Functional Analysis

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Part I

Topological vector spaces

Chapter 1

Locally convex spaces

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1.4 (Seminorms).

$$\bigcap_{i=1}^{m}\{: p() < 1\}$$

Equivalent conditions on the continuity of seminorms

Proof. □

boundedness by seminorms, normability

1.3 Continuous linear functionals

1.5. Let $\overline{x^*} = (x_1^*, \cdots, x_n^*) \in X^{*n}$. $\overline{x^*} : X \to \mathbb{F}^n$. If $x^* \in X^*$ vanishes on $\bigcap_{i=1}^n \ker x_i^*$, then x^* is a linear combination of $\{x_i^*\}$.

1.6 (Hahn-Banach extension). Let X be a real vector space. Suppose V is a linear subspace of X and $l : V \to \mathbb{R}$ is a linear functional dominated by a sublinear functional $q : X \to \mathbb{R}$, that is, $l(v) \le q(v)$ for all $v \in V$.

(a) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l.

(b) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l and is dominated by q if $\dim X/V = 1$.

(c) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l and is dominated by q.

Proof. (a) It can be done by the Hamel basis.

(b) Let $e \in X \setminus V$ so that every vector $x \in X$ can be uniquely written as $x = v + te$ with $v \in V$ and $t \in \mathbb{R}$. For $v_1, v_2 \in V$,

$$l(v_1) + l(v_2) = l(v_1 + v_2) \le q(v_1 + v_2) \le q(v_1 - e) + q(v_2 + e)$$

implies

$$l(v_1) - q(v_1 - e) \le -l(v_2) + q(v_2 + e).$$

Define a linear functional $\widetilde{l} : X \to \mathbb{R}$ such that $\widetilde{l}(v) = v$ and

$$l(v) - q(v - e) \le \widetilde{l}(e) \le -l(v) + q(v + e)$$

for all $v \in V$. Since

$$\widetilde{l}(v + te) = l(v) + t\widetilde{l}(e) \le l(v) + t(-l(t^{-1}v) + q(t^{-1}v + e)) = q(v + te)$$

if $t \ge 0$ and

$$\widetilde{l}(v + te) = l(v) + t\widetilde{l}(e) \le l(v) + t(l(-t^{-1}v) - q(-t^{-1}v - e)) = q(v + te)$$

if $t \le 0$, we have $\widetilde{l}(x) \in q(x)$ for all $x \in X$.

(c) With X and q, Consider a partially ordered set

$$\{(\widetilde{V}, \widetilde{l}) \mid V \le \widetilde{V} \le X,\ \widetilde{l} : \widetilde{V} \to \mathbb{R} \text{ is a linear extension of } l \text{ dominated by } q\}$$

such that $(V_1, l_1) \prec (V_2, l_2)$ if and only if $V_1 \le V_2$ and $l_2|_{V_1} = l_1$. The nonemptyness and the chain condition is easily satisfied, so it has a maximal element $(\widetilde{V}, \widetilde{l})$ by the Zorn lemma. By the part (b), we have $\widetilde{V} = X$. □

1.7 (Complex linear functionals)**.** Let X be a complex vector space. Consider a map

$$\begin{array}{ccc} \{\mathbb{C}\text{-linear functionals on } X\} & \to & \{\mathbb{R}\text{-linear functionals on } X\} \\ l & \mapsto & \operatorname{Re} l. \end{array}$$

Let p be a seminorm on X and l a complex linear functional on X.

(a) The above map is bijective.

(b) $|l(x)| \le p(x)$ if and only if $|\operatorname{Re} l(x)| \le p(x)$.

Proof. (b) There is λ such that $|\lambda| = 1$ and $l(\lambda x) \ge 0$. Then,

$$|l(x)| = |\lambda^{-1} l(\lambda x)| = l(\lambda x) = \operatorname{Re} l(\lambda x) \le p(\lambda x) = p(x).$$

□

1.8 (Hahn-Banach separation)**.**

Exercises

1.9 (Topology of compact convergence)**.**

Chapter 2

Barreled spaces

2.1 Uniform boundedness principle

2.1 (Barreled spaces). Let X be a topological vector space. A *barrel* is an absorbing, balanced, convex, and closed subset of X. A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

2.2 (Uniform boundedness principle). Let X and Y be topological vector spaces. Let $\mathcal{F}$ be a family of continuous linear operator from X to Y. Suppose $\bigcup_{T\in\mathcal{F}} Tx$ is bounded for each $x \in D$, where $D \subset X$.

(a) If D is dense in X, then $\bigcap_{T\in\mathcal{F}} T^{-1}\overline{U}$ is absorbing.

(b) If X is barreled, then $\mathcal{F}$ is equicontinuous.

2.2 Baire category theorem

2.3 (Baire spaces). A topological space is called a *Baire space* if the countable intersection of open dense subsets is always dense.

(a) If a topological vector space is Baire, then it is barreled.

(b) A Baire space is second category in itself.

(c) A topological group that is second category in itself is Baire.

2.4 (Absorbing sets). Let X be a topological vector space that is Baire. A subset $U \subset X$ is said to be *absorbing* if for every $x \in X$ there is a sufficiently large $t > 0$ such that $x \in tU$. Let $U \subset X$.

(a) If U is closed and absorbing, then U has a non-empty open subset.

(b) If U is closed and absorbing, then $U - U$ is a neighborhood of zero.

(c) If U is closed, convex, and absorbing, then U is a neighborhood of zero.

2.5 (Baire category theorem). The Baire category theorem proves many exmples of topological vector space are Baire, in particular barreled.

(a) A complete metric space is Baire.

(b) A locally compact Hausdorff space is Baire.

2.3 Open mapping theorem

2.6 (Open mapping theorem)**.** Let X be a F-space and Y a barreled space. Suppose $T : X \to Y$ is a continuous and surjective linear operator.

(a) $\overline{TU}$ is a neighborhood of zero.

(b) TU is a neighborhood of zero.

Proof. (a) Let U' be a neighborhood of zero such that $U \supset U' - U'$. Because T is surjective, the set $\overline{TU'}$ is a closed absorbing set, so it contains a non-empty open subset, since Y is barreled. Thus, $\overline{TU} \supset \overline{TU'} - \overline{TU'}$ is a neighborhood of zero.

(b) We claim $\overline{TU_{2^{-1}}} \subset TU_1$. Take $y_1 \in \overline{TU_{2^{-1}}}$.

Assume $y_n \in \overline{TU_{2^{-n}}}$. Since $\overline{TU_{2^{-(n+1)}}}$ is a neighborhood of zero, we have

$$(y_n + \overline{TU_{2^{-(n+1)}}}) \cap TU_{2^{-n}} \neq \varnothing.$$

Then, there is $x_n \in U_{2^{-n}}$ such that $Tx_n \in y_n + \overline{TU_{2^{-(n+1)}}}$. Define

$$y_{n+1} := y_n - Tx_n.$$

Then, $\sum_{n=1}^{\infty} x_n$ clearly converges to $x \in U_1$. Therefore,

$$Tx = \sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} (y_n - y_{n+1}) = y_1. \qquad \square$$

Exercises

2.7. Let (T_n) be a sequence in $B(X, Y)$. If T_n coverges strongly then $\|T_n\|$ is bounded by the uniform boundedness principle.

2.8. There is a closed absorbing set in $\ell^2(\mathbb{Z}_{\geq 0})$ that is not a neighborhood of zero;

$$\overline{B}(0,1) \setminus \bigcup_{i=2}^{\infty} B(i^{-1}e_i, i^{-2})$$

is a counterexample.

2.9. There is no metric d on $C([0,1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise as $n \to \infty$ for every sequence f_n. Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

2.10. We show that there is no projection from ℓ^∞ onto c_0.

2.11 (Schur property)**.** ℓ^1

2.12. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

2.13 (Daugavet property)**.** (a) The real Banach space $C([0,1])$ satisfies the Daugavet property.

Proof. Let T be a finite rank operator on $C([0,1])$, and e_i be a basis of $\operatorname{im} T$. Then, for some measures μ_i,

$$Tf(t) = \sum_{i=1}^{n} \int_0^1 f \, d\mu_i e_i(t).$$

Let $M := \max \|e_i\|$.

Take f_0 such that $\|f_0\| = 1$ and $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$. Reversing the sign of f_0 if necessary, take an open interval Δ such that $Tf_0(t) \geq \|T\| - \frac{\varepsilon}{2}$ and $|\mu_i|(\Delta) \leq \frac{\varepsilon}{4nM}$ for all i. Define f_1 such that $f_0 = f_1$ on Δ^c, $f_1(t_0) = 1$ for some $t_0 \in \Delta$, and $\|f_1\| = 1$. Then, $\|Tf_1 - Tf_0\| \leq \frac{\varepsilon}{2}$ shows $Tf_1 \geq \|T\| - \varepsilon$ on Δ. Therefore,

$$\|1 + T\| \geq \|f_1 + Tf_1\| \geq f_1(t_0) + Tf_1(t_0) \leq 1 + \|T\| - \varepsilon.$$

□

2.14 (Bartle-Graves theorem)**.** Let E be a Banach space and N a closed subspace. For $\varepsilon > 0$, there is a continuous homogeneous map $\rho : E/N \to E$ such that $\pi\rho(y) = y$ and $\|\rho(y)\| \leq (1+\varepsilon)\|y\|$ for all $y \in E/N$.

Proof. We want to construct a continuous map $\psi : S_{E/N} \to E$ with $\|\psi(y)\| \leq 1 + \varepsilon$ for all $y \in S_{E/N}$. If then, ρ can be made from ψ.

For each $y_0 \in S_{E/N}$, choose $x_0 \in \pi^{-1}(y_0) \cap B_{1+\varepsilon}$. There is a neighborhood $V_{y_0} \subset S_{E/N}$ of y_0 such that $y \in V_{y_0}$ implies x_0 belongs to $(\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$, which is convex. With a locally finite subcover V_{y_α} and a partition of unity $\eta_\alpha(y)$, define $\psi_1(y) = \sum_\alpha \eta_\alpha(y) x_\alpha$. Then, $\psi_1(y) \in (\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$.

For $i \leq 2$, choose for each y_0 the element x_0 in $\pi^{-1}(y_0) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}})$. Then, we obtain

$$\psi_i(y) \in \left(\pi^{-1}(y) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}})\right) + U_{2^{-i}}.$$

Therefore, $\|\psi_i(y) - \psi_{i-1}(y)\| < 2^{-i-2}$, so it converges uniformly to ψ such that $\psi(y) \in \pi^{-1}(y) \cap B_{1+\varepsilon}$. □

Problems

2.15. Let T be an invertible linear operator on a normed space. Then, $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

Chapter 3

Weak topologies

3.1 Dual spaces

3.1 (Bidual).

3.2. Let X be a locally convex space. The *weak topology* is the topology w on X defined by the family of seminorms $\{x \mapsto |\langle x, \xi\rangle|\}_{\xi\in X^*}$. The *weak* topology* is the topology w^* on X^* defined by the family of seminorms $\{\xi \mapsto |\langle x, \xi\rangle|\}_{x\in X}$. Let $J : X \to X^{**}$ be the canonical embedding.

(a) (X, w) and (X^*, w^*) are locally convex.

(b) $(X, w)^* = X^*$.

(c) $(X^*, w^*)^* = X$. Every locally convex space is a dual of a locally convex space.

Proof. (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show $(X^*, w^*)^* \subset X$. If $u \in (X^*, w^*)^*$, then there are $x_1, \cdots, x_m \in X$ such that

$$|\langle u, \xi\rangle| \le \sum_{i=1}^{m} |\langle x_i, \xi\rangle|$$

for all $\xi \in X^*$. If we let $\ker \vec{x} := \bigcap_{i=1}^{m} \ker x_i$, then it is a closed subspace of X^* such that $\ker \vec{x} \subset \ker u$, so we have $u \in \operatorname{span} \vec{x} \subset X$. □

3.3. closure and weak closure of convex subsets

Proof. Hahn-Banach □

3.4 (Polar).

boundedness, incompleteness

3.5 (Weak convergence by dense set). Let X be a Banach space, D^* a subset of X^*, and $\overline{D^*}$ the norm closure of D^*. For example, if X has a predual $X_* \subset X^*$ and D^* is dense in X_*, then $\sigma(X, \overline{D^*})$ is the weak* topology.

(a) There is a squence $x_n \in X$ converges to zero in $\sigma(X, D^*)$ but not in $\sigma(X, \overline{D^*})$.

(b) A bounded sequence $x_n \in X$ converges to zero in $\sigma(X, \overline{D^*})$ if in $\sigma(X, D^*)$.

Proof. (b) Let $\xi \in \overline{D^*}$ and choose $\eta \in D^*$ such that $\|\xi - \eta\| < \varepsilon$. Then,

$$|\langle x_n, \xi\rangle| \le \|x_n\|\|\xi - \eta\| + |\langle x_n, \eta\rangle| \lesssim \varepsilon + |\langle x_n, \eta\rangle| \to \varepsilon.$$

□

3.2 Weak compactness

3.6 (Banach-Alaoglu theorem).

Proof. Consider

$$B_{X^*} \to \prod_{x\in X} \|x\| B : l \mapsto (l(x))_{x\in X}.$$

Since it is an embedding into a compact space, it suffices to show the closedness of image: for $l(x) := \lim_\alpha l_\alpha(x)$, we have

$$\|l(x)\| \le \|l(x) - l_\alpha(x)\| + \|x\| \xrightarrow{\alpha\to\infty} \|x\|,$$

so l is contained in the range. □

3.7 (Eberlein-Šmulian theorem).

3.8 (James' theorem).

3.3 Weak density

Bishop-Phelps theorem

3.9 (Goldstine's theorem). Let X be a Banach space and $J : X \to X^{**}$ the canonical embedding. Our claim is that $\overline{B}$ is weak*-dense in $\overline{B}_{X^{**}}$. Let $x_0^{**} \in X^{**}$ with $\|x_0^{**}\| \le 1$, and let

$$\bigcap_{i=1}^m \{ x^{**} \in X^{**} : |\langle x^{**} - x_0^{**}, x_i^* \rangle| < \varepsilon \}$$

be an open weak*-neighborhood of zero in X^{**} with $\|x_i^*\| \le 1$ and $\varepsilon > 0$. Let

$$S := \bigcap_{i=1}^m \{ x \in X : \langle x, x_i^* \rangle = \langle x_0^{**}, x_i^* \rangle \}.$$

(a) S is not empty.

(b) $S \cap (1+\varepsilon)\overline{B}_X$ is not empty for all $\varepsilon > 0$.

(c) $\overline{B}_X$ is weak*-dense in $\overline{B}_{X^{**}}$

Proof. (a)

(b) From the part (a), we have $x \in S$. Suppose S does not intersect $(1+\varepsilon)\overline{B}_X$. By the Hahn-Banach theorem, there is $y^* \in X^*$ such that

$$y^*|_{S-x} = 0, \quad \langle x, y^* \rangle > 1+\varepsilon, \quad \text{ and } \quad \|y^*\| = 1.$$

Since $S - x = \bigcap_{i=1}^m \ker x_i^*$, the linear functional y^* is a linear combination of $x_1^*, \cdots, x_m^*$, so we have

$$1+\varepsilon < \langle x, y^* \rangle = \langle x_0^{**}, y^* \rangle \le \|x_0^{**}\| \|y^*\| \le 1.$$

(c) Take $\varepsilon > 0$ such that $\varepsilon \max_{1\le i\le m} \|x_i^*\| < 1$. By the part (b), there is $y \in X$ such that $\|y\| \le 1+\varepsilon$ and $\langle y, x_i^* \rangle = \langle x^{**}, x_i^* \rangle$. If we let $x := (1+\varepsilon)^{-1} y$, then $x \in \overline{B}_X$ so that

$$|\langle x - x_0^{**}, x_i^* \rangle| = |\langle x - y, x_i^* \rangle| = |\langle \varepsilon x, x_i^* \rangle| \le \varepsilon \|x\| \|x_i^*\| < \varepsilon$$

for all i. □

3.4 Krein-Milman theorem

Choquet theory

3.5 Polar topologies

Mackey-Arens

Exercises

3.10 (James' space)**.** not reflexive but isometrically isomorphic to bidual

3.11 (Predual correspondence)**.** Let X be a Banach space. Let

$$\{\,(Y,\varphi)\mid \varphi : X \to Y^* \text{ is an isometric isorphism}\,\}$$

and

$$\{\,Z \le X^* \mid \overline{B_X} \text{ is compact Hausdorff in } (X,\sigma(X,Z))\,\}.$$

$$(Y,\varphi) \mapsto \operatorname{im}\varphi^*|_{J(Y)}$$

(a) The map is well-defined.

(b) The map is surjective. (by Goldstein)

(c) The map is injective up to isomorphism for Y.

3.12. Let X be a closed subspace of a Banach space Y and

$$i : X \to Y$$

the inclusion. Suppose X and Y have preduals X_* and Y_* respectively. Let

$$j := i^*|_{Y_*} : Y_* \to Z \subset X^*,$$

where $Z := i^*(Y_*)^-$. Then we can show

$$j^* : Z^* \subset X^{**} \to Y$$

coincides with i on $X \cap Z^*$. From the existence of X_* we have $X^{**} \to X$, which is restricted to define a map $k : Z^* \to X$.

$$\begin{tikzcd}
& X \arrow[r, "i"] & Y \\
X^{**} \arrow[ur] \arrow[r] & Z^* \arrow[u, "k"] \arrow[ur, "j"'] &
\end{tikzcd}$$

We can show k is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

3.13 (Mazur's lemma)**.**

Part II

Banach spaces

Chapter 4

Operators on Banach spaces

4.1 Bounded operators

4.1 (Bounded belowness in Banach spaces)**.** Let $T \in B(X,Y)$ for Banach spaces X and Y. The following statements are equivalent:

(a) T is bounded below.

(b) T is injective and has closed range.

(c) T is a topological isomorphism onto its image.

4.2 (Bounded belowness in Hilbert spaces)**.** Let $T \in B(H,K)$ for Hilbert spaces H and K. The following statements are equivalent:

(a) T is bounded below.

(b) T is left invertible.

(c) T^* is right invertible.

(d) T^*T is invertible.

4.3 (Injectivity and surjectivity of adjoint)**.** Let $T \in B(X,Y)$ for Banach spaces X and Y.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is bounded below.

4.2 Compact operators

$K(X,Y)$ is closed in $B(X,Y)$. $K(X)$ is an ideal of $B(X)$. adjoint is $K(X,Y) \to K(Y^*,X^*)$. integral operators are compact. riesz operator, quasi-nilpotent operator.

4.3 Fredholm operators

4.4. A bounded linear operator $T : X \to Y$ between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

(a) A Fredholm operator T has closed range.

Proof. (a) Let C be a finite dimensional subsapce of Y such that $\operatorname{im} T \oplus C = Y$. Let $\widetilde{T} : X/\ker T \to Y$ be the induced operator of T. Define $S : (X/\ker T) \oplus C \to Y$ such that $S(x + \ker T, c) := \widetilde{T}(x + \ker T) + c$. Then, S is an topological isomorhpism between Banach spaces by the open mapping theorem, so $S(X/\ker T \oplus \{0\}) = \operatorname{im} \widetilde{T} = \operatorname{im} T$ is closed. □

4.5 (Atkinson's theorem). An operator $T \in B(X, Y)$ is Fredholm if and only if there is $S \in B(Y, X)$ such that $TS - I$ and $ST - I$ is finite rank.

4.6 (Fredholm index). locally constant, in particular, continuous. composition makes the addition of indices.

4.4 Nuclear operators

tensor products

Exercises

4.7 (Completely continuous operators). On reflexive spaces, completely continuous operators are same with compact operators.

4.8 (Dunford-Pettis property). A Banach space X is said to have the *Dunford-Pettis property* if all weakly compact operators $T : X \to Y$ to any Banach space Y is completely continuous.

(a) X has the Dunford-Pettis property if and only if for every sequences $x_n \in X$ and $x_n^* \in X^*$ that converge to x and x^* weakly we have $x_n^*(x_n) \to x^*(x)$.

(b) $C(\Omega)$ for a compact Hausdorff space Ω has the Dunford-Pettis property.

(c) $L^1(\Omega)$ for a probability space Ω has the Dunford-Pettis property.

(d) Infinite dimensional reflexive Banach space does not have the Dunfor-Pettis property.

Problems

1. If $T \in B(L^2([0,1]))$ is a compact operator, then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|Tf\|_{L^2} \le \varepsilon \|f\|_{L^2} + C_\varepsilon \|f\|_{L^1}.$$

Proof. 1. Suppose there is $\varepsilon > 0$ such that we have sequence $f_n \in L^2$ satisfying $\|f_n\|_2 = 1$ and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of T, there is a subsequence Tf_{n_k} converges to $g \neq 0$ in L^2. Then, $\|f_{n_k}\|_1 \to 0$ implies $f_{n_k} \to 0$ weakly in L^2, hence also for Tf_{n_k}. It means $g = 0$, which contradicts to the assumption. □

Chapter 5

Geometry of Banach spaces

5.1 Tensor products

5.2 Approximation property

dual is Banach. Basis problem, Mazur' duck.

5.1 (Approximation property)**.** Every compact operator is a limit of finite-rank operators.

(a) An Hilbert space has the AP.

(b)

Proof. (a) Let H be a Hilbert space and $K \in K(H)$. Since $\overline{KB_H}$ is a compact metric space, it is separable, which means $\overline{KH}$ is separable. Let $(e_i)_{i=1}^\infty$ be an orthonormal basis of $\overline{KH}$, and let P_n be the orthogonal projection on the space spanned by $(e_i)_{i=1}^n$. If we let $K_n := P_nK$, then $K_n \to K$ strongly and K_n has finite rank. Take any $\varepsilon > 0$ and find, using the totally boundedness of KB_H, a finite subset $\{x_j\} \subset B_H$ such that for any $x \in B_H$ there is x_j satisfying $\|Kx - Kx_j\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}\|Kx - K_nx\| &\le \|Kx - Kx_j\| + \|Kx_j - K_nx_j\| + \|P_n(Kx_j - Kx)\| \\ &\le \frac{\varepsilon}{2} + \|Kx_j - K_nx_j\| + \frac{\varepsilon}{2}.\end{aligned}$$

By taking the supremum on $x \in B_H$, we have

$$\|K - K_n\| \le \max_j \|Kx_j - K_nx_j\| + \varepsilon,$$

which deduces $K_n \to K$ in norm.

□

Exercises

Tingley problem

Chapter 6

Part III

Spectral theory

Chapter 7

Operators on Hilbert spaces

7.1 Hilbert spaces

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

7.1. (a) A Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

7.2 (Riesz representation theorem)**.** Let H be a Hilbert space over a field $\mathbb{K}$, which is either $\mathbb{R}$ of $\mathbb{C}$.

We use the bilinear form $\langle -,-\rangle : X \times X^* \to \mathbb{K}$ of canonical duality. The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each $x^* \in H^*$, there is a unique $x \in H$ such that $\langle y, x^*\rangle = \langle y, x\rangle$ for every $y \in H$.

(b) $H \to H^* : x \mapsto \langle -, x\rangle$ is a natural linear and anti-linear isomorphism if $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$, respectively.

Let H be a separable Hilbert space. Find a positive sequence a_n such that every sequence x_n of unit vectors of H satisfying $|\langle x_i, x_j\rangle| \le a_j$ for all $i < j$ converges weakly to zero.

7.3 (Normal operators)**.** For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. Suppose T is normal.

(a) $\ker T = \ker T^*$.

(b) T is bounded below if and only if T is invertible.

(c) If T is surjective, then T is invertible.

7.4 (Invariant and Reducing subsapces)**.** Let K be a closed subspace of H.

(a) K is reducing for T if and only if K is invariant for T and T^*.

(b) K is reducing for T if and only if $TP = PT$, where P is the orthogonal projection on K.

7.5 (Trace class operators)**.** Let $K \in B(H)$ The *trace* of K is

$$\operatorname{Tr}(K) := \sum_i \langle Ke_i, e_i\rangle,$$

where $(e_i) \subset H$ is an orthonormal basis. The operator K is said to be in the *trace-class* if $\operatorname{Tr}(|K|) < \infty$.

(a) The trace does not depend on the choice of (e_i).

(b) K is a trace class if and only if $K = \sum_{i=1}^\infty \lambda_i \theta_{x_i, y_i}$ for some $(\lambda_i)_{i=1}^\infty \subset \ell^1(\mathbb{N})$ and orthogonal sequences $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \subset H$.

(c) $B(H) \to L^1(H)^* : T \mapsto \mathrm{Tr}(T\cdot)$ is an isometric isomorphism.

Proof. (b) Conversely, we can check the diagonalization $K^*K = \sum_{i=1}^\infty |\lambda_i|^2 \theta_{y_i}$, which implies $|K| = \sum_{i=1}^\infty |\lambda_i| \theta_{y_i}$. Thus,

$$Tr(|K|) = \sum_{j=1}^\infty \langle |K| y_j, y_j \rangle = \sum_{i=1}^\infty |\lambda_i| < \infty.$$

□

7.2 Spectral theorems

7.6 (Spectral measure). Let $(\Omega, \mathcal{A})$ be a measurable space and H a Hilbert space. A *projection-valued measure* on Ω for H is a map $E : \mathcal{A} \to B(H)$ with $E(\varnothing) = 0$ such that $E(A)$ is a projection for every $A \in \mathcal{A}$, and one of the following equivalent conditions hold:

(i) the set function $E_{x,y} : \mathcal{A} \to \mathbb{C} : A \mapsto \langle E(A)x, y \rangle$ is a complex measure on Ω for each $x, y \in H$.

(ii) the countable additivity holds, i.e. $E(\bigsqcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty E(A_i)$ in the strong operator topology of $B(H)$ for $(A_i)_{i=1}^\infty \subset \mathcal{M}$.

(a) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{M}$.

7.7. Let $T \in B(H)$ be a normal operator. Then, there exists a projection-valued measure E on $\sigma(T)$ for H such that

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

This spectral measure E is also called the *resolution of the identity*.

Let E be the spectral measure of a normal operator $T \in B(H)$. If we choose $\xi \in E(B(\lambda, n^{-1}))H$, then since $E(B(\lambda, n^{-1})^c)\xi = 0$, or since $E(B(\lambda, n^{-1}))\xi = \xi$, we have

$$\begin{aligned} \|(\lambda - T)\xi\|^2 &= \int |\lambda - z|^2 \, d\langle E(z)\xi, \xi \rangle \\ &= \int_{B(\lambda, n^{-1})} |\lambda - z|^2 \, d\langle E(z)\xi, \xi \rangle \\ &\le n^{-2} \int_{B(\lambda, n^{-1})} d\langle E(z)\xi, \xi \rangle \\ &\le n^{-2} \int d\langle E(z)\xi, \xi \rangle \\ &= n^{-2} \|\xi\|^2. \end{aligned}$$

7.8 (Spectral representation). Let T be a bounded normal operator on a Hilbert space H. Then, the unital C*-algebra $C^*(T)$ generated by T is $*$-isomorphic to $C(\sigma(T))$, and it has a canonical faithful representation $\pi : C(\sigma(T)) \to B(H)$. Decompose $\pi = \bigoplus_\alpha \pi_\alpha$ to cyclic representations $\pi_\alpha : C(\sigma(T)) \to B(H_\alpha)$ with cyclic unit vectors ψ_α. Each vector state ψ_α induces a probability measure μ_α on $\sigma(T)$. It is called the spectral measure associated to the cyclic vector ψ_α. We can check that the GNS-representation of μ_α is unitarily equivalent to π_α. The direct sum $C(\sigma(T)) \to \bigoplus_\alpha B(L^2(\mu_\alpha))$ can be defined.

The bounded normal operator T is always unitarily equivalent to the multiplication operator on a finite measure space. However, in order for T to be unitarily equivalent to the multiplication operator by the identity function of $\mathbb{C}$, T must be multiplicity free, equivalently, T must have a cyclic vector.

On a C*-algebra $\mathcal{A}$, each state ω defines a von Neumann algebra $\pi_\omega(\mathcal{A})''$, which is the start of measure theory.

Two bounded normal operators are unitarily equivalent if and only if the sequence of multiplicity measure classes are identical.

Two spectral theorems: Multiplication operator form(unitary equivalence), Projection-valued measure form(functional calculus).

7.3 Decomposition of spectrum

$$\sigma = \sigma_p \sqcup \sigma_c \sqcup \sigma_r = \overline{\sigma_{pp}} \cup \sigma_{ac}\sigma_{sc} = \sigma_d \sqcup \sigma_{ess,5}.$$

7.4 Operator topologies

7.9. (a) A net T_α converges to T strongly in $B(H)$ if and only if $\|(T_\alpha - T)^{\oplus n}\overline{\xi}\| \to 0$ for all $\overline{\xi} \in H^{\oplus n}$.

(b) A net T_α converges to T σ-strongly in $B(H)$ if and only if $\|(T_\alpha - T)^{\oplus\infty}\overline{\xi}\| \to 0$ for all $\overline{\xi} \in H^{\oplus\infty}$.

7.10 (Continuity of linear functionals)**.** Let f be a linear functional on $B(H)$ for a Hilbert space H.

(a) f is weakly continuous if and only if it is strongly continuous, and in this case we have $f = \sum_{i=1}^n \omega_{x_i,y_i}$ for some $(x_i)_{i=1}^n, (y_i)_{i=1}^n \subset H$.

Proof. Suppose f is strongly continuous. There exists $\overline{x} \in H^{\oplus n}$ such that

$$|f(T)| \le \|T^{\oplus n}\overline{x}\|.$$

The functional $f : A \to \mathbb{C}$ factors through $H^{\oplus n}$ such that

$$A \xrightarrow{\overline{x}} H^{\oplus n} \to \mathbb{C}.$$

□

7.11 (Vigier theorem)**.** Let T_α be a net of bounded self-adjoint operators on H which is increasing and bounded above. Then, T_α converges strongly.

Proof. Define T such that

$$\langle Tx, y\rangle := \lim_\alpha \sum_{k=0}^{3} i^k \langle T_\alpha(x + i^k y), x + i^k y\rangle.$$

The convergence is due to the monotone convergence in $\mathbb{R}$. We can check it is a well-defined bounded linear operator by considering the bounded sesquilinear form. Then, $T_\alpha \to T$ weakly by definition, and strongly by because the net is increasing. □

For $(x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \subset H$ such that $\sum_i \|x_i\|^2 < \infty$ and $\sum_i \|y_i\|^2 < \infty$,

$$p_{\overline{x}}^{\sigma s*}(T) = \left(\sum_i \|Tx_i\|^2 + \|T^*x_i\|^2\right)^{\frac{1}{2}} \qquad p_{\overline{x}}^{\sigma s}(T) = \left(\sum_i \|Tx_i\|^2\right)^{\frac{1}{2}} \qquad p_{\overline{x},\overline{y}}^{\sigma w}(T) = \left|\sum_i \langle Tx_i, y_i\rangle\right|$$

Exercises

7.12 (Strong* operator topology). Let H be a Hilbert space. We provides some conditions for a strongly convergent sequence to converge strongly*. Let $(T_\alpha) \subset B(H)$ and suppose $T_\alpha \to T$ strongly.

7.13 (Strict topology). Let H be a Hilbert space. Let $(T_\alpha) \subset B(H)$ and $K \in K(H)$.

(a) The strong* topology and the strict topology agree on bounded sets of $B(H)$.

7.14 (Unitary group). Let H be a Hilbert space.

(a) The weak topology and the strict topology agree on $U(H)$.

Chapter 8

Unbounded operators

8.1

8.1 (Closed operators)**.**

8.2 (Adjoint operators)**.** Let $T : X \to Y$ be an unbounded linear operator between Banach spaces. Define an unbounded operator $T^* : Y^* \to (\operatorname{dom} T)^*$ by

$$\operatorname{dom} T^* := \{y^* \in Y^* \mid \operatorname{dom} T \to \mathbb{C} : x \mapsto \langle Tx, y^*\rangle \text{ is bounded}\},$$

$$\langle x, T^* y^*\rangle := \langle Tx, y^*\rangle, \qquad x \in \operatorname{dom} T,\ y^* \in \operatorname{dom} T^*.$$

Suppose T is densely defined so that we can write $T^* : Y^* \to X^*$.

(a) If $T \subset S$, then $S^* \subset T^*$.

(b) T^* is closed.

(c) T^* is densely defined if an only if T is closable.

(d) If T is closable, then $\overline{T} = T^{**}$. (Only on Hilbert spaces?)

(e) If T is closable, then $T^* = \overline{T}^*$. Since T^* is a priori closed, we will use the notation $\overline{T}^* := \overline{T}^*$.

Let $L : H \to H$ be a densely defined operator. Here is a routine to find a closure.

1. Compute $\operatorname{dom} L^*$ and check it is dense to show L is closable.
2. Compute $\operatorname{dom} L^{**}$ to find the closure of L.
3. Do work with our densely defined closed operator $\overline{L} = L^{**}$.

8.3. Let $T : X \to Y$ be a densely defined closed operator between Banach spaces.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is bounded below.

Proof. (b) Suppose T is bounded below. Fix $x^* \in X^*$. Since T is bounded below, x^* defines a bounded linear functional on $\operatorname{dom} T$ with respect to $\|x\| + \|Tx\|$, which is embedded in Y as a closed subspace. By the Hahn-Banach extension, we have an element $y^* \in Y^*$ such that $\langle Tx, y^*\rangle = \langle x, x^*\rangle$ for all $x \in X$, which is contained in $\operatorname{dom} T^*$ by the definition of $\operatorname{dom} T^*$. This implies that T is surjective because $T^* y^* = x^*$.

Conversely, suppose T^* is surjective. Let $F := \{x \in \operatorname{dom} T : \|Tx\| \le 1\}$. Since for every $x^* \in X^*$ we have for some $y^* \in \operatorname{dom} T^*$ such that

$$\sup_{x\in F} |\langle x, x^*\rangle| = \sup_{x\in F} |\langle x, T^* y^*\rangle| = \sup_{x\in F} |\langle Tx, y^*\rangle| \le \|y^*\|.$$

By the uniform boundedness principle, we have $\sup_{x\in F}(\|x\| + \|Tx\|)$ is bounded, we are done. □

8.4 (Symmetric operators)**.** An unbounded operator $T : H \to H$ is called *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad x, y \in \operatorname{dom} T.$$

(a) A symmetric operator is always closable and its closure is also symmetric.

(b) If T is symmetric, then $T \subset T^*$. If T is densely defined, then the converse holds.

8.5 (Symmetric extensions)**.**

(a) If T is symmetric, then every symmetric extension is a restriction of T^*.

(b) If T is symmetric, then it has a maximal symmetric extension. Note that T^* is not symmetric in general.

(c) A maximal symmetric operator is closed since the closure of a .

(d) A self-adjoint operator is maximal.

(e) A densely defined symmetric operator is essentially self-adjoint if and only if it has a unique self adjoint extension.

(f) A densely defined symmetric operator may have no or many self-adjoint extensions.

8.6 (Cayley transform)**.**

8.2 Spectral theorem

A self-adjoint operator must be a densely defined and closed.

8.7. For a densely defined closed operator $T : H \to H$, $\sigma(T^*) = \overline{\sigma(T)}$.

8.8. Let $T : H \to H$ be a

(a)

Kato-Rellich theorem
analytic vector theorem

Chapter 9

Operator theory

9.1 Toeplitz operators

invariant subspace problem Beurling theorem Hardy and Bergman and Bloch spaces JB* triple

Part IV

Operator algebras

Chapter 10

Banach algebras

10.1 Spectra of elements

10.1 (Banach algebras)**.** For a Banach algebra A with multiplicative unit, there is a complete renorming such that $\|1\| = 1$, i.e. we can always assume $\|1\| = 1$. It provides a definition of *unital Banach algebra*.

Let A be a unital Banach algebra.

(a) If $\|a\| < 1$, then $1 - a$ is invertible. So $A^\times$ is open.

(b) $A^\times \to A^\times : a \mapsto a^{-1}$ is continuous.

(c) $A^\times \to A^\times : a \mapsto a^{-1}$ is differentiable.

Proof. (a) We can show

$$(1-a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

If a is invertible, then $a + h = a(1 + a^{-1}h)$ has the inverse $(1 + a^{-1}h)^{-1}a^{-1}$ if h is sufficiently small such that $\|a^{-1}h\| < 1$.

(b) Clear from

$$b^{-1} - a^{-1} = b^{-1}(a-b)a^{-1}.$$

(c)

$$\begin{aligned} \frac{\|b^{-1} - a^{-1} - (-a^{-1}(b-a)a^{-1})\|}{\|b-a\|} &= \frac{\|(a^{-1} - b^{-1})(b-a)a^{-1}\|}{\|b-a\|} \\ &\le \|a^{-1} - b^{-1}\| \|a^{-1}\| \xrightarrow{b \to a} 0. \end{aligned}$$

□

10.2 (Spectrum and resolvent)**.** Let a be an element of a unital Banach algebra A. The *spectrum* of a in A is defined to be the set

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A\},$$

and the *resolvent* of a in A is defined to be its complement $\rho_A(a) := \mathbb{C} \setminus \sigma_A(a)$. We can now define the *resolvent map* $R : \rho_A(a) \to A$ such that

$$R(\lambda) = R(\lambda; a) := (\lambda - a)^{-1}.$$

If A is clear in its context, we omit it to just write $\sigma(a)$ and $\rho(a)$.

(a) $\sigma(a)$ is compact.

(b) $\sigma(a)$ is non-empty.

(c) If A is a division ring, then $A \cong \mathbb{C}$. This result is called the *Gelfand-Mazur theorem*.

Proof. (a) If $|\lambda| > \|a\|$, then $\lambda - a$ is always invertible, so the spectrum is bounded. Closedness follows from the fact that the set of invertibles is open.

(b) Suppose the spectrum $\sigma(a) = \varnothing$ so that the resolvent function $R : \mathbb{C} \to A$ is well-defined on the entire $\mathbb{C}$. Note that $a \neq 0$. Since R is continuous and since

$$\|(\lambda - a)^{-1}\| = \|\lambda^{-1}(1 - \lambda^{-1}a)^{-1}\| = \left\|\lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}a)^k\right\| < (2\|a\|)^{-1}\sum_{k=0}^{\infty}2^{-k} = \|a\|^{-1}$$

on $\{\lambda \in \mathbb{C} : |\lambda| > 2\|a\|\}$, the function R is bounded. Also, for every $l \in A^*$ we have that the function $\mathbb{C} \to \mathbb{C} : \lambda \mapsto \langle R(\lambda), l\rangle$ is holomorphic since $a \mapsto a^{-1}$ is differentiable.

Therefore, by the Liouville theorem, the bounded entire function $\lambda \mapsto \langle R(\lambda), l\rangle$ is constant for all $l \in A^*$. Because A^* separates points of A, the function R is constant, which implies $a \in \mathbb{C}$ and contradicts to $\sigma(a) = \varnothing$.

(c) For any $a \in A$, by the part (b), there must be λ such that $\lambda - a$ is not invertible. In a division ring, zero is the only non-invertible element, so $\lambda = a$. $\square$

10.3 (Spectral radius)**.** Let a be an element of a unital Banach algebra A. The *spectral radius* of a in A is defined to be

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$

(a) $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) $\limsup_n \|a^n\|^{\frac{1}{n}} \le r(a)$, i.e. $r(a) = \lim_n \|a^n\|^{\frac{1}{n}}$.

Proof. (a) Since $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1}$ exists if $|\lambda| > \|a\|$, we have $r(a) \le \|a\|$ for all $a \in \mathcal{A}$. For every $\lambda \in \sigma(a)$ and every integer $n \ge 1$ we have

$$|\lambda|^n = |\lambda^n| \le r(a^n) \le \|a^n\|,$$

and it proves $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) Consider a holomorphic function

$$f : \{\lambda \in \mathbb{C} : |\lambda| > r(a)\} \to \mathbb{C} : \lambda \mapsto \langle R(\lambda), l\rangle$$

for each $l \in A^*$. Since on a smaller domain $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$, the function f can be given by

$$f(\lambda) = \left\langle \lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}a)^k, l\right\rangle,$$

we can determine the coefficients of the Laurent series of f at infinity as

$$f(\lambda) = \sum_{k=0}^{\infty}\langle a^k, l\rangle\lambda^{-k-1}$$

on $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$.

Take λ such that $|\lambda| > r(a)$. Then, the sequence $(a^k\lambda^{-k-1})_k \in \mathcal{A}$ is weakly bounded, hence is normly bounded by the uniform boundedness principle. Let $\|a^n\| \le C_\lambda|\lambda^{n+1}|$ for all $n \ge 1$. Then,

$$\limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}} \le \limsup_{n\to\infty} C_\lambda^{\frac{1}{n}}|\lambda^{n+1}|^{\frac{1}{n}} = |\lambda|.$$

If we limit $|\lambda| \downarrow r(a)$, we are done. $\square$

10.4 (Spectrum in closed subalgebras). For fixed element, smaller the ambient algebra, less "holes" in the spectrum. Let $A \subset B$ be a closed subalgebra containing 1_A. Note that A may be unital even for $1_B \notin A$.

(a) $B^\times$ is clopen in $A^\times \cap B$.

10.2 Ideals

10.5 (Ideals). (a) If I is a left ideal, then A/I is a left A-module.

10.6 (Modular left ideals). A left ideal I is called *modular* if there is $e \in \mathcal{A}$ such that $a - ae \in I$ for all $a \in A$. The element e is called a *right modular unit* for I.

(a) I is modular if and only if A/I is unital(?).

(b) A proper modular left ideal is contained in a maximal left ideal.

(c) I is a maximal modular left ideal if and only if I is a modular maximal left ideal.

(d) There is a non-modular maximal ideal in the disk algebra.

10.7 (Closed ideals). (a) closure of proper left ideal is proper left.

(b) maximal modular left ideal is closed.

10.8 (Unitization). Let A be an algebra. Recall that we always assume algebras are associative. Consider an embedding $A \to B(A) : a \mapsto L_a$, where $L_a(b) = ab$. Define

$$\widetilde{A} := \{ L_a + \lambda \operatorname{id}_{B(A)} : a \in A, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital A.

(a) If A is normed, then $\widetilde{A}$ is a normed algebra such that there is an isometric embedding $A \to \widetilde{A}$.

(b) If A is Banach, then $\widetilde{A}$ is a Banach algebra.

(c) $A \oplus \mathbb{C}$ is topologically isomorphic to $\widetilde{A}$ as normed spaces.

Proof. (a) The space of bounded operators $B(A)$ is a normd algebra. Then, $\widetilde{A}$ is a normed $*$-algebra with induced norm

$$\|L_a + \lambda \operatorname{id}_{B(A)}\| = \sup_{b \in A} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then, A is a normed $*$-subalgebra of $\widetilde{A}$ because the norm and involution of A agree with $\widetilde{A}$.

(b) Suppose (x_n, λ_n) is Cauchy in $\widetilde{A}$. Since A is complete so that it is closed in $\widetilde{A}$, we can induce a norm on the quotient $\widetilde{A}/A$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|x\| \le \|(x, \lambda)\| + |\lambda|$ shows that x_n is Cauchy in A.

Since a finite dimensional normed space is always Banach and A is Banach, λ_n and x_n converge. Finally, the inequality $\|(x, \lambda)\| \le \|x\| + |\lambda|$ implies that (x_n, λ_n) converges.

(c) Check the topology on $A \oplus \mathbb{C}$ in detail... □

unitization, homomorphisms, category(direct sum, product, etc.)
$B(\mathbb{C}^n) = M_n(\mathbb{C})$ is simple, but $B(H)$ is not simple.

10.3 Holomorphic functional calculus

10.9. Let a be an element of a unital Banach algebra A. Let f be a holomorphic function on a neighborhood U of $\sigma(a)$. Let C be a positively oriented smooth simple closed curve in U enclosing $\sigma(a)$. Define $f(a) \in A^{**}$ as the Dunford integral

$$\langle f(a), l\rangle := \int_C f(\lambda)\langle R(\lambda), l\rangle \, d\lambda, \qquad l \in A^*.$$

Let $\mathrm{Hol}(\sigma(a))$ be the space of all holomorphic functions on a neighborhood of $\sigma(a)$ endowed with the topology of compact convergence. Note that $\mathrm{Hol}(\sigma(a))$ is not Banach. We define the *holomorphic functional calculus* by the map

$$\mathrm{Hol}(\sigma(a)) \to A : f \mapsto f(a).$$

It is also called the Riesz or the Riesz-Dunford functional calculus.

(a) $f(a) \in A$, i.e. $f(a)$ is given by the Pettis integral.

(b) $f(a)$ is independent of the choice of C.

(c) The functional calculus is an algebra homomorphism.

(d) The functional calculus is bounded.

(e) injective.

(f) unital and $\mathrm{id}_{\mathbb{C}} \mapsto a$.

(g) spectral mapping.

(h) power series.

Proof. (a)

□

10.4 Gelfand theory

Banach algebra of single generator semisimplicity and symmetricity

10.10 (Spectrum of a Banach algebra)**.** Let A be a commutative Banach algbera. A *character* of A is a non-trivial algebra homomorphism $\pi : A \to \mathbb{C}$. Denote by $\sigma(A)$ the set of all characters of A and endow with the weak* topology on $\sigma(A) \subset A^*$. We call this space as the *spectrum* of A.

(a) If A is unital, $\sigma(A)$ is contained in the unit sphere of A^*.

(b) $\sigma(A)$ is locally compact and Hausdorff.

Proof. □

10.11 (Gelfand transform)**.** Let A be a commutative Banach algebra. The *Gelfand transform* or the *Gelfand representation* is the following algebra homomorphism

$$\Gamma : A \to C_0(\sigma(A)) : a \mapsto (\pi \mapsto \pi(a)).$$

(a) Γ has the image separating points by definition.

(b) Γ has closed range if A is a symmetric Banach $*$-algebra.

(c) Γ is injective if and only if A is semisimple.

(d) Γ is isometric if and only if $r(a) = \|a\|$ for all $a \in A$.

Exercises

10.12 (Basic properties of spectrum). Let A be a unital algebra.

(a) $\sigma(ab)\setminus\{0\}=\sigma(ba)\setminus\{0\}$.

(b) If $\sigma(a)$ is non-empty, then $\sigma(p(a))=p(\sigma(a))$.

Proof. (a) Intuitively, the inverse of $1-ab$ is $c=1+ab+abab+\cdots$. Then, $1+bca=1+ba+baba+\cdots$ is the inverse of $1-ba$. □

$C_b(\Omega)$ $\ell^\infty(S)$ $L^\infty(\Omega)$ $B_b(\Omega)$ $A(\mathbb{D})$ $B(X)$

10.13. In $C(\mathbb{R})$, the modular ideals correspond to compact sets.

10.14 (Disk algebra). (a) Every continuous homomorphism is an evaluation.

10.15 (Polynomial convexity). (See Conway)

10.16 (Inclusion relation on spectra). (a) $\sigma(a+b)\subset\sigma(a)+\sigma(b)$ and $\sigma(ab)\subset\sigma(a)\sigma(b)$ for unital cases.

(b) $\sigma(a^{-1})=\sigma(a)^{-1}$ for unital cases.

(c) $r(a)^n=r(a^n)$.

10.17 (Spectral radius function). (a) upper semi-continuous

10.18 (Vector-valued complex function theory). Let Ω be an open subset of $\mathbb{C}$ and X a Banach space. For a vector-valued function $f:\Omega\to X$, we say f is *differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}(f(\lambda)-f(\lambda_0))$$

exists in X for every $\lambda\in\Omega$, and *weakly differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}\langle f(\lambda)-f(\lambda_0),x^*\rangle$$

exists in $\mathbb{C}$ for each $x^*\in X^*$ and every $\lambda\in\Omega$. Then, the followings are all equivalent.

(a) f is differentiable.

(b) f is weakly differentiable.

(c) For each $\lambda_0\in\Omega$, there is a sequence $(x_k)_{k=0}^\infty$ such that we have the power series expansion

$$f(\lambda)=\sum_{k=0}^\infty(\lambda-\lambda_0)^k x_k,$$

where the series on the right hand side converges absolutely and uniformly on any closed ball in Ω centered at λ_0.

10.19 (Exponential of an operator).

Chapter 11

C*-algebras

11.1 C* identity

11.1 ($*$-algebras)**.** normed?

11.2 (C*-identity)**.** A *C*-algebra* is a Banach $*$-algebra A satisfying the C*-identity $\|a^*a\| = \|a\|^2$ for all $a \in A$.

11.3 (Unitization)**.**

$$(L_a + \lambda\,\mathrm{id}_{B(A)})^* = L_{a^*} + \overline{\lambda}\,\mathrm{id}_{B(A)}\,.$$

Proof. The C*-identity easily follows from the following inequality:

$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\
&= \sup_{\|b\|=1} \|(ab+\lambda b)^*(ab+\lambda b)\| \\
&= \sup_{\|b\|=1} \|b^*((a^*a + \lambda a^* + \overline{\lambda}a)b + |\lambda|^2 y)\| \\
&\le \sup_{\|b\|=1} \|(a^*a + \lambda a^* + \overline{\lambda}a)b + |\lambda|^2 b\| \\
&= \|(a,\lambda)^*(a,\lambda)\|.
\end{aligned}$$

□

11.2 Continuous functional calculus

11.4 (Gelfand-Naimark representation for C*-algebras)**.** For a commutative C*-algebra A, consider the Gelfand transform $\Gamma : A \to C_0(\sigma(A))$.

(a) Γ is a $*$-homomorphism.

(b) Γ is an isometry.

(c) Γ is a $*$-isomorphism.

Proof. (a)

(b) Note that we have

$$\|\Gamma a\| = \sup_{\varphi\in\sigma(A)} |\Gamma a(\varphi)| = \sup_{\varphi\in\sigma(A)} |\varphi(a)| = r(a)$$

for all $a \in A$. If we assume a is self-adjoint, then since $\|a\|^2 = \|a^*a\| = \|a^2\|$, the spectral radius coincides with the norm by the Beurling formula for spectral radius in Banach algebras:

$$\|\Gamma a\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Hence we have for all $a \in A$ that

$$\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|(\Gamma a)^*(\Gamma a)\| = \|\Gamma a\|^2.$$

(c) By the part (a) and (b), the image $\Gamma(A)$ is a closed unital $*$-subalgebra of $C(\sigma(A))$, and it separates points by definition. Then, $\Gamma(A)$ is dense in $C(\sigma(A))$ by the Stone-Weierstrass theorem, which implies $\Gamma(A) = C(\sigma(A)$. □

11.5 (Generators of a C*-algebra)**.** joint spectrum.

11.6 (Continuous functional calculus)**.** Let A be a unital C*-algebra, and $a \in A$ a normal element. Then, we have a $*$-isomorphism

$$C(\sigma(a)) \to \widetilde{C}^*(1,a) : \mathrm{id}_{\sigma(a)} \mapsto a$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

(a) spectral mapping: $\lambda \in \sigma_p(a)$ implies $f(\lambda) \in \sigma_p(f(a))$, $\lambda \in \sigma(a)$ iff $f(\lambda) \in \sigma(f(a))$, composition, ...

11.7 (Normal elements)**.** Let a be an element of a unital C*-algebra A. We say a is *normal*, *unitary*, and *self-adjoint* if $a^*a = aa^*$, $a^*a = aa^* = e$, and $a^* = a$ respectively. For normality and self-adjointness, the definitions can be extended to non-unital C*-algebras.

(a) If a is normal, then a is unitary if and only if $\sigma(a) \subset \mathbb{T}$.

(b) If a is normal, then a is self-adjoint if and only if $\sigma(a) \subset \mathbb{R}$.

Proof. (a)

(b) We may assume A is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \in A,$$

and the inverse of e^{ia} is e^{-ia}. Since the involution on A is continuous, we can check e^{ia} is unitary by

$$(e^{ia})^* = \sum_{n=1}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia}.$$

For every $\varphi \in \sigma(A)$, then by the part (a) the equality

$$e^{-\operatorname{Im}\varphi(a)} = |e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$$

proves $\varphi(a) \in \mathbb{R}$, hence $\sigma(a) \subset \mathbb{R}$. □

11.8 ($*$-homomorphism)**.** Let $\varphi : A \to B$ be a $*$-homomorphism between C*-algerbas.

(a) φ is determined by self-adjoint elements.

(b) $\|\varphi\| = 1$ if φ is non-trivial.

(c) The iamge of φ is closed.

(d) The induced map $A/\ker\varphi \to B$ is an isometry.

11.3 Positive elements

11.9 (Positive elements)**.** Let a, b be elements of a C*-algebra A. We say a is *positive* and write $a \geq 0$ if it is normal and $\sigma(a) \subset \mathbb{R}_{\geq 0}$. If we define a relation $a \leq b$ as $b - a \geq 0$, then we can see that it is a partial order on A.

(a) $a \geq 0$ if and only if $\|\lambda - a\| \leq \lambda$ for some $\lambda \geq \|a\|$.

(b) If $a \geq 0$ and $\sigma(b) \subset \mathbb{R}_{\geq 0}$, then $\sigma(a + b) \subset \mathbb{R}_{\geq 0}$.

(c) $a \geq 0$ if and only if $a = b^* b$ for some $b \in A$.

Proof. Let $a := b^* b$. Let $a = a_+ - a_-$. Then we have $(ba_-)^*(ba_-) = a_- a a_- = -a_-^3 \leq 0$, which also implies $(ba_-)(ba_-)^* \leq 0$ and

$$0 \leq (ba_-)^*(ba_-) + (ba_-)(ba_-)^* \leq 0.$$

Thus we have $ba_- = 0$ and $a_-^3 = 0$.

□

11.10 (Operator monotone operations)**.** (a) If $0 \leq a \leq b$, then $a^{-1} \geq b^{-1}$.

(b) If $a \leq b$, then $cac^* \leq cbc^*$.

11.11 (Positive linear functionals)**.** (a) Jordan decomposition: A self-adjoint linear functional is the difference of two positive linear functional.

Proof.

$$\{\omega \in A^* : \omega(a^*) = \overline{\omega(a)},\ \|\omega\| \leq 1\} = \operatorname{conv}(S(A) \cup -S(A)).$$

Krein-Milman

□

11.12 (Approximate identity)**.** separable Let e_α be an approximate identity of A.

(a) For a positive linear functional ω, we have $\lim_\alpha \omega(e_\alpha) = \|\omega\|$.

(b)

(c) separable.

11.4 Representations of C*-algebras

11.13 (Representation of C*-algebras)**.** Let A be a C*-algebra. A *representation* of A is a $*$-homomorphism $\pi : A \to B(H)$ for a Hilbert space H. We say a representation $\pi : A \to B(H)$ is *non-degenerate* if $\pi(A)H$ is dense in H, *cyclic* if there is $\psi \in H$ such that $A\psi$ is dense in H, and *irreducible* if there is no proper closed subspace $K \subset H$ such that $\pi(A)K \subset K$.

(a) The following statements are equivalent:

  (i) π is non-degenerate.
  (ii) For each $\xi \in H$ there is $a \in A$ such that $\pi(a)\xi \neq 0$.
  (iii) $\pi(e_\alpha) \to \mathrm{id}_H$ strongly for an approximate identity e_α of A.

(b) The following statements are equivalent:

  (i) π is irreducible
  (ii) $\pi(A)' = \mathbb{C}\,\mathrm{id}_H$.

(iii) $\pi(A)$ is strongly dense in $B(H)$.

(iv) Every non-zero vector is cyclic.

11.14 (Gelfand-Naimark-Segal representation)**.** Let A be a C^*-algebra, and ω be a state on A. The *left kernel* of ω is defined to be

$$N_\omega := \{a \in A : \omega(a^*a) = 0\}.$$

(a) N_ω is a left ideal of A.

(b) $\langle a + N, b + N\rangle := \omega(b^*a)$ is an inner product on A/N_ω.

(c) There is a unique representation $\pi_\omega : A \to B(H_\omega)$ such that $\pi_\omega(a)(b+N_\omega) := ab+N_\omega$ for $a, b \in A$.

(d) $\pi_\omega : A \to B(H_\omega)$ is a cyclic representation.

11.15 (Left ideals)**.**

11.16 (Primitive ideals)**.** hull kernel topology

$$PS(A) \twoheadrightarrow \widehat{A} \twoheadrightarrow \mathrm{Prim}(A).$$

$$PS(A) \cong \{(\pi, \psi)\}/\sim_u, \qquad \widehat{A} \cong \{\pi\}/\sim_u .$$

A	$PS(A)$	$\widehat{A}$	$\mathrm{Prim}(A)$
$C(X)$	X	X	X
$K(H)$	PH	$*$	$*$
$\widetilde{K}(H)$	?	?	$\{0, K(H)\}$
$B(H)$			

(a) $\mathrm{Prim}(A)$ is locally compact T_0 space.

(b) Two maps $PS(A) \to \widehat{A} \to \mathrm{Prim}(A)$ are continuous surjective open maps

(c) If A is postliminal, then $\widehat{A} \to \mathrm{Prim}(A)$ is an homeomorphism.

Exercises

11.17 (Operator monotone square)**.** Let A be a C^*-algebra in which the square function is operator monotone, that is, $0 \le a \le b$ implies $a^2 \le b^2$ for any positive elements a and b in A. We are going to show that A is necessarily commutative. Let a and b denote arbitrary positive elements of A.

(a) Show that $ab + ba \ge 0$.

(b) Let $ab = c + id$ where c and d are self adjoints. Show that $d^2 \le c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \le c^2$. Show that $c^2d^2 + d^2c^2 - 2\lambda d^4 \ge 0$.

(d) Show that $\lambda(cd + dc)^2 \le (c^2 - d^2)^2$.

(e) Show that $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \le c^2$ and deduce $d = 0$.

(f) Extend the result for general exponent: A is commitative if $f(x) = x^\beta$ is operator monotone for $\beta > 1$.

11.18 (States on unitization)**.** Let A be a non-unital C^*-algebra and $\widetilde{A}$ be its unitization. Let $\widetilde{\omega} = \omega \oplus \lambda$ be a bounded linear functional on $\widetilde{A}$, where $\omega \in A^*$ and $\lambda \in \mathbb{C}^* = \mathbb{C}$.

Since A is hereditary in $\widetilde{A}$, the extension defines a well-defined injective map $S(A) \to S(\widetilde{A})$. We can identify $PS(A)$ as a subset of $PS(\widetilde{A})$ whose complement is a singleton.

(a) $\widetilde{\rho}$ is positive if and only if $\lambda \geq 0$ and $0 \leq \rho \leq \lambda$.

(b) $\widetilde{\omega}$ is a state if and only if $\lambda = 1$ and $0 \leq \omega \leq 1$.

(c) $\widetilde{\omega}$ is a pure state if and only if $\lambda = 1$ and ω is either a pure state or zero.

11.19 (Representations of $C_0(X)$). Let $A = C_0(X)$ and μ be a state on A, a regular Borel probability measure on a locally compact Hausdorff space X.

(a) The left kernel of μ is $N_\mu = \{f \in A : f|_{\operatorname{supp}\mu} = 0\}$.

(b) $H_\mu = L^2(X, \mu)$.

(c) The canonical cyclic vector is the unity function on X.

11.20 (Representations of $K(H)$).

11.21 (Automorphism group of $K(H)$ and $B(H)$).

11.22 (Approximate eigenvectors).

11.23 (Kadison transitivity theorem).

11.24 (Hereditary C*-algebras).

11.25 (Extreme points of the ball). Let A be a C*-algebra and let B_A be the closed unit ball of A.

(a) Extreme points of $A_+ \cap B_A$ is the projections in A.

(b) Extreme points of $A_{sa} \cap B_A$ is the self-adjoint unitaries in A.

(c) Every extreme point of B_A is a partial isometry.

Problems

1. A C-algebra is commutative if and only if a function $f(x) = x(1+x)^{-1}$ is operator subadditive.

Chapter 12

Von Neumann algebras

12.1 Density theorems

12.1 (Von Neumann algebras)**.** Let H be a Hilbert space. A $*$-subalgebra M of $B(H)$ is called a *von Neumann algebra* if it is closed weakly.

12.2 (Double commutant theorem)**.** Let A be a non-degenerate $*$-subalgebra of $B(H)$.

(a) $\overline{A}^{\sigma s^*} \subset \overline{A}^{w} \subset A''$.

(b) If $x \in A''$, for any $\varepsilon > 0$ and $\xi \in H$ there is $a \in A$ such that $\|(x-a)\xi\| < \varepsilon$.
(If we can find such $a \in A$ for any *finite subset* $F \subset H$ not only for a single ξ, then we can construct a net a_α that converges to x strongly, i.e. $\overline{A}^{s} = A''$. We will show, more strongly, that we can do this for any *square-summable countable subset* F in the part (c))

(c) For $\overline{A}^{\sigma s^*} = A''$.

Proof. (b) We claim $x\xi \in \overline{A\xi}$ for each $\xi \in H$. Let p be the projection onto $\overline{A\xi}$. Then, the image of ap is contained in $\overline{A\xi}$, we have $pap = ap$ and $pa^*p = a^*p$ for all $a \in A$ by the self-adjointness of A. It implies $ap = pa$, which deduces $p \in A'$ so $xp = px$. Observe that $a(1-p)\xi = (1-p)a\xi = 0$ for all $a \in A$. Then, $\langle (1-p)\xi, \eta\rangle = 0$ for any $\eta \in H = \overline{AH}$ by the non-degeneracy, so $p\xi = \xi$. Hence $x\xi = xp\xi = px\xi$ so that $x\xi \in \overline{A\xi}$.

(c) We suffices to show $A'' \subset \overline{A}^{\sigma s}$ because A is self-adjoint. Take a finite set $\{(\xi_{ij})_{j=1}^{\infty} \subset H\}_{i=1}^{n}$ of sequences such that $\sum_{j=1}^{\infty} \|\xi_{ij}\|^2 < \infty$ for each i. Then, $x \mapsto \left(\sum_{j=1}^{\infty} \|x\xi_{ij}\|^2\right)^{\frac{1}{2}}$ defines a finite set of seminorms indexed by i which makes a base element of the σ-strong topology. Consider the diagonal map $\Delta : B(H) \to B(H^{\oplus\infty})$ and let $\overline{\xi} := (\xi_{ij})_{i,j} \in (H^{\oplus\infty})^{\oplus n} = H^{\oplus\infty}$. Then, the seminorm for σ-strong topology on H factor through the seminorm defined by ξ on $H^{\oplus\infty}$ as follows:

$$\begin{tikzcd} B(H) \arrow[d, "\Delta"'] \arrow[dr, "x\mapsto\left(\sum_{i,j}\|x\xi_{ij}\|^2\right)^{\frac{1}{2}}"] & \\ B(H^{\oplus\infty}) \arrow[r, "\overline{x}\mapsto\|\overline{x}\overline{\xi}\|"'] & \mathbb{R}_{\ge 0}. \end{tikzcd}$$

Suppose $x \in A''$. Since $\Delta(x) \in \Delta(A)''$ and $\Delta(A)$ is a non-degnerate $*$-subalgebra of $B(H^{\oplus\infty})$, by the part (b), there is $\Delta(a) \in \Delta(A)$ such that

$$\|(\Delta(x) - \Delta(a))\overline{\xi}\| = \left(\sum_{i,j} \|(x-a)\xi_{ij}\|^2\right)^{\frac{1}{2}} < \varepsilon.$$

Thus

$$\Delta(A'') \subset \Delta(A)'' \subset \Delta(\overline{A}^{\sigma s}).$$

□

12.3 (Kaplansky density theorem).

12.4 (Vigier theorem). Increasing bounded net is convergent in strong operator topology. The boundedness is important because we have to construct a bounded sesquilinear form using the monotone convergence in $\mathbb{R}$.

Proof. Existence of range projections. Let $x \in M$. Since $\operatorname{im} x = \operatorname{im}(xx^*)^{\frac{1}{2}}$, we may assume $0 \le x \le 1$. Then, $x^{2^{-n}}$ is an increasing sequence in M bounded by one, so it converges strongly to some $p \in M_+$. We can check $p^2 = p$ by... We can check p is the range projection of x by... □

every von neumann algebra is unital: also can be proved by Krein-Milman
polar decomposition
normal linear functionals form a predual of M.

12.2 Borel functional calculus

12.5 (Sherman-Takeda theorem). Let A be a C*-algebra. Define $M(\pi) := \pi(A)''$ for $\pi : A \to B(H)$ a representation. Let $\pi_u : A \to B(H_u)$ be the universal representation of A, the direct sum of all the GNS-representations of states of A. Consider the following three maps

$$\pi_u : A \to (M(\pi_u), \sigma w), \qquad \pi_u^* : M(\pi_u)_* \to A^*, \qquad \pi_u^{**} : A^{**} \to M(\pi_u),$$

constructed by adjoints, where $M(\pi_u)_*$ denotes the set of normal linear functionals on $M(\pi_u)$.

(a) π_u^* is isometric.

(b) π_u^* is surjective.

(c) π_u^{**} is an isometric isomorhpism with respect to norms, and is an homeomorphism with respect to weak*-topologies.

(d) A^{**} enjoys a universal property in the sense that for every $*$-homomorphism $\varphi : A \to M$ to a von Neumann algebra M, there exists a unique normal extension $\widetilde{\varphi} : A^{**} \to M$ of φ.

Proof. (a) It holds for any representation of $\pi : A \to B(H)$. For each $l \in M(\pi)_*$ we have

$$\|\pi^*(l)\| = \sup_{\substack{\|a\|\le 1\\ a\in A}} |l(\pi(a))| = \sup_{\substack{\|x\|\le 1\\ x\in M(\pi)}} |l(x)| = \|l\|$$

by the Kaplansky density theorem and the σ-weak continuity of l.

(b) The injective $*$-homomorphism π_u is isometric so that its dual $M(\pi_u)^* \to A^*$ is surjective by the Hahn-Banach extension, however, it does not guarantee that the extended linear functional is normal. We claim that every state of A has a normal extension on $M(\pi_u)$. If the claim is true, then the Jordan decomposition can be applied to show that every bounded linear functional has a normal extension.

Let ω be a state of A. If we let ψ be the canonical cyclic vector of the GNS representation $\pi_\omega : A \to B(H_\omega)$, then the state ω can be represented as a vector state ω_ψ in $B(H)$. Since π_ω is a subrepresentation of π_u, the unit vector ψ can be seen as an element of H_u, and it defines a normal state of $M(\pi_u)$.

(c) It is is clear from (a) and (b).

(d) We can define $\widetilde{\varphi}$ as the bitranspose of $\varphi : A \to (M, \sigma w)$, and it is a unique extension because A is σ-weakly dense in A^{**}. □

Remark 12.2.1. The bidual A^{**} is frequently viewed as a von Neumann algebra, and we call it the *enveloping von Neumann algebra* of a C*-algebra A. By the universal property, we have a normal $*$-homomorphism $M(\pi_u) \to M(\pi)$ that is in fact surjective for every representation π of A, and it fails to be injective even if π is faithful.

12.6 (Bounded Borel functions)**.** Let X be a compact Hausdorff space and denote by $B^\infty(X)$ the space of bounded Borel functions on X. The linear combinations of projections in $B^\infty(X)$ are called *simple functions*. (Stonean and hyperstonean spaces?)

(a) There are natural inclusions $C(X) \subset B^\infty(X) \subset C(X)^{**}$ among C*-algebras.

(b) $B^\infty(X)$ is the norm closure of simple functions.

(c) $B^\infty(X)$ factors through all $L^\infty(X,\mu) := M(\pi_\mu)$ for GNS-representations π_μ of $C(X)$.

12.7 (Borel functional calculus)**.** Let $x \in B(H)$ be a normal operator. Consider

$$B^\infty(\sigma(x)) \subset C(\sigma(x))^{**} \to W^*(x) \subset B(H).$$

(a) If we endow the topology of pointwise convergence on $B^\infty(\sigma(a))$ and the strong operator topology on M, then the Borel functional calculus is continuous.

(b) Every von Neumann algebra is the norm closed span of projections.

Proof. (a) By the bounded convergence theorem.

(b) This is because $\sigma(a) \subset \mathbb{C}$ is compact so that it is separable and metrizable; every bounded measurable function is a pointwise limit of simple functions.

□

For normal $a \in B(H)$, the continuous functional calculus for a is just a non-degenerate representation

$$C(\sigma(a)) \to B(H)$$

which maps $\mathrm{id}_{\sigma(a)}$ to a. Also, a projection valued-measure on a compact Hausdorff space X is just a non-degenerate representation

$$C(X) \to B(H).$$

To show this, note that a projection-valued measure defines a "normal" unital $*$-homomorphism

$$\operatorname{span} P(B^\infty(X)) \to B(H).$$

Then, mimick the definition of Lebesgue integral to construct a unital $*$-homomorphism $C(X) \to B(H)$.

12.3 Predual

12.8 (Conditional expectations)**.** Let A be a closed subalgebra of a C*-algebra B. Let $\varphi : B \to A$ be a contractive idempotent surjective linear map. Such a map is called a *conditional expectation*.

(a) φ is an A-bimodule map.

(b) φ is completely positive.

Proof. Since each conclusion of (a) and (b) still holds for restriction, we may assume A and B are von Neumann algebras by thinking of the bitranspose $\varphi^{**} : B^{**} \to A^{**}$.

(a) Since the linear span of projections is σ-weakly dense in a von Neumann algebra, we are enough to show $p\varphi(b) = \varphi(pb)$ and $\varphi(bp) = \varphi(b)p$ for any projection $p \in A$.

Let $p \in A$ be a projection and let $b \in B$. Note that the surjectivity of φ implies that $p\varphi$ is also idempotent. Then, where $1 = 1_B$,

$$\begin{aligned}(1+t)^2\|p\varphi((1-p)b)\|^2 &= \|p\varphi((1-p)b) + tp\varphi(p\varphi((1-p)b))\|^2 \\ &\leq \|(1-p)b + tp\varphi((1-p)b)\|^2 \\ &= \|(1-p)b\|^2 + t^2\|p\varphi((1-p)b)\|^2\end{aligned}$$

implies $p\varphi((1-p)b) = 0$ by letting $t \to \infty$. Putting $1_A - p$ and 1_A instead of p, we obtain

$$(1-p)\varphi((1-1_A+p)b) = 0, \qquad \varphi((1-1_A)b) = 0$$

respectively, which imply $(1-p)\varphi(pb) = 0$. Hence for any $b \in B$ we have

$$p\varphi(b) = p\varphi(pb) = \varphi(pb).$$

Similarly we can show $\varphi(b(1-p))p = 0$ and $\varphi(bp)(1-p) = 0$ for $b \in B$, we are done.

(b) Let $[b_{ij}] \in M_n(B)_+$. Let $\pi : A \to B(H)$ be a cyclic representation with a cyclic vector ψ. Then, $[\xi_i] \in H^n$ can be replaced to $[\pi(a_i)\psi]$, so we can check the positivity of inflations φ_n as

$$\sum_{i,j}\langle \pi(\varphi(b_{ij}))\pi(a_j)\psi, \pi(a_i)\psi\rangle = \langle \pi(\varphi(\sum_{i,j} a_i^* b_{ij} a_j))\psi, \psi\rangle \geq 0,$$

because it follows $\sum_{i,j} a_i^* b_{ij} a_j \geq 0$ by the positivity of b_{ij} from

$$\langle \pi_B(\sum_{i,j} a_i^* b_{ij} a_j)\xi, \xi\rangle = \sum_{i,j}\langle \pi_B(b_{ij})\pi_B(a_j)\xi, \pi_B(a_i)\xi\rangle \geq 0,$$

where π_B is any representation of B. □

12.9 (Sakai theorem)**.** Suppose A is a C*-algebra which admits a predual F.

(a) There is an injective $*$-homomorphism $\pi : A \to A^{**}$ with weakly* closed image.

(b) π is a topological embedding with respect to $\sigma(A, F)$ and $\sigma(A^{**}, A^*)$.

(c) The predual F is unique in A^*.

In particular, there is a faithful representation $A \to B(H)$ whose image is (σ-)weakly closed.

Proof. (a) By taking the adjoint for the inclusion $i : F \hookrightarrow A^*$, we have a conditional expectation $\varepsilon : A^{**} \twoheadrightarrow A$. Its kernel is a A-bimodule, and by the σ-weak density of A in A^{**} and the continuity of ε between weak* topologies, so it is in fact a A^{**}-bimodule, which means it is a σ-weakly closed ideal of A^{**}. Thus we have a central projection $z \in A^{**}$ such that $\ker \varepsilon = (1-z)A^{**}$.

Define $\pi : A \to A^{**}$ such that $\pi(a) := za$. It is clearly a $*$-homomorphism. The injectivity follows from $a = \varepsilon(a) = \varepsilon(za)$ for $a \in A$. The image is weakly* closed because $\varepsilon(x - \varepsilon(x)) = 0$ implies $z(x - \varepsilon(x)) = 0$ for $x \in A^{**}$ so that $zA^{**} = zA$.

(b) Since $\langle a, f\rangle = \langle \varepsilon(za), f\rangle = \langle za, f\rangle$ for $a \in A$ and $f \in F$, in which the second equality holds by the definition of ε, it is enough to show $\sigma(zA, A^*) = \sigma(zA, F)$.

For $l \in A^*$, we claim there exists f such that $\langle za, l\rangle = \langle za, f\rangle$. Define $\widetilde{l} \in A^*$ such that $\langle x, \widetilde{l}\rangle := \langle zx, l\rangle$ for $x \in A^{**}$. Then, $\langle zx, l\rangle = \langle z^2x, l\rangle = \langle zx, \widetilde{l}\rangle$ for $x \in A^{**}$. Suppose $\widetilde{l} \notin F$. Because F is closed in A^*, there is $x \in A^{**}$ such that $\langle x, \widetilde{l}\rangle \neq 0$ and $\langle x, f\rangle = 0$ for all $f \in F$ by the Hahn-Banach separation. Then, $0 = \langle x, f\rangle = \langle x, i(f)\rangle = \langle \varepsilon(x), f\rangle$ implies $\varepsilon(x) = 0$ so that $zx = 0$, which leads a contradiction $\langle x, \widetilde{l}\rangle = \langle zx, l\rangle = 0$, so we have $\widetilde{l} \in F$.

(c) If closed subspaces F_1 and F_2 of A^* are preduals of A, then $\sigma(A, F_1) = \sigma(A, F_2)$ by the part (b). If $l \in F_1$, which is obviously continuous on $\sigma(A, F_1)$, and the continuity in $\sigma(A, F_2)$ implies that l is contained in a linear span of some finitely many elements of F_2, hence $F_1 \subset F_2$. □

Exercises

12.10 (Extremally disconnected space). $\sigma(B^\infty(\Omega))$ is extremally disconnected.

resolution of identity normal operator theories: multiplicity, invariant subspaces L^∞ representation

σ-weakly closed left ideal has the form Mp. II.3.12

Let $\mathfrak{m}$ be an algebraic ideal of a von Neumann algebra M, and $\overline{\mathfrak{m}}$ be its σ-weak closure. If $x \in (\overline{\mathfrak{m}})_+$, then there is an increasing net $(x_i) \subset \mathfrak{m}$ converges to x strongly. II.3.13

binary expansion and hereditary subalgebras