Category Theory

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set theoretical issues duality morphisms monic

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full, faithful natural transformations and equivalence 2-category

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Universal property

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products, equalizers, pullbacks

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2.1 (Yoneda lemma). Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor from a locally small category $\mathcal{C}$. Fix $c \in \mathrm{Ob}(\mathcal{C})$. we can define a function

$$\mathrm{Nat}(\mathrm{Hom}(c,-),F) \to F(c).$$

A *representation* of F is a pair (c,η) of an object $c \in \mathcal{C}$ and a natural isomorphism $\eta : \mathrm{Hom}(c,-) \to F$.

A *universal element* of F is a pair (c,x) with $x \in F(c)$ such that for any pair (d,y) with $y \in F(d)$ there is a unique morphism $f \in \mathrm{Hom}(c,d)$ satisfying $F(f) : x \mapsto y$.

(a)

Proof. (a) Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(c,c) \arrow[rrr, "\eta_c"] \arrow[ddd] \arrow[dr, phantom, "\ni"] & & & F(c) \arrow[ddd, "F(f)"] \arrow[dl, phantom, "\in"] \\
& \mathrm{id}_c \arrow[r, mapsto] \arrow[d, mapsto] & x := \eta_c(\mathrm{id}_c) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \eta_d(f) := F(f)(x) & \\
\mathrm{Hom}(c,d) \arrow[rrr, "\eta_d"'] \arrow[ur, phantom, "\ni"] & & & F(d). \arrow[ul, phantom, "\in"]
\end{tikzcd}$$

For a natural transformation $\eta : \mathrm{Hom}(c,-) \to F$, define $x := \eta_c(\mathrm{id}_c)$ in $F(c)$. For $x \in F(c)$, conversely, define a $\eta_d : \mathrm{Hom}(c,d) \to F(d)$ by $\eta_d(f) := F(f)(x)$ for $d \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}(c,d)$. Then, the collection $\eta = \{\eta_d : d \in \mathrm{Ob}(\mathcal{C})\}$ provides a natural transformation because for each $g \in \mathrm{Hom}(d,e)$ we can check the diagram

$$\begin{tikzcd}
\mathrm{Hom}(c,d) \arrow[r, "\eta_d"] \arrow[d, "g\circ-"'] & F(d) \arrow[d, "F(g)"] \\
\mathrm{Hom}(c,e) \arrow[r, "\eta_e"] & F(e)
\end{tikzcd}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \qquad f \in \mathrm{Hom}(c,d).$$

The correspondences $\eta \mapsto x$ and $x \mapsto \eta$ are inverses of each other, hence the bijection. $\square$

Chapter 3

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Part II

Chapter 4

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Chapter 5

Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

5.1 Enriched categories

5.1 (Pointed category)**.** A *pointed category* is a category with a zero object.

(a) A category is $\mathbf{Set}_*$-enriched if and only if it admits a zero morphism.

(b) Every pointed category is $\mathbf{Set}_*$-enriched.

Chapter 6

Abelian categories

6.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f).

A category is called *regular* if every kernel pair admits a coequalizer.

6.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category **Grp** is regular but not coregular, since there is a monomorphism which is not regular.

6.2 Additive and abelian categories

6.2 (Pre-additive categories)**.** A *pre-additive category* is another name of **Ab**-enriched category.

(a) a

6.3 (Semi-additive cateogries)**.** A *semi-additive category* is a category with binary biproducts.

(a) A category is semiadditive if and only if it is pointed **CMon**-enriched.

6.4 (Additive categories)**.** (a) additive completion by formally adjoining finite biproducts.

(b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a $\mathbf{Set}_*$-enriched category.

6.5 (Pre-abelian categories)**.** A *pre-abelian category* is an additive category in which every morphism has the kernel and cokernel. Equivalently, it is a finitely bicomplete pre-additive category.

(a)

6.6 (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen)**.** A pointed, Baar-exact, proto-modular, with binary coproudcts.

(a) short five lemma, 3×3 lemma, snake lemma, noehter isomorphism hold.

(b) long exact homology sequence

(c) Every semi-abelian category is exact.

(d) Every semi-abelian category is finitely bicomplete.

(e) In general, a semi-abelian category is not pre-additive nor semi-additive.

6.7 (Abelian categories)**.** We say $\mathcal{C}$ is *abelian* if every morphism has the kernel and cokernel, and every mono and epi is normal. Roughly, an abelian category is a **Ab**-enriched category such that it is finitely bicomplete and the first isomorphism holds.

(a) A category is abelian if and only if it is additive and exact.

6.8 (Freyd-Mitchell embedding)**.**

$$\begin{tikzcd}
\text{abelian} \arrow[r] \arrow[d] & \text{pre-abelian} \arrow[r] & \text{additive} \arrow[r] \arrow[d] & \text{pre-additive} \\
\text{semi-abelian (JMT)} & & \text{semi-additive} &
\end{tikzcd}$$

- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C*-algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

The first isomorphism theorem states that coim → im is an isomorphism. The normal subobjects and the first isomorphism theorem is generalized in the context of protomodular categories. The cokernel may not be defined. The category of unital rings is not semi-abelian but protomodular.

- A *protomodular category*
- A *homological category* is a pointed regular protomodular category. (five, nine, snake, long exact sequence, noether isomorphism)
- A *semi-abelian category* is a homological category that is Barr-exact and finite coproducts(free products).

Chapter 7

site, topos $(\infty, 1)$-category