Category Theory

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Part I

Categories

set theoretical issues morphisms monic

1.1 Functors

fully faithful, essentially surjective natural transformations and equivalence 2-category

1.2 Categorical constructions

opposite category product category disjoint union category comma category(slice category, morphism category)

Universal property

2.1 Construction

products, equalizers, pullbacks

2.2 Representable functors

Yoneda embedding gives fully faithful functors $h: \mathcal{C} \to PSh(\mathcal{C})$ and $k: \mathcal{C}^{op} \to coPSh(\mathcal{C})$. A presheaf $\mathcal{C}^{op} \to Set$ is representable if and only if it is essentially contained in the image of the Yoneda embedding.

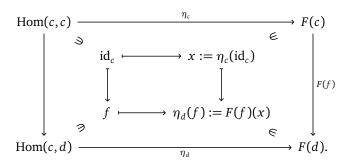
2.1 (Yoneda lemma). Let $F : \mathcal{C} \to \operatorname{Set}$ be a functor from a locally small category \mathcal{C} . Fix $c \in \operatorname{Ob}(\mathcal{C})$. we can define a function

$$Nat(Hom(c, -), F) \rightarrow F(c)$$
.

A representation of F is a pair (c, η) of an object $c \in C$ and a natural isomorphism $\eta : \operatorname{Hom}(c, -) \to F$. A universal element of F is a pair (c, x) with $x \in F(c)$ such that for any pair (d, y) with $y \in F(d)$ there is a unique morphism $f \in \operatorname{Hom}(c, d)$ satisfying $F(f) : x \mapsto y$.

(a)

Proof. (a) Consider the diagram



For a natural transformation $\eta: \operatorname{Hom}(c,-) \to F$, define $x:=\eta_c(\operatorname{id}_c)$ in F(c). For $x \in F(c)$, conversely, define a $\eta_d: \operatorname{Hom}(c,d) \to F(d)$ by $\eta_d(f):=F(f)(x)$ for $d \in \operatorname{Ob}(\mathcal{C})$ and $f \in \operatorname{Hom}(c,d)$. Then, the collection $\eta=\{\eta_d: d \in \operatorname{Ob}(\mathcal{C})\}$ provides a natural transformation because for each $g \in \operatorname{Hom}(d,e)$ we can check the diagram

$$\begin{array}{ccc} \operatorname{Hom}(c,d) & \stackrel{\eta_d}{\longrightarrow} & F(d) \\ & g \circ - \Big\downarrow & & \Big\downarrow^{F(g)} \\ \operatorname{Hom}(c,e) & \stackrel{\eta_e}{\longrightarrow} & F(e) \end{array}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \qquad f \in \text{Hom}(c,d).$$

The correspondences $\eta \mapsto x$ and $x \mapsto \eta$ are inverses of each other, hence the bijection.

Limits

preservation, reflection, creation completeness functoriality limit-preserving filtered limit-preserving product-preserving mono-preserving

Part II

- 4.1 Adjunctions
- 4.2 Monads
- 4.3 Kan extensions

Abelian categories

5.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f).

A category is called regular if every kernel pair admits a coequalizer.

5.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category Grp is regular but not coregular, since there is a monomorphism which is not regular.

5.2 Additive and abelian categories

- **5.2** (Additive categories). A *pre-additive category* is an Ab-enriched category. A *semi-additive category* is one of the followings:
 - (i) a pointed CMon-enriched category.
 - (ii) a category with finite biproducts.

An additive category is one of the followings:

- (i) a pointed Ab-enriched category.
- (ii) Ab-enriched category with finite biproducts.
- (a) additive completion by formally adjoining finite biproducts.
- (b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a Set_* -enriched category. In additive category, we have a natural Set_* -enrichment.

- **5.3** (Pre-abelian categories). A *pre-abelian category* is one of the followings:
 - (i) an additive category in which every morphism has the kernel and cokernel.
 - (ii) a finitely bicomplete pre-additive category.

(a)

- **5.4** (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen). A pointed, Baar-exact, protomodular, with binary coproudcts.
 - (a) short five lemma, 3×3 lemma, snake lemma, noehter isomorphism hold.
 - (b) long exact homology sequence
 - (c) Every semi-abelian category is exact.
 - (d) Every semi-abelian category is finitely bicomplete.
 - (e) In general, a semi-abelian category is not pre-additive nor semi-additive.
- **5.5** (Abelian categories). An *abelian category* is a Ab-enriched category which is finitely bicomplete and satisfies the first isomorphism theorem.
 - (a) A category is abelian if and only if it is additive and exact.
- **5.6** (Freyd-Mitchell embedding).



- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C*-algebras, compact Hausdorff (profinite) spaces.
- · Additive: projective modules

The first isomorphism theorem states that coim \rightarrow im is an isomorphism. The normal subobjects and the first isomorphism theorem is generalized in the context of protomodular categories. The cokernel may not be defined. The category of unital rings is not semi-abelian but protomodular.

- A protomodular category
- A *homological category* is a pointed regular protomodular category. (five, nine, snake, long exact sequence, noether isomorphism)
- A *semi-abelian category* is a homological category that is Barr-exact and finite coproducts(free products).

Tensor categories

6.1 Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

- **6.1** (Monoidal categories). A *monoidal category* is a category \mathcal{C} equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that
 - (i) for each triple $A, B, C \in \mathcal{C}$ there is an isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ called the *associator*, satisfying the pentagon identity

$$((A \otimes B) \otimes C) \otimes D) \xrightarrow{\alpha_{A \otimes B,C,D}} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D))$$

$$\downarrow^{\alpha_{A,B,C} \otimes \mathrm{id}_{D}} \qquad \qquad \qquad \mathrm{id}_{A} \otimes \alpha_{B,C,D} \uparrow$$

$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)$$

commutes for each $A, B, C, D \in \mathcal{C}$.

(ii) there is a specified object $I \in \mathcal{C}$ called the *unit object*, and for each $A \in \mathcal{C}$ there are isomorphisms $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$ called the *left unitor* and the *right unitor*, satisfying the triangle identity

$$(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B)$$

$$\rho_{A} \otimes \mathrm{id}_{B} \xrightarrow{Id_{A} \otimes \lambda_{B}} A \otimes B$$

commutes for each $A, B \in \mathcal{C}$.

We say a monoidal category is *strict* if the associators and unitors are all identity morphisms. A *cartesian* monoidal category is a monoidal category whose monoidal structure \otimes is given by the categorical product.

6.2 (Coherence theorem). Let C be a monoidal category.

(a)
$$(I \otimes A) \otimes B \xrightarrow{\alpha_{I,A,B}} I \otimes (A \otimes B)$$

$$\lambda_{A} \otimes \mathrm{id}_{B} \xrightarrow{\lambda_{A} \otimes B}$$

$$A \otimes B$$

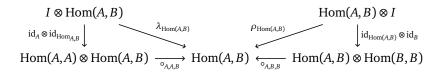
- (b) $\lambda_I = \rho_I$
- (c) The endomorphism monoid End(I) is commutative.

- (d) *I* is unique up to unique isomoprhism.
- **6.3** (Monoidal functors). coherence maps lax, strong, strict
- **6.4** (Enriched categories). Let \mathcal{M} be a monoidal category. A category \mathcal{C} is said to be *enriched* over \mathcal{M} if for each $A, B \in \mathcal{C}$ there is $\text{Hom}(A, B) \in \mathcal{M}$ such that
 - (i) for each $A, B, C \in \mathcal{C}$ there is a morphism $\circ_{AB,C} : \text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, satisfying

$$\begin{split} \operatorname{Hom}(A,B) \otimes \operatorname{Hom}(B,C) \otimes \operatorname{Hom}(C,D) & \xrightarrow{\operatorname{id}_{\operatorname{Hom}(A,B)} \otimes \circ_{B,C,D}} \operatorname{Hom}(A,B) \otimes \operatorname{Hom}(B,D) \\ \circ_{A,B,C} \otimes \operatorname{id}_{\operatorname{Hom}(C,D)} \Big\downarrow & & \downarrow \circ_{A,B,D} \\ \operatorname{Hom}(A,C) \otimes \operatorname{Hom}(C,D) & \xrightarrow{\circ_{A,C,D}} & \operatorname{Hom}(A,D) \end{split}$$

commutes for each $A, B, C, D \in \mathcal{C}$.

(ii) for each $A \in \mathcal{C}$ there is a morphism $id_A : I \to Hom(A, A)$, satisfying



- 6.5 (Pointed category). A pointed category is a category with a zero object.
 - (a) A category is Set_{*}-enriched if and only if it admits a zero morphism.
 - (b) Every pointed category is Set*-enriched.

rigid?

- 6.2 Braided and ribbon categories
- 6.3 Internalization
- 6.4 Tensor and fusion categories