POSITIVE HAHN-BANACH SEPARATIONS IN OPERATOR ALGEBRAS

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ABSTRACT.

1. INTRODUCTION

- definition and properties of $f_\varepsilon(t) := (1+\varepsilon t)^{-1}t$
- relation between $\{\omega' \in M_*^+ : \omega' \le \omega\}$ and $\{h \in \pi(M)'^+ : h \le 1\}$

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Definition 2.1 (Hereditary subsets)**.** Let E be a partially ordered real vector space. We say a subset F of the positive cone E^+ is *hereditary* if $0 \le x \le y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. A $*$-subalgebra B of a $*$-algbera A is called *hereditary* if and only if the positive cone B^+ is a hereditary subset of A^+. We define the *positive polar* of F as the positive part of the real polar

$$F^{\circ+} := \{x^* \in (E^*)^+ : \sup_{x \in F} x^*(x) \le 1\}.$$

An example that is a non-hereditary closed convex subset of a C*-algebra is $\mathbb{C}1$ in any unital C*-algebra.

Theorem 2.2 (Positive Hahn-Banach separation for von Neumann algebras)**.** *Let M be a von Neumann algebra.*

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{\circ+\circ+}$. In particular, if $x \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega \le 1$ on F.*

(2) *If F_* is a weakly closed convex hereditary subset of M_*^+, then $F_* = F_*^{\circ+\circ+}$. In particular, if $\omega \in M_*^+ \setminus F_*$, then there is $x \in M^+$ such that $\omega(x) > 1$ and $x \le 1$ on F_*.*

Proof. (1) Since the positive polar is represented as the real polar

$$F^{\circ+} = F^\circ \cap M_*^+ = F^\circ \cap (-M^+)^\circ = (F \cup -M^+)^\circ = (F - M^+)^\circ,$$

the positive bipolar can be written as $F^{\circ+\circ+} = (F - M^+)^{\circ\circ+} = \overline{F - M^+}^+$ by the usual bipolar theorem. Because $F = (F - M^+)^+ \subset (\overline{F - M^+})^+$, it suffices to prove the opposite inclusion $(\overline{F - M^+})^+ \subset F$.

Let $x \in \overline{F - M^+}^+$. Take a net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly, and take a net $y_i \in F$ such that $x_i \le y_i$ for each i. Suppose we may assume that the net x_i is bounded. Define strongly continuous functions $f_\varepsilon : [-(2\varepsilon)^{-1}, \infty) \to \mathbb{R} : z \mapsto z(1 + \varepsilon z)^{-1}$ parametrized by $\varepsilon > 0$. Then, for sufficiently small ε so that the bounded net x_i has the spectra in $[-(2\varepsilon)^{-1}, \infty)$, we have $f_\varepsilon(x_i) \to f_\varepsilon(x)$ σ-strongly, and hence σ-weakly. On the other hand, by the hereditarity and the σ-weak compactness of F, we may asumme that the bounded net $f_\varepsilon(y_i) \in F$ converges σ-weakly to a point of F by taking a subnet. Then, we have $f_\varepsilon(x) \in F - M^+$ by

$$0 \le f_\varepsilon(x) = \lim_i f_\varepsilon(x_i) \le \lim_i f_\varepsilon(y_i) \in F,$$

thus we have $x \in F$ since $f_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. What remains is to prove the existence of a bounded net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly.

Define a convex set

$$G := \{x \in \overline{F - M^+} : \exists\, x_m \in F - M^+,\ -2x \le x_m \uparrow x\} \subset M^{sa},$$

where x_m denotes a sequence. In fact, it has no critical issue for allowing x_m to be uncountably indexed. Since we clearly have $F - M^+ \subset G$ and every non-decreasing net with supremum is bounded and σ-strongly convergent, it suffices to show that G, or equivalently its intersection with the closed unit ball by the Krein-Smůlian theorem, is σ-strongly closed. Let $x_i \in G$ be a net such that $\sup_i \|x_i\| \le 1$ and $x_i \to x$ σ-strongly. For each i, take a sequence $x_{im} \in F - M^+$ such that $-2x_i \le x_{im} \uparrow x_i$ σ-strongly as $m \to \infty$, and also take $y_{im} \in F$ such that $x_{im} \le y_{im}$. Since $\|x_{im}\| \le 2\|x_i\| \le 2$ is bounded, it implies that there is a bounded net x_j in $F - M^+$ such that $x_j \to x$ σ-strongly, and we can choose arbitrarily small $\varepsilon > 0$ such that $\sigma(x_j) \subset [-(2\varepsilon)^{-1}, \infty)$ for all j. Since $f_\varepsilon(x_j)$ converges to $f_\varepsilon(x)$ σ-strongly and $f_\varepsilon(y_j)$ is a bounded net for each $\varepsilon > 0$ so that we may assume that the net $f_\varepsilon(y_j)$ is σ-weakly covergent by taking a subnet, we have $f_\varepsilon(x) \in F - M^+$ by

$$f_\varepsilon(x) = \lim_j f_\varepsilon(x_j) \le \lim_j f_\varepsilon(y_j) \in F,$$

where the limits are in the σ-weak sense. By taking ε as any decreasingly convergent sequence to zero, we have $x \in G$, hence the closedness of G.

(2) It suffices to prove $(\overline{F_* - M_*^+})^+ \subset F_*$, so we begin our proof by fixing $\omega \in (\overline{F_* - M_*^+})^+$. Since the norm closure and the weak closure of the convex set $F_* - M_*$ coincide, we have a sequence $\omega_n \in F_* - M_*^+$ such that $\omega_n \to \omega$ in norm of M_*, and we can take $\varphi_n \in F_*$ such that $\omega_n \le \varphi_n$ for all n. By modifying ω_n into $\omega_n - (\omega_n - \omega)_+ \in F_* - M_*^+$ and taking a rapidly convergent subsequence, we may assume $\omega_n \le \omega$ and $\|\omega - \omega_n\| \le 2^{-n}$ for all n. If we consider the Gelfand-Naimark-Segal representation $\pi : M \to B(H)$ associated to a positive normal linear functional $\omega + \sum_n(\omega - \omega_n)$ on M with the canonical cyclic vector Ω, we can construct commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(M)'$ of $\omega, \omega_n, \varphi_n$ respectively. Since $-1 \le h_n \le h$ is bounded, $h_n \to h$ in the weak operator topology of $\pi(M)'$. By the Mazur lemma, we can take a net h_i by convex combinations of h_n such that $h_i \to h$ strongly in $\pi(M)'$, and the corresponding linear functionals ω_i and φ_i satisfy $\omega_i \le \varphi_i$ with $\varphi_i \in F_*$ by the convexity of F_* so that $\omega_i \in F_* - M_*^+$. The net h_i can be taken to be a sequence in fact because $\pi(M)'$ is σ-finite, but it is not necessary. For each i and $0 < \varepsilon < 1$, define

$$h_\varepsilon := f_\varepsilon(h), \quad h_{i,\varepsilon} := f_\varepsilon(h_i), \quad k_{i,\varepsilon} := f_\varepsilon(k_i)$$

in $\pi(M)'$, where the functional calculus $f_\varepsilon(h_i)$ can be defined because $h_i \ge -1$ for all i, and define k_ε as the σ-weak limit of the bounded net $f_\varepsilon(k_i)$, which may be assumed to be σ-weakly convergent. Define $\omega_\varepsilon, \omega_{i,\varepsilon}, \varphi_{i,\varepsilon}$, and φ_ε as the corresponding normal linear functionals on M to $h_\varepsilon, h_{i,\varepsilon}, k_{i,\varepsilon}$, and k_ε. Note that $\varphi_i \in F_*$. The hereditarity of F_* and $0 \le \varphi_{i,\varepsilon} \le \varphi_i$ imply $\varphi_{i,\varepsilon} \in F_*$, and the weak closedness of F_* and the weak convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$ in M_* imply $\varphi_\varepsilon \in F^*$. From $\omega_i \le \varphi_i$, we can deduce $0 \le \omega_\varepsilon \le \varphi_\varepsilon$ by considering the operator monotonicity f_ε and taking the weak limit on i. Thus again, the hereditarity of F_* implies $\omega_\varepsilon \in F^*$, and the weak closedness of F_* and the weak convergence $\omega_\varepsilon \to \omega$ in M_* imply $\omega \in F^*$. □

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Lemma 3.1. *Let A be a C^*-algebra, and let F^* be a weakly* closed convex hereditary subset of A^{*+}. If $\omega_i \in F^* - A^{*+}$ is a net such that $\omega_i \to \omega$ weakly* in A^*, and if there is $\widetilde{\omega} \in A^{*+}$ such that $-\widetilde{\omega} \le \omega_i$ for all i, then for any $\delta > 0$ there is a sequence $\omega_n \in F^* - A^{*+}$ such that $\omega - \delta\widetilde{\omega} \le \omega_n \uparrow \omega$ weakly in A^*.*

Proof. Let $\varphi_i \in F^*$ be a net such that $\omega_i \le \varphi_i$ for all i. Consider the Gelfan-Naimark-Segal representation $\pi : A \to B(H)$ of $\widetilde{\omega}$ with the canonical cyclic vector $\Omega \in H$. Since $0 \le \widetilde{\omega} + \omega$, the Friedrichs extension theorem defines a positive self-adjoint operator $\widetilde{h}$ affiliated with $\pi(A)'$ such that $\pi(A)\Omega \subset \operatorname{dom}\widetilde{h}$ and $(\widetilde{\omega} + \omega)(a) = \langle \widetilde{h}\pi(a)\Omega, \Omega\rangle$ for all $a \in A$, and we can define the commutant Radon-Nikodym derivative of ω with respect to $\widetilde{\omega}$ by $h := \widetilde{h} - 1$, which satisfies $\omega(a) = \langle h\pi(a)\Omega, \Omega\rangle$ for all $a \in A$. Similarly, we can define the commutant Radon-Nikodym derivatives h_i, k_i of ω_i, φ_i with respect to $\widetilde{\omega}$, by the Friedrichs extension theorem.

Fix $0 < \varepsilon < 1$. Since $-1 \le h_i$ and $0 \le k_i$, the functional calculus $h_{i,\varepsilon} := f_\varepsilon(h_i)$ and $k_{i,\varepsilon} := f_\varepsilon(k_i)$ are well-defined. Taking a subnet, we may assume $h_{i,\varepsilon}$ and $k_{i,\varepsilon}$ are σ-weakly convergent in $\pi(A)'$, and denote by h_ε and k_ε the limits. By the operator concavity of f_ε, we may assume $h_{i,\varepsilon} \to h_\varepsilon$ and $k_{i,\varepsilon} \to k_\varepsilon$ σ-stronlgy. (It is true, but I will write more details later) Since

$$f_\varepsilon(h) - f_\varepsilon(h_i) = \varepsilon^{-1}((1+\varepsilon h_i)^{-1} - (1+\varepsilon h)^{-1}) = (1+\varepsilon h_i)^{-1}(h-h_i)(1+\varepsilon h)^{-1},$$

and since $(1+\varepsilon h)\pi(A)\Omega$ is dense in H because $\pi(A)\Omega$ is a core(?) of h and $1+\varepsilon h$ is surjective, we have $f_\varepsilon(h) = h_\varepsilon$ from

$$\begin{aligned}
&\|(f_\varepsilon(h) - h_\varepsilon)(1+\varepsilon h)\pi(a)\Omega\| \\
&\le \|(f_\varepsilon(h) - f_\varepsilon(h_i))(1+\varepsilon h)\pi(a)\Omega\| + \|(f_\varepsilon(h_i) - h_\varepsilon)(1+\varepsilon h)\pi(a)\Omega\| \\
&\le \|(1+\varepsilon h_i)^{-1}\|\|(h-h_i)\pi(a)\Omega\| + \|(h_{i,\varepsilon} - h_\varepsilon)(1+\varepsilon h)\pi(a)\Omega\| \\
&\le (1-\varepsilon)^{-1}\|(h-h_i)\pi(a)\Omega\| + \|(h_{i,\varepsilon} - h_\varepsilon)(1+\varepsilon h)\pi(a)\Omega\|
\end{aligned}$$

$$\begin{aligned}
&\langle (1+\varepsilon h_i)^{-1}(h-h_i)\pi(a)\Omega, \pi(b)\Omega\rangle \\
&= \langle (1-\varepsilon h_i(1+\varepsilon h_i)^{-1})(h-h_i)\pi(a)\Omega, \pi(b)\Omega\rangle \\
&= \langle (h-h_i)\pi(a)\Omega, \pi(b)\Omega\rangle + \langle \varepsilon h_i(1+\varepsilon h_i)^{-1}(h-h_i)\pi(a)\Omega, \pi(b)\Omega\rangle
\end{aligned}$$

When we denote by $\omega_{i,\varepsilon}, \omega_i, \varphi_{i,\varepsilon}, \varphi_i$ the linear functionals in A^{*sa} corresponded to $h_{i,\varepsilon}, h_i, \varphi_{i,\varepsilon}, \varphi_i$, it follows clearly that $\omega_{i,\varepsilon} \to \omega_\varepsilon$ and $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$ weakly* in A^*. The inequality $\omega_{i,\varepsilon} \le \varphi_{i,\varepsilon} \in F^*$ implies $\omega_\varepsilon \le \varphi_\varepsilon \in F^*$. Considering the normal extension $\pi^{**} : A^{**} \to B(H)$ of the representation π, we have $\omega_\varepsilon \uparrow \omega$ weakly in A^* as $\varepsilon \to 0$. Then, we obtain a desired sequence ω_n by taking ε to be a decreasing sequence that converges to zero and less than $\delta\|h\|^{-1}(\|h\|-\delta)^{-1}$ so that $t-\delta \le f_\varepsilon(t)$ on $|t| \le \|h\|$, where $\delta < \|h\|$ is assumed without loss of generality. $\square$

The following lemma is a slight generalization of the Krein-Šmulian theorem, and it can be proved as similar as the original theorem.

Lemma 3.2. *Let A be a C^*-algebra, and C_n^* be a non-decreasing sequence of weakly*-closed convex subsets of A^{*sa}, whose union C_∞^* contains A^{*+}. If a norm closed convex subset G^* of A^{*sa} has the property that $G^* \cap C_n^*$ is weakly* closed for each n, then $G^* \cap C_\infty^*$ is relatively weakly* closed in C_∞^*.*

Proof. Fix an element ω_0 of $C_\infty^* \setminus G^*$. It is enough to construct an element a of A^{sa} separating a norm open ball centered at ω_0 from G^*. Since G^* is norm closed, there exists $r > 0$ such that $G^* \cap B(\omega_0, r) = \varnothing$. By replacing G^* to $r^{-1}(G^* - \omega_0)$ and C_n^* to $r^{-1}(C_n^* - \omega_0)$, we may assume $G^* \cap B(0,1) = \varnothing$, and the claim follows if we prove there is $a \in A^{sa}$ separating $B(0,1)$ and G^*. The condition $A^{*+} \subset C_\infty^*$ becomes $A^{*+} - \omega_0 \subset C_\infty^*$. Letting the index n start from one, we may also replace C_n^* to $n(C_n^* \cap B(0,1))$ since its union is still C_∞^*. Note that C_n^* is bounded for each n, and we can easily see that $G^* \cap C_1^* = \varnothing$ and $n^{-1}C_n^* \subset (n+1)^{-1}C_{n+1}^*$.

Note that for any Banach space X, if F is a bounded subset of X, then by endowing with the discrete topology on F, we have a natural bounded linear operator $\ell^1(F) \to X$ with its dual $X^* \to \ell^\infty(F)$. We will construct a bounded subset F of X such that the subset $G^* \cap C_\infty^*$ of X^* induces a subset of the smaller subspace $c_0(F)$ of $\ell^\infty(F)$ via the map $X^* \to \ell^\infty(F)$, and also such that it satisfies $G^* \cap C_\infty^* \cap F^\circ = \varnothing$, where $F^\circ := \{x^* \in X^* : \sup_{x \in F} |x^*(x)| \le 1\}$ denotes the complex polar of F. If such a set $F \subset X$ exists, then the image of $G^* \cap C_\infty^*$ in $c_0(F)$ is a convex set disjoint to the closed unit ball $B_{c_0(F)}$ by the condition $G^* \cap C_\infty^* \cap F^\circ = \varnothing$. Therefore, there exists a separating linear functional $l \in B_{\ell^1(F)}$ by the Hahn-Banach separation, and it induces a linear functional separating G^* and B_{X^*}. Then, we are done.

Let $F_0 := \{0\} \subset X$. As an induction hypothesis on n, suppose for each $0 \le k \le n-1$ we already have a finite subset F_k of $(C_k^*)^\circ$ such that

$$G^* \cap C_n^* \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ = \varnothing.$$

If every finite subset F_n of $(C_n^*)^\circ$ satisfies

$$G^* \cap C_{n+1}^* \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ \cap F_n^\circ \neq \varnothing,$$

then since the intersection $G^* \cap C_{n+1}^*$ is weakly* compact, the finite intersection property leads a contradiction because the intersection of all complex polars F_n° of finite subsets F_n of $(C_n^*)^\circ$ is C_n^*, which is the polar of all union of finite subsets F_n of $(C_n^*)^\circ$ by the bipolar theorem. Thus, we can take a finite subset F_n of $(C_n^*)^\circ$ such that

$$G^* \cap C_{n+1}^* \cap \left(\bigcup_{k=0}^{n} F_k\right)^\circ = \varnothing.$$

Let $F := \bigcup_{k=0}^\infty F_k$. Then, we have $G^* \cap C_\infty^* \cap F^\circ = \varnothing$, and every element of C_∞^* is restricted to F to define an element of $c_0(F)$ because for each $\omega \in C_n^*$ and $k \ge 0$ we

have
$$\omega(F_{n+k})\subset\omega((C_{n+k}^*)^\circ)\subset\frac{n}{n+k}\omega((C_n^*)^\circ)\subset[-\frac{n}{n+k},\frac{n}{n+k}].$$
Finally, for any $\omega\in A^{*sa}$, if we enumerate F as a sequence f_m, then
$$|\omega(f_m)|\le|(\omega_+-\omega_0)(f_m)|+|(\omega_--\omega_0)(f_m)|\to 0,$$
so the uniform boundedness principle concludes that F is bounded. Therefore, the set F satisfies the properties we desired. □

Lemma 3.3. *Let M be a von Neumann algebra, and let $\mathfrak{M}$ be a σ-weakly dense hereditary $*$-subalgebra of M. If $\omega_i\in M_*^{sa}$ is a dominated net such that $\omega_i\to\omega\in M_*^{sa}$ pointwisely on $\mathfrak{M}$, then $\omega_i\to\omega$ weakly in M_*.*

Proof. Let $\widetilde{\omega}\in M_*^+$ be a dominating functional of ω_i such that $-\widetilde{\omega}\le\omega_i\le\widetilde{\omega}$ for all i. If $e_j\in\mathfrak{M}^+$ is a net such that $\|e_j\|\le 1$ and $e_j\to 1$ σ-strongly in M taken by the Kaplansky density theorem, then we have $e_jxe_j\in\mathfrak{M}$ by the hereditarity of $\mathfrak{M}$ and we can check $e_jxe_j\to x$ σ-strongly for each $x\in A^{**}$. For any $x\in A^{**+}$ and $\varepsilon>0$, since the absolute value function is a strongly continuous function, we can fix j such that $\widetilde{\omega}(|x-e_jxe_j|)<\varepsilon$, so the convergence $\omega_i\to\omega$ on $\mathfrak{M}$ is enhanced to the weak convergence in M_* by
$$\begin{aligned}(\omega-\omega_i)(x)&=(\omega-\omega_i)((x-e_jxe_j)_+)-(\omega-\omega_i)((x-e_jxe_j)_-)+(\omega-\omega_i)(e_jxe_j)\\&\le(\omega+\widetilde{\omega})((x-e_jxe_j)_+)+(\omega+\widetilde{\omega})((x-e_jxe_j)_-)+(\omega-\omega_i)(e_jxe_j)\\&\le 2\widetilde{\omega}(|x-e_jxe_j|)+(\omega-\omega_i)(e_jxe_j)\to 2\varepsilon+0.\end{aligned}$$
□

The main difficulty in dominating an approximating net ω_i of ω in the weak* closure is that we cannot modify ω_i to $\omega_i-(\omega_i-\omega)_+$ because $(\omega_i-\omega)_+$ may not coverge to zero weakly*.

Theorem 3.4 (Positive Hahn-Banach separation for C*-algebras)**.** *Let A be a C*-algebra.*

(1) *If F is a weakly closed convex hereditary subset of A^+, then $F=F^{\circ+\circ+}$. In particular, if $a\in A^+\setminus F$, then there is $\omega\in A^{*+}$ such that $\omega(a)>1$ and $\omega\le 1$ on F.*
(2) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then $F^*=(F^*)^{\circ+\circ+}$. In particular, if $\omega\in A^{*+}\setminus F^*$, then there is $a\in A^+$ such that $\omega(a)>1$ and $a\le 1$ on F^*.*

Proof. (1) We directly prove the separation without invoking the arguments of positive bipolars. Denote by F^{**} the σ-weak closure of F in the universal von Neumann algebra A^{**}. We first show that F^{**} is hereditary subset of A^{**+}. Suppose $0\le x\le y$ in A^{**} and $y\in F^{**}$. Then, there is $z\in A^{**}$ such that $x^{\frac12}=zy^{\frac12}$. Take bounded nets b_i in F and c_i in A such that $b_i\to y$ and $c_i\to z$ σ-strongly* in A^{**} using the Kaplansky density. We may assume the indices of these two nets are same. Since both the multiplication and the involution of a von Neumann algebra on bounded parts are continuous in the σ-strong* topology, and since the square root on a positive bounded interval is a strongly continuous function, we have the σ-strong* limit
$$x=y^{\frac12}z^*zy^{\frac12}=\lim_i b_i^{\frac12}c_i^*c_ib_i^{\frac12},$$
so we obtain $x\in F^{**}$ from $b_i^{\frac12}c_i^*c_ib_i^{\frac12}\in F$. Thus, F^{**} is hereditary in A^{**+}.

Let $a \in A^+ \setminus F$. Observe that we have $a \in A^{**+} \setminus F^{**}$ because if $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, meaning that $a_i \to a$ weakly in A and by the weak closedness of F in A we get a contradiction $a \in F^{**} \cap A = F$. By Theorem, there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega \le 1$ on $F \subset F^{**}$, so it completes the proof.

(2) As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, so take $\omega \in (\overline{F^* - A^{*+}})^+$. We first prove it when A is separable, which makes the weak* topology on any bounded part of A^{*sa} metrizable. Consider the following convex set

$$G^* := \left\{ \omega \in \overline{F^* - A^{*+}} : \begin{array}{c} \text{there is a sequence } \omega_n \in F^* - A^{*+} \text{ and } \widetilde{\omega} \in A^{*+} \text{ such that} \\ -\widetilde{\omega} \le \omega_n \to \omega \text{ weakly* in } A^* \end{array} \right\}.$$

In the spirit of the Krein-Šmulian theorem, let ω_n be a bounded sequence in G^* such that $\omega_n \to \omega$ weakly* in A^*, and claim $\omega \in G^*$. Since ω belongs to the relative weak* closure of $G^* \cap C^*_\infty$ in C^*_∞, where

$$C^*_n := \{\omega' \in A^{*sa} : -\sum_{k\le n} \omega_k - \omega_- \le \omega'\}, \quad C^*_\infty := \bigcup_n C^*_n,$$

if we prove G^* is norm closed and $G^* \cap C^*_n$ is weakly* closed for each n, then we obtain $\omega \in G^*$ by Lemma, which completes the proof.

Since the limit of a norm convergent sequence in G^* can be approximated by a dominated sequence in G^* as in the proof of Theorem, and every sequence in C^*_n is dominated, it is enough to show $\omega \in G^*$ when it is the weak* limit of a dominated sequence ω_n in G^*. Since A is σ-unital, when we denote by e a strictly positive element of A, we can take a sequence $\omega_{nm} \in F^* - A^{*+}$ such that $\omega_{nm}(e) \uparrow \omega_n(e)$, which implies the weak* convergence $\omega_{nm} \uparrow \omega_n$ for each n by boundedness. Associated to ω, ω_n, ω_{nm}, and φ_{nm}, the commutant Radon-Nikodym derivatives h, h_n, h_{nm}, and k_{nm} are defined. Note that h_n is a bounded sequence, and h_{nm} are bounded increasing sequences for each n, and k_{nm} are self-adjoint operators constructed by Friedrichs extension with $h_{nm} \le k_{nm}$ for every n and m. The boundedness implies that $h_n \to h$ as $n \to \infty$ and $h_{nm} \uparrow h_n$ as $m \to \infty$ for each n in the weak operator topology. By applying the Mazur lemma, we can take a convergent diagonal sequence such that $h_{nn} \to h$ in the strong operator topology, after taking rapidly convergent subsequence from h_{nm} for each n, which can be done because the existence of a cyclic vector implies that the commutant is a σ-finite von Neumann algebra and the strong operator topology is metrizable. Then, h_{nn} is bounded by the uniform boundedness principle, we can take f_ε to show $\omega \in F^* - A^{*+}$.

Now we consider a general C*-algebra A. For a C*-subalgebra B of A, we define a set

$$F^*_B := \{\omega \in B^{*+} : \text{there is } \varphi \in F^* \text{ such that } \omega \le \varphi \text{ on } B^+\}.$$

It is clearly a convex hereditary subset of B^{*+}. If $\omega_i \in F^*_B$ is a net such that $\omega_i \to \omega$ weakly* in B^* and $\omega_i \le \varphi_i \in F^*$ on B^+, then there are positive extensions $\widetilde{\omega}_i \in A^{*+}$ of ω_i such that $\widetilde{\omega}_i \le \varphi_i$ on B^+.

If $\widetilde{\omega}_i$ is a positive norm preserving extension of ω_i, then $\widetilde{\omega}_i \le \varphi_i$? no.

$\omega_n \to \omega$ in norm and $\omega_n \le \omega$. Let

Considering $k_{i,\varepsilon} \to k_\varepsilon$ so that $\varphi_\varepsilon \in F^*$. We need weak* convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$.

$\widetilde{\omega}_{i,\varepsilon} \le \varphi_{i,\varepsilon}$

$\widetilde{\omega}_\varepsilon \le \varphi_\varepsilon$

$\varphi_i = \omega_i^\sim + (\varphi_i - \omega_i)^\sim$ on B

$\|\omega_i^\sim + (\varphi_i - \omega_i)^\sim\| \le \|\omega_i^\sim\| + \|(\varphi_i - \omega_i)^\sim\| = \|\omega_i\| + \|\varphi_i|_B - \omega_i\| \le \|\varphi_i|_B\|$

Let $\omega \in (\overline{F^* - A^{*+}})^+$, where the closure is taken in the weak* topology. Take a net $\omega_i \in F^* - A^{*+}$ and $\varphi_i \in F^*$ such that $\omega_i \to \omega$ weakly* in A^* and $\omega_i \le \varphi_i$ for each i. If we denote by $\omega_B, \omega_{i,B}, \varphi_{i,B}$ the restrictions of $\omega, \omega_i, \varphi_i$ on B, then we have $\varphi_{i,B} \in F_B^*$ and $\omega_{i,B} \in F_B^* - B^{*+}$, with the weak* convergence $\omega_{i,B} \to \omega_B$ in B^*, thus we have $\omega_B \in (\overline{F_B^* - B^{*+}})^+ = F_B^*$ because B is separable. If we consider the non-decreasing net of all separable C*-subalgebras B_j of A, then the restriction ω_{B_j} of ω on B_j belongs to the set $F_{B_j}^*$ as we have seen just now, so there is $\varphi_j \in F^*$ such that $\omega \le \varphi_j$ on B_j^+ for each j. Here we let ψ be a faithful semi-finite normal weight on A^{**}, and let $\pi : A^{**} \to B(H)$ be the Gelfand-Naimark-Segal representation associated to ψ, together with the left A^{**}-liner map $\Lambda : \mathfrak{N}_\psi \to H$ of dense range such that $\psi(x^*x) = \|\Lambda(x)\|^2$ for all $x \in \mathfrak{N}_\psi$. Note that because the weight ψ is faithful and semi-finite, Λ is injective and σ-weakly densely defined, meaning that $\mathfrak{M}_\psi$ is a hereditary σ-weakly dense $*$-subalgebra of A^{**}. Construct the commutant Radon-Nikodym derivatives h, k_j of ω, φ_j with respect to ψ. Here k_j is a positive self-adjoint operator defined by the Friedrichs extension such that $\operatorname{ran}\Lambda \subset \operatorname{dom} k_j$ for all j. Taking a subnet, we may assume that there is $k_\varepsilon \in \pi(A)'^+$ satisfying $f_\varepsilon(k_j) \to k_\varepsilon$ σ-weakly. Because of the operator concavity of f_ε (more detail), we can take a net $\varphi_l \in F^*$ such that $f_\varepsilon(k_l) \to k_\varepsilon$ σ-strongly, where k_l are again the commutant Radon-Nikodym derivatives of φ_l defined by the Friedrichs extension. Since is a strongly continuous function, we have $(f_\varepsilon(k_l) - k_\varepsilon) \to 0$ σ-strongly, so if we define $\varphi_{l,\varepsilon} \in F^* - A^{*+}$ and $\varphi_\varepsilon \in A^{*+}$ such that

$$\varphi_{l,\varepsilon}(x^*x) := \langle (f_\varepsilon(k_l) - (f_\varepsilon(k_l) - k_\varepsilon)_+)\Lambda(x), \Lambda(x)\rangle, \quad \varphi_\varepsilon(x^*x) := \langle k_\varepsilon \Lambda(x), \Lambda(x)\rangle$$

for each $x \in \mathfrak{N}_\psi$, then we have $\varphi_{l,\varepsilon} \to \varphi_\varepsilon$ pointwisely on $\mathfrak{M}_\psi$ and $\varphi_{l,\varepsilon} \le \varphi_\varepsilon$ for all l.

How to dominate $\varphi_{l,\varepsilon}$ from below?

By Lemma we have $\varphi_{l,\varepsilon} \to \varphi_\varepsilon$ weakly in A^*, so Theorem implies that $\varphi_\varepsilon \in (\overline{F^* - A^{*+}}^w)^+ = F^*$. If we define $\omega_\varepsilon \in A^{*+}$ and $\varphi_{j,\varepsilon} \in F^*$ by

$$\omega_\varepsilon(x^*x) := \langle f_\varepsilon(h)\Lambda(x), \Lambda(x)\rangle, \quad \varphi_{j,\varepsilon}(x^*x) := \langle f_\varepsilon(k_j)\Lambda(x), \Lambda(x)\rangle$$

for each $x \in \mathfrak{N}_\psi$, then since $\omega \le \varphi_j$ on B_j^+ implies $\omega_\varepsilon \le \varphi_{j,\varepsilon}$ on B_j^+, the weak* limit $\omega_\varepsilon \le \lim_j \varphi_{j,\varepsilon} = \varphi_\varepsilon$ deduces $\omega_\varepsilon \in F^* - A^{*+}$. Since $\omega_\varepsilon \to \omega$ pointwisely on $\mathfrak{M}_\psi$ and $0 \le \omega_\varepsilon \le \omega$ for all $0 < \varepsilon$, we have $\omega \in (\overline{F^* - A^{*+}}^w)^+ = F^*$ by Lemma and Theorem. □

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Corollary 4.1. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\begin{Bmatrix} \text{subadditive normal} \\ \text{weights of } M \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{hereditary closed} \\ \text{convex subsets of } M_*^+ \end{Bmatrix}$$

$$\varphi \quad \mapsto \quad \{\omega \in M_*^+ : \omega \le \varphi\}$$