# Linear Partial Differential Equations

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November 27, 2021

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# Part I

# Distribution theory

# Chapter 1

# Distributions

## 1.1 Extension of linear operators

Let $T : \mathcal{D} \to \mathcal{D}'$ be a contiuous linear operator. We can always define the adjoint $T^* : \mathcal{D} \subset \mathcal{D}'' \to \mathcal{D}'$. The most reasonable extension of $T$ is $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$. For $f \in (T^*(\mathcal{D}))'$, we can define $\langle T(f), \varphi \rangle := \langle f, T^*\varphi \rangle$ for $\varphi \in \mathcal{D}$.

Suppose $T : (\mathcal{D}, \mathcal{T}) \to (T(\mathcal{D}), \mathcal{S})$ is proved to be continuous. If $(\mathcal{D}, \mathcal{T}) \to (T^*(\mathcal{D}))'$ and $(T(\mathcal{D}), \mathcal{S}) \to \mathcal{D}'$ are embeddings, then the extension of $T$ to the completion of $(\mathcal{D}, \mathcal{T})$ agrees with $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$.

## 1.2 Convolutions

For example, if $\Phi$ is locally integrable, then since $(T_\Phi)^* = T_{\widetilde{\Phi}}$ and $\Phi * \varphi \in \mathcal{E} = C^\infty$ for $\varphi \in \mathcal{D}$, the convolution operator $T_\Phi : \mathcal{E}' \to \mathcal{D}'$ can be defined on the space of compactly supported distributions.

**Problem:** If $g * f$ is well-defined, is $f * g$ also well-defined? In other words, if $f \in (T_{\widetilde{g}}(\mathcal{D}))'$ so that $g * f \in \mathcal{D}'$, then $g \in (T_{\widetilde{f}}(\mathcal{D}))'$? Are they same?

$$\langle g, \widetilde{f} * \varphi \rangle =$$

**1.1.** (a) If a test function $\varphi$ satisfies $\langle 1, \varphi \rangle = 0$, then there is $v \in \mathbb{R}^d$ and a test function $\psi$ such that $\varphi = v \cdot \nabla \psi$.

(b) If a distribution has zero derivative, then it is a constant.

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# Chapter 3

# Sobolev spaces (2)

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## 4.1 Existence results of Poisson's equation

**4.1** (Fundamental solution of the Laplace equation)**.** Consider a boundary problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \mathbb{R}^d_x, \\ \quad u(x) = 0 & \text{on } |x| = \infty. \end{cases}$$

A function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

defined on $\mathbb{R}^d_x$ for $d \geq 2$ is called *fundamental solution of Laplace's equation*.

(a) $\Phi$ and $\nabla\Phi$ are locally integrable on $\mathbb{R}^d_x$ but $\Delta\Phi$ is not.

(b) $\Delta\Phi$ is a tempered distribution on $\mathbb{R}^d_x$.

(c) $-\Delta\Phi(x) = \delta(x)$ in $\mathbb{R}^d_x$.

(d) $u$ solves the boundary problem if and only if it satisfies a representation formula $u = \Phi * f$, if $\Phi * f$ is a well-defined distribution on $\mathbb{R}^d_x$.

*Proof.* (c) Let $\varphi \in \mathcal{D}(\mathbb{R}^d_x)$. Then, $\nabla\Phi(x) \cdot \nabla\varphi(x) \in L^1(\mathbb{R}^d_x)$ gives

$$\begin{aligned} -\int \Phi(x)\Delta\varphi(x)\,dx &= -\lim_{\varepsilon\to\infty} \int_{|x|\geq\varepsilon} \nabla\Phi(x)\cdot\nabla\varphi(x)\,dx \\ &= -\lim_{\varepsilon\to\infty} \int_{|x|=\varepsilon} \nabla\Phi(x)\varphi(x)\cdot\nu\,dS + \lim_{\varepsilon\to\infty} \int_{|x|\geq\varepsilon} \Delta\Phi(x)\varphi(x)\,dx. \end{aligned}$$

Since

$$\nabla\Phi(x) = -\frac{1}{\omega_d}\frac{x}{|x|^d}, \qquad \nu = \frac{x}{|x|},$$

and $\Delta\Phi(x) = 0$ for $x \neq 0$, we get

$$-\int \Phi(x)\Delta\varphi(x)\,dx = \lim_{\varepsilon\to\infty} \frac{1}{\omega_d \varepsilon^{d-1}} \int_{|x|=\varepsilon} \varphi(x)\,dS = \varphi(x).$$

(d) Note that $\Phi = \widetilde{\Phi}$. If $u$ is a solution of the boundary problem, then

$$\langle \Phi * f, \varphi\rangle = \langle f, \Phi * \varphi\rangle = \langle u, -\Delta(\Phi * \varphi)\rangle = \langle u, \Phi * (-\Delta\varphi)\rangle = \langle u, \varphi\rangle.$$

Conversely, if we let $u = \Phi * f$, then

$$\langle u, -\Delta\varphi\rangle = \langle \Phi * f, -\Delta\varphi\rangle = \langle f, \widetilde{\Phi} * (-\Delta\varphi)\rangle = \langle f, \Phi * (-\Delta\varphi)\rangle = \langle f, \varphi\rangle$$

and $\square$

**4.2** (Green's function)**.** Let $U$ be a bounded open subset of $\mathbb{R}^d_x$ with $C^1$ boundary. Consider a boundary value problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } U, \\ \quad u(x) = g(x) & \text{on } \partial U. \end{cases}$$

A *corrector* is a function $\phi(x,y)$ on $U \times U$ defined as the solution of the boundary value problem

$$\begin{cases} -\Delta_y \phi(x,y) = 0 & \text{in } y \in U, \\ \quad \phi(x,y) = \Phi(x-y) & \text{on } y \in \partial U, \end{cases}$$

for each $x \in U$. We assume a well-known fact that the solution $\phi$ uniquely exists and $\phi \in H^1(U)$, proved later. Then, *Green's function* for $U$ is a function on $U \times U$ defined by

$$G(x,y) := \Phi(x-y) - \phi(x,y).$$

(a) If $g(x) = 0$ on $\partial U$, then for $x \in U$,

$$u(x) = -\int_U G(x,y)\Delta u(y)\,dy.$$

(b) If $f(x) = 0$ in $U$, then for $x \in U$,

$$u(x) = \int_{\partial U} u(y)\nabla_y G(x,y)\cdot\nu\,dS(y).$$

(c) $u$ solves the boundary problem if and only if it satisfies a representation formula

$$u(x) = \int_U G(x,y) f(y)\, dy + \int_{\partial U} g(y) \nabla_y G(x,y) \nu \cdot dS(y),$$

if the right-hand side is well defined distribution on $\mathbb{R}^d_x$.

*Proof.* $\square$

# Chapter 5

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