Operator Algebra Seminar Note II

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Contents

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1 October 18

Definition 1.1 (Countably decomposable von Neumann algebras)**.** Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal non-zero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is.

Proposition 1.2. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M is countably decomposable.*

(b) *M admits a faithful normal state.*

(c) *M admits a faithful normal non-degenerate representation with a cyclic and separating vector.*

(d) *The unit ball of M is metrizable in the six locally convex topology.*

Proof. (a)$\iff$(b) Suppose M is countably decomposable. Let $\{\xi_i\} \subset H$ be a maximal family of unit vectors such that $\overline{M'\xi_i}$ are mutually orthogonal subspaces, taken by Zorn's lemma. If we let p_i be the projection on $\overline{M'\xi_1}$, then $p_i z p_i = z p_i$ for $z \in M'$ implies $p_i \in M'' = M$. By the assumption, the family $\{\xi_i\}$ is countable. Define a state ω of M such that

$$\omega(x) := \sum_{i=1}^{\infty} \omega_{2^{-i}\xi_i}(x), \qquad x \in M.$$

It converges due to $\|\omega_{2^{-i}\xi_i}\| = 2^{-i+1}$. It is normal since the sequence $(2^{-i}\xi_i)$ belongs to $\ell(\mathbb{N}, H)$, and it is faithful because $\omega(x^*x) = 0$ implies $x\xi_i = 0$ for all i, which deduces that $x = \sum_i x p_i = 0$.

Conversely, if ω is a faithful normal state, then for a mutually orthogonal family of non-zero projections $\{p_i\} \subset M$, we have

$$\{p_i\} = \bigcup_{n=1}^{\infty} \{p_i : \varphi(p_i) > n^{-1}\}$$

the countable union of finite sets. Thus M is countable decomposable.

(b)$\iff$(c) Let ω be a faithful normal state of M. Consider any faithful normal nondegenerate representation in which ω is a vector state so that the corresponding vector is a separating vector. Examples include the GNS representation of ω, and the composition with the diagonal map $B(H) \to B(\ell^2(\mathbb{N}, H))$. Then, $\overline{M\Omega}$ admits a cyclic and separating vector Ω of M. The converse is immediate, i.e. the vector state ω_Ω is a faithful normal state of M.

(a)$\iff$(d) Suppose M is countably decomposable and take $\{\xi_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ as we did. Define

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \|(x - y)\xi_i\|.$$

Clearly it generates a topology coarser than strong topology. It is also finer because if a bounded net x_α in M converges to zero in the metric d so that $x\xi_i \to 0$ for all i, then $H = \bigoplus_i M'\xi_i$ implies that for every $\xi \in H$ and $\varepsilon > 0$ we have $\|\xi - \sum_{k=1}^{n} z_k \xi_{i_k}\| < \varepsilon$ for some $z_k \in M'$ so that

$$\|x_\alpha \xi\| \le \|x_\alpha(\xi - \sum_{k=1}^{n} z_k \xi_{i_k})\| + \sum_{k=1}^{n} \|x_\alpha z_k \xi_{i_k}\| < \varepsilon + \sum_{k=1}^{n} \|z_k\| \|x_\alpha \xi_{i_k}\| \to \varepsilon.$$

Since on the bounded part the strong and σ-strong topologies coincide, the two topologies on the unit ball are metrizable. We can do similar for the weak and strong* topologies.

Conversely, for a mutually orthogonal family of non-zero projections $\{p_i\}_{i \in I} \subset M$, since the net of finite partial sums $p_F := \sum_{i \in F} p_i$ is an increasing net in the closed unit ball whose supremum is the identity of M, there is a convergent subsequence $p_{F_n} \uparrow 1$ by the metrizability, which implies $I = \bigcup_{n=1}^{\infty} F_n$, the countable union of finite sets. $\square$

1.1 Semi-cyclic representations

Definition 1.3 (Weights)**.** Let M be a von Neumann algebra. A *weight* is a function $\varphi : M^+ \to [0,\infty]$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\lambda x) = \lambda\varphi(x), \qquad x, y \in M^+,\ \lambda \in \mathbb{R}^{\geq 0},$$

where we use $0 \cdot \infty = 0$. A weight φ is said to be *normal* if

$$\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha)$$

for any bounded increasing net (x_α) in M^+.

Definition 1.4. Let φ be a weight on a von Neumann algebra M. Define a left ideal of M

$$\mathfrak{n} := \{x \in M : \varphi(x^*x) < \infty\},$$

and a hereditary $*$-subalgebra of M

$$\mathfrak{m} := \mathfrak{n}^*\mathfrak{n} = \{\sum_{i=1}^n y_i^* x_i : (x_i), (y_i) \in \mathfrak{n}^n\}.$$

Lemma 1.5. *If $x, y \in M$ satisfies $y^*y \leq x^*x$, then there is a unique $s \in B(H)$ such that $y = sx$ and $s = sp$, where p is the range projection of x, and $s \in M$.*

Proof. Suppose $\mathrm{id}_H \in M \subset B(H)$. The operator $s_0 : \overline{xH} \to \overline{yH} : x\xi \mapsto y\xi$ is well defined because

$$\|y\xi\|^2 = \langle y^*y\xi, \xi\rangle \leq \langle x^*x\xi, \xi\rangle = \|x\xi\|^2.$$

Let p be the range projection of x and let $s := s_0 p$. Then, $y\xi = sx\xi$ for all $\xi \in H$. If $y = s'x$ and $s' = s'p$, then

$$x^*(s-s')^*(s-s')x = (y-y)^*(y-y) = 0$$

implies

$$0 = p(s-s')^*(s-s')p = (s-s')^*(s-s').$$

Therefore, s is unique in $B(H)$. If $u \in M'$ is unitary, then usu^* satisfies the same property $y = usu^*x$ and $usu^* = usu^*p$, so $us = su$. Since the unitary span the whole C*-algebra, we have $s \in M'' = M$. $\square$

Proposition 1.6. *Let φ be a weight on a von Neumann algebra M.*

(a) *Every element of $\mathfrak{m}^+$ can be written to be x^*x for some $x \in \mathfrak{n}$.*

(b) *Every element of $\mathfrak{m}$ can be written to be y^*x for some $x, y \in \mathfrak{n}$.*

Proof. (a) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}^+$ for some $x_i, y_i \in \mathfrak{n}$. The polarization writes

$$a = \frac{1}{4}\sum_{i=1}^n \sum_{k=0}^3 i^k |x_i + i^k y_i|^2$$

and $a^* = a$ implies

$$a = \frac{1}{2}\sum_{i=1}^n (|x_i + y_i|^2 - |x_i - y_i|^2) \leq \frac{1}{2}\sum_{i=1}^n |x_i + y_i|^2$$

implies

$$\varphi(a) \leq \frac{1}{2}\sum_{i=1}^n \varphi(|x_i + y_i|^2) < \infty.$$

Therefore, if $x := a^{\frac{1}{2}} \in \mathfrak{n}$, then $a = x^*x$.

(b) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}$ for some $x_i, y_i \in \mathfrak{n}$. Let $x := (\sum_{i=1}^n x_i^* x_i)^{\frac{1}{2}} \in \mathfrak{n}$. Since $x_i^* x_i \leq x^2$, we have $s_i \in M$ such that $x_i = s_i x$. If we let $y := \sum_{i=1}^n s_i^* y_i \in \mathfrak{n}$, then

$$a = \sum_{i=1}^n y_i^* x_i = \sum_{i=1}^n y_i^* s_i x = (\sum_{i=1}^n s_i^* y_i)x = y^*x. \qquad \square$$

Definition 1.7 (Semi-cyclic representations)**.** Let φ be a weight on a von Neumann algebra. Let H be the Hilbert space defined by the separation and completion of a sesquilinear form

$$\mathfrak{n}\times\mathfrak{n}\to\mathbb{C}:(x,y)\mapsto\varphi(y^*x)$$

and let $\psi:\mathfrak{n}\to H$ be the canonical image map. The pair (π,ψ) is called the *semi-cyclic representation* associated to φ.

Proposition 1.8. *Let φ be a weight on a von Neumann algebra and (π,ψ) be the associated semi-cyclic representation to φ. Consider a map*

$$\Theta:\mathfrak{m}\times\pi(M)'\to\mathbb{C}:(y^*x,z)\mapsto\langle z\psi(x),\psi(y)\rangle$$

and define

$$\theta:\mathfrak{m}\to(\pi(M)')_*,\qquad\theta^*:\pi(M)'\to\mathfrak{m}^\#$$

such that $\Theta(x,z)=\theta(x)(z)=\theta^*(z)(x)$ for $x\in\mathfrak{m}$ and $z\in\pi(M)'$.

(a) *Θ is a well-defined bilinear form.*

(b) *θ^* is bijective onto the space of linear functionals on $\mathfrak{m}$ whose absolute value is majorized by φ. (bounded Radon-Nikodym theorem)*

Proof. (a) The linearity in the second argument is obvious. Fix $z\in\pi(M)'$. We first check the well-definedness on $\mathfrak{m}^+$. Let $x^*x=y^*y\in\mathfrak{m}^+$ for $x,y\in\mathfrak{n}$. Then, there is $s\in M$ such that $y=sx$ and $s=sp$, where p is the range projection of x, so

$$x^*(1-s^*s)x=x^*x-y^*y=0$$

implies

$$0=p(1-s^*s)p=p-s^*s$$

and $x=px=s^*sx=s^*y$. The well-definedness follows from

$$\Theta(x^*x,z)=\langle z\psi(x),\psi(x)\rangle=\langle\pi(s)z\pi(s^*)\psi(y),\psi(y)\rangle=\langle z\psi(ss^*y),\psi(y)\rangle=\Theta(y^*y,z).$$

The homogeneity is clear, so now we prove the addivitiy. Let $x^*x,y^*y\in\mathfrak{m}^+$ for some $x,y\in\mathfrak{n}$. Let $a:=(x^*x+y^*y)^{\frac{1}{2}}$ and take $s,t\in M$ such that $x=sa$, $y=ta$, $s=sa$, and $t=ta$, where p is the range projection of a. Then,

$$a(1-s^*s-t^*t)a=a^*a-x^*x-y^*y=0$$

implies

$$p(1-s^*s-t^*t)p=p-s^*s-t^*t.$$

It follows that

$$\begin{aligned}\Theta(x^*x+y^*y,z)&=\langle z\psi(a),\psi(a)\rangle=\langle z\pi(p)\psi(a),\psi(a)\rangle\\&=\langle z\pi(s^*s)\psi(a),\psi(a)\rangle+\langle z\pi(t^*t)\psi(a),\psi(a)\rangle\\&=\langle z\psi(x),\psi(x)\rangle+\langle z\psi(y),\psi(y)\rangle\\&=\Theta(x^*x,z)+\Theta(y^*y,z).\end{aligned}$$

Now the $\Theta(\cdot,z)$ is linearly extendable to $\mathfrak{m}$.

(b) The linear map θ^* is injective since ψ has dense range. Take $z\in\pi(M)'$ and consider $\theta^*(z)$, which maps x^*x to $\langle z\psi(x),\psi(x)\rangle$ for $x\in\mathfrak{n}$. The image is majorized by φ as

$$|\langle z\psi(x),\psi(x)\rangle|\le\|z\|\|\psi(x)\|^2=\|z\|\varphi(x^*x).$$

Conversely, let $l \in \mathfrak{m}^\#$ is a linear functional majorized by φ, i.e. there is a constant $C > 0$ such that

$$|l(x^*x)| \le C\varphi(x^*x), \qquad x \in \mathfrak{n}.$$

Define a sesquilinear form $\sigma : \mathfrak{n} \times \mathfrak{n} \to \mathbb{C}$ such that $\sigma(x,y) := l(y^*x)$. It is well-defined after separation of $\mathfrak{n}$ and is bounded by the Cauhy-Schwartz inequality

$$|\sigma(x,y)|^2 = |l(y^*x)|^2 \le \|l(x^*x)\|\|l(y^*y)\| \le \varphi(x^*x)\varphi(y^*y) = \|\psi(x)\|^2\|\psi(y)\|^2.$$

Therefore, σ defines a bounded linear operator $z \in \pi(M)'$ such that

$$\sigma(x,y) = \langle z\psi(x), \psi(y)\rangle,$$

exactly meaning $\theta^*(z)(y^*x) = l(y^*x)$ for $x, y \in \mathfrak{n}$. $\square$

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, "\psi"] \arrow[dd, "|\cdot|^2"'] & H \arrow[d, "\omega"] \\
 & B(H)_* \arrow[d, "\mathrm{res}"] \\
\mathfrak{m}^+ \arrow[r, "\theta"] & (\pi(M)')_*.
\end{tikzcd}$$

In particular, for $x \in \mathfrak{n}^+$ we have

$$\|\theta(x^2)\| = \|\omega_{\psi(x)}\| = \|\psi(x)\|^2 = \varphi(x^2).$$

Lemma 1.9. *Let For $z \in \mathfrak{m}^{sa}$, we have*

$$\inf\{\varphi(a) : z \le a \in \mathfrak{m}^+\} \le \|\theta(z)\|.$$

In particular, for $x, y \in \mathfrak{n}^+$ and for any $\varepsilon > 0$ there is $a \in \mathfrak{m}^+$ such that $x^2 - y^2 \le a$ and

$$\varphi(a) \le \|\theta(x^2 - y^2)\| + \varepsilon = \|\omega_{\psi(x)} - \omega_{\psi(y)}\| + \varepsilon.$$

Proof. Denote by $p(z)$ the left-hand side of the inequality. Then, we can check $p : \mathfrak{m}^{sa} \to \mathbb{R}_{\ge 0}$ is a semi-norm such that $p(z) = \varphi(z)$ for $z \ge 0$. (If we take $p(z) := \varphi(z^+)$, then it seems to be dangerous when checking the sublinearity. I could not find the counterexample.)

Fix any non-zero $z_0 \in \mathfrak{m}^{sa}$. By the Hahn-Banach extension, there is an algebraic real linear functional $l : \mathfrak{m}^{sa} \to \mathbb{R}$ such that

$$l(z_0) = p(z_0), \qquad |l(z)| \le p(z), \qquad z \in \mathfrak{m}^{sa}.$$

Extend linearly l to be $l : \mathfrak{m} \to \mathbb{C}$. Since $|l(z)| \le \varphi(z)$ for $z \in \mathfrak{m}^+$, the linear functional l is contained in the image of the closed unit ball under the injective map

$$\theta^* : \pi(M)' \to \mathfrak{m}^\#.$$

If we let $a \in (\pi(M)')_1$ be the corresponding operator such that $\theta^*(a) = l$, then we get

$$p(z_0) = l(z_0) = \theta^*(a)(z_0) = \theta(z_0)(a) \le \|\theta(z_0)\|.$$

Since $z_0 \in \mathfrak{m}^{sa}$ is aribtrary, we are done. $\square$

1.2 σ-weak lower semi-continuity

Theorem 1.10. *Let M be a countably decomposable von Neumann algebra. Then, normal weight on M is σ-weakly lower semi-continuous.*

Proof. Let φ be a normal weight on M and let (π, ψ) be the associated semi-cyclic representation.

In the spirit of the Krein-Šmulian theorem, the σ-weak lower semi-continuity is equivalent to the σ-weak closedness of the intersection with the ball

$$\begin{aligned}\varphi^{-1}([0,1])_1 &= \{x \in M^+ : \varphi(x) \leq 1,\ \|x\| \leq 1\}\\ &= \{x \in M^+ : \|\psi(x^{\frac{1}{2}})\| \leq 1,\ \|x^{\frac{1}{2}}\| \leq 1\}.\end{aligned}$$

Since that the σ-weak and σ-strong closedness of a convex set are equivalent and that the square root operation on M_1^+ is σ-strongly continuous, we are enough to show the set

$$(\varphi^{-1}([0,1])_1)^{\frac{1}{2}} = \{x \in M^+ : \|\psi(x)\| \leq 1,\ \|x\| \leq 1\}$$

is σ-weakly closed. This set, if we denote the graph of $\psi : \mathfrak{n} \to H$ by Γ_ψ, is the image of the positive part of the unit ball

$$(\Gamma_\psi)_1^+ = \{(x, \psi(x)) \in M^+ \oplus_\infty H : \|\psi(x)\| \leq 1,\ \|x\| \leq 1\}$$

under the projection $M \oplus_\infty H \to M$. Observing $M \oplus_\infty H \cong (M_* \oplus_1 H)^*$, if we prove $(\Gamma_\psi)_1^+$ is weakly* closed, then we are done by its compactness.

Consider a linear functional $l : M \oplus_\infty H \to \mathbb{C}$ that is continuous with respect to $(\sigma s, \|\cdot\|)$. If we define $l_1 : M \to \mathbb{C}$ and $l_2 : H \to \mathbb{C}$ such that $l_1(x) := l(x,0)$ and $l_2(\xi) = (0,\xi)$, then they satisfy $l(x,\xi) = l_1(x) + l_2(\xi)$, and are continuous in σ-strong and norm topologies, hence to σ-weak and weak topologies, respectively. Since a net (x_α, ξ_α) converges to (x, ξ) weakly* if and only if $x_\alpha \to x$ σ-weakly and $\xi_\alpha \to \xi$ weakly, l is weakly* continuous. Because $(\Gamma_\psi)_1^+$ is convex, we will now show that $(\Gamma_\psi)_1^+$ is closed in $(M, \sigma s) \times (H, \|\cdot\|)$.

Note that the unit ball M_1 is metrizable in σ-strong topology since M is countably decomposable. Suppose a sequence $x_n \in \mathfrak{n}_1^+$ satisfies $x_n \to x$ σ-strongly and $\psi(x_n) \to \xi$ in H. Then, it suffices to show the following two statements: $x \in \mathfrak{n}_1^+$ and $\psi(x) = \xi$. We first observe that since $\psi(x_n)$ is Cauchy, so is $\omega_{\psi(x_n)}$ in $(\pi(M)')_*$.

Consider for a while, a family of functions

$$f_a(t) := \frac{t}{1+at}, \qquad t \in (-a^{-1}, \infty),$$

parametrized by $a > 0$. They have several properties. At first, they are operator monotone. Next, they are σ-strongly continuous on a closed subset of its domain due to the boundedness of f_a, as we can see in the proof of the Kaplansky density theorem. Finally, for each $x \in M_+$, the increasing limit $f_a(x) \uparrow x$ in norm as $a \to 0$ implies that $\sup_a f_a(x) = x$.

First we show $x \in \mathfrak{n}_1^+$. It is clear that $x \in M_1^+$, so it is enough to show $\varphi(x^2) < \infty$. By taking a subsequence, we may assume $\|\omega_{\psi(x_{n+1})} - \omega_{\psi(x_n)}\| < \frac{1}{2^n}$. In order to dominate x_n with an increasing sequence, find $a_n \in \mathfrak{m}^+$ such that

$$x_{n+1}^2 - x_n^2 \leq a_n, \qquad \varphi(a_n) < \frac{1}{2^n},$$

using the previous lemma. Then, we can write

$$x_{n+1}^2 \leq x_1^2 + \sum_{k=1}^{n} (x_{k+1}^2 - x_k^2) \leq x_1 + \sum_{k=1}^{n} a_k.$$

Here the right-hand side is increasing but not a bounded sequence so we take f_a to get the σ-strong limit

$$f_a(x^2) \le \sup_n f_a(x_1^2 + \sum_{k=1}^{n} a_k).$$

Then, by the normality of φ, we have

$$\begin{aligned}\varphi(f_a(x^2)) &\le \sup_n \varphi(f_a(x_1^2 + \sum_{k=1}^{n} a_k)) \\ &\le \sup_n \varphi(x_1^2 + \sum_{k=1}^{n} a_k) \\ &= \varphi(x_1^2) + \sum_{k=1}^{\infty} \varphi(a_k) \\ &< \varphi(x_1^2) + 1 < \infty\end{aligned}$$

which implies by sending $a \to 0$ that $\varphi(x^2) < \infty$, whence $x \in \mathfrak{n}$.

Next we show $\psi(x) = \xi$. If we prove $\varphi((x_n - x)^2) \to 0$, then

$$\|\xi - \psi(x)\| \le \|\xi - \psi(x_n)\| + \|\psi(x_n) - \psi(x)\| = \|\xi - \psi(x_n)\| + \varphi((x_n - x)^2)^{\frac{1}{2}} \to 0$$

deduces the desired result. By taking a subsequence, since $\psi(x_n - x)$ is Cauchy, we may assume

$$\|\omega_{\psi(x_n - x)} - \omega_{\psi(x_{n+1} - x)}\| < \frac{1}{2^n}.$$

Let $b_n \in \mathfrak{m}^+$ such that

$$(x_n - x)^2 - (x_{n+1} - x)^2 \le b_n, \qquad \varphi(b_n) < \frac{1}{2^n}$$

As we did previously, we have

$$f_a((x_n - x)^2) \le f_a((x_{m+1} - x)^2) + f_a(\sum_{k=n}^{m} b_k) \to \sup_m f_a(\sum_{k=n}^{m} b_k)$$

as $m \to \infty$ and

$$\varphi(f_a((x_n - x)^2)) \le \sup_m \varphi(f_a(\sum_{k=n}^{m} b_k)) \le \sup_m \varphi(\sum_{k=n}^{m} b_k) < \frac{1}{2^{n-1}}.$$

Therefore,

$$\varphi((x_n - x)^2) \le \frac{1}{2^{n-1}} \to 0.$$

□

Theorem 1.11. *Let M be an arbitrary von Neumann algebra. Then, a normal weight on M is σ-weakly lower semi-continuous.*

Proof. Let φ be a normal weight of M. Let Σ be the set of all countably decomposable projections of M and let $M_0 := \bigcup_{p \in \Sigma} pMp$. The equivalent condition for $x \in M$ to belong to M_0 is that the left and right support projections are countably decomposable. Since then the left support projection p and the right support projection are Murray-von Neumann equivalent so that there is a $*$-isomorphism between pMp and qMq, the countable decomposability is equivalent for p and q. It implies that M_0 is an algebraic ideal of M. Moreover, M_0 is σ-weakly sequentially closed in M since if a sequence $x_n \in M_0$ converges to $x \in M$ σ-weakly, then for $p_n \in \Sigma$ such that $x_n = p_n x_n p_n$, we have $p \in \Sigma$ with $p_n \le p$ so that $x_n = p x_n p$ converges to $x = pxp$ σ-weakly.

We claim that $\varphi^{-1}([0,1])_1$ is relatively σ-weakly closed in M_0. Let $y \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0$ so that there is a net $y_\alpha \in \varphi^{-1}([0,1])_1$ converges σ-weakly to y, and there is $p \in \Sigma$ such that $pyp = y$. Since

$$p y_\alpha p \in \varphi^{-1}([0,1])_1 \cap pMp$$

also converges σ-weakly to $pyp = y$ and by the previous theorem $\varphi^{-1}([0,1]) \cap pMp$ is σ-weakly closed for each $p \in \Sigma$, we have $y \in \varphi^{-1}([0,1])$. The claim proved.

Suppose $x_\alpha \in \varphi^{-1}([0,1])_1$ converges to $x \in M_1^+$ σ-weakly. Let $\{p_i\}_{i\in I}$ be a maximal mutually orthogonal projections in Σ, and let $p_F := \sum_{i\in F} p_i$ for finite sets $F \subset I$ so that $\sup_F p_F = 1$. It clearly follows that

$$x_\alpha^{\frac{1}{2}} p_F x_\alpha^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

Because M_0 is an ideal of M,

$$x^{\frac{1}{2}} p_F x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0.$$

By the above claim,

$$x^{\frac{1}{2}} p_F x^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

By the normality of φ, we finally obtain

$$x \in \varphi^{-1}([0,1])_1.$$

Therefore, $\varphi^{-1}([0,1])_1$ is σ-weakly closed. □

1.3 Supremum of positive linear functionals

2 November 10

3 December 20

4 January 17

5 February 9