Topological Algebraic Structures

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Topological vector spaces

2.1 Locally convex spaces

categorical aspects, bornology, tensor products,

Generalized Pettis integral

2.1 (Properties of dual pairs)**.** Let (E, E^*) be a dual pair. We say (E, E^*) has the *Krein property* if the closed balanced convex hull of a compact subset of X is compact in the topology $\sigma(E, E^*)$, and say (E, E^*) has the *Goldstine property* if E is $\beta(E, E^*_\beta)$-closed in the strong bidual $(E^*_\beta)^*_\beta$.

Let E a Banach space. The weak dual pair (E, E^*) satisfies the Krein property by the Krein-Šmulian theorem, and the Goldstine property by the closedness of E in E^{**}. If there is a predual E_* of E, then the weak* dual pair (E, E_*) satisfies the Krein property by the fact that the closed convex hull of a bounded set is bounded, and the Golstine property because the norm topology and $\beta(E, (E_*)_\beta)$ coincide by the Goldstine theorem. In particular, a dual pair (E, F) with $F \subset E^*$ has the Goldstine property if and only if the closed unit ball $F_1 = F \cap E^*_1$ is weakly* dense in the closed ball E^*_1.

2.2 (Well-definedness of Pettis integral)**.** Let (Ω, μ) be a localizable measure space and (X, F) is a dual pair. Let $x : \Omega \to X$ be a $\sigma(X, F)$-bounded $\sigma(X, F)$-measurable function in the sense that it determines a linear operator $F \to L^\infty(\mu)$. By the transpose and restriction, we have a linear operator $\phi_x : L^1(\mu) \to F^\#$, which satisfies

$$\langle \phi_x(f), x^* \rangle := \int_\Omega \langle x(s), x^* \rangle f(s)\, d\mu(s), \qquad f \in L^1(\mu),\ x^* \in F.$$

We usually write as

$$\phi_x(f) = \int_\Omega x(s) f(s)\, d\mu(s).$$

(a) $\phi_x(L^1(\mu)) \subset (F_\beta)^*$ and ϕ_x is always weak-$\sigma((F_\beta)^*, F)$-continuous.

(b) Suppose (X, F) has the Krein property. If x is $\sigma(X, F)$-compactly valued, then $\phi_x(L^1(\mu)) \subset X$.

(c) Suppose (X, F) has the Krein and Goldstine property. Suppose Ω is a locally compact Hausdorff space with a Radon measure μ. If x is $\sigma(X, F)$-continuous, then $\phi_x(L^1(\mu)) \subset X$. (In fact, the continuity of x defines $F \to C_b(\Omega)$, we can prove $\phi_x(M(\beta\Omega)) \subset X$. It does not require the data of μ.)

(d) Suppose we have $\phi_x(L^1(\mu)) \subset X$. Let Y be another topological vector space and G is a weakly* dense subspace of Y^*. If $T : X \to Y$ is a $\sigma(X, F)$-$\sigma(Y, G)$-continuous linear operator, then $T\phi_x =$

$\phi_{T\circ x}$. In other words,

$$T\int_\Omega f(s)x(s)\,d\mu(s) = \int_\Omega f(s)Tx(s)\,d\mu(s), \qquad f \in L^1(\mu).$$

(e) Suppose we have $\phi_x(L^1(\mu)) \subset X$, (X,F) has the Goldstine property, and X is a Banach space. Then,

$$\|\int f(s)x(s)\,d\mu(s)\| \le \int \|f(s)x(s)\|\,d\mu(s), \qquad f \in L^1(\mu).$$

Proof. (a) Let $B^* \subset F$ be a $\beta(F,X_\sigma)$-bounded set. For $x^* \in F$ we have an inequality

$$|\langle \phi_x(f), x^*\rangle| \le \int_\Omega |f(s)\langle x(s), x^*\rangle|\,d\mu(s) \le \|f\|_{L^1} \sup_{y\in x(\Omega)} |\langle y, x^*\rangle|,$$

and a bound

$$\sup_{x_*\in B^*}\sup_{y\in x(\Omega)} |\langle y, x^*\rangle| < \infty$$

due to the $\sigma(X,F)$-boundedness of $x(\Omega)$, so $\phi_x(f) \in (F_\beta)^*$. If $f_\alpha \in L^1(\mu)$ converges weakly to zero, then

$$\langle \phi_x(f_\alpha), x^*\rangle = \int_\Omega f(s)\langle x(s), x^*\rangle\,d\mu(s) \to 0, \qquad x^* \in F$$

because x is $\sigma(X,F)$-integrable so that $(s \mapsto \langle x(s), x^*\rangle) \in L^\infty(\mu)$, so the continuity of ϕ_x.

(b) Fix $p \in L^\infty(\mu)$ and let C be the $\sigma(X,F)$-closed balanced convex hull of $x(\Omega) \subset X$. Then C is $\sigma(X,F)$-compact by the Krein property. Since for every $x^* \in F$ we have

$$|\langle \phi_x(f), x^*\rangle| \le \int_\Omega |f(s)\langle x(s), x^*\rangle|\,d\mu(s) \le \|f\|_{L^1} \sup_{y\in x(\Omega)} |\langle y, x^*\rangle| \le \|f\|_{L^1} \sup_{y\in C}|\langle y, x^*\rangle|,$$

the linear functional $\phi_x(f)$ on F is continuous with respect to the Mackey topology $\tau(F,X)$, which is a dual topology so that $\phi_x(f)$ can be naturally identified with a vector in $(F_\tau)^* = X$.

(c) Fix $f \in L^1(\mu)$. By the tightness of μ, there is a sequence of compact sets $K_n \subset \Omega$ such that $\int_{\Omega\setminus K_n} |f(s)|\,d\mu(s) < n^{-1}$. Since for each $x^* \in F$ we have

$$|\langle \phi_x(f) - \phi_{x|_{K_n}}(f), x^*\rangle| \le \int_{\Omega\setminus K_n} |f(s)|\,d\mu(s)\cdot \sup_{s\in\Omega}|\langle x(s), x^*\rangle| < n^{-1} \sup_{y\in x(\Omega)} |\langle y, x^*\rangle|$$

so that

$$\sup_{x^*\in B^*} |\langle \phi_x(f) - \phi_{x|_{K_n}}(f), x^*\rangle| \le n^{-1} \sup_{x_*\in B^*}\sup_{y\in x(\Omega)} |\langle y, x^*\rangle| \to 0, \qquad n \to \infty,$$

which means that $\phi_{x|_{K_n}}(f)$ converges to $\phi_x(f)$ in $\beta((F_\beta)^*, F_\beta)$. Since $\phi_{x|_{K_n}}(f) \in X$ by the part (b) and X is closed in $\beta((F_\beta)^*, F_\beta)$ by the Goldstine property, we have $\phi_x(f) \in X$.

(d) By the continuity of T, the adjoint $T^* : G \to F$ is well-defined. The measurability of T and the existence of the adjoint T^* imply that the composition $T\circ x : \Omega \to Y$ is $\sigma(Y,G)$-bounded and $\sigma(Y,G)$-measurable, so the operator $\phi_{T\circ x} : L^1(\mu) \to G^\#$ is well-defined. Then,

$$\begin{aligned}\langle T\phi_x(f), y^*\rangle = \langle \phi_x(f), T^*y^*\rangle &= \int_\Omega f(s)\langle x(x), T^*y^*\rangle\,d\mu(s)\\ &= \int_\Omega f(s)\langle Tx(s), y^*\rangle\,d\mu(s) = \langle \phi_{T\circ x}(f), y^*\rangle, \qquad f \in L^1(\mu),\ y^* \in G.\end{aligned}$$

In particular, $\phi_{T\circ x} : L^1(\mu) \to Y$.

(e) By the Goldstine property,

$$\|\int f(s)x(s)\,d\mu(s)\| = \sup_{x^*\in F_1} |\int f(s)x(s)\,d\mu(s)| \le \sup_{x^*\in F_1} \int |f(s)x(s)|\,d\mu(s)$$
$$\le \int \sup_{x^*\in F_1} |f(s)x(s)|\,d\mu(s) \le \int \|f(s)x(s)\|\,d\mu(s). \qquad \square$$

2.3 (Topological tensor products). Let X and Y be locally convex spaces. The *projective tensor product* is the completion $X\widehat{\otimes}_\pi Y$ of $X\otimes Y$ with the finest locally convex topology such that the canonical bilinear map $X\times Y\to X\otimes Y$ is continuous. We can also describe it with semi-norms. We have

$$B_{\mathrm{jnt}}(X,Y)\cong (X\widehat{\otimes}_\pi Y)^*.$$

Note that we have

$$X\otimes Y\cong B_{\mathrm{jnt}}(X_\sigma^*,Y_\sigma^*)\subset B_{\mathrm{sep}}(X_\sigma^*,Y_\sigma^*).$$

The space $B_{\mathrm{sep}}(X_\sigma^*,Y_\sigma^*)$ of separately continuous bilinear forms, which has a natural topology of uniform convergence on the products of equicontinuous sets in X_σ^* and Y_σ^*, and this topology is complete if and only if X and Y are complete. The induced topology on $X\otimes Y$ is called the *injective tensor product* topology. We have $C^k(\Omega,E)\cong C^k(\Omega)\widehat{\otimes}_\varepsilon E$ if E is complete.

Note that the projective tensor product reflects the original topologies of locally convex spaces, while the injective tensor product only depends on the dual pair structure.

The dual of $X\widehat{\otimes}_\pi Y\to X\widehat{\otimes}_\varepsilon Y$ defines an injection $J(X,Y)\to B_{\mathrm{jnt}}(X,Y)$. A bilinear form in $J(X,Y)$ is called to be *integral*.

2.4 (Vector-valued continuous functions). Let X be a locally compact Hausdorff space, and (E,E^*) be a dual pair satisfying the two properties.

We claim there is an embedding [Tre 44.1]

$$C_0(X,E)\to C_0(X,E_\sigma)\subset L(E_\tau^*,C_0(X))=L(E_\sigma^*,C_0(X)_\sigma)=L(M(X)_\sigma,E_\sigma).$$

How about C_c, C_0, C_b, C? See [Tre 42.2] for $L(E_\tau^*,F)=L(E_\sigma^*,F_\sigma)$. Since $C_0(X)\odot E$ is dense in $C_0(X,E)$ for any locally convex space E, the above embedding gives rise to a dense embedding $C_0(X,E_\sigma)\subset C_0(X)\widehat{\otimes}_\varepsilon E\subset B_{\mathrm{sep}}(M(X)_\sigma,E_\sigma^*)$.

2.5 (Vector-valued measurable functions). We need to investigate the natural topology and its weak topology on $L^0_{\mathrm{loc}}(\mu)$. I want to do this in measure theory.

Continuous approximations

2.6 (Vector-valued differentiable functions). Hölder, Sobolev, etc.

2.7 (Vector-valued distributions).

2.8 (Relations to Bochner and Pettis integrals). Bochner integral can be justified in terms of projective tensor products.

A weakly measurable function on (Ω,μ) valued in E gives rise to a linear map $E^*\to L^0_{\mathrm{loc}}(\mu)$. Is it continuous?

2.2 Direct limit

distribution theory LF,LB spaces

2.3 Differentiable spaces

Chapter 3

Topological algebras

Part II

Chapter 4

Continuous fields

Part III

Fréchet and Banach spaces

Chapter 5

5.1 Universal properties

Notation

$L(X,Y)$	the set of bounded linear operators from X to Y
$B(X,Y)$	the set of bounded bilinear forms on $X \times Y$
$F(X,Y)$	the set of continuous finite-rank linear operators from X to Y
B_X	closed unit ball of a normed space X
S_X	unit sphere of a normed space X
$X \otimes Y$	algebraic tensor product of X and Y
X^*	continuous dual space
$X^\#$	algebraic dual space

5.1 (Algebraic tensor product of vector spaces)**.** Let X and Y be vector spaces. The *algebraic tensor product* is a vector space $X \otimes Y$ with a bilinear map $\otimes : X \times Y \to X \otimes Y$ such that the following universal property: for any vector space Z and any bilinear map $\sigma : X \times Y \to Z$, there exists a unique linear map $\widetilde{\sigma} : X \otimes Y \to Z$ such that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "\sigma"'] & X \otimes Y \arrow[d, dashed, "\widetilde{\sigma}"] \\ & Z \end{tikzcd}$$

is commutative.

(a) The tensor product $X \otimes Y$ always exists.

(b) We have linear maps $L(X,Z) \otimes L(Y,W) \to L(X \otimes Y, Z \otimes W)$ and $B(L(X,Z), L(Y,Z)) \to L(X \otimes Y, Z)$.

(c) Every element $t \in X \otimes Y$ is represented as $t = \sum_{i=1}^n x_i \otimes y_i$ such that $\{x_i\}$ is linearly indpendent. In this case, if $t = 0$ then $y_i = 0$ for all i.

Proof. (a) Let T be the set of formal linear combinations of $X \times Y$, that is, an element of T has the form $\sum_{i=1}^n a_i \cdot (x_i, y_i)$ for $x_i \in X$, $y_i \in Y$, and scalars a_i. Define $T_0 \subset T$ to be a linear space spanned by the elements of the following four types:

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(ax, y) - a(x, y), \quad (x, ay) - a(x, y).$$

Then, the quotient space T/T_0 satisfies the universal property with the bilinear map $X \times Y \to T/T_0 : (x, y) \mapsto (x, y) + T_0$. □

5.2 (Algebraic tensor product of involutive algebras)**.**

5.2 Banach spaces

5.3 (Subcross norms).

5.4 (Injective tensor products). Let X and Y be Banach spaces. Define the *injective norm* ε on $X \otimes Y$ such that

$$\varepsilon\left(\sum_{i=1}^{n} x_i \otimes y_i\right) := \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{Y^*}}} \left|\sum_{i=1}^{n} \langle x_i, x^* \rangle \langle y_i, y^* \rangle\right|.$$

We denote by $X \otimes_\varepsilon Y$ the algebraic tensor product with the injective norm, and by $X \widehat{\otimes}_\varepsilon Y$ its completion.

(a) $X \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F((X^*, w^*), (Y, w))$.

(b) $X^* \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F(X, Y)$.

5.5 (Projective tensor products). Let X and Y be Banach spaces. Define the *projective norm* π on $X \otimes Y$ such that

$$\pi(t) := \inf\left\{\sum_{i=1}^{n} \|x_i\| \|y_i\| : t = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

We denote by $X \otimes_\pi Y$ the algebraic tensor product with the projective norm, and by $X \widehat{\otimes}_\pi Y$ its completion.

(a) There are natural isometric isomorphisms $(X \otimes_\pi Y)^* \cong B(X, Y) \cong L(X, Y^*)$.

(b)

5.6 (Hilbert space tensor product). Let $\varphi : H \otimes K \to L(H^*, K)$. Then, $\lambda(\xi) = \|\varphi(\xi)\|$, $\gamma(\xi) = \mathrm{tr}(|\varphi(\xi)|)$, so $H \widehat{\otimes}_\lambda K \cong K(H^*, K)$ and $H \widehat{\otimes}_\gamma K \cong L^1(H^*, K)$.

5.7 (Nuclear operators).

$$X^* \otimes_\pi Y \to X^* \otimes_\varepsilon Y \xrightarrow{\sim} F(X, Y) \xrightarrow{1} K(X, Y)$$

defines

$$J : X^* \widehat{\otimes}_\pi Y \to K(X, Y).$$

Define $N(X, Y) := \mathrm{im}\, J$.

5.8 (Grothendieck theorem). Let Y^* be an RNP space. Then, there is an isometric isomorphism $(X \widehat{\otimes}_\varepsilon Y)^* \cong N(X, Y^*)$.

5.3 Approximation property

5.9 (Approximation property of locally convex spaces).

5.10 (Approximation property of Banach spaces).

5.11 (Approximation property of dual Banach spaces).

5.12 (Mazur's goose). (a) If X has a Schauder basis, then it has the approximation property.

5.4 Nuclear spaces

Part IV

Fréchet and Banach algebras

Chapter 6

Fréchet algebras

Chapter 7

Banach algebras