Sheaves and Bundles

Ikhan Choi

January 25, 2024

Contents

Part I

Bundles

Chapter 1

Fiber bundles

1.1 Principal bundles

1.1 (Locally trivial bundles). A *fiber bundle* is a map $p : E \to B$ such that $p^{-1}(b)$ is homeomorphic to F for each $b \in B$, where E, B, F are topological spaces called the *total space*, *base space*, and *fiber*. We say a fiber bundle ξ is *locally trivial* if it admits an *atlas* $\{\varphi_i\}$, a family of homeomorphisms $\varphi_i : p^{-1}(U_i) \to U_i \times F$ which indexed by an open cover $\{U_i\}$ of B such that

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

commutes. In this note, we are only concerned with locally trivial bundles and every fiber bundle will be assumed to be locally trivial.

A *bundle map* between fiber bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ is a map of pairs $(\widetilde{u}, u) : (E_1, B_1) \to (E_2, B_2)$ such that

$$\begin{tikzcd} E_1 \arrow[r, "\widetilde{u}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "u"] & B_2 \end{tikzcd}$$

commutes.

(a) p is surjective and open.

1.2 (Structure groups). Let F be a left G-space for a topological group G. A fiber bundle $p : E \to B$ with fiber F is said to have a *structure group* G if it admits a *G-atlas*, an atlas $\{\varphi_i\}$ that has a set $\{g_{ij}\}$ of maps $g_{ij} : U_i \cap U_j \to G$ such that the transition maps $\varphi_j \varphi_i^{-1}$ are given by

$$\varphi_j \varphi_i^{-1}(b, f) = (b, g_{ij}(b) f), \qquad b \in U_i \cap U_j,\ f \in F.$$

A *G-bundle* with fiber F is a fiber bundle $p : E \to B$ together with a *G-structure*, a maximal G-atlas.

A *G-bundle map* is a bundle map $(\widetilde{u}, u) : (E_1, B_1) \to (E_2, B_2)$ between G-bundles such that there is a set $\{h_{ij}\}$ of maps $h_{ij} : U_{1,i} \cap u^{-1}(U_{2,j}) \to G$ such that

$$\varphi_{2,j} \widetilde{u} \varphi_{1,i}^{-1}(b, f) = (u(b), h_{ij}(b) f), \qquad b \in U_{1,i} \cap u^{-1}(U_{2,j}),\ f \in F.$$

(a) If F is a locally connected locally compact Hausdorff space, then every fiber bundle with fiber F has the structure group $\mathrm{Homeo}(F)$ with respect to the compact-open topology.

(b) A G-bundle map $(\widetilde{u}, u)$ is an isomorphism if and only if u is a homeomorphism.

(c) A bundle map $(\widetilde{u}, \mathrm{id}_B) : (E_1, B) \to (E_2, B)$ is a G-bundle map if and only if there is a set $\{h_i\}$ of maps $h_i : U_i \to G$ such that

$$\varphi_{2,i}\widetilde{u}\varphi_{1,i}^{-1}(b, f) = (b, h_i(b)f), \qquad b \in U_i,\ f \in F.$$

Proof. (a)

(b) ($\Rightarrow$) Clear.

($\Leftarrow$) The total map $\widetilde{u}$ is continuous bijection because u is a bijection, so it suffices to show $\widetilde{u}^{-1}$ is continuous. Fix $U_i \subset B$ and $U_j' \subset B'$. By substitution of $b' := u(b)$, $f' := h_{ij}(b)f$, we can write

$$\varphi_i \circ \widetilde{u}^{-1} \circ \varphi_{j'}'^{-1}(b', f') = (u^{-1}(b'), h_{ij'}(u^{-1}(b'))^{-1} f').$$

Since the local trivializations, the inverse operation of G, and the inverse u^{-1} are all continuous, $\widetilde{u}^{-1}$ is also continuous. □

1.3 (Fiber bundle construction theorem)**.** Let $\{U_i\}_i$ be an open cover of a topological space B, and let G be a topological group. Let $Z^1(\{U_i\}, G)$ be the set of every *Čech 1-cocyle* on $\{U_i\}$ with coefficients in G, a set $\{g_{ij}\}$ of maps $g_{ij} : U_i \cap U_j \to G$ satisfying the *cocycle condition*:

$$g_{ik}(b) = g_{jk}(b)g_{ij}(b), \qquad b \in U_i \cap U_j \cap U_k.$$

Also let $C^0(\{U_i\}, G)$ be the set of every *Čech 0-cochain* on $\{U_i\}$ with coefficients in G, a collection $\{h_i\}$ of maps $h_i : U_i \to G$ of maps without any conditions.

The *first Čech cohomology* $\check{H}^1(\{U_i\}, G)$ of $\{U_i\}$ with coefficients in G is defined to be the orbit space of an action $\check{C}^0(\{U_i\}, G) \curvearrowright \check{Z}^1(\{U_i\}, G)$ defined as

$$(\{h_i\}\{g_{ij}\})_{ij}(b) := h_j(b)g_{ij}(b)h_i(b)^{-1}, \qquad b \in U_i \cap U_j.$$

We define the *first Čech cohomology* of B with coefficients in G as the direct limit of sets

$$\check{H}^1(B, G) := \varinjlim_{\{U_i\}} \check{H}^1(\{U_i\}, G).$$

Let F be a left G-space, and let $\mathrm{Bun}_F(B)$ be the set of isomorphism classes of G-bundles over B with fiber F.

(a) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G)$ is well-defined.

(b) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G)$ is surjective.

(c) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G/\ker\sigma)$ is injective, where $\sigma : G \to \mathrm{Homeo}(F)$.

Proof. (a) Suppose $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are isomorphic G-bundles with fiber F, and $\widetilde{u} : E_1 \to E_2$ is a G-bundle isomorphism. By considering the refinement, we can find an open cover $\{U_i\}$ of B on which E_1 and E_2 are simultaneously locally trivialized.

(b) Let F be a left G-space and let $\{g_{ij}\} \in \check{Z}^1(B, G)$ that is defined on an open cover $\{U_i\}$. Define

$$E := \left(\coprod_i (U_i \times F)\right) / \sim,$$

where $\sim$ is an equivalence relation generated by

$$(i, b, f) \sim (j, b, g_{ij}(b)f), \qquad b \in U_i \cap U_j,\ f \in F.$$

Also define $p : E \to B : [i, b, f] \mapsto b$ and $\varphi_i^{-1} : U_i \times F \to p^{-1}(U_i) : (b, f) \mapsto [i, b, f]$, which are clearly continuous and surjective without the cocycle condition.

We first claim that φ_i^{-1} is injective. Suppose $\varphi_i^{-1}(b,f)=\varphi_i^{-1}(b',f')$. Since $(i,b,y)\sim(i,b',y')$, we have $b=b'$ and there is a sequence of open sets $U_{i_0},\cdots,U_{i_n}$ in $\{U_i\}$ such that $i_0=i_n=i$ and

$$f'=g_{i_{n-1}i_n}(b)g_{i_{n-2}i_{n-1}}(b)\cdots g_{i_0i_1}(b)f.$$

By applying the cocycle condition inductively, we obtain $f=f'$, which implies the injectivity of φ_i^{-1}.

Next we claim that φ_i^{-1} is open. The map φ_i^{-1} factors through $\coprod_i(U_i\times F)$ such that

$$\varphi_i^{-1}:U_i\times F\to\coprod_i(U_i\times F)\xrightarrow{\pi}p^{-1}(U_i).$$

Since the canonical inclusion to disjoint union is open, it suffices to show the quotient map $\pi:\coprod_i(U_i\times F)\to E$ is open. Let $V\subset\coprod_i(U_i\times F)$ be open. Observe that

$$\pi^{-1}\pi(V\cap(U_i\times F))\cap(U_j\times F)$$

is open for each pair of i and j because it is exactly same as the inverse image of the open set $V\cap(U_i\times F)$ under the map

$$(U_i\cap U_j)\times F\subset U_j\times F\to U_i\times F:(b,f)\mapsto(b,g_{ij}(b)f).$$

Here we used the cocycle condition of $\{g_{ij}\}$. Therefore,

$$\pi^{-1}\pi(V)=\bigcup_{i,j}\pi^{-1}\pi(V\cap(U_i\times F))\cap(U_j\times F)$$

is open, hence the open π.

The transition maps of the G-atlas $\{\varphi_i\}$ coincides with the cocycle $\{g_{ij}\}$ by the cocycle condition. $\square$

1.4 (Principal bundles). Let G be a topological group, and X be a left *principal homogeneous G-space*, i.e. a free and transitive left G-space such that the shear map $G\times X\to X\times X:(g,x)\mapsto(gx,x)$ is a homeomorphism.

A *principal G-bundle* is a G-bundle $p:P\to B$ with fiber X, often together with a fiber-preserving continuous right action $\rho:P\times G\to P$ such that each chart $\varphi_i:p^{-1}(U_i)\to U_i\times X$ induces a principal homogeneous right action on $\{b\}\times X\subset U_i\times X$ which commutes with the left action. The right action ρ is called the *principal right action* or *(global) gauge transformation*. Note that for each $b\in B$ the *fiber* $\{b\}\times X$ has commuting left and right actions, but the *fiber* $p^{-1}(b)$ can admit only the principal right action.

The category of principal G-bundles over B is denoted by $\mathbf{Prin}_G(B)$, and the morphisms are usually defined as right G-equivariant maps with respect to the pricipal right actions. Then, we may consider the forgetful functor $\mathbf{Prin}_G(B)\to\mathbf{Bun}_X(B)$.

(a) $\mathbf{Prin}_G(B)\to\mathbf{Bun}_X(B)$ is fully faithful, i.e. a bundle map $u:P\to P'$ over B is a G-bundle map if and only if it is a right G-equivariant map.

(b) $\mathbf{Prin}_G(B)\to\mathbf{Bun}_X(B)$ is surjective, i.e. every G-bundle with fiber X has a principal right action.

(c) A principal bundle is trivial if it has a global section.

Proof. (a) ($\Rightarrow$) Let $u:P\to P'$ be a G-bundle map over B so that there is a set $\{h_i:U_i\to G\}_i$ of maps such that

$$\varphi_i\circ u\circ\varphi_i^{-1}(b,x)=(b,h_i(b)x),\qquad b\in U_i,\ x\in X.$$

If we write $\rho_s:P\to P:e\mapsto\rho(e,s)$ for $s\in G$, then the induced right action $\varphi_i\circ\rho_s\circ\varphi_i^{-1}$ commutes with the left action $\varphi_i\circ u\circ\varphi_i^{-1}$ on $U_i\times X$. Now for every $e\in P_1$, we have

$$\begin{aligned}\rho_s\circ u(e)&=\varphi_i^{-1}\circ(\varphi_i\circ\rho_s\circ\varphi_i^{-1})\circ(\varphi_i\circ u\circ\varphi_i^{-1})\circ\varphi_i(e)\\&=\varphi_i^{-1}\circ(\varphi_i\circ u\circ\varphi_i^{-1})\circ(\varphi_i\circ\rho_s\circ\varphi_i^{-1})\circ\varphi_i(e)\\&=u\circ\rho_s(e),\end{aligned}$$

therefore u is right G-equivariant.

($\Leftarrow$) let $u : P \to P'$ be a right G-equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \to G$ such that

$$(b, g_i(b)x_0) := \varphi_i \circ u \circ \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xmapsto{\varphi_i \circ u \circ \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The continuity of the last map is due to the assumption that the map $(g, x) \mapsto (gx, x)$ is a homeomorphism.

Then, for every $(b, x) \in U_i \times X$ there is a unique $s \in G$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned}
\varphi_i \circ u \circ \varphi_i^{-1}(b, x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b, x_0) \\
&= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) g_i(b)(b, x_0) \\
&= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= g_i(b)(b, x) \\
&= (b, g_i(b)x).
\end{aligned}$$

Hence, u is a G-bundle map.

(b) Fix $x_0 \in X$ and consider the homeomorphism $G \to X : g \to gx_0$. Define a right action

$$X \times G \to X : (gx_0, s) \mapsto gx_0 s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X.

Define $\rho : P \times G \to P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b, the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge tranformation.

(c) ($\Rightarrow$) Clear.

($\Leftarrow$) Let $s : B \to E$ be a global section and define

$$\widetilde{u} : B \times X \to E : (b, gx_0) \mapsto s(b)g$$

for any fixed $x_0 \in X$. Then, the continuous map $(\widetilde{f}, \mathrm{id}_B)$ preserves fibers and defines a right G-equivariant isomorphism. □

1.5 (Quotient principal bundles)**.**

1.6 (Reduction of structure groups)**.** Let H be a closed subgroup of G. Then, there is a map $\check{H}^1(B, H) \to \check{H}^1(B, G)$, which is neither in general injective nor surjective. If a G-bundle ξ is contained in the image of $\check{H}^1(B, H)$ through the correspondence $\mathrm{Bun}_F(B) \twoheadrightarrow \check{H}^1(B, G)$, then we may give a H-bundle structure on ξ.

A *reduction* of G to H is a H-structure on a principal G-bundle.

1.2 Classifying spaces

Let G be a topological group. Let $\mathrm{Prin}_G(B)$ be the set of isomorphism classes of principal G-bundles over a topological B. Then, we have a contravariant functor

$$\mathrm{Prin}_G : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}.$$

On paracompact spaces:

1. The functor Prin_G is homotopy invariant.
2. The functor Prin_G is representable.
3. The universal elements can be computed using contractibility.

1.7 (Homotopy properteis)**.** Let $p : E \to B$ be a vector bundle

(a) If $p : E \to B \times [0, \frac{1}{2}]$ and $p' : E' \to B \times [\frac{1}{2}, 1]$ are trivial, then

(b) If $f, g : B' \to B$ are homotopic, then $f^*\xi \cong g^*\xi$.

1.3 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

1.8 (Vector bundles)**.** Let $p : E \to B$ and $p : E' \to B$ be vector bundles.

(a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \le n \le \infty$. If $f, g : B \to G_k(\mathbb{F}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{F}^n) \to G_k(\mathbb{F}^{2n})$ is the natural inclusion.

1.9. Riemannian and Hermitian metrics spin structures

Exercises

1.10. Let G be a topological group, and X be a free right G-space.

(a) If the action is proper and the projection $X \to X/G$ admits local sections, then $X \to X/G$ is a principal G-bundle.

1.11 (Clutching functions)**.**

1.12. Suppose $F \to E \to B$ is a principal

(a) If X is contractible, then $X \to$

1.13 (Group quotients)**.** Sufficient conditions for principal bundles. Let G be a Lie group and, X be a free right smooth G-manifold.

(a) If G is compact, then $X \to X/G$ is a principal G-bundle. (Gleason)

(b) The irrational slope provides a counterexample if G is not compact.

(c) Suppose X is a Lie group. If G is a closed subgroup of X, then $X/ \to X/G$ is a principal G-bundle. (Samelson) In particular, if M is a transitive left smooth X-manifold such that G is the isotropy group, then $X \to M$ is a principal G-bundle.

1.14 (Homogeneous spaces)**.** They are all principal bundles.

$$\begin{aligned} \mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n), \\ \mathrm{O}(n-k)\times \mathrm{O}(k) \to \mathrm{O}(n) \to G_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k)\times \mathrm{U}(k) \to \mathrm{U}(n) \to G_k(\mathbb{C}^n), \\ T(n)\cap \mathrm{O}(n) \to \mathrm{O}(n) \to F(\mathbb{R}^n), &\qquad T(n)\cap \mathrm{U}(n) \to \mathrm{U}(n) \to F(\mathbb{C}^n), \\ &\qquad T(n) \to \mathrm{GL}(n,\mathbb{C}) \to F(\mathbb{C}^n), \end{aligned}$$

where $T(n)$ is the group of invertible upper triangular matrices.

$$\mathrm{SO}(n) \to \mathrm{SO}^+(1,n) \to \mathbb{H}^n, \qquad \mathrm{PSO}(2) \to \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2, \qquad ?? \to \mathrm{PSL}(2,\mathbb{C}) \to \mathbb{H}^3,$$

where $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)^+$ is the modular group and $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(1,3)^+$ is the restricted Lorentz group, also called the Möbius group.

1.15 (Hopf fibration)**.** A principal S^1-bundle $S^1 \to S^3 \to S^2$, where we see S^1 as the circle group. The Hopf fibrations are used in describing universal principal bundles off orthogonal or unitary groups. We have principal bundles:

(a) The quaternionic construction gives $S^3 \to S^7 \to S^4$ and the octonianic construction gives $S^7 \to S^{15} \to S^8$. Adams' theorem.

(b) $\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$. In particular, $\mathbb{Z}/2\mathbb{Z} \to S^n \to \mathbb{RP}^n$ for $k=1$.

(c) $\mathrm{U}(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$. In particular, $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $k=1$.

Hopf fibration(real, complex, quaternionic, but not octonianic)
In the category of smooth manifolds, if f diffeomorphic, then $\widetilde{f}$ diffeomorphic.

1.16 (Associated bundles)**.**

$$\mathrm{Prin}_G(B) \xrightarrow{\sim} \mathrm{Bun}_X(B) \xrightarrow{\sim} \check{H}^1(B,G) \hookrightarrow \mathrm{Bun}_F(B)$$

can be given in a more simple way.

Part II

Sheaves

Chapter 2

General sheaf theory

2.1 (Étale bundles)**.** Let X be a topological space. An *étale bundle* over a X is a local homeomorphism $p : E \to X$, where E is a topological space. An étale bundle over X is also often called a *sheaf* on X, but we let this term be reserved for a different but equivalent notion to the étale bundles, which will be introduced later. Étale bundles over X form a category, with morphisms defined as continuous maps $\varphi : E_1 \to E_2$ satisfying $p_1 = \varphi p_2$ for two sheaves $p_i : E_i \to X$ with $i \in \{1, 2\}$.

germs and stalks, section, basis of étale space

(a) A subset $F \subset E$ is defines a subsheaf if and only if F is open in E.

(b) A covering space is nothing but a locally constant sheaf of sets.

(c)

Proof. □

2.2 (Presheaves)**.** Let X be a topological space. A *presheaf* (of sets) on X is a contravariant functor $\mathcal{F}$ from the category of open sets of X with inclusions as its morphisms to the category of sets. Presheaves on X form a category with natural transformations as its morphisms.

We construct a functor from the category of presheaves on X to the category of étale bundles over X, sometimes called the *étale space construction* or the *sheafification*. For a presheaf $\mathcal{F}$ on X, define

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U), \quad U \text{ open in } X, \qquad E(\mathcal{F}) := \bigsqcup_{x \in X} \mathcal{F}_x$$

and let $p : E(\mathcal{F}) \to X$ be such that $p(e) := x$ for $e \in \mathcal{F}_x$. The set $\mathcal{F}_x$ and its element are called the *stalk* and a *germ* at x respectively, and the set $E(\mathcal{F})$ is called the *étale space* of $\mathcal{F}$.

(a) There exists a unique natural topology on $E(\mathcal{F})$ such that $p : E \to X$ is a local homeomorphism.

(b) There exists a unique natural function $\mathcal{F}(U) \to \Gamma(U, E(\mathcal{F}))$ such that .. for open subsets $U \subset X$.

Proof. (a) We endow a topology on E generated by a base $\{s(U) : s \in \mathcal{F}(U),\ U \text{ open}\}$.

(b) For $x \in U$ and $s \in \mathcal{F}(U)$, we define $s_x \in p^{-1}(x)$ by the image of $\mathcal{F}(U) \to \mathcal{F}_x$ at s, and it defines a morphism of presheaves $\mathcal{F} \to \Gamma(E(\mathcal{F}))$. □

2.3 (Sheaves)**.** Let X be a topological space. A *sheaf* (of sets) on X is a presheaf on X that satisfies the following two conditions:

(i) (Locality)

(ii) (Gluing)

The category of sheaves on X is defined to be the full subcategory of the category of presheaves on X.

(a) For an étale bundle $p : E \to X$ over X, the functor $\mathcal{F} : U \mapsto \Gamma(U, E)$ defines a sheaf on X.

(b) The étale space construction defines an equivalence of the category of sheaves on X and the category of étale bundles over X.

2.4 (Morphism of sheaves)**.** epic and monic. The description for $\mathcal{F}(U)$ and the description for $\mathcal{F}_x$.

2.5 (Operations on sheaves)**.** inverse image functor(restriction, pullback) direct image functor(push-forward) Whitney sum, constant sheaf, subsheaf

étale space descriptions

(a)

2.6 (Sheaves of rings)**.** Let X be a topological space.

A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf $\mathcal{O}_X$ of rings on X.

2.7 (Sheaves of module)**.** Let X be a topological space.

A sheaf $\mathcal{F}_X$ of $\mathcal{O}_X$-modules is called *locally finite* or *finite type* if there is an open cover $\{U_i\}$ of X together with surjective ring homomorphisms $\mathcal{O}_X(U_i) \twoheadrightarrow \mathcal{F}_X(U_i)$ for all i. The section space $\Gamma(U, \mathcal{O}_X)$ will be also denoted by $\mathcal{O}_X(U)$.

(a) The category of sheaves of modules over $\mathcal{O}_X$ is abelian.

2.8 (Coherent sheaves)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. Consider a quasi-coherent sheaf with an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{F}_U \to 0.$$

The *generating system* is the basis of $\mathcal{O}_X^p$, and the *relation sheaf* is the kernel of $\mathcal{O}_X^p \to \mathcal{F}_X$. We say $\mathcal{F}_X$ is *coherent* if $\mathcal{F}_X$ is of finite type and the relation sheaf is also of finite type.

(a) If $\mathcal{O}_X$ is itself a coherent module, then every locally finitely presented $\mathcal{O}_X$-module is coherent.

2.9 (Yoga of coherent sheaves)**.**

(a) extension principle?

(b) If a ring $\mathcal{O}$ has a split epi $\mathcal{O} \to \mathcal{O}'$ to a coherent $\mathcal{O}'$, then $\mathcal{O}$ is coherent.

Proof. Consider the following diagram in which every row is exact and K, K_1, K_2 are kernels:

$$\begin{tikzcd}
K \arrow[r] & \mathcal{O}^p \arrow[r] \arrow[d, equal] & \mathcal{O} \arrow[r] \arrow[d, two heads, "\circlearrowright"] & 0 \\
K_1 \arrow[r] & \mathcal{O}^p \arrow[r] \arrow[d, two heads] & \mathcal{O}' \arrow[r] \arrow[d, equal] & 0 \\
K_2 \arrow[r] & \mathcal{O}'^p \arrow[r] & \mathcal{O}' \arrow[r] & 0.
\end{tikzcd}$$

Then, K_2 is finitely generated by the coherence of $\mathcal{O}'$, K_1 is finitely generated by the Schanuel lemma, and K is finitely generated by the snake lemma. □

2.1

Note that affine schemes, complex model spaces, Euclidean open subsets are all locally ringed spaces.

A *scheme* is a ringed space modeled on affine schemes. A *complex (analytic) space* is a Hausdorff ringed space modeled on complex model spaces. A *manifold* is a second countable Hausdorff ringed space modeled on Euclidean open subsets.

They are all locally ringed. For the latter two, the residue field at every stalk is isomorphic to $\mathbb{C}$ or $\mathbb{R}$. For schemes over a field k, the relation between residue fields and k is related to the Nullstellensatz at closed points.