Algebraic quantum field theory

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Contents

Chapter 1

Axiomatic: Osterwalder-Schrader, Wightman, Haag-Kastler

CFT

Statistical physics: Gibbs state by DLR equation, Lieb-Robinson bound, quantum theory

1.1

1.1 (Wightman axioms)**.** Let $\mathbb{R}^{1,d-1}_x$ be the Minkowski space and $\mathcal{P}^\uparrow_+$ the connected component of the Poincaré group. A *Wightman field* is a linear map $\phi : \mathcal{S}(\mathbb{R}^{1,d-1}_x) \to \mathrm{End}(\mathcal{D})$, where $\mathcal{D}$ is an inner product space with completion $\mathcal{H}$, such that

(i) Covariance: there is a representation $U : \mathcal{P}^\uparrow_+ \to U(\mathcal{H})$ such that $\mathrm{Ad}\, U(\gamma)\phi(f) = \phi(\gamma^* f)$,

(ii) Causality: if the supports of f and g are space-like separated, then $[\phi(f), \phi(g)] = 0$ on D,

(iii) Conicality:

(iv) Cyclicity: there is a Poincaré invaraint cyclic vector Ω in the sense that the span of the set $\{\phi(f_1)\cdots\phi(f_n)\Omega\}$ is dense in $\mathcal{D}$.

1.2 (Free massive bosonic fields)**.** Let $m > 0$, called the mass of a scalar particle, and let $d = 1+1$ be a positive integer, called the dimension. Note $\mathcal{P}^\uparrow_+ = \mathbb{R}^{1,1} \rtimes \mathbb{R}$. On the mass shell $H_m := \{p \in \mathbb{R}^{1,1}_p : (p,p) = p_0^2 - p_1^2 = m^2,\ p_0 > 0\}$, the induced metric is Riemannian with the volume form $(p_1^2 + m^2)^{-\frac{1}{2}}\, dp_1$, so we can define $L^2(H_m)$.

For $f \in \mathcal{S}(\mathbb{R}^{1,1}_x)$, consider the restriction of the Fourier transform $\hat{f} \in L^2(H_m)$.

$$\hat{f}(p) = \int_{\mathbb{R}^{1,1}} e^{i(x,p)} f(x)\, d^2x, \qquad p \in H_m,$$

where d^2x is the Lebesgue measure on $\mathbb{R}^{1,1}$, which is Lorentz invariant. Via the Bosonic Fock space construction $\mathcal{F}^+(L^2(H_m))$, we define a operator-valued distribution

$$\phi : f \mapsto a^\dagger(\hat{f}) + a(\hat{f}),$$

$\phi(f)$ is defined densely on $\mathcal{F}^+(L^2(H_m))$.

(a) ϕ is covariant.

(b) ϕ is local.

(c) ϕ has positive energy.

(d) ϕ admits a vaccum.

(e) ϕ has linear energy bound. In particular, it defines a Araki-Haag-Kastler net.

Proof. (a) Consider a representation $U_m : \mathcal{P}_+^\uparrow \to U(L^2(H_m))$ on $L^2(H_m)$, defined by

$$(U_m(a,\Lambda)\Psi)(p) := e^{i(a,p)}\Psi(\Lambda^{-1}p), \qquad (a,\Lambda) \in \mathcal{P}_+^\uparrow,\ \Psi \in L^2(H_m).$$

The action $U_m : \mathcal{P}_+^\uparrow \to U(L^2(H_m))$ is extended to $\Gamma(U_m) : \mathcal{P}_+^\uparrow \to U(\mathcal{F}^+(L^2(H_m)))$, called the second quantization. Then, since $\mathcal{F}(a,\Lambda)\mathcal{F}^{-1}$ maps

$$(p \mapsto \int e^{i(x,p)} f(x)\, d^2x) \quad \text{to} \quad (p \mapsto \int e^{i(x,p)} f(\Lambda^{-1}(x-a))\, d^2x = e^{i(a,p)} \int e^{i(x,\Lambda^{-1}p)} f(x)\, d^2x),$$

so it is covariant.

(b) Define the left and right wedges

$$W_L := \{x \in \mathbb{R}^{1,1} : |x_0| \le -x_1\}, \qquad W_R := \{x \in \mathbb{R}^{1,1} : |x_0| \le x_1\}.$$

Suppose f and g are Schwartz functions supported on W_L and W_R respectively. Write

$$[\phi(f), \phi(g)] = [a^\dagger(\hat{f}), a(\hat{g})] + [a(\hat{f}), a^\dagger(\hat{g})].$$

analytic continuation and residue theorem... If $f(x) \neq 0$ and $g(y) \neq 0$, then x and y are contained in the interior of W_L and W_R, so $(x,y) > 0$.

□