Functional Analysis

Ikhan Choi

September 20, 2022

Contents

Part I

Topological vector spaces

Chapter 1

Locally convex spaces

1.1 General vector topologies

canonical uniformity. canonical bornology. metrizability(Birkhoff-Kakutani). boundedness and continuity

1.2 Seminorms and convex sets

1.1 (Seminorms)**.**

$$\bigcap_{i=1}^{m}\{: p() < 1\}$$

Equivalent conditions on the continuity of seminorms

Proof. □

boundedness by seminorms, normability

1.3 Continuous linear functionals

1.2. Let $\{x_i^*\}_{i=1}^n \subset X^*$. If $x^* \in X^*$ vanishes on $\bigcap_{i=1}^n \ker x_i^*$, then x^* is a linear combination of $\{x_i^*\}$.

1.4 Hahn-Banach theorem

1.3 (Hahn-Banach theorem)**.** Let X be a real vector space and Y a linear subspace of X. Suppose that $l : Y \to \mathbb{R}$ is a linear functional dominated by a sublinear functional $q : X \to \mathbb{R}$.

(a) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l.

(b) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l.

Proof. (a) It can be done by the Hamel basis.
(b)

□

1.4 (Complex linear functionals)**.** Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$\{\mathbb{C}\text{-linear functionals on } X\} \xrightarrow{\sim} \{\mathbb{R}\text{-linear functionals on } X\}.$$

Let p be a seminorm on X and l a linear functional on X.

(a)

(b) If $\mathbb{F} = \mathbb{C}$, then $|l(x)| \leq p(x)$ if and only if $|\operatorname{Re} l(x)| \leq p(x)$.

(c) If $\mathbb{F} = \mathbb{R}$, then

Proof. (b) There is λ such that $|\lambda| = 1$ and $l(\lambda x) \geq 0$. Then,

$$|l(x)| = |\lambda^{-1} l(\lambda x)| = l(\lambda x) = \operatorname{Re} l(\lambda x) \leq p(\lambda x) = p(x).$$

□

1.5. Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and Y a linear subspace of X. Suppose that $l : Y \to \mathbb{R}$ is a linear functional dominated by a seminorm $p : X \to \mathbb{R}$. Then, there is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that is an extension of l and dominated by p.

Exercises

1.6 (Topology of compact convergence).

Chapter 2

Barreled spaces

2.1 Uniform boundedness principle

2.1 (Barreled spaces). A *barrel* is an absorbing, balanced, convex, and closed subset of X. A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

2.2 (Uniform boundedness principle). Let $\mathcal{T}$ be a set of continuous linear operators from X to Y. Suppose $\bigcup_{T\in\mathcal{T}} Tx$ is bounded for each $x \in D$, where $D \subset X$.

(a) If D is dense in X, then $\bigcap_{T\in\mathcal{T}} T^{-1}\overline{U}$ is absorbing.

(b) If X is barreled, then $\mathcal{T}$ is equicontinuous.

2.2 Baire category theorem

2.3 (Baire spaces). A topological space is called a *Baire space* if the intersection of countable open dense subsets is dense.

2.4 (Absorbing set). Let X be a topological vector space that is Baire.

(a) A closed and absorbing set has non-empty interior.

(b) A closed, convex, and absorbing set is a neighborhood of zero.

2.5 (The Baire category theorem).

2.3 Open mapping theorem

2.6 (Open mapping theorem). Let X be a F-space and Y a barreled space. Suppose $T : X \to Y$ is continuous and surjective.

(a) $\overline{TB}$ is a neighborhood of zero.

(b) TB is a neighborhood of zero.

Proof. (a) Let $B = B_1$ be an open ball in X. There is an open neighborhood U of zero such that $U - U \subset B$. The set $\overline{TU}$ is clearly closed, and the surjectivity of T implies $\overline{TU}$ is absorbing. Since Y is barreled, $\overline{TU}$ has a non-empty interior in Y. Thus, $\overline{TB}$ is a neighborhood of zero.

(b) We claim $\overline{TB_{1/2}} \subset TB$. Take $y_1 \in \overline{TB_{1/2}}$. To construct $x \in B$ such that $Tx = y_1$, we use the metrizability and completeness of X. Since $\overline{TB_{1/2^{n+1}}}$ are neighborhoods of zero, we can inductively construct sequences $x_n \in B_{1/2^n}$ and $y_n \in \overline{TB_{1/2^n}}$ such that $Tx_n \in y_n + \overline{TB_{1/2^{n+1}}}$ and $y_{n+1} := Tx_n - y_n$. Let $x := \sum_{n=1}^\infty x_n \in B$. Then,

$$Tx = \lim_{n\to\infty} \sum_{i=1}^n Tx_i = \lim_{n\to\infty} \sum_{i=1}^n y_{i+1} - y_i = y_1. \qquad \square$$

Exercises

2.7. Let (T_n) be a sequence in $B(X,Y)$. If T_n coverges strongly then $\|T_n\|$ is bounded by the uniform boundedness principle.

2.8. There is a closed absorbing set in $\ell^2(\mathbb{Z}_{\ge 0})$ that is not a neighborhood of zero;

$$\overline{B}(0,1) \setminus \bigcup_{i=2}^\infty B(i^{-1}e_i, i^{-2})$$

is a counterexample.

Exercises

2.9. There is no metric d on $C([0,1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise as $n \to \infty$ for every sequence f_n. Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

2.10. We show that there is no projection from ℓ^∞ onto c_0.

2.11 (Schur property)**.** ℓ^1

2.12. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

2.13 (Daugavet property)**.** (a) The real Banach space $C([0,1])$ satisfies the Daugavet property.

Proof. Let T be a finite rank operator on $C([0,1])$, and e_i be a basis of $\operatorname{im} T$. Then, for some measures μ_i,

$$Tf(t) = \sum_{i=1}^n \int_0^1 f\, d\mu_i e_i(t).$$

Let $M := \max \|e_i\|$.

Take f_0 such that $\|f_0\| = 1$ and $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$. Reversing the sign of f_0 if necessary, take an open interval Δ such that $Tf_0(t) \ge \|T\| - \frac{\varepsilon}{2}$ and $|\mu_i|(\Delta) \le \frac{\varepsilon}{4nM}$ for all i. Define f_1 such that $f_0 = f_1$ on Δ^c, $f_1(t_0) = 1$ for some $t_0 \in \Delta$, and $\|f_1\| = 1$. Then, $\|Tf_1 - Tf_0\| \le \frac{\varepsilon}{2}$ shows $Tf_1 \ge \|T\| - \varepsilon$ on Δ. Therefore,

$$\|1 + T\| \ge \|f_1 + Tf_1\| \ge f_1(t_0) + Tf_1(t_0) \le 1 + \|T\| - \varepsilon.$$

$\square$

Problems

2.14. Let T be an invertible linear operator on a normed space. Then, $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

Chapter 3

Weak topologies

3.1 Dual spaces

3.1 (Bidual).

3.2. Let X be a locally convex space. The *weak topology* is the topology w on X defined by the family of seminorms $\{x \mapsto |\langle x,\xi\rangle|\}_{\xi\in X^*}$. The *weak* topology* is the topology w^* on X^* defined by the family of seminorms $\{\xi \mapsto |\langle x,\xi\rangle|\}_{x\in X}$. Let $J : X \to X^{**}$ be the canonical embedding.

(a) (X,w) and (X^*,w^*) are locally convex.

(b) $(X,w)^* = X^*$.

(c) $(X^*,w^*)^* = X$. Every locally convex space is a dual of a locally convex space.

Proof. (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show $(X^*,w^*)^* \subset X$. If $u \in (X^*,w^*)^*$, then there are $x_1,\cdots,x_m \in X$ such that

$$|\langle u,\xi\rangle| \le \sum_{i=1}^m |\langle x_i,\xi\rangle|$$

for all $\xi \in X^*$. If we let $\ker\vec{x} := \bigcap_{i=1}^m \ker x_i$, then it is a closed subspace of X^* such that $\ker\vec{x} \subset \ker u$, so we have $u \in \operatorname{span}\vec{x} \subset X$. □

3.3. closure and weak closure of convex subsets

Proof. Hahn-Banach □

3.4 (Polar).

boundedness, incompleteness

3.5 (Weak convergence by dense set). Let X be a Banach space, D^* a subset of X^*, and $\overline{D^*}$ the norm closure of D^*. For example, if X has a predual $X_* \subset X^*$ and D^* is dense in X_*, then $\sigma(X,\overline{D^*})$ is the weak* topology.

(a) There is a squence $x_n \in X$ converges to zero in $\sigma(X,D^*)$ but not in $\sigma(X,\overline{D^*})$.

(b) A bounded sequence $x_n \in X$ converges to zero in $\sigma(X,\overline{D^*})$ if in $\sigma(X,D^*)$.

Proof. (b) Let $\xi \in \overline{D^*}$ and choose $\eta \in D^*$ such that $\|\xi-\eta\| < \varepsilon$. Then,

$$|\langle x_n,\xi\rangle| \le \|x_n\|\|\xi-\eta\| + |\langle x_n,\eta\rangle| \lesssim \varepsilon + |\langle x_n,\eta\rangle| \to \varepsilon.$$

□

3.2 Weak compactness

3.6 (Banach-Alaoglu theorem).

3.7 (Eberlein-Šmulian theorem).

3.8 (James' theorem).

3.3 Weak density

Bishop-Phelps theorem

3.9 (Goldstine's theorem). Let X be a Banach space and $J : X \to X^{**}$ the canonical embedding. Our claim is that $\overline{B}$ is weak*-dense in $\overline{B}_{X^{**}}$. Let $x_0^{**} \in X^{**}$ with $\|x_0^{**}\| \le 1$, and let

$$N_{\vec{x}^*}(x_0^{**}, \varepsilon) := \bigcap_{i=1}^{m} \{ x^{**} \in X^{**} : |\langle x^{**} - x_0^{**}, x_i^* \rangle| < \varepsilon \}$$

be an open weak*-neighborhood of zero in X^{**} with $\|x_i^*\| \le 1$ and $\varepsilon > 0$.

(a) There is $x \in X$ such that $\langle x, x_i^* \rangle = \langle x_0^{**}, x_i^* \rangle$ for all i.

(b) $\overline{B}_X$ is weak*-dense in $\overline{B}_{X^{**}}$

Proof. (a)

By the Hahn-Banach theorem on $\operatorname{span}\{x\} \cup \ker$, there is $\eta \in X^*$ such that

$$\eta|_{\ker \vec{\xi}} = 0, \quad \langle x, \eta \rangle > 1 + \varepsilon, \quad \text{and} \quad \|\eta\| = 1.$$

Since η is a linear combination of $\xi_1, \cdots, \xi_m$, we have

$$1 + \varepsilon < \langle x, \eta \rangle = \langle x_0^{**}, \eta \rangle \le \|x_0^{**}\| \|\eta\| \le 1.$$

(c) Take $\varepsilon > 0$ such that $\varepsilon \max_{1 \le i \le m} \|x_i^*\| < 1$. By the part (b), there is $y \in X$ such that $\|y\| \le 1 + \varepsilon$ and $\langle y, x_i^* \rangle = \langle x^{**}, x_i^* \rangle$. If we let $x := (1+\varepsilon)^{-1} y$, then $x \in \overline{B}_X$ so that

$$|\langle x - x_0^{**}, x_i^* \rangle| = |\langle x - y, x_i^* \rangle| = |\langle \varepsilon x, x_i^* \rangle| \le \varepsilon \|x\| \|x_i^*\| < 1$$

for all i implies $y \in x + U$, hence we get $\overline{B}_X \cap (u + U) \neq \varnothing$. □

3.4 Krein-Milman theorem

Choquet theory

Exercises

3.10 (James' space). not reflexive but isometrically isomorphic to bidual

3.11 (Predual correspondence). Let X be a Banach space. Let

$$\{ (Y, \varphi) \mid \varphi : X \to Y^* \text{ is an isometric isorphism} \}$$

and

$$\{ Z \le X^* \mid \overline{B_X} \text{ is compact Hausdorff in } (X, \sigma(X, Z)) \}.$$

$$(Y, \varphi) \mapsto \operatorname{im} \varphi^*|_{J(Y)}$$

(a) The map is well-defined.

(b) The map is surjective. (by Goldstein)

(c) The map is injective up to isomorphism for Y.

3.12. Let X be a closed subspace of a Banach space Y and

$$i : X \to Y$$

the inclusion. Suppose X and Y have preduals X_* and Y_* respectively. Let

$$j := i^*|_{Y_*} : Y_* \to Z \subset X^*,$$

where $Z := i^*(Y_*)^-$. Then we can show

$$j^* : Z^* \subset X^{**} \to Y$$

coincides with i on $X \cap Z^*$. From the existence of X_* we have $X^{**} \to X$, which is restricted to define a map $k : Z^* \to X$.

$$\begin{tikzcd} & X \arrow[r, "i"] & Y \\ X^{**} \arrow[ur] \arrow[r] & Z^* \arrow[u, "k"] \arrow[ur, "j"] & \end{tikzcd}$$

We can show k is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

3.13 (Mazur's lemma).

3.14 (Dunford-Pettis property).

3.5 Polar topologies

Mackey-Arens

Part II

Banach spaces

Chapter 4

Fréchet, Banach, Hilbert spaces

4.1 Banach spaces

dual is Banach. Basis problem, Mazur' duck.

4.2 Hilbert spaces

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

4.1. (a) A Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

4.2 (Riesz representation theorem)**.** Let H be a Hilbert space over a field $\mathbb{F}$, which is either $\mathbb{R}$ of $\mathbb{C}$.

We use the bilinear form $\langle -,- \rangle : X \times X^* \to \mathbb{F}$ of canonical duality. *Dirac* notation $\langle - | - \rangle$ for the inner product of a complex Hilbert spaces such that $\langle x, y \rangle = \langle y | x \rangle$. The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each $x^* \in H^*$, there is a unique $x \in H$ such that $\langle y, x^* \rangle = \langle y, x \rangle$ for every $y \in H$.

(b) $H \to H^* : x \mapsto \langle -, x \rangle$ is a natural linear and anti-linear isomorphism if $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, respectively.

Chapter 5

Bounded linear operators

5.1 (Bounded belowness in Banach spaces). Let $T \in B(X, Y)$ for Banach spaces X and Y. The following statements are equivalent:

(a) T is bounded below.

(b) T is injective and has closed range.

(c) T is a topological isomorphism onto its image.

5.2 (Bounded belowness in Hilbert spaces). Let $T \in B(H, K)$ for Hilbert spaces H and K. The following statements are equivalent:

(a) T is bounded below.

(b) T is left invertible.

(c) T^* is right invertible.

(d) T^*T is invertible.

5.3 (Injectivity and surjectivity of adjoint). Let $T \in B(X, Y)$ for Banach spaces X and Y.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is bounded below.

5.4 (Normal operators). For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. Suppose T is normal.

(a) $\ker T = \ker T^*$.

(b) T is bounded below if and only if T is invertible.

(c) If T is surjective, then T is invertible.

5.5 (Invariant and Reducing subsapces). Let K be a closed subspace of H.

(a) K is reducing for T if and only if K is invariant for T and T^*.

(b) K is reducing for T if and only if $TP = PT$, where P is the orthogonal projection on K.

Chapter 6

Compact operators

$K(X,Y)$ is closed in $B(X,Y)$. $K(X)$ is an ideal of $B(X)$. adjoint is $K(X,Y) \to K(Y^*,X^*)$. integral operators are compact. riesz operator, quasi-nilpotent operator.

6.1 Finite-rank operators

6.2 Fredholm operators

6.1. A bounded linear operator $T : X \to Y$ between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

(a) A Fredholm operator T has closed range.

Proof. (a) Let C be a finite dimensional subsapce of Y such that $\operatorname{im} T \oplus C = Y$. Let $\widetilde{T} : X/\ker T \to Y$ be the induced operator of T. Define $S : (X/\ker T) \oplus C \to Y$ such that $S(x + \ker T, c) := \widetilde{T}(x + \ker T) + c$. Then, S is an topological isomorhpism between Banach spaces by the open mapping theorem, so $S(X/\ker T \oplus \{0\}) = \operatorname{im} \widetilde{T} = \operatorname{im} T$ is closed. □

6.2 (Atkinson's theorem)**.** An operator $T \in B(X,Y)$ is Fredholm if and only if there is $S \in B(Y,X)$ such that $TS - I$ and $ST - I$ is finite rank.

6.3 (Fredholm index)**.** locally constant, in particular, continuous. composition makes the addition of indices.

6.3 Nuclear operators

tensor products

Exercises

Problems

1. If $T \in B(L^2([0,1]))$ is a compact operator, then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|Tf\|_{L^2} \leq \varepsilon\|f\|_{L^2} + C_\varepsilon\|f\|_{L^1}.$$

Proof. 1. Suppose there is $\varepsilon > 0$ such that we have sequence $f_n \in L^2$ satisfying $\|f_n\|_2 = 1$ and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of T, there is a subsequence Tf_{n_k} converges to $g \neq 0$ in L^2. Then, $\|f_{n_k}\|_1 \to 0$ implies $f_{n_k} \to 0$ weakly in L^2, hence also for Tf_{n_k}. It means $g = 0$, which contradicts to the assumption. □

Part III

Spectral theory

Chapter 7

Chapter 8

Normal operators

8.1 Spectral theorem for compact normal operators

There is an orthonormal basis $E \subset H$ such that

$$T = \sum_{e \in E} \lambda_e |e\rangle\langle e|.$$

8.2 Spectral theorem for bounded normal operators

8.1 (Projection valued measure)**.** Let $(\Omega, \mathcal{M})$ be a measurable space and H a Hilbert space. A *projection valued measure* on Ω for H is a map $E : \mathcal{M} \to B(H)$ such that $E(A)$ is an orthogonal projection with $E(\varnothing) = 0$ and the set function $\mathcal{M} \to \mathbb{C} : A \mapsto \langle E(A)\xi, \eta\rangle$ is a complex measure on Ω for each ξ and $\eta \in H$. (regularity, it has also two definitions)

(a) The last condition is equivalent to the countable additivity: $E(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} E(A_i)$ in the strong operator topology of $B(H)$ for $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$.

(b) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{M}$.

Let $T \in B(H)$ be a normal operator. Then, there exists a spectral measure E on $\sigma(T)$ for H such that

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

This spectral measure E is also called the *resolution of the identity*.

8.3 Operator topologies

8.2 (Compact left multiplications and SOT)**.** Let T_n be a sequence of bounded linear operators on a Hilbert space that converges in SOT. For compact K, $T_n K$ converges in norm, but $K T_n$ generally does not unless T is self-adjoint.

8.3. Let f be a linear functional on $B(H)$ for a Hilbert space H. Then, TFAE:

(a) f is WOT-continuous,

(b) f is SOT-continuous,

(c) $f(T) = \sum_{i=1}^{n} \langle T x_i, y_i\rangle$ for some x_i, y_i.

Proof. (2) $\Rightarrow$ (3) is the only nontrivial implication. By the definition of SOT, there exists $v \in \mathcal{H}^n$ such that

$$|f(T)| \leq \|T^{\oplus n} v\|.$$

The functional $f : \mathcal{A} \to \mathbb{C}$ factors through $\mathcal{H}^n$ such that

$$\mathcal{A} \to v\mathcal{H}^n \to \mathbb{C}.$$

$\square$

Chapter 9

Unbounded operators

Kato-Rellich theorem

Part IV

Operator algebras

Chapter 10

Banach algebras

10.1 Spectral theory of unital Banach algebras

10.1 (Unital Banach algebras). (a) If $\|a\| < 1$, then $1 - a$ is invertible. So $\mathcal{A}^\times$ is open.

(b) $\mathcal{A}^\times \to \mathcal{A} : a \mapsto a^{-1}$ is differentiable.

(c) $\mathbb{C} \setminus \sigma(a) \to \mathcal{A} : \lambda \mapsto (\lambda - a)^{-1}$ is differentiable.

10.2 (Vector-valued complex function theory). Let Ω be an open subset of $\mathbb{C}$ and X a Banach space. For a vector-valued function $f : \Omega \to X$, we say f is *differentiable* if the limit

$$\lim_{\lambda \to \lambda_0} \mu^{-1}(f(\lambda) - f(\lambda_0))$$

exists in X, and *weakly differentiable* if the limit

$$\lim_{\lambda \to \lambda_0} \mu^{-1}\langle f(\lambda) - f(\lambda_0), x^* \rangle$$

exists in $\mathbb{C}$ for each $x^* \in X^*$. Then, the followings are all equivalent.

(a) f is differentiable.

(b) f is weakly differentiable.

(c) For each $\lambda_0 \in \Omega$, there is a sequence $(x_k)_{k=0}^\infty$ such that the power series

$$\sum_{k=0}^\infty (\lambda - \lambda_0)^k x_k$$

converges to $f(\lambda)$ absolutely and uniformly on any closed ball $\overline{B(\lambda_0, r)} \subset \Omega$.

10.3 (Gelfand-Mazur). $\sigma(a)$ is non-empty. In particular, if $\mathcal{A}^\times = \mathcal{A} \setminus \{0\}$, then $\mathcal{A} \cong \mathbb{C}$.

10.4 (Beurling).

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n} \leq \|a\|.$$

Proof. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < r(a)^{-1}$. Then we have $\lambda^{-1} \notin \sigma(a)$ so that $1 - \lambda a = \lambda(\lambda^{-1} - a)$ is invertible.

Then, $1 - \lambda a = \sum_{i=0}^\infty (\lambda a)^i$.

If $|\lambda| < \|a\|^{-1} \leq r(a)^{-1}$, then the inverse of $1 - \lambda a$ is given by the power series. If $|\lambda| < r(a)^{-1}$, then we can only deduce the invertibility of $1 - \lambda a$. Complex function theory let us to write the inverse even if we have only $|\lambda| < r(a)^{-1}$. Also, the radius of convergence is exactly $r(a)^{-1}$. □

10.5 (Spectrum in closed subalgebras). For fixed element, smaller the ambient algebra, less "holes" in the spectrum. Let $\mathcal{B} \subset \mathcal{A}$ be a closed subalgebra containing $1_{\mathcal{A}}$. Note that $\mathcal{B}$ may be unital even for $1_{\mathcal{A}} \notin \mathcal{B}$.

(a) $\mathcal{B}^\times$ is clopen in $\mathcal{A}^\times \cap \mathcal{B}$.

10.2 Ideals

10.6 (Ideals). (a) If I is a left ideal, then $\mathcal{A}/I$ is a left $\mathcal{A}$-module.

10.7 (Modular left ideals). A left ideal I is called *modular* if there is $e \in \mathcal{A}$ such that $a - ae \in I$ for all $a \in \mathcal{A}$. The element e is called a *right modular unit* for I.

(a) I is modular if and only if $\mathcal{A}/I$ is unital(?).

(b) A proper modular left ideal is contained in a maximal left ideal.

(c) I is a maximal modular left ideal if and only if I is a modular maximal left ideal.

(d) There is a non-modular maximal ideal in the disk algebra.

10.8 (Closed ideals). (a) closure of proper left ideal is proper left.

(b) maximal modular left ideal is closed.

10.9 (Unitization). Let $\mathcal{A}$ be an algebra. Recall that we always assume algebras are associative. Consider an embedding $\mathcal{A} \to B(\mathcal{A}) : a \mapsto L_a$, where $L_a(b) = ab$. Define

$$\widetilde{\mathcal{A}} := \{ L_a + \lambda \, \mathrm{id}_{B(\mathcal{A})} : a \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital $\mathcal{A}$.

(a) If $\mathcal{A}$ is normed, then $\widetilde{\mathcal{A}}$ is a normed algebra such that there is an isometric embedding $\mathcal{A} \to \widetilde{\mathcal{A}}$.

(b) If $\mathcal{A}$ is Banach, then $\widetilde{\mathcal{A}}$ is a Banach algebra.

(c) $\mathcal{A} \oplus \mathbb{C}$ is topologically isomorphic to $\widetilde{\mathcal{A}}$ as normed spaces.

Proof. (a) The space of bounded operators $B(\mathcal{A})$ is a normd algebra. Then, $\widetilde{\mathcal{A}}$ is a normed $*$-algebra with induced norm

$$\|L_a + \lambda \, \mathrm{id}_{B(\mathcal{A})}\| = \sup_{b \in \mathcal{A}} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then, $\mathcal{A}$ is a normed $*$-subalgebra of $\widetilde{\mathcal{A}}$ because the norm and involution of $\mathcal{A}$ agree with $\widetilde{\mathcal{A}}$.

(b) Suppose (x_n, λ_n) is Cauchy in $\widetilde{\mathcal{A}}$. Since $\mathcal{A}$ is complete so that it is closed in $\widetilde{\mathcal{A}}$, we can induce a norm on the quotient $\widetilde{\mathcal{A}}/\mathcal{A}$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|x\| \le \|(x, \lambda)\| + |\lambda|$ shows that x_n is Cauchy in $\mathcal{A}$.

Since a finite dimensional normed space is always Banach and $\mathcal{A}$ is Banach, λ_n and x_n converge. Finally, the inequality $\|(x, \lambda)\| \le \|x\| + |\lambda|$ implies that (x_n, λ_n) converges.

(c) Check the topology on $\mathcal{A} \oplus \mathbb{C}$ in detail... $\square$

unitization, homomorphisms, category(direct sum, product, etc.)
$B(\mathbb{C}^n)$ is simple, but $B(X)$ is not simple.

10.3 Gelfand theory of commutative Banach algebras

also important spectrum for non-unital banach algebras Banach algebra of single generator semisimplicity and symmetricity

10.10 (Spectrum of a Banach algebra)**.** Let $\mathcal{A}$ be a commutative Banach algbera. A *character* of $\mathcal{A}$ is a non-zero algebra homomorphism $\varphi : \mathcal{A} \to \mathbb{C}$. Denote by $\sigma(\mathcal{A})$ the set of all characters of $\mathcal{A}$. We will show that all characters are bounded. Then, endow with the weak* topology on $\sigma(\mathcal{A})$ from the inclusion $\sigma(\mathcal{A}) \subset \mathcal{A}^*$. We call this space as the *spectrum* of $\mathcal{A}$. Let $\varphi \in \sigma(\mathcal{A})$.

(a) $\|\varphi\| = 1$.

(b) If $\mathcal{A}$ is unital, then $\sigma(\mathcal{A})$ is compact and Hausdorff.

(c) Even if $\mathcal{A}$ is non-unital, $\sigma(\mathcal{A})$ is locally compact and Hausdorff.

10.11 (Gelfan-Naimark representation)**.** Let $\mathcal{A}$ be a commutative Banach algebra.

$$\Gamma : \mathcal{A} \to C_0(\sigma(\mathcal{A})).$$

(a) $\Gamma(\mathcal{A})$ separates points.

(b) Γ has closed range if

(c) Γ is injective if

(d) Γ is isometric if $r(a) = \|a\|$ for all $a \in \mathcal{A}$.

10.4 Holomorphic functional calculus

Dunford-Reisz functional calculus

Exercises

10.12. Let $\mathcal{A}$ be a unital algebra.

(a) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

(b) If $\sigma(a)$ is non-empty, then $\sigma(p(a)) = p(\sigma(a))$.

Proof. (a) Intuitively, the inverse of $1-ab$ is $c = 1+ab+abab+\cdots$. Then, $1+bca = 1+ba+baba+\cdots$ is the inverse of $1-ba$. □

$C_b(\Omega)$ $\ell^\infty(S)$ $L^\infty(\Omega)$ $B_b(\Omega)$ $A(\mathbb{D})$ $B(X)$

10.13. In $C(\mathbb{R})$, the modular ideals correspond to compact sets.

10.14 (Disk algebra)**.** (a) Every continuous homomorphism is an evaluation.

10.15 (Polynomial convexity)**.** (conway)

10.16 (Inclusion relation on spectra)**.** (a) $\sigma(a+b) \subset \sigma(a)+\sigma(b)$ and $\sigma(ab) \subset \sigma(a)\sigma(b)$ for unital cases.

(b) $\sigma(a^{-1}) = \sigma(a)^{-1}$ for unital cases.

(c) $r(a)^n = r(a^n)$.

spectral radius is upper semi-continuous

Chapter 11

C*-algebras

11.1 C* identity

Banach $*$-algebra: $\|a^*\| = \|a\|$.

11.1 (C* identity). A normed $*$-algebra $\mathcal{A}$ is called a *C*-algebra* if

(a) $\mathcal{A}$ is Banach,

(b) $\mathcal{A}$ satisfies the C*-identity: $\|x^*x\| = \|x\|^2$.

11.2 (Unitization of C*-algebras).

$$(L_a + \lambda \operatorname{id}_{B(\mathcal{A})})^* = L_{a^*} + \overline{\lambda} \operatorname{id}_{B(\mathcal{A})}.$$

Proof. The C*-identity easily follows from the following inequality:

$$\begin{aligned}
\|(x,\lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\
&= \sup_{\|y\|=1} \|(xy+\lambda y)^*(xy+\lambda y)\| \\
&= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y)\| \\
&\le \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y\| \\
&= \|(x,\lambda)^*(x,\lambda)\|.
\end{aligned}$$

□

11.3 (Spectra of normal elements). Let $\mathcal{A}$ be a C*-algebra, and $\widetilde{\mathcal{A}}$ be its unitization. We say an element $a \in \widetilde{\mathcal{A}}$ is *unitary* if $a^*a = aa^* = e$, and say an element $a \in \mathcal{A}$ is *self-adjoint* if $a^* = a$.

(a) If $a \in \widetilde{\mathcal{A}}$ is unitary, then $\sigma(a) \subset \mathbb{T}$.

(b) If $a \in \mathcal{A}$ is self-adjoint, then $\sigma(a) \subset \mathbb{R}$.

(c) The converses of the parts (a) and (b) are not generally true.

Proof. (a)

(b) We may assume $\mathcal{A}$ is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \in \mathcal{A},$$

and the inverse of e^{ia} is e^{-ia}. Since the involution $^* : \mathcal{A} \to \mathcal{A}$ is continuous, we can check e^{ia} is unitary by

$$(e^{ia})^* = \sum_{n=1}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia}.$$

For every $\varphi \in \sigma(\mathcal{A})$, then by the part (a) the equality

$$e^{-\operatorname{Im}\varphi(a)} = |e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$$

proves $\varphi(a) \in \mathbb{R}$, hence $\sigma(a) \subset \mathbb{R}$.

(c) Let $\mathcal{A} = M_2(\mathbb{C})$ and $a = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then, $\sigma(a) = \{1\}$ but a is neither unitary nor self-adjoint. We will show in the next section that the converses hold if we assume a is normal. □

11.4 (∗-homomorphisms). (a) determined by self-adjoint elements

(b) norm-decreasing

(c)

11.2 Continuous functional calculus

11.5 (Gelfand-Naimark representation for C*-algebras). For a commutative unital C*-algebra $\mathcal{A}$, consider the Gelfand transform $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$.

(a) Γ is a $*$-homomorphism.

(b) Γ is an isometry.

(c) Γ is a $*$-isomorphism.

Proof. (a)

(b) Note that we have

$$\|\Gamma a\| = \sup_{\varphi \in \sigma(\mathcal{A})} |\Gamma a(\varphi)| = \sup_{\varphi \in \sigma(\mathcal{A})} |\varphi(a)| = r(a)$$

for all $a \in \mathcal{A}$. If we assume a is self-adjoint, then since $\|a\|^2 = \|a^*a\| = \|a^2\|$, the spectral radius coincides with the norm by the Beurling formula for spectral radius in Banach algebras:

$$\|\Gamma a\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Hence we have for all $a \in \mathcal{A}$ that

$$\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|(\Gamma a)^*\Gamma a\| = \|\Gamma a\|^2.$$

(c) By the part (a) and (b), the image $\Gamma(\mathcal{A})$ is a closed unital $*$-subalgebra of $C(\sigma(\mathcal{A}))$, and it separates points by definition. Then, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$ by the Stone-Weierstrass theorem, which implies $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A})$. □

11.6 (Finitely generated C*-algebras). joint spectrum.

11.7 (Continuous functional calculus). Let $\mathcal{A}$ be a C*-algebra, and $a \in \mathcal{A}$ a normal element. Then, we have an isometric $*$-homomorphism

$$C(\sigma(a)) \to \mathcal{A}$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

(a) $\mathrm{id} \mapsto a$.

(b) $(f+g)(a) = f(a) + g(a)$ and $(fg)(a)$.

(c) $(f \circ g)(a) = f(g(a))$.

11.3 Positive linear functionals

11.8. (a) If $a, b \geq 0$, then $a + b \geq 0$.

(b) If $a^*a \leq 0$, then $a^*a = 0$.

(c) $a^*a \geq 0$ for all $a \in \mathcal{A}$.

11.9 (Operator monotone functions)**.** (a) inverse

(b) conjugation

11.10 (Operator monotonicity of square and commitativity)**.** Let $\mathcal{A}$ be a C*-algebra in which the square function is operator monotone, that is, $0 \leq a \leq b$ implies $a^2 \leq b^2$ for any positive elements a and b in $\mathcal{A}$. We are going to show that $\mathcal{A}$ is necessarily commutative. Let a and b denote arbitrary positive elements of $\mathcal{A}$.

(a) Show that $ab + ba \geq 0$.

(b) Let $ab = c + id$ where c and d are self adjoints. Show that $d^2 \leq c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \leq c^2$. Show that $c^2d^2 + d^2c^2 - 2\lambda d^4 \geq 0$.

(d) Show that $\lambda(cd + dc)^2 \leq (c^2 - d^2)^2$.

(e) Show that $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \leq c^2$ and deduce $d = 0$.

(f) Extend the result for general exponent: $\mathcal{A}$ is commitative if $f(x) = x^\beta$ is operator monotone for $\beta > 1$.

11.11 (Injective $*$-homomorphism is an isometry)**.**

11.12 (States on unitization)**.** Let $\mathcal{A}$ and $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be a C*-algebra and its unitization respectively. Let $\widetilde{\rho} = \rho \oplus \lambda$ be a bounded linear functional on $\widetilde{\mathcal{A}}$, where $\rho \in \mathcal{A}^*$ and $\lambda \in \mathbb{C}^* = \mathbb{C}$.

(a) $\widetilde{\rho}$ is positive if and only if $\lambda \geq 0$ and $0 \leq \rho \leq \lambda$.

(b) $\widetilde{\rho}$ is a state if and only if $\lambda = 1$ and ρ is positive with $\|\rho\| \leq 1$.

(c) $\widetilde{\rho}$ is a pure state if and only if $\lambda = 1$ and ρ is either a pure state or zero.

Exercises

11.13. Let $\mathcal{B}$ be a hereditary C*-subalgebra of a C*-algebra $\mathcal{A}$. Let $a \in \mathcal{A}^+$. If for any $\varepsilon > 0$ there is $b \in \mathcal{B}^+$ such that $a - \varepsilon \leq b$, then $a \in B^+$.

Proof. To catch the idea, suppose $\mathcal{A}$ is abelian. We want to approximate a by the elements of $\mathcal{B}$ in norm. To do this, for each $\varepsilon > 0$, we want to construct $b' \in \mathcal{B}^+$ such that $a - \varepsilon \leq b' \leq a + \varepsilon$ using b. Taking $b' = \min\{a, b\}$ is impossible in non-abelian case, but we can put $b' = \frac{a}{b+\varepsilon} b$. For a simpler proof, $b' = (\frac{\sqrt{ab}}{\sqrt{b}+\sqrt{\varepsilon}})^2$ is a better choice.

Define

$$b' := \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} a \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}}.$$

Then,

$$\|\sqrt{a} - \sqrt{a} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}}\|^2 = \|\frac{\sqrt{\varepsilon}}{\sqrt{b} + \sqrt{\varepsilon}} a \frac{\sqrt{\varepsilon}}{\sqrt{b} + \sqrt{\varepsilon}}\| \leq \varepsilon$$

implies

$$\lim_{\varepsilon \to 0} b' = \lim_{\varepsilon \to 0} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} \sqrt{a} \cdot \sqrt{a} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} = \sqrt{a} \cdot \sqrt{a} = a.$$

□

Problems

*1. A C*-algebra is commutative if and only if a function $f(x) = x(1+x)^{-1}$ is operator subadditive.

Chapter 12

Von Neumann algebras

12.1 Von Neumann algebras

12.1 (Von Neumann algebras)**.** A C*-algebra $\mathcal{A}$ is called a *von Neumann algebra* if there is a isometric $*$-homomorphism $\mathcal{A} \to B(H)$ for a Hilbert space H whose image is closed in the weak operator topology.

12.2 (Vigier theorem)**.** Increasing bounded net is convergent in strong operator topology. The boundedness is important because we have to construct a bounded sesquilinear form using the monotone convergence in $\mathbb{R}$.

12.3 (Bicommutant theorem)**.** Let $\mathcal{A}$ be a non-degenerate C*-subalgebra of $B(H)$.

(a) $\mathcal{A}'$ and $\mathcal{A}''$ are weakly closed.

(b) For $a \in \mathcal{A}''$ and $\xi \in H$, there is a sequence $a_n \in \mathcal{A}$ such that $a_n(\xi) \to a(\xi)$.

(c) For $a \in \mathcal{A}''$ and $\xi_1, \cdots, \xi_m \in H$, there is a sequence $a_n \in \mathcal{A}$ such that $a_n(\xi_i) \to a(\xi_i)$ for all i.

(d) $\mathcal{A}$ is von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.

Proof. (b) Let $K := \overline{\mathcal{A}\xi}$ be the cyclic subspace of ξ in H and p its orthogonal projection. We claim $a\xi \in K$. For every $b \in \mathcal{A}$, we have $bK \subset K$ because the multiplication by b is continuous on H, and $b^*K \subset K$ because $\mathcal{A}$ is self-adjoint. It means that K reduces all $b \in \mathcal{A}$, and then $bp = pb$ implies $ap = pa$, so K also reduces a. Therefore, $aK \subset K$ proves $a\xi = \lim_\alpha e_\alpha a\xi \in K$, where e_α is an approximate identity of $\mathcal{A}$.

(e) Since $\overline{\mathcal{A}}^{\mathrm{WOT}}$ is closed convex, $\overline{\mathcal{A}}^{\mathrm{SOT}} = \overline{\mathcal{A}}^{\mathrm{WOT}}$. Also, $\mathcal{A}''$ is weakly closed, $\overline{\mathcal{A}}^{\mathrm{WOT}} \subset \mathcal{A}''$. □

12.4 (Kaplansky density theorem)**.**

12.2 Borel functional calculus

resolution of identity normal operator theories: multiplicity, invariant subspaces L^∞ representation

12.5 (Borel functional calculus)**.** Let $\mathcal{A}$ be a von Neumann algebra.

$$B^\infty(\sigma(a)) \to \mathcal{A}.$$

(a) The Borel functional calculus is in general not injective.

(b) If we endow the topology of pointwise convergence on $B^\infty(\sigma(a))$ and the strong operator topology on $\mathcal{A}$, then the Borel functional calculus is continuous.

(c) not isometric, even if it is injective.

(d) Every von Neumann algebra is the closed span of projections.

12.6. (b) By the bounded convergence theorem.

(d) This is because $\sigma(a) \subset \mathbb{C}$ is compact so that it is separable and metrizable; every bounded measurable function is a pointwise limit of simple functions.

12.3 Representations of C*-algebras

12.7 (Representation of C*-algebras). A *representation* of a C*-algebra is a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$ for a Hilbert space H.

12.8 (Non-degenerate representation). Let $\pi : \mathcal{A} \to B(H)$ be a representation of a C*-algebra $\mathcal{A}$. We say π is *non-degenerate* if $\pi(\mathcal{A})H$ is dense in H.

(a) π is non-degenerate.

(b) For each $\xi \in H$ there is $a \in \mathcal{A}$ such that $\pi(a)\xi \neq 0$.

(c) $\pi(e_\alpha) \to \mathrm{id}_H$ strongly for every approximate identity e_α of $\mathcal{A}$.

12.9 (Cyclic representation). Let $\pi : \mathcal{A} \to B(H)$ be a representation of a C*-algebra $\mathcal{A}$.

(a)

12.10 (Irreducible representation). Let $\pi : \mathcal{A} \to B(H)$ be a representation of a C*-algebra $\mathcal{A}$. We say π is irreducible if there is no proper closed subspace $K \subset H$ such that $\pi(a)K \subset K$.

(a) π is irreducible.

(b) $\pi(\mathcal{A})' = \mathbb{C}\,\mathrm{id}_H$.

(c) $\pi(\mathcal{A})$ is strongly dense in $B(H)$.

(d) Every non-zero vector is cyclic.

12.11 (Gelfand-Naimark-Segal representation). Let $\mathcal{A}$ be a C*-algebra, and ρ be a state on $\mathcal{A}$.

(a) The left kernel $L_\rho := \{a \in \mathcal{A} : \rho(a^*a) = 0\}$ is a left ideal of $\mathcal{A}$.

(b) $\langle a + L, b + L\rangle := \rho(b^*a)$ is an inner product on $\mathcal{A}/L_\rho$.

(c) There is a unique representation $\pi_\rho : \mathcal{A} \to B(H_\rho)$ such that $\pi_\rho(a)(b + L) := ab + L$ for $a, b \in \mathcal{A}$.

(d) $\pi_\rho : \mathcal{A} \to B(H_\rho)$ is a cyclic representation.

12.12 (Representations of $C_0(\Omega)$). Let $\mathcal{A} = C_0(\Omega)$ and μ be a state on $\mathcal{A}$, a regular Borel probability measure on Ω.

(a) The left kernel of μ is $L_\mu = \{f \in \mathcal{A} : f|_{\mathrm{supp}\,\mu} = 0\}$.

(b) The quotient is $\mathcal{A}/L_\mu \cong C(\mathrm{supp}\,\mu)$ so that $H_\mu = L^2(\mathrm{supp}\,\mu, \mu)$.

(c) The canonical cyclic vector is the unity function.

12.13 (Representations of $K(H)$).

12.14 (Kadison transitivity theorem).

12.15 (Left ideals).

12.16 (Primitive ideals).

12.17 (Hull-kernel topology).

12.4 Factors and traces

Every trace of factor is faithful

12.18. Normal states is a state in which the monotone convergence theorem holds. Precisely, a state ρ is *normal* if a monotone net a_α strongly converges to a then $\rho(a_\alpha) \to \rho(a)$.