Category Theory

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Chapter 1

Categories

set theoretical issues morphisms monic

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fully faithful, essentially surjective natural transformations and equivalence 2-category

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opposite category product category disjoint union category comma category(slice category, morphism category)

Chapter 2

Universal property

2.1 Construction

products, equalizers, pullbacks

2.2 Representable functors

Yoneda embedding gives fully faithful functors $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ and $k : \mathcal{C}^{\mathrm{op}} \to \mathrm{coPSh}(\mathcal{C})$. A presheaf $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable if and only if it is essentially contained in the image of the Yoneda embedding.

2.1 (Yoneda lemma)**.** Let $F : \mathcal{C} \to \mathrm{Set}$ be a functor from a locally small category $\mathcal{C}$. Fix $c \in \mathrm{Ob}(\mathcal{C})$. we can define a function

$$\mathrm{Nat}(\mathrm{Hom}(c,-), F) \to F(c).$$

A *representation* of F is a pair (c, η) of an object $c \in \mathcal{C}$ and a natural isomorphism $\eta : \mathrm{Hom}(c,-) \to F$.

A *universal element* of F is a pair (c, x) with $x \in F(c)$ such that for any pair (d, y) with $y \in F(d)$ there is a unique morphism $f \in \mathrm{Hom}(c,d)$ satisfying $F(f) : x \mapsto y$.

(a)

Proof. (a) Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}(c,c) \arrow[rrr, "\eta_c"] \arrow[ddd] \arrow[dr, phantom, "\ni"] & & & F(c) \arrow[ddd, "F(f)"] \arrow[dl, phantom, "\in"] \\
& \mathrm{id}_c \arrow[r, mapsto] \arrow[d, mapsto] & x := \eta_c(\mathrm{id}_c) \arrow[d, mapsto] & \\
& f \arrow[r, mapsto] & \eta_d(f) := F(f)(x) & \\
\mathrm{Hom}(c,d) \arrow[rrr, "\eta_d"'] \arrow[ur, phantom, "\ni"] & & & F(d). \arrow[ul, phantom, "\in"]
\end{tikzcd}$$

For a natural transformation $\eta : \mathrm{Hom}(c,-) \to F$, define $x := \eta_c(\mathrm{id}_c)$ in $F(c)$. For $x \in F(c)$, conversely, define a $\eta_d : \mathrm{Hom}(c,d) \to F(d)$ by $\eta_d(f) := F(f)(x)$ for $d \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}(c,d)$. Then, the collection $\eta = \{\eta_d : d \in \mathrm{Ob}(\mathcal{C})\}$ provides a natural transformation because for each $g \in \mathrm{Hom}(d,e)$ we can check the diagram

$$\begin{tikzcd}
\mathrm{Hom}(c,d) \arrow[r, "\eta_d"] \arrow[d, "g\circ-"'] & F(d) \arrow[d, "F(g)"] \\
\mathrm{Hom}(c,e) \arrow[r, "\eta_e"] & F(e)
\end{tikzcd}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \qquad f \in \mathrm{Hom}(c, d).$$

The correspondences $\eta \mapsto x$ and $x \mapsto \eta$ are inverses of each other, hence the bijection. $\square$

Chapter 3

Limits

preservation, reflection, creation completeness functoriality

limit-preserving filtered limit-preserving product-preservig mono-preserving

Part II

Chapter 4

4.1 Adjunctions

4.2 Monads

4.3 Kan extensions

Chapter 5

Abelian categories

5.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f).

A category is called *regular* if every kernel pair admits a coequalizer.

5.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category Grp is regular but not coregular, since there is a monomorphism which is not regular.

5.2 Additive and abelian categories

5.2 (Additive categories)**.** A *pre-additive category* is an Ab-enriched category. A *semi-additive category* is one of the followings:

(i) a pointed CMon-enriched category.
(ii) a category with finite biproducts.

An *additive category* is one of the followings:

(i) a pointed Ab-enriched category.
(ii) Ab-enriched category with finite biproducts.

(a) additive completion by formally adjoining finite biproducts.
(b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a Set_*-enriched category. In additive category, we have a natural Set_*-enrichment.

5.3 (Pre-abelian categories)**.** A *pre-abelian category* is one of the followings:

(i) an additive category in which every morphism has the kernel and cokernel.
(ii) a finitely bicomplete pre-additive category.

(a)

5.4 (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen). A pointed, Baar-exact, protomodular, with binary coproudcts.

(a) short five lemma, 3×3 lemma, snake lemma, noehter isomorphism hold.

(b) long exact homology sequence

(c) Every semi-abelian category is exact.

(d) Every semi-abelian category is finitely bicomplete.

(e) In general, a semi-abelian category is not pre-additive nor semi-additive.

5.5 (Abelian categories). An *abelian category* is a Ab-enriched category which is finitely bicomplete and satisfies the first isomorphism theorem.

(a) A category is abelian if and only if it is additive and exact.

5.6 (Freyd-Mitchell embedding).

$$\begin{tikzcd}
\text{abelian} \arrow[r] \arrow[d] & \text{pre-abelian} \arrow[r] & \text{additive} \arrow[r] \arrow[d] & \text{pre-additive} \\
\text{semi-abelian (JMT)} & & \text{semi-additive} &
\end{tikzcd}$$

- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C^*-algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

The first isomorphism theorem states that coim $\to$ im is an isomorphism. The normal subobjects and the first isomorphism theorem is generalized in the context of protomodular categories. The cokernel may not be defined. The category of unital rings is not semi-abelian but protomodular.

- A *protomodular category*
- A *homological category* is a pointed regular protomodular category. (five, nine, snake, long exact sequence, noether isomorphism)
- A *semi-abelian category* is a homological category that is Barr-exact and finite coproducts(free products).

Chapter 6

Tensor categories

6.1 Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

6.1 (Monoidal categories)**.** A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that

(i) for each triple $A, B, C \in \mathcal{C}$ there is an isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ called the *associator*, satisfying the pentagon identity

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C\otimes D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

commutes for each $A, B, C, D \in \mathcal{C}$.

(ii) there is a specified object $I \in \mathcal{C}$ called the *unit object*, and for each $A \in \mathcal{C}$ there are isomorphisms $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$ called the *left unitor* and the *right unitor*, satisfying the triangle identity

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes for each $A, B \in \mathcal{C}$.

We say a monoidal category is *strict* if the associators and unitors are all identity morphisms. A *cartesian* monoidal category is a monoidal category whose monoidal structure $\otimes$ is given by the categorical product.

6.2 (Coherence theorem)**.** Let $\mathcal{C}$ be a monoidal category.

(a)

$$\begin{tikzcd}
(I \otimes A) \otimes B \arrow[rr, "\alpha_{I,A,B}"] \arrow[dr, "\lambda_A \otimes \mathrm{id}_B"'] & & I \otimes (A \otimes B) \arrow[dl, "\lambda_{A\otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

(b) $\lambda_I = \rho_I$

(c) The endomorphism monoid $\mathrm{End}(I)$ is commutative.

(d) I is unique up to unique isomoprhism.

6.3 (Monoidal functors). coherence maps lax, strong, strict

6.4 (Enriched categories). Let $\mathcal{M}$ be a monoidal category. A category $\mathcal{C}$ is said to be *enriched* over $\mathcal{M}$ if for each $A, B \in \mathcal{C}$ there is $\mathrm{Hom}(A,B) \in \mathcal{M}$ such that

(i) for each $A, B, C \in \mathcal{C}$ there is a morphism $\circ_{A,B,C} : \mathrm{Hom}(A,B) \otimes \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$, satisfying

$$\begin{tikzcd} \mathrm{Hom}(A,B)\otimes\mathrm{Hom}(B,C)\otimes\mathrm{Hom}(C,D) \arrow[r, "\mathrm{id}_{\mathrm{Hom}(A,B)}\otimes\circ_{B,C,D}"] \arrow[d, "\circ_{A,B,C}\otimes\mathrm{id}_{\mathrm{Hom}(C,D)}"'] & \mathrm{Hom}(A,B)\otimes\mathrm{Hom}(B,D) \arrow[d, "\circ_{A,B,D}"] \\ \mathrm{Hom}(A,C)\otimes\mathrm{Hom}(C,D) \arrow[r, "\circ_{A,C,D}"] & \mathrm{Hom}(A,D) \end{tikzcd}$$

commutes for each $A, B, C, D \in \mathcal{C}$.

(ii) for each $A \in \mathcal{C}$ there is a morphism $\mathrm{id}_A : I \to \mathrm{Hom}(A,A)$, satisfying

$$\begin{tikzcd} I\otimes\mathrm{Hom}(A,B) \arrow[d, "\mathrm{id}_A\otimes\mathrm{id}_{\mathrm{Hom}_{A,B}}"'] \arrow[dr, "\lambda_{\mathrm{Hom}(A,B)}"] & & \mathrm{Hom}(A,B)\otimes I \arrow[dl, "\rho_{\mathrm{Hom}(A,B)}"'] \arrow[d, "\mathrm{id}_{\mathrm{Hom}(A,B)}\otimes\mathrm{id}_B"] \\ \mathrm{Hom}(A,A)\otimes\mathrm{Hom}(A,B) \arrow[r, "\circ_{A,A,B}"'] & \mathrm{Hom}(A,B) & \mathrm{Hom}(A,B)\otimes\mathrm{Hom}(B,B) \arrow[l, "\circ_{A,B,B}"] \end{tikzcd}$$

6.5 (Pointed category). A *pointed category* is a category with a zero object.

(a) A category is Set_*-enriched if and only if it admits a zero morphism.

(b) Every pointed category is Set_*-enriched.

rigid?

6.2 Braided and ribbon categories

6.3 Internalization

6.4 Tensor and fusion categories