

Harmonic Analysis

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Part I

Fourier analysis

Chapter 1

Fourier series

1.1 Fourier series in L^p spaces

1.1.

$$\|\widehat{f}\|_{\ell^1(\mathbb{Z})} \lesssim \|f\|_{W^{1,1+\varepsilon}(\mathbb{T})}.$$

Inversion theorem is an approximation problem given by $\mathcal{F}^*\mathcal{F} = \lim_{n \rightarrow \infty} \mathcal{F}_n^*\mathcal{F}$. The condition $\widehat{f} \in \ell^1(\mathbb{Z})$ is a condition just for defining $\mathcal{F}^*\widehat{f}$ without using distribution theory, and it does not affect the inversion phenomena. The approximation, in other words, can be seen as an extension method for $\mathcal{F}^* : \ell^1(\mathbb{Z}) \rightarrow C(\mathbb{T})$ on $c_0(\mathbb{Z})$. Note that \mathcal{F}_n^* on $c_0(\mathbb{Z})$ cannot be bounded directly without distribution theory, but $\mathcal{F}_n^*\mathcal{F}$ on $L^p(\mathbb{T})$ can be bounded well.

1.2 Summability methods

- If \mathcal{F}_n^* is the standard partial sum, then $\mathcal{F}_n^*\mathcal{F}$ is the Dirichlet kernel.
- If \mathcal{F}_n^* is the Cesàro mean, then $\mathcal{F}_n^*\mathcal{F}$ is the Fejér kernel.
- If \mathcal{F}_r^* is the Abel sum, then $\mathcal{F}_r^*\mathcal{F}$ is the Poisson kernel.
- In Fourier transform, we often use the Gauss-Weierstrass kernel.

The injectivity of \mathcal{F} is not an easy problem, which comes from the inversion theorem.

1.2 (Dirichlet kernel). The *Dirichlet kernel* is a function $D_n : \mathbb{T} \rightarrow \mathbb{R}$ defined by

$$D_n = \widehat{\mathbf{1}_{|k| \leq n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = \mathbf{1}_{|k| \leq n}.$$

This is because they are invariant under inverse, in other words, they are even.

(a)

$$D_n(x) = \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.$$

(b) If $f \in \text{Lip}(\mathbb{T})$, then $D_n * f \rightarrow f$ pointwisely as $n \rightarrow \infty$.

(c)

$$\|D_n\|_{L^1(\mathbb{T})} \gtrsim \log n.$$

Proof.

$$\begin{aligned}
D_n(x) &= \sum_{k=-n}^n e^{ikx} \\
&= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.
\end{aligned}$$

(c) By (2) $\sin x \leq x$ for $x \in [0, \pi/2]$, (3) change of variable,

$$\begin{aligned}
\|D_n\|_{L^1(\mathbb{T})} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x} \right| dx \\
&\geq \frac{2}{\pi} \int_0^{\pi} \frac{|\sin \frac{2n+1}{2}x|}{x} dx \\
&= \frac{2}{\pi} \int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x} dx \\
&= \frac{2}{\pi} \sum_{k=0}^{2n} \int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x} dx \\
&\geq \frac{2}{\pi} \sum_{k=0}^{2n} \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi} dx \\
&\geq \frac{4}{\pi^2} \sum_{k=0}^{2n} \frac{1}{1+k} \\
&\geq \frac{4}{\pi^2} \log(2n+2).
\end{aligned}$$

..?

□

1.3 (Fejér kernel). The *Fejér kernel* is

(a)

$$K_n(x) = \frac{1}{n+1} \frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}.$$

Proof. Since

$$\begin{aligned}
D_n(x) &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2},
\end{aligned}$$

by telescoping, we get

$$\begin{aligned}
\sum_{k=0}^n D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}.
\end{aligned}$$

□

Two important results from Fejér kernel:

1. If $f(x-)$, $f(x+)$ exist and $S_n f(x)$ converges, then $S_n f(x) \rightarrow \frac{1}{2}(f(x-) + f(x+))$.
2. (If $f \in L^1(\mathbf{T})$, then $\sigma_n f \rightarrow f$ a.e.)
3. If $f \in L^1(\mathbf{T})$, then $S_n f \rightarrow f$ in L^1 and L^2 .
4. If f is continuous and $\hat{f} \in L^1(\mathbb{Z})$, then $S_n f \rightarrow f$ uniformly.
5. Since $\sigma_n f$ is a trigonometric polynomial, the set of trigonometric polynomials are dense in $L^1(\mathbf{T})$ and $L^2(\mathbf{T})$.

1.3 Pointwise convergence of Fourier series

BV function: Dini, Jordan's criterion

1.4 (Riemann localization principle).

Exercises

1.5 (Gibbs phenomenon).

1.6 (Du Bois-Reymond function).

1.7 (Compactly supported functions). Let f be an integrable compactly supported function. Using the Morera to prove \hat{f} is analytic.

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

There are three conventions for the Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \begin{cases} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx, \\ (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx, \\ \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x) dx. \end{cases}$$

We will accept the second one.

2.1 (Riemann-Lebesgue lemma). basic estimates

$$\begin{aligned} \|\hat{f}\|_{L^\infty(\mathbb{R}^d)} &\leq (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)}, \quad f \in L^1(\mathbb{R}^d). \\ \|\hat{f}\|_{L^1(\mathbb{R}^d)} &\lesssim \|f\|_{W^{d+1,1}(\mathbb{R}^d)}, \quad f \in W^{d+1,1}(\mathbb{R}^d). \end{aligned}$$

L^p extension

2.2 (Fourier inversion). inversion theorem Plancherel

$$\|\hat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}, \quad f \in L^2(\mathbb{R}^d).$$

unitarity

2.3 (Properties). If $D := -i\partial$, then

$$\mathcal{F}D_x \mathcal{F}^* = M_\xi, \quad D_\xi \mathcal{F} = -\mathcal{F}M_x.$$

$$\mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}} \hat{f} * \hat{g}.$$

2.2 Tempered distributions

2.4 (Cauchy principal value). indented contour, imaginary shift, Feynman's trick

Exercises

2.5 (Sampling theorem).

$$\mathcal{F}\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) = \text{sinc}(\xi/2)$$

$$\text{sinc} \in L^{1+\varepsilon}(\mathbb{R}).$$

2.6 (Poisson summation formula).

2.7 (Uncertainty principle).

2.8. Gaussian function computation: differential equation method, contour integral method, imaginary shift

$$\mathcal{F}(e^{-\frac{1}{2}xQx}) = \frac{e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}}{|\det Q|^{\frac{1}{2}}} e^{-\frac{1}{2}xQ^{-1}x}.$$

2.9 (Multipole expansion). Let ρ be a compactly supported distribution on \mathbb{R}^d . We want to investigate the limit behavior of $\rho(\varepsilon^{-1}x)$ as $\varepsilon \rightarrow 0$. More precisely, we want to compute an integer $k \geq d$ such that $\lim_{\varepsilon \rightarrow 0+} \varepsilon^{-k} \rho(\varepsilon^{-1}x)$ defines a distribution supported at $\{0\}$, and the coefficients of derivatives of Dirac measures.

We need to introduce quantities called monopole, dipole, quadrapole, octupole, etc.

- (a) A distribution supported on $\{0\}$ is a linear combination of the Dirac measure and its derivatives.
- (b)

Problems

1. Find all $\alpha > 0$ such that

$$\lim_{x \rightarrow \infty} x^{-\alpha} \int_0^x f(y) dy = 0$$

for all $f \in L^3([0, \infty))$.

Chapter 3

Hilbert transform

3.1 Harmonic conjugate

3.2 Kernel representation

3.3 Fourier series in L^p space

Part II

Singular integral operators

Chapter 4

Calderón-Zygmund theory

4.1 Convolution type operators

4.1 (Calderón-Zygmund decomposition of sets). Let $f \in L^1(\mathbb{R}^d)$. Let $E_n f$ be the conditional expectation with respect to the σ -algebra generated by dyadic cubes with side length 2^{-n} . Let $Mf := \sup_n E_n |f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|} \int_{Q_x} |f| > \lambda.$$

- (a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω .
- (b) We have an weak type estimate $|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1}$.
- (c) $\|f\|_{L^\infty(\mathbb{R}^d \setminus \Omega)} \leq \lambda$.
- (d) For $x \in \Omega$

$$\frac{1}{|Q_x|} \int_{Q_x} |f| \leq 2^d \lambda.$$

4.2 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|} \int_{Q_x} |f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

- (a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.
- (b) $\|b\|_{L^1} \leq 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof.

□

4.3 (L^p boundedness of Calderón-Zygmund operators). Let $T : C_c^\infty(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that $Tf(x) = K * f(x)$ for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \text{supp } f$. We say T is called a *Calderón-Zygmund operator* if

- (i) T is L^2 -bounded: we have

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2},$$

(ii) T satisfies the *Hörmander condition*: we have

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \lesssim 1$$

for every $y > 0$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2 -boundedness implies

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2} \|Tg\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda} \int_{\mathbb{R}^d \setminus \Omega^*} |Tb(x)| dx \leq \frac{2}{\lambda} \sum_i \int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\text{supp } b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y) b_i(y) dy = \int_{Q_i} [K(x-y) - K(x)] b_i(y) dy,$$

and

$$\int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d \setminus Q_i^*} |K(x-y) - K(x)| dx dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.4 (Hölder boundedness of Calderón-Zygmund operators).

4.2 Truncated integrals

Homogeneous kernels

4.3 A_p weights

4.4 Bounded mean oscillation

Exercises

4.5 (Size and cancellation condition). Let $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$. We say the condition $|K(x)| \lesssim |x|^{-d}$ for $x \neq 0$ as the *size condition*, and say the condition $\int_{r < |x| < R} K(x) dx = 0$ for all $0 < r < R < \infty$ as the *cancellation condition*. If K satisfies the size, cancellation, and Hörmander condition, then it is L^2 bounded, hence Calderón-Zygmund.

4.6 (Gradient size condition). Let $|\nabla K(x)| \lesssim |x|^{-d-1}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

4.7 (Riesz potential).

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Almost orthogonality

Carleson measures, paraproducts

6.1 Coltar lemma

6.2 $T(1)$ theorem

Part III

Oscillatory integral operators

Chapter 7

Oscillatory integrals

7.1 (Justification of oscillatory integral). For a function ϕ with fast growth toward infinity, we want to define a linear functional I_ϕ such that

$$I_\phi(a) := \int_{\mathbb{R}^d} e^{i\phi(x)} a(x) dx, \quad a \in S(\mathbb{R}^d).$$

A linear functional of the above form is called the *oscillatory integral* with *phase function* ϕ . As a notation, we will use the above integral in the right-hand side to denote the value of I_ϕ even for $a \notin L^1(\mathbb{R}^d)$. Then, we have pointwise justifications for integral calculus.

- (a) $I_\phi : A_\phi^m(\mathbb{R}^d) \rightarrow \mathbb{C}$ is well-defined and continuous, if ϕ .
- (b) The change of variables is justified as follows:
- (c) The integral by parts is justified as follows:

$$\int_{\mathbb{R}^d} e^{i\phi(y)} i \partial \phi(y) a(x+y) dy = - \int_{\mathbb{R}^d} e^{i\phi(y)} \partial a(x+y) dy, \quad x \in \mathbb{R}^d, a \in A_\phi^m(\mathbb{R}^d).$$

- (d) The Fubini theorem is justified as follows:
- (e) The Fourier inversion is justified as follows:

$$a(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(y) dy d\xi, \quad x \in \mathbb{R}^d, a \in A_\phi^m(\mathbb{R}^d).$$

Proof. (a) Note that $A_\phi^m(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is dense in $A_\phi^m(\mathbb{R}^d)$. The most difficult part is the construction and the computation of L and its transpose.

(e) Note that the function $(y, \xi) \mapsto a(y)$ belongs to $A_\phi^{m'}(\mathbb{R}^{2d})$ since \square

7.2 (Point evaluation of multiplier). Let $\phi \in \mathcal{S}'$ be a phase function. We want to show the following point evaluation holds with previously justified oscillatory integral:

$$\Phi(D)a(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\phi(y)} a(x+y) dy, \quad x \in \mathbb{R}^d, a \in A_\phi^m(\mathbb{R}^d),$$

where $\Phi := \mathcal{F}^* e^{i\phi}$. Which condition for ϕ makes Φ be able to act on \mathcal{S}' by multiplication?

7.3 (Stationary phase approximation).

Proof. \square

7.4 (Van der Corput lemma).

Dispersive equations and strichartz estimates

Exercises

7.5 (Fresnel phase). We compute L with a specific example

Proof.

$$(1 + xQ^{-1}D)e^{\frac{i}{2}xQx} = \langle x \rangle^2 e^{\frac{i}{2}xQx}.$$

The transpose of $\langle x \rangle^{-2}(1 + xQ^{-1}D)$ is $\langle x \rangle^{-2}(1 + di - 2ix^2 - xD)$ for $Q = I$.

Note that $\langle x \rangle^{-2n} \langle D \rangle^{2n}$ is self-adjoint.

Let Q be a non-degenerate symmetric bilinear form on \mathbb{R}^d . Consider a multiplier operator $e^{\frac{i}{2}DQD} : S \rightarrow S$ such that

$$e^{\frac{i}{2}DQD}a(x) := \mathcal{F}^* e^{\frac{i}{2}\xi Q \xi} \mathcal{F}a(x).$$

(a) The pointwise evaluation is given by the oscillatory integral.

$$e^{\frac{i}{2}DQD}a(x) = (2\pi)^{-d} \frac{e^{\frac{i\pi}{4}} \operatorname{sgn}(Q)}{|\det Q|^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{-\frac{i}{2}yQ^{-1}y} a(x+y) dy, \quad x \in \mathbb{R}^d, \quad a \in A_\delta^m.$$

(b)

$$e^{\frac{i}{2}DQD}a(x) = \sum_{k=0}^n \frac{i^k}{2^k k!} (DQD)^k a(x) + r_n(x)$$

□

Chapter 8

Foureir restriction

Takeya Bochner-Riesz Geometric measure theory

Chapter 9

Part IV

Pseudo-differential operators

Chapter 10

Pseudo-differential calculus

10.1

10.1 (Hörmander symbol classes). Let $m, \rho, \delta \in \mathbb{R}$. The Hörmander class $S_{\rho, \delta}^m(\mathbb{R}^{2d})$ of symbols is the set of smooth functions $a \in C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim_{\alpha, \beta} \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|}$$

for each $\alpha, \beta \in \mathbb{Z}_{\geq 0}^d$.

(a) Fréchet space

10.2 (Asymptotic expansion). Let $\rho, \delta \in \mathbb{R}$. Let $a_k \in S_{\rho, \delta}^{m_k}(\mathbb{R}^{2d})$ for a sequence $(m_k)_{k=0}^\infty \subset \mathbb{R}$ with m_0 and $m_k \downarrow -\infty$. We want to construct $a \in S_{\rho, \delta}^{m_0}(\mathbb{R}^{2d})$ such that

$$a - \sum_{k=0}^{n-1} a_k \in S_{\rho, \delta}^{m_n}(\mathbb{R}^{2d}). \quad (\dagger)$$

The symbol a_0 is called the *principal symbol* of a , or the operator $\text{Op}^t(a)$.

Let $\chi \in C_c^\infty(\mathbb{R}_\xi^d, [0, 1])$ be a cutoff function such that

$$\chi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1 \\ 0, & \text{if } |\xi| \geq 2 \end{cases}.$$

(a) If $a \in S_{\rho, \delta}^m$, then $\chi(\varepsilon\xi)a(x, \xi)$ is uniformly bounded in $S_{\rho, \delta}^m$ for $\varepsilon \in (0, 1)$ if $\rho \leq 1$.

(b) There is $a \in S_{\rho, \delta}^{m_0}$ such that (\dagger) if $\rho \leq 1$.

Proof. (a) On the support of $\xi \mapsto \chi(\varepsilon\xi)$ holds $\langle \xi \rangle < 2|\xi| \leq 4\varepsilon^{-1}$ because $1 < \varepsilon^{-1}$, so for each $\alpha, \beta \in \mathbb{Z}_{\geq 0}^d$ we have

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (\chi(\varepsilon\xi)a(x, \xi))| &= \left| \sum_{\tau} \binom{\beta}{\tau} \partial_\xi^{\beta-\tau} (\chi(\varepsilon\xi)) \partial_x^\alpha \partial_\xi^\tau a(x, \xi) \right| \\ &= \left| \sum_{\tau} \binom{\beta}{\tau} \varepsilon^{|\beta| - |\tau|} \partial_\xi^{\beta-\tau} \chi(\varepsilon\xi) \partial_x^\alpha \partial_\xi^\tau a(x, \xi) \right| \\ (\because \langle \xi \rangle \leq 4\varepsilon^{-1}) &\leq \sum_{\tau} \binom{\beta}{\tau} (4\langle \xi \rangle^{-1})^{|\beta| - |\tau|} |\partial_\xi^{\beta-\tau} \chi(\varepsilon\xi)| |\partial_x^\alpha \partial_\xi^\tau a(x, \xi)| \\ &\lesssim \sum_{\tau} \binom{\beta}{\tau} \langle \xi \rangle^{-(|\beta| - |\tau|)} \langle \xi \rangle^{m + \delta|\alpha| - \rho|\tau|} \\ (\because \rho \leq 1) &\leq \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|}. \end{aligned}$$

(b) Because we have $\varepsilon^{-1} \leq \langle \xi \rangle$ on the support of $1 - \chi(\varepsilon \xi)$, for each k we can take a sequence ε_k small enough such that

$$\max_{\substack{\alpha, \beta \in \mathbb{Z}_{\geq 0}^d \\ |\alpha| + |\beta| \leq k}} |\partial_x^\alpha \partial_\xi^\beta ((1 - \chi(\varepsilon_k \xi)) a_k(x, \xi))| \leq 2^{-k} \langle \xi \rangle^{m_k + 1 + \delta|\alpha| - \rho|\beta|}.$$

We may assume $\varepsilon_k \downarrow 0$ so that the following sum is locally finite:

$$a(x, \xi) := \sum_{k=0}^{\infty} (1 - \chi(\varepsilon_k \xi)) a_k(x, \xi).$$

If we choose n such that $m_0 \geq m_n + 1$, then in the expansion

$$a(x, \xi) = \sum_{k=0}^{n-1} (1 - \chi(\varepsilon_k \xi)) a_k(x, \xi) + \sum_{k=n}^{\infty} (1 - \chi(\varepsilon_k \xi)) a_k(x, \xi)$$

the first sum clearly belongs to $S_{\rho, \delta}^{m_0}$ and so is the second sum because

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \sum_{k=n}^{\infty} (1 - \chi(\varepsilon_k \xi)) a_k(x, \xi)| &\leq \sum_{k=n}^{\infty} 2^{-k} \langle \xi \rangle^{m_{k+1} + 1 + \delta|\alpha| - \rho|\beta|} \\ &\leq \langle \xi \rangle^{m_n + 1 + \delta|\alpha| - \rho|\beta|} \\ &\leq \langle \xi \rangle^{m_0 + \delta|\alpha| - \rho|\beta|} \end{aligned}$$

for every $\alpha, \beta \in \mathbb{Z}_{\geq 0}^d$. Therefore, $a \in S_{\rho, \delta}^{m_0}$.

Write

$$(a - \sum_{k=0}^{n-1} a_k)(x, \xi) = \sum_{k=0}^{n-1} \chi(\varepsilon_k \xi) a_k(x, \xi) + \sum_{k=n}^{\infty} (1 - \chi(\varepsilon_k \xi)) a_k(x, \xi).$$

The first sum belongs to $S^{-\infty}$ because it is compactly supported, and we can also show that the second sum belongs to $S_{\rho, \delta}^{m_n}$ by decomposing with n' such that $m_n \geq m_{n'} + 1$ and by considering the multiplication with a cutoff remains in the same symbol class. \square

10.3 (Quantization). For a symbol a defined on \mathbb{R}^{2d} and $t \in [0, 1]$, we want to define a pseudo-differential operator $\text{Op}^t(a)$ such that

$$\text{Op}^t(a)f(x) := (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a((1-t)x + ty, \xi) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The operator $\text{Op}^t(a)$ is the t -quantization of the symbol a . The analysis of t -quantizations is sometimes called the *Kohn-Nirenberg calculus* for $t = 0$, the *Weyl calculus* for $t = \frac{1}{2}$.

(a) $\text{Op}^0(a) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is well-defined and continuous, if $a \in \mathcal{S}'(\mathbb{R}^{2d})$.

(b) $\text{Op}^0(a) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ is well-defined and continuous, if $a \in S_{\rho, \delta}^m(\mathbb{R}^{2d})$ for $\delta \leq 1$.

Proof. (b) Since $\langle D_y \rangle^2$ is a self-adjoint partial differential operator, for any $n \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \text{Op}^0(a)f(x) &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(x, \xi) f(y) dy d\xi \\ (\cdot \cdot D_y e^{i(x-y)\xi} &= \xi e^{i(x-y)\xi}) \quad = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \langle \xi \rangle^{-2n} \langle D_y \rangle^{2n} e^{i(x-y)\xi} a(x, \xi) f(y) dy d\xi \\ (\cdot \cdot \text{IBP}) \quad &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \langle \xi \rangle^{-2n} a(x, \xi) \langle D_y \rangle^{2n} f(y) dy d\xi. \end{aligned}$$

The derivatives of the integrand is integrable with respect to ξ for a sufficiently large n with $m + |\beta| - 2n < -d$ because

$$\begin{aligned}
& |\partial_x^\beta (e^{i(x-y)\xi} \langle \xi \rangle^{-2n} a(x, \xi) \langle D_y \rangle^{2n} f(y))| \\
&= \left| \sum_{\tau} \binom{\beta}{\tau} (i\xi)^{\beta-\tau} e^{i(x-y)\xi} \langle \xi \rangle^{-2n} \partial_x^\tau a(x, \xi) \langle D_y \rangle^{2n} f(y) \right| \\
&\leq \sum_{\tau} \binom{\beta}{\tau} \langle \xi \rangle^{|\beta| - |\tau|} \langle \xi \rangle^{-2n} |\partial_x^\tau a(x, \xi)| |\langle D_y \rangle^{2n} f(y)| \\
&(\because a \in S_{\rho, \delta}^m) \lesssim \sum_{\tau} \binom{\beta}{\tau} \langle \xi \rangle^{|\beta| - |\tau|} \langle \xi \rangle^{-2n} \langle \xi \rangle^{m + \delta |\tau|} |\langle D_y \rangle^{2n} f(y)| \\
&(\because \delta \leq 1) \lesssim \langle \xi \rangle^{m + |\beta| - 2n} |\langle D_y \rangle^{2n} f(y)|,
\end{aligned}$$

so the partial derivative ∂_x commutes with the integral. Since

$$x^\alpha e^{i(x-y)\xi} = (y + D_\xi)^\alpha e^{i(x-y)\xi} = \sum_{\sigma} \binom{\alpha}{\sigma} y^{\alpha-\sigma} D_\xi^\sigma e^{i(x-y)\xi},$$

we have an expansion

$$\begin{aligned}
x^\alpha \partial_x^\beta \text{Op}^0(a) f(x) &= x^\alpha \partial_x^\beta \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \langle \xi \rangle^{-2n} a(x, \xi) \langle D_y \rangle^{2n} f(y) dy d\xi \\
&= \int_{\mathbb{R}^{2d}} x^\alpha \partial_x^\beta (e^{i(x-y)\xi} \langle \xi \rangle^{-2n} a(x, \xi) \langle D_y \rangle^{2n} f(y)) dy d\xi \\
&= \int_{\mathbb{R}^{2d}} \sum_{\sigma, \tau} \binom{\alpha}{\sigma} \binom{\beta}{\tau} y^{\alpha-\sigma} D_\xi^\sigma e^{i(x-y)\xi} (i\xi)^{\beta-\tau} \langle \xi \rangle^{-2n} \partial_x^\tau a(x, \xi) \langle D_y \rangle^{2n} f(y) dy d\xi \\
&= \int_{\mathbb{R}^{2d}} \sum_{\sigma, \tau} \binom{\alpha}{\sigma} \binom{\beta}{\tau} e^{i(x-y)\xi} (-D_\xi)^\sigma [(i\xi)^{\beta-\tau} \langle \xi \rangle^{-2n} \partial_x^\tau a(x, \xi)] y^{\alpha-\sigma} \langle D_y \rangle^{2n} f(y) dy d\xi.
\end{aligned}$$

Here

$$\sup_{x \in \mathbb{R}^d} |(-D_\xi)^\sigma [(i\xi)^{\beta-\tau} \langle \xi \rangle^{-2n} \partial_x^\tau a(x, \xi)]|$$

is integrable with respect to ξ for sufficiently large n , so with this n we have

$$\sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \text{Op}^0(a) f(x)| \lesssim \sum_{\sigma \leq \alpha} \sup_{y \in \mathbb{R}^d} |y^{\alpha-\sigma} \langle D_y \rangle^{2n} f(y)|$$

for each $\alpha, \beta \in \mathbb{Z}_{\geq 0}^d$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, which implies $\text{Op}^0(a) f \in \mathcal{S}(\mathbb{R}^d)$. \square

10.4 (Change of quantization). Let $m \in \mathbb{R}$, .

- (a) $\text{Op}^t(a) = \text{Op}^0(e^{itD_x D_\xi} a)$. In particular, since $M_{e^{itx\xi}} : \mathcal{S}(\mathbb{R}^{2d}) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$, $\text{Op}^t(a) : \mathcal{S}(\mathbb{R}^{2d}) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$ is well-defined and continuous.
- (b) $a \in S_{\rho, \delta}^m(\mathbb{R}^{2d})$ if and only if $e^{itD_x D_\xi} a \in S_{\rho, \delta}^m(\mathbb{R}^{2d})$, if $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$.
- (c) We have the formal adjoint

$$\text{Op}^t(a)^* = \text{Op}^{1-t}(\bar{a}).$$

In particular, $\text{Op}^t(a) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ is well-defined and continuous for $a \in S_{\rho, \delta}^m(\mathbb{R}^{2d})$.

Proof. (a) Note that

$$\begin{aligned}
\text{Op}^t(a)f(x) &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a((1-t)x + ty, \xi) f(y) dy d\xi \\
(\cdot: \text{Inversion on } \mathbb{R}^{2d}) &= (2\pi)^{-3d} \int_{\mathbb{R}^{4d}} e^{i(x-y)\xi} e^{i((1-t)x+ty)x^* + i\xi\xi^*} \hat{a}(x^*, \xi^*) f(y) dx^* d\xi^* dy d\xi \\
&= (2\pi)^{-3d} \int_{\mathbb{R}^{4d}} e^{i(x-y+\xi^*)\xi} \hat{a}(x^*, \xi^*) e^{i((1-t)x+ty)x^*} f(y) dx^* d\xi^* dy d\xi \\
(\cdot: \text{Inversion on } \mathbb{R}^d) &= -(2\pi)^{-2d} \int_{\mathbb{R}^{2d}} \hat{a}(x^*, y-x) e^{i((1-t)x+ty)x^*} f(y) dx^* dy \\
(\cdot: [\xi^*/y-x]) &= -(2\pi)^{-2d} \int_{\mathbb{R}^{2d}} e^{i(x+t\xi^*)x^*} \hat{a}(x^*, \xi^*) f(x+\xi^*) dx^* d\xi^*.
\end{aligned}$$

(b) We have the oscillatory integral

$$e^{itD_x D_\xi} a(x, \xi) = (2\pi)^{-d} |t|^{-d} \int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta} a(x+y, \xi+\eta) dy d\eta.$$

Enough to show

$$| \int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta} a(x+y, \xi+\eta) dy d\eta | \lesssim \langle \xi \rangle^m.$$

Fix ξ and $\delta \leq \rho$

□

10.5 (Moyal product). Let $a \in S_{\rho, \delta}^m(\mathbb{R}^{2d})$ and $b \in S_{\rho, \delta}^l(\mathbb{R}^{2d})$.

(a) there exists a unique function $a \#^t b \in S_{\rho, \delta}^{m+l}(\mathbb{R}^{2d})$ such that

$$a^t(x, D)b^t(x, D) = (a \#^t b)^t(x, D).$$

(b) It is concretely described by

$$(a \#^t b)(x, \xi) = (2\pi)^{-2} \int_{\mathbb{R}^{4d}} e^{-i(y\eta - z\xi)} a(x + tz, \xi + \eta) b((1-t)y + x, \xi + \zeta) dy d\eta dz d\zeta.$$

(c) If $\delta < \rho$, then

$$a \#^t b(x, \xi) \sim \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{1}{i^k k!} (\partial_y \partial_\eta - \partial_z \partial_\zeta)^k a((1-t)x + tz, \eta) b(tx + (1-t)y, \zeta) \Big|_{\substack{y=z=x \\ \eta=\zeta=\xi}}.$$

10.6 (Parametrix and elliptic operators).

10.7 (Calderón-Vaillancourt theorem).

Exercises

Quantization of linear and quadratic exponential symbols.

Chapter 11

Semiclassical analysis

We define for $h > 0$ and $t \in [0, 1]$

$$\text{Op}_h^t(a)f(x) := (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{h}(x-y)\xi} a((1-t)x + ty, \xi) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

$$\text{Op}_h^w(D_x a) = [D_x, \text{Op}_h^w(a)], \quad \text{Op}_h^w(hD_\xi a) = -[x, \text{Op}_h^w(a)].$$

For example, regardless of $h > 0$ and $t \in [0, 1]$,

$$\text{Op}(\xi)\psi(x) = hD\psi(x) = -ih\psi'(x)$$

and

$$\text{Op}(H)\psi(x) = -\frac{h^2}{2m}\Delta\psi(x) + V(x)\psi(x),$$

where

$$H(x, \xi) := \frac{|\xi|^2}{2m} + V(x).$$

In physics, the operator $\text{Op}(H)$ is frequently written as \hat{H} , which will not be used to avoid the confusion regarding the Fourier transform.

$$\begin{aligned} \frac{d}{dt}a(t) &= \{a(t), H\} = X_H a(t) \\ \frac{d}{dt}\hat{a}(t) &= \frac{d}{dt}e^{\frac{i}{h}t\hat{H}}\hat{a}e^{-\frac{i}{h}t\hat{H}} = -\frac{i}{h}[\hat{a}(t), \hat{H}] \end{aligned}$$

Let $J : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d} : (x, \xi) \mapsto (\xi, -x)$ be a symplectomorphism, the rotation of $\frac{\pi}{2}$ in *clock-wise*. Then, we have

$$\mathcal{F}_h^* \text{Op}_h^w(a) \mathcal{F}_h = \text{Op}_h^w(J^*a).$$

Also,

$$[\text{Op}_h^w(a), \text{Op}_h^w(b)] = \text{Op}_h^w(-ih\{a, b\}) + O(h^2).$$

Since the Weyl quantization has a bound

$$\|\text{Op}_h^w(a)\|_{B(L^2(\mathbb{R}^d))} \lesssim \|a\|_{C_b(\mathbb{R}^{2d})} + O(h^{\frac{1}{2}}), \quad a \in C_b(\mathbb{R}^{2d}) \cap S_{\rho, \delta}^m(\mathbb{R}^{2d}),$$

for a bounded net $f_h \in L^2(\mathbb{R}^d)$, the positive linear functional $C_0(\mathbb{R}^{2d})$ defined by

$$a \mapsto \langle \text{Op}_h^w(a) f_h, f_h \rangle, \quad a \in C_0(\mathbb{R}^{2d}) \cap S_{\rho, \delta}^m(\mathbb{R}^{2d})$$

has a limit point in the weak* topology. If a finite Radon measure μ on \mathbb{R}^{2d} is a limit, then μ is called a *semiclassical defect* of the net f_h .

Let p be a symbol such that $|p(x, \xi)| \gtrsim \langle \xi \rangle^k$ for sufficiently large $|\xi|$. This symbol has an interpretation of the Hamiltonian. Suppose the following two conditions are satisfied:

$$\lim_{h \rightarrow 0} \|\text{Op}_h^w(p)f_h\|_{L^2(\mathbb{R}^d)} = 0, \quad \|f_h\|_{L^2(\mathbb{R}^d)} = 1.$$

Then, the support of any semiclassical defect measure μ is contained in $p^{-1}(0)$, called the *characteristic variety* or the *zero energy surface* of the symbol p . We can understand this support restriction as saying that in the semiclassical limit $h \rightarrow 0$ all the mass of solution coalesces onto a specific set in phase space. Also we have the flow invariance $\{p, \mu\} = 0$, i.e. $\int_{\mathbb{R}^{2d}} \{p, a\} d\mu = 0$ for all $a \in \mathcal{D}(\mathbb{R}^{2d})$, which means that μ is invariant under the Hamiltonian flow generated by p .

11.1 Heisenberg group

11.2 Phase space transforms

Chapter 12

Microlocal analysis