Characterizations of Harmonic Functions Vanishing at Infinity on the Punctured Domain

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Theorem. *Let $d \geq 3$. A distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity if and only if there is a distribution $\rho \in \mathcal{D}'(\mathbb{R}^d)$ such that $u = \Phi * \rho$ and $\operatorname{supp}(\rho) \subset \{0\}$, where Φ denotes the fundamental solution of Laplace's equation.*

Proof. ($\Rightarrow$) Define a distribution ρ by

$$\langle \rho, \varphi \rangle := -\langle u, \Delta\varphi \rangle$$

for $\varphi \in C_c^\infty(\mathbb{R}^d)$. In other words, $\rho = -\Delta u$ in distributional sense. Then, ρ has the support contained in $\{0\}$ because if $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ then

$$\langle \rho, \varphi \rangle = -\langle u, \Delta\varphi \rangle = -\int u(x)\Delta\varphi(x)\,dx = -\int \Delta u(x)\varphi(x)\,dx = 0.$$

Therefore, we only need to verify $u = \Phi * \rho$ to complete the proof.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Be cautious that the argument

$$\langle \Phi * \rho, \varphi \rangle = \langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle = \langle u, \varphi \rangle$$

fails to provide a proof because the function $\Phi*\rho$ is not compactly supported so that we cannot deduce $\langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$, and here we use the condition that u vanishes at infinity to justify the equality. Define a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{5}{4} \\ 0 & \text{if } |x| > \frac{7}{4} \end{cases}.$$

If we denote $\chi_r(x) := \chi(\frac{x}{r})$, then we have

$$\langle \rho, (\Phi\chi_r) * \varphi \rangle = -\langle u, \Delta((\Phi\chi_r) * \varphi) \rangle$$

by the definition of ρ. We have the limit of the left-hand side

$$\lim_{r\to\infty}\langle\rho,(\Phi\chi_r)*\varphi\rangle=\langle\rho,\Phi*\varphi\rangle$$

because

$$\begin{aligned}\operatorname{supp}((\Phi(1-\chi_r)*\varphi)&\subset\operatorname{supp}(\Phi(1-\chi_r))+\operatorname{supp}(\varphi)\\&\subset\mathbb{R}^d\setminus B(0,2R)+\overline{B}(0,R)=\mathbb{R}^d\setminus B(0,R)\end{aligned}$$

for all $r>2R$ so that the supports of $\Phi(1-\chi_r)*\varphi$ and ρ are disjoint, where we define $R:=\sup_{x\in\operatorname{supp}(\varphi)}|x|$. However, the right-hand limit

$$-\lim_{r\to\infty}\langle u,\Delta((\Phi\chi_r)*\varphi)\rangle=-\langle u,\Delta(\Phi*\varphi)\rangle$$

is not a trivial result.

Assuming $\chi(x)=\chi(-x)$ without loss of generality, we have

$$\langle u,\Delta(\Phi(1-\chi_r)*\varphi)\rangle=\langle u*\Delta(\Phi(1-\chi_r)),\varphi\rangle.$$

Because

$$\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]=0$$

for $|y|<R$ and $x\in\operatorname{supp}(\varphi)$ if $r>2R$, we can write

$$\langle u*\Delta(\Phi(1-\chi_r)),\varphi\rangle=\int\varphi(x)\int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]dy\,dx.$$

We compute

$$\begin{aligned}\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]&=2\nabla\Phi(x-y)\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r})-\Phi(x-y)\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r})\\&=-\frac{2}{\omega_d}\frac{x-y}{|x-y|^d}\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r})-\frac{1}{(d-2)\omega_d}\frac{1}{|x-y|^{d-2}}\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r}).\end{aligned}$$

Then, since $\frac{5}{4}r\le|x-y|\le\frac{7}{4}r$ if $\nabla\chi(\frac{x-y}{r})\neq0$ and $\Delta\chi(\frac{x-y}{r})\neq0$, we obtain

$$\left|\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\right|\le C\frac{1}{r^d}\psi(\frac{x-y}{r})$$

for some constant $C>0$, where

$$\psi(y):=|\nabla\chi(y)|+|\Delta\chi(y)|.$$

For each $x \in \operatorname{supp}(\varphi)$, since we have $\frac{5}{4}r \leq |x-y| \leq \frac{7}{4}r$ implies $r \leq |y| \leq 2r$ if $r > 4R$, it follows that

$$
\begin{aligned}
\Big|\int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\,dy\Big| &\leq C\int |u(y)\frac{1}{r^d}\psi(\frac{x-y}{r})|\,dy \\
&\leq C \max_{r\leq|y|\leq 2r} u(y)
\end{aligned}
$$

converges to zero as $r \to \infty$. By the bounded convergence theorem, we can deduce

$$\lim_{r\to\infty}\int \varphi(x)\int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\,dy\,dx = 0,$$

so we are done.

($\Leftarrow$) Let $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Since

$$\langle \Phi * \rho, \Delta\varphi\rangle = \langle \rho, \Phi * (\Delta\varphi)\rangle = \langle \rho, \varphi\rangle = 0,$$

the distribution $\Phi * \rho$ on $\mathbb{R}^d \setminus \{0\}$ is weakly harmonic, and by Weyl's lemma for distributions, it is a smooth harmonic function on $\mathbb{R}^d \setminus \{0\}$.

Since ρ is supported at zero, we have a positive integer k and constants a_α such that

$$|\langle \rho, \varphi\rangle| \leq \sum_{|a|\leq k} |a_\alpha D^\alpha \varphi(0)|$$

for $\varphi \in C^\infty(\mathbb{R}^d)$. Then, for $x \in \mathbb{R}^d$ with $|x| = r > 0$, by taking $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on $B(0, 2r)$, we have

$$|\Phi*\rho(x)| = |(\Phi\chi)*\rho(x)| = |\langle \rho(y), \Phi(x-y)\chi(x-y)\rangle_y| \leq \sum_{|a|\leq k} |a_\alpha D^\alpha \Phi(x)| = O(r^{2-d})$$

as $r \to \infty$. Therefore, $\Phi * \rho$ vanishes at infinity. $\square$

Lemma. *Let ρ be a distribution on $\mathbb{R}^d$ such that* $\operatorname{supp}(\rho) \subset \{0\}$. *Then, there is a constant coefficient partial differential operator $P(D)$ such that $\rho = P(D)\delta$.*

Corollary. *Let $d \geq 3$. If a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity, then there are an integer $k \geq 0$ and constants a_α such that*

$$u(x) = \sum_{|a|\leq k} a_\alpha D^\alpha \Phi(x)$$

for $x \neq 0$, where Φ denotes the fundamental solution of Laplace's equation.