

# Measure Theory

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**Part I**

**Measures**

# Chapter 1

## Measure spaces

### 1.1 Measurable spaces

1.1 (Measurable spaces).

### 1.2 Measure spaces

1.2 (Definition of measures). Let  $(\Omega, \mathcal{M})$  be a measurable space. A *measure* on  $\mathcal{M}$  is a set function  $\mu : \mathcal{M} \rightarrow [0, \infty] : \emptyset \mapsto 0$  that is *countably additive*: we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for  $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$ . Here the squared cup notation reads the disjoint union.

1.3 (Continuity of measures).

1.4 (Pushforward measures).

1.5 (Complete measures).

### 1.3 Carathéodory extension

1.6 (Outer measures). Let  $\Omega$  be a set. An *outer measure* on  $\Omega$  is a set function  $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty] : \emptyset \mapsto 0$  such that

(i)  $\mu^*$  is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \leq \mu^*(S_2)$$

for  $S_1, S_2 \in \mathcal{P}(\Omega)$ ,

(ii)  $\mu^*$  is *countably subadditive*: we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \sum_{i=1}^{\infty} \mu^*(S_i)$$

for  $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$ .

Comparing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

- (a) A set function  $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty] : \emptyset \mapsto 0$  is an outer measure if and only if  $\mu^*$  is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \leq \sum_{i=1}^{\infty} \mu^*(S_i)$$

for  $S \in \mathcal{P}(\Omega)$  and  $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$ .

- (b) For  $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$ , let  $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  be a set function. We can associate an outer measure  $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  by defining as

$$\mu^*(S) := \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i, B_i \in \mathcal{A} \right\},$$

where we use the convention  $\inf \emptyset = \infty$ .

*Proof.*

□

**1.7** (Carathéodory measurability). Let  $\mu^*$  be an outer measure on a set  $\Omega$ . We want to construct a measure by restriction of  $\mu^*$  on a properly defined  $\sigma$ -algebra. A subset  $E \subset \Omega$  is called *Carathéodory measurable* relative to  $\mu^*$  if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every  $S \in \mathcal{P}(\Omega)$ . Let  $\mathcal{M}$  be the collection of all Carathéodory measurable subsets relative to  $\mu^*$ .

- (a)  $\mathcal{M}$  is an algebra and  $\mu^*$  is finitely additive on  $\mathcal{M}$ .
- (b)  $\mathcal{M}$  is a  $\sigma$ -algebra and  $\mu^*$  is countably additive on  $\mathcal{M}$ .
- (c) The measure  $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \rightarrow [0, \infty]$  is complete.

*Proof.*

□

**1.8** (Carathéodory extension theorem). The Carathéodory extension is a construction method for a measure extending a given set function  $\rho$ . The idea is to restrict the outer measure  $\mu^*$  associated to  $\rho$  in order to obtain a measure  $\mu$ . We want to find a sufficient condition for  $\mu$  to be a measure on a  $\sigma$ -algebra containing  $\mathcal{A}$ .

For  $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$ , let  $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  be a set function. Let  $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  be the associated outer measure of  $\rho$ , and  $\mu : \mathcal{M} \rightarrow [0, \infty]$  the measure defined by the restriction of  $\mu^*$  on Carathéodory measurable subsets.

- (a) We have  $\mu^*|_{\mathcal{A}} = \rho$  if  $\rho$  satisfies the monotone countable subadditivity:

$$A \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \rho(A) \leq \sum_{i=1}^{\infty} \rho(B_i)$$

for  $A \in \mathcal{A}$  and  $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$ .

- (b) We have  $\mathcal{A} \subset \mathcal{M}$  if  $\rho$  satisfies the following property: for every  $B, A \in \mathcal{A}$ , and for any  $\varepsilon > 0$ , there are  $\{C_j\}_{j=1}^{\infty}$  and  $\{D_j\}_{j=1}^{\infty} \subset \mathcal{A}$  such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} C_j \quad \text{and} \quad B \setminus A \subset \bigcup_{j=1}^{\infty} D_j,$$

and

$$\rho(B) + \varepsilon > \sum_{j=1}^{\infty} \rho(C_j) + \sum_{j=1}^{\infty} \rho(D_j).$$

*Proof.* (a) Clearly  $\mu^*(A) \leq \rho(A)$  for  $A \in \mathcal{A}$ . We may assume  $\mu^*(A) < \infty$ . For arbitrary  $\varepsilon > 0$  there is  $\{B_i\}_{i=1}^\infty$  such that  $A \subset \bigcup_{i=1}^\infty B_i$  and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(B_i) \geq \rho(A).$$

Limiting  $\varepsilon \rightarrow 0$ , we get  $\mu^*(A) \geq \rho(A)$ .

(b) Let  $S \in \mathcal{P}(\Omega)$  and  $A \in \mathcal{A}$ . It is enough to check the inequality  $\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \setminus A)$  for  $S$  with  $\mu^*(S) < \infty$ , so we may assume there is a countable family  $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$  such that  $S \subset \bigcup_{i=1}^\infty B_i$ . Then, we have  $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$  and  $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$  satisfying

$$\mu^*(S) + \varepsilon > \sum_{i=1}^\infty \left( \rho(B_i) + \frac{\varepsilon}{2^{i+1}} \right) > \sum_{i,j=1}^\infty \rho(C_{i,j}) + \sum_{i,j=1}^\infty \rho(D_{i,j}) \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Therefore,  $A$  is Carathéodory measurable relative to  $\mu^*$ .  $\square$

**1.9 (Uniqueness of extension of measures).** The existence of the Carathéodory extension provides a uniqueness theorem for the extension of measures. The important property here is  $\sigma$ -finiteness: for  $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$ , let  $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  be a set function. Then, we say  $\rho$  is  $\sigma$ -finite if there is a countable cover  $(B_i)_{i=1}^\infty \subset \mathcal{A}$  of  $\Omega$  such that  $\rho(B_i) < \infty$  for each  $i$ .

Let  $\mu^*$  be the outer measure associated to  $\rho$ . Let  $\mathcal{M}$  be a  $\sigma$ -algebra such that the restriction  $\mu^*|_{\mathcal{M}} : \mathcal{M} \rightarrow [0, \infty]$  is a measure, and  $\mu : \mathcal{M} \rightarrow [0, \infty]$  be any measure. Suppose further that  $\mu^*(A) = \rho(A) = \mu(A)$  for all  $A \in \mathcal{A}$ . Let  $E \in \mathcal{M}$ .

- (a)  $\mu(E) \leq \mu^*(E)$ .
- (b) If  $E_1, E_2 \in \mathcal{M}$  satisfy  $\mu(E_1) = \mu^*(E_1)$  and  $\mu(E_2) = \mu^*(E_2)$ , then  $\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2)$ .
- (c)  $\mu(E) = \mu^*(E)$  if  $\mu^*(E) < \infty$ .
- (d) If  $\rho$  is  $\sigma$ -finite, then  $\mu(E) = \mu^*(E)$  for  $\mu^*(E) = \infty$ .

*Proof.* (a) If  $\mu^*(E) = \infty$ , then  $\mu(E) \leq \mu^*(E)$  trivially. Suppose  $\mu^*(E) < \infty$ . By the definition of the outer measure, there is  $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$  such that  $E \subset \bigcup_{i=1}^\infty B_i$ . Also, we have

$$\mu(E) \leq \mu\left(\bigcup_{i=1}^\infty B_i\right) \leq \sum_{i=1}^\infty \mu(B_i) = \sum_{i=1}^\infty \rho(B_i)$$

whenever  $E \subset \bigcup_{i=1}^\infty B_i$ , so  $\mu(E) \leq \mu^*(E)$ .

(b) In the light of the inclusion-exclusion principle,

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2) - \mu^*(E_1 \cap E_2) \leq \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \mu(E_1 \cup E_2)$$

proves the identity we want.

(c) Because  $\mu^*(E) < \infty$ , for any  $\varepsilon > 0$  we have a sequence  $(B_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $E \subset \bigcup_{i=1}^\infty B_i$  and

$$\mu^*(E) + \varepsilon > \sum_{i=1}^\infty \rho(B_i).$$

Applying the part (b) inductively, we have for every  $n$  that

$$\mu\left(\bigcup_{i=1}^n B_i\right) = \mu^*\left(\bigcup_{i=1}^n B_i\right),$$

and by limiting  $n \rightarrow \infty$  the continuity from below gives

$$\mu\left(\bigcup_{i=1}^\infty B_i\right) = \mu^*\left(\bigcup_{i=1}^\infty B_i\right).$$

Then, we have

$$\mu^*(E) \leq \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) + \mu(E)$$

and

$$\mu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) \leq \mu^*\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) = \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) - \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(B_i) - \mu^*(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu^*(E) < \varepsilon,$$

we get  $\mu^*(E) < \mu(E) + \varepsilon$  and  $\mu^*(E) \leq \mu(E)$  by limiting  $\varepsilon \rightarrow 0$ .

(d) Let  $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$  be such that  $\rho(B_i) < \infty$  and  $\Omega = \bigcup_{i=1}^{\infty} B_i$ . Define  $E_1 := B_1$  and  $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$  for  $n \geq 2$ . Then,  $(E_i)_{i=1}^{\infty}$  is a pairwise disjoint cover of  $\Omega$  with

$$\mu^*(E \cap E_i) \leq \mu^*(E_i) \leq \mu^*(B_i) = \rho(B_i) < \infty$$

for each  $i$ , so we have by the part (c) that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \sum_{i=1}^{\infty} \mu^*(E \cap E_i) = \mu^*(E). \quad \square$$

## Exercises

**1.10** (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let  $\mathcal{A}$  be a collection of subsets of a set  $\Omega$  such that  $\emptyset \in \mathcal{A}$ . We say  $\mathcal{A}$  is a semi-ring if it is closed under finite intersection, and the complement is a finite union of elements of  $\mathcal{A}$ . We say  $\mathcal{A}$  is a semi-algebra

Let  $\mathcal{A}$  be a semi-ring of sets over  $\Omega$ . Suppose a set function  $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  satisfies

(i)  $\rho$  is *disjointly countably subadditive*: we have

$$\rho\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \rho(A_i)$$

for  $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$ ,

(ii)  $\rho$  is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for  $A_1, A_2 \in \mathcal{A}$ .

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

**1.11** (Monotone class lemma). A collection  $\mathcal{C} \subset \mathcal{P}(\Omega)$  is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let  $H$  be a vector space closed under bounded monotone convergence. If  $\text{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$  then  $B^\infty(\sigma(\mathcal{A})) \subset H$ .



## Chapter 2

# Measures on the real line

2.1 (Distribution functions).

2.2 (Helly selection theorem).

2.3 (Non-Lebesgue measurable set).

### Exercises

2.4 (Steinhaus theorem). Let  $\lambda$  denote the Lebesgue measure on  $\mathbb{R}$  and let  $E \subset \mathbb{R}$  be a Lebesgue measurable set with  $\lambda(E) > 0$ .

(a) For any  $0 < \alpha < 1$ , there is an interval  $I = (a, b)$  such that  $\lambda(E \cap I) > \alpha\lambda(I)$ .

(b)  $E - E = \{x - y : x, y \in E\}$  contains an open interval containing zero.

*Proof.* (a) We may assume  $\lambda(E) < \infty$ . Since  $\lambda$  is outer measure and  $\lambda(E) \neq 0$ , we have an open subset  $U$  of  $\mathbb{R}$  such that  $\lambda(U) < \alpha^{-1}\lambda(E)$ . Because  $U$  is a countable disjoint union of open intervals  $U = \bigsqcup_{i=1}^{\infty} (a_i, b_i)$ , we have

$$\sum_{i=1}^{\infty} \lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1} \sum_{i=1}^n \lambda(E \cap (a_i, b_i)).$$

Therefore, there is  $i$  such that  $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$ . □

### Problems

- \*1. Every Lebesgue measurable set in  $\mathbb{R}$  of positive measure contains an arbitrarily long arithmetic progression.

## Chapter 3

# Measurable functions

### 3.1 Simple functions

**3.1** (Measurability of pointwise limits). Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

*Proof.* Let  $f(x) = \lim_{n \rightarrow \infty} s_n(x)$ .

□

### 3.2 Almost everywhere convergence

**3.2** (Almost everywhere convergence). Let  $(\Omega, \mu)$  be a measure space and let  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$  and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be measurable functions. The set of convergence of the sequence  $f_n$  is defined as the set

$$\{x \in \Omega : \lim_{n \rightarrow \infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say  $f_n$  converges to  $f$  *almost everywhere* with respect to  $\mu$  if the set of divergence is a null set in  $\mu$ . We simply write

$$f_n \rightarrow f \text{ a.e.}$$

if  $f_n$  converges to  $f$  almost everywhere, and we frequently omit the measure  $\mu$  if it has no confusion.

(a) If  $\mu$  is complete and, if  $f_n \rightarrow f$  a.e., then  $f$  is measurable.

**3.3** (Borel-Cantelli lemma). Let  $(\Omega, \mu)$  be a measure space and let  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$  and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon > 0} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a)  $f_n \rightarrow f$  a.e. if and only if for each  $\varepsilon > 0$  we have

$$\mu(\{x : \limsup_{n \rightarrow \infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b)  $f_n \rightarrow f$  a.e. if and only if for each  $\varepsilon > 0$  we have

$$\mu(\limsup_{n \rightarrow \infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(c)  $f_n \rightarrow f$  a.e. if for each  $\varepsilon > 0$  we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \infty.$$

*Proof.* (b) The set of divergence of the sequence  $f_n$  is given by

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \frac{1}{m}\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} (\Omega \setminus E_n^m).$$

(c) Since

$$\mu\left(\bigcup_{i=1}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \leq \sum_{i=1}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned} \mu(\limsup_{n \rightarrow \infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) &= \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0. \end{aligned} \quad \square$$

**3.4 (Convergence in measure).** Let  $(\Omega, \mu)$  be a measure space and let  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions. We say  $f_n$  converges to a measurable function  $f : \Omega \rightarrow \overline{\mathbb{R}}$  in measure if for each  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(a) If  $f_n \rightarrow f$  in measure, then there is a subsequence  $f_{n_k}$  such that  $f_{n_k} \rightarrow f$  a.e.

(b) If every subsequence  $f_{n_k}$  of  $f_n$  has a further subsequence  $f_{n_{k_j}}$  such that  $f_{n_{k_j}} \rightarrow f$  a.e., then  $f_n \rightarrow f$  in measure.

*Proof.* (a) Since for each positive integer  $k$  we have  $\mu(\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}) \rightarrow 0$  as  $n \rightarrow \infty$ , there exists  $n_k$  such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}) = 0.$$

Then, for each  $\varepsilon > 0$ ,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \varepsilon\} &= \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \varepsilon\} \\ &\subset \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \frac{1}{k}\} \\ &= \limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\} \end{aligned}$$

implies the limit superior of the tail events is a null set, hence  $f_{n_k} \rightarrow f$  a.e.

(b) □

**3.5 (Egorov theorem).** Egorov's theorem informally states that an almost everywhere convergent functional sequence is “almost” uniformly convergent. Through this famous theorem, we introduce a convenient “ $\varepsilon/2^m$ ” argument”, occasionally used throughout measure theory to construct a measurable set having a special property.

Let  $(\Omega, \mu)$  be a finite measure space and let  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions such that  $f_n \rightarrow f$  a.e. For each positive integer  $m$ , which indexes the tolerance  $1/m$ , consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty} \{x : |f_i(x) - f(x)| < \frac{1}{m}\}.$$

- (a)  $E_n^m$  converges to a full set for each  $m$ .
- (b) For every  $\varepsilon > 0$  there is a measurable  $K \subset \Omega$  such that  $\mu(\Omega \setminus K) < \varepsilon$  and for each  $m$  there is finite  $n$  satisfying  $K \subset E_n^m$ .
- (c) For every  $\varepsilon > 0$  there is a measurable  $K \subset \Omega$  such that  $\mu(\Omega \setminus K) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $K$ .

*Proof.* (a) Recall that the a.e. convergence  $f_n \rightarrow f$  means that for every fixed  $m$  the intersection

$$\bigcap_{n=1}^{\infty} (\Omega \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \frac{1}{m}\}$$

is a null set. Since  $\mu(\Omega) < \infty$ , it is equivalent to  $E_n^m$  converges to a full set for each  $m$  by the continuity from above.

(b) For each  $m$ , we can find  $n_m$  such that

$$\mu(\Omega \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(\Omega \setminus K) = \mu\left(\bigcup_{m=1}^{\infty} (\Omega \setminus E_{n_m}^m)\right) \leq \sum_{m=1}^{\infty} \mu(\Omega \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix  $m > 0$ . Since  $n \geq n_m$  implies  $K \subset E_{n_m}^m \subset E_n^m$ , we have

$$n \geq n_m \Rightarrow \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \quad \square$$

## Exercises

**3.6 (Cauchy's functional equation).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Cauchy's functional equation refers to the equation  $f(x + y) = f(x) + f(y)$ , satisfied for all  $x, y \in \mathbb{R}$ . Suppose  $f$  satisfies the Cauchy functional equation. We ask if  $f$  is linear, that is  $f(x) = ax$  for all  $x \in \mathbb{R}$ , where  $a := f(1)$ .

- (a)  $f(x) = ax$  for all  $x \in \mathbb{Q}$ , but there is a nonlinear solution of Cauchy's functional equation.
- (b) If  $f$  is continuous at a point, then  $f$  is linear.
- (c) If  $f$  is Lebesgue measurable, then  $f$  is linear.

**Part II**

**Lebesgue integral**

## Chapter 4

# Convergence theorems

### 4.1 Definition of Lebesgue integral

### 4.2 Convergence theorems

4.1 (Monotone convergence theorem).

### 4.3 Radon-Nikodym theorem

An integrable function as a measure  $\sigma$ -finite measures

## **Chapter 5**

# **Product measures**

### **5.1 Fubini-Tonelli theorem**

### **5.2 Lebesgue measure on Euclidean spaces**

## Chapter 6

# Measures on metric spaces

### 6.1 Borel measures

### 6.2 Riesz-Markov-Kakutani representation theorem

locally compact

### 6.3 Hausdorff measures



## **Part III**

# **Linear operators**

## Chapter 7

# Lebesgue spaces

### 7.1 $L^p$ spaces

7.1 (Hölder inequality).

*Proof.*

$$\int f g \leq C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take  $C$  such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \left( \int |f|^p \right)^{\frac{1}{p}} \left( \int |g|^q \right)^{\frac{1}{q}}.$$

Note that we can show that  $1 \leq 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \leq 2$  and the minimum is attained only if  $p = q = 2$ , so this method does not provide the sharpest constant.  $\square$

### 7.2 $L^1$ spaces

7.2 (Convolution?).

7.3 (Approximate identity?).

7.4 (Continuity of translation?).

### 7.3 $L^2$ spaces

### 7.4 $L^\infty$ spaces

## Chapter 8

# Bounded linear operators

### 8.1 Continuity

Schur test

### 8.2 Density arguments

extension of operators

### 8.3 Interpolation

weak  $L_p$ , marcinkiewicz

**Definition 8.3.1.** Let  $f$  be a measurable function on a measure space  $(X, \mu)$ . The *distribution function*  $\lambda_f : [0, \infty) \rightarrow [0, \infty)$  is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}).$$

Do not use  $\mu(\{x : |f(x)| \geq \alpha\})$ . The strict inequality implies the *lower semi-continuity* of  $\lambda_f$ .

(a) For  $p > 0$ , we have

$$\|f\|_{L^p}^p = p \int_0^\infty [\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}]^p \frac{d\alpha}{\alpha}.$$

**Definition 8.3.2.**

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty [\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0 < \alpha < \infty} [\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}].$$

**Theorem 8.3.1.** For  $p \geq 1$  we have  $\|f\|_{p,\infty} \leq \|f\|_p$ .

*Proof.* By the Chebyshev inequality,

$$\sup_{0 < \alpha < \infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p \alpha^{p-1} \cdot \mu(|f| > \alpha) d\alpha = \|f\|_{L^p}^p.$$

□

**8.1** (Marcinkiewicz interpolation). Let  $X$  be a  $\sigma$ -finite measure space and  $Y$  be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator  $T : L^{p_0}(X) + L^{p_1}(X) \rightarrow M(Y)$  has two weak-type estimates

$$\|T\|_{L^{p_0}(X) \rightarrow L^{p_0, \infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X) \rightarrow L^{p_1, \infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X) \rightarrow L^p(X)} < \infty.$$

*Proof.* Let  $f \in L^p(X)$  and denote  $f_h = \chi_{|f| > \alpha} f$  and  $f_l = \chi_{|f| \leq \alpha} f$ . It is easy to show  $f_h \in L^{p_0}$  and  $f_l \in L^{p_1}$ . Then,

$$\begin{aligned} \|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha) \frac{d\alpha}{\alpha} \\ &\lesssim \int \alpha^p \cdot \mu(|T(f \cdot \mathbf{1}_{|f| > \alpha})| > \alpha) \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha) \frac{d\alpha}{\alpha} \\ &\leq \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0, \infty}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}} \|Tf_l\|_{L^{p_1, \infty}}^{p_1} \frac{d\alpha}{\alpha} \\ &\lesssim \int \alpha^{p-p_0} \|f_h\|_{L^{p_0}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{L^{p_1}}^{p_1} \frac{d\alpha}{\alpha} \\ &\sim \|f\|_p^p. \end{aligned}$$

by (1) Fubini, (2) Sublinearity, (3) Chebyshev, (4) Boundedness, (5) Fubini.  $\square$

**8.2** (Hadamard's three line lemma). Let  $f$  be a bounded holomorphic function on the vertical unit stripe  $\{z : 0 < \operatorname{Re} z < 1\}$ . Then, for  $0 < \theta < 1$ ,

$$\|f\|_{L^\infty(\operatorname{Re}=\theta)} \leq \|f\|_{L^\infty(\operatorname{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\operatorname{Re}=1)}^\theta.$$

*Proof.* Define

$$g(z) := \frac{f(z)}{\|f\|_{L^\infty(\operatorname{Re}=0)}^{1-z} \|f\|_{L^\infty(\operatorname{Re}=1)}^z}, \quad g_n(z) = g(z) e^{\frac{z^2-1}{n}}.$$

Then we have

1.  $g_n \rightarrow g$  pointwisely as  $n \rightarrow \infty$ ,
2.  $g_n(z) \rightarrow 0$  uniformly as  $\operatorname{Im} z \rightarrow \infty$ .

The second one is because  $g$  is bounded and for  $z = x + yi$  we have

$$|g_n(z)| \lesssim |e^{\frac{z^2-1}{n}}| = e^{\operatorname{Re} \frac{z^2-1}{n}} = e^{\frac{x^2-y^2-1}{n}} \leq e^{\frac{-y^2}{n}}.$$

By (1), it is enough to bound  $g_n$  for each  $n$ . Truncating the stripe, the outer region is controlled by (2) and the interior region is controlled by the maximum modulus principle.  $\square$

**8.3** (Riesz-Thorin interpolation). Let  $X, Y$  be  $\sigma$ -finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta) \frac{1}{p_0} + \theta \frac{1}{p_1}, \quad \frac{1}{q_\theta} = (1-\theta) \frac{1}{q_0} + \theta \frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \rightarrow q_\theta} \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta.$$

*Proof.* Note that

$$\|T\|_{p_\theta \rightarrow q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g \rangle|}{\|f\|_{p_\theta} \|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z \rangle = \int \overline{g_z(y)} Tf_z(y) dy,$$

where  $f_z$  and  $g_z$  are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z) + \frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have  $f_\theta = f$  and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for  $\operatorname{Re} z = x$ .

Then,

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0} \|f_z\|_{p_0} \|g_z\|_{q'_0} = \|T\|_{p_0 \rightarrow q_0} \|f\|_{p_\theta}^{p_\theta/p_0} \|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for  $\operatorname{Re} z = 0$ , and

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_1 \rightarrow q_1} \|f_z\|_{p_1} \|g_z\|_{q'_1} = \|T\|_{p_1 \rightarrow q_1} \|f\|_{p_\theta}^{p_\theta/p_1} \|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for  $\operatorname{Re} z = 1$ . By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta \|f\|_{p_\theta} \|g\|_{q'_\theta}$$

for  $\operatorname{Re} z = \theta$ . Putting  $z = \theta$  in the last inequality, we get the desired result.  $\square$

## Chapter 9

# Convergence of linear operators

### 9.1 Translation and multiplication operators

### 9.2 Convolution type operators

approximation of identity Fejér, Poisson, box?

### 9.3 Computation of integral transforms

## **Part IV**

# **Fundamental theorem of calculus**

## Chapter 10

# Weak derivatives

The space of weakly differentiable functions with respect to all variables  $= W_{\text{loc}}^{1,1}$ .

**10.1** (Product rule for weakly differentiable functions). We want to show that if  $u$ ,  $v$ , and  $uv$  are weakly differentiable with respect to  $x_i$ , then  $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$ .

(a) If  $u$  is weakly differentiable with respect to  $x_i$  and  $v \in C^1$ , then  $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$ .

**10.2** (Interchange of differentiation and integration). Let  $f : \Omega_x \times \Omega_y \rightarrow \mathbb{R}$  be such that  $\partial_{x_i}f$  is well-defined. Suppose  $f$  and  $\partial_{x_i}f$  are locally integrable in  $x$  and integrable in  $y$ .

Then,

$$\partial_{x_i} \int f(x, y) dy = \int \partial_{x_i} f(x, y) dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if  $F$  is  $C^2$  then the *classical* partial differentiation commute, and if  $F$  is not  $C^2$  then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.



# Chapter 11

## Absolutely continuity

- (a)  $f$  is  $\text{Lip}_{\text{loc}}$  iff  $f'$  is  $L_{\text{loc}}^{\infty}$
- (b)  $f$  is  $\text{AC}_{\text{loc}}$  iff  $f'$  is  $L_{\text{loc}}^1$
- (a)  $f$  is  $\text{Lip}$  iff  $f'$  is  $L^{\infty}$
- (b)  $f$  is  $\text{AC}$  iff  $f'$  is  $L^1$
- (c)  $f$  is  $\text{BV}$  iff  $f'$  is a finite regular Borel measure

### 11.1 Absolute continuous measures

### 11.2 Absolute continuous functions

### 11.3 Functions of bounded variation

# Chapter 12

## Lebesgue differentiation theorem

### 12.1 Hardy-Littlewood maximal function

Let  $T_m$  be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m, \varepsilon: |\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

**12.1** (Hardy-Littlewood maximal function). The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

**12.2** (Weak type estimate).

$$\|Mf\|_{1,\infty} \leq 3^d \|f\|_{L^1(\Omega)}.$$

(a) Proof by covering lemma.

*Proof.* (a) By the inner regularity of  $\mu$ , there is a compact subset  $K$  of  $\{|Mf| > \lambda\}$  such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every  $x \in K$ , since  $|Mf(x)| > \lambda$ , we can choose an open ball  $B_x$  such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover  $\{B_i\}_i$  of  $K$ . Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection  $\{B_k\}_k$  such that

$$K \subset \bigcup_i B_i \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \leq \sum_k 3^d \mu(B_k) \leq \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \leq \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of  $B_k$ 's. □

**12.3** (Radially bounded approximate identity). If an approximate identity  $K_n$  is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every  $n$  and  $x$ , hence has a weak type estimate.

**12.4** (Almost everywhere convergence of operators). Suppose  $T_m$  is a sequence of linear operators such that the maximal function  $T^*f$  is dominated by  $Mf$ . If  $f \in L^1(\Omega)$  and  $T_m g \rightarrow g$  pointwise for  $g \in C(\Omega)$ , then  $T_m f \rightarrow f$  a.e.

*Proof.* Take  $\varepsilon > 0$  and  $g \in C(\Omega)$  such that  $\|f - g\|_{L^1(\Omega)} < \varepsilon$ . Since  $T_m g(x) \rightarrow g(x)$  pointwise, we have

$$\begin{aligned} & \mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\ & \leq \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \frac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \frac{\lambda}{2}\}) \\ & \leq \mu(\{x : M(f - g)(x) > \frac{\lambda}{2}\}) + \frac{2}{\lambda} \|f - g\|_{L^1(\Omega)} \\ & \lesssim \frac{1}{\lambda} \varepsilon \end{aligned}$$

for every  $\lambda > 0$ . Limiting  $\varepsilon \rightarrow 0$ , we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every  $\lambda > 0$ , hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

**Definition 12.1.1.**

$$f^*(x) := \lim_{r \rightarrow 0^+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| dy.$$

**Theorem 12.1.1** (Lebesgue differentiation).  $f^* = 0$  a.e.

*Proof.* Note that  $f^* \leq Mf + |f|$  implies

$$\|f^*\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that  $g^* = 0$  for  $g \in C_c$ . Approximate using  $f^* = (f - g)^*$ .

□

## Exercises

**12.5** (Doubling measure).