

Linear Partial Differential Equations

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Part I

Distributions

Chapter 1

Distribution space

1.1 Extension of linear operators

Let $T : \mathcal{D} \rightarrow \mathcal{D}'$ be a continuous linear operator. We can always define the adjoint $T^* : \mathcal{D} \subset \mathcal{D}'' \rightarrow \mathcal{D}'$. The most reasonable extension of T is $T : (T^*(\mathcal{D}))' \rightarrow \mathcal{D}'$. For $f \in (T^*(\mathcal{D}))'$, we can define $\langle T(f), \varphi \rangle := \langle f, T^*\varphi \rangle$ for $\varphi \in \mathcal{D}$.

Suppose $T : (\mathcal{D}, \mathcal{T}) \rightarrow (T(\mathcal{D}), \mathcal{S})$ is proved to be continuous. If $(\mathcal{D}, \mathcal{T}) \rightarrow (T^*(\mathcal{D}))'$ and $(T(\mathcal{D}), \mathcal{S}) \rightarrow \mathcal{D}'$ are embeddings, then the extension of T to the completion of $(\mathcal{D}, \mathcal{T})$ agrees with $T : (T^*(\mathcal{D}))' \rightarrow \mathcal{D}'$.

1.2 Convolutions

For example, if Φ is locally integrable, then since $(T_\Phi)^* = T_{\tilde{\Phi}}$ and $\Phi * \varphi \in \mathcal{E} = C^\infty$ for $\varphi \in \mathcal{D}$, the convolution operator $T_\Phi : \mathcal{E}' \rightarrow \mathcal{D}'$ can be defined on the space of compactly supported distributions.

Problem: If $g * f$ is well-defined, is $f * g$ also well-defined? In other words, if $f \in (T_{\tilde{g}}(\mathcal{D}))'$ so that $g * f \in \mathcal{D}'$, then $g \in (T_{\tilde{f}}(\mathcal{D}))'$? Are they same?

$$\langle g, \tilde{f} * \varphi \rangle =$$

- 1.1. (a) If a test function φ satisfies $\langle 1, \varphi \rangle = 0$, then there is $v \in \mathbb{R}^d$ and a test function ψ such that $\varphi = v \cdot \nabla \psi$.
- (b) If a distribution has zero derivative, then it is a constant.

Chapter 2

Sobolev spaces

2.1 Sobolev inequalities

Part II

Elliptic equations

Chapter 3

The Laplace and Poisson equations

3.1 Existence results of Poisson's equation

3.1 (Fundamental solution of the Laplace equation). Consider a boundary problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \mathbb{R}_x^d, \\ u(x) = 0 & \text{on } |x| = \infty. \end{cases}$$

A function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

defined on \mathbb{R}_x^d for $d \geq 2$ is called *fundamental solution of Laplace's equation*.

- (a) Φ and $\nabla \Phi$ are locally integrable on \mathbb{R}_x^d but $\Delta \Phi$ is not.
- (b) $\Delta \Phi$ is a tempered distribution on \mathbb{R}_x^d .
- (c) $-\Delta \Phi(x) = \delta(x)$ in \mathbb{R}_x^d .
- (d) u solves the boundary problem if and only if it satisfies a representation formula $u = \Phi * f$, if $\Phi * f$ is a well-defined distribution on \mathbb{R}_x^d .

Proof. (c) Let $\varphi \in \mathcal{D}(\mathbb{R}_x^d)$. Then, $\nabla \Phi(x) \cdot \nabla \varphi(x) \in L^1(\mathbb{R}_x^d)$ gives

$$\begin{aligned} - \int \Phi(x) \Delta \varphi(x) dx &= - \lim_{\varepsilon \rightarrow \infty} \int_{|x| \geq \varepsilon} \nabla \Phi(x) \cdot \nabla \varphi(x) dx \\ &= - \lim_{\varepsilon \rightarrow \infty} \int_{|x|=\varepsilon} \nabla \Phi(x) \varphi(x) \cdot \nu dS + \lim_{\varepsilon \rightarrow \infty} \int_{|x| \geq \varepsilon} \Delta \Phi(x) \varphi(x) dx. \end{aligned}$$

Since

$$\nabla \Phi(x) = -\frac{1}{\omega_d} \frac{x}{|x|^d}, \quad \nu = \frac{x}{|x|},$$

and $\Delta \Phi(x) = 0$ for $x \neq 0$, we get

$$-\int \Phi(x) \Delta \varphi(x) dx = \lim_{\varepsilon \rightarrow \infty} \frac{1}{\omega_d \varepsilon^{d-1}} \int_{|x|=\varepsilon} \varphi(x) dS = \varphi(x).$$

(d) Note that $\Phi = \tilde{\Phi}$. If u is a solution of the boundary problem, then

$$\langle \Phi * f, \varphi \rangle = \langle f, \Phi * \varphi \rangle = \langle u, -\Delta(\Phi * \varphi) \rangle = \langle u, \Phi * (-\Delta \varphi) \rangle = \langle u, \varphi \rangle.$$

Conversely, if we let $u = \Phi * f$, then

$$\langle u, -\Delta \varphi \rangle = \langle \Phi * f, -\Delta \varphi \rangle = \langle f, \tilde{\Phi} * (-\Delta \varphi) \rangle = \langle f, \Phi * (-\Delta \varphi) \rangle = \langle f, \varphi \rangle$$

and □

3.2 (Green's function). Let U be a bounded open subset of \mathbb{R}_x^d with C^1 boundary. Consider a boundary value problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } U, \\ u(x) = g(x) & \text{on } \partial U. \end{cases}$$

A *corrector* is a function $\phi(x, y)$ on $U \times U$ defined as the solution of the boundary value problem

$$\begin{cases} -\Delta_y \phi(x, y) = 0 & \text{in } y \in U, \\ \phi(x, y) = \Phi(x - y) & \text{on } y \in \partial U, \end{cases}$$

for each $x \in U$. We assume a well-known fact that the solution ϕ uniquely exists and $\phi \in H^1(U)$, proved later. Then, *Green's function* for U is a function on $U \times U$ defined by

$$G(x, y) := \Phi(x - y) - \phi(x, y).$$

(a) If $g(x) = 0$ on ∂U , then for $x \in U$,

$$u(x) = - \int_U G(x, y) \Delta u(y) dy.$$

(b) If $f(x) = 0$ in U , then for $x \in U$,

$$u(x) = \int_{\partial U} u(y) \nabla_y G(x, y) \cdot \nu dS(y).$$

(c) u solves the boundary problem if and only if it satisfies a representation formula

$$u(x) = \int_U G(x, y) f(y) dy + \int_{\partial U} g(y) \nabla_y G(x, y) \nu \cdot dS(y),$$

if the right-hand side is well defined distribution on \mathbb{R}_x^d .

Proof.

□

3.2 Uniqueness results of Poisson's equation

3.3 Regularity results of Poisson's equation

Part III

Evolution equations