

# Foundations of Calculus

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# Preface

the main objectives the audience the structure of the book how to use this book acknowledgements  
references

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**Part I**

**Sequences**

# Chapter 1

## Metric spaces

### 1.1 Metric spaces

**1.1** (Definition of metric spaces). Let  $X$  be a set. A *metric* is a function  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  such that

- (i)  $d(x, y) = 0$  if and only if  $x = y$ , (nondegeneracy)
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , (symmetry)
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ . (triangle inequality)

A pair  $(X, d)$  of a set  $X$  and a metric on  $X$  is called a *metric space*. We often write it simply  $X$ .

- (a) A normed space  $X$  is a metric space with a metric defined by  $d(x, y) := \|x - y\|$ .
- (b) A subset of a metric space is a metric space with a metric given by restriction.

**1.2** (System of open balls). A metric is often misunderstood as something that measures a distance between two points and belongs to the study of geometry. The main function of a metric is to make a system of small balls, sets of points whose distance from specified center points is less than fixed numbers. The balls centered at each point provide a concrete images of “system of neighborhoods at a point” in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly.

Note that taking either  $\varepsilon$  or  $\delta$  in analysis really means taking a ball of the very radius. Investigation of the distribution of open balls centered at a point is now an important problem.

Let  $X$  be a metric space. A set of the form

$$\{y \in X : d(x, y) < \varepsilon\}$$

for  $x \in X$  and  $\varepsilon > 0$  is called an *open ball centered at  $x$  with radius  $\varepsilon$*  and denoted by  $B(x, \varepsilon)$  or  $B_\varepsilon(x)$ .

**1.3** (Convergence and continuity in metric spaces). Let  $\{x_n\}_n$  be a sequence of points on a metric space  $(X, d)$ . We say that a point  $x$  is a *limit* of the sequence or the sequence *converges to  $x$*  if for arbitrarily small ball  $B(x, \varepsilon)$ , we can find  $n_0$  such that  $x_n \in B(x, \varepsilon)$  for all  $n > n_0$ . If it is satisfied, then we write

$$\lim_{n \rightarrow \infty} x_n = x,$$

or simply  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

A function  $f : X \rightarrow Y$  between metric spaces is called *continuous at  $x \in X$*  if for any ball  $B(f(x), \varepsilon) \subset Y$ , there is a ball  $B(x, \delta) \subset X$  such that  $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ . The function  $f$  is called *continuous* if it is continuous at every point on  $X$ .

- (a) A sequence  $x_n$  in a metric space  $X$  converges to  $x \in X$  if and only if  $d(x_n, x)$  converges to zero.
- (b) Let  $f : X \rightarrow Y$  be a function between two metric spaces. If there is a constant  $C$  such that  $d(x, y) \leq Cd(f(x), f(y))$  for all  $x$  and  $y$  in  $X$ , then  $f$  is continuous. In this case,  $f$  is particularly called *Lipschitz continuous* with the *Lipschitz constant*  $C$ .

1.4 (Separable metric spaces). separable iff second countable iff lindelof

## 1.2 Normed spaces

banach space

## 1.3 Open sets and closed sets

convergence, limit point

## 1.4 Compact sets

Bolzano-Weierstrass

## 1.5 Connected sets

## Exercises

## Problems

# Chapter 2

## Real sequences

### 2.1 Monotone sequences

preserving inequalities limsup and liminf monotone convergence

### 2.2 Extended real numbers

2.1 (Operations in the extended real numbers). We can extend addition (except  $\infty + (-\infty)$ ), subtraction, multiplication (except  $\infty \times 0$ ), division (except dividing by zero).

2.2 (Limits in the extended real numbers).

### 2.3 Asymptotic analysis

sufficiently large asymptotic expressions growth and decay

Approximate sequences( $\varepsilon/3$ )

2.3 (Change of limits).

$$|a_n - a| \leq |a_n - b_{mn}| + |b_{mn} - b_m| + |b_m - a|$$

$$\limsup_m \sup_n |a_n - b_{mn}| = 0$$

$$\lim_n |b_{mn} - b_m| = 0$$

$$a_n = b_{mn} + c_{mn} \leq b_{mn} + \varepsilon$$

## Exercises

2.4.

2.5 (Newton method).

## Problems

1. Every real sequence  $(a_n)_{n=1}^{\infty}$  has a subsequence  $(a_{n_k})_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n$ .



# Chapter 3

## Series

### 3.1 Absolute convergence

3.1 (Unconditional convergence).

### 3.2 Convergence tests

comparison limit comparison cauchy condensation integral....

ratio root

3.2 (Abel transform).

$$A_n(B_n - B_{n-1}) + (A_n - A_{n-1})B_{n-1} = A_n B_n - A_{n-1} B_{n-1}$$
$$\sum_{m < k \leq n} A_k b_k = A_n B_n - A_m B_m - \sum_{m < k \leq n} a_k B_{k-1}.$$

abel test

3.3 (Dirichlet test).

3.4 (Mertens' theorem). If  $\sum_{k=0}^{\infty} a_k$  converges to  $A$  absolutely and  $\sum_{k=0}^{\infty} b_k$  converges to  $B$ , then their Cauchy product  $\sum_{k=0}^{\infty} c_k$  with  $c_k := \sum_{l=0}^k a_l b_{k-l}$  converges to  $AB$ . Let

$$A_n := \sum_{k=0}^n a_k, \quad B_n := \sum_{k=0}^n b_k, \quad \text{and} \quad C_n := \sum_{k=0}^n c_k.$$

*Proof.* Write

$$|C_n - AB| \leq |C_n - A_n B_n| + |A_n B_n - AB|.$$

Since the limit of the second term  $|A_n B_n - AB| \rightarrow 0$  is clear, we claim  $|C_n - A_n B_n| \rightarrow 0$ .

Fix any  $\varepsilon > 0$ . Note that  $|B_n|$  is bounded by some  $M > 0$ . Write for some  $m$ ,

$$\begin{aligned} |C_n - A_n B_n| &= \left| \sum_{k=0}^n a_k (B_n - B_{n-k}) \right| \\ &\leq \left| \sum_{k=0}^m a_k (B_n - B_{n-k}) \right| + \left| \sum_{k=m+1}^n a_k (B_n - B_{n-k}) \right| \\ &\leq \sum_{k=0}^m |a_k| |B_n - B_{n-k}| + \sum_{k=m+1}^n |a_k| \cdot 2M. \end{aligned}$$

Since  $\sum_k a_k$  converges absolutely, we can take  $m$  such that

$$\sum_{k=m+1}^{\infty} |a_k| < \frac{\varepsilon}{2M}.$$

By taking limit  $n \rightarrow \infty$ , we have

$$\limsup_{n \rightarrow \infty} |C_n - A_n B_n| \leq 0 + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have  $\lim_n |C_n - A_n B_n| = 0$ .

□

## Exercises

3.5 (Cesàro mean).

3.6 (Recursive sine sequence). Let  $a_{n+1} = \sin a_n$  and  $a_n = 1$ . We can use  $\sin x = x - \frac{x^3}{6} + O(x^5)$ .

$$a_n = \sqrt{3}n^{-\frac{1}{2}} - \frac{3\sqrt{3}}{20}n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}).$$

3.7 (Convergence rates of recursive sequences). If  $a_{n+1} = a_n - f(a_n)$ ,  $f(0) = 0$ ,  $f(x) > 0$  for  $0 < x < \varepsilon$ ,  $f \in C^2$ ? then

$$f'(a_n) \sim \lim_{x \rightarrow 0+} \frac{f'(x)^2}{f''(x)f(x)} \frac{1}{n}.$$

## Problems

1. If  $a_n \rightarrow 0$ , then  $\frac{1}{n} \sum_{k=1}^n a_k \rightarrow 0$ . (Cesàro mean)
2. If  $a_n \geq 0$  and  $\sum a_n$  diverges, then  $\sum \frac{a_n}{1+a_n}$  also diverges.
3. If  $a_n \geq 0$  and  $\sum a_n < \infty$ , then there are sequences  $b_n \downarrow 0$  and  $\sum c_n < \infty$  such that  $a_n = b_n c_n$ .  
(Very special case of the Cohen factorization)

# **Part II**

# **Functions**

# Chapter 4

## Continuity

### 4.1 Intermediate and extreme value theorems

left and right limits semicontinuous

### 4.2 Various continuities

Lipschitz uniform cauchy

### Exercises

### Problems

1. The set of local minima of a convex real function is connected.
2. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. The equation  $f(x) = c$  cannot have exactly two solutions for every constant  $c \in \mathbb{R}$ .
3. A continuous function that takes on no value more than twice takes on some value exactly once.
4. Let  $f$  be a function that has the intermediate value property. If the preimage of every singleton is closed, then  $f$  is continuous.
5. If a continuous function  $f : [0, \infty) \rightarrow \mathbb{R}$  has a limit at infinity, then it is uniformly continuous.
6. If  $f : [0, 1]^2 \rightarrow \mathbb{R}$  is continuous, then  $g : [0, 1] \rightarrow \mathbb{R}$  defined by  $g(x) := \max_{y \in [0, 1]} f(x, y)$  is continuous.

## Chapter 5

# Differentiation

### 5.1 Differentiability

5.1 (L'hospital's theorem).

### 5.2 Monotonicity and convexity

### 5.3 Taylor expansion

5.2 (Rolle's theorem). Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function that is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

- (a) If  $f(a) = f(b) = 0$ , then there is  $c \in (a, b)$  such that  $f'(c) = 0$ .
- (b) Suppose  $f$  is  $(n + 1)$ -times differentiable. If  $f(a) = f'(a) = \dots = f^{(n)}(a) = 0$  and  $f(b) = 0$ , then there is  $c \in (a, b)$  such that  $f^{(n+1)}(c) = 0$ .

*Proof.* (a) If  $f \equiv 0$ , then it is clear. If not, we may assume there is  $x \in (a, b)$  such that  $f(x) > 0$  by multiplying  $-1$ . Since  $f$  is continuous, by the extreme value theorem, there is  $c \in (a, b)$  such that  $c$  attains the maximum of  $f$ . Then,  $f'(c) = 0$ .

(b) By the induction, we have  $c_n \in (a, b)$  such that  $f^{(n)}(c) = 0$ . By applying Rolle's theorem (the part (a)) for  $f^{(n)}$ , we have  $c_{n+1} \in (a, c_n)$  such that  $f^{(n+1)}(c_{n+1}) = 0$ .  $\square$

5.3 (Mean value theorem).

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

5.4 (Taylor theorem).

### 5.4 Smooth functions

### Exercises

5.5 (Variations on the mean value theorem). Let  $f$  be a differentiable function on the unit closed interval.

- (a) If  $f(0) = 0$  there is  $c$  such that  $cf'(c) = f(c)$ . (Flett)
- (b) If  $f(0) = 0$  there is  $c$  such that  $cf(c) = (1 - c)f'(c)$ .

5.6 (Dini derivatives).

5.7 (Darboux theorem).

## Problems

1. If  $\lim_{x \rightarrow \infty} f(x) = a$  and  $\lim_{x \rightarrow \infty} f'(x) = b$ , then  $a = 0$ .
2. Let  $f$  be a real  $C^2$  function with  $f(0) = 0$  and  $f''(0) \neq 0$ . Define a function  $\xi$  such that  $f(x) = xf'(\xi(x))$  with  $|\xi| \leq |x|$ , we have  $\xi'(0) = 1/2$ .
3. Let  $f$  be a  $C^2$  function such that  $f(0) = f(1) = 0$ . We have  $\|f\| \leq \frac{1}{8}\|f''\|$ .
4. A smooth function such that for each  $x$  there is  $n$  having the  $n$ th derivative vanish is a polynomial.
5. If a real  $C^1$  function  $f$  satisfies  $f(x) \neq 0$  for  $x$  such that  $f'(x) = 0$ , then in a bounded set there are only finite points at which  $f$  vanishes.
6. Let a real function  $f$  be differentiable. For  $a < a' < b < b'$  there exist  $a < c < b$  and  $a' < c' < b'$  such that  $f(b) - f(a) = f'(c)(b - a)$  and  $f(b') - f(a') = f'(c')(b' - a')$ .
7. Let  $f : [1, \infty) \rightarrow \mathbb{R}$  satisfy that  $f(1) = 1$  and  $f'(x) = (x^2 + f(x)^2)^{-1}$ . Show that  $\lim_{x \rightarrow \infty} f(x)$  exists in the open interval  $(1, 1 + \frac{\pi}{4})$ .
8. If  $f : (0, \infty) \rightarrow \mathbb{R}$  is  $C^2$  and satisfies  $f'(x) \leq 0 < f(x)$  for all  $x > 0$ , then the boundedness of  $f''$  implies  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$ .
9. If a function  $f : [0, 1] \rightarrow \mathbb{R}$  that is continuous on  $[0, 1]$  and differentiable on  $(0, 1)$  satisfies  $f(0) = 0$  and  $0 \leq f'(x) \leq 2f(x)$ , then  $f$  is identically zero.
10. For  $C^2$  function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we have  $\|f'\|^2 \leq 4\|f\|\|f''\|$ .
11. For a smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f'''(x) < 0$ , we have  $\frac{f'(x)+f'(y)}{2} < \frac{f(x)-f(y)}{x-y}$  for all  $x \neq y \in \mathbb{R}$ .

# Chapter 6

## Integration

### 6.1 Riemann integral

tagged partition

### 6.2 Henstock-Kurzweil intergral

bounded compact support  $\leftrightarrow$  lebesgue

### 6.3 Improper integral

### 6.4 Fundamental theorem of calculus for continuous functions

### Exercises

### Problems

1. Find the value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x) dx \right)$ .
2. Find all  $a > 0$  and  $b > 0$  such that  $\int_0^\infty x^{-b} |\tan x|^a dx$  converges.
- \*3. If  $xf'(x)$  is bounded and  $x^{-1} \int_0^x f \rightarrow L$  then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .
4. Show that for a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  we have  $\int_0^1 x^2 f(x) dx = \frac{1}{3} f(c)$  for some  $c \in [0, 1]$ .

## **Part III**

# **Functional sequences**



# Chapter 7

## Continuous functions

### 7.1 Uniform convergence

7.1. Let  $X$  be a compact metric space.

(a)  $C(X)$  is complete.

*Proof.* (a) Suppose  $f_n$  is a Cauchy sequence in  $C(X)$ . Since  $f_n$  is pointwise Cauchy, we have a function  $f$  on  $X$  such that  $f_n \rightarrow f$  pointwisely. We first claim that  $f_n \rightarrow f$  uniformly. Fix  $\varepsilon > 0$ . Write

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\| + |f_m(x) - f(x)|, \quad n, m \geq 0, x \in X. \end{aligned}$$

Since  $f_n$  is uniformly Cauchy, there is  $n_0$  such that

$$|f_n(x) - f(x)| < \varepsilon + |f_m(x) - f(x)|, \quad n, m > n_0, x \in X.$$

Taking the pointwise limit  $m \rightarrow \infty$ , we have

$$|f_n(x) - f(x)| \leq \varepsilon, \quad n > n_0, x \in X.$$

Taking the supremum over  $x \in X$  and limit superior  $n \rightarrow \infty$ , we obtain

$$\limsup_{n \rightarrow \infty} \|f_n - f\| \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we have the uniform limit  $f_n \rightarrow f$ .

Now we claim  $f$  is continuous. Let  $a \in X$  and fix  $\varepsilon > 0$ . Divide the error as

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &\leq 2\|f - f_n\| + |f_n(x) - f_n(a)|, \quad n \geq 0, x \in X. \end{aligned}$$

Using the uniform convergence  $f_n \rightarrow f$ , we can fix  $n$  such that

$$|f(x) - f(a)| < \varepsilon + |f_n(x) - f_n(a)|, \quad x \in X.$$

Then, taking limit superior  $x \rightarrow a$  on the both-hand sides, we get

$$\limsup_{x \rightarrow a} |f(x) - f(a)| \leq \varepsilon.$$

Since  $\varepsilon > 0$  has been arbitrarily taken,

$$\lim_{x \rightarrow a} |f(x) - f(a)| = 0,$$

hence the continuity.

(b)

□

## 7.2

7.2 (Partition of unity).

7.3 (Urysohn lemma).

7.4 (Tietze extension).

## 7.3 Arzela-Ascoli theorem

## 7.4 Stone-Weierstrass theorem

7.5 (Bernstein polynomial). We want to show  $\mathbb{R}[x]$  is dense in  $C([0, 1], \mathbb{R})$ . Let  $f \in C([0, 1], \mathbb{R})$  and define *Bernstein polynomials*  $B_n(f) \in \mathbb{R}[x]$  for each  $n$  such that

$$B_n(f)(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

(a)  $B_n(f)$  uniformly converges to  $f$  on  $[0, 1]$ .

(b) There is a sequence  $p_n \in \mathbb{R}[x]$  with  $p_n(0) = 0$  uniformly convergent to  $x \mapsto |x|$  on  $[-1, 1]$ .

*Proof.* (b) Let

$$B_n(x) := \sum_{k=0}^n \left|1 - \frac{2k}{n}\right| \binom{n}{k} (1-2x)^k (2x-1)^{n-k}.$$

Since  $B_n(x) \rightarrow |x|$  uniformly on  $[-1, 1]$  and  $B_n(0) \rightarrow 0$ , we have  $B_n(x) - B_n(0) \rightarrow |x|$  uniformly on  $[-1, 1]$ .  $\square$

7.6 (Taylor series of square root). We want to show the absolute value is approximated by polynomials in  $C([-1, 1], \mathbb{R})$  in another way. Let

$$f_n(x) := \sum_{k=0}^n a_k (x-1)^k$$

be the partial sum of the Taylor series of the square root function  $\sqrt{x}$  at  $x = 1$ .

(a) By Abel's theorem,  $f_n$  uniformly converges to  $\sqrt{x}$  on  $[0, 1]$

(b) There is a sequence  $p_n \in \mathbb{R}[x]$  with  $p_n(0) = 0$  uniformly convergent to  $x \mapsto |x|$  on  $[-1, 1]$ .

7.7 (Proof of Stone-Weierstrass theorem). Let  $X$  be a compact Hausdorff space and  $S \subset C(X, \mathbb{R})$ . We say that  $S$  *separates points* if for every distinct  $x$  and  $y$  in  $X$  there is  $f \in S$  such that  $f(x) \neq f(y)$ , and that  $S$  *vanishes nowhere* if for every  $x$  in  $X$  there is  $f \in S$  such that  $f(x) \neq 0$ .

Let  $\mathcal{A} = \overline{S\mathbb{R}[S]}$  be the real Banach subalgebra of  $C(X, \mathbb{R})$  generated by  $S$ .

(a)  $\mathcal{A}$  is a lattice.

(b)  $\mathcal{A}$  is dense in  $C(X, \mathbb{R})$ .

Locally compact version, complex version

## 7.8. Some examples

(a)  $z\mathbb{R}[z]$  is dense in  $C([1, 2], \mathbb{R})$ .

(b)  $\mathbb{C}[z]$  is dense in  $C([0, 1], \mathbb{C})$ .

(c)  $z\mathbb{C}[z, \bar{z}]$  is dense in  $C(\mathbb{T}, \mathbb{C})$ .

## Exercises

7.9 (Weierstrass' nowhere differentiable function).

## Problems

- \*1. Show that a sequence of functions  $f_n : [0, 1] \rightarrow [0, 1]$  that satisfies  $|f(x) - f(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$  has a uniformly convergent subsequence.
2. Show that for a sequence of differentiable functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  that satisfies  $|f'_n(x)| \leq 1$  for all  $n \geq 1$  and  $x \in \mathbb{R}$  its pointwise limit is continuous if it exists.
3. Show that a sequence of  $C^1$  functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $|f'_n(x)| \leq x^{-\frac{1}{2}}$  for  $x \in (0, 1]$  and  $\int_0^1 f_n(x) dx = 0$  for all  $n \geq 1$  has a uniformly convergent subsequence.

## Chapter 8

# Differentiable functions

### 8.1 Differentiable class

$C^1$  is Banach: Let a sequence  $f_n \in C^1$  satisfies  $f_n \rightarrow f$  and  $f'_n \rightarrow g$  uniformly. Write

$$\begin{aligned} \left| \frac{f(x) - f(a)}{x - a} - g(a) \right| &\leq \left| \frac{f(x) - f(a)}{x - a} - \frac{f_n(x) - f_n(a)}{x - a} \right| + \left| \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) \right| + |f'_n(a) - g(a)| \\ &\leq \frac{2\|f_n - f\|}{|x - a|} + \left| \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) \right| + \|f'_n - g\|, \quad n \geq 0, x \neq a. \end{aligned}$$

For the second term, by the mean value theorem, there is  $c \in [x, a] \cup [a, x]$  such that

$$\left| \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) \right| = |f'_n(c) - f'_n(a)| \leq 2\|f'_n - g\| + |g(c) - g(a)|, \quad n \geq 0, x \neq a.$$

Thus,

$$\left| \frac{f(x) - f(a)}{x - a} - g(a) \right| \leq \frac{2\|f_n - f\|}{|x - a|} + |g(c) - g(a)| + 3\|f'_n - g\|, \quad n \geq 0, x \neq a.$$

Taking limit superior  $n \rightarrow \infty$  and  $x \rightarrow a$ , from the continuity of  $g$  it follows that

$$\lim_{x \rightarrow a} \left| \frac{f(x) - f(a)}{x - a} - g(a) \right| = 0.$$

Therefore,  $f' = g$ .

### 8.2 Hölder spaces

### 8.3 Analytic functions

Power series uniform convergence and absolute convergence, Abel theorem? differentiation convergence of radius, complex domain sum, product, composition, reciprocal? closed under uniform convergence identity theorem

### Problems

1. Show that if  $f : (-1, 1) \rightarrow \mathbb{R}$  is a smooth function such that  $|f^{(n)}(x)| \leq 1$  for all  $n \geq 1$  uniformly then  $f$  is analytic.

## Chapter 9

# Integrable functions

### 9.1

9.1 (Lebesgue criterion of Riemann integrability).

**Part IV**

**Multivariable Calculus**

## Chapter 10

# Fréchet derivatives

### 10.1 Tangent spaces

10.1 (Vector fields).

### 10.2 Inverse function theorem

# Chapter 11

## Differential forms

### 11.1 Multilinear algebra

11.1 (Tensor product).

11.2 (Wedge product).

11.3 (One-forms).

11.4 (Multiple integral). volume forms, stone weierstrass and fubini

### 11.2 Vector calculus

11.5 (Exterior derivative).

11.6 (Musical isomorphisms).

11.7 (Inner product of differential forms). ONB

11.8 (Hodge star operator). Identification of 2-forms and vector fields

11.9 (Gradient, curl, and divergence).

11.10 (Potentials).

11.11 (Vector calculus identities).

### Exercises

11.12 (Multivariable Taylor's theorem). Symmetric product

11.13 (Vector analysis in two dimension).

11.14 (Geometric algebra).



## Chapter 12

# Stokes theorems

### 12.1 Local coordinates

12.1 (Spherical coordinates). Let  $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = 0, y \geq 0\}$ .

$$(x, y, z) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$$

for  $(r, \theta, \varphi) \in (0, \infty) \times (0, \pi) \times (0, 2\pi)$ . Orthonormal bases are

$$\left( \partial_r, \frac{1}{r} \partial_\theta, \frac{1}{r \sin \theta} \partial_\varphi \right),$$

$$(dr, r d\theta, r \sin \theta d\varphi),$$

$$(r^2 \sin \theta d\theta \wedge d\varphi, r \sin \theta d\varphi \wedge dr, r dr \wedge d\theta).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}.$$

*Proof.* Write  $df$  in the orthonormal basis

$$\begin{aligned} df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi \\ &= \left( \frac{\partial f}{\partial r} \right) dr + \left( \frac{1}{r} \frac{\partial f}{\partial \theta} \right) r d\theta + \left( \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) r \sin \theta d\varphi. \end{aligned}$$

After taking the Hodge star operator

$$\begin{aligned} *df &= \left( \frac{\partial f}{\partial r} \right) r^2 \sin \theta d\theta \wedge d\varphi + \left( \frac{1}{r} \frac{\partial f}{\partial \theta} \right) r \sin \theta d\varphi \wedge dr + \left( \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) r dr \wedge d\theta \\ &= r^2 \sin \theta \frac{\partial f}{\partial r} d\theta \wedge d\varphi + \sin \theta \frac{\partial f}{\partial \theta} d\varphi \wedge dr + \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} dr \wedge d\theta, \end{aligned}$$

the differential is computed as

$$\begin{aligned} d * df &= d \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right) d\theta \wedge d\varphi + d \left( \sin \theta \frac{\partial f}{\partial \theta} \right) d\varphi \wedge dr + d \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right) dr \wedge d\theta \\ &= \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right] dr \wedge d\theta \wedge d\varphi, \end{aligned}$$

so that we have

$$\begin{aligned}\Delta f &= *d*df = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}\end{aligned}$$

□

## 12.2 Integration on curves and surfaces

12.2 (Line integral).

12.3 (Surface integral).

## 12.3 Stokes theorems

12.4 (Bump functions).

12.5 (Partition of unity).

12.6.