vN (or  $W^*$ -) algebra can be defined as a  $C^*$ -algebra  $M$  that admits a predual  $M_*$ , which is unique if it exists. The canonical weak\* topology on  $M$  is conventionally called the  $\sigma$ -weak topology, and normality usually means the  $\sigma$ -weak continuity. We focus on the dual pairs  $(A, A^*)$  and  $(M, M_*)$  and their weak/weak\* topologies. Note that for a convex subset of  $A$  or  $M_*$  it is norm closed iff it is weakly closed by the Hahn-Banach separation.

To see the measure theoretic analogues in today's talk, the following notes would be helpful to keep in mind.

### Example (Commutative $C^*$ -algebras)

A commutative  $A$  is  $*$ -isomorphic to  $C_0(X)$  for a locally compact Hausdorff space  $X$ . The positive part of its dual  $A^{*+}$  is given by the set of finite regular Borel measures on  $X$ .

### Example (Commutative vN algebras)

A commutative  $M$  is  $*$ -isomorphic to  $L^\infty(X, \mu)$  for a localizable measure space  $(X, \mu)$ . The positive part of its predual  $M_*^+$  is isomorphic to  $L^1(X, \mu)$ , which can be identified with the set of finite measures on  $X$  absolutely continuous with respect to  $\mu$ . Note that every  $\sigma$ -finite measure is a localizable.

# Definitions on weights

## Definition (Weights and subadditive weights)

A *weight* on  $A$  is a homogeneous additive functional  $\varphi : A^+ \rightarrow [0, \infty]$ .

A *subadditive weight* on  $A$  is a homogeneous subadditive functional  $\varphi : A^+ \rightarrow [0, \infty]$ .

## Definition (Properties of weights)

For a weight  $\varphi$  on  $A$ , we say it is

- (i) *faithful* if  $\varphi(a) = 0$  implies  $a = 0$  for  $a \in A^+$ ,
- (ii) *densely defined* if  $\varphi^{-1}([0, \infty))$  is norm dense in  $A^+$ ,
- (iii) *lower semi-continuous* if  $\varphi^{-1}([0, 1])$  is norm closed in  $A^+$ .

For a weight  $\varphi$  on  $M$ , we say it is

- (ii') *semi-finite* if  $\varphi^{-1}([0, \infty))$  is  $\sigma$ -weakly dense in  $M^+$ ,
- (iii') *normal* if  $\varphi^{-1}([0, 1])$  is  $\sigma$ -weakly closed in  $M^+$ .

A positive linear functional gives rise to a weight. Normality can be regarded as the generalization of the countable additivity of measures. Note that the  $\sigma$ -weak lower semi-continuity can be understood as a restatement of the Fatou lemma.

# Motivating examples for weights

## Example (Localizable measures)

A localizable measure  $\mu$  is always a faithful semi-finite normal weight on  $L^\infty(\mu)$ . In fact, every  $M$  admits a faithful semi-finite normal weight.

## Example (Radon measures)

Densely defined lower semi-continuous weights on  $C_0(X)$  for a locally compact Hausdorff  $X$  are exactly positive linear functionals on  $C_c(X)$ , and are exactly locally finite inner regular Borel measures on  $X$ . Every densely defined subadditive weight on a unital  $A$  is bounded, so there is  $A$  without a faithful densely defined lower semi-continuous weight.

## Example (Gelfand-Naimark-Segal representations)

There is a one-to-one correspondence between weights on  $A$  and unitary equivalence classes of *semi-cyclic representations* of  $A$ , which is defined as a representation  $\pi : A \rightarrow B(H)$  equipped with a partially defined left  $A$ -linear map  $\Lambda : A \rightarrow H$  of dense range. The corresponding weight  $\varphi$  on  $M$  is normal iff  $\pi$  is normal and  $\Lambda$  is  $\sigma$ -weakly closed. A densely defined lower semi-continuous weight  $\varphi$  on  $A$  gives rise to a faithful semi-finite normal weight on the  $\sigma$ -weak closure  $A''$  in the associated semi-cyclic representation to  $\varphi$ .

# Equivalent characterizations for normality of weights

## Theorem ([Haa75])

For a weight  $\varphi$  on  $M$ , the followings are all equivalent.

- (1)  $\varphi$  is completely additive for positive elements;

$$\varphi\left(\sum_i x_i\right) = \sum_i \varphi(x_i), \quad x_i \in M^+.$$

- (2)  $\varphi$  preserves directed suprema;

$$\varphi\left(\sup_i x_i\right) = \sup_i \varphi(x_i), \quad x_i \uparrow \sup_i x_i \text{ in } M^+.$$

- (3)  $\varphi$  is  $\sigma$ -weakly lower semi-continuous;

$$\varphi\left(\lim_i x_i\right) \leq \liminf_i \varphi(x_i), \quad x_i \rightarrow \lim_i x_i \text{ } \sigma\text{-weakly in } M^+.$$

- (4)  $\varphi$  is given by the pointwise supremum of normal positive linear functionals;

$$\varphi(x) = \sup_{\omega \leq \varphi, \omega \in M_*^+} \omega(x), \quad x \in M^+.$$

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear.



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## First problem

### Problem (1.10)

*For a subadditive weight on  $M$ , is it  $\sigma$ -weakly lower semi-continuous if it preserves directed suprema?*

Haagerup proved  $(1) \Rightarrow (3)$  directly without an intermediate step  $(2)$ . He proved first for  $\sigma$ -finite  $\text{vN}$  algebras, and extended the result to general  $\text{vN}$  algebras.

### Definition

A  $\text{vN}$  algebra is called  $\sigma$ -finite or *countably decomposable* if every orthogonal family of non-zero projections is countable, or equivalently, it admits a faithful normal state.

### Theorem ([Haa75])

*For a weight on  $\sigma$ -finite  $M$ ,  $(1) \Rightarrow (3)$  holds.*

*For a subadditive weight on general  $M$ ,  $(1) \Rightarrow (3)$  holds if every restriction on  $\sigma$ -finite  $\text{vN}$  subalgebras satisfies  $(1) \Rightarrow (3)$ .*

*Therefore, for a weight on general  $M$ ,  $(1) \Rightarrow (3)$  holds.*

Thus, it is enough to solve the problem in the  $\sigma$ -finite case.

## Second problem

### Problem (1.11)

*For a weight on  $M$ , is it normal if it is normal on every commutative  $vN$  subalgebra?*

### Theorem ([Dix53])

*For a positive linear functional  $\omega$  on  $M$ , the followings are all equivalent.*

- (0)  $\omega$  is completely additive for orthogonal projections.
- (1)  $\omega$  is completely additive for positive elements.
- (2)  $\omega$  preserves directed suprema.
- (3)  $\omega$  is  $\sigma$ -weakly continuous.

*In particular, a (positive) linear functional on  $M$  is normal if it is normal on every commutative  $vN$  subalgebra by (0) $\Rightarrow$ (1), and it is used to prove some equivalent characterizations for weak compactness in  $M_*$ .*

### Example

(0) $\Rightarrow$ (1) is false for weights. Define a weight  $\varphi$  for  $x = (x_n)_n \in \ell^\infty(\mathbb{N})^+$  such that  $\varphi(x) := \sum_n x_n$  if  $x \in c_c(\mathbb{N})$  and  $\varphi(x) := \infty$  otherwise. Then, it gives a counterexample.

## Third problem

### Problem (2.7)

*Does the positive bipolar theorem hold for dual  $C^*$ -algebras? See (d) in the below.*

### Definition

Let  $(E, E^*)$  be a dual pair of (directed partially) ordered real vector spaces such that  $E^+$  and  $E^{*+}$  are mutually dual cones, i.e.  $E^{*+} = \{x^* \in E^* : x^*(x) \geq 0 \text{ for } x \in E\}$ . For  $F \subset E^+$ , we say it is *hereditary* if  $0 \leq x \leq y \in F$  implies  $x \in F$ , and its *positive polar* is the set

$$F^{r+} := (F^r)^+ = \{x^* \in E^* : \sup_{x \in F} x^*(x) \leq 1\}^+ = \{x^* \in E^{*+} : \sup_{x \in F} x^*(x) \leq 1\}.$$

### Theorem ((a)~(c) in [Haa75], (d) in [Cho25])

*Consider the ordered real dual pairs  $(M^{sa}, M_*^{sa})$  and  $(A^{sa}, A^{sa*})$  of self-adjoint parts.*

- (a) *If  $F$  is a  $\sigma$ -weakly closed convex hereditary subset of  $M^+$ , then  $F = F^{r+r+}$ .*
- (b) *If  $F_*$  is a norm closed convex hereditary subset of  $M_*^+$ , then  $F_* = F_*^{r+r+}$ .*
- (c) *If  $F$  is a norm closed convex hereditary subset of  $A^+$ , then  $F = F^{r+r+}$ .*
- (d) ***If  $F^*$  is weakly\* closed convex hereditary subset of  $A^{*+}$ , then  $F^* = (F^*)^{r+r+}$ .***

## Corollaries of third problem

### Corollary ([Haa75])

*For a subadditive weight  $\varphi$  on  $M$ , (3) $\Rightarrow$ (4) holds.*

(3)  $\varphi$  is  $\sigma$ -lower semi-continuous.

(4)  $\varphi$  is given by the pointwise supremum of normal positive linear functionals.

*Proof.* Define

$$F := \{x \in M^+ : \varphi(x) \leq 1\}, \quad F_* := \{\omega \in M_*^+ : \omega \leq \varphi\}.$$

Then,  $F_* = F^{r+}$  by definition, and (4) is equivalent to  $F = F_*^{r+}$ . Since  $F$  is  $\sigma$ -weakly closed by the  $\sigma$ -weak lower semi-continuity of  $\varphi$ , and since  $F$  is clearly convex and hereditary by definition of subadditive weights, so we are done by the part (a).  $\square$

Using (c) instead of (a), we can simplify the proof of the following theorem.

### Corollary (Combes, 1968)

*A lower semi-continuous subadditive weight on  $A$  is given by the pointwise supremum of positive linear functionals.*

# Corollaries of third problem

## Corollary

There are one-to-one correspondences

$$\left\{ \begin{array}{c} \text{normal} \\ \text{subadditive} \\ \text{weights on } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \sigma\text{-weakly closed} \\ \text{convex hereditary} \\ \text{subsets of } M^+ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{norm closed} \\ \text{convex hereditary} \\ \text{subsets of } M_*^+ \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{lower semi-} \\ \text{continuous} \\ \text{subadditive} \\ \text{weights on } A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{norm closed} \\ \text{convex hereditary} \\ \text{subsets of } A^+ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{weakly}^* \text{ closed} \\ \text{convex hereditary} \\ \text{subsets of } A^{*+} \end{array} \right\}$$

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## Proof of (a)

## Definition (Suppression by the one-parameter family of functional calculi)

For  $\delta > 0$ , we define a function  $f_\delta : (-\delta^{-1}, \infty) \rightarrow \mathbb{R}$  such that

$$f_\delta(t) := t(1 + \delta t)^{-1}, \quad t > -\delta^{-1}.$$

They are operator monotone,  $\sigma$ -strongly continuous, and has the semi-group property.

*Proof sketch of (a) by Haagerup.* Since

$$\begin{aligned} F^{r+} &= F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r, \\ F^{r+r+} &= (F - M^+)^{rr+} = (\overline{F - M^+})^+ \end{aligned}$$

by the usual real bipolar theorem, it suffices to show  $(\overline{F - M^+})^+ \subset F$ .

**Heuristics.** Let  $x \in (\overline{F - M^+})^+$  with nets  $x_i$  and  $y_i$  such that  $x_i \rightarrow x$   $\sigma$ -weakly in  $M$  and  $x_i \leq y_i \in F$  for all  $i$ . Observe that if  $x_i$  were bounded by  $r > 0$ , then assuming  $x_i \rightarrow x$   $\sigma$ -strongly and  $f_\delta(y_i) \rightarrow y_\delta$   $\sigma$ -weakly for each  $0 < \delta < r^{-1}$ , we get

$$x_i \leq y_i \in F, \quad f_\delta(x_i) \leq f_\delta(y_i) \in F, \quad 0 \leq f_\delta(x) \leq y_\delta \in F, \quad f_\delta(x) \in F, \quad x \in F.$$

The boundedness of  $x_i$  is necessary to define  $f_\delta(x_i)$  for  $\delta$  independently of  $i$ .



## Proof of (a)

**Question.** How can we remove the boundedness condition of  $x_i$ ?

**Solution.** We use the Krein-Šmulian theorem. Define

$$G := \{x \in M^{sa} : \text{for any sufficiently small } \delta > 0, f_\delta(x) \in F - M^+\}.$$

It is enough to show

$$F - M^+ \subset G, \quad G^+ \subset F, \quad \overline{G} \subset G,$$

and the first two are clear. To apply the Krein-Šmulian theorem, fix  $r > 0$  and let  $M_r := \{x \in M : \|x\| \leq r\}$ . The proof of the  $\sigma$ -weak closedness is divided into the two steps:  $G \cap M_r$  is  $\sigma$ -strongly closed,  $G \cap M_r$  is convex. After proving these,  $\overline{G} \subset G$  by the Krein-Šmulian theorem. □

## Proof of (b)

### Definition (Bounded commutant Radon-Nikodym derivatives)

Let  $\psi \in M_*^+$  and let  $\pi : M \rightarrow B(H)$  be associated cyclic representation to  $\psi$  with the canonical cyclic vector  $\Omega \in H$ . Then, we have a positive bounded linear map  $\theta : \pi(M)' \rightarrow M_*$ , which we call the *RN map*, defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega \rangle, \quad h \in \pi(M)', \quad x \in M.$$

If  $\omega \in M_*$  satisfies  $|\omega(x)| \leq \psi(x)$  for all  $x \in M^+$ , then  $\theta^{-1}(\omega)$  is uniquely defined and  $\|\theta^{-1}(\omega)\| \leq 1$ , which has  $\theta^{-1}(\omega) = d\omega/d\psi$  when  $M$  is commutative.

*Proof sketch of (b).* It is enough to prove  $(\overline{F_* - M_*^+})^+ \subset F_*$ . Let  $\omega \in (\overline{F_* - M_*^+})^+$  with sequences  $\omega_n$  and  $\varphi_n$  such that  $\omega_n \rightarrow \omega$  in norm of  $M_*$  and  $\omega_n \leq \varphi_n \in F_*$  for all  $n$ . We may assume  $\|\omega_n - \omega\| \leq 2^{-n}$  for all  $n$  by passing to a subsequence. Define

$$\psi := \omega + \sum_n [\omega_n - \omega] + \sum_n 2^{-n} \frac{\varphi_n}{1 + \|\varphi_n\|} \in M_*^+,$$

and let  $\theta : \pi(M)' \rightarrow M_*$  be the RN map associated to  $\psi$ . Since  $-\psi \leq \omega_n \leq \psi$  implies the boundedness  $\|\theta^{-1}(\omega_n)\| \leq 1$  for all  $n$ , the weak convergence  $\omega_n \rightarrow \omega$  in  $M_*$  implies the convergence  $\theta^{-1}(\omega_n) \rightarrow \theta^{-1}(\omega)$  in the weak operator topology of  $\pi(M)'$ .

By the Mazur lemma, we can take a net  $\omega_i$  in the convex hull of  $\omega_n$  such that  $\theta^{-1}(\omega_i) \rightarrow \theta^{-1}(\omega)$  strongly in  $\pi(M)'$ , and the corresponding  $\varphi_i \in F^*$  can be defined such that  $\omega_i \leq \varphi_i$  for all  $i$ . (In fact, the net  $\omega_i$  can be taken to be a sequence because the commutant is  $\sigma$ -finite by the existence of the separating vector, but it is not necessary in here.) For each  $i$  and  $0 < \delta < 1$ , define

$$\omega_\delta := \theta(f_\delta(\theta^{-1}(\omega))), \quad \omega_{i,\delta} := \theta(f_\delta(\theta^{-1}(\omega_i))), \quad \varphi_{i,\delta} := \theta(f_\delta(\theta^{-1}(\varphi_i))).$$

Then, the strong convergence  $f_\delta(\theta^{-1}(\omega_i)) \rightarrow f_\delta(\theta^{-1}(\omega))$  in  $\pi(M)'$  implies  $\omega_{i,\delta} \rightarrow \omega_\delta$  weakly in  $M_*$ , and the strong convergence  $f_\delta(\theta^{-1}(\omega)) \rightarrow \theta^{-1}(\omega)$  in  $\pi(M)'$  implies  $\omega_\delta \rightarrow \omega$  weakly in  $M_*$  as  $\delta \rightarrow 0$ . If we define  $\varphi_\delta := \theta(\lim_i f_\delta(\theta^{-1}(\varphi_i)))$  by taking a subnet or a cofinal ultrafilter, then  $\varphi_{i,\delta} \rightarrow \varphi_\delta$  weakly in  $M_*$ . Since  $\omega_i \leq \varphi_i$  and  $0 \leq \varphi_{i,\delta} \leq \varphi_i \in F_*$ , we get

$$\omega_{i,\delta} \leq \varphi_{i,\delta} \in F_*, \quad 0 \leq \omega_\delta \leq \varphi_\delta \in F_*, \quad \omega_\delta \in F_*, \quad \omega \in F_*.$$

This completes the proof. □

## Strategies for (d)

Let  $\omega_i$  and  $\varphi_i$  be nets in  $A^{*sa}$  such that  $\omega_i \rightarrow \omega$  weakly\* in  $A^*$  and  $\omega_i \leq \varphi_i \in F^*$  for all  $i$ .

**Question 1.** How can we choose the reference  $\psi$  for the Radon-Nikodym?

**Solution 1.** Take  $\psi_i$  dynamically depending on  $\omega_i$ .

**Question 2.** How can we commute the weak\* limit of  $\omega_i$  and  $f_\delta$  without strong topology?

**Solution 2.** Approximate  $f_\delta$  with affine functions by

$$t - \delta^{\frac{1}{2}} \leq f_\delta(t) \leq t, \quad |t| \leq 2^{-1} \delta^{-\frac{1}{4}},$$

$$(1 + \delta^{-1})t \leq f_\delta(t) \leq t, \quad 0 \leq t \leq 1.$$

## Proof of (d)

*Proof of (d).* It suffices to show  $(\overline{F^* - A^{*+}})^+ \subset F^*$ . Define

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{l} \text{there is } \psi \in A^{*+}, \text{ and there is } \varphi_\delta \in F^* \\ \text{for any sufficiently small } \delta > 0, \text{ such that} \\ \|\psi\| \leq 1, \|\varphi_\delta\| \leq \delta^{-1}, \text{ and } \omega \leq \varphi_\delta + \delta^{\frac{1}{2}} \psi \end{array} \right\}.$$

It suffices to show  $F^* - A^{*+} \subset G^*$ ,  $G^{*+} \subset F^*$ , and  $\overline{G^*} \subset G^*$ .

## Proof of (d)

*Step 1.* Let  $\omega \in F^* - A^{*+}$ . Take  $\varphi \in F^*$  such that  $\omega \leq \varphi$ . Define, for  $\delta > 0$ ,

$$\psi := \frac{[\omega]}{1 + \|\omega\|} + \frac{\varphi}{(1 + \|\omega\|)(1 + \|\varphi\|)}, \quad \varphi_\delta := \theta(f_\delta(\theta^{-1}(\varphi))),$$

where  $\theta$  is the RN map associated to  $\psi$ . The norm conditions  $\|\psi\| \leq 1$  and  $\|\varphi_\delta\| \leq \delta^{-1}$  are easily checked. For sufficiently small  $\delta > 0$  such that  $\|\theta^{-1}(\omega)\| \leq 1 + \|\omega\| \leq 2^{-1}\delta^{-\frac{1}{4}}$  and  $\delta \leq 1$ , we have

$$\theta^{-1}(\omega) \leq f_\delta(\theta^{-1}(\omega)) + \delta^{\frac{1}{2}} \leq f_\delta(\theta^{-1}(\varphi)) + \delta^{\frac{1}{2}},$$

so  $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$  and  $\omega \in G^*$ .

## Proof of (d)

*Step 2.* Let  $\omega \in G^{*+}$ . Take  $\psi \in A^{*+}$  and  $\varphi_\delta \in F^*$  such that  $\|\psi\| \leq 1$ ,  $\|\varphi_\delta\| \leq \delta^{-1}$ ,  $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$ , for any sufficiently small  $\delta > 0$ . Let  $\psi_\delta := \omega + \delta\varphi + \psi$ , and let  $\theta_\delta$  be the associated RN map. For any fixed  $\delta' > 0$ , since  $0 \leq \theta_\delta^{-1}(\omega) \leq 1$ , we have

$$\begin{aligned} 0 &\leq (1 + \delta')^{-1} \theta_\delta^{-1}(\omega) \leq f_{\delta'}(\theta_\delta^{-1}(\omega)) \leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta + \delta^{\frac{1}{2}}\psi)) \\ &\leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta) + \delta^{\frac{1}{2}}) \leq f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta)) + \delta^{\frac{1}{2}}, \end{aligned}$$

and it implies

$$0 \leq (1 + \delta')^{-1} \omega \leq \theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) + \delta^{\frac{1}{2}}\psi_\delta.$$

Since  $\|\psi_\delta\| \leq \|\omega\| + 2$  is bounded and  $\theta_\delta(f_{\delta'}(\theta_\delta^{-1}(\varphi_\delta))) \in F^*$  is also bounded for fixed  $\delta'$  as  $\delta \rightarrow 0$ , by considering the limit along a cofinal ultrafilter on the set of  $\delta$ , we have  $(1 + \delta')^{-1} \omega \in F^*$ , so  $\delta' \rightarrow 0$  gives  $\omega \in F^*$ .

## Proof of (d)

Step 3. To show  $G^*$  is weakly\* closed, we claim for any  $r > 0$  that

$$\overline{(F^* - A^{*+}) \cap A_{2r}^*} \subset G^*, \quad G^* \cap A_r^* \subset \overline{(F^* - A^{*+}) \cap A_{2r}^*},$$

where  $A_r^* := \{\omega \in A^* : \|\omega\| \leq r\}$ . If these are true, then

$$G^* \cap A_r^* = \overline{(F^* - A^{*+}) \cap A_{2r}^*} \cap A_r^*$$

is weakly\* closed and convex in  $A^*$  for all  $r > 0$ , so the Krein-Šmulian theorem shows the claim.



## Proof of (d)

Let  $\omega_i \in (F^* - A^{*+}) \cap A_{2r}^*$  be a net such that  $\omega_i \rightarrow \omega$  weakly\* in  $A^*$ . Following the proof of  $F^* - A^{*+} \subset G^*$ , we can take  $\psi_i \in A^{*+}$  and  $\varphi_{i,\delta} \in F^*$  such that  $\|\psi_i\| \leq 1$ ,  $\|\varphi_{i,\delta}\| \leq \delta^{-1}$ ,  $\omega_i \leq \varphi_{i,\delta} + \delta^{\frac{1}{2}}\psi_i$ , for uniformly sufficiently small  $\delta$  such that  $1 + 2r \leq 2^{-1}\delta^{-\frac{1}{4}}$  because  $\|\omega_i\|$  is bounded by  $2r$ . Since the three conditions are preserved by the weak\* convergence, taking the limit along a cofinal ultrafilter on the index set of  $i$ , we can obtain limit points  $\psi$  and  $\varphi_\delta$  so that  $\omega \in G^*$ .

Let  $\omega \in G^* \cap A_r^*$ . Take  $\psi \in A^{*+}$  and  $\varphi_\delta \in F^*$  with  $\|\psi\| \leq 1$ ,  $\|\varphi_\delta\| \leq \delta^{-1}$ ,  $\omega \leq \varphi_\delta + \delta^{\frac{1}{2}}\psi$ , for any sufficiently small  $\delta > 0$ . If  $\delta^{\frac{1}{2}} < r$ , then  $\omega - \delta^{\frac{1}{2}}\psi \in (F^* - A^{*+}) \cap A_{2r}^*$  converges to  $\omega$  weakly\* in  $A^*$  as  $\delta \rightarrow 0$ , we have  $\omega \in \overline{(F^* - A^{*+}) \cap A_{2r}^*}$ .  $\square$

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