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1 Topological group action

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1.1. Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map.

- (a) $p^{-1}(p(A)) = \bigcup_{g \in G} gA$ for any $A \subset X$.
- (b) p is open.
- (c) If $x \neq gx$, then there is an open neighborhood U of x such that gU is disjoint to U .

Proof. (c) Since X is Hausdorff, there is disjoint open neighborhoods U_0 and U_1 respectively of x and gx . Then, $U := g^{-1}(gU_0 \cap U_1) \subset U_0$ and $gU = gU_0 \cap U_1 \subset U_1$ are disjoint. \square

1.2. Let $f : X \rightarrow Y$ be continuous. We say f is *proper* if $f^{-1}(K)$ is compact for compact K . We say f is *Bourbaki-proper* if it is closed and proper. If X is Hausdorff and Y is locally compact, then two notions are equivalent.

1.3 (Properly discontinuous actions). Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map. This action is called *properly discontinuous* if for every compact $K \subset X$ only finite gK intersect K .

- (a) If Γ is discrete, then orbits are locally finite.
- (b) If orbits are locally finite, then Γ acts properly discontinuously.
- (c) Suppose the stabilizer is always finite. If Γ act properly discontinuously then Γ is discrete.

compact stabilizer condition: If Γ is discrete, then for $z \in \mathbb{H}^2$ and a compact set $K \subset \mathbb{H}^2$, $\{g \in \Gamma : gz \in K\}$ is finite, so properly discontinuous.

1.4 (Covering space actions). Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map. This action is called a *covering space action* if every $x \in X$ has a neighborhood U such that gU are all disjoint for $g \in G$.

- (a) A properly discontinuous and free action is a covering space action, if X is locally compact and Hausdorff.
- (b) A covering space action is properly discontinuous.
- (c) A covering space action is free.

Proof. (a) Fix $x \in X$ and let K be a compact neighborhood of x . By the proper discontinuity, there is a finite subset $F \subset G$ such that gK intersects K only for $g \in F$. Because the action is free, for every $g \in F \setminus \{1\}$ there is an open neighborhood U_g of x such that $gU_g \cap U_g = \emptyset$. Then, $U := K^\circ \cap \bigcap_{g \in F \setminus \{1\}} U_g$ satisfies $gU \cap U = \emptyset$.

(b)

\square

2 Hyperbolic plane geometry

2.1 Fuchsian groups

Classification of elements. An abelian Fuchsian group is cyclic. Elliptic point is discrete
free action \iff no elliptic element \iff torsion free \iff manifold

2.2 Fundamental domain

2.1 (Fundamental domain). Let Γ be a Fuchsian group. An open set $D \subset \mathbb{H}^2$ is called a *fundamental domain* of Γ if

- (i) $\{g(D) : g \in \Gamma\}$ are pairwise disjoint,
- (ii) $\{g(\overline{D}) : g \in \Gamma\}$ covers \mathbb{H}^2 .

2.2 (Dirichlet domain). Let Γ be a Fuchsian group. Let $z_0 \in \mathbb{H}^2$ be a point that is not fixed by any isometry in $\Gamma \setminus \{e\}$, i.e. a non-elliptic point. The *Dirichlet domain* of Γ with center z_0 is defined as the set

$$D := \bigcap_{g \in \Gamma \setminus \{e\}} \{z \in \mathbb{H}^2 : d(z, z_0) < d(z, gz_0)\}.$$

We denote by \overline{D} and ∂D the closure and the boundary of D in $\overline{\mathbb{H}^2}$.

- (a) There exists a non-elliptic point in \mathbb{H}^2 .
- (b) $\{g(\overline{D}) : g \in \Gamma\}$ is a locally finite. It is called the *Dirichlet tessellation*.
- (c) D is a convex fundamental domain of Γ .

Proof. (a) Elliptic points are countably many.

(b) There are finitely many $g \in \Gamma$ satisfying $B(z_0, r) \cap g(\overline{D}) \neq \emptyset$, since this condition implies $gz_0 \in B(z_0, 2r)$. \square

2.3 (Boundary and edges of Dirichlet domain). Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . A subset $l \subset \overline{\mathbb{H}^2}$ is called an *edge* of D if $l = g(\overline{D}) \cap \overline{D}$ for some $g \in \Gamma \setminus \{e\}$ and $|l| > 1$.

- (a) For $g \in \Gamma \setminus \{e\}$, the set $g(\overline{D}) \cap \overline{D}$ has the three cases: the null set, one point, or a geodesic segment.
- (b) If l is an edge, then there is unique $g \in \Gamma \setminus \{e\}$ such that $l = g(\overline{D}) \cap \overline{D}$.
- (c) The intersection of two distinct edges is one point or the null set.
- (d) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{g \in \Gamma \setminus \{e\}} g(\overline{D}) \cap \overline{D}.$$

- (e) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{l: \text{edge}} l.$$

Proof. (d) Let $z \in \partial D \cap \mathbb{H}^2$. Since $d(z, z_0) \leq d(z, gz_0)$ for all $g \in \Gamma \setminus \{e\}$ but $d(z, z_0) \geq d(z, gz_0)$ for some $g \in \Gamma \setminus \{e\}$, there is $g \in \Gamma \setminus \{e\}$ such that $d(z, z_0) = d(z, gz_0)$. By sending z_0 and gz_0 to $\pm 1 + i$ with an isometry so that z is sent to a point on a imaginary axis, we can check for each n that we have $B(z, 1/n) \cap (\mathbb{H}^2 \setminus \overline{D}) \neq \emptyset$. Since $B(z, 1/n) \setminus \overline{D}$ is a non-empty open set in $\mathbb{H}^2 \setminus \overline{D}$, and since

$$\mathbb{H}^2 \setminus \overline{D} \subset \mathbb{H}^2 \setminus D = \overline{\bigcup_{g \in \Gamma \setminus \{e\}} g(D)},$$

we can deduce that $B(z, 1/n)$ intersects with $g(D)$ for some $g \in \Gamma \setminus \{e\}$.

Combining this result with the local finiteness of $\{g(D) : g \in \Gamma\}$, the sequence of sets

$$\{g \in \Gamma \setminus \{e\} : B(z, 1/n) \cap g(D) \neq \emptyset\}$$

indexed by n consists of non-empty finite subsets of $\Gamma \setminus \{e\}$ that are non-increasing. By the pigeonhole principle, there exists $g \in \Gamma \setminus \{e\}$ such that $B(z, 1/n) \cap g(D) \neq \emptyset$ for all n , which allows to extract a sequence $z_n \in g(D)$ that converges to z , which implies $z \in g(\overline{D})$.

(e) Suppose $z \in \partial D \cap \mathbb{H}^2$ is not contained in any edges. Let Z be the set of all $g \in \Gamma \setminus \{e\}$ such that $\{z\} = g(\overline{D}) \cap \overline{D}$. For $g \in \Gamma \setminus (Z \cup \{e\})$, $g(\overline{D}) \cap \overline{D}$ is the null set, one point, or an edge, and any of possibility does not contain z . Therefore,

$$(\partial D \setminus \{z\}) \cap \mathbb{H}^2 = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \mathbb{H}^2$$

by the part (d). Change the restriction \mathbb{H}^2 to a compact ball as

$$(\partial D \setminus \{z\}) \cap \overline{B(z, 1)} = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \overline{B(z, 1)}.$$

Then, the left-handed side is homeomorphic to $[-1, 0) \cup (0, 1]$ or $(-1, 1)$ since ∂D is homeomorphic to S^1 , but the right-handed side is compact because the union becomes finite due to the local finiteness. This is a contradiction, so z is contained in an edge. \square

2.4 (Finitely generated Fuchsian group). Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . Let W be the set of all $g \in \Gamma \setminus \{e\}$ such that $g(\overline{D}) \cap \overline{D}$ is an edge.

- (a) W generates Γ .
- (b) If Γ is finitely generated, then W is finite.
- (c) If W is finite, then Γ is finitely generated.

2.5 (Siegel's theorem). Finite area iff finitely generated.

- (a) If Γ is finitely generated, then

$$\partial D = \bigcup_{l: \text{edge}} l.$$

2.3 Side pairing and cycle conditions

2.6 (Side pairing condition). Let Γ be a finitely generated Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . We have seen that ∂D consists of finitely many edges. A point $v \in \partial D$ is called a *vertex* if it either

- (i) the intersection of two edges, or
- (ii) the fixed point of elliptic isometry $g \in \Gamma$ of order two.

Let $v_0, v_1, \dots, v_n = v_0$ be vertices, indexed along the boundary counterclockwise. A *side* is geodesic segments s_i connecting v_i and v_{i+1} .

- (a) For each side s of D , there is unique $g_s \in \Gamma$ such that $g_s^{-1}(s)$ is another side of D . The isometry g_s is called the *side pairing isometry* of the side s .
- (b) The side pairing isometry of $g_s^{-1}(s)$ is g_s^{-1} .
- (c) The number of sides n is always even.

Proof. \square

2.7 (Cycle condition). Let Γ be a finitely generated Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . Let V and S be the set of all vertices and sides of D , respectively. Define $\sigma : V \rightarrow V$ and $\sigma : S \rightarrow S$ which use same notation such that $\sigma(v_i) = v_{j+1}$ and $\sigma(s_i) := s_{j+1}$ where $s_j = g_s^{-1}(s_i)$. The map σ can be seen as an element of the symmetric group S_n .

- (a) Suppose $v_0 \in \mathbb{H}^2$ and $s = s_0$. Let m be the minimal positive integer such that $\sigma^m(s) = s$. Then, $g_{\sigma^{m-1}(s)} \cdots g_{\sigma(s)} g_s$ is either the identity or elliptic.

(b) Suppose $v_0 \in \partial \mathbb{H}^2$.

2.8 (Genus two surface).

2.9 (Modular group). Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ be the modular group and choose the origin $2i$. $v_0 = i$, $v_1 = e^{\pi i/3}$, $v_2 = \infty$, $v_3 = e^{2\pi i/3}$. $g_{s_0} = S$, $g_{s_1} = T$, $g_{s_2} = T^{-1}$, $g_{s_3} = S^{-1}$. $\sigma = (13)$. The elliptic cycle condition: (0) defines $SS = 1$, (13) defines $(S^{-1}T)^3 = 1$

2.4 The Poincaré polygon theorem

2.10 (Definition of polygon). (a)

2.11 (Side pairing identification). Let P be an open polygon. Define cycles of each vertex. Let $Y := \bar{P}/\sim$ and $\tilde{Y} := (\Gamma \times \bar{P})/\sim$. Define $\Pi : \tilde{Y} \rightarrow Y$.

(a)

2.12 (Elliptic cycle condition). Let P be an open polygon with a side pairing identification. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P .

(a) P satisfies the elliptic cycle condition.

(b) D is a local homeomorphism.

(c) D is a covering map onto its image.

Proof. (a) \Rightarrow (b)

(b) \Rightarrow (c) We claim p has the path lifting property, which is unique because it is a local homeomorphism. Let $w : [0, 1] \rightarrow \text{im } D$, and $\tilde{w} : [0, \tau] \rightarrow \tilde{Y}$ its maximal extension. Write $\tilde{w}(t) = [g(t), z(t)]$ and $w(\tau) = gz$. Define $\tilde{w}(\tau) := [g, z]$. Then,

$$D\tilde{w}(\tau) = D([g, z]) = gz = w(\tau).$$

Let U be an open neighborhood of $[g, z]$ in \tilde{Y} such that $D|_U$ is a homeomorphism and $D(U)$ is open in \mathbb{H}^2 . Then, as $t \rightarrow \tau$,

$$p\tilde{w}(t) = w(t) \rightarrow w(\tau) = p\tilde{w}(\tau)$$

implies

$$\tilde{w}(t) \rightarrow \tilde{w}(\tau),$$

so $\tilde{w} : [0, \tau] \rightarrow \tilde{Y}$ is a continuous extension of $w : [0, \tau] \rightarrow \mathbb{H}^2$. Therefore, D is a local homeomorphism that has the unique path lifting property, so it is a covering map onto its image. \square

2.13 (Finite cycle condition). Let P be an open polygon with a side pairing identification. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P . Consider D and Π such that

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{D} & \mathbb{H}^2 \\ \downarrow \Pi & & \\ Y & & \end{array}.$$

(a) If every cycle of finite points is finite, then $\text{im } D$ is open.

(b) If every cycle is finite, then there is a metric ρ on Y such that $[z_n] \rightarrow [z]$ in ρ if and only if $h_n z_n \rightarrow z$ in \mathbb{H}^2 for a sequence $h_n \in \Gamma$.

Proof.

$$\rho(x, y) := \inf \sum, \\ \inf_{h \in \Gamma} d(h^{-1}z, z') = \rho([z], [z'])$$

\square

2.14 (Parabolic cycle condition). Let P be an open polygon with a side pairing identification such that every cycle is finite. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P . Suppose

- (a) P satisfies the parabolic cycle condition,
- (b) M is a complete metric space.
- (c) D is surjective.

Proof. (b) \Rightarrow (c) Let $w \in \partial(\text{im } D)$ so that we have $[g_n, z_n] \in \tilde{Y}$ such that $g_n z_n \rightarrow w$ in \mathbb{H}^2 . Since $g_n z_n$ is Cauchy, $[z_n]$ is also Cauchy, so we have a limit $[z_n] \rightarrow [z]$ in Y . Then, there exists a sequence $h_n \in \Gamma$ such that $h_n z_n \rightarrow z$ in \mathbb{H}^2 , which implies $g_n h_n^{-1} z \rightarrow w$ in \mathbb{H}^2 and $w \in \overline{\Gamma z}$. Since $\text{im } D$ is open and $\overline{P} \subset \text{im } D$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \text{im } D$. There is $g \in \mathbb{H}^2$ such that $d(gz, w) < \varepsilon$, which implies $g^{-1}w \in B(z, \varepsilon)$. Because Γ acts on $\text{im } D$, we can conclude $w \in \text{im } D$. \square

If P satisfies the cycle conditions,

- (a) Γ is discrete.
- (b) Γ is given by the presentation $\langle S | R \rangle$, where S is the set of side-pairing isometries and R is the set of cycle relations.
- (c) P is a fundamental domain of Γ
- (d) $Y \cong \mathbb{H}^2 / \Gamma$.

2.5 Geometric structures

A geodesically connected and geodesically complete space is rigid.

2.15 (Continuation of path). Let M be a (G, X) -manifold, $\varphi : U \rightarrow X$ a chart, and $\gamma : [0, 1] \rightarrow M$ a path. There is a partition $(t_i)_{i=0}^m$ of the interval $[0, 1]$ with $t_0 = 0$, $t_m = 1$ and a sequence of chart $(\varphi_i : U_i \rightarrow X)_{i=0}^{m-1}$ with $\varphi_0 = \varphi$ such that $\gamma([t_{i-1}, t_i]) \subset U_i$. Since $\tau_{i+1,i} \circ \varphi_{i+1} \circ \gamma(t_{i+1}) = \varphi_i \circ \gamma(t_{i+1})$, we can define a path $\hat{\gamma} : [0, 1] \rightarrow X$ by

$$\hat{\gamma}(t) := \tau_{1,0} \circ \cdots \circ \tau_{i,i-1} \circ \varphi_i \circ \gamma|_{[t_i, t_{i+1}]}(t)$$

for $t \in [t_i, t_{i+1}]$, where $\tau_{i+1,i} = \varphi_i \circ \varphi_{i+1}^{-1}$ are transition maps. The path $\hat{\gamma}$ is called the *continuation* of $\varphi \circ \gamma$.

- (a) $\hat{\gamma}$ does not depend on the choice of the sequence of charts when the partition is given.
- (b) $\hat{\gamma}$ does not depend on the choice of the partition.
- (c) If γ_0 and γ_1 are homotopic fixing endpoints, then their continuations are also homotopic fixing endpoints.

2.16 (Developing map). Let M be a connected (G, X) -manifold, \tilde{M} the universal covering of M , and $\varphi : U \rightarrow X$ a chart on \tilde{M} .

2.17 (Holonomy). Let M be a connected (G, X) -manifold, \tilde{M} the universal covering of M , and $\varphi : U \rightarrow X$ a chart on \tilde{M} .

$$h : \pi_1(M) \rightarrow G.$$

- (a) If $f_0, f_1 : \tilde{M} \rightarrow X$ are (G, X) -maps, then there is a unique $g \in G$ such that $f_1 = g f_0$.
- (b) For $H \leq G$, M admits a (H, X) -structure if and only if $\text{im } h \subset H$.

surjectivity of a map from torsion-free discrete subgroups of G to complete (G, X) -manifolds? (up to homeomorphism, up to geometric structure)

Definition 2.1 (Several definitions of hyperbolic manifolds). Let $G = \text{Isom}^+(\mathbb{H}^n)$ and X a n -manifold. Then, X is a hyperbolic manifold if one of the following satisfied...?:

1. It admits a hyperbolic atlas, and it is “complete”
2. It is homeomorphic to \mathbb{H}^n/Γ for a torsion-free discrete subgroup Γ of G .
3. It is a geodesically complete Riemannian manifold with constant sectional curvature -1 .

Thurston geometry is a three-dimensional model geometry on which a closed 3-manifold has a geometric structure modelled.

oriented prime closed 3-manifolds

3 Universal coefficient theorem

Lemma 3.1. *Suppose we have a flat resolution*

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

Then, we have a exact sequence

$$\cdots \rightarrow 0 \rightarrow \text{Tor}_1^R(A, B) \rightarrow P_1 \otimes B \rightarrow P_0 \otimes B \rightarrow A \otimes B \rightarrow 0.$$

Theorem 3.2. *Let R be a PID. Let C_\bullet be a chain complex of flat R -modules and G be a R -module. Then, we have a short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0,$$

which splits, but not naturally.

1. We have a short exact sequence of chain complexes

$$0 \rightarrow Z_\bullet \rightarrow C_\bullet \rightarrow B_{\bullet-1} \rightarrow 0$$

where every morphism in Z_\bullet and B_\bullet are zero. Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence

$$0 \rightarrow Z_\bullet \otimes G \rightarrow C_\bullet \otimes G \rightarrow B_{\bullet-1} \otimes G \rightarrow 0$$

and the associated long exact sequence

$$\rightarrow H_n(B; G) \rightarrow H_n(Z; G) \rightarrow H_n(C; G) \rightarrow H_{n-1}(B; G) \rightarrow H_{n-1}(Z; G) \rightarrow$$

where the connecting homomorphisms are of the form $(i_n: B_n \rightarrow Z_n) \otimes 1_G$ (It is better to think diagram chasing than a natural construction). Since morphisms in B and Z are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n(C; G) \rightarrow B_{n-1} \otimes G \rightarrow Z_{n-1} \otimes G \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(H_n, G) \rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(H_{n-1}, G) \rightarrow 0.$$

For splitting, □

2. Since R is PID, we can construct a flat resolution of G

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow G \rightarrow 0.$$

Since modules in C_\bullet are flat so that the tensor product functors are exact and $P_1 \rightarrow P_0$ and $P_0 \rightarrow G$ induce the chain maps, we have a short exact sequence of chain complexes

$$0 \rightarrow C_\bullet \otimes P_1 \rightarrow C_\bullet \otimes P_0 \rightarrow C_\bullet \otimes G \rightarrow 0.$$

Then, we have the associated long exact sequence

$$\rightarrow H_n(C; P_1) \rightarrow H_n(C; P_0) \rightarrow H_n(C; G) \rightarrow H_{n-1}(C; P_1) \rightarrow H_{n-1}(C; P_0) \rightarrow .$$

Since flat tensor product functor commutes with homology functor from chain complexes, we have

$$\rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n(C; G) \rightarrow H_{n-1} \otimes P_1 \rightarrow H_{n-1} \otimes P_0 \rightarrow .$$

Since

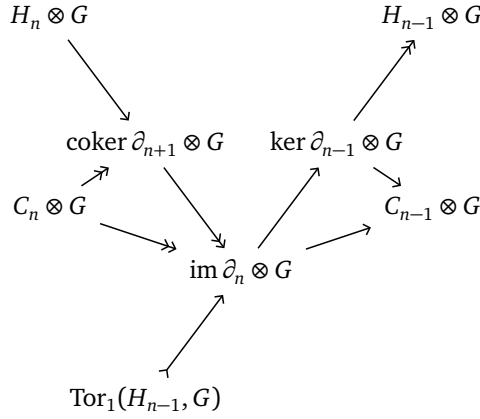
$$0 \rightarrow \text{Tor}_1^R(G, H_n) \rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(G, H_{n-1}) \rightarrow 0.$$

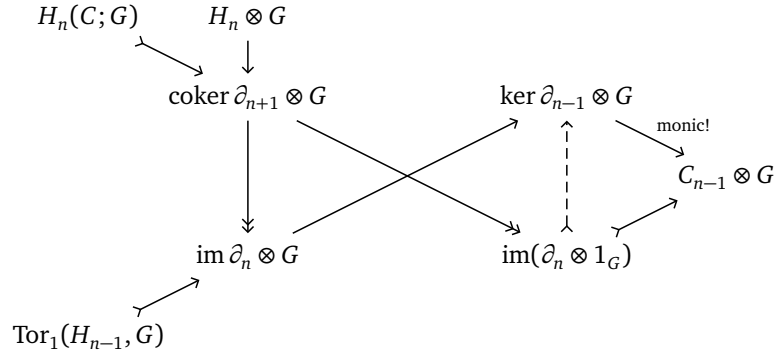
□

Proof 3. By tensoring G , we get the following diagram.



Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernels are preserved, but monomorphisms and kernels are not. Especially, $\text{coker } \partial_{n+1} \otimes G = \text{coker}(\partial_{n+1} \otimes 1_G)$ is important.

Consider the following diagram.



Since $\ker \partial_{n-1}$ is free,

If we show $\text{im}(\partial_n \otimes 1_G) \rightarrow \ker \partial_{n-1} \otimes G$ is monic, then we can get

$$\begin{aligned} H_n(C; G) &= \ker(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{im}(\partial_n \otimes 1_G)) \\ &= \ker(\text{coker } \partial_{n+1} \otimes G \rightarrow \ker \partial_{n-1} \otimes G). \end{aligned}$$

4 Fundamental differential geometry

4.1 Manifold and Atlas

Definition 4.1. A locally Euclidean space M of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to an open subset of \mathbb{R}^d .

Definition 4.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 4.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of \mathbb{R}^d . A chart is often written by a pair (U, φ) .

Definition 4.4. An *atlas* \mathcal{F} is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 4.5. A *differentiable manifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

4.2 Definition of Differentiable Structure

Definition 4.6. An atlas \mathcal{F} is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each transition function $\tau_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 4.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 4.8. An differentiable atlas \mathcal{F} is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in \mathcal{F} , then $(U, \varphi) \in \mathcal{F}$.

Definition 4.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manifold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M . When the charts is already equipped on M , it is natural to define a function $f : M \rightarrow \mathbb{R}$ differentiable if the functions $f \circ \varphi^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas \mathcal{F} , then f is also differentiable on any atlases which is equivalent to \mathcal{F} by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 4.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 4.10. A continuous function $f : M \rightarrow N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

4.3 Curves

Definition 4.11. For $f : M \rightarrow \mathbb{R}$ and (U, ϕ) a chart,

$$df \left(\frac{\partial}{\partial x^\mu} \right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 4.12. Let $\gamma: I \rightarrow M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi: M \rightarrow N$ be a smooth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f: M \rightarrow \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt}f \circ \gamma.$$

Proposition 4.1. Let $\gamma: I \rightarrow M$ be a smooth curve on a manifold M . The notation $\dot{\gamma}^\mu$ is not confusing thanks to

$$(\dot{\gamma})^\mu = (\dot{\gamma}^\mu).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt}x^\mu \circ \gamma.$$

4.4 Connection computation

$$\begin{aligned}\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\ &= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\ &= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma_{\mu\nu}^\lambda \partial_\lambda) \\ &= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda \right) \partial_\nu.\end{aligned}$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ .

$$Y^\nu_{;\mu} = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \quad Y^\nu_{;\mu} = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda.$$

Theorem 4.2. For Levi-civita connection for g ,

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Proof.

$$\begin{aligned}(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} \\ (\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma_{jk}^l g_{li} - \Gamma_{jl}^l g_{kl} \\ (\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}\end{aligned}$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$. Thus,

$$\begin{aligned}\Gamma_{ij}^l g_{kl} &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}). \\ \Gamma_{ij}^l &= \frac{1}{2}g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).\end{aligned}$$

□

4.5 Geodesic equation

Theorem 4.3. *If c is a geodesic curve, then components of c satisfies a second-order differential equation*

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0.$$

Proof. Note

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\lambda \partial_\lambda) = (\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu + \dot{\gamma}^\nu \dot{\gamma}^\lambda \Gamma_{\nu\lambda}^\mu) \partial_\mu.$$

Since

$$\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu = \dot{\gamma}(\dot{\gamma}^\mu) = d\dot{\gamma}^\mu(\dot{\gamma}) = d\dot{\gamma}^\mu \circ d\gamma\left(\frac{\partial}{\partial t}\right) = d\dot{\gamma}^\mu\left(\frac{\partial}{\partial t}\right) = \ddot{\gamma}^\mu,$$

we get a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0$$

for each μ . □

5 Vector calculus on spherical coordinates

$$\begin{aligned} V &= (V_r, V_\theta, V_\phi) \\ &= V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi} \quad (\text{normalized coords}) \\ &= V_r \frac{\partial}{\partial r} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} V_\phi \frac{\partial}{\partial \phi} \quad (\Gamma(TM)) \\ &= V_r dr + r V_\theta d\theta + r \sin \theta V_\phi d\phi \quad (\Omega^1(M)) \\ &= r^2 \sin \theta V_r d\theta \wedge d\phi + r \sin \theta V_\theta d\phi \wedge dr + r V_\phi dr \wedge d\theta \quad (\Omega^2(M)). \end{aligned}$$

$$\nabla \cdot V = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (r \sin \theta V_\theta) + \frac{\partial}{\partial \phi} (r V_\phi) \right]$$

$$\Delta u = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} u \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} u \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} u \right) \right]$$

Let (ξ, η, ζ) be an orthogonal coordinate that is *not* normalized. Then,

$$\sharp = g = \text{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2)$$

$$\hat{x} = \|\partial_x\|^{-1} \partial_x = \|\partial_x\| dx = \|\partial_y\| \|\partial_z\| dy \wedge dz$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r d\theta, \quad r \sin \theta d\phi, \quad (r d\theta) \wedge (r \sin \theta d\phi), \quad (r \sin \theta d\phi) \wedge (dr), \quad (dr) \wedge (r d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[\frac{1}{\|\partial_x\|} \frac{\partial}{\partial x} \cdot -, \frac{1}{\|\partial_y\|} \frac{\partial}{\partial y} \cdot -, \frac{1}{\|\partial_z\|} \frac{\partial}{\partial z} \cdot - \right] \\
\text{curl} : \nabla &= \left[\frac{1}{\|\partial_y\| \|\partial_z\|} \left(\frac{\partial}{\partial y} (\|\partial_z\| \cdot -) - \frac{\partial}{\partial z} (\|\partial_y\| \cdot -) \right), \right. \\
&\quad \frac{1}{\|\partial_z\| \|\partial_x\|} \left(\frac{\partial}{\partial z} (\|\partial_x\| \cdot -) - \frac{\partial}{\partial x} (\|\partial_z\| \cdot -) \right), \\
&\quad \left. \frac{1}{\|\partial_x\| \|\partial_y\|} \left(\frac{\partial}{\partial x} (\|\partial_y\| \cdot -) - \frac{\partial}{\partial y} (\|\partial_x\| \cdot -) \right) \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} (\|\partial_y\| \|\partial_z\| \cdot -), \frac{\partial}{\partial y} (\|\partial_z\| \|\partial_x\| \cdot -), \frac{\partial}{\partial z} (\|\partial_x\| \|\partial_y\| \cdot -) \right] \\
\Delta &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} \left(\frac{\|\partial_y\| \|\partial_z\|}{\|\partial_x\|} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\|\partial_z\| \|\partial_x\|}{\|\partial_y\|} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\|\partial_x\| \|\partial_y\|}{\|\partial_z\|} \frac{\partial}{\partial z} \right) \right]
\end{aligned}$$

$$\text{grad} = \frac{1}{\|\cdot\|^1} (\nabla) \cdot \|\cdot\|^0$$

$$\text{curl} = \frac{1}{\|\cdot\|^2} (\nabla \times) \cdot \|\cdot\|^1$$

$$\text{div} = \frac{1}{\|\cdot\|^3} (\nabla \cdot) \cdot \|\cdot\|^2$$

6 Bundles

Show that S^n has a nonvanishing vector field if and only if n is odd.

Solution. Since S^n is embedded in \mathbb{R}^{n+1} , the tangent bundle TS^n can be considered as an embedded manifold in $S^n \times \mathbb{R}^{n+1}$ which consists of (x, v) such that $\langle x, x \rangle = 1$ and $\langle x, v \rangle = 0$, where the inner product is the standard one of \mathbb{R}^{n+1} .

Suppose n is odd. We have a vector field $(x_1, x_2, \dots, x_{n+1}; x_2, -x_1, \dots, -x_n)$ which is nonvanishing. Conversely, suppose we have a nonvanishing vector field X . Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \rightarrow S^n \times \mathbb{R}^{n+1} \rightarrow \phi \mathbb{R}^{n+1} \rightarrow S^n.$$

The last map can be defined since X is nowhere zero. Since this map satisfies $\langle x, \phi(x) \rangle = 0$ for all $x \in S^n$, we can define homotopies from ϕ to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree, $+1$, so n is odd. \square

Proposition 6.1. *Independent commuting vector fields are realized as partial derivatives in a chart.*

Proposition 6.2. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent involutive vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent. (Maybe)*

Proposition 6.3. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent commuting vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

Proposition 6.4. *An immersed manifold is locally an inverse image of a regular value.*

Proposition 6.5. *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

Proof. It uses tubular neighborhood. Pontryagin construction? □

Proposition 6.6. *An immersed manifold is locally a linear subspace in a chart.*

Proposition 6.7. *Distinct two points on a connected manifold are connected by embedded curve.*

Proof. Let $\gamma : I \rightarrow M$ be a curve connecting the given two points, say p, q .

Step [.1]Constructing a piecewise linear curve For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\text{im } \gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$ for all $1 \leq i \leq n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i, j) : U_{t_i} \cap U_{t_j} \neq \emptyset\}$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \leq i \leq n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q .

Step [.2]Smoothing the curve □

Proposition 6.8. *Let M is an embedded manifold with boundary in N . Any kind of sections on M can be extended on N .*

Proposition 6.9. *Every ring homomorphism $C^\infty(M) \rightarrow \mathbb{R}$ is obtained by an evaluation at a point of M .*

Proof. Suppose $\phi : C^\infty(M) \rightarrow \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Proposition 6.10. *The set of points that is geodesically connected to a point is open.*