Lebesgue Theory

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Contents

Part I

Measure theory

Chapter 1

Measures and σ-algebras

1.1 Carathéodory extension

1.1 (Outer measures). Let Ω be a set. An *outer measure* on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ such that

(i) $E_1 \subset E_2 \Rightarrow \mu^*(E_1) \le \mu^*(E_2)$ in $\mathcal{P}(\Omega)$, (monotonicity)

(ii) $\mu^*(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu^*(E_i)$ in $\mathcal{P}(\Omega)$. (countable subadditivity)

(a) A set function $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*, that is, $E \subset \bigcup_{i=1}^\infty E_i \Rightarrow \mu^*(E) \le \sum_{i=1}^\infty \mu^*(E_i)$ in $\mathcal{P}(\Omega)$.

(b) For a set function $\rho : \mathcal{A} \to [0,\infty]$ with $\rho(\varnothing) = 0$, where $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, we can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ by defining as

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^\infty \rho(A_i) : E \subset \bigcup_{i=1}^\infty A_i,\ A_i \in \mathcal{A} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

Proof. (a)

(b)

□

1.2 (Carathéodory measurable sets). Let μ^* be an outer measure on a set Ω. A subset $A \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

for every subset $E \subset \Omega$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is complete.

1.3 (Carathéodory extension theorem). Let $\rho : \mathcal{A} \to [0,\infty]$ with $\rho(\varnothing) = 0$, where $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$. Consider two conditions

(i) $A \subset \bigcup_{i=1}^\infty A_i \Rightarrow \rho(A) \le \sum_{i=1}^\infty \rho(A_i)$ in $\mathcal{A}$, (monotonically countably subadditive)

(ii) Every $A \in \mathcal{A}$ satisfies that for each $B \in \mathcal{A}$ and $\varepsilon > 0$ there are $\{B_j'\}_{j=1}^\infty$ and $\{B_j''\}_{j=1}^\infty \subset \mathcal{A}$ such that

$$\rho(B) + \varepsilon > \sum_{j=1}^\infty \rho(B_j') + \sum_{j=1}^\infty \rho(B_j'')$$

with $B \cap A \subset \bigcup_{j=1}^\infty B_j'$ and $B \setminus A \subset \bigcup_{j=1}^\infty B_j''$.

Let $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0, \infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) $\mu^*|_{\mathcal{A}} = \rho$ if (i) is satisfied.

(b) $\mathcal{A} \subset \mathcal{M}$ if (ii) is satisfied.

Proof. (a) Clearly $\mu^*(A) \le \rho(A)$ for $A \in \mathcal{A}$. We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{A_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty A_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(A_i) \ge \rho(A).$$

(b) Let $E \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. Since it is enough to check the inequality $\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \setminus A)$ for E with finite $\mu^*(E)$, we may assume there is a countable family $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Then, we have $B_i \cap A \subset \bigcup_{j=1}^\infty B_{i,j}'$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty B_{i,j}''$ satisfying

$$\mu^*(E) + \varepsilon > \sum_{i=1}^\infty (\rho(B_i) + \frac{\varepsilon}{2^{i+1}}) > \sum_{i,j=1}^\infty \rho(B_{i,j}') + \sum_{i,j=1}^\infty \rho(B_{i,j}'') \ge \mu^*(E \cap A) + \mu^*(E \setminus A).$$

□

1.4 (Carathéodory extension from semi-ring)**.** Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a semi-ring of sets on a set X. A function $\rho : \mathcal{A} \to [0, \infty]$ with $\rho(\varnothing) = 0$ is called a *pre-measure* if

(i) $\rho(\bigsqcup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \rho(A_i)$ in $\mathcal{A}$, (disjoint countable subadditivity)

(ii) $\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$ in $\mathcal{A}$. (finite additivity)

(a)

1.5 (Uniqueness of Carathéodory extensions)**.** The Carathéodory extension theorem provides with a uniqueness theorem for measures.

Monotone class lemma: alternative direct proof method without using Carathéodory extension.

Chapter 2

Measures on the real line

distribution functions helly's selection non-measurable set

Exercises

2.1. * A Lebesgue measurable set in $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression.

Chapter 3

Measurable functions

3.1 Extended real numbers

3.2 Simple functions

Pointwise limit of simple functions is measurable.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

$\square$

Every measurable extended real-valued function is a pointwise limit of simple functions.

3.1 (Egorov's theorem)**.** Let (Ω, μ) be a finite measure space. Let $(f_n : \Omega \to \mathbb{R})_n$ be a sequence of a.e. convergent measurable functions. For $\varepsilon > 0$, there exists a measurable $E_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ and f_n uniformly convergent on E_ε.

Proof. Assume $f_n \to 0$. The set of convergence is

$$\bigcap_{k>0} \bigcup_{n_0>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \},$$

which is a full set. We want to get rid of the dependence on the point x of n_0 in the union $\bigcup_{n_0>0}$. Since

$$\bigcap_{n\geq n_0} \{ x : |f_n(x)| < \frac{1}{k} \}$$

is increasing as $n_0 \to \infty$ to a full set for each $k > 0$, we can find $n_0(k, \varepsilon)$ such that

$$\mu(\bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \}) > \mu(\Omega) - \frac{\varepsilon}{2^k}.$$

Then,

$$\mu(\bigcap_{k>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \}) > \mu(\Omega) - \varepsilon.$$

If we define

$$E_\varepsilon := \bigcap_{k>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \},$$

then for any $k > 0$ and $x \in E_\varepsilon$, and with the $n_0(k, \varepsilon)$ we have chosen, we have

$$n \geq n_0 \quad \Rightarrow \quad |f_n(x)| < \frac{1}{k}.$$

$\square$

Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0, \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > \tfrac{1}{k},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n - f| > \tfrac{1}{k}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n - f| > \tfrac{1}{k}\}.\end{aligned}$$

Since for every k we have

$$\begin{aligned}\limsup_n \{x : |f_n - f| > \tfrac{1}{k}\} &\subset \limsup_{n>k} \{x : |f_n - f| > \tfrac{1}{n}\} \\ &= \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\},\end{aligned}$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\}.$$

Part II

Lebesgue integral

Chapter 4

Convergence theorems

4.1 Definition of Lebesgue integral

4.2 Convergence theorems

Stein: Egorov → BCT → Fatou → MCT → L1 is a measure
Stein: BCT + L1 is a measure → DCT
Folland: MCT → Fatou → DCT → BCT

4.3 Radon-Nikodym theorem

4.4 Modes of convergence

4.1 (Convergence in measure)**.** Let (X, μ) be a measure space. Let f_n be a sequence of measurable functions. If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.

Proof. We can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) > \tfrac{1}{2^k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k} - f| > \tfrac{1}{k}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. □

Chapter 5

Product measures

5.1 Fubini-Tonelli theorem

5.2 Lebesgue measure on Euclidean spaces

Chapter 6

Measures on metric spaces

6.1 Compact metric spaces

Part III

Linear operators

Chapter 7

Lebesgue spaces

7.1 L^p spaces

7.2 L^1 spaces

7.3 L^2 spaces

7.4 L^∞ spaces

Chapter 8

Bounded linear operators

8.1 Continuity

Schur test

8.2 Density arguments

extension of operators

8.3 Interpolation

weak Lp, marcinkiewicz

Chapter 9

Convergence of linear operators

9.1 Translation and multiplication operators

9.2 Convolution type operators

approximation of identity

9.3 Computation of integral transforms

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega \to \mathbb{R}$ such that $f(x, y)$ and $\partial_{x_i} f(x, y)$ are both locally integrable in x and integrable y. Then,

$$\partial_{x_i} \int f(x, y)\, dy = \int \partial_{x_i} f(x, y)\, dy$$

where ∂_{x_i} denotes the weak partial derivative.

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}

(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞

(b) f is AC iff f' is L^1

(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

Lebesgue differentiation theorem