Abstract Harmonic Analysis

Ikhan Choi

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Contents

Part I

Chapter 1

Hopf $*$-algebras

1.1

Multiplier Hopf $*$-algebras
  Algebraic quantum groups
  Hopf C*-algebras
  idempotent ring assumption

Chapter 2

Locally compact groups

2.1

2.1 (Non-σ-finite measures)**.** Following technical issues are important

(a) The Fubini theorem

(b) The Radon-Nikodym theorem

(c) The dual space of L^1 space

2.2 (Existence of the Haar measure)**.**

2.3 (Left and right uniformities)**.**

2.4 (Modular functions)**.**

2.5 (Uniformly continuous functions)**.** G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$. $g \in C_c(G)$ is two-sided uniformly continuous.

2.6 (Structures on a locally compact group)**.** For a locally compact group G, consider $A := C_c(G)$. It is a left Hilbert algebra by the existence of the left Haar measure

$$(f * g)(s) := \int f(t)g(t^{-1}s)\,dt, \qquad \langle f, g\rangle := \int \overline{g(s)}f(s)\,ds, \qquad f^\sharp(s) := \delta(s^{-1})\overline{f(s^{-1})}.$$

and is a commutative counital multiplier Hopf $*$-algebra by the group structure.

$$(fg)(s) := f(s)g(s), \qquad \Delta f(s,t) = f(st), \qquad f^*(s) := \overline{f(s)}, \qquad Sf(s) = f(s^{-1}).$$

Since the image of the comultiplication does not belong to $C_c(G) \otimes C_c(G)$, we need to do something unless G is finite. They satisfy a compatibility condition $\langle fg, h\rangle = \langle f, g^*h\rangle$.

With the integral notation $\lambda(f) = \int f(s)\lambda_s\,ds$, we can write

We start from this structures.

From now on, we are going to exclude any measure theory and the theory of non-commutative L^p spaces. First, we have the completion $H =: L^2(G)$. Consider two representations

$$\lambda : (C_c(G), *, ^\sharp) \to B(L^2(G)), \qquad m : (C_c(G), \cdot, ^*) \to B(L^2(G)).$$

(a) λ is well-defined.

(b) m is well-defined.

Proof. The multiplication representation m is well-defined because for $f \in C_c(G)$ we have $f^*f \in C_c(G) \subset L^2(G)$ so

$$\|m(f)g\|^2 = \langle fg, fg\rangle = \langle f^*fg, g\rangle, \qquad g \in C_c(G).$$

□

2.2

We use the notation $L^p(G)$ for the non-commutative L^p-spaces constructed with the left Haar measure on G, which is a faithful semi-finite normal weight of $L^\infty(G)$. The predual of $L^\infty(G)$ can be identified with $L^1(G)$. The regular representation on $L^2(G)$ is the Gelfand-Naimark-Segal representation associated with the left Haar measure.

Density of $C_c(G)$?

2.7 (Convolution algebra)**.** Let G be a locally compact group. Then, $L^1(G)$ is a hermitian Banach $*$-algebra such that

$$(f*g)(x) := (f\otimes g)\Delta(x), \qquad f,g \in L^1(G),\ x \in L^\infty(G).$$

Importance of L^1 instead of C_c: representation equivalence and predual.

(a) $L^1(G)$ has a two-sided approximate unit in $C_c(G)$.

(b) $\alpha : G \to \mathrm{Aut}(L^1(G))$ is point-norm continuous.

(c) $\lambda : G \to U(L^2(G))$ and $\lambda : L^1(G) \to B(L^2(G))$ are strongly continuous.

(d) Convolution inequalities.

(e) Representation theory equivalence.

Proof. Let (U_α) be a directed set of open neighborhoods of the identity e of G. By the Urysohn lemma, there is $e_\alpha \in C_c(U)^+$ such that $\|e_\alpha\|_1 = 1$ for each α. We claim that e_α is a two-sided approximate unit for $L^1(G)$. Suppose $g \in C_c(G)$, which is two-sided uniformly continuous. For any $\varepsilon > 0$, take α_0 such that $\|g-\lambda_s g\| < \varepsilon$ and $\|g-\rho_s g\| < \varepsilon$ for all $s \in U_\alpha$ for $\alpha \succ \alpha_0$. Then, we have

$$\begin{aligned}\|e_\alpha * g - g\|_1 &= \int |e_\alpha * g(t) - g(t)|\,dt \le \iint e_\alpha(s)|g(s^{-1}t)-g(t)|\,ds\,dt \\ &= \int_{U_\alpha} e_\alpha(s)\|\lambda_s g - g\|_1\,ds < \varepsilon \int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|g*e_\alpha - g\|_1 &= \int |g*e_\alpha(s) - g(s)|\,ds \le \iint |g(t)-g(s)|e_\alpha(t^{-1}s)\,dt\,ds \\ &= \iint |g(t)-g(ts)|e_\alpha(s)\,dt\,ds = \int \|g-\rho_s g\|_1 e_\alpha(s)\,ds < \varepsilon\int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and they imply $\lim_\alpha \|e_\alpha * g - g\|_1 = \lim_\alpha \|g * e_\alpha - g\|_1 = 0$. We can approximate $f \in L^1(G)$ with compactly supported continuous functions by the $\varepsilon/3$ argument. □

Note that we have

$$\begin{aligned}|\langle\lambda(\xi)\eta,\zeta\rangle|^2 &= |\iint \xi(t)\eta(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\ &\le \iint |\xi(t)||\eta(t^{-1}s)|^2\,ds\,dt \cdot \iint |\xi(t)||\zeta(s)|^2\,ds\,dt \\ &= \|\xi\|_1^2\|\eta\|_2^2\|\zeta\|_2^2\end{aligned}$$

and

$$
\begin{aligned}
|\langle \rho(\xi)\eta, \zeta\rangle|^2 &= |\iint \eta(t)\xi(t^{-1}s)\overline{\zeta(s)}\, ds\, dt|^2 \\
&\le \iint |\xi(t^{-1}s)||\eta(t)|^2\, ds\, dt \cdot \iint |\xi(t^{-1}s)||\zeta(s)|^2\, ds\, dt \\
&= \|\xi\|_1 \|F\xi\|_1 \|\eta\|_2^2 \|\zeta\|_2^2
\end{aligned}
$$

imply

$$\|\lambda(\xi)\|_{2\to 2} \le \|\xi\|_1, \qquad \|\rho(\xi)\|_{2\to 2} \le \sqrt{\|\xi\|_1 \|F\xi\|_1}.$$

The equalities do not hold, consider $\|\lambda(\xi)\| = \|\widehat{\xi}\|_\infty$ if $G = \mathbb{R}$.

$$
\begin{tikzcd}
G \arrow[r] & M(G) & & \\
L_1(G) \arrow[ur, hook] \arrow[r, hook] \arrow[d, dashed, "*"] & C^*(G) \arrow[r, two heads] \arrow[d, dashed, "*"] & C_r^*(G) \arrow[r, hook] \arrow[d, dashed, "*"] & L(G) \arrow[d, dashed, "* \text{ with } \sigma w"] \\
L^\infty(G) & B(G) \arrow[l, hook'] & C_r^*(G)^* \arrow[l, hook'] & A(G) \arrow[l, hook'] \\
 & C_0(G) \arrow[ul, hook'] & &
\end{tikzcd}
$$

2.3

2.8 (Plancherel theorem)**.** With the left Haar measure on a Banach $*$-algebra $L^1(G)$ or $M(G)$, we want to construct a faithful semi-finite normal weight called the *Plancehrel weight*, and describe the corresponding semi-cyclic representation and left Hilbert algebra for $C_r^*(G)$ and $W_r^*(G)$.

By analyze the decomposition of the canonical representation of $C_r^*(G)$ and $W_r^*(G)$ in $B(L^2(G))$....? Then, we can consider a unitary operator from $L^2(G)$ to the square integrable section space of a bundle on $\widehat{G}$...

Proof. □

2.9 (Fourier algebra)**.** The Fourier algebra is the algebra $A(G)$ of matrix coefficients of the regular representation, i.e. the space spanned by functions $s \mapsto \langle \lambda(s)\xi, \xi\rangle$ for $\xi \in L^2(\widehat{G})$.

It is a dense Banach subalgebra of $C_0(G)$ such that $A(G) \to W_r^*(G)_* : \eta^*\xi \mapsto \omega_{\xi,\eta}$ is an isometric isomorphism.

positive definite functions

Proof. □

2.10 (Locally compact abelian groups)**.** Let G be a locally compact abelian group. Since every irreducible representation of a locally compact abelian group is one-dimensional, we introduce the notation $\langle s, p\rangle = p_s \in \mathbb{T}$. The *Fourier transform* of an integrable function $f \in L^1(\widehat{G})$ is defined as

$$\mathcal{F}f(p) := \int_G \overline{\langle s, p\rangle} f(s)\, ds, \qquad , p \in \widehat{G},$$

and the *Fourier-Stieltjes transform* of a finite complex measure $\mu \in M(G)$ is defined as

$$\mathcal{F}\mu(p) := \int_G \overline{\langle s, p\rangle}\, d\mu(s), \qquad p \in \widehat{G}.$$

(a) The compact open topology of $C(G)$ and the weak* topology of $L^\infty(G)$ coincide on $\widehat{G}$, which provides a locally compact abelian group.

(b) The Fourier transform defines a $*$-homomorphism $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ which is injective with norm dense image.

(c) The Fourier-Stieltjes transform defines a $*$-homomorphism $\mathcal{F} : M(G) \to L^\infty(\widehat{G})$ which is weakly* continuous and injective with weakly* dense image in $C_b(\widehat{G})$.

(d) The canonical homomorphism $\Phi : G \to \widehat{\widehat{G}}$ defined such that $\Phi(s)(p) = \langle s, p\rangle$ for $s \in G$ and $p \in \widehat{G}$ is a topological isomorphism.

(e) Fourier inversion..?

Proof. (b) The Fourier transform is realized as the composition

$$\mathcal{F} : L^1(G) \to C_r^*(G) \to C^*(G) \to C_0(\widehat{G}) \to C_0(\widehat{G}).$$

The first map is the extension of the regular representation $\lambda : C_c(G) \to B(L^2(G))$ using the inequality $\|\lambda(f)\| \le \|f\|_{L^1}$. It has dense image by the definition of $C_r^*(G)$, the norm closure of the image of λ. It is also injective because if $f \in L^1(G)$ satisfies $\langle \lambda(f)\xi, \eta\rangle = 0$ for all $\xi, \eta \in L^2(G)$, then it means that $\langle f, a\rangle = 0$ for every $a \in A(G)$ by definition of the Fourier algebra, which implies that $f = 0$ because $L^1(G) \subset M(G) = C_0(G)^*$ and $A(G)$ is dense in $C_0(G)$.

The second map is the inverse of the canonical map $C^*(G) \to C_r^*(G)$ taken thanks to the amenability of locally compact abelian groups. The third map is the Gelfand transform, which is a $*$-isomorphism for commutative C*-algebras. The last map is the induced map from the inverse map of the domain $\widehat{G}$, clearly a $*$-isomorphism.

Therefore, the Fourier transform $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ is an injective $*$-homomorphism with with dense image.

(c) The Fourier-Stieltjes transform is realized as the composition

$$\mathcal{F} : M(G) \to W_r^*(G) \to L^\infty(\widehat{G}).$$

The first map is the weakly* continuous extension of the regular representation $\lambda : C_c(G) \to B(L^2(G))$, and the weak* continuity follows from the fact that the Fourier algebra $A(G)$ belong to $C_0(G)$. The injectivity follows from the density of $A(G)$ in $C_0(G)$ as we did in the part (b), and the weakly* dense image is also by the definition of $W_r^*(G)$.

The second map is obtained by taking double commutant for the $*$-isomorphism $C_r^*(G) \to C_0(\widehat{G})$ composed from the last three maps described in the part (b), which is in fact the restriction the $*$-isomorphism $B(L^2(G)) \to B(L^2(\widehat{G}))$ defined by the Fourier transform $L^2(G) \to L^2(\widehat{G})$ in the Plancherel theorem and the reflection on the domain $\widehat{G}$.

Therefore, the Fourier-Stieltjes transform $\mathcal{F} : M(G) \to L^\infty(\widehat{G})$ is an injective $*$-homomorphism with with weakly* dense image. The continuous and boundedness of $\mathcal{F}(\mu)$ is because it is a multiplier of $C_0(\widehat{G})$.

(d) Consider the inverse $\mathcal{F}^{-1} : \mathcal{F}(L^1(\widehat{G})) \to L^1(\widehat{G})$ of the Fourier transform $\mathcal{F} : L^1(\widehat{G}) \to C_0(\widehat{\widehat{G}})$ and the weak* transpose $\mathcal{F}^t : L^1(\widehat{G}) \to C_0(G)$ of the Fourier-Stieltjes transform $\mathcal{F} : M(G) \to L^\infty(\widehat{G})$. Their composition is equal to the restriction of the restriction map $C_0(\widehat{\widehat{G}}) \to C_0(G)$ along Φ.

isometry? □

2.4

2.11 (Fell absorption principle)**.** Structure operator $w \in U(L^2(G,G))$ such that $w\xi(s,t) = \xi(s,st)$ or $w \in L^\infty(G)\overline{\otimes}L(G)$ such that $w(\lambda_s \otimes \lambda_s)w^* = \lambda_s \otimes 1$. If $w(x \otimes x)w^* = x \otimes 1$, then $x = \lambda_s$ for some $s \in G$.

2.5 Spectral synthesis

Part II

Topological quantum groups

Chapter 3

Kac algebras

Chapter 4

Compact quantum groups

Chapter 5

Locally compact quantum groups

5.1 Multiplicative unitaries

Part III

Representation categories

Chapter 6

Representations of compact groups

6.1 Peter-Weyl theorem

6.2 Tannaka-Krein duality

6.3 Mackey machine

Example of non-compact Lie groups, Wigner classification