Category Theory

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Part I

Categories

set theoretical issues duality morphisms monic

1.1 Functors

full, faithful natural transformations and equivalence 2-category

Universal property

2.1 Construction

products, equalizers, pullbacks

2.2 Representable functors

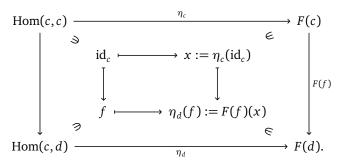
2.1 (Yoneda lemma). Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor from a locally small category \mathcal{C} . Fix $c \in \mathsf{Ob}(\mathcal{C})$. we can define a function

$$Nat(Hom(c, -), F) \rightarrow F(c)$$
.

A representation of F is a pair (c, η) of an object $c \in C$ and a natural isomorphism $\eta : \operatorname{Hom}(c, -) \to F$. A universal element of F is a pair (c, x) with $x \in F(c)$ such that for any pair (d, y) with $y \in F(d)$ there is a unique morphism $f \in \operatorname{Hom}(c, d)$ satisfying $F(f) : x \mapsto y$.

(a)

Proof. (a) Consider the diagram



For a natural transformation $\eta: \operatorname{Hom}(c,-) \to F$, define $x:=\eta_c(\operatorname{id}_c)$ in F(c). For $x \in F(c)$, conversely, define a $\eta_d: \operatorname{Hom}(c,d) \to F(d)$ by $\eta_d(f):=F(f)(x)$ for $d \in \operatorname{Ob}(\mathcal{C})$ and $f \in \operatorname{Hom}(c,d)$. Then, the collection $\eta=\{\eta_d: d \in \operatorname{Ob}(\mathcal{C})\}$ provides a natural transformation because for each $g \in \operatorname{Hom}(d,e)$ we can check the diagram

$$\begin{array}{ccc} \operatorname{Hom}(c,d) & \stackrel{\eta_d}{\longrightarrow} F(d) \\ & g \circ - \bigvee & & \bigvee F(g) \\ \operatorname{Hom}(c,e) & \stackrel{\eta_e}{\longrightarrow} F(e) \end{array}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \qquad f \in \text{Hom}(c, d).$$

The correspondences $\eta \mapsto x$ and $x \mapsto \eta$ are inverses of each other, hence the bijection.
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Limits

preservation, reflection, creation completeness functoriality

Part II

- 4.1 Adjunctions
- 4.2 Monads
- 4.3 Kan extensions

Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

5.1 Enriched categories

- **5.1** (Pointed category). A pointed category is a category with a zero object.
 - (a) A category is \mathbf{Set}_* -enriched if and only if it admits a zero morphism.
 - (b) Every pointed category is **Set***-enriched.

Abelian categories

6.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f).

A category is called *regular* if every kernel pair admits a coequalizer.

6.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category **Grp** is regular but not coregular, since there is a monomorphism which is not regular.

6.2 Additive and abelian categories

- **6.2** (Pre-additive categories). A pre-additive category is another name of Ab-enriched category.
 - (a) a
- **6.3** (Semi-additive cateogries). A semi-additive category is a category with binary biproducts.
 - (a) A category is semiadditive if and only if it is pointed CMon-enriched.
- **6.4** (Additive categories). (a) additive completion by formally adjoining finite biproducts.
 - (b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a Set,-enriched category.

6.5 (Pre-abelian categories). A *pre-abelian category* is an additive category in which every morphism has the kernel and cokernel. Equivalently, it is a finitely bicomplete pre-additive category.

(a)

- **6.6** (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen). A pointed, Baar-exact, protomodular, with binary coproudcts.
 - (a) short five lemma, 3×3 lemma, snake lemma, noehter isomorphism hold.
 - (b) long exact homology sequence
 - (c) Every semi-abelian category is exact.
 - (d) Every semi-abelian category is finitely bicomplete.

- (e) In general, a semi-abelian category is not pre-additive nor semi-additive.
- **6.7** (Abelian categories). We say \mathcal{C} is *abelian* if every morphism has the kernel and cokernel, and every mono and epi is normal. Roughly, an abelian category is a **Ab**-enriched category such that it is finitely bicomplete and the first isomorphism holds.
 - (a) A category is abelian if and only if it is additive and exact.
- 6.8 (Freyd-Mitchell embedding).



- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C*-algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

The first isomorphism theorem states that $coim \rightarrow im$ is an isomorphism. The normal subobjects and the first isomorphism theorem is generalized in the context of protomodular categories. The cokernel may not be defined. The category of unital rings is not semi-abelian but protomodular.

- A protomodular category
- A *homological category* is a pointed regular protomodular category. (five, nine, snake, long exact sequence, noether isomorphism)
- A *semi-abelian category* is a homological category that is Barr-exact and finite coproducts(free products).

site, topos (∞ , 1)-category