Contents

1 Topological group action

1.1 Discontinuous action

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1.1. Let G be a topological group acting on a topological space X. Let $p : X \to X/G$ be the quotient map.

(a) $p^{-1}(p(A)) = \bigcup_{g\in G} gA$ for any $A \subset X$.

(b) p is open.

(c) If $x \neq gx$, then there is an open neighborhood U of x such that gU is disjoint to U.

Proof. (c) Since X is Hausdorff, there is disjoint open neighborhoods U_0 and U_1 respectively of x and gx. Then, $U := g^{-1}(gU_0 \cap U_1) \subset U_0$ and $gU = gU_0 \cap U_1 \subset U_1$ are disjoint. □

1.2 (Proper maps)**.** Let $f : X \to Y$ be continuous. We say f is *proper* if $f^{-1}(K)$ is compact for every compact K. We say f is *Bourbaki-proper* if it is closed and proper. If X is Hausdorff and Y is locally compact, then two notions are equivalent. For this we only need to prove a proper map is closed.

Proof. □

1.3 (Proper actions)**.** Let $G \times X \to X : (g,x) \mapsto gx$ be a continuous group action.

(i) The *shear map* $s : G \times X \to X \times X : (g,x) \mapsto (x, gx)$ is and proper. (Bourbaki properness)

(ii) For every compact $K \subset X$, $\{g \in G : gK \cap K \neq \varnothing\}$ is compact. (Borel properness)

(iii) Every $x, y \in X$ have open neighborhoods U_x, U_y such that $\{g \in G : gU_x \cap U_y \neq \varnothing\}$ is relatively compact. (Wandering property)

(a) (i) implies (ii).

(b) (ii) implies (i) if X is Hausdorff.

(c) (i) implies (iii) if X is locally compact and Hausdorff.

(d) (iii) implies (i) if G is locally compact and Hausdorff.

Proof. Write $\pi_G : G \times X \to G : (g,x) \mapsto g$. Then, for $g \in G$ and subsets $A, B \subset X$, we can see

$$\{g \in G : gA \cap B \neq \varnothing\} = \pi_G(s^{-1}(A \times B)).$$

We note that (i) holds if and only if for every compact $K \subset X$ the set $s^{-1}(K \times K)$ is compact, and that (ii) holds if and only if for every compact $K \subset X$ the set $\pi_G(s^{-1}(K \times K))$ is compact.

(a) Clear.

(b) A compact set K is closed in X. Then,

$$s^{-1}(K \times K) \subset \pi_G(s^{-1}(K \times K)) \times K$$

implies that $s^{-1}(K \times K)$ is closed in a compact set, so we have (i). □

half disk in Euclidean half plane has compact completion but not compact closure If a subset of a topological Hausdorff group has compact completion, then it has compact closure..?

Properties of proper actions:

(a) The orbit space X/G is Hausdorff.

(b) Every orbit is closed.

(c) Every stabilizer is compact

(d) The orbit-stabilizer bijection is a homeomorphism.

(e) If X is locally compact and Hausdorff, then so are G and X/G.

(f) If X is compact and Hausdorff, then so are G and X/G.

1.4 (Properly discontinuous actions)**.** Let $\Gamma \times X \to X : (g, x) \mapsto gx$ be a continuous group action.

(i) Γ is discrete.

(ii) A family of singleton subsets of an orbit of the action $\Gamma \times X \to X$ is locally finite.

(iii) For every compact $K \subset X$, $\{g \in \Gamma : gK \cap K \neq \varnothing\}$ is finite.

(a)

(b) (iii) implies (i) if the stabilizer is finite..?

Proof. □

1.5 (Covering space actions)**.** Let $G \times X \to X : (g, x) \mapsto gx$ be a continuous group action. Let $p : X \to X/G$ be the quotient map. This action is called a *covering space action* if every $x \in X$ has a neighborhood U such that gU are all disjoint for $g \in G$.

(a) A properly discontinuous and free action is a covering space action, if X is locally compact and Hausdorff.

(b) A covering space action is properly discontinuous.

(c) A covering space action is free.

Proof. (a) Fix $x \in X$ and let K be a compact neighborhood of x. By the proper discontinuity, there is a finite subset $F \subset G$ such that gK intersects K only for $g \in F$. Because the action is free, for every $g \in F \setminus \{1\}$ there is an open neighborhood U_g of x such that $gU_g \cap U_g = \varnothing$. Then, $U := K^\circ \cap \bigcap_{g \in F \setminus \{1\}} U_g$ satisfies $gU \cap U = \varnothing$.

(b)

□

1.2 Fundamental domain

1.6 (Fundamental domain)**.** Let G be the group of isometries of a metric space X. Let Γ be a discrete subgroup of G. An open set $D \subset X$ is called a *fundamental domain* of Γ if

(i) $\{g(D) : g \in \Gamma\}$ are pairwise disjoint,

(ii) $\{g(\overline{D}) : g \in \Gamma\}$ covers X.

1.7 (Sides of Dirichlet domain)**.** Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0. A *side* of a polytope is a non-empty maximal convex subset of its boundary.

(a) For $g \in \Gamma \setminus \{e\}$, the set $g(\overline{D}) \cap \overline{D}$ has the three cases: the null set, one point, or a geodesic segment.

(b) If s is a side of $\overline{D}$, then there is unique $g \in \Gamma \setminus \{e\}$ such that $s = g(\overline{D}) \cap \overline{D}$.

(c) The intersection of two distinct edges is one point or the null set.

(d) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{g \in \Gamma \setminus \{e\}} g(\overline{D}) \cap \overline{D}.$$

(e) We have
$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{s:\text{ side}} s.$$

Proof. (d) Let $z \in \partial D \cap \mathbb{H}^2$. Since $d(z, z_0) \le d(z, gz_0)$ for all $g \in \Gamma \setminus \{e\}$ but $d(z, z_0) \ge d(z, gz_0)$ for some $g \in \Gamma \setminus \{e\}$, there is $g \in \Gamma \setminus \{e\}$ such that $d(z, z_0) = d(z, gz_0)$. By sending z_0 and gz_0 to $\pm 1 + i$ with an isometry so that z is sended to a point on a imaginary axis, we can check for each n that we have $B(z, 1/n) \cap (\mathbb{H}^2 \setminus \overline{D}) \ne \varnothing$. Since $B(z, 1/n) \setminus \overline{D}$ is a non-empty open set in $\mathbb{H}^2 \setminus \overline{D}$, and since
$$\mathbb{H}^2 \setminus \overline{D} \subset \mathbb{H}^2 \setminus D = \overline{\bigcup_{g \in \Gamma \setminus \{e\}} g(D)},$$
we can deduce that $B(z, 1/n)$ intersects with $g(D)$ for some $g \in \Gamma \setminus \{e\}$.

Combining this result with the local finiteness of $\{g(D) : g \in \Gamma\}$, the sequence of sets
$$\{\, g \in \Gamma \setminus \{e\} : B(z, 1/n) \cap g(D) \ne \varnothing \,\}$$
indexed by n consists of non-empty finite subsets of $\Gamma \setminus \{e\}$ that are non-increasing. By the pigeonhole principle, there exists $g \in \Gamma \setminus \{e\}$ such that $B(z, 1/n) \cap g(D) \ne \varnothing$ for all n, which allows to extract a sequence $z_n \in g(D)$ that converges to z, which implies $z \in g(\overline{D})$.

(e) Suppose $z \in \partial D \cap \mathbb{H}^2$ is not contained in any edges. Let Z be the set of all $g \in \Gamma \setminus \{e\}$ such that $\{z\} = g(\overline{D}) \cap \overline{D}$. For $g \in \Gamma \setminus (Z \cup \{e\})$, $g(\overline{D}) \cap \overline{D}$ is the null set, one point, or an edge, and any of possibility does not contain z. Therefore,
$$(\partial D \setminus \{z\}) \cap \mathbb{H}^2 = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \mathbb{H}^2$$
by the part (d). Change the restriction $\mathbb{H}^2$ to a compact ball as
$$(\partial D \setminus \{z\}) \cap \overline{B(z, 1)} = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \overline{B(z, 1)}.$$
Then, the left-handed side is homeomorphic to $[-1, 0) \cup (0, 1]$ or $(-1, 1)$ since ∂D is homeomorphic to S^1, but the right-handed side is compact because the union becomes finite due to the local finiteness. This is a contradiction, so z is contained in an edge. $\square$

1.8 (Dirichlet domain)**.** Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. Let $z_0 \in \mathbb{H}^n$ be a point that is not fixed by any isometry in $\Gamma \setminus \{e\}$. The *Dirichlet domain* of Γ with *center* z_0 is defined as the set
$$D := \bigcap_{g \in \Gamma \setminus \{e\}} \{\, z \in \mathbb{H}^2 : d(z, z_0) < d(z, gz_0) \,\}.$$
We denote by $\overline{D}$ and ∂D the closure and the boundary of D in $\overline{\mathbb{H}}^2$.

(a) There exists a non-elliptic point in $\mathbb{H}^2$.

(b) $\{g(\overline{D}) : g \in \Gamma\}$ is a locally finite. It is called the *Dirichlet tesselation*.

(c) D is a geodesically convex locally finite fundamental domain of Γ.

Proof. (a) Elliptic points are countably many.

(b) There are finitely many $g \in \Gamma$ satisfying $B(z_0, r) \cap g(\overline{D}) \ne \varnothing$, since this condition implies $gz_0 \in B(z_0, 2r)$. $\square$

1.9 (Convex polytope)**.** See Ratcliffe Section 6.3. Convexity is not really necessary, but it is extremely useful and sufficient in developing the theory of fundamental domains.

Let P be a non-empty closed subset of a metric space X. A *side* of P is a non-empty maximal convex subset of ∂P. We say P is a *polytope* if the set of sides of P is locally finite.

(a) dimension..? vertices, ridges..

(b) property of sides: cover boundary, closed, polytope again etc.

(c) a point in the boundary of a side is in the boundary of another side.

1.10 (Convex fundamental polytope). (a) The closure of a convex and locally finite fundamental domain is a convex polytope.

(b) (side pairing) Suppose P is a convex fundamental polytope of Γ having finitely many sides. Let $v_0, v_1, \cdots, v_n = v_0$ be vertices, indexed along the boundary counterclockwise. Let s_i be the side of P connecting v_i and v_{i+1}.

For each side s of P, there is unique $g_s \in \Gamma$ such that $g_s^{-1}(s)$ is another side of D. The isometry g_s is called the *side pairing isometry* of the side s.

The side parining isometry of $g_s^{-1}(s)$ is g_s^{-1}.

(c) (cycles) Let V and S be the set of all vertices and sides of P, respectively. Define $\sigma : V \to V$ such that $\sigma(v_i) = v_{j+1}$, where $s_j = g_{s_i}^{-1}(s_i)$. The map σ can be seen as an element of the symmetric group S_n.

Suppose $v_0 \in \mathbb{H}^2$ and $s = s_0$. Let m be the minimal positive integer such that $\sigma^m(s) = s$. Then, $g_{\sigma^{m-1}(s)} \cdots g_{\sigma(s)} g_s$ is either the identity or elliptic. Suppose $v_0 \in \partial\mathbb{H}^2$.

1.11 (Examples). (a) (Genus two surface)

(b) (Modular group) Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ be the modular group and choose the origin $2i$ to consider the Dirichlet domain D.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$v_0 := \rho = e^{\pi i/3}, \quad v_1 := \infty, \quad v_2 := \rho^2 = e^{2\pi i/3}.$$

$$g_0 = T, \quad g_1 = T^{-1}, \quad g_2 = S = S^{-1}.$$

$\sigma = (02)$. The elliptic cycle condition: (02) defines $(g_2 g_0)^3 = (ST)^3 = 1$.

2 Hyperbolic plane geometry

2.1 Fuchsian groups and Kleinian groups

Classification of elements. An abelian Fuchsian group is cyclic. Elliptic point is discrete

free action <=> no elliptic element <=> torsion free <=> manifold

2.1 (Finitely generated Fuchsian group). Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0. Let W be the set of all $g \in \Gamma \setminus \{e\}$ such that $g(\overline{D}) \cap \overline{D}$ is a side of D.

(a) W generates Γ.

(b) If Γ is finitely generated, then W is finite.

(c) If W is finite, then Γ is finitely generated.

2.2 (Siegel's theorem). Finite area then finite sides.

(a)

2.2 The Poincaré polygon theorem

2.3 (Side pairing identification)**.** Let P be a convex polygon. Define cycles of each vertex. Let

$$Y := P/\sim, \quad \text{and} \quad \widetilde{Y} := (\Gamma \times P)/\sim.$$

Define $\Pi : \widetilde{Y} \to Y$.

(a)

2.4 (Elliptic cycle condition)**.** Let P be a convex polygon with a side pairing identification. Let Γ be a subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P. Consider D and Π such that

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "D"] \arrow[d, "\Pi"] & \mathbb{H}^2 \\ Y & \end{tikzcd}.$$

(a) P satisfies the elliptic cycle condition.

(b) D is a local homeomorphism.

(c) D is a covering map onto its image.

Proof. (a)⇒(b)

(b)⇒(c) We claim p has the path lifting property, which is unique because it is a local homeomorphism. Let $w : [0,1] \to \operatorname{im} D$, and $\widetilde{w} : [0,\tau) \to \widetilde{Y}$ its maximal extension. Write $\widetilde{w}(t) = [g(t), z(t)]$ and $w(\tau) = gz$. Define $\widetilde{w}(\tau) := [g,z]$. Then,

$$D\widetilde{w}(\tau) = D([g,z]) = gz = w(\tau).$$

Let U be an open neighborhood of $[g,z]$ in $\widetilde{Y}$ such that $D|_U$ is a homeomorphism and $D(U)$ is open in $\mathbb{H}^2$. Then, as $t \to \tau$,

$$p\widetilde{w}(t) = w(t) \to w(\tau) = p\widetilde{w}(\tau)$$

implies

$$\widetilde{w}(t) \to \widetilde{w}(\tau),$$

so $\widetilde{w} : [0,\tau] \to \widetilde{Y}$ is a continuous extension of $w : [0,\tau] \to \mathbb{H}^2$. Therefore, D is a local homeomorphism that has the unique path lifting property, so it is a covering map onto its image. □

2.5 (Finite cycle condition)**.** Let P be a convex polygon with a side pairing identification. Let Γ be a subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P. Consider D and Π such that

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "D"] \arrow[d, "\Pi"] & \mathbb{H}^2 \\ Y & \end{tikzcd}.$$

(a) If every cycle of finite points is finite, then $\operatorname{im} D$ is open.

(b) If every cycle is finite, then there is a metric ρ on Y such that $[z_n] \to [z]$ in ρ if and only if $h_n z_n \to z$ in $\mathbb{H}^2$ for a sequence $h_n \in \Gamma$.

Proof.

$$\rho(x,y) := \inf \sum,$$

$$\inf_{h\in\Gamma} d(h^{-1}z, z') = \rho([z],[z'])$$

□

2.6 (Parabolic cycle condition)**.** Let P be a convex polygon with a side pairing identification. Let Γ be a subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P. Consider D and Π such that

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "D"] \arrow[d, "\Pi"] & \mathbb{H}^2 \\ Y & \end{tikzcd}.$$

Suppose every cycle is finite.

(a) P satisfies the parabolic cycle condition,

(b) M is a complete metric space.

(c) D is surjective.

Proof. (b)⇒(c) Let $w \in \partial(\operatorname{im} D)$ so that we have $[g_n, z_n] \in \widetilde{Y}$ such that $g_n z_n \to w$ in $\mathbb{H}^2$. Since $g_n z_n$ is Cauchy, $[z_n]$ is also Cauchy, so we have a limit $[z_n] \to [z]$ in Y. Then, there exists a sequence $h_n \in \Gamma$ such that $h_n z_n \to z$ in $\mathbb{H}^2$, which implies $g_n h_n^{-1} z \to w$ in $\mathbb{H}^2$ and $w \in \overline{\Gamma z}$. Since $\operatorname{im} D$ is open and $\overline{P} \subset \operatorname{im} D$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \operatorname{im} D$. There is $g \in \mathbb{H}^2$ such that $d(gz, w) < \varepsilon$, which implies $g^{-1} w \in B(z, \varepsilon)$. Because Γ acts on $\operatorname{im} D$, we can conclude $w \in \operatorname{im} D$. □

If P satisfies the cycle conditions,

(a) Γ is discrete.

(b) Γ is given by the presentation $\langle S | R \rangle$, where S is the set of side-pairing isometries and R is the set of cycle relations.

(c) P is a fundamental domain of Γ

(d) $Y \cong \mathbb{H}^2/\Gamma$.

2.3 Geometric structures

A geodesically connected and geodesically complete space is rigid.

2.7 (Continuation of path)**.** Let M be a (G,X)-manifold, $\varphi : U \to X$ a chart, and $\gamma : [0,1] \to M$ a path. There is a partition $(t_i)_{i=0}^m$ of the interval $[0,1]$ with $t_0 = 0$, $t_m = 1$ and a sequence of chart $(\varphi_i : U_i \to X)_{i=0}^{m-1}$ with $\varphi_0 = \varphi$ such that $\gamma([t_{i-1}, t_i]) \subset U_i$. Since $\tau_{i+1,i} \circ \varphi_{i+1} \circ \gamma(t_{i+1}) = \varphi_i \circ \gamma(t_{i+1})$, we can define a path $\widehat{\gamma} : [0,1] \to X$ by

$$\widehat{\gamma}(t) := \tau_{1,0} \circ \cdots \circ \tau_{i,i-1} \circ \varphi_i \circ \gamma|_{[t_i, t_{i+1}]}(t)$$

for $t \in [t_i, t_{i+1}]$, where $\tau_{i+1,i} = \varphi_i \circ \varphi_{i+1}^{-1}$ are transition maps. The path $\widehat{\gamma}$ is called the *continuation* of $\varphi \circ \gamma$.

(a) $\widehat{\gamma}$ does not depend on the choice of the sequence of charts when the partition is given.

(b) $\widehat{\gamma}$ does not depend on the choice of the partition.

(c) If γ_0 and γ_1 are homotopic fixing endpoints, then their continuations are also homotopic fixing endpoints.

2.8 (Developing map)**.** Let M be a connected (G,X)-manifold, $\widetilde{M}$ the universal covering of M, and $\varphi : U \to X$ a chart on $\widetilde{M}$.

2.9 (Holonomy)**.** Let M be a connected (G,X)-manifold, $\widetilde{M}$ the universal covering of M, and $\varphi : U \to X$ a chart on $\widetilde{M}$.

$h : \pi_1(M) \to G$.

(a) If $f_0, f_1 : \widetilde{M} \to X$ are (G,X)-maps, then there is a unique $g \in G$ such that $f_1 = g f_0$.

(b) For $H \leq G$, M admits a (H,X)-structure if and only if $\operatorname{im} h \subset H$.

surjectivity of a map from torsion-free discrete subgroups of G to complete (G,X)-manifolds? (up to homeomorphism, up to geometric structure)

Definition 2.1 (Several definitions of hyperbolic manifolds)**.** Let $G = \mathrm{Isom}^+(\mathbb{H}^n)$ and X a n-manifold. Then, X is a hyperbolic manifold if one of the following satisfied...?:

1. It admits a hyperbolic atlas, and it is "complete"
2. It is homeomorphic to $\mathbb{H}^n/\Gamma$ for a torsion-free discrete subgroup Γ of G.
3. It is a geodesically complete Riemannian manifold with constant sectional curvature -1.

Thurston geometry is a three-dimensional model geoemtry on which a closed 3-manifold has a geometric structure modelled.

oriented prime closed 3-manifolds

3 Universal coefficient theorem

Lemma 3.1. *Suppose we have a flat resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

Then, we have a exact sequence

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Tor}_1^R(A,B) \longrightarrow P_1 \otimes B \longrightarrow P_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0.$$

Theorem 3.2. *Let R be a PID. Let $C_\bullet$ be a chain complex of flat R-modules and G be a R-module. Then, we have a short exact sequence*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \operatorname{Tor}(H_{n-1}(C),G) \longrightarrow 0,$$

which splits, but not naturally.

1. We have a short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

where every morphism in $Z_\bullet$ and $B_\bullet$ are zero. Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence

$$0 \longrightarrow Z_\bullet \otimes G \longrightarrow C_\bullet \otimes G \longrightarrow B_{\bullet-1} \otimes G \longrightarrow 0$$

and the associated long exact sequence

$$\longrightarrow H_n(B;G) \longrightarrow H_n(Z;G) \longrightarrow H_n(C;G) \longrightarrow H_{n-1}(B;G) \longrightarrow H_{n-1}(Z;G) \longrightarrow$$

where the connecting homomomorphisms are of the form $(i_n : B_n \to Z_n) \otimes 1_G$ (It is better to think diagram chasing than a natural construction). Since morhpisms in B and Z are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C;G) \longrightarrow B_{n-1} \otimes G \longrightarrow Z_{n-1} \otimes G \longrightarrow .$$

Since

$$0 \longrightarrow \operatorname{Tor}_1^R(H_n,G) \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n \otimes G \longrightarrow 0$$

for all n, the exact sequence splits into short exact sequence by images

$$0 \longrightarrow H_n \otimes G \longrightarrow H_n(C;G) \longrightarrow \operatorname{Tor}_1^R(H_{n-1},G) \longrightarrow 0.$$

For splitting, □

2. Since R is PID, we can construct a flat resolution of G

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0.$$

Since modules in $C_\bullet$ are flat so that the tensor product functors are exact and $P_1 \to P_0$ and $P_0 \to G$ induce the chain maps, we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet \otimes P_1 \longrightarrow C_\bullet \otimes P_0 \longrightarrow C_\bullet \otimes G \longrightarrow 0.$$

Then, we have the associated long exact sequence

$$\longrightarrow H_n(C;P_1) \longrightarrow H_n(C;P_0) \longrightarrow H_n(C;G) \longrightarrow H_{n-1}(C;P_1) \longrightarrow H_{n-1}(C;P_0) \longrightarrow .$$

Since flat tensor product functor commutes with homology funtor from chain complexes, we have

$$\longrightarrow H_n \otimes P_1 \longrightarrow H_n \otimes P_0 \longrightarrow H_n(C;G) \longrightarrow H_{n-1} \otimes P_1 \longrightarrow H_{n-1} \otimes P_0 \longrightarrow \; .$$

Since

$$0 \longrightarrow \mathrm{Tor}_1^R(G, H_n) \longrightarrow H_n \otimes P_1 \longrightarrow H_n \otimes P_0 \longrightarrow H_n \otimes G \longrightarrow 0$$

for all n, the exact sequence splits into short exact sequence by images

$$0 \longrightarrow H_n \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}_1^R(G, H_{n-1}) \longrightarrow 0.$$

□

Proof 3. By tensoring G, we get the following diagram.

$$\begin{tikzcd}
H_n \otimes G \arrow[dr] & & & & H_{n-1} \otimes G \\
& \operatorname{coker} \partial_{n+1} \otimes G \arrow[ddr, two heads] & & \ker \partial_{n-1} \otimes G \arrow[ur, two heads] \arrow[dr] & \\
C_n \otimes G \arrow[ur, two heads] \arrow[drr, two heads] & & & & C_{n-1} \otimes G \\
& & \operatorname{im} \partial_n \otimes G \arrow[uur] \arrow[urr] & & \\
& \mathrm{Tor}_1(H_{n-1}, G) \arrow[ur, tail] & & &
\end{tikzcd}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernals are preserved, but monomorphisms and kernels are not. Especially, $\operatorname{coker} \partial_{n+1} \otimes G = \operatorname{coker}(\partial_{n+1} \otimes 1_G)$ is important.

Consider the following diagram.

$$\begin{tikzcd}
H_n(C;G) \arrow[dr, tail] & H_n \otimes G \arrow[d] & & & \\
& \operatorname{coker} \partial_{n+1} \otimes G \arrow[dd, two heads] \arrow[ddrr, two heads] & & \ker \partial_{n-1} \otimes G \arrow[dr, "\text{monic!}"] & \\
& & & & C_{n-1} \otimes G \\
& \operatorname{im} \partial_n \otimes G \arrow[uurr] & & \operatorname{im}(\partial_n \otimes 1_G) \arrow[uu, dashed, tail] \arrow[ur, tail] & \\
\mathrm{Tor}_1(H_{n-1}, G) \arrow[ur, tail] & & & &
\end{tikzcd}$$

Since $\ker \partial_{n-1}$ is free,

If we show $\operatorname{im}(\partial_n \otimes 1_G) \to \ker \partial_{n-1} \otimes G$ is monic, then we can get

$$\begin{aligned}
H_n(C;G) &= \ker(\operatorname{coker} \partial_{n+1} \otimes G \to \operatorname{im}(\partial_n \otimes 1_G)) \\
&= \ker(\operatorname{coker} \partial_{n+1} \otimes G \to \ker \partial_{n-1} \otimes G).
\end{aligned}$$

4 Fundamental differential geometry

4.1 Manifold and Atlas

Definition 4.1. A *locally Euclidean space M* of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{R}^d$.

Definition 4.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 4.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of $\mathbb{R}^d$. A chart is often written by a pair (U, φ).

Definition 4.4. An *atlas* $\mathcal{F}$ is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 4.5. A *differentiable maifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

4.2 Definition of Differentiable Structure

Definition 4.6. An atlas $\mathcal{F}$ is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each *transition function* $\tau_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 4.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 4.8. An differentiable atlas $\mathcal{F}$ is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in $\mathcal{F}$, then $(U, \varphi) \in \mathcal{F}$.

Definition 4.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manofold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M. When the charts is already equipped on M, it is natural to define a function $f : M \to \mathbb{R}$ differentiable if the functions $f \circ \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas $\mathcal{F}$, then f is also differentiable on any atlases which is equivalent to $\mathcal{F}$ by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 4.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 4.10. A continuous function $f : M \to N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

4.3 Curves

Definition 4.11. For $f : M \to \mathbb{R}$ and (U, ϕ) a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 4.12. Let $\gamma\colon I \to M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi : M \to N$ be a smoth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f : M \to \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \to \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt} f \circ \gamma.$$

Proposition 4.1. *Let $\gamma\colon I \to M$ be a smooth curve on a manifold M. The notation $\dot{\gamma}^\mu$ is not confusing thanks to*

$$(\dot{\gamma})^\mu = (\dot{\gamma^\mu}).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt} x^\mu \circ \gamma.$$

4.4 Connection computation

$$\begin{aligned}
\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\
&= X^\mu (\nabla_\mu Y^\nu)\partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu}\right)\partial_\nu + X^\mu Y^\nu (\Gamma^\lambda_{\mu\nu}\partial_\lambda) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda\right)\partial_\nu.
\end{aligned}$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ.

$$Y^\nu_{,\mu} = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \qquad Y^\nu_{;\mu} = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda.$$

Theorem 4.2. *For Levi-civita connection for g,*

$$\Gamma^l_{ij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Proof.

$$\begin{aligned}
(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} \\
(\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma^l_{jk} g_{li} - \Gamma^l_{ji} g_{kl} \\
(\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}
\end{aligned}$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$. Thus,

$$\Gamma^l_{ij} g_{kl} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

□

4.5 Geodesic equation

Theorem 4.3. *If c is a geodesic curve, then components of c satisfies a second-order differential equation*

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0.$$

Proof. Note

$$0 = \nabla_{\dot\gamma}\dot\gamma = \dot\gamma^\mu\nabla_\mu(\dot\gamma^\lambda\partial_\lambda) = (\dot\gamma^\nu\partial_\nu\dot\gamma^\mu + \dot\gamma^\nu\dot\gamma^\lambda\Gamma^\mu_{\nu\lambda})\partial_\mu.$$

Since

$$\dot\gamma^\nu\partial_\nu\dot\gamma^\mu = \dot\gamma(\dot\gamma^\mu) = d\dot\gamma^\mu(\dot\gamma) = d\dot\gamma^\mu \circ d\gamma\left(\frac{\partial}{\partial t}\right) = d\dot\gamma^\mu\left(\frac{\partial}{\partial t}\right) = \ddot\gamma^\mu,$$

we get a second-order differential equation

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0$$

for each μ. □

5 Bundles

Show that S^n has a nonvanishing vector field if and only if n is odd.

Solution. Since S^n is embedded in $\mathbb{R}^{n+1}$, the tangent bundle TS^n can be considered as an embedded manifold in $S^n \times \mathbb{R}^{n+1}$ which consists of (x, v) such that $\langle x, x\rangle = 1$ and $\langle x, v\rangle = 0$, where the inner product is the standard one of $\mathbb{R}^{n+1}$.

Suppose n is odd. We have a vector field $(x_1, x_2, \cdots, x_{n+1}; x_2, -x_1, \cdots, -x_n)$ which is nonvanishing.

Conversely, suppose we have a nonvanishing vector field X. Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \to S^n \times \mathbb{R}^{n+1} \to \phi\mathbb{R}^{n+1} \to S^n.$$

The last map can be defined since X is nowhere zero. Since this map satisfies $\langle x, \phi(x)\rangle = 0$ for all $x \in S^n$, we can define homotopies from ϕ to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree, +1, so n is odd. □

Proposition 5.1. *Independent commuting vector fields are realized as partial derivatives in a chart.*

Proposition 5.2. *Let $\{\partial_1, \cdots, \partial_k\}$ be an independent involutive vector fields. We can find independent commuting $\{\partial_{k+1}, \cdots, \partial_n\}$ such that union is independent. (Maybe)*

Proposition 5.3. *Let $\{\partial_1, \cdots, \partial_k\}$ be an independent commuting vector fields. We can find independent commuting $\{\partial_{k+1}, \cdots, \partial_n\}$ such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

Proposition 5.4. *An immersed manifold is locally an inverse image of a regular value.*

Proposition 5.5. *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

Proof. It uses tubular neighborhood. Pontryagin construction? □

Proposition 5.6. *An immersed manifold is locally a linear subspace in a chart.*

Proposition 5.7. *Distinct two points on a connected manifold are connected by embedded curve.*

Proof. Let $\gamma : I \to M$ be a curve connecting the given two points, say p, q.

*Step [.1]*Constructing a piecewise linear curve For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\operatorname{im}\gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \varnothing$ for all $1 \leq i \leq n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i,j) : U_{t_i} \cap U_{t_j} \neq \varnothing$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \leq i \leq n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q.

*Step [.2]*Smoothing the curve □

Proposition 5.8. *Let M is an embedded manifold with boundary in N. Any kind of sections on M can be extended on N.*

Proposition 5.9. *Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.*

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Proposition 5.10. *The set of points that is geodesically connected to a point is open.*