

# Contents

<b>1</b>	<b>Topological group action</b>	<b>2</b>
<b>2</b>	<b>Hyperbolic plane geometry</b>	<b>3</b>
2.1	Fuchsian groups . . . . .	3
2.2	Fundamental domain . . . . .	3
2.3	Side paring and cycle conditions . . . . .	5
2.4	The Poincaré polygon theorem . . . . .	6
2.5	Geometric structures . . . . .	8
<b>3</b>	<b>Universal coefficient theorem</b>	<b>10</b>
<b>4</b>	<b>Fundamental differential geometry</b>	<b>12</b>
4.1	Manifold and Atlas . . . . .	12
4.2	Definition of Differentiable Structure . . . . .	13
4.3	Curves . . . . .	14
4.4	Connection computation . . . . .	14
4.5	Geodesic equation . . . . .	15
<b>5</b>	<b>Vector calculus on spherical coordinates</b>	<b>16</b>
<b>6</b>	<b>Bundles</b>	<b>17</b>

# 1 Topological group action

mit 18.786 2018 number theory II

**1.1.** Let  $G$  be a topological group acting on a topological space  $X$ . Let  $p : X \rightarrow X/G$  be the quotient map.

- (a)  $p^{-1}(p(A)) = \bigcup_{g \in G} gA$  for any  $A \subset X$ .
- (b)  $p$  is open.
- (c) If  $x \neq gx$ , then there is an open neighborhood  $U$  of  $x$  such that  $gU$  is disjoint to  $U$ .

*Proof.* (c) Since  $X$  is Hausdorff, there is disjoint open neighborhoods  $U_0$  and  $U_1$  respectively of  $x$  and  $gx$ . Then,  $U := g^{-1}(gU_0 \cap U_1) \subset U_0$  and  $gU = gU_0 \cap U_1 \subset U_1$  are disjoint.  $\square$

**1.2.** Let  $f : X \rightarrow Y$  be continuous. We say  $f$  is *proper* if  $f^{-1}(K)$  is compact for compact  $K$ . We say  $f$  is *Bourbaki-proper* if it is closed and proper. If  $X$  is Hausdorff and  $Y$  is locally compact, then two notions are equivalent.

**1.3** (Properly discontinuous actions). Let  $G$  be a topological group acting on a topological space  $X$ . Let  $p : X \rightarrow X/G$  be the quotient map. This action is called *properly discontinuous* if for every compact  $K \subset X$  only finite  $gK$  intersect  $K$ .

- (a) If  $\Gamma$  is discrete, then orbits are locally finite.
- (b) If orbits are locally finite, then  $\Gamma$  acts properly discontinuously.
- (c) Suppose the stabilizer is always finite. If  $\Gamma$  act properly discontinuously then  $\Gamma$  is discrete.

compact stabilizer condition: If  $\Gamma$  is discrete, then for  $z \in \mathbb{H}^2$  and a compact set  $K \subset \mathbb{H}^2$ ,  $\{g \in \Gamma : gz \in K\}$  is finite, so properly discontinuous.

**1.4** (Covering space actions). Let  $G$  be a topological group acting on a topological space  $X$ . Let  $p : X \rightarrow X/G$  be the quotient map. This action is called a *covering space action* if every  $x \in X$  has a neighborhood  $U$  such that  $gU$  are all disjoint for  $g \in G$ .

- (a) A properly discontinuous and free action is a covering space action, if  $X$  is locally compact and Hausdorff.

(b) A covering space action is properly discontinuous.

(c) A covering space action is free.

*Proof.* (a) Fix  $x \in X$  and let  $K$  be a compact neighborhood of  $x$ . By the proper discontinuity, there is a finite subset  $F \subset G$  such that  $gK$  intersects  $K$  only for  $g \in F$ . Because the action is free, for every  $g \in F \setminus \{1\}$  there is an open neighborhood  $U_g$  of  $x$  such that  $gU_g \cap U_g = \emptyset$ . Then,  $U := K^\circ \cap \bigcap_{g \in F \setminus \{1\}} U_g$  satisfies  $gU \cap U = \emptyset$ .

(b)

□

## 2 Hyperbolic plane geometry

### 2.1 Fuchsian groups

Classification of elements. An abelian Fuchsian group is cyclic. Elliptic point is discrete

### 2.2 Fundamental domain

**2.1 (Fundamental domain).** Let  $\Gamma$  be a Fuchsian group. An open set  $D \subset \mathbb{H}^2$  is called a *fundamental domain* of  $\Gamma$  if

(i)  $\{g(D) : g \in \Gamma\}$  are pairwise disjoint,

(ii)  $\{g(\overline{D}) : g \in \Gamma\}$  covers  $\mathbb{H}^2$ .

**2.2 (Dirichlet domain).** Let  $\Gamma$  be a Fuchsian group. Let  $z_0 \in \mathbb{H}^2$  be a point that is not fixed by any isometry in  $\Gamma \setminus \{e\}$ , i.e. a non-elliptic point. The *Dirichlet domain* of  $\Gamma$  with *center*  $z_0$  is defined as the set

$$D := \bigcap_{g \in \Gamma \setminus \{e\}} \{z \in \mathbb{H}^2 : d(z, z_0) < d(z, gz_0)\}.$$

We denote by  $\overline{D}$  and  $\partial D$  the closure and the boundary of  $D$  in  $\overline{\mathbb{H}^2}$ .

(a) There exists a non-elliptic point in  $\mathbb{H}^2$ .

(b)  $\{g(\overline{D}) : g \in \Gamma\}$  is a locally finite. It is called the *Dirichlet tessellation*.

(c)  $D$  is a convex fundamental domain of  $\Gamma$ .

*Proof.* (a) Elliptic points are countably many.

(b) There are finitely many  $g \in \Gamma$  satisfying  $B(z_0, r) \cap g(\bar{D}) \neq \emptyset$ , since this condition implies  $gz_0 \in B(z_0, 2r)$ .  $\square$

**2.3** (Boundary and edges of Dirichlet domain). Let  $\Gamma$  be a Fuchsian group, and let  $D$  be a Dirichlet domain of  $\Gamma$  with center  $z_0$ . A subset  $l \subset \mathbb{H}^2$  is called an *edge* of  $D$  if  $l = g(\bar{D}) \cap \bar{D}$  for some  $g \in \Gamma \setminus \{e\}$  and  $|l| > 1$ .

(a) For  $g \in \Gamma \setminus \{e\}$ , the set  $g(\bar{D}) \cap \bar{D}$  has the three cases: the null set, one point, or a geodesic segment.

(b) If  $l$  is an edge, then there is unique  $g \in \Gamma \setminus \{e\}$  such that  $l = g(\bar{D}) \cap \bar{D}$ .

(c) The intersection of two distinct edges is one point or the null set.

(d) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{g \in \Gamma \setminus \{e\}} g(\bar{D}) \cap \bar{D}.$$

(e) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{l: \text{edge}} l.$$

*Proof.* (d) Let  $z \in \partial D \cap \mathbb{H}^2$ . Since  $d(z, z_0) \leq d(z, gz_0)$  for all  $g \in \Gamma \setminus \{e\}$  but  $d(z, z_0) \geq d(z, gz_0)$  for some  $g \in \Gamma \setminus \{e\}$ , there is  $g \in \Gamma \setminus \{e\}$  such that  $d(z, z_0) = d(z, gz_0)$ . By sending  $z_0$  and  $gz_0$  to  $\pm 1 + i$  with an isometry so that  $z$  is sent to a point on a imaginary axis, we can check for each  $n$  that we have  $B(z, 1/n) \cap (\mathbb{H}^2 \setminus \bar{D}) \neq \emptyset$ . Since  $B(z, 1/n) \setminus \bar{D}$  is a non-empty open set in  $\mathbb{H}^2 \setminus \bar{D}$ , and since

$$\mathbb{H}^2 \setminus \bar{D} \subset \mathbb{H}^2 \setminus D = \overline{\bigcup_{g \in \Gamma \setminus \{e\}} g(D)},$$

we can deduce that  $B(z, 1/n)$  intersects with  $g(D)$  for some  $g \in \Gamma \setminus \{e\}$ .

Combining this result with the local finiteness of  $\{g(D) : g \in \Gamma\}$ , the sequence of sets

$$\{g \in \Gamma \setminus \{e\} : B(z, 1/n) \cap g(D) \neq \emptyset\}$$

indexed by  $n$  consists of non-empty finite subsets of  $\Gamma \setminus \{e\}$  that are non-increasing. By the pigeonhole principle, there exists  $g \in \Gamma \setminus \{e\}$  such that  $B(z, 1/n) \cap g(D) \neq \emptyset$  for all  $n$ , which allows to extract a sequence  $z_n \in g(D)$  that converges to  $z$ , which implies  $z \in g(\bar{D})$ .

(e) Suppose  $z \in \partial D \cap \mathbb{H}^2$  is not contained in any edges. Let  $Z$  be the set of all  $g \in \Gamma \setminus \{e\}$  such that  $\{z\} = g(\overline{D}) \cap \overline{D}$ . For  $g \in \Gamma \setminus (Z \cup \{e\})$ ,  $g(\overline{D}) \cap \overline{D}$  is the null set, one point, or an edge, and any of possibility does not contain  $z$ . Therefore,

$$(\partial D \setminus \{z\}) \cap \mathbb{H}^2 = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \mathbb{H}^2$$

by the part (d). Change the restriction  $\mathbb{H}^2$  to a compact ball as

$$(\partial D \setminus \{z\}) \cap \overline{B(z, 1)} = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \overline{B(z, 1)}.$$

Then, the left-handed side is homeomorphic to  $[-1, 0) \cup (0, 1]$  or  $(-1, 1)$  since  $\partial D$  is homeomorphic to  $S^1$ , but the right-handed side is compact because the union becomes finite due to the local finiteness. This is a contradiction, so  $z$  is contained in an edge.  $\square$

**2.4 (Finitely generated Fuchsian group).** Let  $\Gamma$  be a Fuchsian group, and let  $D$  be a Dirichlet domain of  $\Gamma$  with center  $z_0$ . Let  $W$  be the set of all  $g \in \Gamma \setminus \{e\}$  such that  $g(\overline{D}) \cap \overline{D}$  is an edge.

- (a)  $W$  generates  $\Gamma$ .
- (b) If  $\Gamma$  is finitely generated, then  $W$  is finite.
- (c) If  $W$  is finite, then  $\Gamma$  is finitely generated.

**2.5 (Siegel's theorem).** Finite area iff finitely generated.

- (a) If  $\Gamma$  is finitely generated, then

$$\partial D = \bigcup_{l: \text{edge}} l.$$

## 2.3 Side pairing and cycle conditions

**2.6 (Side pairing condition).** Let  $\Gamma$  be a finitely generated Fuchsian group, and let  $D$  be a Dirichlet domain of  $\Gamma$  with center  $z_0$ . We have seen that  $\partial D$  consists of finitely many edges. A point  $v \in \partial D$  is called a *vertex* if it either

- (i) the intersection of two edges, or
- (ii) the fixed point of elliptic isometry  $g \in \Gamma$  of order two.

Let  $v_0, v_1, \dots, v_n = v_0$  be vertices, indexed along the boundary counterclockwise. A side is geodesic segments  $s_i$  connecting  $v_i$  and  $v_{i+1}$ .

- (a) For each side  $s$  of  $D$ , there is unique  $g_s \in \Gamma$  such that  $g_s^{-1}(s)$  is another side of  $D$ . The isometry  $g_s$  is called the *side pairing isometry* of the side  $s$ .
- (b) The side pairing isometry of  $g_s^{-1}(s)$  is  $g_s^{-1}$ .
- (c) The number of sides  $n$  is always even.

*Proof.* □

**2.7 (Cycle condition).** Let  $\Gamma$  be a finitely generated Fuchsian group, and let  $D$  be a Dirichlet domain of  $\Gamma$  with center  $z_0$ . Let  $V$  and  $S$  be the set of all vertices and sides of  $D$ , respectively. Define  $\sigma : V \rightarrow V$  and  $\sigma : S \rightarrow S$  which use same notation such that  $\sigma(v_i) = v_{j+1}$  and  $\sigma(s_i) := s_{j+1}$  where  $s_j = g_s^{-1}(s_i)$ . The map  $\sigma$  can be seen as an element of the symmetric group  $S_n$ .

- (a) Suppose  $v_0 \in \mathbb{H}^2$  and  $s = s_0$ . Let  $m$  be the minimal positive integer such that  $\sigma^m(s) = s$ . Then,  $g_{\sigma^{m-1}(s)} \cdots g_{\sigma(s)} g_s$  is either the identity or elliptic.
- (b) Suppose  $v_0 \in \partial \mathbb{H}^2$ .

**2.8 (Genus two surface).**

**2.9 (Modular group).** Let  $\Gamma = \text{PSL}(2, \mathbb{Z})$  be the modular group and choose the origin  $2i$ .  $v_0 = i$ ,  $v_1 = e^{\pi i/3}$ ,  $v_2 = \infty$ ,  $v_3 = e^{2\pi i/3}$ .  $g_{s_0} = S$ ,  $g_{s_1} = T$ ,  $g_{s_2} = T^{-1}$ ,  $g_{s_3} = S^{-1}$ .  $\sigma = (13)$ . The elliptic cycle condition:  $(0)$  defines  $SS = 1$ ,  $(13)$  defines  $(S^{-1}T)^3 = 1$

## 2.4 The Poincaré polygon theorem

**2.10 (Definition of polygon).** (a)

**2.11 (Side pairing identification).** Let  $P$  be an open polygon. Define cycles of each vertex. Let  $Y := \bar{P} / \sim$  and  $\tilde{Y} := (\Gamma \times \bar{P}) / \sim$ . Define  $\Pi : \tilde{Y} \rightarrow Y$ .

(a)

**2.12 (Elliptic cycle condition).** Let  $P$  be an open polygon with a side pairing identification. Let  $\Gamma$  be a subgroup of  $\text{Isom}^+(\mathbb{H}^2)$  generated by side pairing isometries of  $P$ .

- (a)  $P$  satisfies the elliptic cycle condition.

(b)  $D$  is a local homeomorphism.

(c)  $D$  is a covering map onto its image.

*Proof.* We claim  $p$  has the path lifting property, which is unique because it is a local homeomorphism. Let  $w : [0, 1] \rightarrow \text{im } p$ , and  $\tilde{w} : [0, \tau) \rightarrow (\Gamma \times \overline{D}) / \sim$  its maximal extension. Write  $\tilde{w}(t) = [g(t), z(t)]$  and  $w(\tau) = gz$ . Define  $\tilde{w}(\tau) := [g, z]$ . Then,

$$p\tilde{w}(\tau) = p([g, z]) = gz = w(\tau).$$

Let  $U$  be an open neighborhood of  $[g, z]$  such that  $p|_U$  is a homeomorphism and  $p(U)$  is open in  $\mathbb{H}^2$ . Then, as  $t \rightarrow \tau$ ,

$$p\tilde{w}(t) = w(t) \rightarrow w(\tau) = p\tilde{w}(\tau)$$

implies

$$\tilde{w}(t) \rightarrow \tilde{w}(\tau),$$

so  $\tilde{w} : [0, \tau] \rightarrow (\Gamma \times \overline{D}) / \sim$  is a continuous extension of  $w : [0, \tau] \rightarrow \mathbb{H}^2$ . Therefore,  $p$  is a local homeomorphism that has the unique path lifting property, so it is a covering map onto its image.  $\square$

**2.13** (Finite cycle condition). Let  $P$  be an open polygon with a side pairing identification. Let  $\Gamma$  be a subgroup of  $\text{Isom}^+(\mathbb{H}^2)$  generated by side pairing isometries of  $P$ . Consider  $D$  and  $\Pi$  such that

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{D} & \mathbb{H}^2 \\ \downarrow \Pi & & \\ Y & & . \end{array}$$

(a) If every cycle of finite points is finite, then  $\text{im } D$  is open.

(b) If every cycle is finite, then there is a metric  $\rho$  on  $Y$  such that  $[z_n] \rightarrow [z]$  in  $\rho$  if and only if  $h_n z_n \rightarrow z$  in  $\mathbb{H}^2$  for a sequence  $h_n \in \Gamma$ .

*Proof.*

$$\begin{aligned} \rho(x, y) &:= \inf \sum, \\ \inf_{h \in \Gamma} d(h^{-1}z, z') &= \rho([z], [z']) \end{aligned}$$

$\square$

**2.14** (Parabolic cycle condition). Let  $P$  be an open polygon with a side pairing identification such that every cycle is finite. Let  $\Gamma$  be a subgroup of  $\text{Isom}^+(\mathbb{H}^2)$  generated by side pairing isometries of  $P$ . Suppose

- (a)  $P$  satisfies the parabolic cycle condition,
- (b)  $M$  is a complete metric space.
- (c)  $D$  is surjective.

*Proof.* Let  $w \in \partial(\text{im } D)$  so that we have  $[g_n, z_n] \in (\Gamma \times \overline{P}) / \sim$  such that  $g_n z_n \rightarrow w$  in  $\mathbb{H}^2$ . Since  $g_n z_n$  is Cauchy,  $[z_n]$  is also Cauchy, so we have a limit  $[z_n] \rightarrow [z]$  in  $\overline{P} / \sim$ . Then, there exists a sequence  $h_n \in \Gamma$  such that  $h_n z_n \rightarrow z$  in  $\mathbb{H}^2$ , which implies  $g_n h_n^{-1} z \rightarrow w$  in  $\mathbb{H}^2$  and  $w \in \overline{\Gamma z}$ . Since  $\text{im } D$  is open and  $\overline{P} \subset \text{im } D$ , there is  $\varepsilon > 0$  such that  $B(z, \varepsilon) \subset \text{im } D$ . There is  $g \in \mathbb{H}^2$  such that  $d(gz, w) < \varepsilon$ , which implies  $g^{-1}w \in B(z, \varepsilon)$ . Because  $\Gamma$  acts on  $\text{im } D$ , we can conclude  $w \in \text{im } D$ .  $\square$

If  $P$  satisfies the cycle conditions, then  $\Gamma$  is discrete,  $\Gamma$  is given by the presentation. Also,  $P$  is a fundamental domain of  $\Gamma$  and  $Y = \mathbb{H}^2 / \Gamma$ .

## 2.5 Geometric structures

*Model geometry* is a minimal one among  $G$ -space  $X$  spaces that are connected, simply connected, transitive, and has compact stabilizers. Is the action analytic?

**Definition 2.1** (Pseudogroup). cover, restriction, locality composition, inverse

**Definition 2.2** ( $(G, X)$ -structure). Let  $X$  be a model geometry of  $G$ . For an analytic action.

**Definition 2.3** (Development and holonomy). Let  $M$  be a  $(G, X)$ -manifold and  $\tilde{M}$  the universal covering.

- (a)  $D : \tilde{M} \rightarrow X$ .
- (b)  $h : \pi_1(M) \rightarrow G$ .

surjectivity of a map from torsion-free discrete subgroups of  $G$  to complete  $(G, X)$ -manifolds? (up to homeomorphism, up to geometric structure)

**Definition 2.4** (Several definitions of hyperbolic manifolds). Let  $G = \text{Isom}^+(\mathbb{H}^n)$  and  $X$  a  $n$ -manifold. Then,  $X$  is a hyperbolic manifold if one of the following satisfied...?:

1. It admits a hyperbolic atlas, and it is “complete”
2. It is homeomorphic to  $\mathbb{H}^n / \Gamma$  for a torsion-free discrete subgroup  $\Gamma$  of  $G$ .



3. It is a geodesically complete Riemannian manifold with constant sectional curvature  $-1$ .

*Thurston geometry* is a three-dimensional model geometry on which a closed 3-manifold has a geometric structure modelled.  
oriented prime closed 3-manifolds

### 3 Universal coefficient theorem

**Lemma 3.1.** *Suppose we have a flat resolution*

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

*Then, we have a exact sequence*

$$\cdots \rightarrow 0 \rightarrow \operatorname{Tor}_1^R(A, B) \rightarrow P_1 \otimes B \rightarrow P_0 \otimes B \rightarrow A \otimes B \rightarrow 0.$$

**Theorem 3.2.** *Let  $R$  be a PID. Let  $C_\bullet$  be a chain complex of flat  $R$ -modules and  $G$  be a  $R$ -module. Then, we have a short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \operatorname{Tor}(H_{n-1}(C), G) \rightarrow 0,$$

*which splits, but not naturally.*

1. We have a short exact sequence of chain complexes

$$0 \rightarrow Z_\bullet \rightarrow C_\bullet \rightarrow B_{\bullet-1} \rightarrow 0$$

where every morphism in  $Z_\bullet$  and  $B_\bullet$  are zero. Since modules in  $B_{\bullet-1}$  are flat, we have a short exact sequence

$$0 \rightarrow Z_\bullet \otimes G \rightarrow C_\bullet \otimes G \rightarrow B_{\bullet-1} \otimes G \rightarrow 0$$

and the associated long exact sequence

$$\rightarrow H_n(B; G) \rightarrow H_n(Z; G) \rightarrow H_n(C; G) \rightarrow H_{n-1}(B; G) \rightarrow H_{n-1}(Z; G) \rightarrow$$

where the connecting homomorphisms are of the form  $(i_n: B_n \rightarrow Z_n) \otimes 1_G$  (It is better to think diagram chasing than a natural construction). Since morphisms in  $B$  and  $Z$  are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n(C; G) \rightarrow B_{n-1} \otimes G \rightarrow Z_{n-1} \otimes G \rightarrow .$$

Since

$$0 \rightarrow \operatorname{Tor}_1^R(H_n, G) \rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n \otimes G \rightarrow 0$$

for all  $n$ , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \operatorname{Tor}_1^R(H_{n-1}, G) \rightarrow 0.$$

For splitting,

□

2. Since  $R$  is PID, we can construct a flat resolution of  $G$

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow G \rightarrow 0.$$

Since modules in  $C_\bullet$  are flat so that the tensor product functors are exact and  $P_1 \rightarrow P_0$  and  $P_0 \rightarrow G$  induce the chain maps, we have a short exact sequence of chain complexes

$$0 \rightarrow C_\bullet \otimes P_1 \rightarrow C_\bullet \otimes P_0 \rightarrow C_\bullet \otimes G \rightarrow 0.$$

Then, we have the associated long exact sequence

$$\rightarrow H_n(C; P_1) \rightarrow H_n(C; P_0) \rightarrow H_n(C; G) \rightarrow H_{n-1}(C; P_1) \rightarrow H_{n-1}(C; P_0) \rightarrow .$$

Since flat tensor product functor commutes with homology functor from chain complexes, we have

$$\rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n(C; G) \rightarrow H_{n-1} \otimes P_1 \rightarrow H_{n-1} \otimes P_0 \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(G, H_n) \rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n \otimes G \rightarrow 0$$

for all  $n$ , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(G, H_{n-1}) \rightarrow 0.$$

□

Proof 3. By tensoring  $G$ , we get the following diagram.

$$\begin{array}{ccccc}
H_n \otimes G & & & & H_{n-1} \otimes G \\
& \searrow & & & \nearrow \\
& \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G & \\
& \nearrow & \searrow & \nearrow & \searrow \\
C_n \otimes G & & \text{im } \partial_n \otimes G & & C_{n-1} \otimes G \\
& \nearrow & \nwarrow & \nearrow & \\
& \text{Tor}_1(H_{n-1}, G) & & & 
\end{array}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernels are preserved, but monomorphisms and kernels are not. Especially,  $\text{coker } \partial_{n+1} \otimes G = \text{coker}(\partial_{n+1} \otimes 1_G)$  is important.

Consider the following diagram.

$$\begin{array}{ccccc}
 H_n(C; G) & & H_n \otimes G & & \\
 \searrow & & \downarrow & & \\
 & & \text{coker } \partial_{n+1} \otimes G & \xrightarrow{\quad} & \text{ker } \partial_{n-1} \otimes G \\
 & & \downarrow & \nearrow & \uparrow \text{monic!} \\
 & & \text{im } \partial_n \otimes G & \xrightarrow{\quad} & \text{im}(\partial_n \otimes 1_G) \\
 & \nearrow & & & \uparrow \\
 \text{Tor}_1(H_{n-1}, G) & & & & C_{n-1} \otimes G
 \end{array}$$

Since  $\text{ker } \partial_{n-1}$  is free,

If we show  $\text{im}(\partial_n \otimes 1_G) \rightarrow \text{ker } \partial_{n-1} \otimes G$  is monic, then we can get

$$\begin{aligned}
 H_n(C; G) &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{im}(\partial_n \otimes 1_G)) \\
 &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{ker } \partial_{n-1} \otimes G).
 \end{aligned}$$

## 4 Fundamental differential geometry

### 4.1 Manifold and Atlas

**Definition 4.1.** A *locally Euclidean space*  $M$  of dimension  $m$  is a Hausdorff topological space  $M$  for which each point  $x \in M$  has a neighborhood  $U$  homeomorphic to an open subset of  $\mathbb{R}^d$ .

**Definition 4.2.** A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

**Definition 4.3.** A *chart* or a *coordinate system* for a locally Euclidean space is a map  $\varphi$  is a homeomorphism from an open set  $U \subset M$  to an open subset of  $\mathbb{R}^d$ . A chart is often written by a pair  $(U, \varphi)$ .

**Definition 4.4.** An *atlas*  $\mathcal{F}$  is a collection  $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$  of charts on  $M$  such that  $\bigcup_{\alpha \in A} U_\alpha = M$ .

**Definition 4.5.** A *differentiable manifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

## 4.2 Definition of Differentiable Structure

**Definition 4.6.** An atlas  $\mathcal{F}$  is called *differentiable* if any two charts  $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$  is *compatible*: each *transition function*  $\tau_{\alpha\beta}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  which is defined by  $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$  is differentiable.

It is called a *gluing condition*.

**Definition 4.7.** For two differentiable atlases  $\mathcal{F}, \mathcal{F}'$ , the two atlases are *equivalent* if  $\mathcal{F} \cup \mathcal{F}'$  is also differentiable.

**Definition 4.8.** An differentiable atlas  $\mathcal{F}$  is called *maximal* if the following holds: if a chart  $(U, \varphi)$  is compatible to all charts in  $\mathcal{F}$ , then  $(U, \varphi) \in \mathcal{F}$ .

**Definition 4.9.** A *differentiable structure* on  $M$  is a maximal differentiable atlas.

To differentiate a function on a flexible manifold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on  $M$ . When the charts is already equipped on  $M$ , it is natural to define a function  $f: M \rightarrow \mathbb{R}$  differentiable if the functions  $f \circ \varphi^{-1}: \mathbb{R}^d \rightarrow \mathbb{R}$  is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because  $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$ . If a function  $f$  is differentiable on an atlas  $\mathcal{F}$ , then  $f$  is also differentiable on any atlases which is equivalent to  $\mathcal{F}$  by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

**Example 4.1.** While the circle  $S^1$  has a unique smooth structure,  $S^7$  has 28 smooth structures. The number of smooth structures on  $S^4$  is still unknown.

**Definition 4.10.** A continuous function  $f: M \rightarrow N$  is differentiable if  $\psi \circ f \circ \varphi^{-1}$  is differentiable for charts  $\varphi, \psi$  on  $M, N$  respectively.

### 4.3 Curves

**Definition 4.11.** For  $f : M \rightarrow \mathbb{R}$  and  $(U, \phi)$  a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

**Definition 4.12.** Let  $\gamma : I \rightarrow M$  be a smooth curve. Then,  $\dot{\gamma}(t)$  is defined by a tangent vector at  $\gamma(t)$  such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let  $\phi : M \rightarrow N$  be a smooth map. Then,  $\phi(t)$  can refer to a curve on  $N$  such that

$$\phi(t) := \phi(\gamma(t)).$$

Let  $f : M \rightarrow \mathbb{R}$  be a smooth function. Then,  $\dot{f}(t)$  is defined by a function  $\mathbb{R} \rightarrow \mathbb{R}$  such that

$$\dot{f}(t) := \frac{d}{dt}f \circ \gamma.$$

**Proposition 4.1.** Let  $\gamma : I \rightarrow M$  be a smooth curve on a manifold  $M$ . The notation  $\dot{\gamma}^\mu$  is not confusing thanks to

$$(\dot{\gamma})^\mu = (\dot{\gamma}^\mu).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt}x^\mu \circ \gamma.$$

### 4.4 Connection computation

$$\begin{aligned} \nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\ &= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\ &= X^\mu \left( \frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma_{\mu\nu}^\lambda \partial_\lambda) \\ &= X^\mu \left( \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda \right) \partial_\nu. \end{aligned}$$

The covariant derivative  $\nabla_X Y$  does not depend on derivatives of  $X^\mu$ .

$$Y^\nu_{;\mu} = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \quad Y^\nu_{;\mu} = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda.$$

**Theorem 4.2.** For Levi-civita connection for  $g$ ,

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

*Proof.*

$$(\nabla_i g)_{jk} = \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl}$$

$$(\nabla_j g)_{kl} = \partial_j g_{kl} - \Gamma_{jk}^l g_{li} - \Gamma_{jl}^l g_{kl}$$

$$(\nabla_k g)_{ij} = \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}$$

If  $\nabla$  is a Levi-civita connection, then  $\nabla g = 0$  and  $\Gamma_{ij}^k = \Gamma_{ji}^k$ . Thus,

$$\Gamma_{ij}^l g_{kl} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

$$\Gamma_{ij}^l = \frac{1}{2}g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

□

## 4.5 Geodesic equation

**Theorem 4.3.** If  $c$  is a geodesic curve, then components of  $c$  satisfies a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0.$$

*Proof.* Note

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\lambda \partial_\lambda) = (\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu + \dot{\gamma}^\nu \dot{\gamma}^\lambda \Gamma_{\nu\lambda}^\mu) \partial_\mu.$$

Since

$$\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu = \dot{\gamma}(\dot{\gamma}^\mu) = d\dot{\gamma}^\mu(\dot{\gamma}) = d\dot{\gamma}^\mu \circ d\gamma \left( \frac{\partial}{\partial t} \right) = d\dot{\gamma}^\mu \left( \frac{\partial}{\partial t} \right) = \ddot{\gamma}^\mu,$$

we get a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0$$

for each  $\mu$ .

□

## 5 Vector calculus on spherical coordinates

$$\begin{aligned}
V &= (V_r, V_\theta, V_\phi) \\
&= V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi} \quad (\text{normalized}) \\
&= V_r \frac{\partial}{\partial r} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} V_\phi \frac{\partial}{\partial \phi} \quad (\Gamma(TM)) \\
&= V_r dr + r V_\theta d\theta + r \sin \theta V_\phi d\phi \quad (\Omega^1(M)) \\
&= r^2 \sin \theta V_r d\theta \wedge d\phi + r \sin \theta V_\theta d\phi \wedge dr + r V_\phi dr \wedge d\theta \quad (\Omega^2(M)) \\
\nabla \cdot V &= \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (r \sin \theta V_\theta) + \frac{\partial}{\partial \phi} (r V_\phi) \right] \\
\Delta u &= \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial}{\partial r} u \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} u \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} u \right) \right]
\end{aligned}$$

Let  $(\xi, \eta, \zeta)$  be an orthogonal coordinate that is *not* normalized. Then,

$$\begin{aligned}
\sharp &= g = \text{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2) \\
\hat{x} &= \|\partial_x\|^{-1} \partial_x = \|\partial_x\| dx = \|\partial_y\| \|\partial_z\| dy \wedge dz
\end{aligned}$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r d\theta, \quad r \sin \theta d\phi, \quad (r d\theta) \wedge (r \sin \theta d\phi), \quad (r \sin \theta d\phi) \wedge (dr), \quad (dr) \wedge (r d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[ \frac{1}{\|\partial_x\|} \frac{\partial}{\partial x} \cdot -, \frac{1}{\|\partial_y\|} \frac{\partial}{\partial y} \cdot -, \frac{1}{\|\partial_z\|} \frac{\partial}{\partial z} \cdot - \right] \\
\text{curl} : \nabla &= \left[ \frac{1}{\|\partial_y\| \|\partial_z\|} \left( \frac{\partial}{\partial y} (\|\partial_z\| \cdot -) - \frac{\partial}{\partial z} (\|\partial_y\| \cdot -) \right), \right. \\
&\quad \frac{1}{\|\partial_z\| \|\partial_x\|} \left( \frac{\partial}{\partial z} (\|\partial_x\| \cdot -) - \frac{\partial}{\partial x} (\|\partial_z\| \cdot -) \right), \\
&\quad \left. \frac{1}{\|\partial_x\| \|\partial_y\|} \left( \frac{\partial}{\partial x} (\|\partial_y\| \cdot -) - \frac{\partial}{\partial y} (\|\partial_x\| \cdot -) \right) \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[ \frac{\partial}{\partial x} (\|\partial_y\| \|\partial_z\| \cdot -), \frac{\partial}{\partial y} (\|\partial_z\| \|\partial_x\| \cdot -), \frac{\partial}{\partial z} (\|\partial_x\| \|\partial_y\| \cdot -) \right] \\
\Delta &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[ \frac{\partial}{\partial x} \left( \frac{\|\partial_y\| \|\partial_z\|}{\|\partial_x\|} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\|\partial_z\| \|\partial_x\|}{\|\partial_y\|} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\|\partial_x\| \|\partial_y\|}{\|\partial_z\|} \frac{\partial}{\partial z} \right) \right]
\end{aligned}$$



$$\begin{aligned}\text{grad} &= \frac{1}{\|\cdot\|^1} (\nabla) \|\cdot\|^0 \\ \text{curl} &= \frac{1}{\|\cdot\|^2} (\nabla \times) \|\cdot\|^1 \\ \text{div} &= \frac{1}{\|\cdot\|^3} (\nabla \cdot) \|\cdot\|^2\end{aligned}$$

## 6 Bundles

Show that  $S^n$  has a nonvanishing vector field if and only if  $n$  is odd.

*Solution.* Since  $S^n$  is embedded in  $\mathbb{R}^{n+1}$ , the tangent bundle  $TS^n$  can be considered as an embedded manifold in  $S^n \times \mathbb{R}^{n+1}$  which consists of  $(x, v)$  such that  $\langle x, x \rangle = 1$  and  $\langle x, v \rangle = 0$ , where the inner product is the standard one of  $\mathbb{R}^{n+1}$ .

Suppose  $n$  is odd. We have a vector field  $(x_1, x_2, \dots, x_{n+1}; x_2, -x_1, \dots, -x_n)$  which is nonvanishing.

Conversely, suppose we have a nonvanishing vector field  $X$ . Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \rightarrow S^n \times \mathbb{R}^{n+1} \rightarrow \phi \mathbb{R}^{n+1} \rightarrow S^n.$$

The last map can be defined since  $X$  is nowhere zero. Since this map satisfies  $\langle x, \phi(x) \rangle = 0$  for all  $x \in S^n$ , we can define homotopies from  $\phi$  to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree,  $+1$ , so  $n$  is odd.  $\square$

**Proposition 6.1.** *Independent commuting vector fields are realized as partial derivatives in a chart.*

**Proposition 6.2.** *Let  $\{\partial_1, \dots, \partial_k\}$  be an independent involutive vector fields. We can find independent commuting  $\{\partial_{k+1}, \dots, \partial_n\}$  such that union is independent. (Maybe)*

**Proposition 6.3.** *Let  $\{\partial_1, \dots, \partial_k\}$  be an independent commuting vector fields. We can find independent commuting  $\{\partial_{k+1}, \dots, \partial_n\}$  such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

**Proposition 6.4.** *An immersed manifold is locally an inverse image of a regular value.*

**Proposition 6.5.** *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

*Proof.* It uses tubular neighborhood. Pontryagin construction?  $\square$

**Proposition 6.6.** *An immersed manifold is locally a linear subspace in a chart.*

**Proposition 6.7.** *Distinct two points on a connected manifold are connected by embedded curve.*

*Proof.* Let  $\gamma : I \rightarrow M$  be a curve connecting the given two points, say  $p, q$ .

*Step [.1]*Constructing a piecewise linear curve For  $t \in I$ , take a convex chart  $U_t$  at  $\gamma(t)$ . Since  $I$  is compact, we can choose a finite  $\{t_i\}_i$  such that  $\bigcup_i \gamma^{-1}(U_{t_i}) = I$ . This implies  $\text{im } \gamma \subset \bigcup_i U_{t_i}$ . Reorganize indices such that  $\gamma(t_1) = p$ ,  $\gamma(t_n) = q$ , and  $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$  for all  $1 \leq i \leq n-1$ . It is possible since the graph with  $V = \{i\}_i$  and  $E = \{(i, j) : U_{t_i} \cap U_{t_j} \neq \emptyset\}$  is connected. Choose  $p_i \in U_{t_i} \cap U_{t_{i+1}}$  such that they are all dis for  $1 \leq i \leq n-1$  and let  $p_0 = p$ ,  $p_n = q$ .

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from  $p$  to  $q$ .

*Step [.2]*Smoothing the curve  $\square$

**Proposition 6.8.** *Let  $M$  is an embedded manifold with boundary in  $N$ . Any kind of sections on  $M$  can be extended on  $N$ .*

**Proposition 6.9.** *Every ring homomorphism  $C^\infty(M) \rightarrow \mathbb{R}$  is obtained by an evaluation at a point of  $M$ .*

*Proof.* Suppose  $\phi : C^\infty(M) \rightarrow \mathbb{R}$  is not an evaluation. Let  $h$  be a positive exhaustion function. Take a compact set  $K := h^{-1}([0, \phi(h)])$ . For every  $p \in K$ , we can find  $f_p \in C^\infty(M)$  such that  $\phi(f_p) \neq f_p(p)$  by the assumption. Summing  $(f_p - \phi(f_p))^2$  finitely on  $K$  and applying the extreme value theorem, we obtain a function  $f \in C^\infty(M)$  such that  $f \geq 0$ ,  $f|_K > 1$ , and  $\phi(f) = 0$ . Then, the function  $h + \phi(h)f - \phi(h)$  is in kernel of  $\phi$  although it is strictly positive and thereby a unit. It is a contradiction.  $\square$

**Proposition 6.10.** *The set of points that is geodesically connected to a point is open.*