Three-dimensional Topology

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Contents

1 Day 1: April 11

Plan:

1. Fundamental groups of manifolds
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7. JSJ composition
8. Geometrization
9. Virtually special theorems

References:

1. J. Hempel, 3-manifolds
2. W. Jaco, Lectures on three-manifold topology
3. B. Martelli, An introduction to geometric topology
4. Morimoto, An introduction to three-dimensional manifolds (Japanses)

Grading: Submit a report for any three among the exercises given in the lecture (ITC-LMS Kadai).
Cancellation of class: 5/2, 7/11(maybe)
Convention:

- manifold = connected compact orientable smooth manifold
- surface = connected compact orientable smooth 2-dimensional manifold
- tub nbd, isotopy extension, transversality, triangulation,$\cdots$

1. Fundamental group

1.1 Fundamental groups of higher dimensional manifolds

Proposition 1.1. *Let π be a finitely presented group. Then, for every $d \geq 4$ there is a d-manifold X such that $\pi_1(X) \cong \pi$.*

Proof. Let $\pi = \langle x_1, \cdots, x_n \mid r_1, \cdots r_m \rangle$ be given. If $Y = (S^1 \times S^{d-1})^{\# n}$, then $\pi_1(Y) \cong \langle x_1, \cdots x_n \rangle$ by the van Kampen theorem. Let

$$Z = Y \setminus (\coprod_{i=1}^{m} \nu(l_i)),$$

where $l_i \subset Y$ is embedded loops representing r_i and ν denotes the open tubular neighborhood. Then, $\partial Z = \coprod_{i=1}^{m} l_i \times S^{d-2}$. Since $d \geq 4$, any loops and disks can be pushed off $l_1 \cdots, l_n$, we have an isomorphism $\pi_1(Z) \to \pi_1(Y)$. Then, if we let

$$X = Z \cup_\partial (\coprod_{i=1}^{m} D^2 \times S^{d-2}),$$

then $l_i \times * = \partial(D^2 \times *)$, we have $\pi_1(X) \cong \pi_1(Y)/\langle [l_1], \cdots, [l_m] \rangle \cong \pi$. □

1.2 Surfaces and their groups

Theorem 1.2 (Radó, Whitehead)**.** *Every topological surface admits a unique smooth and PL structure.*

Theorem 1.3. *Every surface is diffeomorphic to only one of $\Sigma_{g,b}$, where $\Sigma_{g,b} = (T^2)^{\# g} \# (D^2)^{\# b}$.*

Corollary 1.4. *$S = S^2, T^2, D^2$ are prime, that is, $S = S_1 \# S_2$ implies $S_i \approx S^2$ for $i = 1$ or $i = 2$.*

Remark 1.5. If the orientability is reduced out, then $\mathbb{RP}^2$ is prime. Also note that $T^2 \# \mathbb{RP}^2 \approx (\mathbb{RP}^2)^{\# 3}$.

Theorem 1.6 (Uniformization)**.** *For every surface $S \neq D^2$, its interior admits a complete Riemannian metric of constant curvature*

$$\begin{cases} 1, & \chi(S) > 0 \\ 0, & \chi(S) = 0 \\ -1, & \chi(S) < 0 \end{cases}$$

with universal covering S^2, $\mathbb{R}^2$, $\mathbb{H}^2$, respectively.

The hyperbolic plane is $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, and $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$.

Proposition 1.7. *If a surface S has $\chi(S) < 0$, then there is a discrete group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ such that $S \approx \mathbb{H}^2/\Gamma$. In particular, $\pi_1(S)$ is isomorphic to Γ.*

We have

$$\pi_1(\Sigma_{g,b}) \cong F_{2g+b-1} \quad \text{and} \quad \pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1], \cdots, [a_g, b_g] \rangle.$$

Proposition 1.8. *$\pi_1(\Sigma_g)$ is torsion free.*

Exercise 1. Prove Proposition 1.7.

Theorem 1.9 (Newman)**.** *$\pi_1(\Sigma_{g,b})$ is linear over $\mathbb{Z}$, that is, is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$. For example, we can check $F_n \hookrightarrow F_2 \hookrightarrow \mathrm{SL}_2(\mathbb{Z})$ according to the pingpong lemma.*

Over $\mathbb{R}$, we may embed $\pi_1(S) \hookrightarrow \mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{SO}^+_{1,2}(\mathbb{R}) < \mathrm{GL}_3(\mathbb{R})$ if $\chi(S) < 0$,

Definition. A group π is called residually finite(RF) if for every $1 \neq \gamma \in \pi$ there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \neq 1$. A subgroup $\pi' < \pi$ is called separable if there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \notin \varphi(\pi')$. In particular, π is residually finite if the trivial subgroup is separable in π. A group π is called locally extended residually finite(LERF) if every finitely generated subgroup of π is separable.

Theorem 1.10 (Mal'cev)**.** *Every finitely generated linear group over a field is residually finite.*

Over $\mathbb{Z}$, if $1 \neq (a_{ij}) \in \pi < \mathrm{GL}_n(\mathbb{Z})$ given, then for $m > \max_{i,j} |a_{ij}|$ if we let $\varphi_m : \pi \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, then $\varphi_m((a_{ij})) \neq 1$.

Theorem 1.11 (Scott)**.** *$\pi_1(\Sigma_{g,b})$ is LERF.*

2 Day 2: April 18

2. Examples and constructions of 3-manifolds

Theorem 2.1 (Moise)**.** *Every topological 3-manifold(not neccesarily compact, connected, orientable) admits a unique smooth and PL structure.*

2.1 Spherical manifolds

Recall

$$\begin{aligned} S^3 :&= \{x \in \mathbb{R}^4 : |x| = 1\} \\ &= \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \\ &= \{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}. \end{aligned}$$

Lens spaces

Let $p, q \in \mathbb{Z}$, $p > 0$, $(p,q) = 1$. Then, $\mathbb{Z}/p\mathbb{Z} = \langle \zeta = \exp(2\pi\sqrt{-1}/p) \rangle$ acts on S^3 such that $\zeta \cdot (z,w) = (\zeta z, \zeta^q w)$. Then, the Lens spaces are defined as

$$L(p,q) := S^3/(\mathbb{Z}/p\mathbb{Z}) \quad \text{with} \quad \pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}.$$

For example, $L(1,1) = S^3$ and $L(2,1) = \mathbb{RP}^3$.

Theorem 2.2 (Reidemeister)**.**

(a) $L(p,q) \simeq L(p,q')$ *(homotopy equiv) if and only if there is* $a \in \mathbb{Z}$ *such that* $qq' \equiv \pm a^2 \pmod p$.

(b) $L(p,q) \approx L(p,q')$ *(diffeo) if and only if* $q' \equiv \pm q^{\pm 1} \pmod p$.

For example, $L(7,1) \simeq L(7,2)$ since $1 \cdot 2 \equiv 3^2 \pmod 7$, but $L(7,1) \approx L(7,2)$ since $2 \not\equiv \pm 1 \pmod 7$.

Sketch of ($\Leftarrow$). Direct construction. (a) With the linking form $H_1(L) \times H_1(L) \to \mathbb{Q}/\mathbb{Z}$. (b) Reidemeister torsion. $\square$

General quotients

A spherical manifold is the orbit space S^3/Γ, where Γ is a finite subgroup of SO(4) and $\Gamma \curvearrowright S^3$ freely.

Example. With an action $\langle -1, i, j, k \rangle \curvearrowright S^3$, we obtain the prism manifold.

Example. With an action of the binary icosahedral group $\Gamma = \mathbb{Z}/2\mathbb{Z} \rtimes A_5$ on S^3, we obtain the Poincaré sphere. We have $H_*(S^3/\Gamma) \cong H_*(S^3)$. If we take 3/10 turn instead of 1/10, we have the Seifert-Weber space.

2.2 Fibered manifolds

Twisted bundles

$$N_{g,b} = (\mathbb{RP}^2)^{\#g} \# (D^2)^{\#b}.$$

Let D be a polygon with oriented sides $a_1, a_1', a_2, a_2', \cdots, a_g, a_g'$.

$$N_g \widetilde{\times} [0,1] := D \times [0,1]/\sim,$$

where $(x,t) \sim (x', 1-t)$ for $x \in a_i$, $x' \in a_i'$, $t \in [0,1]$ with $[x] = [x'] \in N_g$, and it is orientable.

$$N_g \widetilde{\times} S^1 := N_g \widetilde{\times} [0,1]/(x,0) \sim (x,1),\ x \in N_g$$

$$N_{g,b} \widetilde{\times} S^1 := N_g \widetilde{\times} S^1 \setminus \nu(b \text{ fibers}).$$

Exercise 2. Show the following:

(a) $\mathbb{RP}^2 \widetilde{[}0,1] \approx \mathbb{RP}^3 \setminus$ (open ball).

(b) $\mathbb{RP}^2 \widetilde{\times} S^1 \approx \mathbb{RP}^3 \# \mathbb{RP}^3$.

(c) $N_{1,1} \widetilde{\times} S^1 \approx N_2 \widetilde{\times} [0,1]$.

Mapping tori

The mapping class group is

$$\mathcal{M}_{g,b} := \mathrm{Diff}^+(\Sigma_{g,b}, \partial\Sigma_{g,b})/\text{isotopy relative to } \partial.$$

Theorem 2.3 (Dehn, Lickorish). *$\mathcal{M}_{g,b}$ is finitely generated by Dehn twists.*

For examples, $\mathcal{M}_0 = \mathcal{M}_{0,1} = 1$ by the Alexander trick, and $\mathcal{M}_{0,2} \cong \mathbb{Z}$, $\mathcal{M}_1 = \mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})$. Let $\varphi \in \mathcal{M}_{g,b}$. Then, a mapping torus is defined by

$$M_\varphi := \Sigma_{g,b} \times [0,1]/(\varphi(x),0) \sim (x,1).$$

2.3 Heegaard decomposition

A manifold with a boundary

$$H_g := D^3 \cup (D^2 \times [0,1])^{\sqcup g}$$

is called the handle body with genus g. Then, $\pi_1(H_g) \cong F_g$ and $\partial H_g \approx \Sigma_g$. Let $\varphi : \partial H_g \to \partial H_g$ be an orientation-preserving diffeomorphism, i.e. an element of the mapping class group $\mathcal{M}_g$. If a 3-manifold M satisfies

$$M \approx H_g \approx H_g,$$

then the right-hand side is called the Heegaard decomposition(splitting) of M.

Proposition 2.4. *Every closed 3-manifold admitting a Heegaard decomposition of genus 0 is diffeomorphic to S^3.*

Proof. $\mathcal{M}_0 = 1$. □

Proposition 2.5. *Every closed 3-manifold admitting a Heegaard decomposition of genus 1 is diffeomorphic to S^3, $S^2\times$, or $L(p,q)$.*

Exercise 3. Prove the above proposition.

Theorem 2.6. *Every closed 3-manifold M admits a Heegaard decomposition along some Σ_g.*

Proof. Pick a triangulation T of M. Then, $H := \overline{\nu(T^{(1)})}$ is a handlebody. Then, $H' := M \setminus \mathrm{Int}H$ is also a handlebody. Since M is orientable, so are H and H', thus we are done. □

There is another proof using Morse theory.

Corollary 2.7. *The fundamental group of every closed 3-manifold admits a finite presentation of deficiency 0, i.e. the number of generators is equal to the number of relations.*

Proof. Apply the van Kampen theorem to

$$M = H_g \cup H_g = H_g \cup (D^2 \times [-\varepsilon,\varepsilon])^{\sqcup g} \cup D^3.$$ □

2.4 Dehn surgery

Let L be a link. The link exterior is the set $E_L = S^3 \setminus \nu(L)$.

Proposition 2.8. *Let M be a 3-manifold, and $T \subset \partial M$ a torus component. Let $h : \partial(D^2 \times S^1) \to T$ be a diffeomorphism. Then, $M \cup_h (D^2 \times S^1)$ is determined only by $\pm[h(\partial D^2 \times *)] \in H_1(T)$.*

Proof. Write

$$M \cup_h (D^2 \times S^1) = M \cup_h (D^2 \times (-\varepsilon,\varepsilon)) \cup D^3.$$ □

For a knot K, there are two generators μ and λ, called the meridian and the longitude, of $H_1(\partial E_K)$ such that $\ker(H_1(\partial E_K) \to H_1(E_K) = \mathbb{Z}\mu)$ is generated by λ.

Exercise 4. Show that $L(p,q)$ with $p \neq 0$ and $S^2 \times S^1$ are obtained by the p/q-Dehn surgery along the unknot.

Theorem 2.9 (Lickorish-Wallace)**.** *Every closed 3-manifold can be obtained by an (integral)-Dehn surgery along some link in S^3.*

Sketch. Heegaard decomposition and $\mathcal{M}_g = \langle \text{Dehn twists} \rangle$. Each Dehn twist realizes the Dehn surgery steps. $\square$

3 Day 3: April 25

3. Prime decomposition

3.1 Alexander's theorem

Theorem 3.1. *Every (smooth) embedding $S^2 \subset \mathbb{R}^3$ bounds some (smooth) embedding $D^3 \subset \mathbb{R}^3$.*

Remark. The above theorem does not hold in the category of topological spaces. Alexander's horned sphere is one of the counterexamples.

If $\mathbb{R}^d$ for $d \geq 5$, then more complicated result such as h-cobordism theorem must be used to obtain the same conclusion.

Sketch. Isotope such a sphere S so that the coordinate $z : S \to \mathbb{R}$ is a Morse function. Assume that for all $p \neq q \in \mathrm{Crit}(z)$, then $z(p) \neq z(q)$. We use induction on (m, n), where m is the number of saddles and

$$n := \min\{\#\pi_0(S \cap z^{-1}(r)) : r \text{ is a regular value s.t. } z(p) < r < z(q) \text{ for some saddles } p, q\}.$$

Note that $\#(\text{minima of } z) - m + \#(\text{maxima of } z) = \chi(S) = 2$.

For the case $m = 0$ so that there are only one minimum and maximum, then we can construct a ball by applying the Jordan-Schönflies theorem to each level.

For the case $m = 1$, then only four types appear: a jelly bean, a red blood cell, and their upside down versions. Apply the Jordan Schönflies again.

For the case $m \geq 2$, let r be a regular value realizing the value of n. Let D be union of the closure of the interior of the innermost circles of $S \cap z^{-1}(r)$. Replace S by $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{-\varepsilon, \varepsilon\})$. Then, each connected component has lower (m, n) so that it bounds a ball. Attaching all balls bounded by the components with balls $D \times [-\varepsilon, \varepsilon]$, S also bounds a ball. □

3.2 Irreducible manifolds

The connected sum is defined as

$$M \# N := (M \setminus (\text{open ball})) \cup_\partial (N \setminus (\text{open ball})).$$

Proposition 3.2.

(a) $M \# N \approx N \# M$.

(b) $(M_1 \# M_2) \# M_3 \approx M_1 \# (M_2 \# M_3)$.

(c) $M \# S^3 \approx M$.

We say a manifold M is *prime* if $M = N_1 \# N_2$ implies $N_1 \approx S^3$ or $N_2 \approx S^3$. We say a 3-manifold M is *irreducible* if every embedding $S^2 \subset M$ bounds some embedding $D^3 \subset M$. In other words, in a prime manifold every separating sphere bounds a ball, in an irreducible manifold every sphere bounds a ball.

Corollary 3.3. *Every irreducible 3-manifold is prime.*

Corollary 3.4. *By Theorem 3.1, S^3 is irreducible.*

Theorem 3.5.

(a) *$S^2 \times S^1$ is prime, but is not irreducible.*

(b) *Every closed prime 3-manifold which is not irreducible is diffeomorphic to $S^2 \times S^1$.*

Proof. (a) A sphere $S^2 \times *$ cannot bound any $D^3 \subset S^2 \times S^1$ because $[S^2 \times *] \neq 0 \in H_2(S^2 \times S^1)$, so $S^2 \times S^1$ is not irreducible.

Suppose $S^2 \times S^1 = N_1 \# N_2$. Since $\pi_1(N_1) * \pi_1(N_2) \cong \mathbb{Z}$, one of $\pi_1(N_1)$ or $\pi_1(N_1)$ is trivial. Assume $\pi_1(N_1)$ is trivial and let $B := N_1 \setminus$(open ball). Since B is also simply connected, it lifts diffeomorphically into the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. Because $S^2 \times \mathbb{R} \approx \mathbb{R}^3 \setminus \{0\}$, we have an embedding $B \subset \mathbb{R}^3$. Because $\partial B \approx S^2$, by Theorem 3.1 we have $B \approx D^3$, so $N_1 \approx S^3$.

(b) If every sphere in M is separating, then it has to be irreducible since M is prime, so such M contains a nonseparating sphere S. Let γ be an arc connecting the inside and the outside of $\partial\, \nu(S)$. If we let $M' := \overline{\nu(S)} \cup \overline{\nu(\gamma)}$, then $\partial M' \approx S^2 \#(S^1 \times I) \# S^2 \approx S^2$ is a separating sphere. Since $M \setminus \mathrm{Int} M' \approx D^3$ because M' is not simply connected and M is prime, M is diffeomorphic to $S^2 \times S^1$. $\square$

Proposition 3.6. *If a covering space $\widetilde{M}$ of M is irreducible, then so is M.*

Exercise 4. Prove Proposition 3.6.

Remark. The converse of Proposition 3.6 is also known to be true.

3.3 Normal surfaces

Fix a 3-manifold M and its triangulation T. A (possibly disconnected) subsurface $S \subset M$ is called a *normla surface* with respect to T if S is a union of *normal disks*, defined as seven types of disks in a given tetrahedron: four triangles and three quadrilaterals.

Proposition 3.7. *Every (possibly disconnected) subsurface $S \subset M$ becomes a normal surface with respect to T by isotopies and the following operations:*

(i) Replace S by $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{\pm\varepsilon\})$ for a disk D satisfying $D \cap (S \cup \partial M) = \partial D$.

(ii) Remove a components of S contained in a ball in M.

Proof. Isotope S so that $S \pitchfork T$. It is sufficient to realize the following:

(a) For every tetrahedron $\Delta^{(3)} \subset T^{(3)}$, $S \cap \Delta^{(3)} = \coprod$(disks).

(b) For every disk component D in (a) and for every edge $\Delta^{(1)} \in T^{(1)}$, $\#(D \cap \Delta^{(1)}) \leq 1$.

(c) For every triangle $\Delta^{(2)} \subset T^{(2)}$, $S \cap \Delta^{(2)} = \coprod$(arcs).

For (a), if there is a non-disk connected component of $S \cap \Delta^{(3)}$, perform (i) along innermost loops in $S \cap \partial \Delta^{(3)}$. Then, perform (ii) for closed components of S contained in $\Delta^{(3)}$.

For (b), if there is a disk component of $S \cap \Delta^{(3)}$ which intersects an edge $\Delta^{(1)}$ more than twice, then for an inner most pair of two points connected by an arc in D, push the arc to the outside $\Delta^{(3)}$ with an ambient isotopy.

For (c) $\square$

4 Day 4: May 9

3.4 Prime decomposition theorem

Proposition 3.8. *Let T be a triangulation of M and let $S \subset M$ be a normal surface with respect to T. If no component of $M \setminus \nu(S)$ is a trivial $[0,1]$-bundle, then*

$$\#\pi_0(S) \leq 6 \cdot \#T^{(3)} + \mathrm{rk}\, H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}).$$

Proof. Let M_0 be a component of $M \setminus \nu(S)$ such that for each tetrahedron $\Delta^{(3)}$ the intersection $M_0 \cap \Delta^{(3)}$ is a trivial $[0,1]$-bundle over a normal disk $D_0(\Delta^{(3)}) \subset \Delta^{(3)}$. Let $S' = (S \setminus \sqcup \partial M_0) \sqcup (\sqcup D_0)$. Then,

$$\#\pi_0(S) = \#\pi_0(S') = \mathrm{rk}\, H_2(S', \partial S'; \mathbb{Z}/2\mathbb{Z}),$$

and since

$$H_3(M, \partial M \cup S'; \mathbb{Z}/2\mathbb{Z}) \to H_2(S', \partial S; \mathbb{Z}/2\mathbb{Z}) \to H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$$

is exact, we have

$$\mathrm{rk}\, H_2(S', \partial S'; \mathbb{Z}/2\mathbb{Z}) \leq \mathrm{rk}\, H_3(M, \partial M \cup S'; \mathbb{Z}/2\mathbb{Z}) + \mathrm{rk}\, H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}).$$

If we let $M' := (M \setminus \nu(S)) \setminus \sqcup M_0$, then by the excision theorem,

$$\mathrm{rk}\, H_3(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = \mathrm{rk}\, H_3(M', \partial M'; \mathbb{Z}/2\mathbb{Z}) = \pi_0(M').$$

Every component of M' is not a trivial $[0,1]$-bundle over a normal disk in some tetrahedron in T. The number of such non-trivial components of the complement $\Delta^{(3)} \setminus \nu(S)$ is at most 6. $\square$

Theorem 3.9 (Prime decomposition theorem; Kneser, Milnor)**.** *Every 3-manifold $M \neq S^3$ admits a decomposition*

$$M = N_1 \# \cdots \# N_n, \quad N_i : \text{ prime } \neq S^3.$$

Moreover, such a decomposition is unique up to diffeomorphisms and permutations.

Proof. (Existence) If M contains a non-separating sphere, then there is M' such that $M = M' \# (S^2 \times S^1)$ as in the proof of Proposition 3.5. Since we have $H_(M) = H_1(M') \oplus \mathbb{Z}$, the number of such $S^2 \times S^1$ factors is finite, so by cutting out all of them, we may assume that every sphere in M is separating.

Let S be an arbitrary union of spheres in M such that

$$\text{no components of } M \setminus \nu(S) \text{ is diffeomorphic to } S^3 \setminus (\text{balls}). \qquad (\dagger)$$

It suffices to show that $\pi_0(S)$ is finite. Let T be a triangulation of M and $S' \subset M$ be a normal surface with respect to T obtained from S as Proposition 3.7. Suppose that a component $S_0 \subset S$ is contained in a ball. Then by the Alexander theorem such an innermost component would bound a ball, which contradicts to (†), hence the operation (ii) in Proposition 3.7 does not occur when we construct S'. Each operation (i) splits a component $S_0 \subset S$ into two spheres S_1 and S_2, and one of them satisfies (†). Therefore, there exists a union of components $S'' \subset S'$ satisfying the same condition with S with $\pi_0(S'') = \pi_0(S)$, which is bounded by Proposition 3.8.

(Uniqueness) Suppose

$$P_1 \# \cdots \# P_m \# (S^2 \times S^1)^{\# k} = Q_1 \# \cdots \# Q_n \# (S^2 \times S^1)^{\# l},$$

where P_i and Q_j are irreducible but not S^3. It suffices to find a union of spheres $T \subset M$ such that

(P) $M \setminus \nu(T) = \coprod_{i=1}^m (P_i \setminus (\text{balls})) \sqcup (\coprod S^3 \setminus (\text{balls}))$,

(Q) $M \setminus \nu(T) = \coprod_{j=1}^n (Q_j \setminus (\text{balls})) \sqcup (\coprod S^3 \setminus (\text{balls}))$.

Pick unions of spheres $R, S \subset M$ such that R satisfies (P), S satisfies (Q), and $R \pitchfork S$. If $R \cap S = \varnothing$, then let $T := R \sqcup S$. If not, pick components $R_0 \subset R$ and $S_0 \subset S$ such that $R_0 \cap S_0 \neq \varnothing$. Split R_0 along an innermost disk in S_0 into two spheres R_1 and R_2, and replace R by $R' := (R \setminus R_0) \sqcup R_1 \sqcup R_2$. Then, R' satisfies (P) and $\#\pi_0(R' \cap S) < \#\pi_0(R \cap S)$ so that such operations reduce to the case of $R \cap S = \varnothing$. □

Exercise 5. Show that the existence of the prime decomposition theorem follows from the Poincaré conjecture and Grushko's theorem: $\operatorname{rk} G * H = \operatorname{rk} G + \operatorname{rk} H$, where the rank is defined to be the minimum number of generators.

Theorem 3.10 (Kneser conjecture, proved by Stallings). *Let M be an oriented 3-manifold such that $\pi_1(M) = G_1 * G_2$. Then, there are oriented 3-manifolds N_1 and N_2 such that $M = N_1 \# N_2$ and $\pi_1(N_i) \cong G_i$ for $i \in \{1, 2\}$.*

For this theorem, it suffices to study irreducible 3-manifolds.

4. Loop and sphere theorems

4.1 Loop theorem

Theorem 4.1 (Loop theorem). *For a subsurface $S \subset M$, there is a continuous embedding $f : (D^2, \partial D^2) \to (M, S)$ such that $f|_{\partial D^2} \neq$ null homotopic in S. In particular, we may assume f is smooth.*

Remark. The loop theorem holds also for noncompact or nonorientable 3-manifolds.

Corollary 4.2 (Dehn's lemma). *Every loop in ∂M null homotopic in M bounds some embedded disk in M.*

Proof. Apply the loop theorem for the tubular neighborhood of the loop in ∂M. □

Theorem 4.3. *Let $K \subset S^3$ be a knot and $E_K := S^3 \setminus \nu(K)$. Then, K is unknot if and only if $\pi_1(E_k) \cong \mathbb{Z}$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Take a longitude $\lambda \in \pi(\partial E_K) = H_1(\partial E_K)$ such that

$$\lambda \in \ker(\pi_1(\partial E_K) \to H_1(E_K)) \subset \iota_*^{-1}([\pi_1(E_K), \pi_1(E_K)]),$$

where $\iota : \partial E_K \hookrightarrow E_K$. Then, $\pi_1(E_K) \cong \mathbb{Z}$ implies $\lambda = 1$, and apply Dehn's lemma. □

Exercise 6. Let M be an irreducible 3-manifold and $T \subset M$ a torus. Show that the followins are equivalent:

(i) $\pi_1(T) \to \pi_1(M)$ is not injective.

(ii) T bounds some embedded solid torus $D^2 \times S^1$ in M or T is contained in some embedded ball in M.

5 Day 5: May 16

Theorem 4.1 (Loop theorem, Papakyriakopoulos). *For a subsurface S of 3-manifold M, if there is a continuous map $f : (D^2, \partial D^2) \to (M, S)$ such that $f|_{\partial D^2}$ is not null-homotopic, then there is a smooth embedding $g : (D^2, \partial D^2) \to (M, S)$ such that $g|_{\partial D^2}$ is not null-homotopic.*

Proof. Step 1: *Tower construction*. Pick triangulations of (M, S) and D^2.

Let $f_0 : (D^2, \partial D) \to (M, S)$ be a simplicial map such that $f_0 \simeq f$. Define $V_0 \subset M$ to be a 3-submanifold deformation retracting to $f_0(D^2)$ and $S_0 \subset \partial V_0$ a component of $S \cap V_0$ containing $f_0(\partial D^2)$.

Given the i-th level, for any double covering $p_{i+1} : M_{i+1} \to V_i$, if it exists, let $f_{i+1} : (D^2, \partial D^2) \to (M_{i+1}, p_{i+1}^{-1}(S_i))$ be a lift of f_i. Similarly as the case $i+1 = 0$, define $V_{i+1} \subset M_{i+1}$ to be a 3-manifold deformation retracting to $f_{i+1}(D^2)$ and $S_{i+1} \subset \partial V_{i+1}$ the component of $p_{i+1}^{-1}(S_i) \cap V_{i+1}$ containing $f_{i+1}(\partial D^2)$.

$$\begin{tikzcd}
& V_n \arrow[r, hook] & M_n \arrow[dl, "p_n"] \\
& \vdots & \vdots \arrow[dl, "p_2"] \\
& V_1 \arrow[r, hook] & M_1 \arrow[dl, "p_1"] \\
D^2 \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uuur, "f_n"] & V_0 \arrow[r, hook] & M_0 = M
\end{tikzcd}
\qquad
\begin{tikzcd}
& S_n \arrow[r, hook] \arrow[d] & \partial V_n \\
& \vdots \arrow[d] & \\
& S_1 \arrow[r, hook] \arrow[d] & \partial V_1 \\
\partial D^2 \arrow[r] \arrow[ur] \arrow[uuur] & S_0 \arrow[r, hook] & \partial V_0
\end{tikzcd}$$

Since

$$\#\text{of simplices of } f_0(D^2) < \#\text{of simplices of } f_1(D^2) < \cdots \le \#\text{of simplices of } D^2,$$

there is n such that V_n has no double cover. If we let $N_i := \ker(\pi_1(S_i) \to \pi_1(S))$, then $f_{i*}([\partial D^2]) \notin N_i$ implies that $N_i \neq \pi_1(S_i)$ for each i.

Step 2: *Embedding on the top*. Consider the following exact sequence:

$$H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) \to H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) \to H_1(V_n; \mathbb{Z}/2\mathbb{Z}).$$

Since V_n has no double cover, the Poincaré duality writes

$$H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) \cong H^1(V_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(V_n), \mathbb{Z}/2\mathbb{Z}) = 0.$$

Also we have

$$H_1(V_n; \mathbb{Z}/2\mathbb{Z}) \cong H_1(V_n) \otimes \mathbb{Z}/2\mathbb{Z} = 0,$$

which implies $H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) = 0$ so that every component of ∂V_n is a sphere.

Note that $S_n \approx S^2 \setminus (\text{disks})$ and $\pi_1(S_n)$ is generated by components of ∂S_n. Since $N_n \neq \pi_1(S_n)$, there is a component $l \subset \partial S_n$ such that $[l] \notin N_n$. Pick

Step 3: *Descending the tower*. We shall show that given an embedding $g_i : (D^2, \partial D^2) \to (V_i, S_i)$ such that $g_{i*}([\partial D^2]) \notin N_i$, there is such an embedding g_{i-1}. Then, g_0 is the desired embedding.

Isotope $p_i \circ g_i$ so that self-intersections are simple double curves. These are modified by the following operations which give g_{i-1}.

Case 2 double arces Therse are cancelled in two ways and at least one of the results satisfies $g_{i-1}([\partial D^2]) \notin N_{i-1}$. $\square$

Let X be a path-connected space and $x_0 \in X$. Then,

$$\pi_n(X) := \pi_n(X, x_0) = \{f : (S^n, *) \to (X, x_0)\}/\text{homotopy fixing base points.}$$

For a covering $p : \widetilde{X} \to X$, the induced homomorphism $p_* : \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for $n \ge 2$. We will use the following two theorems without proofs

Theorem (Whitehead theorem). *Let X, Y be CW-complexes. If $\varphi : X \to Y$ satisfies $\varphi_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for each n, then φ is a homotopy equivalence.*

Theorem (Hurewicz theorem). *For, $h_n : \pi_n(X) \to H_n(X) : [f] \mapsto f_*([S_n])$, h_1 is the abelianization, and h_{n+1} is an isomorphism if $\pi_i(X) = 0$ for $1 \leq i \leq n$.*

Theorem 4.4 (Sphere theorem, Papakyriakopoulos). *For a 3-manifold M with $\pi_2(M) \neq 0$, there exists an embedding $f : S^2 \to M$ such that $[f] \neq 0$.*

Corollary 4.5. *If M is irreducible, then $\pi_2(M) = 0$.*

Theorem 4.6. *Let M be an irreducible 3-manifold with infinite fundamental group.*

(a) *M is aspherical, i.e. $\pi_n(M) = 0$ for $n \geq 2$.*

(b) *$\pi_1(M)$ is torsion-free.*

It is well-known that a manifold M is aspherical if and only if $M \simeq K(\pi_1(M), 1)$.

Lemma 4.7. *Let $m \geq 2$. Just use the identity $H_*(\pi) = H_*(K(\pi, 1))$ as the definition of group homology. Then,*

$$H_n(\mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/m\mathbb{Z} & n : \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $K(\mathbb{Z}/m\mathbb{Z}, 1) \simeq S^\infty/(\mathbb{Z}/m\mathbb{Z})$.

Proof of Theorem 4.6. (a) By Corollary 4.5, we have $\pi_2(M) = 0$. If we let $\widetilde{M}$ be the universal cover of M, which is not compact because $\pi_1(M)$ is infinite, so that we have

$$\pi_3(M) = \pi_3(\widetilde{M}) = H_3(\widetilde{M}) = 0.$$

Inductively, and since $\dim M = 3$, we have for $n \geq 4$ that

$$\pi_n(M) = \pi_n(\widetilde{M}) = H_n(\widetilde{M}) = 0.$$

(b) Suppose that there is a non-trivial finite cyclic subgroup $C \leq \pi_1(M)$. If we let $\widehat{M}$ be the cover of M corresponding to C, then $\pi_1(\widehat{M}) = C$. It follows from (i) that $\widehat{M} = K(C, 1)$. By Lemma 4.7, we have $H_n(\widehat{M}) \neq 0$ for odd n, but $\dim \widehat{M} = 3$ implies that $H_n(\widehat{M}) = 0$ for $n \geq 4$. □

Theorem 4.8. *The only abelian groups appearing as the fundamental group of closed (orientable) 3-manifolds are $1, \mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}/n\mathbb{Z}$.*

Remark. If we also consider non-orientable 3-manifolds, then $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ also appears as $\pi_1(\mathbb{RP}^2 \times S^1)$.

Exercise 7. Show that the only abelian groups admitting presentations of deficiency zero are

$$1, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

(Hint: use Hopf's formula $H_2(\pi) = R \cap [F, F]/[F, R]$, where $\pi = F/R$.)

Proof of Theorem 4.8. By Corollary 2.7 and Exercise 7, it suffices to show $\pi_1(M) \ncong \mathbb{Z}^2, \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. By the prime decomposition theorem, we may assume that M is irreducible. By the part (b) of Theorem 4.6, $\pi_1(M) \ncong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Now suppose that $\pi_1(\mathbb{Z}) \cong \mathbb{Z}^2$.

Since M is aspherical by Theorem 4.6, $M \cong K(\mathbb{Z}^2, 1) \simeq \mathbb{T}^2$. However, $\mathbb{Z} = H^3(M) \cong H^3(\mathbb{T}^2) = 0$ leads to a contradiction. □

6 Day 6: May 23

5. Haken manifolds

In the following, we only consider *irreducible* 3-manifolds, i.e. every embedded sphere bounds a ball.

5.1 Essential surfaces

When we say subsurfaces S, we always assume $S \cap \partial M = \partial S$.

A subsurface $S \subset M$ which is not a sphere is called *incompressible* if there is no (non-trivial) compressing disk. A *compressing disk* of a subsurface S is a disk $D \subset M$ such that $D \cap S = \partial D$ and ∂D bounds no disk in S. The second condition intuitively says that the compression does not generate spheres so that the disk compresses S non-trivially.

A subsurface $S \subset M$ which is not a sphere is called *boundary incompressible* if there is no (non-trivial) boundary compressing disks. A *boundary compressing disk* of S is a disk D such that arcs $\gamma = D \cap S$ and $\gamma' = D \cap \partial M$ form the boundary $\partial D = \gamma \cup_\partial \gamma'$ and $\gamma \cup_\partial \gamma''$ does not bound a disk in S for any arc $\gamma'' \subset \partial S$.

A surface S is called *boundary parallel* if S can be isotoped into ∂M.

A (possibly disconnected) subsurface $S \subset M$ is called an *essential surface* if every component S_0 of S is incompressible, boundary incompressible, and not boundary parallel. (S_0 must not be a sphere)

Proposition 5.1. *For a subsurface $S \subset M$, S is incompressible if and only if $\pi_1(S) \to \pi_1(M)$ is injective.*

Proof. The loop theorem. $\square$

Proposition 5.2. *For nonzero $\alpha \in H_2(M, \partial M)$, there is an essential surface $S \subset M$ with $[S] = \alpha$.*

Let S be a possibly disconnected subsurface of M. We define a complexity $c(S) := \sum_i (2 - \chi(S_i))^2$, where S_i are components of S.

Lemma 5.3. *If S' is obtained from S by the surgery along (boundary) compressing disks, then $c(S') < c(S)$.*

Proof. We may assume S is connected. Note that every component has $\chi < 2$ since we are not allowing spheres after surgery. If S' is connected, then since $\chi(S') = \chi(S) + 2$, we are done. If $S' = S_1 \sqcup S_2$, then since $\chi(S_1) + \chi(S_2) = \chi(S') = \chi(S) + 2$, we are done. $\square$

Proof of Proposition 5.2. Pick a (possibly disconnected) subsurface $S_0 \subset M$ with $[S_0] = \alpha$. (Choose $f : M \to S^1$ so that $[f] \in [M, S^1] = H^1(M)$ is equal to the Poincaré dual of α and let $S_0 := f^{-1}$(a regular value) Perform the surgery S_0 along (boundary) compressing disks until every components of S_0 becomes (boundary) incompressible. By Lemma 5.3, this operation stops, and before and after of the compression are homologous. Removing all spheres and boundary parallel components of S_0, now we have a desired subsurface S with $[S] = [S_0] = \alpha$. Since $\alpha \neq 0$, $S \neq \emptyset$. $\square$

A 3-manifold is a *Haken manifold* if it admits an essential surface in M.

Corollary 5.4. *If $b_1(M) > 0$, then M is Haken.*

Proof. It follows from Proposition 5.2. $\square$

Corollary 5.5. *If there is a non-sphere component of ∂M, then M is Haken.*

Proof. Since $H_1(\partial M)$ contains an element of infinite order, the exact sequence

$$H^1(M) \cong H_2(M, \partial M) \to H_1(\partial M) \to H_1(M)$$

implies $b_1(M) > 0$. $\square$

Theorem 5.6 (Thurston). *For the figure eight knot, the Dehn filling by a rational number $r = p/q$ is Haken if and only if $r = 0, \pm 4$.*

5.2 Haken hierarchy

Proposition 5.7. *Let $S \subset M$ be an essential surface. Let $M' = M \setminus \nu(S)$. Then, (a) M' is irreducible and (b) every closed incompressible surface in M' is incompressible also in M.*

Proof. (a) Let $S' \subset M'$ be arbitrary sphere. Since M is irreducible, S' bounds a ball $B \subset M$. Then, we have $S \subset B$ or $S \cap B = \varnothing$. Since B contains no closed incompressible surface, $B \cap S = \varnothing$, so $B \subset M'$.

(b) Suppose that there is a closed incompressible surface $S' \subset M'$ not incompressible in M. Let $D \subset M$ be a compressing disk for S' such that $D \pitchfork S$ and $\#\pi_0(D \cap S)$ is minimal. Since S is essential, an innermost loop in $S \cap D \subset S$ bounds a disk $D' \subset S$ (if we take any compressible disk of S in M whose boundary is $S \cap D$, then the assumption S is essential implies that the compression gives a sphere; we can take a disk D'). Since M is irreducible, $D \cup D'$ bounds a ball in M, so D can be isotoped such that $\#\pi_0(D \cap S)$ would reduce, which is a contradiction. □

Proposition 5.8. *If $\partial M \neq \varnothing$ and M contains no closed incompressible surface, then M is a handlebody.*

Proof. Cut M along essential disks into $\coprod_i M_i$ until every component of ∂M_i is incompressible. Suppose that there is i_0 such that $M_{i_0} \not\approx D^3$. By Proposition 5.7(a), every component of ∂M_{i_0} is not a sphere. By Proposition 5.7(b), the componenets of ∂M_{i_0} are closed incompressible surfaces in M which contradicts the assumption. Therefore, for every i we have $M_i \approx D^3$, M is a handlebody. □

Theorem 5.9 (Haken hierarchy)**.** *For every Haken manifold M, there is a sequence of irreducible (possibly disconnected) 3-manifolds $M = M_0, M_1, \cdots, M_n$ such that*

(a) *$M_{i+1} = M_i \setminus \nu(S_i)$ for some essential surface $S_i \subset M_i$*

(b) *$M_n \approx \coprod D^3$.*

The strongest feature of the Haken hierarchy is the possibility of using induction.

Theorem 5.10 (Waldhausen)**.** *Two closed Haken manifolds with isomorphic fundamental groups are diffeomorphic.*

Proof. Induction on the Haken hierarchy. □

Remark. When $\partial M \neq \varnothing$, the theorem also holds under a certain additional condition on isomorphisms on π_1.

7 Day 7: June 6

Exercise 8. Show that every connected essential surface in $\Sigma_g \times [0,1]$ ($g \geq 1$) is isotopic to an anulus $\gamma \times [0,1]$ for some simple closed curve γ bounding no disk in Σ_g.

Lemma 5.11. *Let M be an irreducible 3-manifold. Let $\partial M = \coprod_i R_i$, where R_i are components which are not diffeomorphic to sphere. Then, there is $\alpha \in H_2(M, \partial M)$ such that for every i we have $\partial \alpha|_{R_i} \neq 0 \in H_1(R_i)$.*

Exercise 9. Prove Lemma 5.11.

Proof of Theorem 5.9. Note that the theorem holds for handlebodies H_g. We may assume $M \not\approx H_g$. By Proposition 5.8, M contains a union of closed incompressible surfaces S. By Proposition 3.7 and 3.8, S is isotopic to a normal surface with respect to some triangulation of M, and if no component of $M \setminus \nu(S)$ is a trivial $[0,1]$-bundle, then $\pi_0(S)$ is bounded.

Take $S_0 \subset M$ with maximal fundamental group. Let $M_1 = M \setminus \nu(S_0)$. Let R_i are components of ∂M_1 such that $R_i \not\approx S^2$. By Corollary 5.5 and Proposition 5.7(a), M_1 is Haken. By Lemma 5.11, pick $\alpha \in H_2(M, \partial M)$ such that $\partial \alpha|_{R_i} \neq 0 \in H_1(R_i)$. By Proposition 5.2, pick an essential surface $S_1 \subset M$ with $[S_1] = \alpha$. Since $\pi_0(S_0)$ is maximal, there is a component M' of $M \setminus (S_0 \sqcup S')$ is a trivial $[0,1]$-bundle. Since $\partial \alpha|_{R_i} = [S_1 \cap R_i] \neq 0$, some component of S_1 intersects only one component of $\partial M'$. However, there is not such an essential in $M' \approx S' \times [0,1]$. Then, constradiction(Exercise 8). $\square$

6. Seifert manifolds

6.1 Seifert fibrations

Let S be a (possibly non-orientable) surface. Let $p_i, q_i \in \mathbb{Z}$, $p_i \geq 1$, $(p_i, q_i) = 1$, $1 \leq i \leq k$.

If orientable, then $\times^{(\sim)} = \times$, and if non-orientable, then $\times^{(\sim)} = \widetilde{\times}$, the twisted direct product.

$$M(S;(p_1,q_1),\cdots,(p_k,q_k)) := \left((S \setminus \coprod_{i=1}^{k} D_i) \times^{(\sim)} S^1 \right) \cup_{h_1,\cdots,h_k} \coprod_{i=1}^{k} (D^2 \times S^1),$$

where $h_i : \partial(D^2 \times S^1) \to \partial(D_i \times S^1)$ is a diffeomorphism such that

$$[h(\partial D^2 \times \{*\})] = p_i[\partial D_i \times \{*\}] + q_i[\{*\} \times S^1] \in H_1(\partial D_i \times S^1).$$

For example,

- $M(S;(1,0)) \approx S \times^{(\sim)} S^1$.
- $M(D^2;(p,q)) \approx D^2 \times S^1$.
- $M(S^2;(p,q)) \approx L(q,p)$.
- $M(S^2;(p_1,q_1),(p_2,q_2)) \approx L(p_1q_2 + p_2q_1, q_2r + p_2s$, where $p_1s - q_r = \pm 1$.
- $M(D^2;(p,q)) = D^2 \times [0,1]/(r,\theta,1) \sim (r,\theta + \frac{2\pi q}{p}, 0)$.

Note that we have $M(D^2;(p,q)) \to D^2/(r,\theta) \sim (r,\theta + \frac{2\pi n}{p})$ for $n \in \mathbb{Z}$. As a generallization of this, we have the *Seifert fibration*

$$M(S;(p_1,q_1),\cdots,(p_k,q_k)) \to S(p_1,\cdots,p_k),$$

where $S(p_1,\cdots,p_k)$ is an orbifold constructed from S with cone points of order $p_1,\cdots,p_k$. A *singular fiber* is the fiber of a cone point of order ≥ 2.

Remark. Every Seifert fibration without any singular fiber is a S^1-bundle over S.

Proposition 6.1. *Every S^1-bundle over S with orientable total space M is a Seifert fibration.*

Proof. If $\partial S \neq \emptyset$, then $M \cong S \times^{(\sim)} S^1$. If $\partial S = \emptyset$, then $M \setminus \nu(\text{fiber}) \cong (S \setminus (\text{open disk})) \times^{(\sim)} S^1$. Thus there is $e \in \mathbb{Z}$ such that $M \cong M(S;(1,e))$. □

Exercise 10. Show that if $\partial S = \emptyset$ and $e, e' \in \mathbb{Z}$, then $M(S;(1,e)) \cong M(S;(1,e'))$ if and only if $e = \pm e'$.

Let $\pi : M(S;(p_1,q_1),\cdots,(p_k,q_k)) \to S(p_1,\cdots,p_k)$ be a Seifert fibration. The *Euler number* of π is

$$e(\pi) := \sum_{i=1}^{k} \frac{q_i}{p_i},$$

which is set to sit in $\mathbb{Q}$ if $\partial S = \emptyset$, and in $\mathbb{Q}/\mathbb{Z}$ if $\partial S \neq \emptyset$.

Theorem 6.2. *Two Seifert fibrations*

$$\pi : M(S;(p_1,q_1),\cdots,(p_k,q_k)) \to S(p_1,\cdots,p_k)$$

and

$$\pi' : M(S;(p_1,q_1'),\cdots,(p_k,q_k')) \to S(p_1,\cdots,p_k)$$

with $p_i \geq 2$ for all i are isomorphic by an orientation preserving diffeomorphism if and only if the following two conditions hold:

(i) $q_i \equiv q_i' \pmod{p_i}$ *for all i,*

(ii) $e(\pi) = e(\pi')$.

Remark. We write $M(S;(p_1,-q_1),\cdots,(p_k,-q_k)) = -M(S;(p_1,q_1),\cdots,(p_k,q_k))$.

This theorem follows from the following proposition.

Proposition 6.3. *Isomorphism by an orientation preserving diffeomorphism is obtained by a finite number of the following moves:*

(i) $(p_i,q_i),(p_j,q_j) \longleftrightarrow (p_i,q_i+p_i),(p_j,q_j-p_j)$,

(ii) $(p_i,q_i) \longleftrightarrow (p_i,q_i \pm p_i)$ *if $\partial S \neq \emptyset$.*

(iii) $(1,0) \longleftrightarrow \emptyset$.

Proof. First we prove the moves generate orientation preserving diffeomorphisms. Let $A \subset M$ be a fibered annulus connecting $M(D^2;(p_i,q_i))$ and $M(D^2;(p_j,q_j))$(or resp. ∂M). Then, moves (i) (and resp.!(ii)) correspond to the 3-dimensional version of twist along A. The move (iii) inserts or eliminates $M(D^2,(1,0)) \approx D^2 \times S^1$.

Now we show the converse. First we can reduce the regular fibers $p_i = 1$ with three kinds of moves.

Case (a): no singular fiber. If $\partial S = \emptyset$, then we have $M(S;(1,e))$ with the same e(Exercise 10). Otherwise $\partial S \neq \emptyset$, then we have $S \times^{(\sim)} S^1$.

Case (b): singular fiber. We have $p_i \geq 2$ for all i. Let $\gamma_1,\cdots,\gamma_n \subset S$ disjoint arcs connecting cone points or ∂S, cutting S into a disk D. Note that

$$M(S;(p_1,q_1),\cdots,(p_k,q_k)) = \coprod_{i=1}^{k} M(D^2;(p_i,q_i)) \cup \coprod_{j=1}^{n} \overline{\nu(A_j)} \cup D^2 \times S^1,$$

where $A_j := \gamma_j \times S^1$. Every isomorphism by an orientation preserving diffeomorphism is determined by mapping classes on $\overline{\nu(A_j)}$ since it sends singular fibers to themselves. Such a mapping class is some product of the twist along A_j, which corresponds the moves (i) or (ii). □

6.2 Classifications