

Foundations of Calculus

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Part I

Sequences

Chapter 1

Convergence

Chapter 2

Series

2.1 Convergence tests

2.1 (Mertens' theorem). If $\sum_{k=0}^{\infty} a_k$ converges to a absolutely and $\sum_{k=0}^{\infty} b_k$ converges to b , then their Cauchy product $\sum_{k=0}^{\infty} c_k$ with $c_k := \sum_{l=0}^k a_l b_{k-l}$ converges to ab .

Proof. Let

$$A_n := \sum_{k=0}^n a_k, \quad B_n := \sum_{k=0}^n b_k, \quad \text{and} \quad C_n := \sum_{k=0}^n c_k$$

so that

$$C_n = \sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l} = \sum_{l=0}^n \sum_{k=l}^n a_l b_{k-l} = \sum_{l=0}^n a_l \sum_{k=0}^{n-l} b_k = \sum_{l=0}^n a_l B_{n-l}.$$

For $\varepsilon > 0$ fix k_0 such that $k \geq k_0$ implies

$$|B_k - B| \left(\sum_{l=0}^{\infty} |a_l| \right) < \varepsilon.$$

Then, since we have

$$\begin{aligned} |C_n - AB| &= \left| \sum_{k=0}^n a_{n-k} (B_k - B) + (A_n - A)B \right| \\ &\leq \sum_{k=0}^{k_0-1} |a_{n-k}| |B_k - B| + \sum_{k=k_0}^n |a_{n-k}| |B_k - B| + |A_n - A| |B| \\ &\leq \max_{n-k_0 < k \leq n} |a_k| \left(\sum_{k=0}^{k_0-1} |B_k - B| \right) + \left(\sum_{k=k_0}^n |a_{n-k}| \right) \max_{k_0 \leq k \leq n} |B_k - B| + |A_n - A| |B| \end{aligned}$$

and $|a_n|$ and $|A_n - A|$ tend to zero as $n \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} |C_n - AB| < 0 + \varepsilon + 0.$$

□

2.2. (a) If $a_n \rightarrow 0$, then $\frac{1}{n} \sum_{k=1}^n a_k \rightarrow 0$.

2.3. (a) If $a_n \downarrow 0$, then $\sum_{k=1}^n (-1)^k a_n \rightarrow 0$.

Part II

Functions

Chapter 3

Continuity

Chapter 4

Differentiation

Chapter 5

Functional sequences

Part III

Integration

Chapter 6

Riemann integration