

# Fiber Bundles

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# 1 Day 1: April 10

References: Steenrod, *The topology of fiber bundles*, and Tamaki, *Fiber bundles and homotopy* (Japanese)

## 1. Introduction

An ultimate goal of topology is to classify topological spaces, up to homeomorphism. If you want to show two spaces are homeomorphic, we should construct a homeomorphism: *Shokuninwaza* (wild knot, Casson handle). If you want to show two spaces are not homeomorphic, then we can investigate topological *properties*, and as their quantitative comparison, we can investigate topological *invariants*. Some examples include

- the number of connected components,
- the Euler characteristic,
- homology groups,
- homotopy groups,
- the minimal number of open contractible sets to cover the spaces (Lusternik-Schnirelmann category, topological complexity),
- Gelfand-Naimark theorem:  $C(X) \cong C(Y)$  implies  $X \cong Y$  if they are compact Hausdorff.

We will restrict objects to study. For example, metric spaces, manifolds, CW-complexes. As the assumptions change, invariants may have different appearances. For a manifold  $X$ ,

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rk}_{\mathbb{Z}} H_q(X) = \sum_{q=0}^{\infty} (-1)^q b_q(X).$$

For a CW-complex  $X$ ,

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q (\text{the number of } q\text{-cells}).$$

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Let  $M$  be an connected closed  $n$ -dimensional manifold. Some classification results are as follows (up to both homeomorphisms and diffeomorphisms, because  $d \leq 2$ ):

- $(n=0)$   $M \cong *$ , and  $\chi(*) = 1$ .
- $(n=1)$   $M \cong S^1$ , and  $\chi(S^1) = 0$ .
- $(n=2)$ 
  - If  $M$  is orientable, then  $M \cong \Sigma_g$  for  $g \geq 0$ , and  $\chi(\Sigma_g) = 2 - 2g$ .  
 $\Sigma_0 \cong S^2$ ,  $\Sigma_1 \cong T^2$ .
  - If  $M$  is not orientable, then  $M \cong (\mathbb{RP}^2)^{\#h}$  for  $h \geq 1$ , and  $\chi((\mathbb{RP}^2)^{\#h}) = 2 - h$ .  
 $\mathbb{RP}^2 (\cong \text{Möbius strip} \cup D^2)$ ,  $K = \mathbb{RP}^2 \# \mathbb{RP}^2$

**Problem 1.** Show  $\mathbb{RP}^2 \# T^2 \cong \mathbb{RP}^2 \# K$ .

Here are some facts about triangulability:

- Cairns(1935), Whitehead (1940): every  $C^1$ -manifold is triangulable (unique as a PL-manifold).
- Rado(1925,  $n=2$ ), Moise(1952,  $n=3$ ): for  $n \leq 3$ , every  $C^0$ -manifold is triangulable (unique as a PL-manifold).
- Kirby-Siebermann(1966,  $n \geq 5$ ): for  $n \geq 4$ , there is a non-triangulable PL-manifold.

- Donaldson, Freedman, Casson: for  $n = 4$ , there is a non-triangulable manifold as a topological space.
- Manolescu(2013): for  $n \geq 5$ , there is a non-triangulable manifold as a topological space.

Orientability? For a connected closed surface  $S$ , it is orientable iff  $H_2(S) \cong \mathbb{Z}$ , not orientable iff  $H_2(S) \cong 0$ . The generator of  $H_2(S)$  is called the fundamental class. Orientability asks if the tubular neighborhood of every simple closed curve is homeomorphic to an annulus. It is described by the first Stiefel-Whitney class:

$$w_1(S) \in H^1(S; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(H^1(S), \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(\pi_1(S), \mathbb{Z}/2\mathbb{Z}).$$

## Euler characteristic of manifolds

### (0) Odd-dimensional manifolds

**Theorem.** For an odd-dimensional closed connected manifold,  $\chi(M^{2n+1}) = 0$ .

*Proof.* If orientable, then  $b_0(M) = 1$ ,  $b_3(M) = 1$ ,  $b_1(M) = b_2(M)$  by the Poincaré duality. If not, a double cover is orientable, and  $\chi(\tilde{M}) = 2\chi(M)$ .  $\square$

### (1) Gauss-Bonnet theorem

**Theorem** (Gauss-Bonnet). If a smooth manifold  $M^n$  embeds into  $\mathbb{R}^{n+1}$  (hypersurface), then it is orientable and the Euler characteristic is given by

$$\chi(M) = \frac{2}{\text{vol}(S^n)} \int_M K \, d \, \text{vol}_M.$$

## 2 Day 2: April 17

We have a cohomological interpretation. In the Chern-Weil theory, we have a generalized version of the Gauss-Bonnet theorem for a general compact manifold using the theory of connections. We can interpret  $2 \text{vol}(S^n)^{-1} K \cdot d \, \text{vol}_M$  as a differential form which provides with the Euler characteristic. In the context of the de Rham theorem, we will eventually call the equivalence class of this differential form as the *Euler class*.

### (2) Poincaré-Hopf theorem

Let  $M^n$  be a orientable connected smooth closed manifold. Let  $X$  be a smooth vector field on  $M$  such that there are only finitely many zeros  $\{p_1, \dots, p_m\}$ . For each  $p_j$ , define the index  $\text{Ind}(X, p_j)$  as follows: seeing  $X$  as a vector field on  $\varphi_j(U_j)$  for a chart  $(U_j, \varphi_j)$  not containing zeros of  $X$  but  $p_j$  and mapping  $p_j$  to zero in  $\mathbb{R}^n$ , we define  $\text{Ind}(X, p_j) = \deg f_j$ , where  $f_j : S_\varepsilon(\approx S^{n-1}) \rightarrow S^{n-1} : x \mapsto X_x / \|X_x\|$ .

**Example.** Let  $n = 2$ . We have indices 1, 1, 1, -1, 0, 2 for

$$\begin{aligned} X_1(x, y) &= (x, y), & X_2(x, y) &= (-x, -y), & X_3(x, y) &= (-y, x), \\ X_4(x, y) &= (-x, y), & X_5(x, y) &= \sqrt{x^2 + y^2}(1, 1), & X_6(x, y) &= (x^2 - y^2, 2xy). \end{aligned}$$

**Theorem** (Poincaré-Hopf).

$$\sum_{j=1}^m \text{Ind}(X, p_j) = \chi(M).$$

We have a cohomological interpretation. Let  $c = \sum_{j=1}^m \text{Ind}(X, p_j) p_j$  be a singular 0-cycle on  $M$ . Then, the Poincaré-Hopf theorem states that we have

$$\begin{array}{ccc} H_0(M) & \xrightarrow{\sim} & \mathbb{Z} \\ p_j & \mapsto & 1 \\ c & \mapsto & \chi(M). \end{array}$$

By the Poincaré duality, we can identify the homology class  $[c]$  with a de Rham cohomology class, and the above map is just an integration map.

The cycle  $c$  tells us the information of intersections of  $X$  and zero section (of the tangent bundle). If  $TM$  is trivial, then the zero section does not self-intersect(?) so that  $c = 0$ . The Euler characteristic measures the twist of a bundle, and the characteristic class generalizes this wakugumi.

## 2. Fiber bundles

From now we will only consider paracompact Hausdorff spaces. Recall that a space is paracompact iff for every open cover there is a locally finite refinement.

**Example.** Open sets of  $\mathbb{R}^n$ , metric spaces, CW-complexes, countable inductive limit of compact spaces are paracompact.

**Theorem 2.1.** *For any open cover of a paracompact Hausdorff space  $X$ , there is a partition of unity subordinate to it.*

**Problem 2.** Prove the above theorem.

**Definition 2.2.** Let  $B$  be connected (for simplicity). A map  $E \rightarrow B$  is called a fiber bundle with fiber  $F$ , or just a  $F$ -bundle, if it is locally trivial: every point  $x \in B$  has an open neighborhood  $U_x$  such that there is a homeomorphism  $\varphi : p^{-1}(U_x) \rightarrow U_x \times F$  with  $p = \text{pr}_{U_x} \circ \varphi$ .

For each  $y \in B$   $E_y := p^{-1}(y)$  is homeomorphic to  $F$ , and is called the fiber at  $y$ . Also,  $E$  and  $B$  are called the total space and the base space. We sometimes write as  $\xi = (F \rightarrow E \xrightarrow{p} B)$ .

**Example.**

- (a) We say  $\text{pr}_1 : B \times F \rightarrow B$  is the product or bundle.
- (b)  $p : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z} : t \mapsto [t]$  is a  $\mathbb{Z}$ -bundle. In general, a fiber bundle with a discrete fiber is called a covering space.
- (c)  $p_1 : S^n \rightarrow \mathbb{R}P^n = S^n/(x \sim -x)$  is a  $\mathbb{Z}/2\mathbb{Z}$ -bundle.
- (d)  $p : S^{2n+1} \rightarrow \mathbb{C}P^n = S^{2n+1}/(z \sim uz)$  for  $u \in S^1$  is a  $S^1$ -bundle. (a generalization of Hopf bundles)
- (e) Let  $M^n$  be a smooth manifold. Then, the tangent and the cotangent bundles are  $\mathbb{R}^n$ -bundles.

**Problem 3.** Show that  $p : S^{2n+1} \rightarrow \mathbb{C}P^n$  is a  $S^1$ -bundle by checking concretely its local triviality.

**Definition 2.3.** If  $F, E, B$  are  $C^r$ ,  $p : E \rightarrow B$  is  $C^r$ , and the local trivialization is  $C^r$ , then we say the fiber bundle is  $C^r$ .

**Definition 2.4.** For  $\xi_1 = (F \rightarrow E_1 \xrightarrow{p_1} B_1)$ ,  $\xi_2 = (F \rightarrow E_2 \xrightarrow{p_2} B_2)$ , a bundle map  $\Phi = (\tilde{f}, f) : \xi_1 \rightarrow \xi_2$  is a pair of maps  $\tilde{f} : E_1 \rightarrow E_2$  and  $f : B_1 \rightarrow B_2$  such that  $f \circ p_1 = p_2 \tilde{f}$  and the restriction  $\tilde{f} : p_1^{-1}(b) \rightarrow p_2^{-1}(f(b))$  is a homeomorphism for every  $b \in B$ .

If both  $f$  and  $\tilde{f}$  are homeomorphisms, then  $\Phi$  is called a bundle isomorphism. If a bundle is isomorphic to a product bundle, then it is called to be trivial.

**Problem 4** For a bundle map  $\Phi$ , is  $\tilde{f}$  homeomorphic if  $f$  is homeomorphic? (If we are doing in the category of smooth manifolds, then the inverse function theorem may be helpful.)

### 3 Day 3: April 24

#### Transition maps and structure groups

Let  $\xi = (F \rightarrow E \xrightarrow{p} B)$  be an  $F$ -bundle. We have an open cover  $\{U_\alpha\}$  such that for each  $\alpha$  we have a local trivialization  $p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$ . For  $U_\alpha \cap U_\beta \neq \emptyset$ , we have a map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F,$$

by which we can define  $\tilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \rightarrow F$  such that  $\varphi_\alpha \circ \varphi_\beta^{-1}(b, f) = (b, \tilde{g}_{\alpha\beta}(b, f))$ . The map  $\tilde{g}_{\alpha\beta}$  is continuous, and we have for each  $b$  a homeomorphism

$$g_{\alpha\beta}(b) : F \rightarrow F : f \mapsto \tilde{g}(b, f),$$

that is,  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Homeo}(F)$ . If we endow the compact-open topology on  $\text{Homeo}(F)$ , then  $g_{\alpha\beta}$  is continuous.

From definition,  $g_{\alpha\beta}(b) \circ g_{\beta\alpha}(b) = \text{id}_F$  for  $b \in U_\alpha \cap U_\beta \neq \emptyset$ , and  $g_{\alpha\beta}(b) \circ g_{\beta\gamma}(b) = g_{\alpha\gamma}(b)$  for  $b \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$  (Note that the second relation implies the first.). The second condition is called the cocycle condition. The maps  $\{g_{\alpha\beta}\}$  are called transition maps.

**Theorem 2.5.** *Let  $\{U_\alpha\}$  be an open cover of a connected space  $B$ . Suppose we have a collection of continuous maps*

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Homeo}(F)\}_{(\alpha,\beta): U_\alpha \cap U_\beta \neq \emptyset}$$

*satisfying the cocycle condition.*

(♠) *Suppose also that  $F$  is locally compact, or there exists a topological transformation group  $G$  (i.e.  $G$  is a topological group such that the group action  $G \times F \rightarrow F$  is continuous) with*

$$\bigcup_{\alpha,\beta} g_{\alpha\beta}(U_\alpha \cap U_\beta) \subset G \subset \text{Homeo}(F).$$

*Then, there exists a unique  $F$ -bundle  $(F \rightarrow E \xrightarrow{p} B)$  such that it is locally trivializable over  $\{U_\alpha\}$  and  $\{g_{\alpha\beta}\}$  is the transition maps of the bundle.*

The viewpoint of the above theorem is more likely to be the physicist's way of defining manifolds in the sense that they sometimes define a manifold as a collection of open subsets of a Euclidean space and transition maps between them.

The condition (♠) guarantees for the second map in

$$\begin{aligned} \tilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F &\rightarrow (U_\alpha \cap U_\beta) \times \text{Homeo}(F) \times F \rightarrow (U_\alpha \cap U_\beta) \times F \\ (b, f) &\mapsto (b, g_{\alpha\beta}(b), f) \mapsto (b, g_{\alpha\beta}(f)) \end{aligned}$$

to be continuous.

*Proof.* (Sketch) Define

$$\tilde{E} := \bigsqcup U_\alpha \times F$$

and  $E := \tilde{E} / \sim$ , where the equivalence relation  $\sim$  is generated by: for each  $(b_1, f_1) \in U_\alpha \times F$  and  $(b_2, f_2) \in U_\beta \times F$  we have  $(b_1, f_1) \sim (b_2, f_2)$  iff  $b_1 = b_2$  and  $f_1 = g_{\alpha\beta}(b_2)(f_2)$ . Let  $\pi : \tilde{E} \rightarrow E$  be the canonical projection. Define also

$$\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times F : [(b, f) \in U_\alpha, F] \mapsto (b, f),$$

which are homeomorphisms by the assumption (♠), satisfying  $\text{pr}_1 \circ \varphi_\alpha = p$ . □

For the second condition in ( $\spadesuit$ ),  $G$  is called a structure group of the  $F$ -bundle. From now on, whenever we consider a fiber bundle along with a structure group  $G$ , we assume it includes the data of local trivialization.

*Remark.* We will always think of  $G$  for bundle maps between fiber bundles with structure group  $G$ . We will frequently consider the maximal transition data and compatible (i.e. satisfying the cocycle condition) local trivializations.

**Example.**

1. Let  $F = V \cong \mathbb{R}^n$  be a real vector space, and  $G \in \{GL(V), SL(V)\}$  or  $G \in \{O(V), SO(V)\}$  with a fixed inner product on  $V$ . These fiber bundles are called real vector bundles.
2. Let  $F = V \cong \mathbb{C}^n$  be a complex vector space, and  $G \in \{GL_{\mathbb{C}}(V)\}$  or  $G \in \{U(V)\}$  with a fixed inner product on  $V$ . These fiber bundles are called complex vector bundles.
3.  $F = G$  be a Lie group. Then,  $G$ -bundle with structure group  $G$  is called a principal bundle.
4. Let  $F$  be a nice smooth manifold and  $G = \text{Diff}^{C^\infty}(F)$  be the group of smooth diffeomorphisms together with the Fréchet topology. Then, we have smooth  $F$ -bundles.

**Definition 2.6.** Let  $G$  be a structure group and  $B$  be a topological space. If an  $F$ -bundle  $\xi = (F \rightarrow E \rightarrow B, G)$  and an  $F'$ -bundle  $\xi' = (F' \rightarrow E' \rightarrow B, G)$  has the same transition data, then they are called associated bundles.

**Example.** Let  $F = \mathbb{R}^n$  be a real vector space with the standard inner product. Let  $G = O(n)$ . With  $S^{n-1} \subset F$ , the sphere bundle inside a real vector bundle is an associated bundle of the original real vector bundle. In particular for  $n = 2$  and  $G = SO(2)$ , then the circle bundle can be recognized as a principal  $SO(2)$ -bundle associated to a real plane bundle, and if we see the plane bundle as a complex line bundle, then it corresponds to a principal  $U(1)$ -bundle.

**Proposition 2.7.** Let  $G$  be a topological group and  $\xi = (G \rightarrow E \rightarrow B, G)$  be a principal  $G$ -bundle. Then, there is a natural right action of  $G$  on  $E$  which is free and the orbit space  $E/G$  is homeomorphic to  $B$  (transitively act on each fiber).

*Proof.* Let  $u \in E$  and  $\varphi_\alpha$  a local trivialization containing  $u$  such that

$$\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times G : u \mapsto (p(u), h).$$

We can check the well-definedness of  $ug = \varphi_\alpha^{-1}(p(u), hg)$  by

$$\varphi_\beta(ug) = \varphi_\beta \circ \varphi_\alpha^{-1}(p(u), hg) = (p(u), g_{\beta\alpha}(p(u))(hg)) = (p(u), h'g).$$

The right action of  $G$  on  $G$  is continuous, free, and transitive. The right action of  $G$  on  $E$  is continuous and free, and  $\bar{p} : E/G \rightarrow B$  is continuous and bijective.  $\square$

**Problem 5.** Show that  $\bar{p}^{-1}$  is also continuous.

*Remark.* A principal  $G$ -bundle may also be defined as follows: a  $G$ -bundle such that (1) there is a continuous free right action of  $G$  on  $E$  which is (2) fiber-preserving and fiberwise transitive, and (3) we can choose  $G$ -equivariant local trivialization such that  $\varphi_\alpha(u) = (p(u), h)$  implies  $\varphi_\alpha(ug) = (p(u), hg)$ .

## 4 Day 4: May 1

Let  $G$  be a topological group. A principal  $G$ -bundle  $(G \rightarrow E \rightarrow B, G)$  has a continuous free action of  $G$  on  $E$ .

*Remark.* For two principal  $G$ -bundles,  $(\tilde{f}, f)$  is a bundle map if and only if  $\tilde{f}$  is  $G$ -equivariant.

**Definition 2.8.** Let  $\xi = (F \rightarrow E \xrightarrow{p} B)$  be a fiber bundle. A continuous map  $s : B \rightarrow E$  such that  $p \circ s = \text{id}_B$  is called a section or a cross section.

An important question asks if there is a section globally defined on the whole  $B$ .

**Proposition 2.9.** Let  $\xi = (G \rightarrow E \rightarrow B, G)$  be a principal  $G$ -bundle. Then,  $\xi$  is trivial if and only if it admits a global section.

*Proof.*  $(\Rightarrow)$  Clear.

$(\Leftarrow)$  Let  $s : B \rightarrow E$  be a global section. Define

$$\Phi : B \times G \rightarrow E : (b, g) \mapsto s(b)g.$$

Then, it is an  $G$ -equivariant isomorphism. □

Let  $X$  be a right  $G$ -space which is free. Then, is  $X/G$  a principal  $G$  bundle? We have two problems:

- (a) Is the inverse image(=orbit) of each point of  $X/G$  homeomorphic to  $G$ ? No, the dynamics  $\mathbb{T}^2 \curvearrowright \mathbb{R}$  with irrational slope.
- (b) Does it satisfy the local triviality? No, the translation  $\mathbb{R} \leftarrow \mathbb{Q}$ .

**Proposition 2.10.** Let  $X$  be a right  $G$ -space which is free. The quotient map  $\pi : X \rightarrow X/G$  defines a principal  $G$ -bundle if and only if  $X \curvearrowright G$  strongly freely (i.e.  $X \times X \rightarrow G : (x, xg) \mapsto g$  is continuous) and there is a local trivialization for some  $y \in X/G$ .

*Proof.*  $(\Rightarrow)$  Clear.

$(\Leftarrow)$

$$\pi^{-1}(U) \rightarrow U \times G : s(x)g \mapsto (x, g)$$

is continuous by the strongly free action. It defines local trivializations. □

**Theorem 2.11** (Gleason, 1950). Let  $M$  be a smooth manifold and  $G$  a compact Lie group which gives a free right smooth action on  $M$ . Then,  $M/G$  is a smooth manifold such that  $M \rightarrow M/G$  is a principal  $G$ -bundle.

(Compactness of  $G$  implies the properness of the action, and smoothness implies the local triviality)

**Corollary 2.12** (Samelson, 1941). Let  $H$  be a compact Lie subgroup of a Lie group  $G$ . Then,  $G \rightarrow G/H$  is a principal  $H$ -bundle. In fact, it is sufficient for  $H$  to be a closed subgroup of  $G$ , even if it is not compact.

**Example.**

- (a) With an action  $S^{2n+1} \curvearrowright S^1$  such that  $(z_0, \dots, z_n)w = (z_1w, \dots, z_nw)$ , we have an  $S^1$ -bundle

$$S^{2n+1} \rightarrow \mathbb{CP}^n : (z_0, \dots, z_n) \mapsto [z_0 : \dots : z_n].$$

It is a general Hopf bundle.

(b) For  $k \leq n$ , the Stiefel variety is

$$V_k(\mathbb{R}^n) := \{M \in M_{n,k}(\mathbb{R}) : \text{rk } M = k\}.$$

Also define

$$V_k^O(\mathbb{R}^n) := \{M \in V_k(\mathbb{R}^n) : \text{column vectors of } M \text{ are orthonormal}\}$$

and the Grassmannian manifold

$$G_k(\mathbb{R}^n) := \{k\text{-dimensional subspaces of } \mathbb{R}^n\}.$$

Stiefel varieties can be realized as principal bundles on Grassmannian manifolds.

With an action  $V_k(\mathbb{R}^n) \curvearrowright \text{GL}(k, \mathbb{R})$  such that  $(v_1, \dots, v_k)X = (v_1X, \dots, v_kX)$ , we have  $G_k(\mathbb{R}^n) \cong V_k(\mathbb{R}^n)/\text{GL}(k, \mathbb{R})$  and  $G_k(\mathbb{R}^n) \cong V_k^O(\mathbb{R}^n)/\text{O}(k)$ . Then,  $(\text{O}(k) \rightarrow V_k^O(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n))$  and  $(\text{GL}(k, \mathbb{R}) \rightarrow V_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n))$  are principal bundles.

(c) As a complex version of (b), we have principal bundles  $(\text{U}(k) \rightarrow V_k^U(\mathbb{C}^n) \rightarrow G_k(\mathbb{C}^n))$  and  $(\text{GL}(k, \mathbb{C}) \rightarrow V_k(\mathbb{C}^n) \rightarrow G_k(\mathbb{C}^n))$ .

**Theorem 2.13.** *Let  $M$  be smooth manifold and suppose we have a transitive smooth left action of a Lie group  $G$  on  $M$ . Let  $H$  be the isotropy group. Then,  $G/H \rightarrow M$  defines a diffeomorphism and  $(H \rightarrow G \rightarrow M)$  is a principal bundle. Such  $M$  is called a homogeneous space.*

**Example.** With an action  $\text{SO}(n) \curvearrowright S^{n-1}$ , since the isotropy group is isomorphic to  $\text{SO}(n-1)$ , we have a principal bundle  $\text{SO}(n-1) \rightarrow \text{SO}(n) \rightarrow S^n$ .

We can also see the examples above (Grassmann and Steifel manifolds) as principal bundles on homogeneous spaces with a diffeomorphism  $\text{O}(n-k) \setminus \text{O}(n) \rightarrow V_k^O(\mathbb{R}^n) : [A] \mapsto (Ae_1, \dots, Ae_k)$  and  $\text{O}(n)/\text{O}(n-k) \times \text{O}(k) \cong G_k(\mathbb{R}^n)$ : principal  $\text{O}(k)$ -bundle

$$\begin{array}{ccccc} \text{O}(k) & \longrightarrow & V_k^O(\mathbb{R}^n) & \xrightarrow{\sim} & \text{O}(n)/\text{O}(n-k) \\ & & \downarrow & & \\ & & G_k(\mathbb{R}^n) & \xrightarrow{\sim} & (\text{O}(n)/\text{O}(n-k))/\text{O}(k). \end{array}$$

We also have a complex version.



## 5 Day 5: May 8

### Principal bundles and associated bundles

Let  $G$  be a topological group and  $\xi = (G \rightarrow E \xrightarrow{p} B, G)$  be a principal  $G$ -bundle. Let  $\{U_\alpha\}$  be an open cover of  $B$ . Let  $G$  effectively act on  $F$  from left as a transformation group, i.e. there is an injective group homomorphism  $\sigma : G \rightarrow \text{Homeo}(F)$  such that the action  $G \times F \rightarrow F$  is continuous. Define

$$E \times_G F := E \times F / (eh, f) \sim (e, \sigma(h)f)$$

and

$$\pi : E \times_G F \rightarrow B : [e, f] \mapsto p(e).$$

This map is well-defined and continuous so that  $\eta = (F \rightarrow E \times_G F \xrightarrow{\pi} B, G)$  is a fiber bundle with structure group  $G$  and fiber  $F$ .

In fact, if  $\{g_{\alpha\beta}\}$  is the transition maps of  $\xi$ , then the transition maps of  $\eta$  are given by  $\{\sigma \circ g_{\alpha\beta}\}$ .

Conversely, let  $\tilde{\eta} = (F \rightarrow \tilde{E} \rightarrow B, G)$  be a fiber bundle with structure group  $G$  and fiber  $F$ . If we construct principal  $G$ -bundle  $\xi$  with the transition data  $\{g_{\alpha\beta}\}$  of  $\tilde{\eta}$ , then  $\eta$  and  $\tilde{\eta}$  are isomorphic.

*Remark.* If  $\sigma : G \rightarrow \text{Homeo}(F)$  is not injective, then  $\eta = (F \rightarrow E \times_G F \rightarrow B)$  is a  $G/\ker \sigma$ -bundle with fiber  $F$ . It can be seen as a generalized version of associated bundles.

**Example.** Let  $M^n$  be a smooth manifold and  $p : TM \rightarrow M$  be the tangent bundle with structure group  $\text{GL}(n, \mathbb{R})$ . For each  $x \in M$ , consider

$$F_x := \{[v_1, \dots, v_n] : \text{ordered bases of } T_x M\}$$

and  $FM := \bigcup_{x \in M} F_x \hookrightarrow \text{GL}(n, \mathbb{R})$ . We call  $FM \rightarrow M$  the tangent frame bundle.

**Theorem** (2.14).

$$\left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{real vector bundles of rank } n \text{ on } B \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{principal } \text{GL}(n, \mathbb{R})\text{-bundles on } B \end{array} \right\}.$$

*Proof.* Transition maps. □

**Example.** The tautological vector bundle  $\gamma_k$  is defined as  $\mathbb{R}^k \rightarrow E_k \xrightarrow{\text{pr}_1} G_k(\mathbb{R}^n)$ , where

$$E_k := \{(W, p) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n : p \in W\}.$$

This is the vector bundle associated to the canonical principal  $\text{GL}(n, \mathbb{R})$ -bundle on  $G_k(\mathbb{R}^n)$ .

### Reduction of structure groups

**Definition** (2.15). Let  $H$  be a closed subgroup  $G$ . We say the structure group of a  $G$ -bundle  $\xi$  with fiber  $F$  can be reduced to  $H$  if  $\xi$  is isomorphic to a  $H$ -bundle with fiber  $F$ . In other words, we have a collection of  $H$ -valued transition maps on an appropriately taken open cover on the base space.

**Example.**

- (a) Let  $H := \text{Homeo}^+(F) \subset G := \text{Homeo}(F)$ . A bundle with fiber  $F$  is orientable if and only if the structure group can be reduced to  $H$ .
- (b) Let  $H := \text{O}(n) \subset G := \text{GL}(n, \mathbb{R})$ . A vector bundle of rank  $n$  has a Euclidean metric (it is a Riemannian metric if smooth) if and only if the structure group of the associated principal  $G$ -bundle can be reduced to  $H$ .

( $\Rightarrow$ ) Suppose a vector bundle  $\xi$  has a collection of  $O(n)$ -valued transition maps on a sufficiently refined open cover, and the local trivialization is written by  $\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n$ . Then, for  $x, y \in E_b$  and  $b \in U_\alpha \subset B$ , the symmetric bilinear form

$$(x, y)_b := (\text{pr}_2 \circ \varphi_\alpha(x), \text{pr}_2 \circ \varphi_\alpha(y))_{\mathbb{R}^n}$$

is a well-defined inner product.

( $\Leftarrow$ ) Suppose a Euclidean metric on a vector bundle  $\xi$  of rank  $n$  is given. Since  $p^{-1}(U_\alpha) \rightarrow U_\alpha$  is trivial, we can take sections  $(s_i)_{i=1}^n$  on  $U_\alpha$  which are linearly independent at each point of  $U_\alpha$ . Using the given Euclidean metric, we can apply the Gram-Schmidt algorithm to get another set of sections  $(e_i)_{i=1}^n$  which form an orthonormal basis at each point of  $U_\alpha$ . With these sections we can construct new local trivializations, having  $O(n)$ -valued transition maps.

(Another remark) Since every vector bundle over a paracompact space  $B$  admits a Euclidean metric, the structure group of every principal  $GL(n, \mathbb{R})$ -bundle can be reduced to  $O(n)$ .

- (c) For a complex version, a complex vector bundle of rank  $n$  admits a Hermitian metric if and only if the structure group  $GL(n, \mathbb{C})$  can be reduced to  $U(n)$ . Similarly, the reduction is always possible if  $B$  is paracompact.

## 6 Day 6: May 15

### 3. Classification of principal bundles

#### Pullback bundles

Let  $\xi = (F \rightarrow E \rightarrow B)$  and  $f : A \rightarrow B$  a map. Then, the pullback bundle of  $\xi$  by  $f$  is  $(F \rightarrow f^*E \rightarrow A)$ , where  $f^*E := \{(a, e) \in A \times F : f(a) = p(e)\}$ . For example, if  $\iota : A \hookrightarrow B$  is an inclusion, then the restriction is defined by the pullback  $\xi|_A := \iota^*\xi = (F \rightarrow p^{-1}(A) \rightarrow A)$ . For  $i : \text{Gr}_k(\mathbb{R}^n) \hookrightarrow \text{Gr}_k(\mathbb{R}^n)$  with  $k < n' < n$ , if we define  $E_{n,k} := \{(W, p) \in \text{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \mid p \in W\}$ , then  $i^*E_{n,k} = E_{n',k}$ . If  $\eta$  is a principal  $G$ -bundle and  $F$  is a left  $G$ -space, then  $f^*(\eta \times_G F) = f^*\eta \times_G F$ .

**Lemma (3.1).** *Let  $\xi = (F \rightarrow E_1 \xrightarrow{p_1} B_1, G)$  and  $\eta = (F \rightarrow E_2 \xrightarrow{p_2} B_2, G)$ . For a map  $f : B_1 \rightarrow B_2$ , then a bundle map  $(\tilde{f}, f) : \xi \rightarrow \eta$  exists if and only if  $\xi \cong f^*\eta$  over  $B_1$ .*

*Proof.* ( $\Rightarrow$ )

$$E_1 \rightarrow f^*E_2 : x \mapsto (p_1(x), \tilde{f}(x)).$$

( $\Leftarrow$ )

$$\begin{array}{ccccc} E_1 & \xrightarrow{\quad} & f^*E_2 & \xrightarrow{\quad} & E_2 \\ & \searrow & \swarrow & & \downarrow \\ & & B_1 & \xrightarrow{\quad f \quad} & B_2 \end{array}$$

□

*Exercise. 6.*

- (1) Check that  $f^*\eta$  over  $B_1$  is an  $F$ -bundle which have  $G$  as the structure group.
- (2) Fill the gap of the above proof of Lemma 3.1.

**Theorem (3.2).** *Let  $\xi = (F \rightarrow E \rightarrow B, G)$  be a bundle and  $A$  be a paracompact space. If there is a homotopy  $F : A \times I \rightarrow B : (x, t) \mapsto f_t(x)$ , then  $f_0^*\xi \cong f_1^*\xi$  over  $B_1$ .*

**Corollary.** *Let  $A, B$  be paracompact homotopy equivalent spaces. Then, there is a one-to-one correspondence between the sets of equivalence classes of bundles with fiber  $F$  and structure group  $G$  over  $A$  and  $B$  respectively.*

In the following lemmas, let  $\xi = (F \rightarrow E \rightarrow A \times I, G)$  and  $A$  a paracompact space.

**Lemma (Step 1 of 3.2).** *If  $\xi|_{A \times [a, b]}$  and  $\xi|_{A \times [b, c]}$  are trivial, then  $\xi|_{A \times [a, c]}$  is also trivial, where  $0 \leq a < b < c \leq 1$ .*

*Proof.*

□

**Lemma (Step 2 of 3.2).** *Each point  $a \in A$  has an open neighborhood  $U$  of  $a$  such that  $\xi|_{U \times I}$  is trivial.*

*Proof.*

□

**Lemma (Step 3 of 3.2).** *If we define  $r : A \times I \rightarrow A \times I : (a, t) \mapsto (a, 1)$ , then  $\xi \cong r^*\xi$ .*

*Proof.*

□

## **7 Day 7: May 22**

## 8 Day 8: June 5

The total space  $EG$  is always contractible.

Note that the base space  $BG$  need not be paracompact, but in most cases  $BG$  is paracompact, e.g. compact Lie groups and discrete groups.

**Proposition (3.7).** *Let  $B_1, B_2$  be paracompact spaces and  $\xi_1, \xi_2$  be the universal principal  $G$ -bundles of  $B_1, B_2$ , respectively. Then, there is a homotopy equivalence  $f : B_1 \rightarrow B_2$  such that  $f^*\xi_2 = \xi_1$ . In particular,  $BG$  is uniquely defined for a homotopy equivalence of paracompact spaces.*

*Proof.* By the universality, we have maps  $f_1 : B_1 \rightarrow B_2$  and  $f_2 : B_2 \rightarrow B_1$  such that  $f_1^*\xi_2 = \xi_1$  and  $f_2^*\xi_1 = \xi_2$ . Then,  $\text{id}^*\xi_1 = \xi_1 = (f_2 \circ f_1)^*\xi$  implies  $f_2 \circ f_1 \simeq \text{id}$  by Theorem 3.4.  $\square$

**Theorem (3.8).** *Let  $\xi = (G \rightarrow E \rightarrow B, G)$  be a principal  $G$ -bundle on a paracompact space  $B$ . If  $E$  is contractible, then  $\xi$  is universal principal  $G$ -bundle.*

*Proof.* For a paracompact  $X$ , we want to show the correspondence  $[X, B] \rightarrow \text{Prin}_G(X) : [f] \mapsto f^*\xi$  is bijective.

(Surjectivity) Let  $\eta = (G \rightarrow E_1 \rightarrow X, G)$  be a principal  $G$ -bundle on  $X$ . Define a left action  $G \curvearrowright E$  such that  $ge := eg^{-1}$  using the principal right action on  $E$ . Let  $(\eta, \xi) = (E \rightarrow E_1 \times_G E \rightarrow X)$  be the associated bundle of  $\eta$ . Since  $E$  is contractible, it admits a global section by Theorem 3.9, which implies the existence of a bundle map  $(\tilde{f}, f) : \eta \rightarrow \xi$ . Thus,  $f^*\xi \cong \eta$ .

**Problem 8.** For two principal  $G$ -bundles  $\xi_i = (G \rightarrow E_i \rightarrow B_i, G)$  with  $i = 1, 2$ , show that there is a bundle map  $(\tilde{f}, f) : \xi_1 \rightarrow \xi_2$  if and only if the associated bundle  $(\xi_1, \xi_2) := (E_2 \rightarrow E_1 \times_G E_2 \rightarrow B_1)$  has a global section.

(Injectivity) Suppose  $f_0, f_1 : X \rightarrow B$  satisfies  $f_0^*\xi \cong f_1^*\xi$ . Then, there exist bundle maps  $(\tilde{f}_i, f_i) : f_i^*\xi \rightarrow \xi$  and a bundle isomorphism  $(\tilde{h}, \text{id}_X) : f_0^*\xi \rightarrow f_1^*\xi$ . Define  $\zeta := (G \rightarrow f_0^*E \times [0, 1] \xrightarrow{\pi \times \text{id}} X \times [0, 1], G)$ , where  $\pi : f_0^*\xi \rightarrow X$ . Also define a partial bundle map  $s : \zeta|_{X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])} \rightarrow \xi$  such that

$$s(z, t) := \begin{cases} \tilde{f}_0(z), & t < \frac{1}{2}, \\ \tilde{f}_1(\tilde{h}(z)), & t > \frac{1}{2}. \end{cases}$$

We can see  $s$  as a section of the associated bundle  $(\zeta, \xi)$  on  $X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])$ . By Theorem 3.9, there exists a section  $s' : X \times [0, 1] \rightarrow \zeta$  such that  $s'|_{X \times \{0, 1\}} = s|_{X \times \{0, 1\}}$ . The bundle map  $S : \zeta \rightarrow \xi$  corresponding to  $s'$  induces a map  $X \times [0, 1] \rightarrow B$ , and it is a homotopy between  $f_0$  and  $f_1$ .  $\square$

**Theorem (3.9).** *Let  $A$  be a closed subset of a paracompact Hausdorff space  $B$ , and  $C$  be a contractible space. Let  $\zeta = (C \rightarrow E \rightarrow B)$  be a  $C$ -bundle. For a section  $s : N \rightarrow E$  on an open neighborhood  $N$  of  $A$ , there exists a section  $S : B \rightarrow E$  such that  $S|_A = s|_A$ . In particular, if we take  $A = N = \emptyset$ , then  $S : B \rightarrow E$  always exists.*

*Proof.* Uploaded on ITC-LMS.  $\square$

**Example.** Recall that  $\xi = (\text{O}(k) \rightarrow V_k^{\text{O}}(\mathbb{R}^n) \rightarrow \text{Gr}_k(\mathbb{R}^n), \text{O}(k))$  is a principal  $\text{O}(k)$ -bundle, where

$$V_k^{\text{O}}(\mathbb{R}^n) = \{M \in M_{n \times k}(\mathbb{R}) \mid \text{rk}(M) = k, \text{ orthonormal columns}\} \cong \text{O}(n)/\text{O}(n-k).$$

Here, we refer to a well-known fact:  $\text{Gr}_k(\mathbb{R}^n)$  has a natural cell decomposition (Schubert) so that  $\text{Gr}_k(\mathbb{R}^n) \rightarrow \text{Gr}_k(\mathbb{R}^{n+1})$  induced by the embedding  $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$  is a cellular map.

Let  $\xi_{\text{O}(k)} = (\text{O}(k) \rightarrow V_k^{\text{O}}(\mathbb{R}^\infty) \rightarrow \text{Gr}_k(\mathbb{R}^\infty), \text{O}(k))$ . (space of  $k$ -dimensional subspaces of a separable infinite dimensional Hilbert space..?)

**Theorem (3.10).**  $\xi_{\text{O}(k)}$  is a universal principal  $\text{O}(k)$ -bundle.

*Proof.* By Theorem 3.8, enough to show  $V_k^O(\mathbb{R}^\infty)$  is contractible.

Define unilateral  $k$ -shift  $f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ , where  $\mathbb{R}^\infty = \bigcup_{n \geq 1} \mathbb{R}^n$ . With the inner product on  $\mathbb{R}^\infty$ , we can check

$$h_t(v) := \text{Schmidt}[(1-t)v_1 + tf(v_1), \dots, (1-t)v_k + tf(v_k)]$$

is well-defined homotopy such that  $h_0 = \text{id}$  and  $h_1(V_k^O(\mathbb{R}^\infty)) = V_k^O(\{0\}^k \times \mathbb{R}^\infty)$ . We also have

$$h'_t : V_k^O(\{0\}^k \times \mathbb{R}^\infty) \rightarrow V_k^O(\mathbb{R}^\infty)$$

such that

$$h'_t(w_1, \dots, w_k) \mapsto \text{Schmidt}[te_1 + (1-t)w_1, \dots, te_k + (1-t)w_k],$$

and by connecting them, we are done.  $\square$

With the same reasoning,

$$\xi_{\text{GL}(k, \mathbb{R})} = (\text{GL}(k, \mathbb{R}) \rightarrow V_k(\mathbb{R}^\infty) \rightarrow \text{Gr}_k(\mathbb{R}^\infty), \text{GL}(k, \mathbb{R}),$$

$$\xi_{\text{U}(k)} = (\text{U}(k) \rightarrow V_k^U(\mathbb{C}^\infty) \rightarrow \text{Gr}_k(\mathbb{C}^\infty), \text{U}(k),$$

$$\xi_{\text{GL}(k, \mathbb{C})} = (\text{GL}(k, \mathbb{C}) \rightarrow V_k(\mathbb{C}^\infty) \rightarrow \text{Gr}_k(\mathbb{C}^\infty), \text{GL}(k, \mathbb{C}))$$

are all universal.

**Example.** Let  $G$  be a compact Lie group with closed embedding  $G \hookrightarrow \text{U}(k)$ . Then, the bundle

$$G \rightarrow V_k^U(\mathbb{C}^\infty) \rightarrow V_k^U(\mathbb{C}^\infty)/G$$

has a paracompact base space, hence it is universal since  $V_k^U(\mathbb{C}^\infty)$  is contractible. If we write  $EG = V_k^U(\mathbb{C}^\infty)$  and  $BG = V_k^U(\mathbb{C}^\infty)/G$ , then  $\text{U}(k)/G \rightarrow BG \rightarrow BU(k)$  is a bundle.

In differential geometry, we sometimes stop to limit and take a sufficiently large but finite  $n$  when, for example, considering  $V_k^U(\mathbb{C}^\infty)/G = \lim_{\rightarrow} V_k^U(\mathbb{C}^n)/G$ .

**Definition (3.11).** For  $\xi = (F \rightarrow E \rightarrow B, G)$  with  $B$  paracompact, if  $c(\xi_1) = f^*(c(\xi_2)) \in H^*(B; R)$  for every bundle map  $(\tilde{f}, f) : \xi_1 \rightarrow \xi_2$ , then  $c$  is called a *characteristic class* of bundles with structure group  $G$  and fiber  $F$  with coefficient  $R$ .

From definition,  $\xi_1 \cong \xi_2$  implies  $c(\xi_1) = c(\xi_2)$ , and for trivial  $\xi$ ,  $c(\xi) = 0$ .

**Proposition (3.12).** Let  $G$  be a topological group such that we can take paracompact  $BG$ . Then,

$$\left\{ \begin{array}{c} \text{characteristic classes} \\ \text{of } G, F, R \end{array} \right\} = H^*(BG, R).$$

Let  $a \in H^*(BG, R)$ . For  $\xi = (F \rightarrow E \rightarrow B, G)$ , if we define  $a(\xi) := f^*(a)$  for  $f : B \rightarrow BG$ , then  $a(\xi)$  is a characteristic class.