Differential Topology

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Part I

Category of smooth manifolds

Chapter 1

Tangent bundle

Consider the disjoint union of tangent spaces where the base points run through the whole manifold. This subsection discusses two ways of giving topology on the disjoint union to make it a vector bundle: one uses the local trivializations and coefficients to make a map to a Euclidean space, and the other inherits the topology of the ambient space. Both topologies are so sufficiently smooth that we can settle a smooth structure, which are identical.

The latter case is essentially same with the topologize pullback bundle with respect to the embedding $M \to \mathbb{R}^A$ for some A. We need to show it does not depend on the choice of the index set A. (We have checked that there is a natural choice of $A = C^\infty(M)$ with embedding $i : M \to \mathbb{R}^{C^\infty(M)}$ such that $\pi_f(i(x)) = f(x)$.)

Theorem 1.0.1. *The tangent bundle functor T preserves finitary products*

1.1 Pullbacks

When is the pullback possible?

Chapter 2

Embeddings

2.1 Immersion and submersion

2.2 Submanifolds

Recall that in the category of manifolds a monomorphism is an injective smooth map. A subobject in category theory is usually defined as an equivalence class of monomorphisms. Note that *submanifold is not an equivalent notion to subobject*.

Unlike topological spaces, we cannot start from set-theoretic functions. More precisely, there is no way to give a unique smooth structure as we did in giving initial topology. So, we want to consider a submanifold as a injective smooth map with a ready-made smooth structure on domian, and check if the smooth structures of domain and target are compatiable. In this reason, it is convenient to think that there will be no submanifold tests for a subset, but rather we have for smooth injections.

Example 2.2.1. Consider two smooth structures on a horizontal line in a plane

$$M = \{(x,0) \in \mathbb{R}^2\}$$

generated by two charts $\varphi_1 : (x,0) \mapsto x$ and $\varphi_2 : (x,0) \mapsto \sqrt[3]{x}$ respectively. In a smooth structure by φ_1, the inclusion is immersion, while φ_2 is not. That is, the first condition is not enough.

Example 2.2.2. There are two different smooth structures on a lemniscate. Both make the inclusion an immersion.

However, the following proposition suggests a good definition of submanifolds.

Proposition 2.2.1. *Let $i : M \to N$ be a smooth map. Then, TFAE:*

(a) *i is an injective immersion;*

(b) *a universal property is satisfied: a set-theoretical function $f : L \to M$ on a manifold L is smooth if $i \circ f : L \to N$ is smooth.*

Embedding.

Definition 2.2.1. A *smooth embedding* is a regular monomorphism in the category of smooth manifolds.

Proposition 2.2.2. *A smooth map $i : M \to N$ is a smooth embedding iff it is a topological embedding.*

whitney embedding

2.3 Sheaf theoretical aspects

Consider a (commutative unital) ring A such that every residue field is isomorphic to a field k. Familiar examples include any Banach algebras by the Gelfand-Mazur theorem. The unital condition is attached because we want to treat maximal ideals. Then, maximal ideals correspond to a nonzero multipicative linear functional to k because the residue field is k. Therefore, the set of maximal ideals can be identified with the set of all nonzero multiplicative linear functionals.

$$\begin{tikzcd}
\text{Point} \arrow[r] & \text{Multiplicative functional to } k \arrow[r] \arrow[l, dashed, bend left, "\text{case by case}"] & \text{Maximal ideal} \arrow[l, dashed, bend left, "\text{every residue field is isomorphic}"]
\end{tikzcd}$$

2.4 The ring $C^\infty(M)$

The following theorem is presented as the problem 1-C in the book of Milnor and Stasheff about characteristic classes.

Theorem 2.4.1. *Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.*

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. $\square$

Part II

De Rham Theory

Chapter 3

De Rham theorem

Chapter 4

Čech-de Rham complexes

Part III

Morse Theory

Part IV

Cobordism