Lebesgue Theory

Ikhan Choi

April 7, 2022

Contents

Part I

Measure theory

Chapter 1

Measures and σ-algebras

1.1 Definition of measures

Chapter 2

Carathéodory extension

2.1 (Outer measures)**.** Let Ω be a set. An *outer measure* on Ω is a function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ with $\mu^*(\varnothing) = 0$ such that

(i) if $E_1 \subset E_2$, then $\mu^*(E_1) \le \mu^*(E_2)$, (monotonicity)

(ii) $\mu^*(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu^*(E_i)$, (countable subadditivity)

for any $\{E_i\}_{i=1}^\infty \subset \mathcal{P}(\Omega)$.

(a) A function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ with $\mu^*(\varnothing) = 0$ is an outer measure if and only if $\mu^*(E) \le \sum_{i=1}^\infty \mu^*(E_i)$ whenever $E \subset \bigcup_{i=1}^\infty E_i$.

(b) Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ with $\varnothing \in \mathcal{A}$. If a function $\rho : \mathcal{A} \to [0, \infty]$ satisfies $\rho(\varnothing) = 0$, then we can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ by defining as

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^\infty \rho(A_i) : E \subset \bigcup_{i=1}^\infty A_i,\ A_i \in \mathcal{A} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

2.2 (Carathéodory measure)**.** Let μ^* be an outer measure on a set Ω. A subset $A \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

for every subset $E \subset \Omega$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is complete. We call μ the *Carathéodory measure* constructed from ρ.

2.3 (Carathéodory extension theorem)**.** Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ with $\varnothing \in \mathcal{A}$. Let $\rho : \mathcal{A} \to [0,\infty]$ with $\rho(\varnothing) = 0$. Consider two conditions

(i) $A \subset \bigcup_{i=1}^\infty A_i$ implies $\rho(A) \le \sum_{i=1}^\infty \rho(A_i)$,

(ii) for any $\varepsilon > 0$ and B,A there are A_1,A_2 such that $B \cap A \subset A_1$, $B \setminus A \subset A_2$ and $\rho(B) + \varepsilon > \rho(A_1) + \rho(A_2)$.

Let $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0,\infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) $\mu^*|_{\mathcal{A}} = \rho$ if (i) is satisfied.

(b) $\mathcal{A} \subset \mathcal{M}$ if (ii) is satisfied.

Proof. (a) Clearly $\mu^*(A) \le \rho(A)$ for $A \in \mathcal{A}$.

We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{A_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty A_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(A_i) \ge \rho(A).$$

(b) Let $E \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. Then, $E \subset \bigcup_{i=1}^\infty A_i$ and $A_i \cap A \subset A_{i,1}$ and $A_i \setminus A \subset A_{i,2}$ such that

$$\begin{aligned}\mu^*(E) + \varepsilon > \sum_{i=1}^\infty (\rho(A_i) + \frac{\varepsilon}{2^{i+1}}) &> \sum_{i=1}^\infty \rho(A_{i,1}) + \sum_{i=1}^\infty \rho(A_{i,2}) \\ &\ge \mu^*(E \cap A) + \mu^*(E \setminus A).\end{aligned}$$

□

2.4 (Carathéodory extension from semi-ring)**.** Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a semi-ring of sets on a set X. A function $\rho : \mathcal{A} \to [0,\infty]$ with $\rho(\varnothing) = 0$ is called a *pre-measure* if

(i) $\rho(\bigsqcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \rho(A_i)$, (disjoint countable subadditivity)

(ii) $\rho(\bigsqcup_{i=1}^n A_i) = \sum_{i=1}^n \rho(A_i)$, (finite additivity) for any $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ with $\bigsqcup_{i=1}^\infty A_i \in \mathcal{A}$ and $n \in \mathbb{N}$.

Let $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0, \infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) A pre-measure is a priori countably additive.

2.5 (Uniqueness of Carathéodory extensions)**.** The Carathéodory extension theorem provides with a uniqueness theorem for measures.

Monotone class lemma: alternative direct proof method without using Carathéodory extension.

Chapter 3

Measures on the real line

distribution functions helly's selection non-measurable set

Exercises

3.1. * A Lebesgue measurable set in $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression.

Part II

Lebesgue integral

Chapter 4

Measurable functions

4.1 Extended real numbers

4.2 Simple functions

Pointwise limit of simple functions is measurable.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

□

Every measurable extended real-valued function is a pointwise limit of simple functions.

4.1 (Egorov's theorem)**.** Let (Ω, μ) be a finite measure space. Let $(f_n : \Omega \to \mathbb{R})_n$ be a sequence of a.e. convergent measurable functions. For $\varepsilon > 0$, there exists a measurable $E_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ and f_n uniformly convergent on E_ε.

Proof. Assume $f_n \to 0$. The set of convergence is

$$\bigcap_{k>0} \bigcup_{n_0>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\},$$

which is a full set. We want to get rid of the dependence on the point x of n_0 in the union $\bigcup_{n_0>0}$. Since

$$\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \frac{1}{k} \,\}$$

is increasing as $n_0 \to \infty$ to a full set for each $k > 0$, we can find $n_0(k, \varepsilon)$ such that

$$\mu(\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\}) > \mu(\Omega) - \frac{\varepsilon}{2^k}.$$

Then,

$$\mu\Big(\bigcap_{k>0}\ \bigcap_{n\geq n_0}\{\,x : |f_n(x)| < \tfrac{1}{k}\,\}\Big) > \mu(\Omega) - \varepsilon.$$

If we define

$$E_\varepsilon := \bigcap_{k>0}\ \bigcap_{n\geq n_0}\{\,x : |f_n(x)| < \tfrac{1}{k}\,\},$$

then for any $k > 0$ and $x \in E_\varepsilon$, and with the $n_0(k, \varepsilon)$ we have chosen, we have

$$n \geq n_0 \quad \Rightarrow \quad |f_n(x)| < \frac{1}{k}.$$

□

Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0, \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > \tfrac{1}{k},$$

we have

$$\begin{aligned}
\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0}\bigcap_{n_0>0}\bigcup_{n>n_0}\{x : |f_n - f| > \tfrac{1}{k}\} \\
&= \bigcup_{k>0} \limsup_n \{x : |f_n - f| > \tfrac{1}{k}\}.
\end{aligned}$$

Since for every k we have

$$\begin{aligned}
\limsup_n \{x : |f_n - f| > \tfrac{1}{k}\} &\subset \limsup_{n>k} \{x : |f_n - f| > \tfrac{1}{n}\} \\
&= \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\},
\end{aligned}$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\}.$$

Chapter 5

Convergence theorems

5.1 Definition of Lebesgue integral

5.2 Convergence theorems

Stein: Egorov → BCT → Fatou → MCT → L1 is a measure
Stein: BCT + L1 is a measure → DCT
Folland: MCT → Fatou → DCT → BCT

5.3 Radon-Nikodym theorem

5.4 Modes of convergence

5.1 (Convergence in measure)**.** Let (X,μ) be a measure space. Let f_n be a sequence of measurable functions. If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.

Proof. We can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) > \tfrac{1}{2^k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k} - f| > \tfrac{1}{k}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. $\square$

Chapter 6

Product measures

6.1 Fubini-Tonelli theorem

6.2 Lebesgue measure on Euclidean spaces

Part III

Linear operators

Chapter 7

Lebesgue spaces

7.1 L^p spaces

7.2 L^2 spaces

7.3 Dual spaces

riesz representations

Chapter 8

Bounded linear operators

8.1 Continuity

Schur test

8.2 Density arguments

extension of operators

8.3 Interpolation

weak Lp, marcinkiewicz

Chapter 9

Convergence of linear operators

9.1 Translation and multiplication operators

9.2 Convolution type operators

approximation of identity

9.3 Computation of integral transforms

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega \to \mathbb{R}$ such that $f(x, y)$ and $\partial_{x_i} f(x, y)$ are both locally integrable in x and integrable y. Then,

$$\partial_{x_i} \int f(x, y)\, dy = \int \partial_{x_i} f(x, y)\, dy$$

where ∂_{x_i} denotes the weak partial derivative.

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}

(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞

(b) f is AC iff f' is L^1

(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

Lebesgue differentiation theorem