Harmonic Analysis

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Caldéron-Zygmund theory

Let f be a nonnegative and sufficiently nice function on $\mathbb{R}^d$, and fix $\lambda > 0$.

3.1 (Calderón-Zygmund decomposition of sets). Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf = \sup_n |E_n f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$. For $x \in \Omega$, let Q_x be the maximal dyadic cube satisfying $Q_x \ni x$ and

$$\frac{1}{|Q_x|} \int_{Q_x} f > \lambda.$$

(a) Show that $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) Show the following weak type estimate. $|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1}$.

(c) Show that $f(x) \leq \lambda$ for a.e. $x \notin \Omega$.

(d) Show that for $x \in \Omega$

$$\frac{1}{|Q_x|} \int_{Q_x} f \leq 2^d \lambda.$$

3.2 (Calderón-Zygmund decomposition of a function). Let

$$g(x) := \begin{cases} f(x) & , x \notin \Omega \\ \frac{1}{|Q_x|} \int_{Q_x} f & , x \in \Omega \end{cases}$$

and $b_i := (f - g)\chi_{Q_i}$. Check that we have

(a) $f = g + b$ where $b = \sum_i b_i$.

(b) $\|g\|_{L^1} = \|f\|_{L^1}$.

(c) $\|g\|_{L^\infty} \lesssim_d \lambda$.

(d) $\int_{Q_i} b_i = 0$.

3.3 (Calderón-Zygmund theory for convolution type). Let T be a singular integral operator of convolution type in the sense that there is a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ such that

$$Tf(x) = \int K(x-y)f(y)\,dy$$

whenever $x \notin \text{supp} f$. Suppose the following two conditions are satisfied.

(i) The L^2-boundedness: $\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$.

(ii) The Hörmander condition: $\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx \lesssim 1$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) Use the L^2-boundedness of T to show that

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) Use the Hörmander condition to show that

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c) Deduce that T is weak $(1,1)$.

Chapter 4

Littlewood-Paley theory

Chapter 5

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Part III

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Part IV

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