

# Smooth Surfaces

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# Chapter 1

## Smooth manifolds

### 1.1 Parametrizations and Coordinates

For each text on classical differential geometry, the definitions frequently vary. In this note, we define as follows.

**Definition 1.1.1.** An  $m$ -dimensional parametrization is a smooth map  $\alpha : U \rightarrow \mathbb{R}^n$  such that

- (a)  $U \subset \mathbb{R}^m$  is open,
- (b) the Fréchet derivative  $d\alpha : U \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is injective everywhere, (immersion)
- (c)  $\alpha$  is one-to-one, (immersed submanifold)
- (d)  $\alpha$  is a homeomorphism onto its image. (embedded submanifold)

The Euclidean space  $\mathbb{R}^n$  is called the *ambient space*.

The first condition is necessary to avoid differentiating at points that are not in the interior of domain. Of course, it is possible to generalize the definition of differentiation on boundary points, but we will not introduce the notion because it goes out of the scope of this note.

The second condition is the most important one. This condition is paraphrased as follows: the set of column vectors of  $d\alpha|_x : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , which are exactly the partial derivatives  $\{\partial_i \alpha(x)\}_{i=1}^m \subset \mathbb{R}^n$  of  $\alpha$  with respect to each direction, is linearly independent at every point  $x \in U$ . Differential geometry does not consider parametrizations that fail this. This condition is vital for considering an appropriate and well-defined linear approximation of curves or surfaces. If it is not satisfied, every definition including tangent spaces in differential geometry can suffer.

Above the second condition, we call the image of a parametrization  $\alpha$  a (immersed) submanifold if the third condition is satisfied. If  $\alpha$  is not one-to-one, then two distinct ordered tuples of real numbers – we call the tuples coordinates – may represent the same point. Namely, this condition allows to use the inverse map  $\alpha^{-1} : \text{im } \alpha \rightarrow U$ . By this, we can recognize a parametrization as the inverse of a coordinate map. To describe a geometric object that cannot be covered by a single injective parametrization, such as a circle or a sphere, we can admit several parametrizations: see (3) in Example 1.1.

The fourth condition is stronger than the third, so we may assume just (1), (2), and (4) in Definition 1.1. It is sometimes called a proper patch in some references. This condition is introduced to exclude some exceptional examples such as lemniscates: see (4) Example 1.1.

**Definition 1.1.2.** A subset  $M \subset \mathbb{R}^n$  is called a *regular curve* (resp. *regular surface*) if there exists a one-dimensional (resp. two-dimensional) parametrization whose image is exactly  $M$ .

All curves and surfaces in this note are assumed to be regular: all of the four conditions are satisfied. One can notice that this definition is exactly same as an embedded submanifold of  $\mathbb{R}^n$  that can be covered by a single parametrization.

We also just often say that  $\alpha$  is a regular curve (resp. regular surface) for a *particular parametrization*  $\alpha$ . However, note that a curve or surface admits infinitely many parametrizations. We can solve many geometry or physics problems very easily by choosing an appropriate parametrization. Related to the choice of parametrizations, the following issues are always importantly considered when developing a theory of differential geometry:

- Well-definedness of a structure with respect to the dependency on parametrizations(coordinates).
- Existence of a parametrization(coordinates) that has nice properties we want.

**Definition 1.1.3.** Let  $M$  be the image of a parametrization  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ . The inverse  $\varphi : M \rightarrow U$  of a parametrization is called a *coordinate map*.

Coordinates and parametrizations have perfectly equivalent information except that they are inverses of each other (only if parametrization satisfies the injectivity). A parametrization tends to be taken in order for explicit computations in the ambient space like  $\mathbb{R}^3$  while coordinates are more usefully taken in verifications of abstract propositions. We use the term *reparametrization* to refer to nothing but a choice of another parametrization for the same curve or surface. As said, the choice of coordinate(parametrization) is important in differential geometry.

**Example 1.1.4.** (a) Let  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  be a map given by

$$\alpha(t) = (\cos t, \sin t, t).$$

Since  $d\alpha|_t(1) = \alpha'(t) = (-\sin t, \cos t, 1)$  is always nonzero so that  $d\alpha$  is injective everywhere,  $\alpha$  is a parametrization of the regular curve

$$\{(x, y, z) \in \mathbb{R}^3 : x = \cos z, y = \sin z\}.$$

Notice that it is enough to check  $\alpha'(t) \neq 0$  for a curve parametrization  $\alpha$  to show the injectivity of  $d\alpha$ . This curve is an example of circular helices.

(b) Let  $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$  be a map given by

$$\alpha(t) = (t^3, t^6, t^9).$$

Since  $d\alpha|_t(1) = \alpha'(t) = (3t^2, 6t^5, 9t^8)$  is zero when  $t = 0$ , it would be better to avoid calling  $\alpha$  a parametrization. Instead, the restrictions  $\alpha_+ : (0, \infty) \rightarrow \mathbb{R}^3$  and  $\alpha_- : (-\infty, 0) \rightarrow \mathbb{R}^3$  satisfy the axioms of parametrization at the beginning.

However, by reparametrization, we can show the image of  $\alpha$  is a regular curve, that is, we can find a parametrization that shares the image with  $\alpha$ , even though we sometimes say that  $\alpha$  is not a regular curve according to the fact  $\alpha'$  can vanish. Consider  $\beta : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by

$$\beta(t) = (t, t^2, t^3).$$

This map has the same image  $\text{im } \alpha = \text{im } \beta$ , but  $\beta'(t) = (1, 2t, 3t^2) \neq 0$  for all  $t \in \mathbb{R}$ .

(c) Let  $S^1$  be the unit circle in  $\mathbb{R}^2$ , precisely

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

It cannot be covered by a single parametrization, so we can consider two different parametrizations  $\alpha : (0, 2\pi) \rightarrow \mathbb{R}^2$  and  $\beta : (\pi, 3\pi) \rightarrow \mathbb{R}^2$  for  $S^1$ :

$$\alpha(t) = (\cos t, \sin t), \quad \beta(t) = (\cos t, \sin t).$$

Then, we have  $S^1 = \text{im } \alpha \cup \text{im } \beta$ . If we want to investigate the geometry of  $S^1$  near the point  $(1, 0)$ , we can choose  $\beta$  rather than  $\alpha$  because  $(1, 0) \notin \text{im } \alpha$ .

(d) Let  $\alpha, \beta : (0, 2\pi) \rightarrow \mathbb{R}^2$  be maps given by

$$\alpha(t) = (\sin t, \sin 2t), \quad \beta(t) = (\sin t, -\sin 2t).$$

They are one-to-one smooth maps such that  $d\alpha|_t(1) \neq 0 \neq d\beta|_t(1)$ , and one can check that they have common images; it is shaped like the character ‘ $\infty$ ’. A problem occurs when we think tangent vectors at  $(0, 0)$ :

$$\alpha'(0) = (1, 2), \quad \beta'(0) = (1, -2)$$

imply that the notion of tangent vectors at a point do depend on the choice of parametrizations. Generally, since one philosophy of parametrizations is to view them as *identifications* between curved spaces and flat Euclidean spaces, we want for them to have the images of open sets be open with respect to subspace topology. Thus, we assume parametrizations are homeomorphisms onto its images.

(e) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be any smooth function and  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a map given by

$$\alpha(x, y) = (x, y, f(x, y)).$$

Then,  $\alpha$  is a two-dimensional parametrization because

$$\begin{aligned} d\alpha|_{(x,y)}(1, 0) &= \frac{\partial \alpha}{\partial x}(x, y) = \left(1, 0, \frac{\partial f}{\partial x}(x, y)\right), \\ d\alpha|_{(x,y)}(0, 1) &= \frac{\partial \alpha}{\partial y}(x, y) = \left(0, 1, \frac{\partial f}{\partial y}(x, y)\right) \end{aligned}$$

are linearly independent for every  $(x, y) \in \mathbb{R}^2$ . A parametrization of this form is called a *Monge patch*. Notice that it is enough to check that the two partial derivatives  $\partial_x \alpha$  and  $\partial_y \alpha$  are linearly independent for a surface parametrization  $\alpha$ .

Let  $S = \text{im } \alpha$  be the regular surface determined by  $\alpha$ , and let  $p$  be a point on the surface  $S$  so that we have  $p = (x, y, f(x, y))$ . Associated with  $\alpha$ , a coordinate map  $\varphi : S \rightarrow \mathbb{R}^2$  is defined as the inverse of  $\alpha$ :

$$\varphi(p) := \alpha^{-1}(p) = (x, y).$$

This map  $\varphi$  consists of two real-valued functions on  $S$ ,

$$x : S \rightarrow \mathbb{R} : p \mapsto x, \quad y : S \rightarrow \mathbb{R} : p \mapsto y.$$

In this regard, we often write the coordinates  $\varphi$  as  $(x, y)$ .

(f) Let

$$S = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ or } y \neq 0\}.$$

The set  $S$  is a regular surface. Consider two different coordinates

$$\begin{aligned} (x, y) : S &\rightarrow S : (x, y) \mapsto (x, y), \\ (r, \theta) : S &\rightarrow \mathbb{R}_{>0} \times (-\pi, \pi) : (x, y) \mapsto \left(\sqrt{x^2 + y^2}, 2 \tan^{-1} \frac{y}{\sqrt{x^2 + y^2} + x}\right), \end{aligned}$$

where  $\tan^{-1}(t) := \int_0^t \frac{ds}{1+s^2}$ . They are the inverses of parametrizations  $\alpha : S \rightarrow \mathbb{R}^2$  and  $\beta : (0, \infty) \times (-\pi, \pi) \rightarrow \mathbb{R}^2$  defined by

$$\alpha(x, y) = (x, y), \quad \beta(r, \theta) = (r \cos \theta, r \sin \theta).$$

The coordinate maps  $(x, y)$  and  $(r, \theta)$  are called *Cartesian coordinates* and *polar coordinates* respectively.

## 1.2 Differentiation

Differentiation in differential geometry can be understood in many different viewpoints. We, here, review the two kinds of main usages of differentiation: differentiation of parametrizations, and differentiation by directional vectors. Do not forget that all differentiations in this note will be done thanks to the structure of the ambient space  $\mathbb{R}^n$ .

### 1.2.1 Differentiation of parametrizations

We introduce the notion of tangent spaces, geometrically the spaces of vectors that starts from each base point, by differentiation of parametrization. In this note we define tangent spaces by the image of the Fréchet derivative  $d\alpha$  of a parametrization  $\alpha$ .

**Definition 1.2.1.** Let  $M$  be the image of a parametrization  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $p \in M$  be a point and  $x = \alpha^{-1}(p) \in U$  be its coordinates. The *tangent space* of  $M$  at  $p$ , denoted by  $T_p M$ , is the image of the Fréchet derivative  $d\alpha|_x : \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

Since  $d\alpha_x$  is an injective linear transformation at every  $x$ , the tangent space  $T_p M$  is an  $m$ -dimensional linear subspace of  $\mathbb{R}^n$ . For a parametrization  $\alpha$ , the tangent space  $T_p M$  has a basis  $\{\alpha_i|_p\}_{i=1}^m$ , which is frequently chosen as the standard. Now, after a definition, we must show its consistency.

**Proposition 1.2.2.** Let  $M$  be the image of a parametrization  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $p \in M$  and  $v \in \mathbb{R}^n$ . Then, the two are equivalent:

- (a)  $v \in T_p M$ ;
- (b) there is a regular curve  $\gamma : I \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma'(0) = v$ .

In particular,  $T_p M$  is independent on the parametrization  $\alpha$ .

*Proof.* □

**Remark 1.2.3.** We can easily check that  $T_p \mathbb{R}^n = \mathbb{R}^n$  for any  $p \in \mathbb{R}^n$ . The notation  $T_p \mathbb{R}^n$  will be used to emphasize that a vector in  $\mathbb{R}^n$  is geometrically recognized to cast from the point  $p$ . Since  $T_p \mathbb{R}^n = \mathbb{R}^n = T_q \mathbb{R}^n$  for every pair of points  $p, q \in \mathbb{R}^n$ , summation and inner product of a vector in  $T_p \mathbb{R}^n$  and a vector in  $T_q \mathbb{R}^n$  make sense. This identification of tangent spaces are allowed *only for the case of linear spaces* such as  $\mathbb{R}^n$ . (In fact, the identification  $T_p \mathbb{R}^n = \mathbb{R}^n$  is *natural* in categorical language.)

**Remark 1.2.4.** One way to view tangent spaces is to see them as domains and codomains of Fréchet derivatives. For open sets  $U \subset \mathbb{R}^m$  and  $V \subset \mathbb{R}^n$ , the Fréchet derivative of a smooth map  $F : U \rightarrow V$  at  $x \in U$  is a linear transformation  $dF|_x : T_x U \rightarrow T_{F(x)} V$ . Since  $T_x U = \mathbb{R}^m$  and  $T_{F(x)} V = \mathbb{R}^n$ , the original definition on Euclidean spaces agrees with it. In this reason, the Fréchet derivative  $dF$  is also called a *tangent map*, *pushforward*, or *differential* in differential geometry.

**Notation.** Let  $\alpha$  be a parametrization for a regular curve or surface  $M$ . For derivatives of  $\alpha$ , we will use the following notations:

$$\partial_t \alpha = \alpha', \quad \partial_x \alpha = \alpha_x, \quad \partial_i \alpha = \alpha_i.$$

The set  $\{\alpha_i\}_i$  will be used to denote a basis of tangent space  $T_p M$ .

### 1.2.2 Differentiation by tangent vectors

**Definition 1.2.5.** Let  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a parametrization with  $M = \text{im } \alpha$ .

- (a) A *scalar field*, *smooth function*, or just a *function* is a function  $f : M \rightarrow \mathbb{R}$  such that  $f \circ \alpha : U \rightarrow \mathbb{R}$  is smooth.

(b) A *vector field* is a map  $X : M \rightarrow \mathbb{R}^n$  such that  $X \circ \alpha : U \rightarrow \mathbb{R}^n$  is smooth.

(c) A *tangent vector field* is a vector field  $X : M \rightarrow \mathbb{R}^n$  such that  $X|_p \in T_p M$ .

The set of tangent vector fields is often denoted by  $\mathfrak{X}(M)$ .

*Remark 1.2.6.* In general, the word *vector fields* are basically assumed to be tangent. However, we will distinguish them in this note.

The following proposition proves that the smoothness of functions and vector fields does not depend on parametrizations.

**Proposition 1.2.7.** *Let  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $\beta : V \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be parametrizations with same image  $M = \text{im } \alpha = \text{im } \beta$ . Then, the map  $\beta^{-1} \circ \alpha : U \rightarrow V$  is smooth.*

*Proof.*

□

*Remark 1.2.8.* The map  $\beta^{-1} \circ \alpha$  is called the *transition map*.

**Definition 1.2.9.** Let  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a parametrization  $M = \text{im } \alpha$ .

(a) The coordinate representation of a function  $f : M \rightarrow \mathbb{R}$  is

$$f \circ \alpha : U \rightarrow \mathbb{R}.$$

(b) The (external) coordinate representation of a vector field  $X : M \rightarrow \mathbb{R}^n$  is

$$X \circ \alpha : U \rightarrow \mathbb{R}^n.$$

(c) The coordinate representation of a tangent vector field  $X : M \rightarrow \mathbb{R}^n$  is

$$(X^1 \circ \alpha, \dots, X^m \circ \alpha) : U \rightarrow \mathbb{R}^m$$

$$\text{where } X = \sum_i X^i \alpha_i.$$

**Definition 1.2.10.** Let  $M$  be the image of a parametrization  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $v = \sum_i v^i \alpha_i|_p \in T_p M$  be a tangent vector at  $p = \alpha(x)$ . For a function  $f : M \rightarrow \mathbb{R}$ , its partial derivative is defined by

$$\partial_v f(p) := \sum_{i=1}^m v^i \partial_i (f \circ \alpha)(x) \in \mathbb{R}.$$

For a vector field  $X : M \rightarrow \mathbb{R}^n$ , its partial derivative is defined by

$$\partial_v X|_p := \sum_{i=1}^m v^i \partial_i (X \circ \alpha)(x) \in \mathbb{R}^n.$$

This definition is not dependent on parametrization  $\alpha$ .

**Proposition 1.2.11.** *Let  $M$  be the image of a parametrization. Let  $X$  be a tangent vector field on  $M$ .*

(a) *If  $f$  is a function, then so is  $\partial_X f$ .*

(b) *If  $Y$  is a vector field, then so is  $\partial_X Y$ .*

(c) *If  $Y$  is a tangent vector field, then so is  $\partial_X Y - \partial_Y X$ .*

*Proof.* (1) and (2) are clear. For (3), if we let  $X = \sum_i X^i \alpha_i$  and  $Y = \sum_j Y^j \alpha_j$  for a parametrization  $\alpha : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ , then

$$\begin{aligned} \partial_X Y - \partial_Y X &= \partial_X (\sum_j Y^j \alpha_j) - \partial_Y (\sum_i X^i \alpha_i) \\ &= \sum_j [(\partial_X Y^j) \alpha_j + Y^j \partial_X \alpha_j] - \sum_i [(\partial_Y X^i) \alpha_i + X^i \partial_Y \alpha_i] \\ &= \sum_j [(\partial_X Y^j) \alpha_j + Y^j \sum_i X^i \partial_i \alpha_j] - \sum_i [(\partial_Y X^i) \alpha_i + X^i \sum_j Y^j \partial_i \alpha_j] \\ &= \sum_j (\partial_X Y^j) \alpha_j - \sum_i (\partial_Y X^i) \alpha_i \\ &= \sum_i (\partial_X Y^i - \partial_Y X^i) \alpha_i. \end{aligned} \quad \square$$

**Notation.** Let  $M$  be the image of a parametrization  $\alpha$ . For derivatives of functions on  $M$  by tangent vectors, we will use

$$\partial_{\alpha_i} f = \partial_i f, \quad \partial_{\alpha_i} f = \partial_t f = f', \quad \partial_{\alpha_x} f = \partial_x f = f_x.$$

For derivatives of vector fields on  $M$  by tangent vectors, we will use

$$\partial_{\alpha_i} X = \partial_i X, \quad \partial_{\alpha_t} X = \partial_t X = X', \quad \partial_{\alpha_x} X = \partial_x X = X_x.$$

We will *not* use  $f_i$  or  $X_i$  for  $\partial_i f$  and  $\partial_i X$  because it is confusing with coordinate representations, and *not* use the nabula symbol  $\nabla_v$  in this sense because it will be devoted to another kind of derivatives introduced in Section 4.

**Example 1.2.12.** (a) Let  $\alpha$  be an  $m$ -dimensional parametrization with  $M = \text{im } \alpha$ . The value of  $\partial_i \alpha = \alpha_i : M \rightarrow \mathbb{R}^3$  is always a tangent vector at each point  $p = \alpha(x)$ , and  $\alpha_i$  becomes a vector field.

Let  $s$  be either a smooth function or vector field on  $\alpha$ . Then, we can compute the directional derivative as

$$\partial_i s := \partial_i (s \circ \alpha) = \partial_t (s \circ \gamma)$$

by taking  $\gamma(t) = \alpha(x + t e_i)$ , where  $e_i$  is the  $i$ -th standard basis vector for  $\mathbb{R}^m$ .

(b) Let  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a regular surface given by

$$\alpha(x, y) = \left( \frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2} \right).$$

This map gives a parametrization for the sphere  $S^2$  without the north pole  $(0, 0, 1)$ , and is called the *stereographic projection*. Let  $f : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}$  be the height function of  $\alpha$  defined by

$$f(p) := z$$

for  $p = (x, y, z) \in S^2 \setminus \{(0, 0, 1)\}$ . Its coordinate representation is

$$f \circ \alpha(x, y) = 1 - \frac{2}{1+x^2+y^2}.$$

Then, the directional derivative is

$$\partial_x f = \frac{\partial (f \circ \alpha)}{\partial x} = \frac{\partial}{\partial x} \left( 1 - \frac{2}{1+x^2+y^2} \right) = \frac{4x}{(1+x^2+y^2)^2}.$$

Note that  $\partial_x f \neq \partial_{(1,0,0)} z = 0$ .

### 1.3 Linear algebra on tangent spaces



## Chapter 2

# Local Theory of Curves and Surfaces

### 2.1 Curves

#### 2.1.1 Parametrization

By definition, a regular curve has at least one parametrization. However, a given parametrization may not have useful properties, so we often take a new parametrization. The existence of a parametrization with certain properties is one of the main problems in differential geometry. Practically, the existence proof is usually done by constructing a *diffeomorphism* between open sets in  $\mathbb{R}^m$ ; a bijective smooth map whose inverse is also smooth.

We introduce the arc-length reparametrization. It is the most general choice for the local study of curves.

**Definition 2.1.1.** A parametrization  $\alpha$  of a regular curve is called a *unit speed curve* or an *arc-length parametrization* when it satisfies  $\|\alpha'\| = 1$ .

**Theorem 2.1.2.** Every regular curve may be assumed to have unit speed. Precisely, for every regular curve, there is a parametrization  $\alpha$  such that  $\|\alpha'\| = 1$ .

*Proof.* By the definition of regular curves, we can take a parametrization  $\beta : I_t \rightarrow \mathbb{R}^d$  for a given regular curve. We will construct an arc-length parametrization from  $\beta$ .

Define  $\tau : I_t \rightarrow I_s$  such that

$$\tau(t) := \int_0^t \|\beta'(s)\| ds.$$

Since  $\tau$  is smooth and  $\tau' > 0$  everywhere so that  $\tau$  is strictly increasing, the inverse  $\tau^{-1} : I_s \rightarrow I_t$  is smooth by the inverse function theorem;  $\tau$  is a diffeomorphism. Define  $\alpha : I_s \rightarrow \mathbb{R}^d$  by  $\alpha := \beta \circ \tau^{-1}$ . Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt} \frac{d\tau^{-1}}{ds} = \beta' \left( \frac{d\tau}{dt} \right)^{-1} = \frac{\beta'}{\|\beta'\|}. \quad \square$$

#### 2.1.2 Differentiation of Frenet-Serret frame

The Frenet-Serret frame is a standard frame for a curve, and it is in particular effective when we assume the arc-length parametrization. It is defined for nondegenerate regular curves, i.e. nowhere straight curves. It provides with a useful orthonormal basis of  $T_p \mathbb{R}^3 \supset T_p C$  for points  $p$  on a regular curve  $C$ .

**Definition 2.1.3.** We call a curve parametrized as  $\alpha : I \rightarrow \mathbb{R}^3$  is *nondegenerate* if the normalized tangent vector  $\alpha'/\|\alpha'\|$  is never locally constant everywhere. In other words,  $\alpha$  is nowhere straight.

**Definition 2.1.4** (Frenet-Serret frame). Let  $\alpha$  be a nondegenerate curve. The *tangent unit vector*, *normal unit vector*, *binormal unit vector* are  $T_p\mathbb{R}^3$ -valued vector fields on  $\alpha$  defined by:

$$T(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad N(t) := \frac{T'(t)}{\|T'(t)\|}, \quad B(t) := T(t) \times N(t).$$

The set of vector fields  $\{T, N, B\}$ , which is called *Frenet-Serret frame*, forms an orthonormal basis of  $T_p\mathbb{R}^3$  at each point  $p$  on  $\alpha$ . The Frenet-Serret frame is uniquely determined up to sign as  $\alpha$  changes.

We study the derivatives of the Frenet-Serret frame and their coordinate representations. In the coordinate representations on the Frenet-Serret frame, important geometric measurements such as curvature and torsion come out as coefficients.

**Definition 2.1.5.** Let  $\alpha$  be a nondegenerate curve. The *curvature* and *torsion* are scalar fields on  $\alpha$  defined by:

$$\kappa(t) := \frac{\langle T'(t), N(t) \rangle}{\|\alpha'\|}, \quad \tau(t) := -\frac{\langle B'(t), N(t) \rangle}{\|\alpha'\|}.$$

Note that  $\kappa > 0$  cannot vanish by definition of nondegenerate curve. This definition is independent on  $\alpha$ .

**Theorem 2.1.6** (Frenet-Serret formula). Let  $\alpha$  be a nondegenerate curve. Then,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \|\alpha'\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

*Proof.* Note that  $\{T, N, B\}$  is an orthonormal basis. We first show the first and third rows, and the second row later.

Step1[Show that  $T', B', N$  are parallel] Two vectors  $T'$  and  $N$  are parallel by definition of  $N$ . Since  $\langle T, B \rangle = 0$  and  $\langle B, B \rangle = 1$  are constant, we have

$$\langle B', T \rangle = \langle B, T' \rangle - \langle B, T' \rangle = 0, \quad \langle B', B \rangle = \frac{1}{2} \langle B, B \rangle' = 0,$$

which show  $B'$  and  $N$  are parallel. By the definition of  $\kappa$  and  $\tau$ , we get

$$T' = \|\alpha'\| \kappa N, \quad B' = -\|\alpha'\| \tau N.$$

Step2[Describe  $N'$ ] Since

$$\langle N', T \rangle = -\langle N, T' \rangle = -\|\alpha'\| \kappa,$$

$$\langle N', N \rangle = \frac{1}{2} \langle N, N \rangle' = 0,$$

$$\langle N', B \rangle = -\langle N, B' \rangle = \|\alpha'\| \tau,$$

we have

$$N' = \|\alpha'\| (-\kappa T + \tau B). \quad \square$$

**Remark 2.1.7.** Let  $X(t)$  be the curve of orthogonal matrices  $(T(t), N(t), B(t))^T$ . Then, the Frenet-Serret formula reads

$$X'(t) = A(t)X(t)$$

for a matrix curve  $A(t)$  that is completely determined by  $\kappa(t)$  and  $\tau(t)$ , if we let us only consider arc-length parametrized curves. This is a typical form of an ODE system, so we can apply the Picard-Lindelöf theorem to get the following proposition: if we know  $\kappa(t)$  and  $\tau(t)$  for all time  $t$ , and if  $T(0)$  and  $N(0)$  are given so that an initial condition

$$X(0) = (T(0), N(0), T(0) \times N(0))$$

is established, then the solution  $X(t)$  exists and uniquely determined in a short time range. Furthermore, if  $\alpha(0)$  is given in addition, the integration

$$\alpha(t) = \alpha(0) + \int_0^t T(s) ds$$

provides a complete formula for unit speed parametrization  $\alpha$ .

*Remark 2.1.8.* Skew-symmetry in the Frenet-Serret formula is not by chance. Let  $X(t) = (T(t), N(t), B(t))^T$  and write  $X'(t) = A(t)X(t)$  as we did in the above remark. Since  $X(t+h) = R_t(h)X(t)$  for a family of special orthogonal matrices  $\{R_t(h)\}_h$  with  $R_t(0) = I$ , we can describe  $A(t)$  as

$$A(t) = \left. \frac{dR_t}{dh} \right|_{h=0}.$$

By differentiating the relation  $R_t^T(h)R_t(h) = I$  with respect to  $h$ , we get to know that  $A(t)$  is skew-symmetric for all  $t$ . In other words, the tangent space  $T_t SO(3)$  forms a skew symmetric matrix.

### 2.1.3 Computational problems

The following proposition gives the most effective and shortest way to compute the Frenet-Serret apparatus in general case. If we try to reparametrize the given curve into a unit speed curve or find  $\kappa$  by differentiating  $T$ , then we must encounter the normalizing term of the form  $\sqrt{(-)^2 + (-)^2 + (-)^2}^{-1}$ , and it must be painful when time is limited. The Frenet-Serret frame is useful in proofs of interesting propositions, but not a good choice for practical computation. Instead, a computation from derivatives of parametrization is highly recommended.

**Proposition 2.1.9.** *Let  $\alpha$  be a nondegenerate curve. Then,*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}$$

and

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad N = B \times T.$$

*Proof.* If we let  $s = \|\alpha'\|$ , then

$$\begin{aligned} \alpha' &= sT, \\ \alpha'' &= s'T + s^2\kappa N, \\ \alpha''' &= (s'' - s^3\kappa^2)T + (3ss'\kappa + s^2\kappa')N + (s^3\kappa\tau)B. \end{aligned}$$

Now the formulas are easily derived. □

### 2.1.4 General problems

We are interested in regular curves, not a particular parametrization. By the Theorem 2.1, we may always assume that a parametrization  $\alpha$  has unit speed. Let  $\alpha$  be a nondegenerate unit speed space curve, and let  $\{T, N, B\}$  be the Frenet-Serret frame for  $\alpha$ .

Consider a diagram as follows:

$$\begin{array}{ccccc} \langle \alpha, T \rangle = ? & \longleftrightarrow & \langle \alpha, N \rangle = ? & \longleftrightarrow & \langle \alpha, B \rangle = ? \\ \downarrow & & \downarrow & & \downarrow \\ \langle \alpha', T \rangle = 1 & & \langle \alpha', N \rangle = 0 & & \langle \alpha', B \rangle = 0. \end{array}$$

Here the arrows indicate which term we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t), T(t) \rangle = f(t),$$

then we can obtain

$$\langle \alpha(t), N(t) \rangle = \frac{f'(t) - 1}{\kappa(t)}$$

by direct differentiation since we have known  $\langle \alpha', T \rangle$  but not  $\langle \alpha, N \rangle$ . Further, we get

$$\langle \alpha(t), B(t) \rangle = \frac{\left( \frac{f'(t) - 1}{\kappa(t)} \right)' + \kappa(t)f(t)}{\tau(t)}$$

since we have known  $\langle \alpha, T \rangle$  and  $\langle \alpha', N \rangle$  but not  $\langle \alpha, B \rangle$ . Thus,  $\langle \alpha, T \rangle = f$  implies

$$\alpha(t) = f(t) \cdot T + \frac{f'(t) - 1}{\kappa(t)} \cdot N + \frac{\left( \frac{f'(t) - 1}{\kappa(t)} \right)' + \kappa(t)f(t)}{\tau(t)} \cdot B,$$

when given  $\tau(t) \neq 0$ .

We suggest a strategy for space curve problems:

- Build and differentiate equations of the following form:

$$\langle \text{(interesting vector)}, \text{(Frenet-Serret basis)} \rangle = \text{(some function)}.$$

- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of  $\kappa$  and  $\tau$  by comparing with assumptions.
- Heuristically find a constant vector and show what you want directly.

Here we give example solutions of several selected problems. Always  $\alpha$  denotes a reparametrized unit speed nondegenerate curve in  $\mathbb{R}^3$ .

**2.1.** A curve whose normal lines always pass through a fixed point lies in a circle.

*Solution.* Step1[Formulate conditions] By the assumption, there is a constant point  $p \in \mathbb{R}^3$  such that the vectors  $\alpha - p$  and  $N$  are parallel so that we have

$$\langle \alpha - p, T \rangle = 0, \quad \langle \alpha - p, B \rangle = 0.$$

Our goal is to show that  $\|\alpha - p\|$  is constant and there is a constant vector  $v$  such that  $\langle \alpha - p, v \rangle = 0$ .

Step2[Collect information] Differentiate  $\langle \alpha - p, T \rangle = 0$  to get

$$\langle \alpha - p, N \rangle = -\frac{1}{\kappa}.$$

Differentiate  $\langle \alpha - p, B \rangle = 0$  to get

$$\tau = 0.$$

Step3[Complete proof] We can deduce that  $\|\alpha - p\|$  is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p \rangle' = 2\langle \alpha - p, T \rangle = 0.$$

Also, if we heuristically define a vector  $v := B$ , then  $v$  is constant since

$$v' = -\tau N = 0,$$

and clearly  $\langle \alpha - p, v \rangle = 0$

□

**2.2.** A spherical curve of constant curvature lies in a circle.

*Solution.* Step1[Formulate conditions] The condition that  $\alpha$  lies on a sphere can be given as follows: for a constant point  $p \in \mathbb{R}^3$ ,

$$\|\alpha - p\| = \text{const.}$$

Also we have

$$\kappa = \text{const.}$$

Step2[Collect information] Differentiate  $\|\alpha - p\|^2 = \text{const}$  to get

$$\langle \alpha - p, T \rangle = 0.$$

Differentiate  $\langle \alpha - p, T \rangle = 0$  to get

$$\langle \alpha - p, N \rangle = -\frac{1}{\kappa}.$$

Differentiate  $\langle \alpha - p, N \rangle = -1/\kappa = \text{const}$  to get

$$\tau \langle \alpha - p, B \rangle = 0.$$

There are two ways to show that  $\tau = 0$ .

*Method 1:* Assume that there is  $t$  such that  $\tau(t) \neq 0$ . By the continuity of  $\tau$ , we can deduce that  $\tau$  is locally nonvanishing. In other words, we have  $\langle \alpha - p, B \rangle = 0$  on an open interval containing  $t$ . Differentiate  $\langle \alpha - p, B \rangle = 0$  at  $t$  to get  $\langle \alpha - p, N \rangle = 0$  near  $t$ , which is a contradiction. Therefore,  $\tau = 0$  everywhere.

*Method 2:* Since  $\langle \alpha - p, B \rangle$  is continuous and

$$\langle \alpha - p, B \rangle = \pm \sqrt{\|\alpha - p\|^2 - \langle \alpha - p, T \rangle^2 - \langle \alpha - p, N \rangle^2} = \pm \text{const},$$

we get  $\langle \alpha - p, B \rangle = \text{const}$ . Differentiate to get  $\tau \langle \alpha - p, N \rangle = 0$ . Finally we can deduce  $\tau = 0$  since  $\langle \alpha - p, N \rangle \neq 0$ .

Step3[Complete proof] The zero torsion implies that the curve lies on a plane. A planar curve in a sphere is a circle.  $\square$

**2.3.** A curve such that  $\tau/\kappa = (\kappa'/\tau\kappa^2)'$  lies on a sphere.

*Solution.* Step1[Find the center heuristically] If we assume that  $\alpha$  is on a sphere so that we have  $\|\alpha - p\| = r$  for constants  $p \in \mathbb{R}^3$  and  $r > 0$ , then by the routine differentiations give

$$\langle \alpha - p, T \rangle = 0, \quad \langle \alpha - p, N \rangle = -\frac{1}{\kappa}, \quad \langle \alpha - p, B \rangle = -\left(\frac{1}{\kappa}\right)' \frac{1}{\tau},$$

that is,

$$\alpha - p = -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B.$$

Step2[Complete proof] Let us get started the proof. Define

$$p := \alpha + \frac{1}{\kappa}N + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B.$$

We can show that it is constant by differentiation. Also we can show that

$$\langle \alpha - p, \alpha - p \rangle$$

is constant by differentiation. So we are done.  $\square$

**2.4.** A curve with more than one Bertrand mates is a circular helix.

*Solution.* Step1[Formulate conditions] Let  $\beta$  be a Bertrand mate of  $\alpha$  so that we have

$$\beta = \alpha + \lambda N, \quad N_\beta = \pm N,$$

where  $\lambda$  is a function not vanishing somewhere and  $\{T_\beta, N_\beta, B_\beta\}$  denotes the Frenet-Serret frame of  $\beta$ . We can reformulate the conditions as follows: Note that  $\beta$  is not unit speed.

Step2[Collect information] Differentiate  $\langle \beta - \alpha, N \rangle = \lambda$  to get

$$\lambda = \text{const} \neq 0.$$

Differentiate  $\langle \beta - \alpha, T \rangle = 0$  and  $\langle \beta - \alpha, B \rangle = 0$  to get

$$\langle T_\beta, T \rangle = \frac{1 - \lambda \kappa}{\|\beta'\|}, \quad \langle T_\beta, B \rangle = \frac{\lambda \tau}{\|\beta'\|}.$$

Differentiate  $\langle T_\beta, T \rangle$  and  $\langle T_\beta, B \rangle$  to get

$$\frac{1 - \lambda \kappa}{\|\beta'\|} = \text{const}, \quad \frac{\lambda \tau}{\|\beta'\|} = \text{const}.$$

Thus, there exists a constant  $\mu$  such that

$$1 - \lambda \kappa = \mu \lambda \tau$$

if  $\alpha$  is not planar so that  $\tau \neq 0$ .

We have shown that the torsion is either always zero or never zero at every point:  $\lambda \tau / \|\beta'\| = \text{const}$ . The problem can be solved by dividing the cases, but in this solution we give only for the case that  $\alpha$  is not planar; the other hand is not difficult.

Step3[Complete proof] If

$$\beta = \alpha + \lambda N, \quad \tilde{\beta} = \alpha + \tilde{\lambda} N$$

are different Bertrand mates of  $\alpha$  with  $\lambda \neq \tilde{\lambda}$ , then  $(\kappa, \tau)$  solves a two-dimensional linear system

$$\begin{aligned} \kappa + \mu \tau &= \lambda^{-1}, \\ \kappa + \tilde{\mu} \tau &= \tilde{\lambda}^{-1}. \end{aligned}$$

It is nonsingular since  $\mu = \tilde{\mu}$  implies  $\lambda = \tilde{\lambda}$ , which means we can represent  $\kappa$  and  $\tau$  in terms of constants  $\lambda, \tilde{\lambda}, \mu$ , and  $\tilde{\mu}$ . Therefore,  $\kappa$  and  $\tau$  are constant.  $\square$

Here is a well-prepared problem set for exercises.

**2.5 (Plane curves).** Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  lies on a plane,
- (b)  $\tau = 0$ ,
- (c) the osculating plane contains a fixed point.

**2.6 (Helices).** Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  is a helix,
- (b)  $\tau / \kappa = \text{const}$ ,
- (c) normal lines are parallel to a plane.

**2.7 (Sphere curves).** Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  lies on a sphere,

- (b)  $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$ ,
- (c)  $\tau/\kappa = (\kappa'/\tau\kappa^2)'$ ,
- (d) normal planes contain a fixed point.

**2.8** (Bertrand mates). Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  has a Bertrand mate,
- (b) there are two constants  $\lambda \neq 0, \mu$  such that  $1/\lambda = \kappa + \mu\tau$ .

## 2.2 Curves may be updated

By definition, a regular curve has at least one parametrization. However, a given parametrization may not have useful properties, so we often take a new parametrization. The existence of a parametrization with certain properties is one of the main problems in differential geometry. Practically, the existence proof is usually done by constructing a *diffeomorphism* between open sets in  $\mathbb{R}^m$ ; a bijective smooth map whose inverse is also smooth.

We introduce the arc-length reparametrization. It is the most general choice for the local study of curves.

**Definition 2.2.1.** A parametrization  $\alpha$  of a regular curve is called a *unit speed curve* or an *arc-length parametrization* when it satisfies  $\|\alpha'\| = 1$ .

**Theorem 2.2.2.** Every regular curve may be assumed to have unit speed. Precisely, for every regular curve, there is a parametrization  $\alpha$  such that  $\|\alpha'\| = 1$ .

*Proof.* By the definition of regular curves, we can take a parametrization  $\beta : I_t \rightarrow \mathbb{R}^d$  for a given regular curve. We will construct an arc-length parametrization from  $\beta$ .

Define  $\tau : I_t \rightarrow I_s$  such that

$$\tau(t) := \int_0^t \|\beta'(s)\| ds.$$

Since  $\tau$  is smooth and  $\tau' > 0$  everywhere so that  $\tau$  is strictly increasing, the inverse  $\tau^{-1} : I_s \rightarrow I_t$  is smooth by the inverse function theorem;  $\tau$  is a diffeomorphism. Define  $\alpha : I_s \rightarrow \mathbb{R}^d$  by  $\alpha := \beta \circ \tau^{-1}$ . Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt} \frac{d\tau^{-1}}{ds} = \beta' \left( \frac{d\tau}{dt} \right)^{-1} = \frac{\beta'}{\|\beta'\|}. \quad \square$$

The Frenet-Serret frame is a standard frame for a curve, and it is in particular effective when we assume the arc-length parametrization. It is defined for nondegenerate regular curves, i.e. nowhere straight curves. It provides with a useful orthonormal basis of  $T_p\mathbb{R}^3 \supset T_p\gamma(I)$  for points  $p$  on a regular curve  $\gamma : I \rightarrow \mathbb{R}^3$ .

**2.9.** A regular curve  $\gamma : I \rightarrow \mathbb{R}^3$  is called *non-degenerate* if the normalized tangent vector  $\gamma'/\|\gamma'\|$  is never locally constant everywhere. In other words,  $\gamma$  is nowhere straight.

**Definition 2.2.3** (Frenet-Serret frame). Let  $\alpha$  be a nondegenerate curve. The *tangent unit vector*, *normal unit vector*, *binormal unit vector* are  $T_p\mathbb{R}^3$ -valued vector fields on  $\alpha$  defined by:

$$T(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \quad N(t) := \frac{T'(t)}{\|T'(t)\|}, \quad B(t) := T(t) \times N(t).$$

The set of vector fields  $\{T, N, B\}$ , which is called *Frenet-Serret frame*, forms an orthonormal basis of  $T_p\mathbb{R}^3$  at each point  $p$  on  $\alpha$ . The Frenet-Serret frame is uniquely determined up to sign as  $\alpha$  changes.

We study the derivatives of the Frenet-Serret frame and their coordinate representations. In the coordinate representations on the Frenet-Serret frame, important geometric measurements such as curvature and torsion come out as coefficients.

**Definition 2.2.4.** Let  $\alpha$  be a nondegenerate curve. The *curvature* and *torsion* are scalar fields on  $\alpha$  defined by:

$$\kappa(t) := \frac{\langle T'(t), N(t) \rangle}{\|\alpha'\|}, \quad \tau(t) := -\frac{\langle B'(t), N(t) \rangle}{\|\alpha'\|}.$$

Note that  $\kappa > 0$  cannot vanish by definition of nondegenerate curve. This definition is independent on  $\alpha$ .

**2.10. Frenet-Serret formula.** Let  $\gamma$  be a non-degenerate regular curve. Then,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \|\gamma'\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

- (a)  $T' = \|\gamma'\| \kappa N$ .
- (b)  $B' = -\|\gamma'\| \tau N$ .
- (c)  $N' = -\|\gamma'\| \kappa T + \|\gamma'\| \tau B$ .

*Proof.* Note that  $\{T, N, B\}$  is an orthonormal basis.

- (a) Two vectors  $T'$  and  $N$  are parallel by definition of  $N$ . By the definition of  $\kappa$ , we get  $T' = \|\gamma'\| \kappa N$ .
- (b) Since  $\langle T, B \rangle = 0$  and  $\langle B, B \rangle = 1$  are constant, we have

$$\langle B', T \rangle = \langle B, T' \rangle - \langle B, T' \rangle = 0, \quad \langle B', B \rangle = \frac{1}{2} \langle B, B \rangle' = 0.$$

By the definition of  $\tau$ , we get  $B' = -\|\alpha'\| \tau N$ .

(c) Since

$$\begin{aligned} \langle N', T \rangle &= -\langle N, T' \rangle = -\|\alpha'\| \kappa, \\ \langle N', N \rangle &= \frac{1}{2} \langle N, N \rangle' = 0, \\ \langle N', B \rangle &= -\langle N, B' \rangle = \|\alpha'\| \tau, \end{aligned}$$

we have

$$N' = \|\alpha'\| (-\kappa T + \tau B). \quad \square$$

**Remark 2.2.5.** Let  $X(t)$  be the curve of orthogonal matrices  $(T(t), N(t), B(t))^T$ . Then, the Frenet-Serret formula reads

$$X'(t) = A(t)X(t)$$

for a matrix curve  $A(t)$  that is completely determined by  $\kappa(t)$  and  $\tau(t)$ , if we let us only consider arc-length parametrized curves. This is a typical form of an ODE system, so we can apply the Picard-Lindelöf theorem to get the following proposition: if we know  $\kappa(t)$  and  $\tau(t)$  for all time  $t$ , and if  $T(0)$  and  $N(0)$  are given so that an initial condition

$$X(0) = (T(0), N(0), T(0) \times N(0))$$

is established, then the solution  $X(t)$  exists and uniquely determined in a short time range. Furthermore, if  $\alpha(0)$  is given in addition, the integration

$$\alpha(t) = \alpha(0) + \int_0^t T(s) ds$$

provides a complete formula for unit speed parametrization  $\alpha$ .



*Remark 2.2.6.* Skew-symmetry in the Frenet-Serret formula is not by chance. Let  $X(t) = (T(t), N(t), B(t))^T$  and write  $X'(t) = A(t)X(t)$  as we did in the above remark. Since  $X(t+h) = R_t(h)X(t)$  for a family of special orthogonal matrices  $\{R_t(h)\}_h$  with  $R_t(0) = I$ , we can describe  $A(t)$  as

$$A(t) = \left. \frac{dR_t}{dh} \right|_{h=0}.$$

By differentiating the relation  $R_t^T(h)R_t(h) = I$  with respect to  $h$ , we get to know that  $A(t)$  is skew-symmetric for all  $t$ . In other words, the tangent space  $T_t SO(3)$  forms a skew symmetric matrix.

The following proposition gives the most effective and shortest way to compute the Frenet-Serret apparatus in general case. If we try to reparametrize the given curve into a unit speed curve or find  $\kappa$  by differentiating  $T$ , then we must encounter the normalizing term of the form  $\sqrt{(-)^2 + (-)^2 + (-)^2}^{-1}$ , and it must be painful when time is limited. The Frenet-Serret frame is useful in proofs of interesting propositions, but not a good choice for practical computation. Instead, a computation from derivatives of parametrization is highly recommended.

**Proposition 2.2.7.** *Let  $\alpha$  be a nondegenerate curve. Then,*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}$$

and

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad N = B \times T.$$

*Proof.* If we let  $s = \|\alpha'\|$ , then

$$\begin{aligned} \alpha' &= sT, \\ \alpha'' &= s'T + s^2\kappa N, \\ \alpha''' &= (s'' - s^3\kappa^2)T + (3ss'\kappa + s^2\kappa')N + (s^3\kappa\tau)B. \end{aligned}$$

Now the formulas are easily derived. □

Now we are interested in regular curves, not a particular parametrization. By the Theorem 2.1, we may always assume that a parametrization  $\alpha$  has unit speed. Let  $\alpha$  be a nondegenerate unit speed space curve, and let  $\{T, N, B\}$  be the Frenet-Serret frame for  $\alpha$ .

Consider a diagram as follows:

$$\begin{array}{ccccc} \langle \alpha, T \rangle = ? & \longleftrightarrow & \langle \alpha, N \rangle = ? & \longleftrightarrow & \langle \alpha, B \rangle = ? \\ \downarrow & & \downarrow & & \downarrow \\ \langle \alpha', T \rangle = 1 & & \langle \alpha', N \rangle = 0 & & \langle \alpha', B \rangle = 0. \end{array}$$

Here the arrows indicate which term we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t), T(t) \rangle = f(t),$$

then we can obtain

$$\langle \alpha(t), N(t) \rangle = \frac{f'(t) - 1}{\kappa(t)}$$

by direct differentiation since we have known  $\langle \alpha', T \rangle$  but not  $\langle \alpha, N \rangle$ . Further, we get

$$\langle \alpha(t), B(t) \rangle = \frac{\left( \frac{f'(t) - 1}{\kappa(t)} \right)' + \kappa(t)f(t)}{\tau(t)}$$

since we have known  $\langle \alpha, T \rangle$  and  $\langle \alpha', N \rangle$  but not  $\langle \alpha, B \rangle$ . Thus,  $\langle \alpha, T \rangle = f$  implies

$$\alpha(t) = f(t) \cdot T + \frac{f'(t)-1}{\kappa(t)} \cdot N + \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t)f(t)}{\tau(t)} \cdot B,$$

when given  $\tau(t) \neq 0$ .

We suggest a strategy for space curve problems:

- Build and differentiate equations of the following form:

$$\langle \text{(interesting vector)}, \text{(Frenet-Serret basis)} \rangle = \text{(some function)}.$$

- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of  $\kappa$  and  $\tau$  by comparing with assumptions.
- Heuristically find a constant vector and show what you want directly.

Here we give example solutions of several selected problems. Always  $\alpha$  denotes a reparametrized unit speed nondegenerate curve in  $\mathbb{R}^3$ .

If

$$f = \langle \alpha - p, T \rangle, \quad g = \langle \alpha - p, N \rangle, \quad h = \langle \alpha - p, B \rangle,$$

then

$$f' = 1 + \kappa g, \quad g' = -\kappa f + \tau h, \quad h' = -\tau g.$$

**2.11.** A curve whose normal lines always pass through a fixed point lies in a circle.

*Solution.* By the assumption, we have

$$\alpha - p = 0 \cdot T + \langle \alpha - p, N \rangle \cdot N + 0 \cdot B.$$

$$\alpha' = -\kappa \langle \alpha - p, N \rangle \cdot T + 0 \cdot N + \tau \langle \alpha - p, N \rangle \cdot B.$$

[Formulate conditions] By the assumption, there is a constant point  $p \in \mathbb{R}^3$  such that the vectors  $\alpha - p$  and  $N$  are parallel so that we have

$$\langle \alpha - p, T \rangle = 0, \quad \langle \alpha - p, B \rangle = 0.$$

Our goal is to show that  $\|\alpha - p\|$  is constant and there is a constant vector  $v$  such that  $\langle \alpha - p, v \rangle = 0$ .

[Collect information] Differentiate  $\langle \alpha - p, T \rangle = 0$  to get

$$\langle \alpha - p, N \rangle = -\frac{1}{\kappa}.$$

Differentiate  $\langle \alpha - p, B \rangle = 0$  to get

$$\tau = 0.$$

[Complete proof] We can deduce that  $\|\alpha - p\|$  is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p \rangle' = 2\langle \alpha - p, T \rangle = 0.$$

Also, if we heuristically define a vector  $v := B$ , then  $v$  is constant since

$$v' = -\tau N = 0,$$

and clearly  $\langle \alpha - p, v \rangle = 0$

□

**2.12.** A spherical curve of constant curvature lies in a circle.

*Solution.* [Formulate conditions] The condition that  $\alpha$  lies on a sphere can be given as follows: for a constant point  $p \in \mathbb{R}^3$ ,

$$\|\alpha - p\| = \text{const.}$$

Also we have

$$\kappa = \text{const.}$$

[Collect information] Differentiate  $\|\alpha - p\|^2 = \text{const}$  to get

$$\langle \alpha - p, T \rangle = 0.$$

Differentiate  $\langle \alpha - p, T \rangle = 0$  to get

$$\langle \alpha - p, N \rangle = -\frac{1}{\kappa}.$$

Differentiate  $\langle \alpha - p, N \rangle = -1/\kappa = \text{const}$  to get

$$\tau \langle \alpha - p, B \rangle = 0.$$

There are two ways to show that  $\tau = 0$ .

*Method 1:* Assume that there is  $t$  such that  $\tau(t) \neq 0$ . By the continuity of  $\tau$ , we can deduce that  $\tau$  is locally nonvanishing. In other words, we have  $\langle \alpha - p, B \rangle = 0$  on an open interval containing  $t$ . Differentiate  $\langle \alpha - p, B \rangle = 0$  at  $t$  to get  $\langle \alpha - p, N \rangle = 0$  near  $t$ , which is a contradiction. Therefore,  $\tau = 0$  everywhere.

*Method 2:* Since  $\langle \alpha - p, B \rangle$  is continuous and

$$\langle \alpha - p, B \rangle = \pm \sqrt{\|\alpha - p\|^2 - \langle \alpha - p, T \rangle^2 - \langle \alpha - p, N \rangle^2} = \pm \text{const},$$

we get  $\langle \alpha - p, B \rangle = \text{const}$ . Differentiate to get  $\tau \langle \alpha - p, N \rangle = 0$ . Finally we can deduce  $\tau = 0$  since  $\langle \alpha - p, N \rangle \neq 0$ .

[Complete proof] The zero torsion implies that the curve lies on a plane. A planar curve in a sphere is a circle.  $\square$

**2.13.** A curve such that  $\tau/\kappa = (\kappa'/\tau\kappa^2)'$  lies on a sphere.

*Solution.* [Find the center heuristically] If we assume that  $\alpha$  is on a sphere so that we have  $\|\alpha - p\| = r$  for constants  $p \in \mathbb{R}^3$  and  $r > 0$ , then by the routine differentiations give

$$\langle \alpha - p, T \rangle = 0, \quad \langle \alpha - p, N \rangle = -\frac{1}{\kappa}, \quad \langle \alpha - p, B \rangle = -\left(\frac{1}{\kappa}\right)' \frac{1}{\tau},$$

that is,

$$\alpha - p = -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B.$$

[Complete proof] Let us get started the proof. Define

$$p := \alpha + \frac{1}{\kappa}N + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B.$$

We can show that it is constant by differentiation. Also we can show that

$$\langle \alpha - p, \alpha - p \rangle$$

is constant by differentiation. So we are done.  $\square$

**2.14.** A curve with more than one Bertrand mates is a circular helix.

*Solution.* [Formulate conditions] Let  $\beta$  be a Bertrand mate of  $\alpha$  so that we have

$$\beta = \alpha + \lambda N, \quad N_\beta = \pm N,$$

where  $\lambda$  is a function not vanishing somewhere and  $\{T_\beta, N_\beta, B_\beta\}$  denotes the Frenet-Serret frame of  $\beta$ . We can reformulate the conditions as follows:

Note that  $\beta$  is not unit speed.

[Collect information] Differentiate  $\langle \beta - \alpha, N \rangle = \lambda$  to get

$$\lambda = \text{const} \neq 0.$$

Differentiate  $\langle \beta - \alpha, T \rangle = 0$  and  $\langle \beta - \alpha, B \rangle = 0$  to get

$$\langle T_\beta, T \rangle = \frac{1 - \lambda\kappa}{\|\beta'\|}, \quad \langle T_\beta, B \rangle = \frac{\lambda\tau}{\|\beta'\|}.$$

Differentiate  $\langle T_\beta, T \rangle$  and  $\langle T_\beta, B \rangle$  to get

$$\frac{1 - \lambda\kappa}{\|\beta'\|} = \text{const}, \quad \frac{\lambda\tau}{\|\beta'\|} = \text{const}.$$

Thus, there exists a constant  $\mu$  such that

$$1 - \lambda\kappa = \mu\lambda\tau$$

if  $\alpha$  is not planar so that  $\tau \neq 0$ .

We have shown that the torsion is either always zero or never zero at every point:  $\lambda\tau/\|\beta'\| = \text{const}$ . The problem can be solved by dividing the cases, but in this solution we give only for the case that  $\alpha$  is not planar; the other hand is not difficult.

[Complete proof] If

$$\beta = \alpha + \lambda N, \quad \tilde{\beta} = \alpha + \tilde{\lambda} N$$

are different Bertrand mates of  $\alpha$  with  $\lambda \neq \tilde{\lambda}$ , then  $(\kappa, \tau)$  solves a two-dimensional linear system

$$\begin{aligned} \kappa + \mu\tau &= \lambda^{-1}, \\ \kappa + \tilde{\mu}\tau &= \tilde{\lambda}^{-1}. \end{aligned}$$

It is nonsingular since  $\mu = \tilde{\mu}$  implies  $\lambda = \tilde{\lambda}$ , which means we can represent  $\kappa$  and  $\tau$  in terms of constants  $\lambda, \tilde{\lambda}, \mu$ , and  $\tilde{\mu}$ . Therefore,  $\kappa$  and  $\tau$  are constant.  $\square$

Here is a well-prepared problem set for exercises.

**2.15** (Plane curves). Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  lies on a plane,
- (b)  $\tau = 0$ ,
- (c) the osculating plane contains a fixed point.

**2.16** (Helices). Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  is a helix,
- (b)  $\tau/\kappa = \text{const}$ ,
- (c) normal lines are parallel to a plane.

**2.17** (Sphere curves). Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  lies on a sphere,
- (b)  $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$ ,
- (c)  $\tau/\kappa = (\kappa'/\tau\kappa^2)'$ ,
- (d) normal planes contain a fixed point.

**2.18** (Bertrand mates). Let  $\alpha$  be a nondegenerate curve in  $\mathbb{R}^3$ . TFAE:

- (a) the curve  $\alpha$  has a Bertrand mate,
- (b) there are two constants  $\lambda \neq 0, \mu$  such that  $1/\lambda = \kappa + \mu\tau$ .