

# 複素解析学I演習 2023 年

**問 1** (フックス群としてのモジュラー群). 複素数体  $\mathbb{C}$  の部分集合  $A$  に対して、成分  $a, b, c, d$  が  $A$  の元で  $ad - bc = 1$  を満たす一次分数変換  $f(z) = (az + b)/(cz + d)$  の集合を  $\text{PSL}(2, A)$  と書く. 特に  $\text{PSL}(2, \mathbb{Z})$  を **モジュラー群** と呼ぶ. 上半平面  $\mathbb{H} := \{z \in \mathbb{C} : \text{Im} z > 0\}$  の部分集合  $D := \{z \in \mathbb{H} : |z| > 1, |\text{Re} z| < \frac{1}{2}\}$  を定義する.

- (1)  $\text{PSL}(2, \mathbb{R})$  の元  $f$  は全単射写像  $\mathbb{H} \rightarrow \mathbb{H}$  を定義することを示せ.
- (2)  $\text{PSL}(2, \mathbb{Z})$  は  $S(z) := -1/z$  と  $T(z) := z + 1$  によって生成されることを示せ. つまり、全ての元が  $S^{\pm 1}$  と  $T^{\pm 1}$  の有限回の合成として表れることを示せ.
- (3) 集合  $D$  は  $\text{PSL}(2, \mathbb{Z})$  の **基本領域** であることを示せ. つまり、次の二つが成り立つことを示せ:
  - (a) 任意の点  $z \in \mathbb{H}$  に対して  $f(z) \in \overline{D}$  を満たす  $f \in \text{PSL}(2, \mathbb{Z})$  が少なくとも一つ存在する.
  - (b) 任意の点  $z \in \mathbb{H}$  に対して  $f(z) \in D$  を満たす  $f \in \text{PSL}(2, \mathbb{Z})$  が多くとも一つ存在する.
- (4)  $\text{PSL}(2, \mathbb{Z})$  は  $\mathbb{H}$  に **真性不連続に作用** することを示せ. つまり、任意の点  $z \in \mathbb{H}$  に対して軌道  $\{f(z) : f \in \text{PSL}(2, \mathbb{Z})\}$  が離散集合であることを示せ.

**問 2** (カラテオドリ級関数集合の極点). 開単位円板上で定義された正則関数  $f$  が  $f(0) = 1$  を満たすとする. もし任意の  $|z| < 1$  を満たす複素数  $z$  に対して  $\text{Re} f(z) > 0$  ならば、 $f$  を **カラテオドリ級** の関数という. 関数  $f$  が冪級数展開  $f(z) = 1 + 2 \sum_{k=1}^{\infty} c_k z^k$  を持つとする.

- (1) 正の整数  $k$  と実数  $0 < r < 1$  に対して次の式を示せ:

$$c_k r^k = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} f(re^{i\theta}) e^{-ik\theta} d\theta.$$

- (2) 次の二つの条件が同値であることを示せ:
  - (a) 関数  $f$  がカラテオドリ級である.
  - (b) 任意の正の整数  $n$  に対して点  $(c_1, \dots, c_n) \in \mathbb{C}^n$  は  $\theta \in [0, 2\pi)$  によって媒介変数表示された曲線  $(e^{-i\theta}, \dots, e^{-in\theta}) \in \mathbb{C}^n$  の凸包絡の元である.

**問 3** (アールフォルス・清水標数). 複素平面上の有理型関数  $f$  を考える. 次のように  $r \geq 0$  に対する関数  $A(\cdot, f)$  を定義する:

$$A(r, f) := \frac{1}{\pi} \int_{\sqrt{x^2+y^2} \leq r} f^\#(x+iy)^2 dx dy, \quad \text{ただし、} f^\#(z) := \frac{|f'(z)|}{1+|f(z)|^2}, \quad z \in \mathbb{C}.$$

関数  $f^\#$  を  $f$  の **球面導関数** と呼ぶ.

- (1) 任意の点  $(x, y) \in \mathbb{R}^2$  に対して、

$$\frac{1}{\pi} f^\#(x+iy)^2 = \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)$$

を満たす実平面  $\mathbb{R}^2$  上の実関数  $P$  と  $Q$  を求め、関数  $K(x, y) := 1 + |f(x+iy)|^2$  を用いて表せ.

- (2) グリーンの定理と偏角の原理を用いて  $r \geq 0$  に対して次の式が成り立つことを示せ：

$$\int_0^r A(t, f) \frac{dt}{t} = \int_0^r n(t, f) \frac{dt}{t} + \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |f(re^{i\theta})|^2} d\theta - \log \sqrt{1 + |f(0)|^2}.$$

ただし、 $n(r, f)$  は閉円板  $\overline{B(0, r)}$  内にある重複度を込めて数えた  $f$  の極の数である．左辺の関数を  $f$  の **アールフォルス・清水標数** と呼ぶ．

- (3) 球面導関数  $f^\#$  が有界ならば、ある定数  $C > 0$  が存在して、全ての  $z \in \mathbb{C}$  に対して  $|f(z)| \leq Ce^{|z|^2}$  であることを示せ．特に、 $f$  は  $\mathbb{C}$  全体上正則である．

**問 4** (四分円上のディリクレ問題)．領域  $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0, y > 0\}$  上に定義された調和関数  $v \in C^2(\Omega, \mathbb{R})$  が次の境界値条件を満たすとする：各点  $(x_0, y_0) \in \partial\Omega$  に対して

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = \begin{cases} 1 & \text{if } y_0 > 0, \\ 0 & \text{if } y_0 = 0 \text{ and } 0 < x_0 < 1. \end{cases}$$

- (1) 反射原理を用いて  $v$  は領域  $\tilde{\Omega} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0\}$  上の調和関数  $\tilde{v} \in C^2(\tilde{\Omega}, \mathbb{R})$  に拡張されることを示せ．
- (2) 適切な等角変換とポアソン積分を用いて  $v$  を求めよ．

*Solution of 1.* (1) Let  $f(z) = (az + b)/(cz + d)$  with  $a, b, c, d \in \mathbb{R}$  such that  $ad - bd = 1$ . Since it has the inverse transform  $z \mapsto (dz - b)/(-cz + a)$  that is also an element of  $\text{PSL}(2, \mathbb{R})$ , it is enough to show the well-definedness  $f(z) \in \mathbb{H}$  for  $z \in \mathbb{H}$ . Let  $z = x + iy \in \mathbb{H}$  with  $y > 0$ . Then,

$$\text{Im} f(z) = \text{Im} \frac{ax + b + iay}{cx + d + icy} = \frac{ay(cx + d) - (ax + b)cy}{(cx + d)^2 + (cy)^2} \frac{y}{(cx + d)^2 + (cy)^2} > 0,$$

so  $f(z) \in \mathbb{H}$ .

(2) Let  $f(z) = (az + b)/(cz + d)$  with  $a, b, c, d \in \mathbb{Z}$  such that  $ad - bd = 1$ . Consider the following two kinds of moves of  $f$ :

- When  $|a| < |c|$ , we take

$$Sf(z) = \frac{-cz - d}{az + b}.$$

- When  $|a| \geq |c| > 0$ , with  $q, r \in \mathbb{Z}$  such that  $a = qc + r$  and  $0 \leq r < |c|$ , we take

$$T^{-q}f(z) = \frac{rz + b - qd}{cz + d}.$$

By repeating the two moves alternately, we arrive at  $c = 0$  in finitely many steps because  $|c|$  strictly decreases. Then, since  $ad - bc = 1$ , we may assume  $a = d = 1$  so that  $(az + b)/(cz + d) = z + b = T^b(z)$ .

(3) (a) Let  $z_0 \in \mathbb{H}$ . We may assume  $\text{Re } z_0 \in [-\frac{1}{2}, \frac{1}{2}]$  by taking  $T^q$  on  $z_0$  for appropriate  $q \in \mathbb{Z}$ . Define a sequence  $z_n \in \mathbb{H}$  inductively by

$$z_n := T^{-\lfloor \text{Re } S(z_{n-1}) + \frac{1}{2} \rfloor} S(z_{n-1}), \quad n \geq 1.$$

Then, one can show  $\text{Re } z_n \in [-\frac{1}{2}, \frac{1}{2}]$  for all  $n$ . Since

$$\text{Im } z_n = \text{Im } S(z_{n-1}) = \frac{\text{Im } z_{n-1}}{(\text{Re } z_{n-1})^2 + (\text{Im } z_{n-1})^2} \geq g(\text{Im } z_{n-1}),$$

where  $g(y) := 4y/(1 + 4y^2)$ , and since  $g^n(y) \uparrow \frac{\sqrt{3}}{2}$  for  $0 < y < \frac{\sqrt{3}}{2}$  as  $n \rightarrow \infty$ , there is  $n$  such that

$$-\frac{1}{2} \leq \text{Re } z_n < \frac{1}{2}, \quad \text{Im } z_n > \frac{\sqrt{3}}{4}.$$

If  $|z_n| \geq 1$ , then we are done, so assume  $|z_n| < 1$ . Now we have three possibilities:  $|z_n - 1| < 1$ ,  $|z_n + 1| < 1$ , or  $\min\{|z_n - 1|, |z_n + 1|\} \geq 1$ . For each case, we can check that  $T^{-1}S z_n$ ,  $TS z_n$ ,  $S z_n$  is contained in  $\overline{D}$ , respectively.

(b) For  $z \in D$ , let  $w = (az + b)/(cz + d) \in D$  with  $a, b, c, d \in \mathbb{Z}$  such that  $ad - bd = 1$ . It suffices to show  $c = 0$ . Suppose  $c \neq 0$ . Note that  $|z - n| > 1$  and  $|w - n| > 1$  for every integer  $n$  since  $z, w \in D$ . Write

$$1 < |w - n| = \left| \frac{az + b}{cz + d} - n \right| \leq \left| \frac{az + b}{cz + d} - \frac{a}{c} \right| + \left| n - \frac{a}{c} \right| = \left| \frac{1}{c(cz + d)} \right| + \left| n - \frac{a}{c} \right|, \quad n \in \mathbb{Z}.$$

If  $|c| \geq 2$ , then by taking  $n$  such that  $|n - (a/c)| \leq \frac{1}{2}$ , the estimate  $|c(cz + d)| \geq |c|^2 \text{Im } z > 2\sqrt{3}$  leads a contradiction to the above inequality. If  $|c| = 1$ , then since  $a/c$  is an integer, by letting  $n = a/c$ , we have a contradiction  $|c(cz + d)| = |z + cd| > 1$  from the assumption  $z \in D$ . Thus,  $c = 0$ , and we are done.

(4) Suppose the orbit  $\{f(z) : f \in \text{PSL}(2, \mathbb{Z})\}$  is not discrete. Then, there is  $z_0 \in \mathbb{H}$  and a sequence  $f_n \in \text{PSL}(2, \mathbb{Z})$  such that  $f_n(z) \neq z_0$  for all  $n$  and  $f_n(z) \rightarrow z_0$  as  $n \rightarrow \infty$ . We may assume  $z, z_0 \in \overline{D}$  by the part (a) of (3). Note that there are only finitely many  $f \in \text{PSL}(2, \mathbb{Z})$  such that  $f(\overline{D}) \cap \overline{D} \neq \emptyset$ . Let  $P$  be the set of such  $f$ .

If  $z_0 \in D$ , then  $f_n(z)$  eventually belongs to  $D$ , which is impossible by the part (b) of (3). If  $z_0 \in$

□

*Remark.* A discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{PSL}(2, \mathbb{C})$  is called a *Fuchsian group* and a *Kleinian group* respectively. It is known that a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  is discrete if and only if it properly discontinuously acts on  $\mathbb{H}$ . There is a more generalized theorem used for verifying a group generated by several elements of  $\mathrm{PSL}(2, \mathbb{R})$  is Fuchsian, the *Poincare polygon theorem*. It states that if there is a polygon in  $\mathbb{H}$  satisfying two conditions called a side pairing condition and elliptic cycle condition is realized as a fundamental domain, so the group acts on  $\mathbb{H}$  properly discontinuously.  $\square$

*Solution of 2.* (1) Suppose  $k > 0$  first. The Cauchy integral formula writes

$$2c_k k! = [k]f(z)(0) = \frac{k!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz = \frac{k!}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^k} d\theta,$$

and it implies

$$2c_k r^k = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta.$$

Since  $f(z)z^k$  is analytic, the Cauchy theorem can be applied to get

$$0 = \frac{1}{2\pi i} \int_{|z|=r} f(z) z^k dz = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) r^k e^{ik\theta} d\theta,$$

and it implies

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(re^{i\theta})} e^{-ik\theta} d\theta.$$

By combining the above two equations, we obtain the formula. For  $k = 0$ , applying the Cauchy theorem for  $f$ , we have

$$c_0 = f(0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z} dz = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) d\theta.$$

Alternatively, we can show the same result using the orthogonal relation of complex exponential functions. An easy computation shows the identity

$$\begin{aligned} \operatorname{Re} f(re^{i\theta}) &= \frac{1}{2} [f(re^{i\theta}) + \overline{f(re^{i\theta})}] \\ &= \frac{1}{2} \left[ \left( 1 + \sum_{k=1}^{\infty} 2c_k (re^{i\theta})^k \right) + \overline{\left( 1 + \sum_{k=1}^{\infty} 2c_k (re^{i\theta})^k \right)} \right] \\ &= \frac{1}{2} \left[ \left( 1 + \sum_{k=1}^{\infty} 2c_k r^k e^{ik\theta} \right) + \left( 1 + \sum_{k=1}^{\infty} 2\overline{c_k} r^k e^{-ik\theta} \right) \right] \\ &= \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}. \end{aligned}$$

From the uniform convergence of the power series on the compact set  $\{z : |z| \leq (r+1)/2\}$ , it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) e^{-ik\theta} d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|} \frac{1}{2\pi} \int_0^{2\pi} e^{il\theta} e^{-ik\theta} d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|} \delta_{kl} = c_k r^{|k|}.$$

(2) (b) $\Rightarrow$ (a) Denote by  $K_n$  the convex hull of the curve  $\theta \mapsto (e^{-i\theta}, \dots, e^{-in\theta}) \in \mathbb{C}^n$ . Suppose first that  $(c_1, \dots, c_n) \in K_n$ . For each  $n$ , there exists a finite sequence of pairs  $(\lambda_{n,j}, \theta_{n,j})_j$  having the following convex combination

$$(c_1, \dots, c_n) = \sum_j \lambda_{n,j} (e^{-i\theta_{n,j}}, \dots, e^{-in\theta_{n,j}})$$

with coefficients  $\lambda_{n,j} \geq 0$  such that  $\sum_j \lambda_{n,j} = 1$ . Define

$$f_n(z) := \sum_j \lambda_{n,j} \frac{e^{i\theta_{n,j}} + z}{e^{i\theta_{n,j}} - z},$$

which has positive real part on  $|z| < 1$  because  $\operatorname{Re}(e^{i\theta_{n,j}} + z)/(e^{i\theta_{n,j}} - z) > 0$  for  $|z| < 1$ . Then,

$$f_n(z) = \sum_j \lambda_{n,j} \left( 1 + \sum_{k=1}^{\infty} 2e^{-ik\theta_{n,j}} z^k \right) = 1 + \sum_{k=1}^n 2c_k z^k + \sum_{k=n+1}^{\infty} \left( \sum_j 2\lambda_{n,j} e^{-ik\theta_{n,j}} \right) z^k$$

implies

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=n+1}^{\infty} \left( \sum_j 2\lambda_{n,j} e^{-ik\theta_{n,j}} \right) z^k - \sum_{k=n+1}^{\infty} 2c_k z^k \right| \\ &\leq \sum_{k=n+1}^{\infty} \left| \left( \sum_j 2\lambda_{n,j} e^{-ik\theta_{n,j}} \right) - 2c_k \right| |z|^k \leq \sum_{k=n+1}^{\infty} 4|z|^k \end{aligned}$$

converges to zero for  $|z| < 1$ . Therefore,  $f$  has a non-negative real part on the open unit disk. The non-negativity can be strengthened to positivity by the open mapping theorem, so  $f$  belongs to the Carathéodory class.

(a) $\Rightarrow$ (b) Conversely, suppose that  $f$  is in the Carathéodory class. Let  $(\gamma_1, \dots, \gamma_n)$  be any point on the surface  $\partial K_n$  of  $K_n$  and  $S$  any supporting hyperplane of  $K_n$  tangent at  $(\gamma_1, \dots, \gamma_n)$ . Let  $(u_1, \dots, u_n) \in \mathbb{C}^n$  be the outward unit normal vector of the supporting hyperplane  $S$ . Note that this outward unit normal vector is uniquely determined for each hyperplane  $S$  with respect to the real inner product structure on the  $2n$ -dimensional real vector space  $\mathbb{C}^n$  given by

$$\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \rangle = \sum_{k=1}^n (\operatorname{Re} z_k \operatorname{Re} w_k + \operatorname{Im} z_k \operatorname{Im} w_k) = \operatorname{Re} \sum_{k=1}^n z_k \bar{w}_k.$$

Then, we know that  $\sum_{k=1}^n |u_k|^2 = 1$  and the maximum

$$M := \max_{(x_1, \dots, x_n) \in K_n} \operatorname{Re} \sum_{k=1}^n x_k \bar{u}_k > 0$$

is attained at  $(\gamma_1, \dots, \gamma_n)$ . Our goal is now to verify the bound

$$\operatorname{Re} \sum_{k=1}^n c_k \bar{u}_k \leq M$$

from the assumption that  $f$  is of Carathéodory class. Once the bound is obtained, then it means that  $(c_1, \dots, c_n)$  is contained in the same side as  $K_n$  of arbitrary hyperplanes tangent to  $K_n$ , so we finally conclude  $(c_1, \dots, c_n) \in K_n$ .

Since for any  $\theta \in [0, 2\pi)$  the point  $(e^{-i\theta}, \dots, e^{-in\theta})$  is in  $K_n$ , we have

$$\operatorname{Re} \sum_{k=1}^n e^{-ik\theta} \bar{u}_k \leq M.$$

For  $\varepsilon > 0$ , we have

$$\operatorname{Re} \sum_{k=1}^n \frac{1}{r^k} e^{-ik\theta} \bar{u}_k \leq M + \varepsilon$$

for any  $0 < r < 1$  sufficiently close to 1, thus we can write

$$\begin{aligned} \operatorname{Re} \sum_{k=1}^n c_k \bar{u}_k &= \operatorname{Re} \sum_{k=1}^n \frac{1}{2\pi r^k} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) e^{-ik\theta} \bar{u}_k d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) \operatorname{Re} \sum_{k=1}^n \frac{1}{r^k} e^{-ik\theta} \bar{u}_k d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(re^{i\theta}) d\theta \cdot (M + \varepsilon) \\ &= \operatorname{Re} f(0)(M + \varepsilon) = M + \varepsilon \end{aligned}$$

thanks to the part (1) and the positivity of  $\operatorname{Re} f$ , and by limiting  $r \rightarrow 1$  from left we get the bound we want.  $\square$

Solution of 3. (1)

$$\frac{du \wedge dv}{\pi(1+u^2+v^2)^2} = d\left(-\frac{v}{2\pi(1+u^2+v^2)} du + \frac{u}{2\pi(1+u^2+v^2)} dv\right)$$

$$P = -\frac{K_y}{4\pi K}, \quad Q = \frac{K_x}{4\pi K}.$$

(2) Since the equation holds for  $r = 0$ , it suffices to show the differentiated equation

$$A(r, f) = n(r, f) + \frac{r}{2\pi} \frac{d}{dr} \int_0^{2\pi} \log \sqrt{1 + |f(re^{i\theta})|^2} d\theta, \quad r \geq 0.$$

(3) Since every Taylor coefficient of the logarithm is real, we have

$$\operatorname{Re} \log f(z) = \frac{1}{2}(\log f(z) + \log \overline{f(z)}) = \log |f(z)|.$$

Take  $a \in \mathbb{C}$  and let  $r := 2|a|$ . By the Schwarz integral formula,

$$\begin{aligned} \log |f(a)| &= \operatorname{Re} \log f(a) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{re^{i\theta} + a}{re^{i\theta} - a} \operatorname{Re} \log f(re^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{re^{i\theta} + a}{re^{i\theta} - a} \right| \log |f(re^{i\theta})| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} 3 \log \sqrt{1 + |f(re^{i\theta})|^2} d\theta \\ &\leq 3 \int_0^r A(t, f) \frac{dt}{t} \leq 3 \int_0^r M^2 t^2 \frac{dt}{t} = 6M^2 |a|^2, \end{aligned}$$

so  $C := e^{6M^2}$  proves the theorem, where  $M$  is a bound of the spherical derivative  $f^\#$ . □

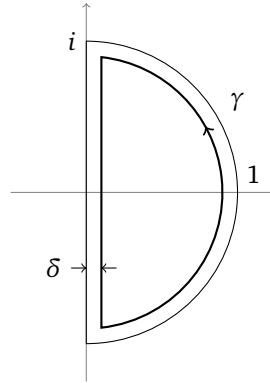
*Solution of 4.* (1) Identify  $\Omega$  and  $\tilde{\Omega}$  as subsets of  $\mathbb{C}$  by letting  $(x, y) = x + iy$ . Consider a harmonic conjugate  $-u$  of  $v$  on  $\Omega$  such that a function  $f(x + iy) := u(x, y) + iv(x, y)$  is holomorphic on  $\Omega$ . If we define

$$\tilde{f}(z) := \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0, \\ \overline{f(\bar{z})} & \text{if } \operatorname{Im} z < 0, \end{cases} \quad z \in \tilde{\Omega},$$

then  $\tilde{f}$  is holomorphic on  $\tilde{\Omega} \setminus (0, 1)$ , and is also continuous on the whole  $\tilde{\Omega}$  because of the boundary condition of  $v$  on the real axis. We claim that  $\tilde{f}$  is in fact holomorphic on  $\tilde{\Omega}$ . If the claim is true, then  $\tilde{v} := \operatorname{Im} \tilde{f}$  is the desired extension of  $v$ , which satisfies in addition that for  $(x_0, y_0) \in \partial \tilde{\Omega}$  we have

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \tilde{v}(x, y) = \begin{cases} 1 & \text{if } y_0 > 0, \\ -1 & \text{if } y_0 < 0. \end{cases}$$

Let  $\gamma$  be a contour defined for sufficiently small  $\delta > 0$  as the following figure:



Denote by  $\tilde{\Omega}_\delta := \{a \in \tilde{\Omega} : \min_{z_0 \in \partial \tilde{\Omega}} |z_0 - a| > \delta\}$  the interior of  $\gamma$ . Define a function  $\tilde{g}$  on  $\tilde{\Omega}_\delta$  such that

$$\tilde{g}(a) := \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(z)}{z - a} dz, \quad a \in \tilde{\Omega}_\delta.$$

Note that the integrand is continuous on the contour  $\gamma$ , and  $\tilde{g}$  is holomorphic on  $\tilde{\Omega}_\delta$  by the Morera theorem, because for every affine triangle  $\sigma$  in the interior of  $\gamma$  we have

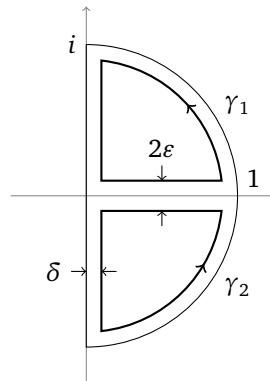
$$\int_\sigma \tilde{g}(a) dz = \int_\sigma \frac{1}{2\pi i} \int_\gamma \frac{\tilde{f}(z)}{z - a} dz da = \frac{1}{2\pi i} \int_\gamma \left[ \int_\sigma \frac{\tilde{f}(z)}{z - a} da \right] dz = 0$$

by the Fubini theorem and the Cauchy theorem for  $\sigma$ .

Moreover, for  $a \in \tilde{\Omega}_\delta \cap \Omega$  we have

$$\tilde{g}(a) = \lim_{\varepsilon \rightarrow 0} \left[ \frac{1}{2\pi i} \int_{\gamma_1} \frac{\tilde{f}(z)}{z - a} dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{\tilde{f}(z)}{z - a} dz \right] = \tilde{f}(a) + 0 = \tilde{f}(a),$$

where  $\gamma_1$  and  $\gamma_2$  are contours given as the following figure for  $\varepsilon > 0$ :





The same result holds also for  $a \in \tilde{\Omega}_\delta \setminus \bar{\Omega}$ , so we can conclude  $\tilde{g}(a) = \tilde{f}(a)$  on  $a \in \tilde{\Omega}_\delta \setminus (0, 1)$ , and by the continuity of  $\tilde{f}$  and  $\tilde{g}$ , we finally have  $\tilde{f} = \tilde{g}$  so that  $\tilde{f}$  is holomorphic on  $\tilde{\Omega}_\delta$ . Since the above arguments make sense for every  $\delta > 0$  small enough, the union  $\tilde{\Omega} = \bigcup_{\delta > 0} \tilde{\Omega}_\delta$  implies that the function  $\tilde{f}$  is holomorphic on  $\tilde{\Omega}$ .

(2) The domain  $\tilde{\Omega}$  is conformally mapped onto the upper half plane  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$  by

$$\varphi : \tilde{\Omega} \rightarrow \mathbb{H} : z \mapsto \left( \frac{z+i}{iz+1} \right)^2.$$

Note that  $\varphi(\Omega) = \{z \in \mathbb{H} : |z| > 1\}$ .

We can compute for  $(x, y) \in \tilde{\Omega}$

$$|\varphi(x+iy)|^2 = \left( \frac{x^2 + (y+1)^2}{x^2 + (y-1)^2} \right)^2, \quad \text{Im } \varphi(x+iy) = \frac{4x(1-x^2-y^2)}{(x^2 + (y-1)^2)^2}.$$

Define a function  $V : \mathbb{H} \rightarrow \mathbb{R}$  such that  $V := \tilde{v} \circ \varphi^{-1}$ . Then,  $V$  is a harmonic function satisfying the boundary condition

$$\lim_{(x,y) \rightarrow (x_0,0)} V(x,y) = \begin{cases} -1 & \text{if } |x_0| < 1, \\ 1 & \text{if } |x_0| > 1. \end{cases}$$

For  $(x, y) \in \varphi(\Omega)$  so that  $x^2 + y^2 > 1$  the Poisson kernel gives that

$$\begin{aligned} \frac{1-V(x,y)}{2} &= \frac{1}{\pi} \int_{-1}^1 \frac{y}{(x-t)^2 + y^2} dt \\ &= \frac{1}{\pi} \left( \tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) \\ &= \frac{1}{\pi} \tan^{-1} \frac{2y}{x^2 + y^2 - 1}, \end{aligned}$$

so

$$V(x,y) = \frac{2}{\pi} \tan^{-1} \frac{x^2 + y^2 - 1}{2y}.$$

Thus we have for  $(x, y) \in \Omega$

$$v(x,y) = V(\varphi(x+iy)) = \frac{2}{\pi} \tan^{-1} \frac{y(1+x^2+y^2)}{x(1-x^2-y^2)}.$$

□