General Topology

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Part I

Topological spaces

Chapter 1

Topologies

1.1 Topologies

A *directed set* is a preordered set in which every finite subset has a upper bound. A *downward directed set* is a preordered set in which every finite subset has a lower bound. A *filter* is a partially ordered set which is non-empty, downward directed, upward closed.

1.1 (Filters)**.** Let X be a set. A *filter base* on X is a family of subsets of X that is non-empty and downward directed. We say a filter base $\mathcal{B}$ is coarser than another filter base $\mathcal{B}'$ if for every $U \in \mathcal{B}$ there is $U' \in \mathcal{B}'$ such that $U' \subset U$. A *filter* on X is a filter base on X that is upward closed. For a filter base $\mathcal{B}$ on X, the smallest filter $\mathcal{F}$ on X containing $\mathcal{B}$ is said to be generated by $\mathcal{B}$.

(a) Let $\mathcal{F}$ be the filter generated by a filter base $\mathcal{B}$. Then, $U \in \mathcal{F}$ if and only if there is $V \subset X$ such that $V \subset U$ and $V \in \mathcal{B}$.

(b) A filter base is a filter if and only if it is the maximal one in its equivalence class.

(c) $\mathcal{B}$ is coarser than $\mathcal{B}'$ if and only if $\mathcal{F} \subset \mathcal{F}'$.

1.2 (Topologies)**.** Let X be a set. A *topological base* on X is a family $\mathcal{B}$ of subsets of X such that the restriction $\mathcal{B}_x := \{U \in \mathcal{B} : x \in U\}$ is a filter base for each $x \in X$. We say a topological base $\mathcal{B}$ is coarser than another topological base $\mathcal{B}'$ if $\mathcal{B}_x$ is coarser than $\mathcal{B}'_x$ for each $x \in X$. A *topology* on X is a topological base on X that is closed under union and contains an empty set. For a topological base $\mathcal{B}$ on X, the smallest topology $\mathcal{T}$ on X containing $\mathcal{B}$ is said to be generated by $\mathcal{B}$.

(a) Let $\mathcal{T}$ be the topology generated by a topological base $\mathcal{B}$. Then, $U \in \mathcal{T}$ if and only if U is the union of elements of $\mathcal{B}$.

(b) A topological base is a topology if and only if it is the maximal one in its equivalence class.

(c) $\mathcal{B}$ is coarser than $\mathcal{B}'$ if and only if $\mathcal{T} \subset \mathcal{T}'$.

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1.3. If $A^\circ \subset B$ and B is closed, then $(A \cup B)^\circ \subset B$.

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A filter is a non-empty downward directed and upward closed partially ordered set. Note that every partially ordered set can be embedded in the power set of a set. Let $\mathcal{F}$ be a filter on a set $\mathcal{A}$ and $\mathcal{A} \to X : \alpha \mapsto x_\alpha$ be a function. We say the function $\alpha \mapsto x_\alpha$ converges to x_0 if the neighborhood filter $\mathcal{F}_{x_0}$ of x_0 is contained in the pushforward filter of $\mathcal{F}$ by x_α.

When we consider the case that $\mathcal{A}$ is a directed set, i.e. x_α is a net, then since a directed set $\mathcal{A}$ has a natural filter generated by the filter base $\{\{\alpha \in \mathcal{A} : \alpha \geq \alpha_0\} : \alpha_0 \in \mathcal{A}\}$, thus every net in X derives a filter on X by pushforward.

3.1 (Ultrafilter convergence theorem). (a) X is compact if and only if every ultrafilter on X converges at least one point.

(b) X is Hausdorff if and only if every ultrafilter on X converges at most one point.

3.2 (Non-principal ultrafilter). (a) Every filter on a set X is contained in an ultrafilter.

(b) Every infinite set X

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3.4 Sequential spaces

3.3 (Sequential spaces). Let X be a topological space. A subset U is called *sequentially open* in X if every sequence that converges to a point in U is eventually in U. A *sequential space* is a topological space in which every sequentially open set is open.

(a) Every open set is sequentially open.

(b)

3.4 (Sequential continuity). Let $f : X \to Y$ be a map between topological spaces.

(a) If X is sequential, then the sequential continuity implies the continuity of f.

Proof. Suppose f is not continuous so that there is an open set V in Y such that $f^{-1}(V)$ is not open. Since X is sequential, $f^{-1}(Y)$ is not sequentially open. Take a seuqence x_n in $X \setminus f^{-1}(V)$ that converges to $x \in f^{-1}(V)$. Then, $f(x_n)$ dose not converges since $f(x_n) \notin V$ and $f(x) \in V$, so f is not sequentially continuous. $\square$

3.5.

important: existence of convergent subnet

no convergent subsequence implies the discreteness(closedness) of the sequence convergent sequence is relatively compact there is a convergent subnet relatively compact if X is locally compact every infinite set is cofinal

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Chapter 7

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7.1 Equivalence of metrics

7.1 (Comparison of metrics)**.** Let d_1 and d_2 be metrics on a set X. We say that d_1 is *stronger than* d_2 (equivalently, d_2 is *weaker than* d_1) or d_1 *refines* d_2 if for any $x \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon),$$

where B_1 and B_2 refer to balls within the metrics d_1 and d_2 respectively.

(a) This refinement relation is a preorder.

(b) d_1 is stronger than d_2 if and only if every sequence that converges to $x \in X$ in d_1 converges to x in d_2.

(c) d_1 is stronger than d_2 if and only if the identity map $\mathrm{id} : (X, d_1) \to (X, d_2)$ is continuous.

Proof. (a) It is enough to show the transitivity. Suppose there are three metric d_1, d_2, and d_3 on a set X such that d_1 is stronger than d_2 and d_2 is stronger than d_3. For $i = 1, 2, 3$, let B_i be a notation for the balls defined with the metric d_i.

Take $x \in X$ and $\varepsilon > 0$ arbitrarily. Then, we can find $\varepsilon' > 0$ such that

$$B_2(x, \varepsilon') \subset B_3(x, \varepsilon).$$

Also, we can find $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon').$$

Therefore, we have $B_1(x, \delta) \subset B_3(x, \varepsilon)$ which implies that d_1 refines d_3.

(b) ($\Rightarrow$) Let $\{x_n\}_n$ be a sequence in X that converges to x in d_1. By the assumption, for an arbitrary ball $B_2(x, \varepsilon) = \{y : d_2(x, y) < \varepsilon\}$, there is $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon),$$

where $B_1(x, \delta) = \{y : d_1(x, y) < \delta\}$. Since $\{x_n\}_n$ converges to x in d_1, there is an integer n_0 such that

$$n > n_0 \implies x_n \in B_1(x, \delta).$$

Combining them, we obtain an integer n_0 such that

$$n > n_0 \implies x_n \in B_2(x, \varepsilon).$$

It means $\{x_n\}$ converges to x in the metric d_2.

($\Leftarrow$) We prove it by contradiction. Assume that for some point $x \in X$ we can find $\varepsilon_0 > 0$ such that there is no $\delta > 0$ satisfying $B_1(x, \delta) \subset B_2(x, \varepsilon_0)$. In other words, at the point x, the difference set $B_1(x, \delta) \setminus B_2(x, \varepsilon_0)$ is not empty for every $\delta > 0$. Thus, we can choose x_n to be a point such that

$$x_n \in B_1\left(x, \tfrac{1}{n}\right) \setminus B_2(x, \varepsilon_0)$$

for each positive integer n by putting $\delta = \frac{1}{n}$.

We claim $\{x_n\}_n$ converges to x in d_1 but not in d_2. For $\varepsilon > 0$, if we let $n_0 = \lceil \frac{1}{\varepsilon} \rceil$ so that we have $\frac{1}{n_0} \leq \varepsilon$, then

$$n > n_0 \quad \Longrightarrow \quad x_n \in B_1\left(x, \tfrac{1}{n}\right) \subset B_1(x, \varepsilon).$$

So $\{x_n\}_n$ converges to x in d_1. However in d_2, for $\varepsilon = \varepsilon_0$, we can find such n_0 like d_1 since

$$x_n \notin B_2(x, \varepsilon_0)$$

for every n. Therefore, $\{x_n\}$ does not converges to x in d_2. □

7.2 (Equivalence of metrics)**.** Let d_1 and d_2 be metrics on a set X. They are said to be *(topologically) equivalent* if they refines each other.

(a) d_1 and d_2 are equivalent if for each x in X there exist two constants C_1 and C_2 such that $d_2(x, y) \leq C_1 d_1(x, y)$ and $d_1(x, y) \leq C_2 d_2(x, y)$ for all y in X.

(b) d_1 and d_2 are equivalent if $d_2 = f \circ d_1$ for a monotonically increasing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is continuous at 0.

Proof. (a) Let d_1 and d_2 be metrics on a set X. Suppose for each point x there exists a constant C which may depend on x such that $d_2(x, y) \leq C d_1(x, y)$ for all $y \in Y$. We will show d_1 is stronger than d_2.

(b) For any ball $B_1(x, \varepsilon)$, we have a smaller ball

$$B_2(x, f(\varepsilon)) \subset B_1(x, \varepsilon)$$

since $f(d(x, y)) < f(\varepsilon)$ implies $d(x, y) < \varepsilon$. Conversely, take an arbitrary ball $B_2(x, \varepsilon)$. Since f is continuous at 0, we can find $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad f(d(x, y)) < \varepsilon,$$

which implies $B_1(x, \delta) \subset B_2(x, \varepsilon)$. □

A metric can be viewed as a function that takes a sequence as input and returns whether the sequence converges or diverges. That is, a metric acts like a criterion which decides convergence of sequences. Take note on the fact that the sequence of real numbers defined by $x_n = \frac{1}{n}$ converges in standard metric but diverges in discrete metric. Like this example, even for the same sequence on a same set, the convergence depends on the attached metrics. What we are interested in is comparison of metrics and to find a proper relation structure. If a sequence converges in a metric d_2 but diverges in another metric d_1, we would say d_1 has stronger rules to decide the convergence. Refinement relation formalizes the idea.

Unlike metrics, there exist two different topologies that have same sequential convergence data. For example, a sequence in an uncountable set with cocountable topology converges to a point if and only if it is eventually at the point, which is same with discrete topology. This means the informations of sequence convergence are not sufficient to uniquely characterize a topology. Instead, convergence data of generalized sequences also called nets, recover the whole topology. For topologies having a property called the first countability, it is enough to consider only usual sequences in spite of nets. What we did in this subsection is not useless because topology induced from metric is a typical example of first countable topologies. These kinds of problems will be profoundly treated in Chapter 3.

One can ask some results for the equivalence of metrics characterized by a same set of continuous functions. However, they are generally difficult problems: is it possible to recover the base space from a continuous function space or a path space?

Topologies are occasionally described by not a single but several metrics. It provides a useful method to construct a metric or topology, which can be applied to a quite wide range of applications. Specifically, in a conventional way, metrics can be summed or taken maximum to make another metric out of olds. The following proposition can be easily generalized to an arbitrary finite number of metrics by mathematical induction.

Proposition 7.1.1. *Let d_1 and d_2 be metrics on a set X. For a sequecne $\{x_n\}_n$ in X, the following statements are all equivalent:*

(a) *it converges to x in both d_1 and d_2,*

(b) *it converges to x in $d_1 + d_2$,*

(c) *it converges to x in $\max\{d_1, d_2\}$.*

In particular, the metrics $d_1 + d_2$ and $\max\{d_1, d_2\}$ are equivalent.

Proof. We skip to prove $d_1 + d_2$ and $\max\{d_1, d_2\}$ are metrics.

(b) or (c)⇒(a) The inequalities $d_i \le d_1 + d_2$ and $d_i \le \max\{d_1, d_2\}$ imply the desired results.

(a)⇒(b) For $\varepsilon > 0$, we may find positive integers n_1 and n_2 such that $n > n_1$ and $n > n_2$ imply $d_1(x_n, x) < \frac{\varepsilon}{2}$ and $d_2(x_n, x) < \frac{\varepsilon}{2}$ respectively. If we define $n_0 := \max\{n_1, n_2\}$, then

$$n > n_0 \quad \Longrightarrow \quad d_1(x_n, x) + d_2(x_n, x) < \varepsilon.$$

(a)⇒(c) Take n_0 as we did previously. Then,

$$n > n_0 \quad \Longrightarrow \quad \max\{d_1(x_n, x), d_2(x_n, x)\} < \frac{\varepsilon}{2} < \varepsilon.$$

□

Remark 7.1.2. In general, for any norm $\|\cdot\|$ on $\mathbb{R}^2$, the function $\|(d_1, d_2)\|$ defines another equivalent metric.

There is also a method for combining not only finite family of metrics, but also infinite family of metrics. Since the sum of infinitely many positive numbers may diverges to infinity, we cannot sum the metrics directly. The strategy is to “bound” the metrics. We call a metric bounded when the range of metric function is bounded.

Proposition 7.1.3. *Every metric possesses an equivalent bounded metric.*

Proof. Let d be a metric on a set. Let f be a bounded, monotonically increasing, and subadditive function on $\mathbb{R}_{\ge 0}$ that is continuous at 0 and satisfies $f^{-1}(0) = \{0\}$. The mostly used examples are

$$f(x) = \frac{x}{1+x} \quad \text{and} \quad f(x) = \min\{x, 1\}.$$

Then, $f \circ d$ is a bounded metric equivalent to d by Example 1.4. □

Definition 7.1.4. Let d be a metric on a set X. A *standard bounded metric* means either metric

$$\min\{d, 1\} \quad \text{or} \quad \frac{d}{d+1},$$

and we will denote it by $\widehat{d}$.

Proposition 7.1.5. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a countable family of metrics on a set X. For a sequence $\{x_n\}_n$ in X, the following statements are all equivalent:*

(a) *it converges in d_i for every i,*

(b) *it converges in a metric*

$$d(x,y) := \sum_{i\in\mathbb{N}} 2^{-i}\widehat{d}_i(x,y),$$

(c) *it converges in a metric*

$$d'(x,y) := \sup_{i\in\mathbb{N}} i^{-1}\widehat{d}_i(x,y).$$

In particular, the metrics d and d' are equivalent.

Proof. The functions d and d' in (b) and (c) are well-defined by the monotone convergence theorem and the least upper bound property. We skip checking for them to satisfy the triangle inequality and be metrics.

(b) or (c)⇒(a) We have ineuqalities $\widehat{d}_i \le 2^i d$ and $\widehat{d}_i \le id'$ for each i, so convergence in d or d' implies the convergence in each $\widehat{d}_i$. The equivalence of $\widehat{d}_i$ and d_i implies the desired result.

(a)⇒(b) Suppose a sequence $\{x_n\}_n$ converges to a point x in d_i for every index i. Take an arbitrary small ball $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad\Longrightarrow\quad \widehat{d}_i(x_n,x) < \frac{\varepsilon}{2}.$$

Define $k := \lceil 1-\log_2\varepsilon\rceil$ so that we have $2^{-k} \le \frac{\varepsilon}{2}$. With this k, define

$$n_0 := \max_{1\le i\le k} n_i.$$

If $n > n_0$, then

$$\begin{aligned} d(x_n,x) &= \sum_{i=1}^{k} 2^{-i}\widehat{d}_i(x_n,x) + \sum_{i=k+1}^{\infty} 2^{-i}\widehat{d}_i(x_n,x) \\ &< \sum_{i=1}^{k} 2^{-i}\frac{\varepsilon}{2} + \sum_{i=k+1}^{\infty} 2^{-i} \\ &< \frac{\varepsilon}{2} + 2^{-k} \le \varepsilon, \end{aligned}$$

so x_n converges to x in the metric d.

(a)⇒(c) Suppose a sequence $\{x_n\}_n$ converges to a point x in each d_i, and take an arbitrary small ball $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad\Longrightarrow\quad \widehat{d}_i(x_n,x) < \varepsilon.$$

Define $k := \lceil \frac{1}{\varepsilon}\rceil$ so that we have $k^{-1} \le \varepsilon$. With this k, define

$$n_0 := \max_{1\le i\le k} n_i.$$

If $n > n_0$, then

$$i^{-1}\widehat{d}_i(x,y) \le \widehat{d}_i(x,y) < \varepsilon \quad \text{for} \quad i \le k$$

and

$$i^{-1}\widehat{d}_i(x,y) \le i^{-1} < k^{-1} \le \varepsilon \quad \text{for} \quad i > k$$

imply $d(x_n,x) < \varepsilon$, which means that x_n converges to x in the metric d. □

Combination of uncountably many metrics does not result in a single metric, but a topology which cannot be induced from a metric in general. It will be discussed in the rest of the note.

Remark 7.1.6. A metric

$$d''(x,y) = \sup_{i\in\mathbb{N}} d_i(x,y)$$

is not used because the convergence in this metric is a stronger condition than the convergence with respect to each metric d_i. In other words, this metric generates a finer(stronger) topology than the topology generated by subbase of balls. For example, the topology on $\mathbb{R}^{\mathbb{N}}$ generated by this metric defined with the projection pseudometrics is exactly what we often call the box topology.

We can also form a metric by summation of generalized metrics, called pseudometrics, which is defined by missing the nondegeneracy condition from the original definition of metric.

Definition 7.1.7. A function $\rho : X \times X \to \mathbb{R}_{\geq 0}$ is called a *pseudometric* if

(a) $\rho(x,x) = 0$ for all $x \in X$,

(b) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$, (symmetry)

(c) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$. (triangle inequality)

For pseudometrics, it is possible to duplicate every definition we studied in metric spaces: convergence of a sequence, continuity between a set endowed with a pseudometric, refinement and equivalence relations, and countable sum of bounded pseudometrics to make a new pseudometric. Furthermore, every statement for metrics can be generalized to pseudometrics since we have not actually used the condition that $d(x,y) = 0$ implies $x = y$. In fact, we have a flaw that the limit of a convergent sequence may not be unique within a pseudometric.

Example 7.1.8. Let $\rho(x,y) = \rho((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$ be a pseudometric on $\mathbb{R}^2$. Consider a sequence $\{(\frac{1}{n},0)\}_n$. Since $(0,c)$ satisfies

$$\rho((\tfrac{1}{n},0),(0,c)) = \tfrac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

for any real number c, the sequence converges to (x_1,x_2) if and only if $x_1 = 0$.

Although sequences may have several limits in each pseudometric, the sum of a family of pseudometrics can allow the sequences to have at most one limit, only if the sum satisfies the axioms of a metric.

Definition 7.1.9. A family of pseudometrics $\{\rho_\alpha\}_\alpha$ on a set X is said to *separate points* if the condition

$$\rho_\alpha(x,y) = 0 \quad \text{for all } \alpha$$

implies $x = y$.

Proposition 7.1.10. (a) *A finite family of pseudometrics $\{\rho_i\}_{i=1}^N$ separates points if and only if the pseudometric $\rho := \sum_{i=1}^N \rho_i$ is a metric.*

(b) *A countable family of pseudometrics $\{\rho_i\}_{i\in\mathbb{N}}$ separates points if and only if the pseudometric defined by*

$$\rho := \sum_{i\in\mathbb{N}} 2^{-i}\widetilde{\rho}_i \quad \textit{or} \quad \sup_{i\in\mathbb{N}} i^{-1}\widetilde{\rho}_i,$$

where $\widetilde{\rho}_i$ is either $\min\{\rho_i, 1\}$ *or* $\rho_i/(\rho_i+1)$*, is a metric.*

7.2 Metrization theorems

Urysohn metrization theorem: regular second countable

Paracompactness Dieudonne's theorem: paracompact Hausdorff is normal.

7.3 Polish spaces

7.3 (Embedding of separable metric spaces)**.** Urysohn's embedding Hilbert cube

Exercises

7.4 (Discrete metrics)**.** Let X be a set, and define a metric as

$$d(x,y) := \begin{cases} 0 & ,x = y \\ 1 & ,x \neq y \end{cases}.$$

This metric is called *discrete*.

(a) The discrete metric is a strongest metric on X.

(b)

(c) A sequence x_n converges to a point x in a discrete metric space if and only if there is n_0 such that $n \geq n_0$ implies $x_n = x$.

7.5. (a) Let d be a metric on a set X. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $f^{-1}(0) = \{0\}$. If f is monotonically increasing and subadditive, then $f \circ d$ satisfies the triangle inequality, hence is another metric on X. Note that a function f is called *subadditive* if $f(x+y) \leq f(x) + f(y)$ for all x, y in the domain.

(b) Let $G = (V, E)$ be a connected graph. Define $d : V \times V \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the distance of two vertices; the length of shortest path connecting two vertices. Then, (V, d) is a metric space.

(c) Let $\mathcal{P}(X)$ be the power set of a finite set X. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the cardinality of the symmetric difference; $d(A,B) := |(A-B) \cup (B-A)|$. Then $(\mathcal{P}(X), d)$ is a metric space.

(d) Let C be the set of all compact subsets of $\mathbb{R}^d$. Recall that a subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded. Then, $d : C \times C \to \mathbb{R}_{\geq 0}$ defined by

$$d(A,B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|\}$$

is a metric on C. It is a little special case of *Hausdorff metric*.

7.6 (Kuratowski embedding)**.** While every subset of a normed space is a metric space, we have a converse statement that every metric space is in fact realized as a subset of a normed space. Let X be a metric space, and denote by $C_b(X)$ the space of continuous and bounded real-valued functions on X with uniform norm given by

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Fix a point $p \in X$, which will serve as the origin.

(a) Show that a map $\phi : X \to C_b(X)$ such that

$$[\phi(x)](t) = d(x,t) - d(p,t)$$

is well-defined.

(b) Show that the map ϕ is an isometry; $d(x,y) = \|\phi(x) - \phi(y)\|$.

7.7 (Equivalence of norms in finite dimension). Let V be a vector space of dimension d over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Fix a basis $\{e_i\}_{i=1}^d$ on V and let $x = \sum_{i=1}^d x_i e_i$ denote an arbitrary element of V. We will prove all norms are equivalent to the standard Euclidean norm defined for this fixed basis:

$$\|x\|_2 := (\sum_{i=1}^{d} |x_i|^2)^{\frac{1}{2}}.$$

With this standard norm any theorems studied in elementary analysis including the Bolzano-Weierstrass theorem are allowed to be applied. Take a norm $\|\cdot\|$ on V which may differ to $\|\cdot\|_2$.

(a) Find a constant C_2 such that $\|x\| \leq C_2\|x\|_2$ for all $x \in V$.

(b) Show that if no constant C satisfies $\|x\|_2 \leq C\|x\|$, then there exists a sequence $\{x_n\}_n$ such that $\|x_n\|_2 = 1$ and $\|x_n\| < \frac{1}{n}$.

(c) Show that there exists a constant C satisfies $\|x\|_2 \leq C\|x\|$.

Chapter 8

Uniform spaces

8.1 Uniformity

8.1 (Uniformity and entourages)**.** A uniformity is a generalization of a pseudometric. An *uniformity* on a topological space X is a filter Φ on $X \times X$ such that

(i) if $E \in \Phi$, there is $F \in \Phi$ such that $\{(x,z) : (x,y),(y,z) \in E\} \in \Phi$, (Triangle inequality)

(ii) if $E \in \Phi$, then $\{(y,x) : (x,y) \in E\} \in \Phi$, (Symmetry)

(iii) if $E \in \Phi$, then $(x,x) \in E$ for every $x \in X$.

For a uniformity Φ on X, an element of Φ is called an *entourage*.

(a) a

8.2 Completely regular spaces

uniformizability Stone-Čech compactification

8.3 Cauchy structure

completion compact uniform space is complete relatively compact

Exercises

8.2. Even for a first countable uniform space, completeness and sequential completeness are not equivalent.

8.3. A uniformity is pseudometrizable if and only if it admits a countable fundamental system of entourages.

Chapter 9

Part IV

Category of topological spaces

Chapter 10

Compact-open topology

10.1

10.1 (Compact-open topology)**.** Let X and Y be topological spaces. The *compact-open topology* is a topology on $C(X,Y)$ generated by

$$V(K,U) := \{f \in C(X,Y) : f(K) \subset U\}$$

for compact $K \subset X$ and open $U \in Y$. Suppose the function space $C(X,Y)$ carries the compact-open topology.

10.2. Let X, Y, and Z be topological spaces. Consider the *transpose* or the *adjoint*

$$\alpha : C(X \times Y, Z) \to C(X, C(Y,Z)), \qquad (\alpha(f)(x))(y) := f(x,y).$$

Suppose all function spaces are endowed with the compact-open topology.

(a) α is well-defined and injective.

(b) α is continuous if X is Hausdorff.

(c) α is surjective if Y is locally compact.

(d) α is an embedding if X and Y are Hausdorff.

(e) The evaluation $C(Y,Z) \times Y \to Z$ is continuous if Y is locally compact.

(f) The composition $C(X,Y) \times C(Y,Z) \to C(X,Z)$ is continuous if X and Y are locally compact.

Proof. (a) Suppose $f \in C(X \times Y, Z)$. We need to check the continuity of the function

$$\alpha(f)(x) : Y \to Z : y \mapsto f(x,y)$$

for each $x \in X$, which is trivial, and the continuity of the function $\alpha(f) : X \to C(Y,Z)$. Take any compact $K \subset Y$ and open $W \subset Z$. The inverse image of $V(K,W)$ under $\alpha(f)$ is the set

$$S := \{x \in X : f(x,K) \subset W\} = \{x \in X : \{x\} \times K \subset f^{-1}(W)\}.$$

Pick any $x \in S$ and apply the tube lemma for the slice $\{x\} \times K$ to obtain an open neighborhood U of x such that $U \times K \subset S$, which implies S is open. The injectivity is clear.

(b) Take compact $K \subset X$, compact $L \subset Y$, and open $W \subset Z$. Then,

$$\alpha^{-1}(V(K,V(L,W))) = \{f \in C(X \times Y, Z) : \forall x \in K\ \forall y \in L,\ f(x,y) \in W\} = V(K \times L, W)$$

is open. Since $V(K,-)$ preserves the intersection, for any base element $B \subset C(Y,Z)$ the set $\alpha^{-1}(V(K,B))$ is also open.

(c)

□

10.3 (Exponential spaces)**.** Let Y and Z be topological spaces and consider a contravariant functor $C(- \times Y, Z) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$. An *exponential space* of Y and Z is a topological space Z^Y such that there is a natural isomorphism $C(- \times Y, Z) \cong C(-, Z^Y)$. We say a topological space Y is *exponentiable* if the exponential space Z^Y exists for every topological space Z. Suppose Y is Hausdorff.

(a) If Z^Y exists, then $Z^Y \cong C(Y,Z)$ as sets.

(b) Z^Y exists if and only if the evaluation map $Z^Y \times Y \to Z$ is continuous.

(c) If Y is locally compact, then Y is exponentiable.

(d) If Y is exponentiable, then Y is locally compact. Moreover, Z^Y is homeomorphic to $C(Y,Z)$ with the compact-open topology for every topological space Z.

Proof. (a) We claim $C(Y,Z)$ is the exponential space of topological spaces Y and Z.

□

10.4 (Topology of compact convergence)**.** Let X be a topological space and Y be a topological and a uniform spaces.

Topologies on $C_c(X)$ for LCH X: weaker to stronger

(a) Topology of compact convergence: $\overline{C_c(X)} = C(X)$.

(b) Topology of uniform convergence: $\overline{C_c(X)} = C_0(X)$.

(c) Inductive topology: $\overline{C_c(X)} = C_c(X)$.

Chapter 11

Topological groups

11.1 Topological properties of groups

left and right uniform structures Birkhoff-Kakutani metrizability

11.2 Transformation groups

11.1. Let G be a topological group. A *left G-space* is a topological space X together with a *continuous left action* of G, a continuous map $G \times X \to X : (g, x) \mapsto gx$ such that $g(hx) = (gh)x$ and $ex = x$. Equivalently we sometimes say G is a *transformation group* of X. We may define right actions and right G-spaces similarly. The left orbit space $G\backslash X$ is topologized with the quotient topology induced by the canonical surjection $X \to G\backslash X$. We frequently write X/G for the left orbit space if there is no confusion on the left-right issues.

G-equivariant map and G-homeomorphism

G-homotopy is a G-equivariant map $h : X \times [0, 1] \to Y$ such that $h(x, i) = f_i(x)$ for $i \in \{0, 1\}$.

11.2. Suppose $\rho : G \times X \to X$ is a continuous left action, and $\overline{\rho} : G \to \mathrm{Homeo}(X)$ is its adjoint map. Here we suppose $\mathrm{Homeo}(X)$ carries the compact-open topology.

(a) The continuity of ρ implies the continuity of $\overline{\rho}$, but the converse holds if X is locally compact.

(b) If X is locally compact, then $\mathrm{Homeo}(X) \times X \to X$ is continuous.

(c) If X is locally compact, $\mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X) : (f, g) \mapsto g \circ f$ is continuous.

(d) If X is locally compact, locally connected, and Hausdorff, then $\mathrm{Homeo}(X) \to \mathrm{Homeo}(X) : f \mapsto f^{-1}$ is continuous, hence $\mathrm{Homeo}(X)$ is a topological group.

The isometry group of a locally compact metric space with finite components is locally compact.

11.3. Let H be a subgroup of a locally compact group G. Then, H is closed if and only if H is locally compact. (We always assume a topological group is Hausdorff.)

- free action of a discrete group: the theory of covering spaces
- transitive action: the theory of homogeneous spaces

classification of homogeneous space realization of spheres by compact connected Lie groups: Montgomery-Samuelson, Borel.

11.4 (Proper actions).

Chapter 12

Implications

12.1 (Metrizable spaces (1)). Let X be a regular Hausdorff space.

(a) (Urysohn) second countable $\Rightarrow$ metrizable.

(b) (Nagata-Smirnov) σ-locally countable base $\Leftrightarrow$ metrizable.

(c) (Smirnov) paracompact locally metrizable $\Leftrightarrow$ metrizable.

12.2 (Metrizable spaces (2)). Let X be a metrizable space.

(a) (countability) separable $\Leftrightarrow$ second countable $\Leftrightarrow$ Lindelöf.
(cf. separable $\Leftarrow$ second countable $\Rightarrow$ Lindelöf in general)

12.3 (Locally compact spaces (2)). Let X be a locally compact Hausdorff space.

(a) completely regular.

(b) metrizable $\Leftrightarrow$ completely metrizable.

(c) σ-compact $\Leftrightarrow$ Lindelöf.
(cf. σ-compact $\Rightarrow$ Lindelöf in general)

(d) σ-compact metrizable $\Leftrightarrow$ $C_0(X)$ is separable $\Leftrightarrow$ second countable $\Leftrightarrow$ Polish.

For a compact Hausdorff space X, $C(X)$ is separable if and only if X is metrizable. For a locally compact Hausdorff space X, $C_0(X)$ is separable if and only if X is second countable.

12.4 (Paracompact spaces (1)). Let X be a topological space.

(a) metrizable $\Rightarrow$ fully normal $\Leftrightarrow$ paracompact. (Stone)

(b) regular Lindelöf $\Rightarrow$ paracompact.

12.5 (Paracompact spaces (2)). Let X be a paracompact Hausdorff space.

(a) normal. (Dieudonné)

(b)