Contents

1 Topological group action

1.1. Let G be a topological group acting on a topological space X. Let $p : X \to X/G$ be the quotient map.

(a) $p^{-1}(p(A)) = \bigcup_{g \in G} gA$ for any $A \subset X$.

(b) p is open.

(c) If $x \neq gx$, then there is an open neighborhood U of x such that gU is disjoint to U.

Proof. (c) Since X is Hausdorff, there is disjoint open neighborhoods U_0 and U_1 respectively of x and gx. Then, $U := g^{-1}(gU_0 \cap U_1) \subset U_0$ and $gU = gU_0 \cap U_1 \subset U_1$ are disjoint. □

1.2. Let $f : X \to Y$ be continuous. We say f is *proper* if $f^{-1}(K)$ is compact for compact K. We say f is *Bourbaki-proper* if it is closed and proper. If X is Hausdorff and Y is locally compact, then two notions are equivalent.

1.3 (Properly discontinuous actions)**.** Let G be a topological group acting on a topological space X. Let $p : X \to X/G$ be the quotient map. This action is called *properly discontinuous* if for every compact $K \subset X$ only finite gK intersect K.

(a) A free and proper action is properly

1.4 (Covering space actions)**.** Let G be a topological group acting on a topological space X. Let $p : X \to X/G$ be the quotient map. This action is called a *covering space action* if every $x \in X$ has a neighborhood U such that gU are all disjoint for $g \in G$.

(a) A properly discontinuous and free action is a covering space action, if X is locally compact and Hausdorff.

(b) A covering space action is properly discontinuous.

(c) A covering space action is free.

Proof. (a) Fix $x \in X$ and let K be a compact neighborhood of x. By the proper discontinuity, there is a finite subset $F \subset G$ such that gK intersects K only for $g \in F$. Because the action is free, for every $g \in F \setminus \{1\}$ there is an open neighborhood U_g of x such that $gU_g \cap U_g = \varnothing$. Then, $U := K^\circ \cap \bigcap_{g \in F\setminus\{1\}} U_g$ satisfies $gU \cap U = \varnothing$.

(b)

$\square$

2 Universal coefficient theorem

Lemma 2.1. *Suppose we have a flat resolution*

$$0 \to P_1 \to P_0 \to A \to 0.$$

Then, we have a exact sequence

$$\cdots \to 0 \to \mathrm{Tor}_1^R(A,B) \to P_1 \otimes B \to P_0 \otimes B \to A \otimes B \to 0.$$

Theorem 2.2. *Let R be a PID. Let $C_\bullet$ be a chain complex of flat R-modules and G be a R-module. Then, we have a short exact sequence*

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0,$$

which splits, but not naturally.

1. We have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

where every morphism in $Z_\bullet$ and $B_\bullet$ are zero. Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence

$$0 \to Z_\bullet \otimes G \to C_\bullet \otimes G \to B_{\bullet-1} \otimes G \to 0$$

and the associated long exact sequence

$$\to H_n(B;G) \to H_n(Z;G) \to H_n(C;G) \to H_{n-1}(B;G) \to H_{n-1}(Z;G) \to$$

where the connecting homomomorphisms are of the form $(i_n \colon B_n \longrightarrow Z_n) \otimes 1_G$ (It is better to think diagram chasing than a natural construction). Since morhpisms in B and Z are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\to B_n \otimes G \to Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to Z_{n-1} \otimes G \to \ .$$

Since

$$0 \to \mathrm{Tor}_1^R(H_n, G) \to B_n \otimes G \to Z_n \otimes G \to H_n \otimes G \to 0$$

for all n, the exact sequence splits into short exact sequence by images

$$0 \to H_n \otimes G \to H_n(C;G) \to \mathrm{Tor}_1^R(H_{n-1}, G) \to 0.$$

For splitting, $\square$

2. Since R is PID, we can construct a flat resolution of G

$$0 \to P_1 \to P_0 \to G \to 0.$$

Since modules in $C_\bullet$ are flat so that the tensor product functors are exact and $P_1 \longrightarrow P_0$ and $P_0 \longrightarrow G$ induce the chain maps, we have a short exact sequence of chain complexes

$$0 \to C_\bullet \otimes P_1 \to C_\bullet \otimes P_0 \to C_\bullet \otimes G \to 0.$$

Then, we have the associated long exact sequence

$$\to H_n(C;P_1) \to H_n(C;P_0) \to H_n(C;G) \to H_{n-1}(C;P_1) \to H_{n-1}(C;P_0) \to \ .$$

Since flat tensor product functor commutes with homology funtor from chain complexes, we have

$$\to H_n \otimes P_1 \to H_n \otimes P_0 \to H_n(C;G) \to H_{n-1} \otimes P_1 \to H_{n-1} \otimes P_0 \to \ .$$

Since

$$0 \to \operatorname{Tor}_1^R(G, H_n) \to H_n \otimes P_1 \to H_n \otimes P_0 \to H_n \otimes G \to 0$$

for all n, the exact sequence splits into short exact sequence by images

$$0 \to H_n \otimes G \to H_n(C;G) \to \operatorname{Tor}_1^R(G, H_{n-1}) \to 0.$$

$\square$

Proof 3. By tensoring G, we get the following diagram.

$$\begin{tikzcd}
H_n \otimes G \arrow[dr] & & & & H_{n-1} \otimes G \\
& \operatorname{coker} \partial_{n+1} \otimes G \arrow[dr, two heads] & & \ker \partial_{n-1} \otimes G \arrow[ur, two heads] \arrow[dr] & \\
C_n \otimes G \arrow[ur, two heads] \arrow[drr, two heads] & & & & C_{n-1} \otimes G \\
& & \operatorname{im} \partial_n \otimes G \arrow[uur] \arrow[urr] & & \\
& \operatorname{Tor}_1(H_{n-1}, G) \arrow[ur, tail] & & &
\end{tikzcd}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernals are preserved, but monomorphisms and kernels are not. Especially, $\operatorname{coker} \partial_{n+1} \otimes G = \operatorname{coker}(\partial_{n+1} \otimes 1_G)$ is important.

Consider the following diagram.

$$\begin{tikzcd}
H_n(C;G) \arrow[dr, tail] & H_n \otimes G \arrow[d] & & & \\
& \operatorname{coker} \partial_{n+1} \otimes G \arrow[dd, two heads] \arrow[ddrr, two heads] & & \ker \partial_{n-1} \otimes G \arrow[dr, "\text{monic!}"] & \\
& & & & C_{n-1} \otimes G \\
& \operatorname{im} \partial_n \otimes G \arrow[uurr] & & \operatorname{im}(\partial_n \otimes 1_G) \arrow[uu, dashed, tail] \arrow[ur, tail] & \\
\operatorname{Tor}_1(H_{n-1}, G) \arrow[ur, tail] & & & &
\end{tikzcd}$$

Since $\ker \partial_{n-1}$ is free,

If we show $\operatorname{im}(\partial_n \otimes 1_G) \longrightarrow \ker \partial_{n-1} \otimes G$ is monic, then we can get

$$\begin{aligned}
H_n(C;G) &= \ker(\operatorname{coker} \partial_{n+1} \otimes G \longrightarrow \operatorname{im}(\partial_n \otimes 1_G)) \\
&= \ker(\operatorname{coker} \partial_{n+1} \otimes G \longrightarrow \ker \partial_{n-1} \otimes G).
\end{aligned}$$

3 Fundamental differential geometry

3.1 Manifold and Atlas

Definition 3.1. A *locally Euclidean space M* of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{R}^d$.

Definition 3.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 3.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of $\mathbb{R}^d$. A chart is often written by a pair (U, φ).

Definition 3.4. An *atlas* $\mathcal{F}$ is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 3.5. A *differentiable maifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

3.2 Definition of Differentiable Structure

Definition 3.6. An atlas $\mathcal{F}$ is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each *transition function* $\tau_{\alpha\beta} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 3.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 3.8. An differentiable atlas $\mathcal{F}$ is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in $\mathcal{F}$, then $(U, \varphi) \in \mathcal{F}$.

Definition 3.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manofold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M. When the charts is already equipped on M, it is natural to define a function $f : M \to \mathbb{R}$ differentiable if the functions $f \circ \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas $\mathcal{F}$, then f is also differentiable on any atlases which is equivalent to $\mathcal{F}$ by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 3.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 3.10. A continuous function $f : M \to N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

3.3 Curves

Definition 3.11. For $f : M \to \mathbb{R}$ and (U, ϕ) a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 3.12. Let $\gamma : I \to M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi : M \to N$ be a smoth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f : M \to \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \to \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt} f \circ \gamma.$$

Proposition 3.1. *Let $\gamma : I \to M$ be a smooth curve on a manifold M. The notation $\dot{\gamma}^\mu$ is not confusing thanks to*

$$(\dot{\gamma})^\mu = (\dot{\gamma^\mu}).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt} x^\mu \circ \gamma.$$

3.4 Connection computation

$$\begin{aligned}
\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\
&= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma^\lambda_{\mu\nu} \partial_\lambda) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda \right) \partial_\nu.
\end{aligned}$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ.

$$Y^\nu_{,\mu} = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \qquad Y^\nu_{;\mu} = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda.$$

Theorem 3.2. *For Levi-civita connection for g,*

$$\Gamma^l_{ij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Proof.

$$\begin{aligned}
(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} \\
(\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma^l_{jk} g_{li} - \Gamma^l_{ji} g_{kl} \\
(\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}
\end{aligned}$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$. Thus,

$$\Gamma^l_{ij} g_{kl} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

□

3.5 Geodesic equation

Theorem 3.3. *If c is a geodesic curve, then components of c satisfies a second-order differential equation*

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0.$$

Proof. Note

$$0 = \nabla_{\dot\gamma}\dot\gamma = \dot\gamma^\mu\nabla_\mu(\dot\gamma^\lambda\partial_\lambda) = (\dot\gamma^\nu\partial_\nu\dot\gamma^\mu + \dot\gamma^\nu\dot\gamma^\lambda\Gamma^\mu_{\nu\lambda})\partial_\mu.$$

Since

$$\dot\gamma^\nu\partial_\nu\dot\gamma^\mu = \dot\gamma(\dot\gamma^\mu) = d\dot\gamma^\mu(\dot\gamma) = d\dot\gamma^\mu\circ d\gamma\left(\frac{\partial}{\partial t}\right) = d\dot\gamma^\mu\left(\frac{\partial}{\partial t}\right) = \ddot\gamma^\mu,$$

we get a second-order differential equation

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0$$

for each μ. □

4 Vector calculus on spherical coordinates

$$\begin{array}{rcccccccccc}
V & = & (V_r, V_\theta, V_\phi) \\
& = & V_r & \widehat{r} & + & V_\theta & \widehat{\theta} & + & V_\phi & \widehat{\phi} & \text{(normalized} \\
& = & V_r & \frac{\partial}{\partial r} & + & \frac{1}{r}\,V_\theta & \frac{\partial}{\partial\theta} & + & \frac{1}{r\sin\theta}\,V_\phi & \frac{\partial}{\partial\phi} & (\Gamma(TM \\
& = & V_r & dr & + & r\,V_\theta & d\theta & + & r\sin\theta\,V_\phi & d\phi & (\Omega^1(M \\
& = & r^2\sin\theta\,V_r & d\theta\wedge d\phi & + & r\sin\theta\,V_\theta & d\phi\wedge dr & + & r\,V_\phi & dr\wedge d\theta & (\Omega^2(M
\end{array}$$

$$\nabla\cdot V = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\,V_r\right) + \frac{\partial}{\partial\theta}(r\sin\theta\,V_\theta) + \frac{\partial}{\partial\phi}\left(r\,V_\phi\right)\right]$$

$$\Delta u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}u\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}u\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}u\right)\right]$$

Let (ξ,η,ζ) be an orthogonal coordinate that is *not* normalized. Then,

$$\sharp = g = \mathrm{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2)$$

$$\widehat{x} = \|\partial_x\|^{-1}\,\partial_x = \|\partial_x\|\,dx = \|\partial_y\|\|\partial_z\|\,dy\wedge dz$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r\,d\theta, \quad r\sin\theta\,d\phi, \quad (r\,d\theta)\wedge(r\sin\theta\,d\phi), \quad (r\sin\theta\,d\phi)\wedge(dr), \quad (dr)\wedge(r\,d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[\ \frac{1}{\|\partial_x\|}\frac{\partial}{\partial x}\cdot - \ ,\ \frac{1}{\|\partial_y\|}\frac{\partial}{\partial y}\cdot - \ ,\ \frac{1}{\|\partial_z\|}\frac{\partial}{\partial z}\cdot - \ \right] \\
\text{curl} : \nabla &= \left[\ \frac{1}{\|\partial_y\|\|\partial_z\|}\left(\frac{\partial}{\partial y}(\|\partial_z\|\cdot -) - \frac{\partial}{\partial z}(\|\partial_y\|\cdot -)\right),\right. \\
&\qquad \frac{1}{\|\partial_z\|\|\partial_x\|}\left(\frac{\partial}{\partial z}(\|\partial_x\|\cdot -) - \frac{\partial}{\partial x}(\|\partial_z\|\cdot -)\right), \\
&\qquad \left.\frac{1}{\|\partial_x\|\|\partial_y\|}\left(\frac{\partial}{\partial x}(\|\partial_y\|\cdot -) - \frac{\partial}{\partial y}(\|\partial_x\|\cdot -)\right)\ \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\|\|\partial_y\|\|\partial_z\|}\left[\ \frac{\partial}{\partial x}\big(\|\partial_y\|\|\partial_z\|\cdot -\big),\ \frac{\partial}{\partial y}\big(\|\partial_z\|\|\partial_x\|\cdot -\big),\ \frac{\partial}{\partial z}\big(\|\partial_x\|\|\partial_y\|\cdot -\big)\ \right] \\
\Delta &= \frac{1}{\|\partial_x\|\|\partial_y\|\|\partial_z\|}\left[\ \frac{\partial}{\partial x}\left(\frac{\|\partial_y\|\|\partial_z\|}{\|\partial_x\|}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\|\partial_z\|\|\partial_x\|}{\|\partial_y\|}\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\|\partial_x\|\|\partial_y\|}{\|\partial_z\|}\frac{\partial}{\partial z}\right)\ \right]
\end{aligned}$$

$$\begin{aligned}
\text{grad} &= \frac{1}{\|\cdot\|^1}(\nabla)\|\cdot\|^0 \\
\text{curl} &= \frac{1}{\|\cdot\|^2}(\nabla\times)\|\cdot\|^1 \\
\text{div} &= \frac{1}{\|\cdot\|^3}(\nabla\cdot)\|\cdot\|^2
\end{aligned}$$

5 Bundles

Show that S^n has a nonvanishing vector field if and only if n is odd.

Solution. Since S^n is embedded in $\mathbb{R}^{n+1}$, the tangent bundle TS^n can be considered as an embedded manifold in $S^n \times \mathbb{R}^{n+1}$ which consists of (x, v) such that $\langle x, x\rangle = 1$ and $\langle x, v\rangle = 0$, where the inner product is the standard one of $\mathbb{R}^{n+1}$.

Suppose n is odd. We have a vector field $(x_1, x_2, \cdots, x_{n+1}; x_2, -x_1, \cdots, -x_n)$ which is nonvanishing.

Conversely, suppose we have a nonvanishing vector field X. Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \to S^n \times \mathbb{R}^{n+1} \to \phi\mathbb{R}^{n+1} \to S^n.$$

The last map can be defined since X is nowhere zero. Since this map satisfies $\langle x, \phi(x)\rangle = 0$ for all $x \in S^n$, we can define homotopies from ϕ to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree, $+1$, so n is odd. □

Proposition 5.1. *Independent commuting vector fields are realized as partial derivatives in a chart.*

Proposition 5.2. *Let $\{\partial_1, \cdots, \partial_k\}$ be an independent involutive vector fields. We can find independent commuting $\{\partial_{k+1}, \cdots, \partial_n\}$ such that union is independent. (Maybe)*

Proposition 5.3. *Let $\{\partial_1, \cdots, \partial_k\}$ be an independent commuting vector fields. We can find independent commuting $\{\partial_{k+1}, \cdots, \partial_n\}$ such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

Proposition 5.4. *An immersed manifold is locally an inverse image of a regular value.*

Proposition 5.5. *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

Proof. It uses tubular neighborhood. Pontryagin construction? □

Proposition 5.6. *An immersed manifold is locally a linear subspace in a chart.*

Proposition 5.7. *Distinct two points on a connected manifold are connected by embedded curve.*

Proof. Let $\gamma : I \to M$ be a curve connecting the given two points, say p, q.

*Step [.1]*Constructing a piecewise linear curve For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\operatorname{im}\gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \varnothing$ for all $1 \le i \le n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i,j) : U_{t_i} \cap U_{t_j} \neq \varnothing$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \le i \le n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q.

*Step [.2]*Smoothing the curve □

Proposition 5.8. *Let M is an embedded manifold with boundary in N. Any kind of sections on M can be extended on N.*

Proposition 5.9. *Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.*

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Proposition 5.10. *The set of points that is geodesically connected to a point is open.*