Representation Theory

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Contents

Part I

Finite group representations

Chapter 1

Character theory

1.1 Irreducible representations

1.1 (Definition of group representations).

1.2 (Interwining maps).

1.3 (Subrepresentations). We say *invariant* or *stable*

1.4 (Irreducible representations). indecomposable and irreducible

1.5 (Maschke's theorem). Let G be a finite group and k a field of characteristic coprime to $|G|$. Let (ρ, V) be a finite-dimensional representation of G over k. Let W be an invariant subspace of V.

(a) There is an invariant subspace $W^\perp$ of V that is a complement of W.

(b) Every finite-dimensional representation of G over k is isomorphic to the direct sum of irreducible representations of G over k.

(c) If $k = \mathbb{R}$ or $\mathbb{C}$, then there is a inner product on V such that $W^\perp$ is orthogonal to W. With this innerproduct, $\rho(g)$ is orthogonal (resp. unitary) for all $g \in G$.

1.6 (Schur's lemma). Let G be a finite group and k a field. Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible representations of G over k. Let $\psi \in \hom_G(V_1, V_2)$ be an interwining map.

(a) If V_1 and V_2 are not isomorphic, then $\psi = 0$.

(b) If V_1 and V_2 are isomorphic, then ψ is a homothety.

1.2 Group algebra

1.7 (Modules and representations). ring <-> group module <-> representation finitely generated <-> finite dimensional

1.8 (Wedderburn's theorem). central idempotents dimension computation

1.9 (Group algebra). regular representation $k[G]$-module and G-representation correspondence

(a) $\mathbb{C}[G]$ is the direct sum of all irreducible representations.

(b) $|G| = \sum_{[V]\in\widehat{G}} (\dim V)^2$.

1.10. The number of irreducible representations and the number of conjugacy classes double counting on $Z(\mathbb{C}[G])$.

1.3 Characters

1.11 (Space of class functions)**.** Ring and inner product structure on the space of class functions.

(a) $\dim \hom_G(V_1, V_2) = \langle \chi_{V_1}, \chi_{V_2} \rangle$.

(b) Irreducible characters form an orthonormal basis of the space of class functions.

1.12 (Characters classify representations)**.** Let G be a finite group and let $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of G over $\mathbb{C}$.

$$\mathrm{Tr} : \mathbf{Rep}(G) \to \{\text{finite sum of irreducible characters}\}$$

surjectivity: trivial injectivity: Suppose two characters are equal. Maschke -> all characters are sum of irreducible characters Schur -> orthogonality, so the coefficients are all equal irreducible-factor-wisely construct an isomoprhism.

1.13 (Character table)**.** computation of matrix elements by character table abelian group, 1dim rep lifting

S^3	e	(12)	(123)
1	1	1	1
ε	1	-1	1
ρ	2	0	-1

tensoring, complex, real symmetric, exterior
the dual inner product: conjugacy check relation to normal subgroups center of rep
algebraic integer dim of irrep divides group order burnside pq theorem

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young tableux

2.2 Linear groups over finite fields

GL2 and SL2 over finite fields

2.3 Induced representations

induction and restriction of reps (from and to subgroup) frobenius reciprocity, mackey theory

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