Homological Algebra

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1. Show that if $n \geq 2$ is an integer which is not a power of a prime, then there is a projective $\mathbb{Z}/n\mathbb{Z}$-module which is not free.

Solution. Suppose p and q are distinct prime divisors of n so that we can write $n = p^a q^b m$. Since p^a divides n, $\mathbb{Z}/p^a\mathbb{Z}$ is a $\mathbb{Z}/n\mathbb{Z}$-module. Then, the decomposition $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/q^b\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ implies that $\mathbb{Z}/p^a\mathbb{Z}$ is a direct summand of a free $\mathbb{Z}/n\mathbb{Z}$-module, which means that $\mathbb{Z}/p^a\mathbb{Z}$ is projective. However, $\mathbb{Z}/p^a\mathbb{Z}$ is not free because its cardinality p^a cannot be the power of $n = |\mathbb{Z}/n\mathbb{Z}|$. □

2. Show that if n is a power of prime, then every projective $\mathbb{Z}/n\mathbb{Z}$-module is free.

Solution. Note that $R := \mathbb{Z}/p^a\mathbb{Z}$ is a local ring with the unqie maximal ideal (p). Write $F := \mathbb{Z}/p\mathbb{Z}$. We claim that every projective module is free over a local ring. Suppose first M is a finitely generated projective R-module. Find the minimal number of generators $(m_i)_{i=1}^n$ such that they form a basis of F-module M/pM, a vector space over F. Then we have a split short exact sequence

$$0 \to N \to R^n \to M \to 0.$$

By taking $\otimes_R F$ on $R^n \cong M \oplus N$, we have an isomorphism $F^n \cong M/pM \oplus N/pN$ between vector spaces over F, and the dimension counting implies that $N/pN = 0$. This means $N = pN$, which deduces that $N = 0$. Thus M is free. □

3. Let p be a prime and M_i are abelian groups, where $i \in \{1,2,3\}$. Suppose that $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are group homomorphisms satisfying $g \circ f = 0$, and that the homomorphisms $M_i \to M_i : x \mapsto px$ are injective for all i. Consider a sequence

$$0 \to M_1/p^n M_1 \xrightarrow{f_n} M_2/p^n M_2 \xrightarrow{g_n} M_3/p^n M_3 \to 0,$$

where f_n and g_n are homomorphisms naturally induced from f and g. Show that the following statements are equivalent:

(i) The above sequence is exact for an integer $n \geq 1$.

(ii) The above sequence is exact for all integer $n \geq 1$.

Solution. □

4. Let $R := \mathbb{Z}/n\mathbb{Z}$ for an integer $n \geq 2$.

(1) Show that an R-module M is injective if and only if for every $a \in M \setminus \{0\}$ there exist $b \in M$ and $m \mid n$ such that the order of a is n/m and $a = mb$.

(2) Let m and l be divisors of n. Using an injective resolution of $\mathbb{Z}/m\mathbb{Z}$ in the category of R-modules, compute $\mathrm{Ext}_R^i(\mathbb{Z}/l\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

Solution. (1) Note that R is not an integral domain, but every ideal of R is principal because $\mathbb{Z}$ is a PID.

(2)

□

5. Let $R = \mathbb{C}[x, y]$.

(1) Compute $\mathrm{Ext}^i_R(R/(x,y), R)$.

(2) Are $\mathbb{C}(x,y)$ and $\mathbb{C}(x,y)/\mathbb{C}[x,y]$ injective R-modules?

Solution. (1) We have a projective resolution

$$0 \to R \xrightarrow{\binom{x}{-y}} R \oplus R \xrightarrow{(y\ x)} R \to R/(x,y) \to 0.$$

We compute the cohomology of

$$0 \to \mathrm{Hom}_R(R,R) \to \mathrm{Hom}_R(R^2,R) \to \mathrm{Hom}_R(R,R) \to 0.$$

For the first cohomology, an element belongs to the kernel is equivalent to that x and y are mapped to zero, so the R-morphism is determined by the value at constants. Therefore $\mathrm{Ext}^1(R/(x,y),R) = H^1 = R$. □

6. For a prime p, is the ideal (p,x) of $\mathbb{Z}[x]$ a flat $\mathbb{Z}[x]$-module?

Solution. No. Consider a short exact sequence

$$0 \to (p,x) \to \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z} \to 0$$

provides the long exact sequence

$$0 = \mathrm{Tor}_2^{\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_2^{\mathbb{Z}[x]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}[x]}((p,z), \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}/p\mathbb{Z}) = 0,$$

so we have $\mathrm{Tor}_2^{\mathbb{Z}[x]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{Tor}_1^{\mathbb{Z}[x]}((p,x), \mathbb{Z}/p\mathbb{Z})$. For a free resolution

$$0 \to \mathbb{Z}[x] \xrightarrow{\binom{x}{-p}} \mathbb{Z}[x] \oplus \mathbb{Z}[x] \xrightarrow{(p\ x)} \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z} \to 0$$

of $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}[x]$-modules, we can compute $\mathrm{Tor}_2^{\mathbb{Z}[x]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ by the second homology of

$$0 \to \mathbb{Z}/p\mathbb{Z}[x] \xrightarrow{\binom{x}{0}} \mathbb{Z}/p\mathbb{Z}[x] \oplus \mathbb{Z}/p\mathbb{Z}[x] \xrightarrow{(0\ x)} \mathbb{Z}/p\mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Therefore, $\mathrm{Tor}_1^{\mathbb{Z}[x]}((p,x), \mathbb{Z}/p\mathbb{Z}) \neq 0$ and (p,x) is not flat. □

7. Let A be a commutative ring and B be a commutative A-algebra. Let d be a positive integer and suppose an A-module M satisfies $\mathrm{Tor}_n^A(B,M) = 0$ for $0 < n \leq d$. Show that for any B-module N we have $\mathrm{Ext}_B^m(B \otimes_A M, N) \cong \mathrm{Ext}_A^m(M,N)$ for $0 \leq m \leq d$.

Solution. Let

$$\cdots \to P_2 \to P_1 \to P_0 \to M$$

be a projective resolution of an A-modules M. Since $P \oplus Q \cong A^{\oplus I}$ implies $B \otimes_A P \oplus B \otimes_A Q \cong B^{\oplus I}$, and since the vanishing of $\mathrm{Tor}_n^A(B,M)$ implies that the sequence

$$\cdots \to B \otimes_A P_2 \to B \otimes_A P_1 \to B \otimes_A P_0 \to B \otimes_A M$$

is exact, this sequence is a projective resolution of $B \otimes_A M$ up to $n \leq d$. Recall that $\mathrm{Ext}_A(M,N)$ is the cohomology of

$$0 \to \mathrm{Hom}_A(P_0,N) \to \mathrm{Hom}_A(P_1,N) \to \cdots$$

and $\mathrm{Ext}_B(B \otimes_A M, N)$ is the cohomology of

$$0 \to \mathrm{Hom}_B(B \otimes_A P_0, N) \to \mathrm{Hom}_B(B \otimes_A P_1, N) \to \cdots,$$

so the desired statement follows from the natural isomorphism $\mathrm{Hom}_A(P,N) \cong \mathrm{Hom}_B(B \otimes_A P, N)$ for A-modules P. □

8. Let $L_\bullet$ be a chain complex of finitely generated free abelian groups. Here we do not assume L is bounded below. For a prime p and an integer n, define $r_{n,p} := \dim_{\mathbb{F}_p} H_n(L_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p)$. Show that the following are equivalent:

(i) The integer $r_{n,p}$ does not depend on p for all n.

(ii) The homology group $H_n(L_\bullet)$ is free for all n.

Solution. □

9. Define a category $\mathcal{C}$ as follows: an object is a tuple $\mathcal{M} = (M_0, M_1, f_0, f_1)$ of abelian groups M_0, M_1 and homomorphisms $f_i : M_0 \to M_1$ with $i \in \{0, 1\}$, and a morphism between $\mathcal{M} = (M_0, M_1, f_0, f_1)$ and $\mathcal{M}' = (M_0', M_1', f_0', f_1')$ is a pair $\varphi = (\varphi_0, \varphi_1)$ of homomorphisms $\varphi_i : M_i \to M_i'$ such that $\varphi_1 \circ f_j = f_j' \circ \varphi_0$ for $i, j \in \{0, 1\}$.

(1) Show that $\mathcal{C}$ is abelian.

(2) For an abelian group N, define objects $r_0(N) := (N, 0, 0, 0)$ and $r_1(N) := (N \otimes N, N, \mathrm{pr}_0, \mathrm{pr}_1)$ in $\mathcal{C}$. Show that for any object $\mathcal{M} = (M_0, M_1, f_0, f_1)$ in $\mathcal{C}$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, r_0(N)) \cong \mathrm{Hom}(M_0, N), \qquad \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, r_1(N)) \cong \mathrm{Hom}(M_1, N).$$

(3) Show that $\mathcal{C}$ has enough injective objects.

(4) Define a functor $F : \mathcal{C} \to \mathbf{Ab}$ such that $F(\mathcal{M}) := \{m \in M_0 : f_0(m) = f_1(m)\}$. Show that $R^1F(\mathcal{M}) = \mathrm{coker}(f_0 - f_1)$ and $R^iF = 0$ for $i \geq 2$, where R^iF denotes the right derived functor.

Solution. □

10. Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $C^{\geq 0}(\mathcal{A})$ be an abelian category of cochain complexes $K^\bullet$ such that $K^n = 0$ for $n < 0$.

(1) For an integer $n \geq 0$, find the right adjoint functor of the functor $e_n^* : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto K^n$.

(2) Show that $C^{\geq 0}(\mathcal{A})$ has enough injective objects.

(3) Show that the right derived functor of the left exact functor $H^0 : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto H^0(K^\bullet)$ is given by $H^n : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto H^n(K^\bullet)$ for $n \geq 0$.

Solution. □

11. Give an example of an abelian category in which the direct product exists and the direct product does not preserve right exact sequences.

Solution. Consider the category of sheaves over the Hawaiian earring M. Let A and B be the set of $\mathbb{R}$ and S^1-valued continuous functions on M. □

12. Give an example of an additive category $\mathcal{C}$ with kernels and cokernels in which a morphism $f : A \to B$ such that $\mathrm{coim}\, f \to \mathrm{im}\, f$ is not epi exists.

Solution. Let $\mathcal{C}$ be the category of filtered real vector spaces with the indexing set $\mathbb{Z}$. More precisely, an object of $\mathcal{C}$ is a real vector space V together with an non-decreasing sequence $(V_n)_{n=-\infty}^{\infty}$ of subspaces of V, and a morphism from $(V, (V_n))$ to $(W, (W_n))$ is a pair $(f, (f_n))$ of a linear map $f : V \to W$ and a sequence of linear maps $f_n : V_n \to W_n$ such that for every $m < n$ the restriction of f_n on V_m is equal to f_m, where we write $f = f_\infty$ by convention. The category $\mathcal{C}$ has kernel $(\ker f, (\ker f \cap V_n))$ and cokernel $(\mathrm{coker}\, f, (\pi(W_n)))$, where $\pi : W \to \mathrm{coker}\, f$.

Let $V := W := \mathbb{R}$, $f : V \to W$ with $f(v) = v$, $f_n : V_n \to W_n$, and

$$V_n := \begin{cases} 0 & \text{if } n \leq 0 \\ \mathbb{R} & \text{if } n \geq 1 \end{cases}, \qquad W_n := \begin{cases} 0 & \text{if } n \leq -1 \\ \mathbb{R} & \text{if } n \geq 0 \end{cases}, \qquad f_n := \begin{cases} 0 & \text{if } n \leq 0 \\ \mathrm{id}_{\mathbb{R}} & \text{if } n \geq 1 \end{cases}.$$

Since the kernel and cokernel of $(f, (f_n))$ are trivial, we have $\operatorname{coim} = \operatorname{coker} \ker = V$ and $\operatorname{im} = \ker \operatorname{coker} = W$, and the morphism $\operatorname{coim} f \to \operatorname{im} f$ is just f, which is not epi.

The category of topological vector spaces also provieds a counterexample. $\square$