Representation Theory

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Contents

Part I

Finite groups

Chapter 1

Character theory

1.1 Irreducible representations

1.1 (Definition of group representations).

1.2 (Intertwining maps).

1.3 (Subrepresentations). We say *invariant* or *stable*

1.4 (Irreducible representations). indecomposable and irreducible

1.5 (Maschke's theorem). Let G be a finite group and k be a field. Suppose the characteristic of k does not divide $|G|$. Let V be a finite-dimensional representation of G over k.

(a) Every invariant subspace W of V has a complement W' in V that is also invariant.

(b) V is isomorphic to the direct sum of irreducible representations of G over k.

(c) If $k = \mathbb{R}$ or $\mathbb{C}$, then V admits an inner product such that $W \perp W'$ and $\rho_V(g)$ is unitary for all $g \in G$.

1.6 (Schur's lemma). Let G be a group and k be a field. Let V and W be irreducible representations of G over k. Let $\psi : V \to W$ be an intertwining map.

(a) If $V \ncong W$, then $\psi = 0$.

(b) If $V \cong W$, then ψ is an isomorphism.

(c) If k is algebraically closed and $\dim V < \infty$, then every intertwining map $\psi : V \to V$ is a homothety.

1.2 Group algebra

1.7 (Modules and representations). ring <-> group module <-> representation finitely generated <-> finite dimensional

1.8 (Wedderburn's theorem). central idempotents dimension computation

1.9 (Group algebra). regular representation $k[G]$-module and G-representation correspondence

(a) $\mathbb{C}[G]$ is the direct sum of all irreducible representations.

(b) $|G| = \sum_{[V]\in\widehat{G}} (\dim V)^2$.

1.10. The number of irreducible representations and the number of conjugacy classes double counting on $Z(\mathbb{C}[G])$.

1.3 Characters

1.11 (Space of class functions)**.** Ring and inner product structure on the space of class functions.

(a) $\dim \hom_G(V, W) = \langle \chi_V, \chi_W \rangle$.

(b) Irreducible characters form an orthonormal basis of the space of class functions.

1.12 (Characters classify representations)**.** Let G be a finite group and let $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of G over $\mathbb{C}$.

$$\mathrm{Tr} : \mathbf{Rep}(G) \to \{\text{finite sum of irreducible characters}\}$$

surjectivity: trivial injectivity: Suppose two characters are equal. Maschke -> all characters are sum of irreducible characters Schur -> orthogonality, so the coefficients are all equal irreducible-factor-wisely construct an isomoprhism.

1.13 (Character table)**.** computation of matrix elements by character table abelian group, 1dim rep lifting

S^3	e	(12)	(123)
1	1	1	1
ε	1	-1	1
ρ	2	0	-1

the dual inner product: conjugacy check relation to normal subgroups center of rep
algebraic integer dim of irrep divides group order burnside pq theorem

Chapter 2

Classification of representations

2.1 Symmetric groups

young tableux

2.2 Linear groups over finite fields

GL2 and SL2 over finite fields

2.3 Induced representations

induction and restriction of reps (from and to subgroup) frobenius reciprocity, mackey theory

tensoring, complex, real symmetric, exterior

Chapter 3

Brauer theory

Part II

Lie algebras

Chapter 4

Semisimple Lie algebras

4.1

group acts on an algebra A(e.g. $\mathrm{End}(V)$). then its group algebra acts on A. Lie algebra acts on A, and this Lie algebra information is enough to recover the group action. Geometric meaning of Lie algebra action?

Lie algebra can only considered as a quantization of Poisson bracket. How can the Poisson bracket embodies the group action?

Following Humphrey's book, let L be always finite dimensional Lie algebra unless stated.

4.1. Every associative algebra is a Lie algebra, where the Lie bracket is given by the commutator. For a Lie algebra, we are

Intuitions of subalgebras, ideals, derivations. Intuitions of solvable, nilpotent, and semisimple Lie algebras. Constructing representations, trace forms,

The *general linear Lie algebra* $\mathfrak{gl}(V)$ is just $\mathrm{End}(V)$ with a Lie bracket $[x,y] := xy - yx$.

4.2 (Derivations)**.** Let L be a Lie algebra. A *derivation* of L is a linear map $\delta : L \to L$ such that

$$\delta([x,y]) = [\delta(x),y)] + [x,\delta(y)]$$

for all $x,y \in L$. The set of derivations $\mathrm{Der}(L)$ of L is a subalgebra of $\mathfrak{gl}(L)$, and we have the *adjoint representation* $L \to \mathrm{Der}(L) \le \mathfrak{gl}(L)$ of L. If I is an ideal, then we have a faithful representation $\mathrm{ad} : L \to \mathrm{ad}\, L \le \mathrm{Der}(I) \le \mathfrak{gl}(I)$.

4.3 (Inner derivations and automorphisms)**.** Let L be a Lie algebra.

The linear map $\mathrm{ad}\, x = [x,-] : L \to L$ for $x \in L$ is derivation, and derivation of this form is called *inner*, and they form an ideal of $\mathrm{Der}(L)$.

Automorphisms of the form $\exp(\mathrm{ad}\, x)$ with nilpotent $\mathrm{ad}\, x$ generates a normal subgroup of $\mathrm{Aut}(L)$, and each generator is called *inner automorphisms*.

4.4 (Solvable and nilpotent Lie algebras)**.** Let L be a Lie algebra. If the *derived series* $L^{(0)} = L$, $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ eventually vanishes, then we call L *solvable*.

If L is solvable, then its subalgebras and quotient algebras are all solvable. If I is a solvable ideal of L such that L/I is solvable, then L is solvable. The sum of two solvable ideals is also solvable.

Let L be a Lie algebra. If the *lower central series* $L^0 = L$, $L^n = [L, L^{n-1}]$ eventually vanishes, then we call L *nilpotent*. It is a stronger notion than solvability.

If L is nilpotent, then its subalgebras and quotient algebras are all nilpotent. If $L/Z(L) \cong \mathrm{ad}(L) \subset \mathfrak{gl}(L)$ is nilpotent, then L is nilpotent. If L is non-zero and nipotent, then $Z(L)$ is non-trivial.

4.5 (Engel's theorem). .

(a) A linear Lie algebra $L \subset \mathfrak{gl}(V)$ consists of nilpotent endomorphisms if and only if $L \subset \mathfrak{n}(V)$ for a certain basis of V.

(b) An abstract Lie algebra L is nilpotent if and only if $\mathrm{ad}(L)$ consists of nilpotent endomorphisms.

(c) If $L \subset \mathfrak{gl}(V)$ is nilpotent in $\mathrm{End}(V)$, then there is a *common eigenvector* $v \in V$ such that $[L, v] = 0$, i.e. there is a flag V_i such that $xV_i \subset V_{i-1}$...?

Proof. Let L be an ad-nilpotent Lie algebra. Then, every element of $\mathrm{ad}\, L \subset \mathfrak{gl}(L)$ is a nilpotent endomorphism, so there is $x \in L$ such that $[L, x] = 0$, which implies $Z(L) \neq 0$. Since $L/Z(L)$ is also ad-nilpotent, and by induction on dimension, $L/Z(L)$ is nilpotent. Therefore, L is nilpotent. □

4.6 (Lie's theorem). Let $\mathbb{F}$ have characteristic zero and be algebraically closed.

(a) A linear Lie algebra $L \subset \mathfrak{gl}(V)$ is solvable if and only if $L \subset \mathfrak{t}(V)$ for a certain basis of V.

(b) If L is solvable, then there is a flag V_i such that $xV_i \subset V_i$.

(c) Let L be an abstract Lie algebra. L is solvable if and only if $[L, L]$ is nilpotent.

(d) Every finite-dimensional irreducible representation of a solvable Lie algebra is one-dimensional.

Proof. Use induction on dimension. Since $L/[L, L]$ is a non-trivial commutative Lie algebra, in which every subspace is an ideal, we can show the existence of an ideal K of L with codimension one by pullback.

By the induction assumption, we have a common eigenvector in V for K so that we have the "eigenvalue" linear functional $\kappa : K \to \mathbb{F}$ such that the "eigenspace" of κ as

$$V_\kappa := \{v \in V : xv = \kappa(x)v \text{ for } x \in K\}$$

is non-trivial.

Let $L = K + \mathbb{F}z$ with $z \in \mathfrak{gl}(V)$. If V_κ is invariant by L, then V_κ contains an eigenvector of z by the fact that $\mathbb{F}$ is algebraically closed, so we can extend κ to obtain $\lambda : L \to \mathbb{F}$ such that $(V_\kappa)_\lambda$ is non-trivial.

We now show that V_κ is invariant by L. Let $v \in V_\kappa$ and $x \in L$. Since

$$yxv - \lambda(y)xv = yxv - xyv = [y, x]v = \lambda([y, x])v$$

for $y \in K$, we have to show $\lambda([y, x]) = 0$. Take n to be largest such that $v, \cdots, x^{n-1}v$ are linearly independent. Since $[x, y]$ is upper triangular matrix relative to the basis $v, \cdots, x^{n-1}v$ and the diagonal entries are $\lambda([x, y])$. Since the trace of $[x, y]$ must be zero, we have $\lambda([x, y]) = 0$ because $\mathbb{F}$ has characteristic zero.

□

There is a linear functional $\lambda : L \to \mathbb{F}$ such that $\lambda|_{[L,L]} = 0$ and V_λ is non-trivial. V_κ

For a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, then a weight of V is a linear functional $\lambda : \pi(\mathfrak{h}) \to \mathbb{F}$ such that the weight space V_λ is non-trivial.

4.7 (Jordan-Chevalley decomposition). Let V be a finite-dimensional vector space over a field K. Let $x \in \mathrm{End}(V)$. Even if $\mathbb{F}$ is not algebraically closed, we have a generalization of Jordan decomposition as follows:

$x = x_s + x_n$ iff x is the product of separable polynomials.

(a) There exist unique $x_s, x_n \in \mathrm{End}(V)$ such that $x = x_s + x_n$ and x_s semisimple, x_n nilpotent.

(b) x_s and x_n are polynomials in x.

(c) If x maps B to A, then x_s and x_n also map B to A for subspaces $A \leq B \leq V$.

Proof. □

4.8 (Cartan criterion). We will show a powerful criterion for solvability. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$, and consider a finite-dimensional faithful representation $\mathfrak{g} \subset \mathfrak{gl}(V)$.

(a) If $\mathrm{tr}(xy) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Proof. Since the nilpotency of $[\mathfrak{g}, \mathfrak{g}]$ implies the solvability of $\mathfrak{g}$, it suffices to show the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Let $A \subset B$ be two linear subspaces of $\mathfrak{gl}(V)$. Let

$$M := \{x \in \mathfrak{gl}(V) : [x, B] \subset A\}.$$

If $x \in M$ satisfies $\mathrm{tr}(xy) = 0$ for all $y \in M$, then x is nilpotent. □

4.9 (Levi decomposition). Therefore, $\mathfrak{g}$ admits a unique maximal solvable ideal, called *radical*. Since the center is a solvable ideal, the center of a semisimple Lie algebra is trivial.

A canonical example of a solvable Lie algebra is the Lie algbera of upper triangular matrices. The radical of $\mathfrak{gl}(n, K)$ is $\mathfrak{sl}(n, K)$.(K cahracteristic zero?) Upper triangular matrices do not form an ideal of $\mathfrak{gl}(n, K)$.

We have $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{n}$, $\mathfrak{t} = \mathfrak{d} \otimes \mathfrak{n}$. $\mathfrak{t}$ is a solvable subalgebra of $\mathfrak{gl}$, but not a solvable ideal.

$\mathfrak{sl}(n, \mathbb{F})$ is simple if $\mathrm{char}\, K = 0$.

(a) $\mathfrak{g}$ is semi-simple if and only if the radical is trivial.

4.10 (Killing form). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$. Since an endomorphism algebra of a finite-dimensional vector space over a field has a canonical symmetric bilinear form defined called the trace form, the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ induces a symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ called the *Killing form* such that $\kappa(x, y) := \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ for $x, y \in \mathfrak{g}$.

(a) The kernel of κ is contained in the radical of L, and triviality is equivalent; L is semisimple if and only if L is non-degenerate. (Here we use Cartan's criterion)

(b) If L is semisimple, then it is the direct sum of simple ideals.

(c) If L is semisimple, then every derivation is inner.

(d) If L is semisimple, then $L = [L, L]$ and every subalgebras and quotients are semisimple.

Proof. Suppose $\mathrm{rad}\, \mathfrak{g} = 0$. The restriction of the Killing form of $\mathfrak{g}$ to an ideal $\mathfrak{i} \subset \mathfrak{g}$ is the Killing form of $\mathfrak{i}$.

□

4.11 (Weyl's theorem on complete reducibility). Finite dimensional representation of a semisimple Lie algebra is completely reducible. Preservation of Jordan decomposition.

4.12 (Toral subalgebras). Cartan subalgebra uniqueness (conjugacy theorem)

Chapter 5

Root systems

root space decomposition Killing form on Cartan subalgebra integrality and rationality Weyl group

Classification: Coxeter graph Dynkin diagram Real forms

Chapter 6

Representations of Lie algebras

6.1 Representations of $\mathfrak{sl}(2,\mathbb{C})$

6.1 (Pauli matrices)**.** Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis of complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, and $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of real Lie algebra $\mathfrak{so}(3)$.

(b) For a unit vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ has eigenvalues ± 1.

6.2 Highest weight theory

Isomorphism and conjugacy theorem?

Existence: Universal enveloping algebra and the PBW theorem Verma module definition and quotient finiteness proof

6.3 Character theory

6.4 Multiplicity formulas

Exercises

6.2 (Triplets and quadraplets)**.** Let (π_2, V_2) be the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of degree two. Consider $V_2 \otimes V_2$. Cartan element S_z. $V_2^{\otimes 3}$.

6.3 (Casimir element)**.** Casimir element decomposes a representation into irreducible representations. For a faithful representation $\varphi : L \to \mathfrak{gl}(V)$, we can associate a non-degenerate trace form since L is semisimple. Then, the *Casimir element* of the representation φ is $C_\varphi := \sum_i \varphi(x_i)\varphi(y_i) \in \operatorname{End}(V)$ where i runs over dual bases relative to the trace form.

Part III

Algebraic groups

Chapter 7

Group schemes

Usually we define a variety as an integral separated scheme of finite type over a field. However, here we define a *variety* as a reduced separated scheme of finite type over a field to allow its reducibility. If a group scheme is a variety over a field, then we call it an *algebraic group*.

$$\text{Reductive group schemes} \subset \text{Affine group schemes} \supset \text{Linear group schemes}$$

$$\text{Reductive group varieties} \subset \text{Affine algebraic groups} = \text{Linear algebraic groups}$$

$$\text{Projective algebraic groups} = \text{Abelian varieties}$$

Chapter 8

Linear algebraic groups

8.1 Affine group schemes

8.2 Reductive groups

Chapter 9

Abelian varieties

9.1 Projective

Part IV

Hopf algebras

Chapter 10

10.1

The category of affine group schemes is the opposite of the category of commutative Hopf algebras.

10.1 (Hopf algebras)**.** Over the complex field, recall that the category of vector spaces is a symmetric monoidal category with the swap map $\sigma_A : A\otimes A \to A\otimes A$ for each vector space A. A unital algebra can be internally defined as a vector space A together with linear maps $\mu : A\otimes A \to A$ and $\eta : \mathbb{C} \to A$ such that we have the following commutative diagrams:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\eta"'] \arrow[dr, "\mathrm{id}"] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

A *counital coalgebra* is a vector space A together with linear maps $\delta : A \to A\otimes A$ and $\varepsilon : A \to \mathbb{C}$ such that we have following commutative diagrams:

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] & A\otimes A \arrow[d, "\delta\otimes\mathrm{id}"] \\
A\otimes A \arrow[r, "\mathrm{id}\otimes\delta"] & A\otimes A\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, "\mathrm{id}"] & A\otimes A \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\
A\otimes A \arrow[r, "\mathrm{id}\otimes\varepsilon"'] & A
\end{tikzcd}$$

The linear maps μ, η, δ, and ε are called the multiplication, unit, comultiplication, and counit.

A *biunital bialgebra*, or just simply a *bialgebra*, is a vector space A which is simultaneously a unital algebra and a counital algebra, satisfying the compatibility condition as the following four commutative diagrams:

$$\begin{tikzcd}
A\otimes A \arrow[r, "\mu"] \arrow[d, "\delta\otimes\delta"'] & A \arrow[r, "\delta"] & A\otimes A \\
A\otimes A\otimes A\otimes A \arrow[rr, "1\otimes\sigma_A\otimes 1"] & & A\otimes A\otimes A\otimes A \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}
\quad
\begin{tikzcd}
A\otimes A \arrow[rr, "\varepsilon\otimes\varepsilon"] \arrow[dr, "\mu"'] & & \mathbb{C} \\
& A \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{C} \arrow[rr, "\eta\otimes\eta"] \arrow[dr, "\delta"'] & & A\otimes A \\
& A \arrow[ur, "\eta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbb{C} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & \mathbb{C} \\
& A \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

This compatibility condition is equivalent to that the comultiplication and the counit are algebra homomorphisms, or that the multiplication and the unit are coalgebra homomorphisms. A *Hopf algebra* is a bialgebra A equipped with an invertible linear map $\kappa : A \to A$, called the *antipode*, satisfying the following hexagonal commutative diagram:

$$\begin{tikzcd}
& A\otimes A \arrow[rr, "\kappa\otimes\mathrm{id}"] & & A\otimes A \arrow[dr, "\delta"] & \\
A \arrow[ur, "\mu"] \arrow[dr, "\mu"'] \arrow[rr, "\varepsilon"] & & \mathbb{C} \arrow[rr, "\eta"] & & A \\
& A\otimes A \arrow[rr, "\mathrm{id}\otimes\kappa"] & & A\otimes A \arrow[ur, "\delta"'] &
\end{tikzcd}$$

A morphism between Hopf algebras is a linear map preserving the five structure maps μ, η, δ, ε, κ.

Chapter 11

Quantum groups