Noncommutative Algebraic Geometry

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1 Algebras

- 1987: Artin-Schelter, regular algebra.
- 1990: Artin-Tate-Bergh, three dimensional, geometrically classified.
- 1994: Artin-Zhang, noncommutative scheme, categorical perspective.

1.1

Let k be an algebraically closed field of characteristic zero. Examples of k-algebras include the free algebra $T := k\langle x_1, \cdots, x_n\rangle$, which is noncommutative for $n \geq 2$. It consists of linear combinations of monomials, and there are 2^n monomials of degree n in T, and T is k-isomorphic to the tensor algebra constructed from n-dimensional vector space k^n. Note that $(x+y)^2 = x^2 + xy + yx + y^2$ in T. An algebra R is finitely generated if and only if $R \cong T/I$ for some n and some ideal I of R. If $n \geq 2$, then T is not right noetherian, $I = \sum_{i=0}^{\infty} x^i yR$ is a right ideal which is not finitely generated for exmaple(not easy to show finitely generatedness). Is $k\langle x, y\rangle/(yx, y^2)$ noetherian? It is known that it is left notherian, but not right noetherian.

1.2

Let R be a ring and let $\sigma \in \operatorname{Aut}(R)$. An additive map $\delta : R \to R$ is called a *σ-derivation* if $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for $a, b \in R$. We define a ring $R[x;\sigma,\delta]$, called the *Ore extension*, as an additive group $R[x]$ together with multiplication defined by

$$xa := \sigma(a)x + \delta(a), \qquad a \in R.$$

Example 1.1.

(a) We can compute

$$\begin{aligned}(ax+b)(cx+d) &= axcx + axd + bcx + bd \\ &= a(\sigma(c)x + \delta(c))x + a(\sigma(d)x + \delta(d)) + bcx + bd \\ &= a\sigma(c)x^2 + (a\delta(c) + a\sigma(d) + bc)x + a\delta(d) + bd.\end{aligned}$$

(b) We have $R[x;\mathrm{id}_R, 0] \cong R[x]$ as rings.

(c) If $\sigma(f(x)) := f(\alpha x)$ for some non-zero $\alpha \in k$, then $k[x][y;\sigma,0] \cong k\langle x, y\rangle/(\alpha xy - yx)$ since $yx = \sigma(x)y - \delta(x) = \alpha xy$.

(d) If $\delta(f(x)) := f'(x)$, then $k[x][y;\mathrm{id}_{k[x]},\delta] \cong k\langle x,y\rangle/(xy-yx+1)$, called the *Weyl algebra*, since $yx = \sigma(x)y+\delta(x) = xy+1$.

(e) How can we find a k-automorphism σ of $k[x]$ and a σ-derivation δ such that $k\langle x,y\rangle/(xy-yx+x^2) \cong k[x][y;\sigma,\delta]$? What should $\delta(x^i)$ be? One answer is $\sigma = \mathrm{id}_{k[x]}$ and $\delta(f(x)) = x^2 f'(x)$.

Theorem 1.2. *Let R be a ring and $S := R[x;\sigma,\delta]$ be an Ore extension.*

(a) *If R is right noetherian, then so is S.*

(b) *If R is a domain, then so is S.*

(c) *If R is of finite global dimension, then so is S.*

As examples, we have $k\langle x,y\rangle/(\alpha xy-yx)$ and $\dim k\langle x,y\rangle/(xy-yx+1)$ are noetherian domains of global dimensions 2 and 1, respectively. There is a result that left and right global dimensions coincide when R is two-sided noetherian.

1.3

Theorem 1.3. *If R is a k-algebra and $a_1,\cdots,a_n \in R$, then there is a unique k-algebra homomorphism $\varphi : k\langle x_1,\cdots,x_n\rangle \to R$ such that $\varphi(x_i) = a_i$. If a k-algebra homomorphism $\varphi : S \to R$ satisfies $\varphi(I) = 0$ for an ideal I of S, then it factors through S/I.*

With the above theorem we can construct an k-algebra isomorphism $k[x] \cong k\langle x,y\rangle/(x^2-y)$. As an another example, for $\operatorname{char} k \neq 2$, then

$$k\langle x,y\rangle/(x^2+y^2,xy+yx) = k\langle x+y,x-y\rangle/((x+y)^2,(x-y)^2) \cong k\langle x,y\rangle/(x^2+y^2).$$

1.4

We now consider grading, a direct sum decomposition over a monoid. The free k-algebra $T = k\langle x_1,\cdots,x_n\rangle$ is $\mathbb{N}$-graded by degree. Let $A = \bigoplus A_i$ be a graded ring. We can define homogeneous ideals of A, and the quotient can be written as $A/I \cong \bigoplus A_i/I_i$, where $I_i := I \cap A_i$. Also, graded homomorphisms between graded rings or graded modules are able to be introduced. Let I and J be homogeneous ideal of $T_n := k\langle x_1,\cdots,x_n\rangle$ and $T_m := k\langle y_1,\cdots,y_m\rangle$ such that $J_0 = J_1 = 0$. Then, a graded algebra homomorphism $\varphi : T_n \to T_m$ is uniquely determined by $\varphi(x_i) = a_{ij}y_j$ for $(a_{ij}) \in M_{nm}(k)$. Let $\mathrm{GrAut}(A)$ be the group of graded algebra automorphisms of A. Then,

$$\mathrm{GrAut}(T_n) \cong \mathrm{GrAut}(k[x_1,\cdots,x_n]) \cong \mathrm{GL}(n,k),$$

and if I is a homogeneous ideal of T_n such that $I_0 = I_1 = 0$, then $\mathrm{GrAut}(T_n/I)$ is a subgroup of $\mathrm{GL}(n,k)$. For example, we have

$$\mathrm{GrAut}(k\langle x,y\rangle/(x^2)) \cong \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a,d \in k^\times\right\}$$

and for $\alpha \neq \pm 1$ we have

$$\mathrm{GrAut}(k\langle x,y\rangle/(\alpha xy-yx)) \cong \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a,d \in k^\times\right\}$$

since $\alpha\varphi(x)\varphi(y) - \varphi(y)\varphi(x) = (\alpha-1)(acx^2+bdy^2)+(\alpha^2-1)bcxy$.

Fix $\theta \in \mathrm{GrAut}(A)$. Define an algebra $A^\theta := A$ as sets and multiplication $a * b := a\theta^i(b)$ on A^θ for $a \in A_i$ and $b \in A$. It is called the *twist* of A by θ, and it is also graded. For example, if we let $A = k[x,y]$, then

$$\text{If } \theta = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \text{ then } \quad A^\theta \cong k\langle x,y\rangle/(\alpha xy-yx)$$

and

$$\text{If } \theta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \text{ then} \quad A^\theta \cong k\langle x, y\rangle/(xy - yx + x^2).$$

Note that $\varphi(xy - yx) = (ad - bc)(xy - yx)$ if $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Theorem 1.4. *Let A be a graded ring and $\theta \in \mathrm{GrAut}(A)$.*

(a) *If A is right noetherian, then so is A^θ.*

(b) *If A is a domain, then so is A^θ.*

(c) *If A is of finite global dimension, then so is A^θ.*

2 Quantum polynomial algebras

2.1

Today, let $A := k\langle x_1, \cdots, x_n\rangle/I$ be a finitely generated graded algebra such that I is a homogeneous ideal satisfying $I_0 = I_1 = 0$, i.e. I is an admissible ideal. Let M be a graded right A-module, $M_{\geq n} := \bigoplus_{i\geq n} M_i$ be a graded submodule of M, and $M(n)$ be a graded module such that $M(n) := M$ as a set but $M(n)_i := M_{n+i}$. With this notation, $\mathfrak{m} := A_{\geq 1}$ is the unique maximal homogeneous ideal of A. A *free* graded right A-module is a graded right A-module of the form $\oplus_s A(n_s)$. A finitely generated graded right A-module is free if and only if projective. A function $\varphi : A(l) \to A(m)$ is a graded right A-module homomorphism if and only if $\varphi = a\cdot$ for some $a \in A_{m-l}$. Therefore, between free right A-modules, $\varphi : \oplus A(l_s) \to \oplus A(m_t)$ is a graded right A-module homomorphism if and only if $\varphi = (a_{st})\cdot$, for some $a_{st} \in A_{m_t - l_s}$. A free resolution

$$\cdots \to F^2 \to F^1 \to F^0 \to M \to 0$$

is called *minimal* if the map $\varphi_i : F^i \to F^{i-1}$ is given by the left multiplication of a matrix whose entries are in A_1. We can define the projective dimension of a module as the minimal length of free resolution, and the global dimension of A as the supremum of the projective dimension of graded right A-modules.

Lemma 2.1. $\mathrm{gldim} A = \mathrm{pd}(k)$.

For example, $A = k\langle x, y\rangle$, then $k = A/(xA + yA)$, so $\mathrm{pd}(k) = 1$, hence $\mathrm{gldim} A = 1$, and in generally $\mathrm{gldim} A = 1$ for $I = 0$.

2.2

Let M be a finitely generated graded right A-module. Suppose further M is locally finite, i.e. $\dim_k M_i < \infty$ for each i. Then,

$$H_M(t) := \sum_{i\in\mathbb{Z}} (\dim_k M_i) t^i \in \mathbb{Z}[[t, t^{-1}]]$$

is called the *Hilbert series* of M. For example, letting $M = A$,

$$H_{k[x_1,\cdots,x_n]}(t) = \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} t^i = (1-t)^{-n},$$

and

$$H_{k\langle x_1,\cdots,x_n\rangle}(t) = \sum_{i=0}^{\infty} n^i t^i = (1-nt)^{-1}.$$

Lemma 2.2. *Let M be a finitely generated graded right A-module.*

(a) $H_{M^{\oplus r}}(t) = rH_M(t)$.

(b) $H_{M(n)}(t) = t^{-n} H_M(t)$.

(c) *If* $0 \to M^r \to \cdots \to M^1 \to M^0 \to 0$ *is exact, then* $\sum_{i=0}^{r} (-1)^i H_{M_i}(t) = 0$.

For example for (c), consider

$$0 \to A(-1)^{\oplus 2} \to A \to k \to 0.$$

Then, we can check $H_A(t) = (1-2t)^{-1}$ from

$$0 = H_k(t) - H_A(t) + H_{A(-1)^{\oplus 2}}(t) = 1 - H_A(t) + 2tH_A(t).$$

2.3

Definition 2.3 (Artin-Schelter)**.** We say A is a d-dimensional *quantum polynomial algebra*(QPA) if $\mathrm{gldim}A = d < \infty$, $H_A(t) = (1-t)^{-d}$, and $\mathrm{Ext}^i_A(k,A) = \delta_{di} \cdot k(d)$. The last condition is called the *Gorenstein condition*.

If a QPA is commutative, then it is isomorphic to the polynomial algebra. The above two conditions are equivalent to have the minimal free resolution of the graded right A-module k

$$0 \to A(-d) \to \oplus A(-d+1) \to \cdots \to \oplus A(-1) \to A \to k \to 0,$$

where $\phi^i : \oplus A(-i) \to \oplus A(-i+1)$ is the left multiplication of a matrix whose components are in A_1. The Gorenstein condition is equivalent to the transpose

$$0 \leftarrow k(d) \leftarrow \oplus A(d) \leftarrow \cdots \leftarrow \oplus A(1) \leftarrow A \leftarrow 0$$

is a minimal free resolution of left A-module $k(d)$, where the arrows are right multiplications of matrices whose compoeenents are in A_1. Ranks of each free modules must be determined by the Hilbert series.

For example, $A = k\langle x,y\rangle/(\alpha xy - yx)$ is a 2-dimensional QPA for all non-zero $\alpha \in k$. The classification up to dimension two is easy:

Lemma 2.4. *Let A be a QPA over an algebraically closed field k.*

(a) $\mathrm{gldim}A = 0$ *iff* $A \cong k$,

(b) $\mathrm{gldim}A = 1$ *iff* $A \cong k[x]$,

(c) $\mathrm{gldim}A = 2$ *iff* $A \cong k[x,y]^\theta$ *for some* $\theta \in \mathrm{GL}(2,k)$.

2.4

We can describe three-dimensional QPAs are classified in terms of derivation quotient algebras.

Definition 2.5. Let $V = k^n$ and let

$$\varphi : V^{\otimes m} \to V^{\otimes m} : v_1 \otimes \cdots \otimes v_m \mapsto v_2 \otimes \cdots \otimes v_1.$$

We say $w \in V^{\otimes m}$ is called a *superpotential*(SP) if $\varphi(w) = w$, and a *twisted superpotential*(TSP) if $(\sigma \otimes \mathrm{id}^{\otimes(m-1)})\varphi(w) = w$ for all $\sigma \in \mathrm{GL}(V)$.

Example 2.6. Let $V = kx + ky$, and $w = \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \in V^{\otimes 2}$. Then, w is SP iff $\beta = \gamma$ and $SP^2(V) = kx^2 + k(xy+yx) + ky^2 \subset V^{\otimes 2}$.

Definition 2.7. For $\dim_k V = n$ and $w \in V^{\otimes m}$, we can define $\partial_i w, w\partial_i \in V^{\otimes(m-1)}$ such that $w = \sum x_i \otimes \partial_i w = \sum w\partial_i \otimes x_i$. *Derivation quotient algebras* are

$$D_l(w) := k\langle x_1,\cdots,x_n\rangle/(\partial_1 w,\cdots,\partial_n w), \qquad D_r(w) := k\langle x_1,\cdots,x_n\rangle/(w\partial_1,\cdots,w\partial_n).$$

Lemma 2.8.

(a) *w is SP iff $\partial_i w = w\partial_i$.*

(b) *w is TSP iff $D_l(w) = D_r(w) =: D(w)$ (ideals quotiented are same as sets.)*

Example 2.9. If $V = kx + ky$, and $w = \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \in V^{\otimes 2}$, then

$$\partial_x w = \alpha x + \beta y, \qquad w\partial_x = \alpha x + \gamma y.$$

Theorem 2.10.

(a) *If ω is TSP with $m = n = 3$, then $D(w)$ is a three-dimensional QPA.*

(b) *The converse holds.*

Example 2.11 (Sklyanin algebra)**.** For $\alpha, \beta, \gamma \in k$,

$$w = \alpha(xyz + yzx + zxy) + \beta(xzy + yxz + zyx) + \gamma(x^3 + y^3 + z^3)$$

is a superpotential. $D(w)$ is called the Sklyanin algebra. We can construct with $M = \begin{pmatrix} \gamma x & \beta z & \alpha y \\ \alpha z & \gamma y & \beta x \\ \beta y & \alpha x & \gamma z \end{pmatrix}$ the minimal free resolutions of k and $k(3)$.

There is $\theta \in \mathrm{GrAut}(k\langle x, y\rangle/(\alpha xy - yx))$ such that

$$(k\langle x, y\rangle/(\alpha xy - yx))^\theta \cong k\langle x, y\rangle/(xy - yx + x^2)$$

if and only if $\alpha = 1$. We can see this for $\alpha = -1$ by computing GrAut. Note that

$$(k\langle x, y\rangle/(\alpha xy - yx))^\theta \cong k\langle x, y\rangle/(\alpha\theta(x)y - \theta(y)x)$$

If $\alpha \neq \pm 1$....?