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1 Universal coefficient theorem

Lemma 1.1. *Suppose we have a flat resolution*

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

Then, we have a exact sequence

$$\cdots \rightarrow 0 \rightarrow \text{Tor}_1^R(A, B) \rightarrow P_1 \otimes B \rightarrow P_0 \otimes B \rightarrow A \otimes B \rightarrow 0.$$

Theorem 1.2. *Let R be a PID. Let C_\bullet be a chain complex of flat R -modules and G be a R -module. Then, we have a short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0,$$

which splits, but not naturally.

Proof 1. We have a short exact sequence of chain complexes

$$0 \rightarrow Z_\bullet \rightarrow C_\bullet \rightarrow B_{\bullet-1} \rightarrow 0$$

where every morphism in Z_\bullet and B_\bullet are zero. Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence

$$0 \rightarrow Z_\bullet \otimes G \rightarrow C_\bullet \otimes G \rightarrow B_{\bullet-1} \otimes G \rightarrow 0$$

and the associated long exact sequence

$$\rightarrow H_n(B; G) \rightarrow H_n(Z; G) \rightarrow H_n(C; G) \rightarrow H_{n-1}(B; G) \rightarrow H_{n-1}(Z; G) \rightarrow$$

where the connecting homomorphisms are of the form $(i_n: B_n \rightarrow Z_n) \otimes 1_G$ (It is better to think diagram chasing than a natural construction). Since morphisms in B and Z are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n(C; G) \rightarrow B_{n-1} \otimes G \rightarrow Z_{n-1} \otimes G \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(H_n, G) \rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(H_{n-1}, G) \rightarrow 0.$$

For splitting, □

Proof 2. Since R is PID, we can construct a flat resolution of G

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow G \rightarrow 0.$$

Since modules in C_\bullet are flat so that the tensor product functors are exact and $P_1 \rightarrow P_0$ and $P_0 \rightarrow G$ induce the chain maps, we have a short exact sequence of chain complexes

$$0 \rightarrow C_\bullet \otimes P_1 \rightarrow C_\bullet \otimes P_0 \rightarrow C_\bullet \otimes G \rightarrow 0.$$

Then, we have the associated long exact sequence

$$\rightarrow H_n(C; P_1) \rightarrow H_n(C; P_0) \rightarrow H_n(C; G) \rightarrow H_{n-1}(C; P_1) \rightarrow H_{n-1}(C; P_0) \rightarrow .$$

Since flat tensor product functor commutes with homology functor from chain complexes, we have

$$\rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n(C; G) \rightarrow H_{n-1} \otimes P_1 \rightarrow H_{n-1} \otimes P_0 \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(G, H_n) \rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(G, H_{n-1}) \rightarrow 0.$$

□

Proof 3. By tensoring G , we get the following diagram.

$$\begin{array}{ccccc}
 H_n \otimes G & & & & H_{n-1} \otimes G \\
 \searrow & & & & \nearrow \\
 & \text{coker } \partial_{n+1} \otimes G & \text{ker } \partial_{n-1} \otimes G & & \\
 \nearrow & \searrow & \nearrow & \searrow & \\
 C_n \otimes G & & \text{im } \partial_n \otimes G & & C_{n-1} \otimes G \\
 \searrow & \nearrow & & \nearrow & \\
 & \text{Tor}_1(H_{n-1}, G) & & &
 \end{array}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernels are preserved, but monomorphisms and kernels are not. Especially, $\text{coker } \partial_{n+1} \otimes G = \text{coker}(\partial_{n+1} \otimes 1_G)$ is important.

Consider the following diagram.

$$\begin{array}{ccccccc}
 H_n(C; G) & & H_n \otimes G & & & & \\
 \searrow & & \downarrow & & & & \\
 & \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G & & \text{monic!} & \\
 & \downarrow & \nearrow & \searrow & \uparrow & \searrow & \\
 & \text{im } \partial_n \otimes G & & \text{im}(\partial_n \otimes 1_G) & & C_{n-1} \otimes G & \\
 \nearrow & & & & & \nearrow & \\
 \text{Tor}_1(H_{n-1}, G) & & & & & &
 \end{array}$$

Since $\text{ker } \partial_{n-1}$ is free,

If we show $\text{im}(\partial_n \otimes 1_G) \rightarrow \text{ker } \partial_{n-1} \otimes G$ is monic, then we can get

$$\begin{aligned}
 H_n(C; G) &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{im}(\partial_n \otimes 1_G)) \\
 &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{ker } \partial_{n-1} \otimes G).
 \end{aligned}$$

2 Fundamental differential geometry

2.1 Manifold and Atlas

Definition 2.1. A *locally Euclidean space* M of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to

an open subset of \mathbb{R}^d .

Definition 2.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 2.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of \mathbb{R}^d . A chart is often written by a pair (U, φ) .

Definition 2.4. An *atlas* \mathcal{F} is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 2.5. A *differentiable manifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

2.2 Definition of Differentiable Structure

Definition 2.6. An atlas \mathcal{F} is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each *transition function* $\tau_{\alpha\beta}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 2.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 2.8. An differentiable atlas \mathcal{F} is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in \mathcal{F} , then $(U, \varphi) \in \mathcal{F}$.

Definition 2.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manifold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M . When the charts is already equipped on M , it is natural to define a function $f: M \rightarrow \mathbb{R}$ differentiable if the functions $f \circ \varphi_\alpha^{-1}: \mathbb{R}^d \rightarrow \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas \mathcal{F} , then f is also differentiable on any atlases

which is equivalent to \mathcal{F} by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 2.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 2.10. A continuous function $f : M \rightarrow N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

2.3 Curves

Definition 2.11. For $f : M \rightarrow \mathbb{R}$ and (U, ϕ) a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 2.12. Let $\gamma : I \rightarrow M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi : M \rightarrow N$ be a smooth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt}f \circ \gamma.$$

Proposition 2.1. Let $\gamma : I \rightarrow M$ be a smooth curve on a manifold M . The notation $\dot{\gamma}^\mu$ is not confusing thanks to

$$(\dot{\gamma})^\mu = (\dot{\gamma}^\mu).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt}x^\mu \circ \gamma.$$

2.4 Connection computation

$$\begin{aligned}
\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\
&= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma_{\mu\nu}^\lambda \partial_\lambda) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda \right) \partial_\nu.
\end{aligned}$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ .

$$Y_{;\mu}^\nu = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \quad Y_{;\mu}^\nu = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda.$$

Theorem 2.2. For Levi-civita connection for g ,

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Proof.

$$\begin{aligned}
(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} \\
(\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma_{jk}^l g_{li} - \Gamma_{jl}^l g_{kl} \\
(\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}
\end{aligned}$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$. Thus,

$$\Gamma_{ij}^l g_{kl} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

$$\Gamma_{ij}^l = \frac{1}{2}g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

□

2.5 Geodesic equation

Theorem 2.3. If c is a geodesic curve, then components of c satisfies a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0.$$

Proof. Note

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\lambda \partial_\lambda) = (\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu + \dot{\gamma}^\nu \dot{\gamma}^\lambda \Gamma_{\nu\lambda}^\mu) \partial_\mu.$$

Since

$$\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu = \dot{\gamma}(\dot{\gamma}^\mu) = d\dot{\gamma}^\mu(\dot{\gamma}) = d\dot{\gamma}^\mu \circ d\gamma \left(\frac{\partial}{\partial t} \right) = d\dot{\gamma}^\mu \left(\frac{\partial}{\partial t} \right) = \ddot{\gamma}^\mu,$$

we get a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0$$

for each μ . □

3 Vector calculus on spherical coordinates

$$\begin{aligned} V &= (V_r, V_\theta, V_\phi) \\ &= V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi} \quad (\text{normalized}) \\ &= V_r \frac{\partial}{\partial r} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} V_\phi \frac{\partial}{\partial \phi} \quad (\Gamma(TM)) \\ &= V_r dr + r V_\theta d\theta + r \sin \theta V_\phi d\phi \quad (\Omega^1(M)) \\ &= r^2 \sin \theta V_r d\theta \wedge d\phi + r \sin \theta V_\theta d\phi \wedge dr + r V_\phi dr \wedge d\theta \quad (\Omega^2(M)) \\ \nabla \cdot V &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (r \sin \theta V_\theta) + \frac{\partial}{\partial \phi} (r V_\phi) \right] \\ \Delta u &= \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} u \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} u \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} u \right) \right] \end{aligned}$$

Let (ξ, η, ζ) be an orthogonal coordinate that is *not* normalized. Then,

$$\sharp = g = \text{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2)$$

$$\hat{x} = \|\partial_x\|^{-1} \partial_x = \|\partial_x\| dx = \|\partial_y\| \|\partial_z\| dy \wedge dz$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r d\theta, \quad r \sin \theta d\phi, \quad (r d\theta) \wedge (r \sin \theta d\phi), \quad (r \sin \theta d\phi) \wedge (dr), \quad (dr) \wedge (r d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[\frac{1}{\|\partial_x\|} \frac{\partial}{\partial x} \cdot -, \frac{1}{\|\partial_y\|} \frac{\partial}{\partial y} \cdot -, \frac{1}{\|\partial_z\|} \frac{\partial}{\partial z} \cdot - \right] \\
\text{curl} : \nabla &= \left[\frac{1}{\|\partial_y\| \|\partial_z\|} \left(\frac{\partial}{\partial y} (\|\partial_z\| \cdot -) - \frac{\partial}{\partial z} (\|\partial_y\| \cdot -) \right), \right. \\
&\quad \frac{1}{\|\partial_z\| \|\partial_x\|} \left(\frac{\partial}{\partial z} (\|\partial_x\| \cdot -) - \frac{\partial}{\partial x} (\|\partial_z\| \cdot -) \right), \\
&\quad \left. \frac{1}{\|\partial_x\| \|\partial_y\|} \left(\frac{\partial}{\partial x} (\|\partial_y\| \cdot -) - \frac{\partial}{\partial y} (\|\partial_x\| \cdot -) \right) \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} (\|\partial_y\| \|\partial_z\| \cdot -), \frac{\partial}{\partial y} (\|\partial_z\| \|\partial_x\| \cdot -), \frac{\partial}{\partial z} (\|\partial_x\| \|\partial_y\| \cdot -) \right] \\
\Delta &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} \left(\frac{\|\partial_y\| \|\partial_z\|}{\|\partial_x\|} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\|\partial_z\| \|\partial_x\|}{\|\partial_y\|} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\|\partial_x\| \|\partial_y\|}{\|\partial_z\|} \frac{\partial}{\partial z} \right) \right]
\end{aligned}$$

$$\text{grad} = \frac{1}{\|\cdot\|^1} (\nabla) \|\cdot\|^0$$

$$\text{curl} = \frac{1}{\|\cdot\|^2} (\nabla \times) \|\cdot\|^1$$

$$\text{div} = \frac{1}{\|\cdot\|^3} (\nabla \cdot) \|\cdot\|^2$$

4 Bundles

Show that S^n has a nonvanishing vector field if and only if n is odd.

Solution. Since S^n is embedded in \mathbb{R}^{n+1} , the tangent bundle TS^n can be considered as an embedded manifold in $S^n \times \mathbb{R}^{n+1}$ which consists of (x, v) such that $\langle x, x \rangle = 1$ and $\langle x, v \rangle = 0$, where the inner product is the standard one of \mathbb{R}^{n+1} .

Suppose n is odd. We have a vector field $(x_1, x_2, \dots, x_{n+1}; x_2, -x_1, \dots, -x_n)$ which is nonvanishing.

Conversely, suppose we have a nonvanishing vector field X . Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \rightarrow S^n \times \mathbb{R}^{n+1} \rightarrow \phi \mathbb{R}^{n+1} \rightarrow S^n.$$

The last map can be defined since X is nowhere zero. Since this map satisfies $\langle x, \phi(x) \rangle = 0$ for all $x \in S^n$, we can define homotopies from ϕ to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree, $+1$, so n is odd. \square

Proposition 4.1. *Independent commuting vector fields are realized as partial derivatives in a chart.*

Proposition 4.2. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent involutive vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent. (Maybe)*

Proposition 4.3. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent commuting vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

Proposition 4.4. *An immersed manifold is locally an inverse image of a regular value.*

Proposition 4.5. *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

Proof. It uses tubular neighborhood. Pontryagin construction? □

Proposition 4.6. *An immersed manifold is locally a linear subspace in a chart.*

Proposition 4.7. *Distinct two points on a connected manifold are connected by embedded curve.*

Proof. Let $\gamma : I \rightarrow M$ be a curve connecting the given two points, say p, q .

*Step [.1]*Constructing a piecewise linear curve For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\text{im } \gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$ for all $1 \leq i \leq n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i, j) : U_{t_i} \cap U_{t_j} \neq \emptyset\}$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \leq i \leq n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q .

*Step [.2]*Smoothing the curve □

Proposition 4.8. *Let M is an embedded manifold with boundary in N . Any kind of sections on M can be extended on N .*

Proposition 4.9. *Every ring homomorphism $C^\infty(M) \rightarrow \mathbb{R}$ is obtained by an evaluation at a point of M .*

Proof. Suppose $\phi : C^\infty(M) \rightarrow \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. \square

Proposition 4.10. *The set of points that is geodesically connected to a point is open.*