Harmonic Analysis

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4.1 Calderón-Zygmund operator of convolution type

4.1 (Calderón-Zygmund decomposition of sets)**.** Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf = \sup_n E_n|f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|}\int_{Q_x} |f| > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \le \frac{1}{\lambda}\|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\mathbb{R}^d\setminus\Omega)} \le \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|}\int_{Q_x} |f| \le 2^d\lambda.$$

4.2 (Calderón-Zygmund decomposition of functions)**.** Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|}\int_{Q_x} |f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \le 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof. □

4.3 (Calderón-Zygmund operator of convolution type). Let $T : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that

$$Tf(x) = \int K(x-y)f(y)\,dy$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \operatorname{supp} f$. If T is L^2-bounded

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$$

and satisfies the *Hörmander condition*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \lesssim 1,$$

then it is called a *Calderón-Zygmund* operator.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \le \frac{4}{\lambda^2}\|Tg\|^2_{L^2(\Omega)} \le \frac{4C}{\lambda^2}\|g\|^2_{L^2(\Omega)} \le \frac{4C}{\lambda^2}\|g\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \le \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega^*} |Tb(x)|\,dx \le \frac{2}{\lambda}\sum_i \int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\operatorname{supp} b_i \subset Q_i$ and $\int b_i = 0$, we have

$$T b_i(x) = \int_{Q_i} K(x-y) b_i(y)\, dy = \int_{Q_i} [K(x-y) - K(x)] b_i(y)\, dy,$$

and

$$\int_{\mathbb{R}^d \setminus Q_i^*} |T b_i(x)|\, dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d \setminus Q_i^*} |K(x-y) - K(x)|\, dx\, dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.2 L^2-boundedness of truncated integrals

4.3 Calderón-Zygmund operator of non-convolution type

standard kernels

Exercises

4.4 (Gradient size condition). Let $|\nabla K(x)| \lesssim \frac{1}{|x|^{d+1}}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

Chapter 5

Littlewood-Paley theory

Chapter 6

Multiplier theorems

Part III

Pseudo-differential operators

Part IV

Oscillatory integral operators