

Representation Theory

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Part I

Finite groups

Chapter 1

Character theory

1.1 Irreducible representations

1.1 (Definition of group representations).

1.2 (Intertwining maps).

1.3 (Subrepresentations). We say *invariant* or *stable*

1.4 (Irreducible representations). indecomposable and irreducible

1.5 (Maschke's theorem). Let G be a finite group and k be a field. Suppose the characteristic of k does not divide $|G|$. Let V be a finite-dimensional representation of G over k .

- (a) Every invariant subspace W of V has a complement W' in V that is also invariant.
- (b) V is isomorphic to the direct sum of irreducible representations of G over k .
- (c) If $k = \mathbb{R}$ or \mathbb{C} , then V admits an inner product such that $W \perp W'$ and $\rho_V(g)$ is unitary for all $g \in G$.

1.6 (Schur's lemma). Let G be a group and k be a field. Let V and W be irreducible representations of G over k . Let $\psi : V \rightarrow W$ be an intertwining map.

- (a) If $V \not\cong W$, then $\psi = 0$.
- (b) If $V \cong W$, then ψ is an isomorphism.
- (c) If k is algebraically closed and $\dim V < \infty$, then every intertwining map $\psi : V \rightarrow V$ is a homothety.

1.2 Group algebra

1.7 (Modules and representations). ring \leftrightarrow group module \leftrightarrow representation finitely generated \leftrightarrow finite dimensional

1.8 (Wedderburn's theorem). central idempotents dimension computation

1.9 (Group algebra). regular representation $k[G]$ -module and G -representation correspondence

- (a) $\mathbb{C}[G]$ is the direct sum of all irreducible representations.
- (b) $|G| = \sum_{[V] \in \hat{G}} (\dim V)^2$.

1.10. The number of irreducible representations and the number of conjugacy classes double counting on $Z(\mathbb{C}[G])$.

1.3 Characters

1.11 (Space of class functions). Ring and inner product structure on the space of class functions.

(a) $\dim \text{hom}_G(V, W) = \langle \chi_V, \chi_W \rangle.$

(b) Irreducible characters form an orthonormal basis of the space of class functions.

1.12 (Characters classify representations). Let G be a finite group and let $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of G over \mathbb{C} .

$$\text{Tr} : \mathbf{Rep}(G) \rightarrow \{\text{finite sum of irreducible characters}\}$$

surjectivity: trivial injectivity: Suppose two characters are equal. Maschke \rightarrow all characters are sum of irreducible characters Schur \rightarrow orthogonality, so the coefficients are all equal irreducible-factor-wisely construct an isomorphism.

1.13 (Character table). computation of matrix elements by character table abelian group, 1dim rep lifting

S^3	e	(12)	(123)
1	1	1	1
ε	1	-1	1
ρ	2	0	-1

the dual inner product: conjugacy check relation to normal subgroups center of rep
algebraic integer dim of irrep divides group order burnside pq theorem

Chapter 2

Classification of representations

2.1 Symmetric groups

young tableaux

2.2 Linear groups over finite fields

GL_2 and SL_2 over finite fields

2.3 Induced representations

induction and restriction of reps (from and to subgroup) frobenius reciprocity, mackey theory
tensoring, complex, real symmetric, exterior

Chapter 3

Brauer theory

Part II

Lie algebras

Chapter 4

Semisimple Lie algebras

4.1 Linear Lie algebras

group acts on an algebra A (e.g. $\text{End}(V)$). then its group algebra acts on A . Lie algebra acts on A , and this Lie algebra information is enough to recover the group action. Geometric meaning of Lie algebra action?

Lie algebra can only considered as a quantization of Poisson bracket. How can the Poisson bracket embodies the group action?

Following Humphrey's book, let L be always finite dimensional Lie algebra unless stated.

4.1. Every associative algebra is a Lie algebra, where the Lie bracket is given by the commutator. For a Lie algebra, we are

Intuitions of subalgebras, ideals, derivations. Intuitions of solvable, nilpotent, and semisimple Lie algebras. Constructing representations, trace forms,

The *general linear Lie algebra* $\mathfrak{gl}(V)$ is just $\text{End}(V)$ with a Lie bracket $[x, y] := xy - yx$.

4.2 (Derivations). Let L be a Lie algebra. A *derivation* of L is a linear map $\delta : L \rightarrow L$ such that

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in L$. The set of derivations $\text{Der}(L)$ of L is a subalgebra of $\mathfrak{gl}(L)$, and we have the *adjoint representation* $L \rightarrow \text{Der}(L) \leq \mathfrak{gl}(L)$ of L . If I is an ideal, then we have a faithful representation $\text{ad} : L \rightarrow \text{ad } L \leq \text{Der}(I) \leq \mathfrak{gl}(I)$.

4.3 (Inner derivations and automorphisms). Let L be a Lie algebra.

The linear map $\text{ad } x = [x, -] : L \rightarrow L$ for $x \in L$ is derivation, and derivation of this form is called *inner*, and they form an ideal of $\text{Der}(L)$.

Automorphisms of the form $\exp(\text{ad } x)$ with nilpotent $\text{ad } x$ generates a normal subgroup of $\text{Aut}(L)$, and each generator is called *inner automorphisms*.

4.2 Solvable Lie algebras

4.4 (Solvable and nilpotent Lie algebras). Let L be a Lie algebra. If the *derived series* $L^{(0)} = L$, $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ eventually vanishes, then we call L *solvable*.

If L is solvable, then its subalgebras and quotient algebras are all solvable. If I is a solvable ideal of L such that L/I is solvable, then L is solvable. The sum of two solvable ideals is also solvable.

Let L be a Lie algebra. If the *lower central series* $L^0 = L$, $L^n = [L, L^{n-1}]$ eventually vanishes, then we call L *nilpotent*. It is a stronger notion than solvability.

If L is nilpotent, then its subalgebras and quotient algebras are all nilpotent. If $L/Z(L) \cong \text{ad}(L) \subset \mathfrak{gl}(L)$ is nilpotent, then L is nilpotent. If L is non-zero and nilpotent, then $Z(L)$ is non-trivial.

4.5 (Engel's theorem). .

- (a) A linear Lie algebra $L \subset \mathfrak{gl}(V)$ consists of nilpotent endomorphisms if and only if $L \subset \mathfrak{n}(V)$ for a certain basis of V .
- (b) An abstract Lie algebra L is nilpotent if and only if $\text{ad}(L)$ consists of nilpotent endomorphisms.
- (c) If $L \subset \mathfrak{gl}(V)$ is nilpotent in $\text{End}(V)$, then there is a *common eigenvector* $v \in V$ such that $[L, v] = 0$, i.e. there is a flag V_i such that $xV_i \subset V_{i-1} \dots$?

Proof. Let L be an ad-nilpotent Lie algebra. Then, every element of $\text{ad } L \subset \mathfrak{gl}(L)$ is a nilpotent endomorphism, so there is $x \in L$ such that $[L, x] = 0$, which implies $Z(L) \neq 0$. Since $L/Z(L)$ is also ad-nilpotent, and by induction on dimension, $L/Z(L)$ is nilpotent. Therefore, L is nilpotent. \square

4.6 (Lie's theorem). Let \mathbb{F} have characteristic zero and be algebraically closed.

- (a) A linear Lie algebra $L \subset \mathfrak{gl}(V)$ is solvable if and only if $L \subset \mathfrak{t}(V)$ for a certain basis of V .
- (b) If L is solvable, then there is a flag V_i such that $xV_i \subset V_i$.
- (c) Let L be an abstract Lie algebra. L is solvable if and only if $[L, L]$ is nilpotent.
- (d) Every finite-dimensional irreducible representation of a solvable Lie algebra is one-dimensional.

Proof. Use induction on dimension. Since $L/[L, L]$ is a non-trivial commutative Lie algebra, in which every subspace is an ideal, we can show the existence of an ideal K of L with codimension one by pullback.

By the induction assumption, we have a common eigenvector in V for K so that we have the “eigenvalue” linear functional $\kappa : K \rightarrow \mathbb{F}$ such that the “eigenspace” of κ as

$$V_\kappa := \{v \in V : xv = \kappa(x)v \text{ for } x \in K\}$$

is non-trivial.

Let $L = K + \mathbb{F}z$ with $z \in \mathfrak{gl}(V)$. If V_κ is invariant by L , then V_κ contains an eigenvector of z by the fact that \mathbb{F} is algebraically closed, so we can extend κ to obtain $\lambda : L \rightarrow \mathbb{F}$ such that $(V_\kappa)_\lambda$ is non-trivial.

We now show that V_κ is invariant by L . Let $v \in V_\kappa$ and $x \in L$. Since

$$yxv - \lambda(y)xv = yxv - xyv = [y, x]v = \lambda([y, x])v$$

for $y \in K$, we have to show $\lambda([y, x]) = 0$. Take n to be largest such that $v, \dots, x^{n-1}v$ are linearly independent. Since $[x, y]$ is upper triangular matrix relative to the basis $v, \dots, x^{n-1}v$ and the diagonal entries are $\lambda([x, y])$. Since the trace of $[x, y]$ must be zero, we have $\lambda([x, y]) = 0$ because \mathbb{F} has characteristic zero. \square

There is a linear functional $\lambda : L \rightarrow \mathbb{F}$ such that $\lambda|_{[L, L]} = 0$ and V_λ is non-trivial. V_κ

For a representation $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, then a weight of V is a linear functional $\lambda : \pi(\mathfrak{h}) \rightarrow \mathbb{F}$ such that the weight space V_λ is non-trivial.

4.7 (Jordan-Chevalley decomposition). Let $\text{char } \mathbb{F}$ be arbitrary. We say $x \in \text{End}(V)$ is called *semisimple* if the roots of its minimal polynomial are all distinct. If \mathbb{F} is algebraically closed, $x \in \text{End}(V)$ is semisimple if and only if it is diagonalizable. Let $x \in \text{End } V$. Even if \mathbb{F} is not algebraically closed, we have a generalization of Jordan decomposition as follows:

- (a) There exist unique $x_s, x_n \in \text{End } V$ such that $x = x_s + x_n$ and x_s semisimple, x_n nilpotent.

- (b) x_s and x_n are polynomials in x .
- (c) If x maps B to A , then x_s and x_n also map B to A for subspaces $A \leq B \leq V$.

4.8 (Cartan's criterion). We will show a powerful criterion for solvability.

- (a) Let $A \subset B$ be two subspaces of $\mathfrak{gl}(V)$, V finite dimensional. Let

$$M := \{x \in \mathfrak{gl}(V) : [x, B] \subset A\}.$$

If $x \in M$ satisfies $\text{Tr}(xy) = 0$ for all $y \in M$, then x is nilpotent.

- (b) Let $L \subset \mathfrak{gl}(V)$, V finite dimensional. If $\text{Tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$, then L is solvable.

4.3 Semisimple Lie algebras

4.9 (Levi decomposition). Therefore, L admits a unique maximal solvable ideal, called *radical*. If the radical is trivial, then we say L is *semisimple*. Since the center is a solvable ideal, the center of a semisimple Lie algebra is trivial.

- (a) A canonical example of a solvable Lie algebra is the Lie algebra of upper triangular matrices.
- (b) The radical of $\mathfrak{gl}(n, \mathbb{F})$ is $\mathfrak{sl}(n, \mathbb{F})$. (\mathbb{F} characteristic zero?) Upper triangular matrices do not form an ideal of $\mathfrak{gl}(n, \mathbb{F})$.
- (c) $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{n}$, $\mathfrak{t} = \mathfrak{d} \otimes \mathfrak{n}$. \mathfrak{t} is a solvable subalgebra of \mathfrak{gl} , but not a solvable ideal.
- (d) $\mathfrak{sl}(n, \mathbb{F})$ is simple if $\text{char } \mathbb{F} = 0$.

4.10 (Killing form). Let L be a Lie algebra.

$$\kappa(x, y) := \text{Tr}(\text{ad } x \text{ ad } y)$$

is a symmetric bilinear form on L , which is called the *Killing form* on L , i.e. it is the trace form for the adjoint representation.

- (a) On an ideal $I \subset L$, the Killing form is given by restriction.
- (b) The kernel of κ is contained in the radical of L , and triviality is equivalent; L is semisimple if and only if L is non-degenerate. (Here we use Cartan's criterion)
- (c) If L is semisimple, then it is the direct sum of simple ideals.
- (d) If L is semisimple, then every derivation is inner.
- (e) If L is semisimple, then $L = [L, L]$ and every subalgebras and quotients are semisimple.

4.11 (Weyl's theorem on complete reducibility). Finite dimensional representation of a semisimple Lie algebra is completely reducible. Preservation of Jordan decomposition.

4.12 (Toral subalgebras). Cartan subalgebra uniqueness (conjugacy theorem)

Chapter 5

Root systems

root space decomposition Killing form on Cartan subalgebra integrality and rationality Weyl group
Classification: Coxeter graph Dynkin diagram Real forms

Chapter 6

Representations of Lie algebras

6.1 Representations of $\mathfrak{sl}(2, \mathbb{C})$

6.1 (Pauli matrices). Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (a) $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis of complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, and $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of real Lie algebra $\mathfrak{so}(3)$.
- (b) For a unit vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ has eigenvalues ± 1 .

6.2 Highest weight theorem

Isomorphism and conjugacy theorem?

Existence: Universal enveloping algebra and the PBW theorem Verma module definition and quotient finiteness proof

6.3 Character theory

6.4 Multiplicity formulas

Exercises

6.2 (Triplets and quadruplets). Let (π_2, V_2) be the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of degree two. Consider $V_2 \otimes V_2$. Cartan element S_z . $V_2^{\otimes 3}$.

6.3 (Casimir element). Casimir element decomposes a representation into irreducible representations. For a faithful representation $\varphi : L \rightarrow \mathfrak{gl}(V)$, we can associate a non-degenerate trace form since L is semisimple. Then, the *Casimir element* of the representation φ is $C_\varphi := \sum_i \varphi(x_i)\varphi(y_i) \in \text{End}(V)$ where i runs over dual bases relative to the trace form.

Part III

Algebraic groups

Part IV

Hopf algebras

Chapter 7

Chapter 8

Quantum groups