Algebraic Geometry

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Contents

Part I

Chapter 1

Schemes

1.1 Morphisms

smooth, finite type, proper, regular, dominant, unramified, flat, complete intersection closed immersion
direct image, inverse image

1.1 (Integral schemes). A scheme X is said to be *integral* if it is non-empty and every non-empty affine open is isomorphic to the spectrum of an integral domain. A *generic point* of a topological space is a point whose closure is the whole space.

(a) A scheme is integral if and only if it is reduced and irreducible.

(b) An integral scheme has a unique generic point.

1.2 (Projective schemes). We say a variety is *projective* if it is isomorphic to a closed subvariety of $\mathbb{P}^n$ for some n.

For a fixed a base ring A, let S be a $\mathbb{Z}_{\ge 0}$-graded ring such that $S_0 = A$, and define the *irrelavent ideal* $S_+ := \bigoplus_{i\ge 1} S_i$ of S. The *Proj construction* of S is a scheme $\operatorname{Proj} S$ constructed as follows. The set $\operatorname{Proj} S$ consists of all homogeneous prime ideals of S not containing S_+, the topology is determined by setting $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{a} \subset \mathfrak{p}\}$ as closed sets where $\mathfrak{a}$ runs through the homogeneous ideals of S, and the structure sheaf defined such that $\mathcal{O}_{\operatorname{Proj} S}(D(f)) := S_{((f))}$ for homogeneous $f \in S_+$, where $S_{(\mathfrak{p})} := (S_{\mathfrak{p}})_0$ denotes the zeroth graded piece of localized $\mathbb{Z}$-graded rings $S_{\mathfrak{p}}$, and the set $D(f) := \operatorname{Proj} S \setminus V(f)$ is called a *standard open* of $\operatorname{Proj} S$, which can be shown to be affine.

There is a canonical $\mathbb{Z}$-graded $\mathcal{O}_{\operatorname{Proj} S}$-modules, of which the graded pieces $\mathcal{O}(i)$ are line bundles called the *Serre twisting sheaves*.

Some analysis of line bundles construct projective embeddings

Chapter 2

Coherent sheaves

Chapter 3

Curves

In general, a variety over k is meant by an integral(=reduced+irreducible) scheme which is separated and of finite type. We wan to classify

- Hartshorne: integral scheme of dimension 1 which is proper and regular.
- Vakil: integral scheme of dimension 1 which is projective and regular.

I think they are equivalent to smooth complete curves.

3.1 Preliminaries

Invariants

- genus: $p_a(X) = p_g(X) = h^1(\mathcal{O}_X)$
- Weil vs Cartier divisor groups: $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$

Computation tools

- $|D| \longleftrightarrow PH^0(X, \mathcal{L}(D))$ so that $|D|$ is identified as a projective space
- $\Omega_X \cong \omega_X$
- Riemann-Roch theorem: $l(D) - l(K - D) = \deg D + 1 - g$
- Hurwitz theorem: $2g(X) - 2 = \deg f \cdot (2g(Y) - 2) + \deg R$

birational iff isomorphic A morphism $f : X \to Y$ induces a field extension $\mathcal{K}(X)/\mathcal{K}(Y)$.

3.2 Lower genus

elliptic: invariants, moduli space, structures hyperelliptic: non-hyperelliptic: canonical embedding

3.3 Classification by genus and moduli spaces

Deligne-Mumford: $\mathcal{M}_g$ for $g \geq 2$ is an irreducible quasi-projective variety of dimension $3g - 3$.

3.4 Classification by degree in $\mathbb{P}^3$

A divisor D is called *very ample* if $\mathcal{L}(D) \cong \mathcal{O}(1)$ in some closed immersion into a projective space. A divisor D is called *ample* if $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for sufficiently large n, for each coherent sheaf $\mathcal{F}$. A *linear system* is a projective subspace of some complete linear system $|D| \cong \mathbb{P}^{l(D)-1}$, the set of all effective divisors linearly equivalent to D, which is identified to a projective space. The *base locus* of a linear system $\mathfrak{d}$ is the set $\bigcap_{D\in\delta} \operatorname{supp} D$. It is known that $|D|$ is base point free if and only if $\mathcal{L}(D)$ is generated by global sections, and a linear system is base point free if and only if some embedding....? For a map $X \to \mathbb{P}^n$, by showing the corresponding linear system is base point free, we can show the map is an immersion or an embedding.

chow variety or hilbert scheme

Chapter 4

Surfaces