Operator Algebra

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Part I

C*-algebras

Chapter 1

Basic concepts

1.1 Multiplier algebra

1.1 (Multiplier algebra)**.** Let $\mathcal{A}$ be a C*-algebra. A *double centralizer* of $\mathcal{A}$ is a pair (L, R) of bounded linear maps on $\mathcal{A}$ such that $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$. The *multiplier algebra* $M(\mathcal{A})$ of $\mathcal{A}$ is defined to be the set of all double centralizers of $\mathcal{A}$.

1.2 (Essential ideals)**.** (a) Hilbert C*-module description

1.3 (Examples of multiplier algebras)**.** (a) $M(K(H)) \cong B(H)$.

(b) $M(C_0(\Omega)) \cong C_b(\Omega)$.

Proof. (a)

(b) First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega) \cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega) \cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective *-homomorphism $C_b(\Omega) \to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g \in M(C_0(\Omega))^+$. □

1.4 (Strict topology)**.**

1.2 Hereditary C*-subalgebras

1.5 (Hereditary C*-subalgebra and state embedding)**.**

Chapter 2

Representation theory

2.1 States and pure states

Chapter 3

Part II

Von Neumann algebras

Chapter 4

Factor classifications

4.1

4.1 (Semi-finite traces)**.** Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x \in M^+$ has a net $x_\alpha \in L^1(M,\tau)^+$ such that $x_\alpha \uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x) = \sup\{\, \tau(y) : y \le x,\ y \in L^1(M,\tau)^+ \,\} \quad \text{for} \quad x \in M^+.$$

4.2

Direct integral of factors.

Type I factors. It possess a minimal projection. It is isomorphic to the whole $B(H)$ for some Hilbert space. Therefore, it is classified by the cardinality of H.

Type II factors. No minimal projection, but there are non-zero finite projections so that every projection can be "halved" by two Murray-von Neumann equivalent projections.

In type II_1 factors, the identity is a finite projection Also, Murray and von Neumann showed there is a unique finite tracial state and the set of traces of projections is $[0,1]$. Free probability theory attacks the free groups factors, which are type II_1.

In type II_∞ factors There is a unique semifinite tracial state up to rescaling and the set of traces of projections is $[0,\infty]$.

In type III factors no non-zero finite projections exists. Classified the $\lambda \in [0,1]$ appeared in its Connes spectrum, they are denoted by III_λ. Tomita-Takesaki theory. It is represented as the crossed product of a type II_∞ factor and $\mathbb{R}$.

Amenability, equivalently hyperfiniteness is a very nice condition in von Neumann algebra theory. Group-measure space construction can construct them. There are unique hyperfinite type II_1 and II_∞ factors, and their property is well-known. Fundamental groups of type II factors, discrete group theory, Kazhdan's property (T) are used.

Tensor product facctors such as Araki-Woods factors and Powers factors.

4.3 Hyperfinite factors

weight, trace, state.

finite trace=tracial state.

4.2 (Uniformly hyperfinite algebras)**.** Let $\mathcal{A}$ be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique finite trace.

(b) Every UHF algebra admits a unique finite trace.

(c) Every hyperfinite

4.3 (Classification of UHF algebras)**.**

Chapter 5

Weight theory

Chapter 6

6.1 Connes' bicentralizer problem

Part III

Operator K-theory

Chapter 7

Brown-Douglas-Fillmore theory

7.1 (Haagerup property)**.**

Baum-Connes conjecture Non-commutative geometry Elliott theorem

Chapter 8

Approximately finite algebras

Elliott conjecture: amenable simple separable C*-algerbas are classified by K-theory.

Chapter 9

Crossed products and dynamical systems

Part IV

Subfactor theory

The way how quantum systems are decomposed. Quite combinatorial! And has Galois analogy.

9.1 (Jones index theorem)**.** A *subfactor* of a factor M is a factor N containing 1_M.

Tensor categories and topological invariants of 3-folds. Ergodic flows.