

Operator Algebra Seminar Note I

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1 April 14

1.1 Completely positive maps

Definition 1.1. Let A and B be C^* -algebras. A linear map $\varphi : A \rightarrow B$ is said to be *completely positive* (c.p.) if the inflation $\varphi_n : M_n(A) \rightarrow M_n(B) : [a_{ij}] \mapsto [\varphi(a_{ij})]$ is positive for each $n \geq 1$.

Remark 1.2. For the positivity in matrix algebras, the following equivalent statements are useful.

- (a) $[a_{ij}] \in M_n(A)$ is positive.
- (b) $[a_{ij}] = [b_{ij}]^* [b_{ij}] = [b_{ji}^*] [b_{ij}] = [\sum_k b_{ki}^* b_{kj}]$ for some $[b_{ij}] \in M_n(A)$.
- (c) $\sum_{i,j} \langle \pi(a_{ij}) \xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for a faithful representation $\pi : A \rightarrow B(H)$.
- (d) $\sum_{i,j} \langle \pi(a_{ij}) \xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for every representation $\pi : A \rightarrow B(H)$.

Every positive element of $M_n(A)$ is the sum of elements of the form $[a_i^* a_j]$, so for $\varphi : A \rightarrow B$ to be completely positive, it is necessary and suffices to have

$$\sum_{i,j} \langle \varphi(a_i^* a_j) \xi_j, \xi_i \rangle \geq 0, \quad [a_i] \in A^n, [\xi_i] \in H^n,$$

where B is embedded in $B(H)$.

Example 1.3.

- (a) A $*$ -homomorphism is c.p.
- (b) A state is c.p.
- (c) A conjugation $B(\hat{H}) \rightarrow B(H) : a \mapsto V^* a V$ is c.p. for every bounded linear $V : H \rightarrow \hat{H}$.
- (d) The transpose $M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ is not c.p.
- (e) The convex combination, composition, restriction of c.p. maps is c.p.

Proof. (a) A $*$ -homomorphism is positive, and its inflations are all $*$ -homomorphisms.

(b) Let $\rho : A \rightarrow \mathbb{C}$ be a state. Then, we have for $[c_i] \in \ell_2^n$ that

$$\sum_{i,j} \langle \rho(a_i^* a_j) c_j, c_i \rangle_{\mathbb{C}} = \rho(\sum_{i,j} \bar{c}_i a_i^* a_j c_j) = \rho((\sum_i c_i a_i)^* (\sum_j c_j a_j)) \geq 0.$$

(c) We have for $[\xi_i] \in H^n$ that

$$\sum_{i,j} \langle V^* a_i^* a_j V \xi_j, \xi_i \rangle = \langle \sum_j a_j V \xi_j, \sum_i a_i V \xi_i \rangle \geq 0.$$

(d) We have a counterexample for $M_2(M_2(\mathbb{C})) \rightarrow M_2(M_2(\mathbb{C}))$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The former has an eigenvalues $\{2, 0\}$, and the latter has $\{\pm 1\}$.

(e) Clear. □

Theorem 1.4 (Stinespring dilation). *Let A be a unital C^* -algebra and $\varphi : A \rightarrow B(H)$ be a c.p. map. Then, there is a representation $\pi : A \rightarrow B(\hat{H})$ and a bounded linear operator $V : H \rightarrow \hat{H}$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B(H) \\ \pi \downarrow & \nearrow V^* \cdot V & \\ B(\hat{H}) & & \end{array}$$

Proof. Define a sesquilinear form on the algebraic tensor product $A \otimes H$ as

$$\langle \sum_j a_j \otimes \xi_j, \sum_i b_i \otimes \eta_i \rangle := \sum_{i,j} \langle \varphi(b_i^* a_j) \xi_j, \eta_i \rangle.$$

It is positive since

$$\sum_{i,j} \langle a_i^* a_j \xi_j, \xi_i \rangle = \sum_{i,j} \langle a_j \xi_j, a_i \xi_i \rangle = \left\| \sum_i a_i \xi_i \right\|^2 \geq 0$$

implies

$$\langle \sum_j a_j \otimes \xi_j, \sum_i a_i \otimes \xi_i \rangle = \sum_{i,j} \langle \varphi(a_i^* a_j) \xi_j, \xi_i \rangle \geq 0.$$

Taking quotient by the left kernel N and completion, we obtain a hilbert space $\hat{H} := (A \otimes H / N)^-$.

Define $\pi : A \rightarrow B(\hat{H})$ such that

$$\pi(a)(b \otimes \xi + N) := ab \otimes \xi + N,$$

and define $V : H \rightarrow \hat{H}$ such that

$$V\xi := 1_A \otimes \xi + N.$$

Then for any $\xi, \eta \in H$,

$$\langle V^* \pi(a) V \xi, \eta \rangle = \langle \pi(a)(1_A \otimes \xi + N), 1_A \otimes \eta + N \rangle = \langle a_A \otimes \xi + N, 1_A \otimes \eta + N \rangle = \langle \varphi(a) \xi, \eta \rangle. \quad \square$$

Remark 1.5.

- (a) If φ is unital, then V is an isometry since $V^* V = V^* \pi(1) V = \varphi(1) = 1$.
- (b) If φ is unital and $H = \mathbb{C}$, then it is just the GNS-construction with the cyclic vector $V1_{\mathbb{C}}$.
- (c) If $\varphi : A \rightarrow B$ is c.p., then by embedding B into $B(H)$ and applying the Stinespring dilation,

$$\|\varphi(a)\| = \|V^* \pi(a) V\| \leq \|V\| \|a\| \|V\| = \|a\| \|V^* V\| = \|a\| \|\varphi(1)\|$$

implies $\|\varphi\| \leq \|\varphi(1)\|$, hence $\|\varphi\| = \|\varphi(1)\|$.

- (d) It has a physical meaning: a unital completely positive map is called quantum channel or quantum operation in quantum information theory. They are interpreted as an evolution in open quantum system, and taking \hat{H} means introducing a closed ambient system in which unitary evolution occurs.

Theorem 1.6 (Completely positive maps for matrix algebras). *Let A be a C^* -algebra. Let $e_i \in \ell_2^n$ be standard orthonormal basis and let $e_{ij} = e_i \otimes e_j = |e_i\rangle\langle e_j| \in M_n(\mathbb{C})$ be unit matrix elements.*

- (a) *There is a 1-1 correspondence*

$$\text{CP}(M_n(\mathbb{C}), A) \rightarrow M_n(A)_+ : \psi \mapsto [\psi(e_{ij})].$$

- (b) *Let A be unital. There is a 1-1 correspondence*

$$\text{CP}(A, M_n(\mathbb{C})) \rightarrow M_n(A)_+^* : \varphi \mapsto (\hat{\varphi} : [a_{ij}] \mapsto \sum_{i,j} \langle \varphi(a_{ij}) e_j, e_i \rangle).$$

Proof. (a) Fix $A \rightarrow B(H)$ a faithful representation and just write $A \subset B(H)$.

Suppose $\psi : M_n(\mathbb{C}) \rightarrow A$ is a c.p. map. Identify $M_n(\mathbb{C}) = B(\ell_2^n)$. Since $[e_{ij}] \in M_n(B(\ell_2^n))_+$ is positive because

$$\sum_{i,j} \langle e_{ij} \xi_j, \xi_i \rangle = \sum_{i,j} \langle e_j, \xi_j \rangle \langle \xi_i, e_i \rangle = \left| \sum_i \langle e_i, \xi_i \rangle \right|^2 \geq 0, \quad \forall [\xi_i] \in (\ell_2^n)^n,$$

it follows that $[\psi(e_{ij})] \in M_n(A)_+$ by the complete positivity of ψ .

Conversely, let $[\psi(e_{ij})] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(H))_+$. For $T = [t_{ij}] \in M_n(\mathbb{C})$ and $\xi, \eta \in H$, write

$$\begin{aligned} \langle \psi(T)\xi, \eta \rangle &= t_{ij} \langle \psi(e_{ij})\xi, \eta \rangle \\ &= t_{ij} \langle b_{kj}\xi, b_{ki}\eta \rangle \\ &= t_{ij} \delta_{kl} \langle b_{lj}\xi, b_{ki}\eta \rangle \\ &= \langle Te_j, e_i \rangle \langle e_l, e_k \rangle \langle b_{lj}\xi, b_{ki}\eta \rangle \\ &= \langle (T \otimes 1 \otimes 1)(e_j \otimes e_l \otimes (b_{lj}\xi)), (e_i \otimes e_k \otimes (b_{ki}\eta)) \rangle. \end{aligned}$$

The summation symbols are omitted in each row. Then, if we define

$$V : H \rightarrow \ell_2^n \otimes \ell_2^n \otimes H : \xi \mapsto \sum_{i,k} e_i \otimes e_k \otimes (b_{ki}\xi),$$

we have an expression

$$\langle \psi(T)\xi, \eta \rangle = \langle V^*(T \otimes 1 \otimes 1)V\xi, \eta \rangle,$$

which implies that ψ is c.p. because $T \mapsto T \otimes 1_{\ell_2^n} \otimes 1_H$ is a $*$ -homomorphism.

(b) Suppose $\varphi : A \rightarrow M_n(\mathbb{C})$ is a c.p. map. Then, $\hat{\varphi}$ is positive since $[a_{ij}] \in M_n(A)_+$ implies

$$\hat{\varphi}([a_{ij}]) = \sum_{i,j} \langle \varphi(a_{ij})e_j, e_i \rangle \geq 0.$$

Conversely, let $\hat{\varphi} \in M_n(A)_+^*$. By the GNS-construction, we have a cyclic representation $\pi : M_n(A) \rightarrow B(H)$ with a cyclic vector $\psi \in H$ such that

$$\hat{\varphi}([a_{ij}]) = \langle \pi([a_{ij}])\psi, \psi \rangle.$$

For $\xi = \sum_j \xi_j e_j, \eta = \sum_i \eta_i e_i \in \ell_2^n$, write

$$\begin{aligned} \langle \varphi(a)\xi, \eta \rangle &= \sum_{i,j} \langle \varphi(a)\xi_j e_j, \eta_i e_i \rangle = \sum_{i,j} \langle \varphi(\overline{\eta_i} a \xi_j) e_j, e_i \rangle \\ &= \hat{\varphi}([\overline{\eta_i} a \xi_j]) = \langle \pi([\overline{\eta_i} a \xi_j])\psi, \psi \rangle = \langle \pi([\delta_{ij} \eta_i 1_A]^* [a] [\delta_{ij} \xi_j 1_A])\psi, \psi \rangle \\ &= \langle \pi([a])\pi([\delta_{ij} \xi_j 1_A])\psi, \pi([\delta_{ij} \eta_i 1_A])\psi \rangle. \end{aligned}$$

If we define

$$V : \ell_2^n \rightarrow H : \xi \mapsto \pi([\delta_{ij} \xi_j 1_A])\psi,$$

then

$$\langle \varphi(a)\xi, \eta \rangle = \langle V^* \pi([a])V\xi, \eta \rangle,$$

so φ is c.p. since $A \rightarrow M_n(A) : a \mapsto [a]$ is a $*$ -homomorphism. \square

Theorem 1.7 (Arveson extension). *Let $A \subset B$ be C^* -algebras such that $1_B \in A$. Then, every c.p. map $\varphi : A \rightarrow B(H)$ has a norm-preserving c.p. extension $\tilde{\varphi} : B \rightarrow B(H)$, i.e. $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Let p_α be the net of projections of finite rank n_α in $B(H)$ with the image V_α , which strongly converges to id_H . Fix α temporarily and let $\varphi_\alpha := p_\alpha \varphi|_{V_\alpha} : A \rightarrow B(V_\alpha)$. Choosing an any orthonormal basis of each V_α , we can rewrite as $\varphi_\alpha : A \rightarrow M_{n_\alpha}(\mathbb{C})$. By the above theorem, we have the associated linear functional $\hat{\varphi}_\alpha \in M_{n_\alpha}(A)$. Then, the Hahn-Banach extension provides an extension $(\hat{\varphi}_\alpha)^\sim \in M_{n_\alpha}(B)$, and we can define $\tilde{\varphi}_\alpha : B \rightarrow M_{n_\alpha}(\mathbb{C})$ as the associated completely positive map. Via the identification $B(V_\alpha) = M_{n_\alpha}(\mathbb{C})$ we used to write $\varphi_\alpha : A \rightarrow M_{n_\alpha}(\mathbb{C})$, we have $\tilde{\varphi}_\alpha : B \rightarrow B(V_\alpha)$. We can check $\tilde{\varphi}_\alpha$ actually extends φ_α , i.e. $\tilde{\varphi}_\alpha(a) = \varphi_\alpha(a)$ for $a \in A$, by putting $[a\delta_{ik}\delta_{jl}]_{i,j} \in M_{n_\alpha}(A)$ and comparing matrix components for each k, l .

Since $\|\tilde{\varphi}_\alpha\| = \|\tilde{\varphi}_\alpha(1)\| = \|\varphi_\alpha(1)\| = \|\varphi_\alpha\| \leq \|\varphi\|$, the net $\tilde{\varphi}_\alpha$ is bounded in $B(B, B(H))$. The norm-closed unit ball is compact in the point- σ -weak topology $\sigma(B(B, B(H)), B \odot L^1(H))$ because it is coarser than the weak* topology $\sigma(B(B, B(H)), B \hat{\otimes}_\pi L^1(H))$. By taking a convergent subnet, we have a limit point $\tilde{\varphi} : B \rightarrow B(H)$. It is easily seen to be completely positive and extend φ , and satisfies $\|\varphi\| = \|\varphi(1)\| = \|\tilde{\varphi}(1)\| = \|\tilde{\varphi}\|$. \square

1.2 Enveloping von Neumann algebras

Definition 1.8. For a representation $\pi : A \rightarrow B(H)$ of a C^* -algebra A , we define a von Neumann algebra $\mathcal{M}(\pi) := \pi(A)''$ associated to π .

Theorem 1.9 (Sherman-Takeda). *Let A be a C^* -algebra and $\pi_u : A \rightarrow B(H_u)$ the universal representation, the direct sum of all the GNS-representations of states of A . Consider the following three maps*

$$\pi_u : A \rightarrow (\mathcal{M}(\pi_u), \sigma w), \quad \pi_u^* : \mathcal{M}(\pi_u)_* \rightarrow A^*, \quad \tilde{\pi}_u := \pi_u^{**} : A^{**} \rightarrow \mathcal{M}(\pi_u),$$

constructed by adjoints, where $\mathcal{M}(\pi_u)_*$ denotes the set of σ -weakly continuous (= normal) linear functionals on $\mathcal{M}(\pi_u)$.

- (a) π_u^* is isometric.
- (b) π_u^* is surjective.
- (c) $\tilde{\pi}_u$ is an isometric isomorphism (w.r.t. norms), and is an homeomorphism (w.r.t. weak*-topologies).
- (d) A^{**} enjoys a universal property in the sense that for every $*$ -homomorphism $\varphi : A \rightarrow \mathcal{M}$ to a von Neumann algebra \mathcal{M} , there exists a unique normal extension $\tilde{\varphi} : A^{**} \rightarrow \mathcal{M}$ of φ .

Proof. (a) It holds for any representation of $\pi : A \rightarrow B(H)$. For each $l \in \mathcal{M}(\pi)_*$, we have

$$\|\pi^*(l)\| = \sup_{\substack{\|a\| \leq 1 \\ a \in A}} |l(\pi(a))| = \sup_{\substack{\|b\| \leq 1 \\ b \in \mathcal{M}(\pi)}} |l(b)| = \|l\|$$

by the Kaplansky density theorem and the σ -weak continuity of l .

(b) Although π_u is an injective $*$ -homomorphism and hence is isometric so that its dual $\mathcal{M}(\pi_u)^* \rightarrow A^*$ is surjective by the Hahn-Banach extension, it does not guarantee the σ -weak continuity of the extended linear functional. We claim that every state of A has a normal extension on $\mathcal{M}(\pi_u)$. If the claim is true, then the Jordan decomposition can be applied to show that every bounded linear functional has a normal extension.

Let ρ be a state of A . If we let ψ be the canonical cyclic vector of the GNS representation $\pi_\rho : A \rightarrow B(H_\rho)$, then the state ρ can be represented as a vector state ω_ψ . Since π_ρ is a subrepresentation of π_u , the unit vector ψ can be seen as an element of H_u , and it defines a normal state of $\mathcal{M}(\pi_u)$.

(c) It is clear from (a) and (b).

(d) We can define $\tilde{\varphi}$ as the bitranspose of $\varphi : A \rightarrow (\mathcal{M}, \sigma w)$, and it is a unique extension because A is σ -weakly dense in A^{**} . \square

Remark 1.10. The bidual A^{**} is frequently viewed as a von Neumann algebra, and we call it the *enveloping von Neumann algebra* of a C^* -algebra A . By the universal property, we have a normal $*$ -homomorphism $\mathcal{M}(\pi_u) \rightarrow \mathcal{M}(\pi)$ that is in fact surjective for every representation π of A , and it fails to be injective even for faithful representations.

Theorem 1.11 (Tomiya). *Let $B \subset A$ be C^* -algebras. Let $\varphi : A \rightarrow B$ be a conditional expectation, i.e. a contractive idempotent linear map.*

- (a) φ is B -bimodule map.

(b) φ is completely positive.

Proof. Since each conclusion of (a) and (b) still holds for restriction, we may assume A and B are von Neumann algebras by thinking of the bitranspose $\varphi^{**} : A^{**} \rightarrow B^{**}$.

(a) Since the linear span of projections is σ -weakly dense in a von Neumann algebra, we are enough to show $p\varphi(a) = \varphi(pa)$ and $\varphi(ap) = \varphi(a)p$ for any projection $p \in B$.

Let $p \in B$ be a projection and let $a \in A$. Note that we have

$$p\varphi(a) = pp\varphi(a) = p\varphi(p\varphi(a))$$

and

$$(a - pa)^*(p\varphi(a - pa)) = (p\varphi(a - pa))^*(a - pa) = 0.$$

Then,

$$\begin{aligned} (1+t)^2 \|p\varphi(a - pa)\|^2 &= \|p\varphi(a - pa) + tp\varphi(a - pa)\|^2 \\ &= \|p\varphi((a - pa) + tp\varphi(a - pa))\|^2 \\ &\leq \|(a - pa) + tp\varphi(a - pa)\|^2 \\ &= \|a - pa\|^2 + t^2 \|p\varphi(a - pa)\|^2 \end{aligned}$$

implies $p\varphi(a - pa) = 0$ by letting $t \rightarrow \infty$. Putting $1_B - p$ and 1_B instead of p , we obtain $(1_B - p)\varphi(a - 1_B a + pa) = 0$ and $\varphi(a - 1_B a) = 0$, so

$$p\varphi(a) = p\varphi(pa) = \varphi(pa).$$

Similarly, we can show $\varphi(a - ap)p = 0$ and $\varphi(ap)(1 - p) = 0$, we are done.

(b) Let $[a_{ij}] \in M_n(A)_+$. Let $\pi : B \rightarrow B(H)$ be a cyclic representation with a cyclic vector ψ . Then, $[\xi_i] \in H^n$ can be replaced to $[\pi(b_i)\psi]$, so we can check the positivity of inflations φ_n as

$$\sum_{i,j} \langle \pi(\varphi(a_{ij}))\pi(b_j)\psi, \pi(b_i)\psi \rangle = \langle \pi(\varphi(\sum_{i,j} b_i^* a_{ij} b_j))\psi, \psi \rangle \geq 0,$$

because it follows $\sum_{i,j} b_i^* a_{ij} b_j \geq 0$ by the positivity of a_{ij} from

$$\langle \pi_A(\sum_{i,j} b_i^* a_{ij} b_j)\xi, \xi \rangle = \sum_{i,j} \langle \pi_A(a_{ij})\pi_A(b_j)\xi, \pi_A(b_i)\xi \rangle \geq 0,$$

where π_A is any representation of A . □

Theorem 1.12 (Sakai). Suppose A is a C^* -algebra which admits a predual F .

- (a) There is an injective $*$ -homomorphism $\pi : A \rightarrow A^{**}$ with weakly* closed image.
- (b) π is a topological embedding w.r.t. $\sigma(A, F)$ and $\sigma(A^{**}, A^*)$.
- (c) The predual F is unique in A^* .

In particular, there is a faithful representation $A \rightarrow B(H)$ whose image is (σ) -weakly closed.

Proof. (a) By taking the adjoint for the inclusion $i : F \hookrightarrow A^*$, we have a conditional expectation $\varepsilon : A^{**} \rightarrow A$. Its kernel is a A -bimodule, and by the σ -weak density of A in A^{**} and the continuity of ε between weak* topologies, so it is in fact a A^{**} -bimodule, which means it is a σ -weakly closed ideal of A^{**} . Thus we have a central projection $z \in A^{**}$ such that $\ker \varepsilon = (1 - z)A^{**}$.

Define $\pi : A \rightarrow A^{**}$ such that $\pi(a) := za$. It is clearly a $*$ -homomorphism. The injectivity follows from $a = \varepsilon(a) = \varepsilon(za)$ for $a \in A$. The image is weakly* closed because $\varepsilon(x - \varepsilon(x)) = 0$ implies $z(x - \varepsilon(x)) = 0$ for $x \in A^{**}$ so that $zA^{**} = zA$.

(b) Since $\langle a, f \rangle = \langle \varepsilon(za), f \rangle = \langle za, f \rangle$ for $a \in A$ and $f \in F$, in which the second equality holds by the definition of ε , it is enough to show $\sigma(zA, A^*) = \sigma(zA, F)$.

For $l \in A^*$, we claim there exists f such that $\langle za, l \rangle = \langle za, f \rangle$. Define $\tilde{l} \in A^*$ such that $\langle x, \tilde{l} \rangle := \langle zx, l \rangle$ for $x \in A^{**}$. Then, $\langle zx, l \rangle = \langle z^2x, l \rangle = \langle zx, \tilde{l} \rangle$ for $x \in A^{**}$. Suppose $\tilde{l} \notin F$. Because F is closed in A^* , there is $x \in A^{**}$ such that $\langle x, \tilde{l} \rangle \neq 0$ and $\langle x, f \rangle = 0$ for all $f \in F$ by the Hahn-Banach separation. Then, $0 = \langle x, f \rangle = \langle x, i(f) \rangle = \langle \varepsilon(x), f \rangle$ implies $\varepsilon(x) = 0$ so that $zx = 0$, which leads a contradiction $\langle x, \tilde{l} \rangle = \langle zx, l \rangle = 0$, so we have $\tilde{l} \in F$.

(c) If closed subspaces F_1 and F_2 of A^* are preduals of A , then $\sigma(A, F_1) = \sigma(A, F_2)$ by the part (b). If $l \in F_1$, which is obviously continuous on $\sigma(A, F_1)$, and the continuity in $\sigma(A, F_2)$ implies that l is contained in a linear span of some finitely many elements of F_2 , hence $F_1 \subset F_2$. \square

2 May 12

2.1 Nuclear C^* -algebras

Proposition 2.1 (Maximal and minimal tensor products). *Let A_1, A_2, B, B_1, B_2 be C^* -algebras and H_1, H_2 be Hilbert spaces.*

- (a) (Continuity of tensor product maps) *For any $*$ -homomorphisms $\varphi_1 : A_1 \rightarrow B_1$ and $\varphi_2 : A_2 \rightarrow B_2$, the $*$ -homomorphisms*

$$\varphi_1 \otimes \varphi_2 : A_1 \otimes_{\max} A_2 \rightarrow B_1 \otimes_{\max} B_2$$

and

$$\varphi_1 \otimes \varphi_2 : A_1 \otimes_{\min} A_2 \rightarrow B_1 \otimes_{\min} B_2$$

are well-defined. The same holds for c.p. maps (see Corollary 3.3 for \otimes_{\max}).

- (b) (Universal property) *For any $*$ -homomorphisms $\varphi_1 : A_1 \rightarrow B$ and $\varphi_2 : A_2 \rightarrow B$ whose images are commuting, there exists a unique $*$ -homomorphism*

$$\varphi_1 \times \varphi_2 : A_1 \otimes_{\max} A_2 \rightarrow B$$

such that $\varphi_1 \times \varphi_2(a_1 \otimes a_2) = \varphi_1(a_1)\varphi_2(a_2)$.

- (c) (Minimal norm is spatial) *There is a natural $*$ -monomorphism $B(H_1) \otimes_{\min} B(H_2) \hookrightarrow B(H_1 \otimes H_2)$.*

Proof. Omitted. □

Definition 2.2. A C^* -algebra A is called *nuclear* if the canonical surjection $A \otimes_{\max} B \rightarrow A \otimes_{\min} B$ is injective for any C^* -algebra B .

Example 2.3.

- (a) Every finite-dimensional C^* -algebra is nuclear.
- (b) Every abelian C^* -algebra is nuclear.
- (c) A non-unital C^* -algebra is nuclear if and only if its unitization is nuclear.
- (d) A quotient of a nuclear C^* -algebra is nuclear.
- (e) The inductive limit of nuclear C^* -algebras is nuclear.
- (f) The tensor product of nuclear C^* -algebra is nuclear.

Proof. (a), (b), (e) See Theorem 6.3.9, 6.4.15, and 6.3.10 of [Murphy]. □

2.2 Completely positive approximation property

Definition 2.4. Let $\theta : A \rightarrow B$ be a c.c.p. map between C^* -algebras. We say θ is *factorable* if it factors through a matrix algebra $M_n(\mathbb{C})$. We say θ is *approximable* or *nuclear* if it is a limit of factorable maps in the point-norm topology. When B is a von Neumann algebra, we say θ is *weakly approximable* or *weakly nuclear* if it is a limit of factorable maps in the point- σ -weak topology.

Proposition 2.5. *Let A and B be C^* -algebras, and $\mathcal{M} \subset B(H)$ a von Neumann algebra. Let $\mathcal{F} \subset B(A, B)$ or $B(A, \mathcal{M})$ be the set of factorable maps.*

- (a) \mathcal{F} is convex.
- (b) In $B(A, B)$, we have

$$\overline{\mathcal{F}}^{co} = \overline{\mathcal{F}}^{pt-\|\cdot\|} = \overline{\mathcal{F}}^{pt-w}.$$

(c) In $B(A, \mathcal{M})$, we have

$$\overline{\mathbb{R}_{\geq 0} \mathcal{F}}^{pt-\sigma w} \cap B = \overline{\mathcal{F}}^{pt-\sigma w} = \overline{\mathcal{F}}^{pt-wot} = \overline{\mathcal{F}}^{pt-sot},$$

where B denotes the closed unit ball of $B(A, \mathcal{M})$.

Sketch. (a) Let $A \xrightarrow{\psi_i} M_{n_i}(\mathbb{C}) \xrightarrow{\varphi_i} B$ be c.c.p. maps for $i \in \{0, 1\}$. Then, for $t \in [0, 1]$ we have a diagram

$$\begin{array}{ccc} A & \xrightarrow{(1-t)\psi_0 \circ \varphi_0 + t\psi_1 \circ \varphi_1} & B \\ \downarrow & & \uparrow \\ A \oplus A & \xrightarrow[\varphi_0 \oplus \varphi_1]{M_{n_0}(\mathbb{C}) \oplus M_{n_1}(\mathbb{C})} & B \oplus B \\ & \xrightarrow{((1-t)\psi_0) \oplus (t\psi_1)} & \end{array}$$

which is commutative, so we are done.

(b), (c) When comparing the strong (operator) topology and the weak (operator) topology, we can use the fact that continuous functionals with respect to both topologies are same and apply the Hahn-Banach separation because \mathcal{F} is convex.

When comparing the weak topology and the σ -weak topology, we can use the compactness of the closed unit ball in the σ -weak topology and use the fact that weak topology is weaker than the σ -weak topology to prove a homeomorphism.

Finally, for the first equality of (b), see Proposition 3.8.2 in [Brown-Ozawa], which assumes unital maps. \square

Theorem 2.6. *Let A be a C^* -algebra. Then, the identity $A \rightarrow A$ is approximable if and only if the inclusion $A \rightarrow A^{**}$ is weakly approximable.*

Proof. (\Rightarrow) Clear.

(\Leftarrow) Let $E \subset A$ and $F \subset A^*$ be any finite subsets and fix $\varepsilon > 0$. We may assume E and F are bounded by one and F is positive. We want to construct c.c.p. maps $A \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} A$ such that

$$|l(a - \psi \circ \varphi(a))| < \varepsilon, \quad a \in E, l \in F.$$

By the assumption, we have a net of c.c.p. maps $A \xrightarrow{\varphi'_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi'_\alpha} A^{**}$ satisfying

$$|l(a - \psi'_\alpha \circ \varphi'_\alpha(a))| \rightarrow 0, \quad a \in A, l \in A^*.$$

If we choose $0 \leq e \leq 1$ in A such that $l(1 - e) < \frac{\varepsilon}{8}$ for all $l \in F$, then

$$\begin{aligned} l(1 - \psi'_\alpha(\text{id})) &= l(1 - e) + l(e - \psi'_\alpha \circ \varphi'_\alpha(e)) + l(\psi'_\alpha(\varphi'_\alpha(e)) - \text{id}) \\ &< \frac{\varepsilon}{8} + l(e - \psi'_\alpha \circ \varphi'_\alpha(e)) + 0 \end{aligned}$$

implies that we have a c.c.p. map $A \xrightarrow{\varphi'} M_n(\mathbb{C}) \xrightarrow{\psi'} A^{**}$ in the net such that

$$|l(a - \psi' \circ \varphi'(a))| < \frac{\varepsilon}{4}, \quad a \in E, l \in F \tag{1}$$

and

$$l(1 - \psi'(\text{id})) < \frac{\varepsilon}{4}, \quad l \in F. \tag{1'}$$

Now we let $\varphi := \varphi'$ and try to deform ψ' so that A contains the codomain.

Define $\psi'' : M_n(\mathbb{C}) \rightarrow A^{**}$ such that

$$\psi''(T) := \frac{1}{n} \text{Tr}(T)(1 - \psi'(\text{id})) + \psi'(T).$$

Then, ψ'' is a u.c.p. map and (1') implies

$$|l(\psi' \circ \varphi(a) - \psi'' \circ \varphi(a))| < \frac{\varepsilon}{4}, \quad a \in E, l \in F. \quad (2)$$

Now consider the associated matrix element $[\psi''(e_{ij})] \in M_n(A^{**})_+$ and the correspondence

$$M_n(A^*) \xrightarrow{\sim} M_n(A)^* : [l_{ij}] \mapsto ([a_{ij}] \mapsto \sum_{i,j} l_{ij}(a_{ij})).$$

Since $M_n(A)$ is σ -weakly dense in $M_n(A)^{**} = M_n(A^*)^* = M_n(A^{**})$, the Kaplansky density theorem implies that the closed ball of $M_n(A)_+$ is σ -weakly dense in the closed ball of $M_n(A^{**})_+$. For each pair $(a, l) \in E \times F$, if we define $[l_{ij}] \in M_n(A^*)$ such that $l_{ij} := t_{ij}l$ and $\varphi(a) = [t_{ij}] \in M_n(\mathbb{C})$, then we can take $[a_{ij}] \in M_n(A)$ such that

$$|l(\psi''(\varphi(a)) - \sum_{i,j} t_{ij}a_{ij})| = |\sum_{i,j} l_{ij}(\psi''(e_{ij}) - a_{ij})| = |[l_{ij}](\psi''(e_{ij}) - a_{ij})| < \frac{\varepsilon}{4}$$

for all $a \in E$ and $l \in F$. Now we define $\psi''' : M_n(\mathbb{C}) \rightarrow A$ by $\psi'''(e_{ij}) := a_{ij}$ to get

$$|l(\psi'' \circ \varphi(a) - \psi''' \circ \varphi(a))| < \frac{\varepsilon}{4}, \quad a \in E, l \in F \quad (3)$$

and $\psi'' : M_n(\mathbb{C}) \rightarrow A$ is c.p. Note that we may not assume ψ'' is contractive because the correspondence $\text{CP}(M_n(\mathbb{C}), A) \cong M_n(A)_+$ does not preserve the norm.

However, when we take ψ''' we can insert an additional condition

$$\|1 - \psi'''(\text{id})\| = \|\psi''(\text{id}) - \psi'''(\text{id})\| < \frac{\varepsilon}{4} \quad (3')$$

using Mazur's lemma. Define $\psi'''' : M_n(\mathbb{C}) \rightarrow A$ by $\psi'''' := \psi''' / \|\psi'''\|$. Then, (3') implies

$$|l((\psi''' - \psi''') \circ \varphi(a))| = |l((\|\psi'''\| - 1)\psi''' \circ \varphi(a))| \leq \|\psi'''\| - 1 < \frac{\varepsilon}{4}, \quad a \in E, l \in F. \quad (4)$$

Combining (1)~(4), we finally obtain $\psi := \psi''''$ such that

$$|l(a - \psi'''' \circ \varphi(a))| < \frac{\varepsilon}{4} \cdot 4 = \varepsilon, \quad a \in E, l \in F,$$

so we are done. In summary, we have constructed the following maps

$$\begin{array}{lll} \psi' & : M_n(\mathbb{C}) \rightarrow A^{**} & \text{c.c.p.} \\ \psi'' & : M_n(\mathbb{C}) \rightarrow A^{**} & \text{u.c.p.} \\ \psi''' & : M_n(\mathbb{C}) \rightarrow A & \text{c.p.} \\ \psi'''' & : M_n(\mathbb{C}) \rightarrow A & \text{u.c.p.} \end{array}$$

□

2.3 Choi-Effros-Kirchberg characterization

Lemma 2.7 (Bounded Radon-Nikodym theorem). *Let A be a C^* -algebra. For F a finite dimensional subspace of A^* , there is a cyclic representation $\pi : A \rightarrow B(H)$ with the cyclic vector Ω such that there is a linear map $\pi' : F \rightarrow \pi(A)'$ satisfying $l(a) = \langle \pi(a)\pi'(l)\Omega, \Omega \rangle$ for every $l \in F$. The operator $\pi'(l)$ is called the Radon-Nikodym derivative of l with respect to the vector state ω_Ω .*

Proof. Choose a basis l_1, \dots, l_n of F . With the Jordan decomposition $l_i = l_{i,1} - l_{i,2} + i(l_{i,3} - l_{i,4})$, define a state

$$l_0 := \frac{1}{4n} \sum_{i=1}^n \sum_{j=1}^4 \frac{l_{i,j}}{\|l_{i,j}\|}$$

by averaging, and let $\pi : A \rightarrow B(H)$ be the GNS-representation of l_0 with cyclic vector $\Omega \in H$. Then for each $l \in F$,

$$\sigma(\pi(a)\Omega, \pi(b)\Omega) := l(b^*a)$$

are extended to well-defined bounded sesquilinear forms on H . If we write by $\pi'(l)$ the bounded linear operator associated to the sesquilinear form σ , then for $l \in F$ we have

$$l(b^*a) = \langle \pi'(l)\pi(a)\Omega, \pi(b)\Omega \rangle = \langle \pi(b^*)\pi'(l)\pi(a)\Omega, \Omega \rangle,$$

and by putting $a = 1$ and $b = 1$ respectively, we can conclude $\pi'(l) \in \pi(A)'$ and the desired result. \square

Lemma 2.8 (Fell's theorem). *Let $\pi : A \rightarrow B(H)$ be a faithful representation of a C^* -algebra. Then, every state of A is a weak* limit of the convex combination of pullbacks of vector states of $B(H)$.*

Proof. Suppose not and let ω be a counterexample. By the Hahn-Banach separation, there is $a \in A$ and $r \in \mathbb{R}$ such that

$$\operatorname{Re} \omega_\xi(a) \leq r < \operatorname{Re} \omega(a)$$

for all unit vector $\xi \in H$. Defining $h = (a + a^*)/2$, rewrite the above inequality as

$$\omega_\xi(h) \leq r < \omega(h).$$

Then, $r - h \geq 0$ by the left inequality, which contradicts to the right inequality. \square

Theorem 2.9 (Choi-Effros-Kirchberg). *Let A be a C^* -subalgebra.*

- (a) *The identity $A \rightarrow A$ is approximable.*
- (b) *A is nuclear.*
- (c) *$\pi \times i : A \otimes_{\min} \pi(A)' \rightarrow B(H)$ is continuous.*

Proof. (a) \Rightarrow (b) Recall that every finite-dimensional C^* -algebra is nuclear. For any C^* -algebra B , we have (by Corollary 3.3) a diagram

$$\begin{array}{ccc} A \otimes_{\max} B & \xrightarrow{\quad} & A \otimes_{\max} B \\ \downarrow & \searrow \text{dashed} & \nearrow \text{dashed} \\ & M_{n_\alpha}(\mathbb{C}) \otimes_{\max} B & \\ & \updownarrow \text{dashed} & \\ & M_{n_\alpha}(\mathbb{C}) \otimes_{\min} B & \\ & \nearrow \text{dashed} & \\ A \otimes_{\min} B & & \end{array}$$

in which the square at lower left side algebraically commutes for each α and the upper triangle approximately commutes in the point-norm topology because

$$\|a \otimes b - \psi_\alpha \circ \varphi_\alpha(a) \otimes b\|_{\max} = \|a - \psi_\alpha \circ \varphi_\alpha(a)\| \|b\| \rightarrow 0$$

for each $a \in A, b \in B$. Here the dashed arrows mean that they vary as α goes to limit. Hence we have an approximately commuting diagram

$$\begin{array}{ccc} A \otimes_{\max} B & \xrightarrow{\quad} & A \otimes_{\max} B \\ & \searrow & \nearrow \text{dashed} \\ & A \otimes_{\min} B & \end{array}$$

so that we can verify the injectivity of $A \otimes_{\max} B \rightarrow A \otimes_{\max} B$ (because it is just the identity, so we do not have to take care of the failure of inclusion for maximal tensor products) implies the injectivity of $A \otimes_{\max} B \rightarrow A \otimes_{\min} B$.

(b) \Rightarrow (c) Clear.

(c) \Rightarrow (a) Let $E \subset A$ and $F \subset A^*$ be finite subsets and fix $\varepsilon > 0$. We want to find c.c.p. maps $A \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} A$ such that

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon$$

for $a \in E$ and $l \in F$. To implement the approximation, we would like to regard the inclusion operator as a state of a tensor product C^* -algebra via the correspondence

$$B(A, A) \subset B(A, A^{**}) \xrightarrow{\sim} (A \otimes_{\pi} A^*)^*,$$

which maps the identity map $A \rightarrow A$ to the linear functional characterized by $a \otimes l \mapsto l(a)$. Since A^* is not a C^* -algebra, we think a “representation” of $\pi' : F \rightarrow \pi(A)'$ through the above Radon-Nikodym type result. Let $\pi : A \rightarrow B(H)$ be the cyclic representation obtained from the above Radon-Nikodym theorem and Ω the cyclic vector such that $l(a) = \langle \pi(a)\pi'(l)\Omega, \Omega \rangle$ for $a \in E$ and $l \in F$.

By the assumption, we have a representation

$$\pi \times i : A \otimes_{\min} \pi(A)' \rightarrow B(H).$$

Consider any faithful representation $\rho : A \rightarrow B(K)$ and the tensor representation

$$\rho \otimes i : A \otimes_{\min} \pi(A)' \rightarrow B(K \otimes H),$$

which is also faithful. By Fell’s theorem, the state $\omega_{\Omega} \circ (\pi \times i)$ on $A \otimes_{\min} \pi(A)'$ can be approximated by convex combinations of vector states in $B(K \otimes H)$. In particular, by the density of $\pi(A)\Omega$ in H , we have tensors $(\tau_k)_{k=1}^m \subset K \otimes \pi(A)\Omega$ such that

$$\left| \omega_{\Omega}((\pi \times i)(a \otimes \pi'(l))) - \sum_{k=1}^m \lambda_k \omega_{\tau_k}((\rho \otimes i)(a \otimes \pi'(l))) \right| < \varepsilon \quad (\dagger)$$

for all $a \in E$ and $l \in F$, where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$.

If we write each element $\tau \in K \otimes \pi(A)\Omega$ as

$$\tau = \sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega,$$

then

$$\begin{aligned} \omega_{\tau}((\rho \otimes i)(a \otimes \pi'(l))) &= \left\langle (\rho(a) \otimes \pi'(l)) \left(\sum_{j=1}^n \eta_j \otimes \pi(b_j)\Omega \right), \left(\sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega \right) \right\rangle \\ &= \sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i \rangle \langle \pi'(l)\pi(b_i^*b_j)\Omega, \Omega \rangle \\ &= l \left(\sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i \rangle b_i^*b_j \right). \end{aligned}$$

If we define c.c.p. maps $A \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} A$ for each τ such that

$$\varphi(a) := [\langle \rho(a)\eta_j, \eta_i \rangle], \quad \psi([e_{ij}]) := b_i^*b_j,$$

then we have $\omega_{\tau}(a \otimes \pi'(l)) = l(\psi \circ \varphi(a))$.

Since $\mu(a \otimes \pi'(l)) = l(a)$ and since the c.c.p. maps that factor through a matrix algebra form a convex set, we have c.c.p. maps $A \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} A$ such that the inequality (\dagger) is rewritten as

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon,$$

so we are done. □

3 May 26

In the next two subsections, we extend our investigation of famous theorems related to completely positive maps: the Stinespring dilation theorem and the Arveson extension theorem.

3.1 Stinespring dilation revisited

Here is a Stinespring dilation theorem for non-unital C^* -algebras. We also discuss shortly the minimal Stinespring representation.

Theorem 3.1 (Stinespring dilation). *Let A be a C^* -algebra and $\varphi : A \rightarrow B(H)$ is a c.p. map. Then, there is a representation $\pi : A \rightarrow B(\hat{H})$ and a bounded linear operator $V : H \rightarrow \hat{H}$ such that $\varphi(a) = V^* \pi(a) V$ for all $a \in A$. Moreover, we can take π to be minimal in the sense that $(\pi(A) V H)^- = \hat{H}$.*

Proof. In the first day we have defined $\pi : A \rightarrow B(\hat{H})$ and $V : H \rightarrow \hat{H}$, where $\hat{H} := (A \otimes H/N)^-$, such that

$$\pi(a)(b \otimes \xi + N) := (ab) \otimes \xi + N, \quad V\xi := 1_A \otimes \xi + N.$$

Since A may not have its unit, we want to adjust the definition of V with approximate unit. For an approximate unit e_α of A , let $V_\alpha : H \rightarrow \hat{H}$ be such that

$$V_\alpha \xi := e_\alpha \otimes \xi + N.$$

Recall that if ω is a positive linear functional, then $\lim_\alpha \omega(e_\alpha) = \omega(1)$. For fixed $\xi \in H$, because $\omega_\xi \circ \varphi$ is a positive linear functional and products of approximate units are also approximate units, we have

$$\begin{aligned} \|(V_\alpha - V_\beta)\xi\|^2 &= \langle \varphi((e_\alpha - e_\beta)^2)\xi, \xi \rangle = \langle \varphi(e_\alpha^2 - e_\alpha e_\beta - e_\beta e_\alpha + e_\beta^2)\xi, \xi \rangle \\ &\rightarrow \langle (\varphi - \varphi - \varphi + \varphi)\xi, \xi \rangle = 0, \end{aligned}$$

so the net $V_\alpha \xi$ is Cauchy. Define $V\xi := \lim_\alpha V_\alpha \xi$. Then,

$$\begin{aligned} \langle V^* \pi(a) V \xi, \eta \rangle &= \lim_\alpha \langle \pi(a) V_\alpha \xi, V_\alpha \eta \rangle \\ &= \lim_\alpha \langle (ae_\alpha) \otimes \xi + N, e_\alpha \otimes \eta + N \rangle \\ &= \lim_\alpha \langle \varphi(e_\alpha a e_\alpha) \xi, \eta \rangle \\ &= \langle \varphi(a) \xi, \eta \rangle. \end{aligned}$$

It is easy to check the condition $(\pi(A) V H)^- = \hat{H}$ holds.

(Alternatively, we can define $V \in B(H, \hat{H}) \cong H^* \otimes \hat{H}$ by

$$\langle V\xi, a \otimes \eta + N \rangle := \langle \varphi(a^*) \xi, \eta \rangle.$$

The vector $V\xi$ in \hat{H} for each $\xi \in H$ is well-defined because

$$|\langle \varphi(a^*) \xi, \eta \rangle|^2 = |\langle \xi, \varphi(a) \eta \rangle|^2 \leq \|\xi\|^2 \langle \varphi(a^*) \varphi(a) \eta, \eta \rangle \leq \|\xi\|^2 \|\varphi\| \langle \varphi(a^* a) \eta, \eta \rangle = \|\xi\|^2 \|\varphi\| \|a \otimes \eta + N\|^2.$$

(For this proof, we have to establish (a) in Proposition 3.2 by applying the Cauchy-Schwarz inequality on the second inflation of φ (Maybe..?).)) \square

Remark. For a c.p. map $\varphi : A \rightarrow B(H)$, we can define a *Stinespring dilation* of φ as a triplet (π, K, V) , where $\pi : A \rightarrow K$ is a representation and $V : H \rightarrow K$ is a bounded linear operator, such that $\varphi(a) = V^* \pi(a) V$. For two Stinespring dilations (π_1, K_1, V_1) and (π_2, K_2, V_2) of φ , we can define morphisms by

a bounded linear operator $U : K_2 \rightarrow K_1$ such that $\pi_2(a) = U^* \pi_1(a) U$ and $V_2 = UV_1$. The two conditions imply that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{\varphi} & B(H) \\
 \pi_2 \searrow & & \nearrow V_2^* \cdot V_2 \\
 & B(K_2) & \\
 \pi_1 \searrow & \uparrow U^* \cdot U & \nearrow V_1^* \cdot V_1 \\
 & B(K_1) &
 \end{array}$$

For arbitrary Stinespring dilation, we may always assume π acts non-degenerately on the subspace VH of K by changing V to $\lim_{\alpha} \pi(e_{\alpha})V$ to make $\text{im } V \subset (\pi(A)VH)^{\perp}$. Even we add this in the definition of a Stinespring dilation, the definition of morphisms still left unchanged. Here, the non-degeneracy condition $(\pi(A)VH)^{\perp} = \hat{H}$ for the *minimal* Stinespring dilation is equivalent to that the Stinespring dilation enjoys the universal property, and hence unique up to morphisms. A nice fact, the isomorphism U is unitary, which means the minimal Stinespring is unique up to unitary equivalence.

Proposition 3.2 (Multiplicative domain). *Let $\varphi : A \rightarrow B$ be a c.c.p. map between C^* -algebras.*

- (a) $|\varphi(a)|^2 \leq \varphi(|a|^2)$ for $a \in A$.
- (b) If $|\varphi(a)|^2 = \varphi(|a|^2)$ and $|\varphi(a^*)|^2 = \varphi(|a^*|^2)$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$, respectively.
- (c) $\{a \in A : |\varphi(a)|^2 = \varphi(|a|^2)\}$ is a closed under the multiplication. In particular, $\{a \in A : |\varphi(a)|^2 = \varphi(|a|^2), |\varphi(a^*)|^2 = \varphi(|a^*|^2)\}$ is a C^* -subalgebra of A on which the restriction of φ is a $*$ -homomorphism.

Proof. (a) Let (π, H, V) be the minimal Stinespring dilation of φ . Since φ is contractive, so is V . Then,

$$\varphi(|a|^2) - |\varphi(a)|^2 = V^* \pi(a)^* (1_H - VV^*) \pi(a) V \geq 0.$$

- (b) For $a \in A$, the equality $|\varphi(a)|^2 = \varphi(|a|^2)$ holds if and only if $(\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) V = 0$. Then,

$$\varphi(ba) - \varphi(b)\varphi(a) = V^* \pi(b)^* (1_H - VV^*) \pi(a) V = 0.$$

Similar for a^* .

- (c) If a and b satisfy the equality, then

$$\begin{aligned}
 (\text{id}_H - VV^*)^{\frac{1}{2}} \pi(ab) V &= (\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) \pi(b) V \\
 &= (\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) \pi(b) V - [(\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) V] V^* \pi(b) V \\
 &= (\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) (\text{id}_H - VV^*) \pi(b) V \\
 &= (\text{id}_H - VV^*)^{\frac{1}{2}} \pi(a) (\text{id}_H - VV^*)^{\frac{1}{2}} [(\text{id}_H - VV^*)^{\frac{1}{2}} \pi(b) V] = 0. \quad \square
 \end{aligned}$$

Lemma 3.3 (Restriction of product representation). *Let A and B be C^* -algebras. and $\Pi : A \otimes B \rightarrow B(H)$ be a $*$ -homomorphism. Then, there are $*$ -homomorphisms $\pi : A \rightarrow B(H)$ and $\pi' : B \rightarrow B(H)$ with commuting ranges such that $\Pi = \pi \times \pi'$.*

Proof. Let $K := (\Pi(A \otimes B)H)^{\perp}$. We first claim that a map $\pi : A \rightarrow B(K)$ defined by

$$\pi(a)(\Pi(\sum_i a_i \otimes b_i)\xi) := \Pi(\sum_i (aa_i) \otimes b_i)\xi$$

for $\xi \in H$ is indeed well-defined and bounded for each $a \in A$. We can define π' in the same manner.

We may assume $\|a\| \leq 1$. It suffices to show the inequality

$$\|\Pi(\sum_i (aa_i) \otimes b_i)\xi\|^2 \leq \|\Pi(\sum_i a_i \otimes b_i)\xi\|^2$$

for $\xi \in H$. The left hand side is

$$\|\Pi(\sum_i (aa_i) \otimes b_i)\xi\|^2 = \langle \Pi(\sum_{i,j} (a_i^* a^* a a_j) \otimes (b_i^* b_j))\xi, \xi \rangle$$

and the right hand side is

$$\|\Pi(\sum_i a_i \otimes b_i)\xi\|^2 = \langle \Pi(\sum_{i,j} (a_i^* a_j) \otimes (b_i^* b_j))\xi, \xi \rangle$$

so that their difference is positive because

$$\sum_{i,j} (a_i^* (1 - a^* a) a_j) \otimes (b_i^* b_j) = \sum_i ((1 - a^* a)^{\frac{1}{2}} a_i) \otimes b_i|^2 \geq 0,$$

where 1 is the unit in the unitization of A for the continuous functional calculus.

Note that the restrictions π and π' are clearly $*$ -homomorphisms, and commuting ranges of π and π' also can be immediately checked. By definition of the restrictions, we have

$$\pi(a)\pi'(b)\eta = \Pi(a \otimes b)\eta$$

for all $a \in A$, $b \in B$, and $\eta \in K$. If we extend the domain from K to H by letting $\pi(a)\eta^\perp = \pi'(b)\eta^\perp := 0$ for $\eta^\perp \in H \ominus K$, then since

$$\langle \Pi(a \otimes b)\eta^\perp, \xi \rangle = \langle \eta^\perp, \Pi(a^* \otimes b^*)\xi \rangle = 0$$

for $\xi \in H$ implies $\Pi(a \otimes b)\eta^\perp = 0$, we have

$$\pi(a)\pi'(b)(\eta + \eta^\perp) = \pi(a)\pi'(b)\eta = \Pi(a \otimes b)\eta = \Pi(a \otimes b)(\eta + \eta^\perp),$$

hence $\pi \times \pi' = \Pi$. □

Proposition 3.4 (Maximal tensor products of c.c.p. maps). *Let A, B and C be C^* -algebras and $\varphi : A \rightarrow B$ be a c.c.p. map. Then, $\varphi \otimes i : A \otimes_{\max} C \rightarrow B \otimes_{\max} C$ is well-defined.*

Proof. First assume $B \otimes_{\max} C \subset B(H)$ by taking a faithful representation $b \otimes c \mapsto bc$. By restriction, B and C can be seen as commuting C^* -subalgebras of $B(H)$. Let $\pi : A \rightarrow B(\hat{H})$ be the minimal Stinespring representation of the c.c.p. map $\varphi : A \rightarrow B \subset B(H)$.

$$\begin{array}{ccccc} A & \xrightarrow{\pi} & B(\hat{H}) & \xleftarrow{\rho} & C \\ \downarrow \varphi & & \downarrow V^* \cdot V & & \parallel \\ B & \hookrightarrow & B(H) & \longleftarrow & C \end{array}$$

Now we want to lift the inclusion $C \subset B' \subset B(H)$ to obtain a representation $\rho : C \rightarrow \pi(A)' \subset B(\hat{H})$.

Since $\pi(A)VH$ is dense in \hat{H} , we can try to define ρ by

$$\rho(c)(\sum_i \pi(a_i)V\xi_i) := \sum_i \pi(a_i)Vc\xi_i.$$

It is indeed well-defined from the following inequality:

$$\begin{aligned}
\left\| \sum_i \pi(a_i) V c \xi_i \right\|^2 &= \sum_{i,j} \langle c^* \varphi(a_i^* a_j) c \xi_j, \xi_i \rangle \\
&= \langle ([c \delta_{ij}]^* [\varphi(a_i^* a_j)] [c \delta_{ij}]) [\xi_i], [\xi_i] \rangle_{H^n} \\
&= \langle ([\varphi(a_i^* a_j)]^{\frac{1}{2}} [c \delta_{ij}]^* [c \delta_{ij}] [\varphi(a_i^* a_j)]^{\frac{1}{2}}) [\xi_i], [\xi_i] \rangle_{H^n} \\
&\leq \| [c \delta_{ij}] \|_{M_n(C)}^2 \langle ([\varphi(a_i^* a_j)]^{\frac{1}{2}} [\varphi(a_i^* a_j)]^{\frac{1}{2}}) [\xi_i], [\xi_i] \rangle_{H^n} \\
&= \|c\|^2 \left\| \sum_i \pi(a_i) V \xi_i \right\|^2.
\end{aligned}$$

Now then we can easily deduce that ρ is linear and preserves the multiplication, and is a $*$ -homomorphism from checking

$$\left\langle \sum_j \pi(a_j) V c \xi_j, \sum_i \pi(a_i) V \xi_i \right\rangle = \left\langle \sum_j \pi(a_j) V \xi_j, \sum_i \pi(a_i) V c^* \xi_i \right\rangle.$$

The commutation with $\pi(A)'$ is clear.

Using the universality, we have a $*$ -homomorphism $\pi \times \rho : A \otimes_{\max} C \rightarrow B \otimes_{\max} C \subset B(\hat{H})$, which satisfies

$$V^*(\pi \times \rho)(a \otimes c) V \xi = V^* \pi(a) \rho(c) V \xi = V^* \pi(a) V c \xi = \varphi(a) c \xi.$$

Since the product $\varphi(a)c$ has been identified with the simple tensor $\varphi(a) \otimes c \in B \otimes_{\max} C$ when we embed B and C into $B(H)$, the above equality implies that $V^*(\pi \times \rho)V = \varphi \otimes i$. \square

Remark. For completely positive maps $\varphi_1 : A_1 \rightarrow \mathbb{B}_1$ and $\varphi_2 : A_2 \rightarrow B_2$, the tensor product map

$$\varphi_1 \otimes \varphi_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2 : a_1 \otimes a_2 \mapsto \varphi_1(a_1) \otimes \varphi_2(a_2)$$

can be extended to both minimal and maximal tensor products: $A_1 \otimes_{\min} A_2 \rightarrow B_1 \otimes_{\min} B_2$ and $A_1 \otimes_{\max} A_2 \rightarrow B_1 \otimes_{\max} B_2$, respectively. For \otimes_{\max} , it is justified by applying Proposition 3.4 twice, and for \otimes_{\min} , we can show it by the Stinespring dilation and the spatial property.

3.2 Arveson extension revisited

Proposition 3.5 (Representation extensioin). *Let A be a closed ideal of a C^* -algebra B . For a representation $\pi : A \rightarrow B(H)$, there is a representation $\tilde{\pi} : B \rightarrow B(H)$ which extends π . If π is non-degenerate, the extension is unique and $\pi(e_\alpha b) \rightarrow \tilde{\pi}(b)$ and $\pi(b e_\alpha) \rightarrow \tilde{\pi}(b)$ strongly for $b \in B$, where e_α is an approximate unit of A .*

Proof. Let $K := (\pi(A)H)^\perp$. Define $\tilde{\pi} : B \rightarrow B(H)$ such that

$$\tilde{\pi}(b)(\pi(a)\xi + \eta) := \pi(ba)\xi$$

for $\xi \in H$, $\eta \in H \ominus K$. The well-definedness is from

$$\| \pi(ba)\xi \|^2 = \langle \pi(a^* b^* ba)\xi, \xi \rangle \leq \|b\|^2 \langle \pi(a^* a)\xi, \xi \rangle = \|b\|^2 \| \pi(a)\xi \|^2.$$

It is clearly a $*$ -homomorphism and extends π since $\langle \pi(a')\eta, \zeta \rangle = \langle \eta, \pi(a'^*)\zeta \rangle = 0$ for all $\zeta \in H$ implies

$$\tilde{\pi}(a')(\pi(a)\xi + \eta) = \pi(a'a)\xi = \pi(a')(\pi(a)\xi + \eta).$$

For the uniqueness, if π is non-degenerate and $\tilde{\pi}$ is a $*$ -homomorphism which extends π , then

$$\tilde{\pi}(b)(\pi(a)\xi) = \tilde{\pi}(b)\tilde{\pi}(a)\xi = \tilde{\pi}(ba)\xi = \pi(ba)\xi,$$

which is unique by the density of $\pi(A)H$ in H . \square

Theorem 3.6 (Arveson extension). *Let $A \subset B$ be C^* -algebras. Let $\varphi : A \rightarrow B(H)$ be a c.p. map and consider the following diagram:*

$$\begin{array}{ccc} & B & \\ \uparrow & \searrow \tilde{\varphi} & \\ A & \xrightarrow{\varphi} & B(H). \end{array}$$

- (a) *The n.p.c.p. (norm preserving c.p.) extension $\tilde{\varphi}$ of φ exists if B is unital and $1_B \in A$.*
- (b) *The n.p.c.p. extension $\tilde{\varphi}$ of φ exists if A is unital and $B = A \oplus \mathbb{C}$.*
- (c) *The n.p.c.p. extension $\tilde{\varphi}$ of φ exists if A is non-unital and $B = \tilde{A}$.*
- (d) *The n.p.c.p. extension $\tilde{\varphi}$ of φ always exists.*

Proof. (a) We have proved on the first day.

(b) Define $\tilde{\varphi}(a + \lambda) = \varphi(a)$. Then, $\tilde{\varphi} : A \oplus \mathbb{C} \rightarrow B(H)$ is a norm-preserving c.p. extension of φ since

$$\|\tilde{\varphi}(a + \lambda)\| = \|\varphi(a)\| \leq \|\varphi\| \|a\| \leq \|\varphi\| \|a + \lambda\|$$

and

$$\|\varphi(a)\| = \|\tilde{\varphi}(a)\| \leq \|\tilde{\varphi}\| \|a\|.$$

(c) Let $\pi : A \rightarrow B(\hat{H})$ be the minimal and hence non-degenerate Stinespring representation of φ . Since A is a closed ideal of $B = \tilde{A}$, we can apply the representation extension for π to get $\tilde{\pi}$ as follows:

$$\begin{array}{ccc} \tilde{A} & & \\ \uparrow & \searrow \tilde{\pi} & \\ A & \xrightarrow{\pi} & B(\hat{H}) \\ \parallel & & \downarrow V^* \cdot V \\ A & \xrightarrow{\varphi} & B(H). \end{array}$$

Then, the c.p. extension is given by $\tilde{\varphi}(b) := V^* \tilde{\pi}(b) V$ for $b \in B$. To check the norm is preserved, take an approximate unit e_α of A so that the net $\varphi(e_\alpha b) = V^* \pi(e_\alpha b) V$ converges to $\tilde{\varphi}(b) = V^* \tilde{\pi}(b) V$ strongly and weakly, which implies

$$\|\tilde{\varphi}(1)\xi\| = \lim_\alpha \|\varphi(e_\alpha)\xi\| \leq \|\varphi\| \|\xi\|.$$

Hence $\|\tilde{\varphi}\| \leq \|\varphi\|$.

(d) By the part (c), we may assume B is unital. Let $\tilde{A} = A + 1_B \mathbb{C}$ be the C^* -subalgebra of B generated by A and 1_B . Since A is an ideal of \tilde{A} with codimension at most one, we have three cases:

- (i) $\tilde{A} = A$, if $1_B \in A$,
- (ii) $\tilde{A} = A \oplus \mathbb{C}$, if A is unital but $1_B \notin A$
- (iii) \tilde{A} is the unitization of A , if A is non-unital.

For the last case reduces to the part (a). By using (b) or (c), we have an extension to \tilde{A} , and the other two cases also reduce to the part (a). \square

We introduce a more controlled version of the Arveson extension, the Trick, named after [Brown-Ozawa]. It is useful when we want to restrict the codomain of an Arveson extension.

Theorem 3.7 (The Trick). *Let $A \subset B$ and C be C^* -algebras, and let $\pi : A \rightarrow B(H)$ be a representation.*

- (i) *If there are C^* -norm α and β on $A \odot C$ and $B \odot C$ respectively such that $A \otimes_\alpha C \rightarrow B \otimes_\beta C$ is injective,*

(ii) and if there is a representation $\pi' : C \rightarrow B(H)$ such that $\pi \times \pi' : A \otimes_\alpha C \rightarrow B(H)$ is continuous,

then there is a norm-preserving c.c.p. extension $\varphi : B \rightarrow \pi'(C)'$ of $\pi : A \rightarrow \pi'(C)'$.

Proof. We first claim that we may assume B and C are unital. To see this, we must verify the existence of a C^* -norm $\tilde{\beta}$ on $B \otimes \tilde{C}$ such that $B \otimes_\beta C \rightarrow B \otimes_{\tilde{\beta}} \tilde{C}$ is injective. We can do same thing for B with symmetry. Take a faithful representation $\rho \times \rho' : B \otimes_\beta C \rightarrow B(K)$, where $\rho : B \rightarrow B(K)$ and $\rho' : C \rightarrow B(K)$ are restrictions of $\rho \times \rho'$. Then, ρ' can be extended to a faithful representation $\tilde{\rho}' : \tilde{C} \rightarrow B(K)$, and it commutes with ρ since the new element $\tilde{\rho}'(1)$ is the strong limit of elements of $\rho'(C)$. Now consider the product representation $\rho \times \tilde{\rho}' : B \otimes \tilde{C} \rightarrow B(K)$. We want this is faithful. Suppose $(\rho \times \tilde{\rho}')(\sum_{i=1}^n b_i \otimes c_i) = 0$ and b_i are linearly independent. For approximate units e_α and f_α of B and C respectively, we have

$$(\rho \times \rho')(\sum_i b_i e_\alpha \otimes c_i f_\alpha) = (\rho \times \tilde{\rho}')(\sum_i b_i \otimes c_i)(e_\alpha \otimes f_\alpha) = 0,$$

which implies $\sum_i b_i e_\alpha \otimes c_i f_\alpha = 0$ for each α . For every sufficiently large α , the finite set $b_i e_\alpha$ should be linearly independent. (Define $W_i := \text{span}(\{b_1, \dots, b_n\} \setminus \{b_i\})$ and $d_{i,\alpha} := \inf\{\|b_i e_\alpha - b e_\alpha\| : b \in W_i\}$. Then, we can deduce $\min_i d_{i,\alpha} > 0$ for every sufficiently small $\|e_\alpha\|$ from the linear independence of b_i .) Therefore, $c_i f_\alpha = 0$. By \lim_α , we can conclude $\rho \times \tilde{\rho}'$ is faithful so that we have $B \otimes_\beta C \rightarrow B \otimes_{\tilde{\beta}} \tilde{C}$ injective with the C^* -norm induced from $B(K)$.

Now let B and C be unital. Then, by the Arveson extension theorem, there is a c.c.p. map $\Phi : B \otimes_\beta C \rightarrow B(H)$ which extends $\pi \times \pi'$. Define $\varphi : B \rightarrow B(H)$ by $\varphi(b) := \Phi|_B(b) = \Phi(b \otimes 1)$. Then, φ clearly extends π . Since $\Phi|_{C \otimes 1} = \pi'$ is a $*$ -homomorphism, $1 \otimes c$ belongs to the multiplicative domain of Φ , hence we can compute

$$\varphi(b)\pi'(c) = \Phi(b \otimes 1)\Phi(1 \otimes c) = \Phi((b \otimes 1)(1 \otimes c)) = \Phi(1 \otimes c)\Phi(b \otimes 1) = \pi'(c)\varphi(b). \quad \square$$

Corollary 3.8 (Inclusion problem). *Let $A \subset B$ be C^* -algebras. Then, the following statements are all equivalent:*

(a) *There is a c.c.p. map $\varphi : B \rightarrow A^{**}$ such that*

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow \varphi & \\ A & \longrightarrow & A^{**} \end{array}$$

is commutative.

(b) *There is a c.c.p. map $\varphi : B \rightarrow \pi(A)''$ such that*

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow \varphi & \\ A & \xrightarrow{\pi} & \pi(A)'' \end{array}$$

is commutative for every representation $\pi : A \rightarrow B(H)$.

(c) *The map $A \otimes_{\max} C \rightarrow B \otimes_{\max} C$ is injective for every C^* -algebra C .*

Proof. (a) \Rightarrow (b) For any representation $\pi : A \rightarrow B(H)$ admits a normal extension $\tilde{\pi} : A^{**} \rightarrow B(H)$ by the universal property of the enveloping von Neumann algebra. Then, our c.c.p. map is $\tilde{\pi} \circ \varphi$, where $\varphi : B \rightarrow A^{**}$ is a c.c.p. map constructed from the assumption (a), because the following diagram

commutes:

$$\begin{array}{ccc}
 & B & \\
 & \uparrow & \searrow \varphi \\
 A & \xrightarrow{\quad} & A^{**} \\
 \parallel & & \downarrow \tilde{\pi} \\
 A & \xrightarrow{\quad \pi \quad} & \pi(A)''
 \end{array}$$

(b) \Rightarrow (a) Clear.

(b) \Rightarrow (c) Taking any faithful representation of $A \otimes_{\max} C$ and by restriction, we may assume A , C , and $A \otimes_{\max} C$ are C^* -subalgebras of $B(H)$ such that $A \subset C'$. Then, by the assumption (b) we can define a c.c.p. map $\varphi \otimes \text{id}$ such that

$$\begin{array}{ccc}
 B \otimes_{\max} C & & \\
 \uparrow & \searrow \varphi \otimes \text{id} & \\
 A \otimes_{\max} C & \longrightarrow & A'' \otimes_{\max} C \subset B(H).
 \end{array}$$

commutes. Since the horizontal arrow is injective, the vertical arrow is also injective.

(c) \Rightarrow (b) This step is a corollary of “The Trick”. □

3.3 Exact C^* -algebras

Let \mathcal{I} be a closed ideal of a C^* -algebra A . For a C^* -algebra C , do we have an exact sequence

$$0 \rightarrow \mathcal{I} \otimes C \rightarrow A \otimes C \rightarrow (A/\mathcal{I}) \otimes C \rightarrow 0,$$

where $\otimes = \otimes_{\min}$ or \otimes_{\max} ? The answer is given as follows:

exact	at left	at middle	at right
\otimes_{\min}	True	False	True
\otimes_{\max}	True	True	True

Definition 3.9. A C^* -algebra C is called *exact* if the following sequence exact for every C^* -algebra A and its closed ideal \mathcal{I} :

$$0 \rightarrow \mathcal{I} \otimes_{\min} C \rightarrow A \otimes_{\min} C \rightarrow (A/\mathcal{I}) \otimes_{\min} C \rightarrow 0.$$

4 June 16

4.1 Group C^* -algebras

Let G be a locally compact group, which is possibly not σ -compact. We denote by $L^p(G)$ for $1 \leq p < \infty$ the L^p space with respect to the *left* Haar measure on G . Let $L^\infty(G)$ be the Kolmogorov quotient of the set of all locally Borel measurable functions on G that is bounded except on locally null subsets together with the norm of essential supremum. Then, we have a representation $L^1(G)^* \cong L^\infty(G)$. The Fubini theorem for integrable functions are always fine since their support are σ -finite.

Definition 4.1 (Left regular representations). We will consider the actions $\alpha : G \rightarrow \text{Aut}(L^1(G))$ and $\lambda : G \rightarrow B(L^2(G))$ defined as

$$\alpha_s f(t) := f(s^{-1}t), \quad \lambda_s \xi(t) := \xi(s^{-1}t), \quad \lambda_s x \lambda_s^*(t) := x(s^{-1}t)$$

for $f \in L^1(G)$, $\xi \in L^2(G)$, $x \in L^\infty(G)$. The actions α and λ , called the *left regular representations*, are continuous with respect to the strong operator topologies, i.e. the point-norm topology and the strict topology.

Definition 4.2 (Group C^* -algebras). Let G be a locally compact group. Note that $C_c(G)$ has a (generally noncommutative) $*$ -algebra structure

$$f * g(s) := \int f(t)g(t^{-1}s) d\mu(t), \quad f^*(s) := \Delta(s^{-1})\overline{f(s^{-1})}.$$

The *reduced group C^* -algebra* $C_r^*(G)$ and the *full group C^* -algebra* $C^*(G)$ are defined as the completion of $C_c(G)$ with the following norms respectively:

$$\|f\|_{C_r^*(G)} := \|\lambda(f)\|_{B(L^2(G))}, \quad \|f\|_{C^*(G)} := \sup\{\|\pi(f)\| \mid \pi : C_c(G) \rightarrow B(H) \text{ } *\text{-hom}\},$$

where $\lambda : C_c(G) \rightarrow B(L^2(G))$ is given by $\lambda(f)\xi := f * \xi$ for $\xi \in L^2(G)$. The *group von Neumann algebra* $L(G)$ is defined to be the double commutant $C_r^*(G)''$.

Remark.

- By convention, we consider $C_r^*(G)$ as a C^* -subalgebra of $B(L^2(G))$.
- We have $\|f\|_{C_r^*(G)} \leq \|f\|_{C^*(G)} \leq \|f\|_{L^1(G)}$ because a homomorphism from a Banach algebra to a C^* -algebra is contractive. We can show the injectivity of $\lambda : L^1(G) \rightarrow C_r^*(G)$ using measure theory, and hence also $L^1(G) \subset C^*(G)$. Due to the closed image of $*$ -homomorphisms, we have surjection $\lambda : C^*(G) \rightarrow C_r^*(G)$.
- There is a natural one-to-one correspondences among (non-degenerate, unitary) representations of G , $L^1(G)$, and $C^*(G)$.
- $L^1(G)$ has two order structure; as a $*$ -subalgebra of group C^* -algebras, as a predual of $L^\infty(G)$.
- If $\rho : C_c(G) \rightarrow B(L^2(G))$ is the right regular representation, then it is known that $L(G) = \rho(C_c(G))'$, so in particular we have $\lambda_s \in L(G)$ for $s \in G$. We will skip the proof. See Theorem 6.1.4 in [Brown-Ozawa].

Lemma 4.3 (Fell's absorption principle). *Let G be a locally compact group.*

- (a) *For every unitary representation $\rho : G \rightarrow U(H)$, we have the unitary equivalence $\lambda \otimes \rho \cong \lambda \otimes \text{id}_H$.*

Definition 4.4 (Positive definite functions). Let G be a locally compact group. A function φ is called *positive definite* if the matrix $[\varphi(s_i^{-1}s_j)]_{i,j} \in M_n(\mathbb{C})$ is always positive semi-definite for all n and $[s_i] \in G^n$. The *Fourier-Stieltjes algebra* $B(G)$ is the linear span of continuous positive definite functions. We can check $B(G)$ is a commutative $*$ -algebra.

Proposition 4.5. Let G be a locally compact group and let $\varphi \in B(G)_{+,1}$, i.e. φ is continuous and positive definite satisfying $\varphi(e) = 1$. Define a linear functional $\omega : C^*(G) \rightarrow \mathbb{C}$ and a linear operator $M : C^*(G) \rightarrow C^*(G)$ called multiplier such that

$$\omega(f) := \int \varphi(s)f(s)\mu(s), \quad M(f)(s) := \varphi(s)f(s), \quad f \in C_c(G).$$

Endow on $B(G)_{+,1}$ the topology of uniform convergence on compact sets.

- (a) If φ is compactly supported, then there is $\xi \in L^2(G)$ such that $\varphi(s) = \langle \lambda_s \xi, \xi \rangle_{L^2(G)}$.
- (b) $\varphi \mapsto \omega$ defines a homeomorphism $B(G)_{+,1} \rightarrow S(C^*(G))$ with respect to the weak* topology.
- (c) $\varphi \mapsto M$ defines a continuous map $B(G)_{+,1} \rightarrow B(C^*(G))$ with respect to the point-norm topology.
- (d) If φ is compactly supported, then ω and M factor through $\lambda : C^*(G) \rightarrow C_r^*(G)$.

Proof. (Since

$$\langle \eta * \overline{\varphi}, \eta \rangle = \int \eta^* * \eta(s) \overline{\varphi(s)} d\mu(s) \geq 0,$$

the convolution operator $(-\ast \overline{\varphi}) \in B(L^2(G))$ is positive. If we assume $\varphi(s) = \langle \lambda_s \xi, \xi \rangle$ as an ansatz, then we have $\langle \eta * \overline{\varphi}, \eta \rangle = \langle \eta * \xi, \eta * \xi \rangle$. For an approximate unit e_α of $C^*(G)$ in $C_c(G)$, the net $(-\ast \overline{\varphi})^{\frac{1}{2}} e_\alpha$ is Cauchy because

$$\|(-\ast \overline{\varphi})^{\frac{1}{2}}(e_\alpha - e_\beta)\|_{L^2(G)}^2 = \langle (e_\alpha - e_\beta) * \overline{\varphi}, e_\alpha - e_\beta \rangle \rightarrow 0,$$

which follows from that we can take large α, β for any $\varepsilon > 0$ such that

$$|\langle e_\alpha * \overline{\varphi}, e_\beta \rangle - \overline{\varphi(e)}| \leq \iint e_\alpha(s) |\varphi(s^{-1}t) - \varphi(e)| e_\beta(t) d\mu(s) d\mu(t) < \varepsilon.$$

Define $\xi := \lim_\alpha (-\ast \overline{\varphi})^{\frac{1}{2}} e_\alpha$ so that $\eta * \xi = (-\ast \overline{\varphi})^{\frac{1}{2}} \eta$. Then we can check $\varphi(s) = \langle \lambda_s \xi, \xi \rangle$ by integration with arbitrarily taken $f \in C_c(G)$.

(b) Skipped. See Theorem A.10 in [Williams].

(c) Suppose $\varphi_\alpha \rightarrow 0$ uniformly on compacts. Then, $\varphi_\alpha f \rightarrow 0$ in $L^1(G)$ for $f \in C_c(G)$, so we have $\|\varphi_\alpha f\|_{C^*(G)} \leq \|\varphi_\alpha f\|_{L^1(G)} \rightarrow 0$. By the density of $C_c(G)$ in $C^*(G)$, the continuity follows.

(d) For the states, it is easy to check from (a) and the Fubini theorem that $\omega(f) = \langle \lambda(f) \xi, \xi \rangle$. For the multipliers, consider the following commutative diagram:

$$\begin{array}{ccccc} C^*(G) & \xrightarrow{\Delta} & M(C^*(G) \otimes_{\min} C^*(G)) & \xrightarrow{\omega \otimes \text{id}} & C^*(G) \\ \downarrow \lambda & & \downarrow \lambda \otimes \text{id} & & \parallel \\ C_r^*(G) & \xrightarrow{\Delta} & M(C_r^*(G) \otimes_{\min} C^*(G)) & \xrightarrow{\omega_\xi \otimes \text{id}} & C^*(G). \end{array}$$

The diagonal map Δ on the second row is due to Fell's absorption principle. Since M can be realized as the composition $(\omega \otimes \text{id})\Delta$ at the first row, we are done. (We need more detail!) \square

4.2 Amenable groups

Definition 4.6. Let G be a locally compact group. A mean on G is a state of $L^\infty(G)$. We say G is amenable if there is an invariant mean m on $L^\infty(G)$, i.e. $m(\lambda_s x \lambda_s^*) = m(x)$ for $x \in L^\infty(G)$ and $s \in G$. It is known to be equivalent to that $m(\lambda(f)x\lambda(f)^*)$ for $x \in L^\infty(G)$ and $f \in C_c(G)_{+,1}$. See [Runde] or [Williams].

Theorem 4.7. Let G be a locally compact group with a left Haar measure μ . Then, the following are equivalent:

(a) G is amenable.

(b) G satisfies the Day-Reiter condition: there is a net $f_\alpha \in L^1(G)_{+,1}$ such that for each compact $K \subset G$,

$$\limsup_{\alpha} \sup_{s \in K} \|\alpha_s f_\alpha - f_\alpha\|_{L^1(G)} = 0$$

(c) G satisfies the Følner condition: there is a net $F_\alpha \subset G$ of Borel sets with $0 < \mu(F_\alpha) < \infty$ such that for each compact $K \subset G$,

$$\limsup_{\alpha} \sup_{s \in K} \frac{\mu(sF_\alpha \Delta F_\alpha)}{\mu(F_\alpha)} = 0.$$

(d) G satisfies the Dixmier condition: there is a net of unit vectors $\xi_\alpha \in L^2(G)$ such that for each compact $K \subset G$,

$$\limsup_{\alpha} \sup_{s \in K} \|\lambda_s \xi_\alpha - \xi_\alpha\|_{L^2(G)} = 0.$$

(e) There is a net of compactly supported continuous positive definite functions $\varphi_\alpha : G \rightarrow \mathbb{C}$ such that for each $K \subset G$,

$$\limsup_{\alpha} \sup_{s \in K} |\varphi_\alpha(s) - 1| = 0.$$

(f) $C^*(G) = C_r^*(G)$.

Proof. (a) \Rightarrow (b) Let $m \in L^\infty(G)_{+,1}^*$ be an invariant mean. Since the set $L^1(G)_{+,1}$ is exactly the set of normal states of $L^\infty(G)$, and since a state of a von Neumann algebra is σ -weakly approximated by normal states (we can show this similarly as Lemma 2.8), there exists a net $f_\alpha \in L^1(G)_{+,1}$ such that

$$\lim_{\alpha} |f_\alpha(x) - m(x)| = 0, \quad x \in L^\infty(G).$$

Fix any $h \in L^1(G)_{+,1}$ so that $h * f \in L^1(G)_{+,1}$ for any $f \in L^1(G)_{+,1}$. Since for every $x \in L^\infty(G)$ we have

$$\begin{aligned} \lim_{\alpha} |h * f_\alpha(x) - f_\alpha(x)| &\leq \lim_{\alpha} |h * f_\alpha(x) - h * m(x)| + \lim_{\alpha} |m(x) - f_\alpha(x)| \\ &= \lim_{\alpha} |f_\alpha(h^* * x) - m(h^* * x)| + \lim_{\alpha} |m(x) - f_\alpha(x)| = 0, \end{aligned}$$

the Mazur lemma implies that the closed convex hull of the set $\{h * f - f : f \in L^1(G)_{+,1}\}$ contains the zero. In other words, we have a net $f_\alpha \in L^1(G)_{+,1}$ such that

$$\lim_{\alpha} \|h * f_\alpha - f_\alpha\|_{L^1(G)} = 0 \quad h \in L^1(G)_{+,1}.$$

Let $K \subset G$ be compact and take $\varepsilon > 0$. Take an open neighborhood U of the unit of G such that $t^{-1}s \in U$ implies $\|\alpha_s h - \alpha_t h\|_{L^1(G)} < \varepsilon$. Using compactness, take a finite sequence $(s_i) \subset K$ such that for each $s \in K$ there is s_i with $s_i^{-1}s \in U$. Then,

$$\begin{aligned} \|\alpha_s(h * f_\alpha) - h * f_\alpha\|_{L^1(G)} &\leq \|(\alpha_s h - \alpha_{s_i} h) * f_\alpha\|_{L^1(G)} + \|(\alpha_{s_i} h) * f_\alpha - f_\alpha\| + \|f_\alpha - h * f_\alpha\| \\ &< \varepsilon + \max_i \|(\alpha_{s_i} h) * f_\alpha - f_\alpha\| + \|f_\alpha - h * f_\alpha\| \end{aligned}$$

implies

$$\limsup_{\alpha} \sup_{s \in K} \|\alpha_s(h * f_\alpha) - h * f_\alpha\|_{L^1(G)} \leq \varepsilon$$

for arbitrary $\varepsilon > 0$, so the net $h * f_\alpha$ satisfies our desired condition.

(b) \Rightarrow (c) For $f \in L^1(G)_{+,1}$ and $r > 0$, define $F(f, r) := \{s \in G : f(s) > r\}$. Then,

$$\begin{aligned}\|\alpha_s f - f\|_{L^1(G)} &= \int |\alpha_s f(t) - f(t)| d\mu(t) \\ &= \int \int_0^\infty |1_{F(\alpha_s f, r)}(t) - 1_{F(f, r)}(t)| dr d\mu(t) \\ &= \int_0^\infty \int |1_{sF(f, r)}(t) - 1_{F(f, r)}(t)| d\mu(t) dr \\ &= \int_0^\infty \mu(sF(f, r) \Delta F(f, r)) dr,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \mu(F(f, r)) dr &= \int_0^\infty \int 1_{F(f, r)}(t) d\mu(t) dr \\ &= \int \int_0^{f(t)} dr d\mu(t) \\ &= \int f(t) d\mu(t) = 1.\end{aligned}$$

Suppose a net f_α satisfies the Day-Reiter condition. Then, for arbitrary $\varepsilon > 0$ we have sufficiently large α such that

$$\int_0^\infty \mu(sF(f_\alpha, r) \Delta F(f_\alpha, r)) dr = \|\alpha_s f - f\|_{L^1(G)} < \varepsilon = \varepsilon \int_0^\infty \mu(F(f_\alpha, r)) dr$$

so there is $r_\alpha > 0$ such that

$$\mu(sF(f_\alpha, r_\alpha) \Delta F(f_\alpha, r_\alpha)) < \varepsilon \mu(F(f_\alpha, r_\alpha)).$$

We are done if we take $F_\alpha := F(f_\alpha, r_\alpha)$.

(c) \Rightarrow (d) Let F_α be a net of subsets satisfying the Følner condition. Define $\xi_\alpha := \mu(F_\alpha)^{-\frac{1}{2}} \in L^2(G)$. Then,

$$\|\lambda_s \xi_\alpha - \xi_\alpha\|_{L^2(G)}^2 = \int \mu(F_\alpha)^{-1} |1_{sF_\alpha}(t) - 1_{F_\alpha}(t)| d\mu(t) = \frac{\mu(sF_\alpha \Delta F_\alpha)}{\mu(F_\alpha)}$$

proves the claim.

(d) \Rightarrow (e) Let ξ_α be a net of unit vectors in $L^2(G)$ satisfying the Dixmier condition. Take compact sets $F_\alpha \subset G$ such that $0 < \mu(F_\alpha) < \infty$ and $\|\xi_\alpha - \xi_\alpha 1_{F_\alpha}\| < \varepsilon_\alpha$ for each α , where $\varepsilon_\alpha \downarrow 0$. Since if we consider $\tilde{\xi}_\alpha := \xi_\alpha 1_{F_\alpha} / \|\xi_\alpha 1_{F_\alpha}\|$, then we have

$$\|\lambda_s \tilde{\xi}_\alpha - \tilde{\xi}_\alpha\| < \|\lambda_s \xi_\alpha - \xi_\alpha\| + 4\varepsilon_\alpha,$$

we may assume ξ_α is compactly supported. Define $\varphi_\alpha : G \rightarrow \mathbb{C}$ such that $\varphi_\alpha := \langle \lambda_s \xi_\alpha, \xi_\alpha \rangle$, which is clearly continuous, positive definite, and compactly supported function. We can also check $|\varphi_\alpha(s)| \leq 1$ and $\|\lambda_s \xi_\alpha - \xi_\alpha\|_{L^2(G)}^2 = 2 - 2\operatorname{Re} \varphi_\alpha(s)$ imply the compact convergence of $\varphi_\alpha \rightarrow 1$.

(e) \Rightarrow (f) Let φ_α be the net of compactly supported continuous positive definite functions satisfying the condition of (e). Since φ_α has the compact support, the associated multiplication operator M_α factors through $\lambda : C^*(G) \rightarrow C_r^*(G)$. Since the constant unity function $1 \in B(G)_{+,1}$ corresponds to the identity operator in $B(C^*(G))$, the net M_α converges to the identity strongly. Therefore, if $a \in C^*(G)$ satisfies $\lambda(a) = 0$, then

$$a = \lim_\alpha M_\alpha(a) = 0.$$

The injectivity of λ implies the equality $C^*(G) = C_r^*(G)$.

(f) \Rightarrow (a) Take any one-dimensional representation $L^1(G) \rightarrow \mathbb{C} : f \mapsto \int f(s) d\mu(s)$ and extend to obtain a $*$ -homomorphism $C_r^*(G) = C^*(G) \rightarrow \mathbb{C}$. It is extended again to a state $\varphi : B(L^2(G)) \rightarrow \mathbb{C}$, whose multiplicative domain contains $C_r^*(G)$. Since $\lambda(f) \in C_r^*(G)$ for any $f \in L^1(G)$, we have

$$\varphi(\lambda(f)x\lambda(f)^*) = \varphi(\lambda(f))\varphi(x)\varphi(\lambda(f)^*) = \varphi(x), \quad x \in L^\infty(G).$$

Therefore, $\varphi|_{L^\infty(G)}$ is a left-invariant mean. \square

Lemma 4.8 (Markov-Kakutani fixed point theorem). *Let K be a compact convex subset of a topological vector space X . Let \mathcal{T} be a commuting family of continuous linear operators $T : X \rightarrow X$ such that $TK \subset K$. Then, K has a point x such that $Tx = x$ for all $T \in \mathcal{T}$.*

Proof. We may assume \mathcal{T} is convex and closed under the multiplication. Let

$$T^{(n)} := \frac{1}{n} \sum_{k=0}^{n-1} T^k$$

Since

$$\{T^{(n)}K : T \in \mathcal{T}, n \in \mathbb{N}\} \subset K$$

satisfies the finite intersection property. Take x in the intersection. We claim that this x is a fixed point.

Fix $T \in \mathcal{T}$ and take any neighborhood U of zero in X . Since $K - K$ is bounded, we have an integer n such that $K - K \subset nU$. With this n , because $x \in T^{(n)}K$, we have $y \in K$ such that $T^{(n)}y = x$ so that

$$Tx - x = \frac{1}{n}(T^n y - y) \in \frac{1}{n}(K - K) \subset U.$$

Since U is arbitrary, we have $Tx = x$. \square

Proposition 4.9. *Let G be a locally compact group.*

- (a) *If G is compact, then it is amenable.*
- (b) *If G is abelian, then it is amenable.*

Proof. (a) The normalized left Haar measure gives rise to a left-invariant state of $L^\infty(G)$.

(b) Define linear operators $T_s : L^\infty(G)^* \rightarrow L^\infty(G)^*$ such that $T_s m(x) := m(\lambda_s x \lambda_s^*)$ for $s \in G$ and $x \in L^\infty(G)$. We can check that $T_s(S(L^\infty(G))) \subset S(L^\infty(G))$ and is continuous with respect to the weak* topology because it is given by the dual operator in $B(L^\infty(G))$. Then, since $\{T_s : s \in G\}$ is commuting, by the Markov-Kakutani fixed point theorem, we have a left-invariant mean for G . \square

Proposition 4.10. *Let G be a locally compact group and let $H \leq G$ and $N \triangleleft G$ be closed. Then, we have the following permanence properties.*

- (a) *If G is amenable, then so is H .*
- (b) *If G is amenable, then so is G/N .*
- (c) *If N and G/N are amenable, then so is G .*
- (d) *The direct limit of amenable groups is amenable.*

Proof. See [Paterson]. \square

4.3 Discrete amenable groups

Let Γ be a discrete group. Then,

- $C_r^*(\Gamma)$ and $C^*(\Gamma)$ are unital.
- $c_0(\Gamma)^* \cong \ell^1(\Gamma)$, $\ell^1(\Gamma)^* \cong \ell^\infty(\Gamma)$, $\ell^\infty(\Gamma)^* \cong ba(\Gamma)$. We skip the proof of this.
- Any function $\Gamma \rightarrow X$ is continuous.
- We can write $C_c(C, A) = A\Gamma$.

A mean on Γ is identified to a finitely additive probability measure on Γ .

Example 4.11 (Subexponential growth).

Example 4.12 (Elementary amenable groups). Every finite group and every abelian group is an discrete amenable group. Their finite number of elementary operations of four types (subgroups, quotients, extensions, direct limits) are also amenable, and they are called *elementary amenable groups*.

Example 4.13 (Nonabelian free groups). The free group F_2 is not amenable. In particular, every discrete group containing F_2 , for example, F_n or $SL(n, \mathbb{Z})$ for $n \geq 2$, is not amenable.

Proof. Suppose there is a left-invariant finitely additive probability measure m on $F_2 = \langle a, b \rangle$. Its value $m(S)$ of finite set S is zero. Let $S(a)$ be the set of all reduced words in F_2 starting with a , and similarly for other generators. Then, $F_2 = S(a) \sqcup aS(a^{-1})$ implies $m(S(a)) + m(S(a^{-1})) = 1$ and

$$F_2 = \{e\} \sqcup S(a) \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1})$$

implies

$$1 = m(F_2) = m(S(a)) + m(S(a^{-1})) + m(S(b)) + m(S(b^{-1})) = 2,$$

a contradiction. □

Remark 4.14. We have the following implications for a discrete group Γ :

$$\text{elementary amenable} \implies \text{amenable} \implies \text{no non-abelian free subgroup}$$

The question about the converses had been a longstanding problem, which is also called the *von Neumann-Day problem*. Grigorchuk's group(1984) is an amenable that is not elementary amenable, and the Tarski monster group(1980) is a non-amenable group without non-abelian free subgroup. The Tits alternative states that the three statements are equivalent for a linear group over a field.

Theorem 4.15 (Nuclearity). *Let Γ be a discrete group. Then, the following are equivalent:*

- Γ is amenable.
- $C_r^*(\Gamma)$ is nuclear.
- $C^*(\Gamma)$ is nuclear.

Proof. (a) \implies (b) Take a Følner sequence $F_k \subset \Gamma$. Define $e_{s,t}, p_k \in B(\ell^2(\Gamma))$ such that

$$e_{s,t} \xi := \langle \xi, \delta_t \rangle \delta_s, \quad p_k := \sum_{s \in F_k} e_{s,s}.$$

Define $C_r^*(\Gamma) \xrightarrow{\varphi_k} p_k B(\ell^2(\Gamma)) p_k \xrightarrow{\psi_k} C_r^*(\Gamma)$ such that

$$\varphi(a) := p_k a p_k, \quad \psi(e_{s,t}) := \frac{1}{|F_k|} \lambda_{st^{-1}}.$$

Since

$$\begin{aligned} (p_k \lambda_s p_k) \delta_t &= p_k \lambda_s \delta_t 1_{F_k}(t) = p_k \delta_{st} 1_{F_k}(t) = \delta_{st} 1_{F_k}(t) 1_{F_k}(st) = 1_{sF_k \cap F_k}(st) \delta_{st} \\ &= \sum_{p \in sF_k \cap F_k} \langle \delta_{st}, \delta_p \rangle \delta_p = \sum_{p \in sF_k \cap F_k} e_{p, s^{-1}p} \delta_t, \end{aligned}$$

we have

$$\psi_k \circ \varphi_k(\lambda_s) = \psi_k \left(\sum_{p \in sF_k \cap F_k} e_{p, s^{-1}p} \right) = \sum_{p \in sF_k \cap F_k} e_{p, s^{-1}p} \frac{1}{|F_k|} \lambda_s = \frac{|sF_k \cap F_k|}{|F_k|} \lambda_s$$

for all $s \in \Gamma$. Therefore, $\psi_k \circ \varphi_k$ converges to the identity in the point-norm topology, and $C_r^*(\Gamma)$ is nuclear.

(b) \Rightarrow (a) Consider a net of c.c.p. maps $C_r^*(\Gamma) \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} C_r^*(\Gamma)$ which converges to the identity in the point-norm topology. By the Arveson extension theorem, we have $\tilde{\varphi}_\alpha : B(\ell^2(\Gamma)) \rightarrow M_{n_\alpha}$ which extends φ_α . Then, the bounded net $\psi \circ \tilde{\varphi}_\alpha : B(\ell^2(\Gamma)) \rightarrow C_r^*(\Gamma) \subset L(\Gamma)$ has a limit point in the point- σ -weak topology. By taking a subnet, denote by Φ the limit. Then, $\Phi(a) = a$ for $a \in C_r^*(\Gamma)$. See the following diagram.

$$\begin{array}{ccccc} & B(\ell^2(\Gamma)) & & & \\ & \uparrow & \searrow \tilde{\varphi}_\alpha & \nearrow \Phi := \lim_\alpha \psi \circ \tilde{\varphi}_\alpha & \\ C_r^*(\Gamma) & \xrightarrow{\quad} & C_r^*(\Gamma) & \hookrightarrow & L(\Gamma) = C_r^*(\Gamma)'' \\ & \searrow \varphi_\alpha & \downarrow M_{n_\alpha}(\mathbb{C}) & \nearrow \psi_\alpha & \end{array}$$

Consider a vector state $\omega_{\delta_e} : x \mapsto \langle x \delta_e, \delta_e \rangle$ on $L(\Gamma) \subset B(\ell^2(\Gamma))$. It is tracial because the right-hand side of

$$\omega_{\delta_e}(ab) = \langle ab \delta_e, \delta_e \rangle = \sum_{s,t} \langle a_s b_t \delta_{st}, \delta_e \rangle = \sum_s a_s b_{s^{-1}}$$

is symmetric. Then, the state $m := \omega_{\delta_e} \circ \Phi$ on $B(\ell^2(\Gamma))$ is left-invariant since $C_r^*(\Gamma)$ belongs to the multiplicative domain of m so that

$$m(\lambda_s x \lambda_s^*) = \omega_{\delta_e}(\Phi(\lambda_s x \lambda_s^*)) = \omega_{\delta_e}(\lambda_s \Phi(x) \lambda_s^*) = \omega_{\delta_e}(\Phi(x)) = m(x)$$

for $x \in B(\ell^2(\Gamma))$. Therefore, $m|_{\ell^\infty(\Gamma)}$ is an invariant mean on Γ .

(a) \Rightarrow (c) If Γ is amenable, then $C_r^*(\Gamma)$ is amenable and $C^*(\Gamma) = C_r^*(\Gamma)$, so we are done.

(c) \Rightarrow (a) It is known that a quotient of a nuclear C^* -algebra is nuclear, which requires a rather difficult proof in this stage, so we give a direct proof using the trick.

Suppose $C^*(\Gamma)$ is nuclear. Here we denote by $C_\lambda^*(\Gamma)$ and $C_\rho^*(\Gamma)$ the reduced group C^* -algebras in $B(\ell^2(\Gamma))$ generated by the image of the left and right regular representations. Then the maximal tensor products in the following diagram can be changed into the minimal tensor products:

$$\begin{array}{ccc} B(\ell^2(\Gamma)) \otimes_{\max} C^*(\Gamma) & & \\ \uparrow & \xleftarrow{i \times \rho} & \\ C_\lambda^*(\Gamma) \otimes_{\max} C_\rho^*(\Gamma) & \xrightarrow{\quad} & B(\ell^2(\Gamma)), \end{array}$$

where $\rho : C^*(\Gamma) \rightarrow C_\rho^*(\Gamma) \subset B(\ell^2(\Gamma))$ is the right regular representation. Then, we have two facts: the vertical arrow is injective because the minimal tensor products can be implemented spatially in a large Hilbert space (i.e. the conclusion (c) of the inclusion problem in Corollary 3.8 always holds for

minimal tensor product), and $\rho(C^*(\Gamma))' = C_\rho^*(\Gamma)' = L(\Gamma)$. By the trick, the embedding $C_\lambda^*(\Gamma) \hookrightarrow L(\Gamma)$ is extended to $B(\ell^2(\Gamma)) \rightarrow L(\Gamma)$. Then, as we did in the proof of (b) \Rightarrow (a), pullback the tracial vector state ω_{δ_e} of $L(\Gamma)$ to get a left-invariant state of $B(\ell^2(\Gamma))$, which contains $\ell^\infty(\Gamma)$. The restriction on $\ell^\infty(\Gamma)$ implies that Γ is amenable. \square

Remark 4.16. It is known that a locally compact group G is amenable if and only if $C_r^*(G)$ is nuclear and admits a tracial state [Ng, 2015]. Note that if Γ is discrete, then the reduced group C^* -algebra $C_r^*(\Gamma)$ has a tracial state $a \mapsto \langle a\delta_e, \delta_e \rangle$.

5 August 4

5.1 Hilbert C^* -modules

Definition 5.1. Let A be a C^* -algebra. An (right) *inner product A -module* is a complex vector space E together with

- (i) a ring anti-homomorphism $A^{\text{op}} \rightarrow \text{End}_{\mathbb{C}}(E)$
- (ii) an A -valued inner product $\langle -, - \rangle : E \times E \rightarrow A$.

On an inner product A -module E we have a scalar-valued norm given by $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$ for $\xi \in E$, and we say E is a *Hilbert A -module* if the norm is complete.

Note that we have the sesquilinear map to be linear in the second argument because E is a right A -module. Boundedness follows from the following proposition.

Proposition 5.2. A -valued norm $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$.

- (a) $|\langle \xi, \eta \rangle| \leq \|\xi\| \|\eta\|$ in A . In particular, $\|\langle \xi, \eta \rangle\| \leq \|\xi\| \|\eta\|$
- (b) $|\xi a| \leq \|\xi\| \|a\|$ in A . In particular, $\|\xi a\| \leq \|\xi\| \|a\|$.

Proof. (a) If we let $\|\xi\| = 1$ and $a := \langle \xi, \eta \rangle$, then it is clear from

$$\begin{aligned} 0 &\leq \langle \xi a - \eta, \xi a - \eta \rangle \\ &= a^* \langle \xi, \xi \rangle a - \langle \eta, \xi \rangle a - a^* \langle \xi, \eta \rangle + \langle \eta, \eta \rangle \\ &\leq a^* a - \langle \eta, \xi \rangle a - a^* \langle \xi, \eta \rangle + \langle \eta, \eta \rangle \\ &\leq -|\langle \xi, \eta \rangle|^2 + \langle \eta, \eta \rangle. \end{aligned}$$

- (b) If we let $\|\xi\| = 1$, then $\langle \xi a, \xi a \rangle \leq a^* \langle \xi, \xi \rangle a \leq a^* a$. □

Therefore, a right Hilbert A -module is a right Banach A -module, i.e. we have a homomorphism $A^{\text{op}} \rightarrow \mathcal{L}(E)$. Also, the right action $A^{\text{op}} \rightarrow \mathcal{L}(E)$ is always non-degenerate, and faithful if and only if E is full, i.e. $\langle E, E \rangle$ is dense in A .

Definition 5.3 (Adjointable operators). Let A be a C^* -algebra. Let E and E' be Hilbert A -modules. An *adjointable operator* is a linear operator $t : E \rightarrow E'$ such that there is a linear operator $t^* : E' \rightarrow E$ satisfying $\langle t\xi, \eta \rangle = \langle \xi, t^*\eta \rangle$ for all $\xi \in E$ and $\eta \in E'$. The set of adjointable operators are denoted by $B(E, E')$.

Proposition 5.4 (Adjointable operators). Let A be a C^* -algebra. Let E and E' be Hilbert A -modules.

- (a) The sum, scalar multiplication, composition, and adjoint of adjointable operators are adjointable.
- (b) An adjointable operator is A -linear and bounded.
- (c) $B(E) := B(E, E)$ is a C^* -algebra.
- (d) $\|t\xi\| \leq \|t\| \|\xi\|$ in A for $t \in B(E, E')$. In particular, $\|t\xi\| \leq \|t\| \|\xi\|$.

Proof. (a) Clear.

- (b) Let t be an adjointable operator from E to E' . It is A -linear because

$$\langle t(\xi a), \eta \rangle = \langle \xi a, t^*\eta \rangle = a^* \langle \xi, t^*\eta \rangle = a^* \langle t\xi, \eta \rangle = \langle (t\xi)a, \eta \rangle$$

for any $\eta \in E'$ implies $t(\xi a) = (t\xi)a$.

Consider $\{\langle t\xi, - \rangle : \|\xi\| \leq 1\} \subset B(E', A)$ a family of operators between Banach spaces. Since it is pointwise bounded because $\|\langle t\xi, \eta \rangle\| \leq \|t^*\eta\|$ for each $\eta \in E'$, by the uniform boundedness principle, $\|\langle t\xi, \eta \rangle\| \leq C\|\eta\|$. Then, t is bounded because $\|t\xi\|^2 \leq \|\langle t\xi, t\xi \rangle\| \leq C\|t\xi\|$ for $\|\xi\| \leq 1$.

(c) Note that

$$\|t^*t\| \geq \sup \frac{\|\langle t^*t\xi, \xi \rangle\|}{\|\xi\|^2} = \sup \frac{\|\langle t\xi, t\xi \rangle\|}{\|\xi\|^2} = \|t\|^2.$$

Therefore, the C^* -identity holds and we have $\|t\| = \|t^*\|$. Since if a sequence of adjointable operators t_n is Cauchy then the sequence of their adjoints t_n^* is also Cauchy, $B(E, E)$ is a closed subspace of $L(E, E)$.

(d) (geometric series of Cauchy-Schwarz inequality). \square

Example 5.5 (Hilbert $C(X)$ -modules). Let X be a compact Hausdorff space. A *Banach bundle* over X is a continuous map $p : E \rightarrow X$ from a topological space E such that each fiber $p^{-1}(x)$ is given a compatible Banach space structure and satisfies

- (i) the summation $E \times_X E \rightarrow E$ is continuous,
- (ii) the scalar multiplication $E \times \mathbb{F} \rightarrow E$ is continuous,
- (iii) the norm function $X \rightarrow \mathbb{R}_{\geq 0}$ is continuous,
- (iv) for a net $e_i \in E$ if $p(e_i) \rightarrow x$ and $\|e_i\| \rightarrow 0$, then $e_i \rightarrow 0 \in E_x$.

If each fiber is a Hilbert space, then we call the Banach bundle as a *Hilbert bundle*.

- (a) There is an equivalence between the category of Hilbert bundles over X is equivalent to the category of Hilbert $C(X)$ -modules, via the functor $(E \rightarrow X) \mapsto \Gamma(E)$, the set of continuous sections of E .
- (b) Not every Hilbert bundle is locally trivial.
- (c) If the fibers have a common finite dimension, then the Hilbert bundle is locally trivial.

Proof sketch for (a). For every Hilbert bundle E over X , we can easily check the space of sections $\Gamma(E)$ is a Hilbert $C(X)$ -module. Conversely, let E be a Hilbert $C(X)$ -module. Consider the evaluation map $\text{ev}_x : C(X) \rightarrow \mathbb{C}$ and let $\mathfrak{m}_x := \text{ev}_x^{-1}(0)$ be the maximal ideal at $x \in X$. Then, we can show that $E\mathfrak{m}_x$ is a Hilbert $C(X)$ -submodule of E . If $\xi_n f_n$ converges to $\eta \in E$, then we can show that

$$\eta = \left(\lim_{n \rightarrow \infty} \eta(n^{-1} + |\eta|^{\frac{1}{2}})^{-1} \right) |\eta|^{\frac{1}{2}} \in E\mathfrak{m}_x.$$

Alternitively, we may apply the Cohen factorization theorem.

Now we define $E_x := E/E\mathfrak{m}_x$. We topologize the set $E := \bigcup_{x \in X} E_x$ by the quotient topology with quotient map $E \times X \rightarrow E$. Then, we can check the four conditions for Banach bundles. \square

Example 5.6. Let $A := C([0, 1])$ and $\mathcal{I} := C_0((0, 1])$ be Hilbert A -modules. The inclusion $\iota : \mathcal{I} \rightarrow A$ is not adjointable because $i = \langle \iota i, 1 \rangle = \langle i, \iota^* 1 \rangle$ implies $\iota^* 1 = 1$, which is impossible. The $A \oplus \mathcal{I} \rightarrow A \oplus \mathcal{I}$ that maps (ξ_1, ξ_2) to $(\xi_2, 0)$ is bounded but not adjointable.

Example 5.7 (Hilbert modules for noncommutative C^* -algebras). Let A be a C^* -algebra.

- (a) (Scalar change) Every Hilbert space H is exactly same to the Hilbert \mathbb{C} -module. The algebraic vector space tensor product $H \otimes_{\mathbb{C}} A$ has a natural inner product A -module structure. The completion $H_A := H \otimes A = H \otimes_{\mathbb{C}} A$ is a Hilbert A -module.
- (b) (Trivial Hilbert module) A^n is a A -module.
- (c) (Standard Hilbert module) $\ell^2(\mathbb{N}, A)$.

Definition 5.8 (Compact adjointable operators). Define $K(E, E')$ to be the closed linear subspace of $B(E, E')$ spanned by adjointable operators of the form $\theta_{\xi, \eta} := \xi \langle \eta, - \rangle$. The elements of $K(E, E')$ are often called *compact operators* even though they are not compact operators between Banach spaces.

Proposition 5.9 (Compact adjointable operators).

- (a) $K(E)$ is a closed ideal of $B(E)$.
- (b) $K(A) \cong A$.

Proof. (a) Note that

$$\theta_{\xi,\eta} \theta_{\xi',\eta} = \theta_{\xi(\eta,\xi'),\eta'}, \quad t \theta_{\xi,\eta} = \theta_{t\xi,\eta}, \quad \theta_{\xi,\eta} t = \theta_{\xi,t^*\eta}.$$

It shows that the linear span of $\theta_{\xi,\eta}$ is an ideal, and its closure is closed ideal of $B(E)$.

(b) Let $\mathcal{T} \subset A$ be the linear span of $\theta_{a,b}$. Consider the isometry $\mathcal{T} \rightarrow A : \theta_{a,b} \mapsto ab^*$. Since $\{ab^*\}$ is dense in A , we are done. □

5.2 Multipliers

Let A be a non-unital C^* -algebra.

Definition 5.10 (Multiplier algebra). Let A be a C^* -algebra. The *multiplier algebra* is a maximal C^* -algebra denoted by $M(A)$ which contains A as an essential ideal. We will show that $M(A)$ always uniquely exists up to $*$ -isomorphism.

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Proposition 5.11 (Ideal extension of representation). Let A, B, C be C^* -algebras, and suppose A is a closed ideal of B . Let E be a Hilbert C -module and $\pi : A \rightarrow B_C(E)$ be a representation.

- (a) There is a representation $\tilde{\pi} : B \rightarrow B_C(E)$ which extends π .
- (b) If π is non-degenerate, then the extension is unique.
- (c) If π is faithful and A is essential in B , then the extension is faithful.

Proof. (a) □

Proposition 5.12 (Idealizer description of multiplier algebra). Let $\pi : A \rightarrow B_A(E)$ be a non-degenerate injective representation. Let

$$B := \{b \in B_A(E) : b\pi(a), \pi(a)b \in \pi(A) \forall a \in A\}.$$

Then, π extends to a $*$ -isomorphism between $M(A)$ and B

Proof. □

Example 5.13.

- (a) Since the inclusion representation $K_A(E) \rightarrow B_A(E)$ is non-degenerate, the above proposition implies that $B_A(E)$ is the multiplier algebra of $K_A(E)$ (Kasparov).
- (b) $M(K(H)) \cong B(H)$.
- (c) $M(C_0(X)) \cong C_b(X)$ from $C_0(X) \rightarrow B(\ell^2(X))$.

Definition 5.14. A *morphism* from A to B is a non-degenerate $*$ -homomorphism $A \rightarrow M(B)$. Every morphism uniquely extends to a $*$ -homomorphism $M(A) \rightarrow M(B)$, and the injectivity is preserved. The composition is well-defined in the sense of extensions, and the non-degeneracy is proved by the following proposition.

Lemma 5.15 (Non-degeneracy). *For a representation $\pi : A \rightarrow B_B(E)$, TFAE:*

- (a) π is non-degenerate
- (b) π is the restriction to A of a unital $*$ -homomorphism $\tilde{\pi} : M(A) \rightarrow B_B(E)$ which is strictly continuous on the unit ball.
- (c) $\pi(e_\alpha) \rightarrow \text{id}_E$ strictly.

complemented Miščenko's closed range and complemented submodule theorem isometric surjective \Leftrightarrow unitary

tensor product

We have $K_A(H \otimes A) \cong K(H) \otimes A$. Note that $K(H) = \lim_{\rightarrow} M_n(\mathbb{C})$ is nuclear.

5.3 C^* -correspondence

Definition 5.16 (C^* -correspondences). Let A and B be C^* -algebras. A C^* -correspondence over A and B , or a Hilbert A - B -bimodule, is a Hilbert B -module E together with a $*$ -homomorphism $A \rightarrow B_B(E) : a \mapsto (\xi \mapsto a\xi)$, called the *left action*. The C^* -correspondence E is called *faithful* or *non-degenerate* if the left action is faithful or non-degenerate respectively.

Definition 5.17 (Representations of a C^* -correspondence). For a C^* -correspondence E over A , a *representation* of E on B is a pair (π, τ) of a $*$ -homomorphism $\pi : A \rightarrow B$ and a complex linear map $\tau : E \rightarrow B$ such that

$$\pi(\langle \xi, \eta \rangle) = \tau(\xi)^* \tau(\eta), \quad \tau(a\xi) = \pi(a)\tau(\xi), \quad \tau(\xi a) = \tau(\xi)\pi(a), \quad \xi \in E, a \in A.$$

In fact, the third condition is redundant because it follows from the first two, and τ is automatically contractive. The C^* -subalgebra of B generated by $\pi(A)$ and $\tau(E)$ is denoted by $C^*(\pi, \tau)$.

From now on, let E be a C^* -correspondence over a C^* -algebra A , and let (π, τ) be a representation of E on a C^* -algebra B .

Definition 5.18. We define complex linear maps

$$\tau_n : E^{\otimes n} \rightarrow B, \quad \psi : K(E) \rightarrow B, \quad \psi_n : K(E^{\otimes n}) \rightarrow B,$$

defined by

$$\tau_n(\xi_1 \otimes \cdots \otimes \xi_n) := \tau(\xi_1) \cdots \tau(\xi_n), \quad \xi_1, \dots, \xi_n \in E$$

and

$$\psi(\theta_{\xi, \eta}) := \tau(\xi)\tau(\eta)^*, \quad \xi, \eta \in E.$$

We have a representation (π, τ_n) of the interior tensor product $E^{\otimes n}$. In particular, $\tau_0 = \pi$ and $\tau_1 = \tau$. Since τ_n is also a representation, we can also define $\psi_n : K(E^{\otimes n}) \rightarrow B$.

Remark 5.19. For the well-definedness of ψ , see Proposition 4.6.3 in [Brown-Ozawa].

Lemma 5.20 (Injectivity on representations). *If π is injective, then τ_n and ψ_n are isometric.*

Definition 5.21 (Core). Let $\psi_n : K(E^{\otimes n}) \rightarrow B$ be the associated $*$ -homomorphism for $n \geq 0$. We define

$$B_n := \text{im } \psi_n = \psi_n(K(E^{\otimes n})) \subset B.$$

In particular, $B_0 = \pi(A)$ and $B_1 = \psi(K(E))$.

For a representation (π, τ) , we define the *Katsura ideal*

$$I_{(\pi, \tau)} := \pi^{-1}(B_0 \cap B_1) = \pi^{-1}(\pi(A) \cap \psi(K(E))).$$

5.4 Pimsner algebras

Proposition 5.22 (Universal representation). *We say a representation $(\tilde{\pi}, \tilde{\tau})$ of E on $C^*(\tilde{\pi}, \tilde{\tau})$ is universal if for every representation (π, τ) of E there is a unique $*$ -homomorphism $C^*(\tilde{\pi}, \tilde{\tau}) \rightarrow C^*(\pi, \tau)$ such that $\tilde{\pi}(a)$ and $\tilde{\tau}(a)$ are mapped to $\pi(a)$ and $\tau(a)$.*

- (a) A universal representation always exists.
- (b) A universal representation satisfies $I_{(\pi, \tau)} = 0$, that is, it is faithful and $B_0 \cap B_1 = 0$.
- (c) A universal representation admits a gauge action $\beta : \mathbb{T} \rightarrow \text{Aut}(C^*(\pi, \tau))$.

Proof. (a) Consider the ℓ^∞ -direct sum

$$\bigoplus_{(\pi, \tau)} C^*(\pi, \tau)$$

where (π, τ) runs through all representations of E . Then, it satisfies the universal property.

(b), (c) Assume that there is a special representation (π, τ) that satisfies each condition. (We will see the existence later) Let $\phi : C^*(\tilde{\pi}, \tilde{\tau}) \rightarrow C^*(\pi, \tau)$ be the $*$ -homomorphism obtained by the universality.

Suppose $I_{(\pi, \tau)} = 0$. Since (π, τ) is faithful, $(\tilde{\pi}, \tilde{\tau})$ is also faithful. Since $\pi(A) \cap \psi(K(E)) = 0$ and

$$\phi(\tilde{\pi}(a)) = \pi(a), \quad \phi(\tilde{\psi}(\theta_{\xi, \eta})) = \phi(\tilde{\tau}(\xi)\tilde{\tau}(\eta)^*) = \tau(\xi)\tau(\eta)^* = \theta_{\xi, \eta}$$

imply that $\tilde{\pi}(A) \cap \tilde{\psi}(K(E)) = 0$. □

Definition 5.23 (Fock representation). Let

$$\mathcal{F}(E) := \bigoplus_{n=0}^{\infty} E^{\otimes n}.$$

Then, we have a representation (π, τ) of E on $B(\mathcal{F}(E))$ as follows:

Theorem 5.24 (Gauge-invariant uniqueness theorem I). *If a representation (π, τ) satisfies $I_{(\pi, \tau)} = 0$ and admits a gauge action $\beta : \mathbb{T} \rightarrow \text{Aut}(C^*(\pi, \tau))$, then (π, τ) is universal, i.e. $C^*(\pi, \tau) \cong \mathcal{T}(E)$.*

Sketch. Step 1: B_{n+1} is a closed ideal of B_n and $B_{\leq n} \cap B_{n+1} = 0$.

$$0 \rightarrow B_{\leq n} \rightarrow B_{\leq n+1} \rightarrow B_{n+1} \rightarrow 0.$$

Step 2:

$$C^*(\pi, \tau)^\beta = \overline{\bigcup_{n \geq 0} B_{\leq n}}$$

Step 3: Enough to show injectivity only on the fixed point algebra. □

Denote the left action map by $\varphi : A \rightarrow B(E)$. We define the ideal

$$I(E) := \{a \in A : \varphi(a) \in K(E), ab = 0 \text{ for all } b \in \ker \varphi\}.$$

In other words, $I(E)$ is the largest ideal on which φ is faithfully mapped into $K(E)$.

Definition 5.25 (Covariant representation). We say a representation (π, τ) of E is said to be *covariant* if $\pi(a) = \psi(\varphi(a))$ on $a \in I(E)$.

Theorem 5.26 (Gauge-invariant uniqueness theorem II). *If a covariant representation (π, τ) is faithful and admits a gauge action $\beta : \mathbb{T} \rightarrow \text{Aut}(C^*(\pi, \tau))$, then (π, τ) is universal, i.e. $C^*(\pi, \tau) \cong \mathcal{O}(E)$.*

second quantization representation $B_A(E) \rightarrow B_A(\mathcal{F}(E))$.

Definition 5.27. Let E be a C^* -correspondence over A . The *Toeplitz-Pimsner algebra* $\mathcal{T}(E)$ is defined as the image of the universal representation of E . The *Cuntz-Pimsner algebra* $\mathcal{O}(E)$ is defined as the C^* -generated by the image of the universal covariant representation of E .

The name of Toeplitz from the C^* -correspondence $C(S^1)$ over $C(S^1)$. The name of Cuntz from ${}_C\ell_{2C}^n$.
gauge action, universality, examples(crossed products, graph and Cuntz-Krieger, groupoid), nuclearity and exactness