Positive Hahn-Banach separation theorems in operator algebras

Ikhan Choi

The University of Tokyo

Tokyo, January 2025

Contents

Positive Hahn-Banach separation theorems in operator algebras

In E an ordered vector space, $F \subset E^+$ is called *hereditary* if $0 \le x \le y \in F$ implies $x \in F$.

Theorem (Haagerup '75, C. '25)

Let M be a von Neumann algebra, and let A be a C^*-algebra.

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then for any $x \in M^+ \setminus F$ there exists $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega(x') \le 1$ for all $x' \in F$.*

(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then for any $\omega \in M_*^+ \setminus F_*$ there exists $x \in M^+$ such that $\omega(x) > 1$ and $\omega'(x) \le 1$ for all $\omega' \in F_*$.*

(3) *If F is a norm closed convex hereditary subset of A^+, then for any $a \in A^+ \setminus F$ there exists $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for all $a' \in F$.*

(4) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then for any $\omega \in A^{*+} \setminus F^*$ there exists $a \in A^+$ such that $\omega(a) > 1$ and $\omega'(a) \le 1$ for all $\omega' \in F^*$.*

Positive Hahn-Banach separation theorems in operator algebras

In E an ordered vector space, $F \subset E^+$ is called *hereditary* if $0 \le x \le y \in F$ implies $x \in F$.

Theorem (Haagerup '75, C. '25)

Let M be a von Neumann algebra, and let A be a C^*-algebra.

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then for any $x \in M^+ \setminus F$ there exists $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega(x') \le 1$ for all $x' \in F$.*

(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then for any $\omega \in M_*^+ \setminus F_*$ there exists $x \in M^+$ such that $\omega(x) > 1$ and $\omega'(x) \le 1$ for all $\omega' \in F_*$.*

(3) *If F is a norm closed convex hereditary subset of A^+, then for any $a \in A^+ \setminus F$ there exists $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for all $a' \in F$.*

(4) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then for any $\omega \in A^{*+} \setminus F^*$ there exists $a \in A^+$ such that $\omega(a) > 1$ and $\omega'(a) \le 1$ for all $\omega' \in F^*$.*

Haagerup proved (1)~(3) in his master's thesis [Haa75], and asked if (4) holds. The part (1) plays a major role in the proof of some equivalence conditions for normal weights on a von Neumann algebra. The difficulty is (3)<(2)≈(1)<(4). I proved (1) and (2) in different ways, and solved (4).

Suppression by the one-parameter family of functional calculi

Since $F^{r+r+} = (F - E^+)^{rr+} = (\overline{F - E^+})^+$ by the usual real bipolar theorem, where r denotes the real polar, each statement is equivalent to $(\overline{F - E^+})^+ \subset F$. For example in (1), fixing $\delta > 0$, we want to suppress y_i to get $y_{i,\delta}$ with $\|y_{i,\delta}\| \le \delta^{-1}$ so that $y_{i,\delta} \to y_\delta$ σ-weakly.

$$\begin{array}{ccccccc} F - M^+ & \ni & x_i & \le & y_i & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x & \le & ? & \in & F \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} F - M^+ & \ni & x_{i,\delta} & \le & y_{i,\delta} & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x_\delta & \le & y_\delta & \in & F \end{array}$$

It implies $x_\delta \in F$, and if $x_\delta \to x$, then $x \in F$.

Suppression by the one-parameter family of functional calculi

Since $F^{r+r+} = (F - E^+)^{rr+} = (\overline{F - E^+})^+$ by the usual real bipolar theorem, where r denotes the real polar, each statement is equivalent to $(\overline{F - E^+})^+ \subset F$. For example in (1), fixing $\delta > 0$, we want to suppress y_i to get $y_{i,\delta}$ with $\|y_{i,\delta}\| \le \delta^{-1}$ so that $y_{i,\delta} \to y_\delta$ σ-weakly.

$$\begin{array}{ccccccc} F - M^+ & \ni & x_i & \le & y_i & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x & \le & ? & \in & F \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} F - M^+ & \ni & x_{i,\delta} & \le & y_{i,\delta} & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x_\delta & \le & y_\delta & \in & F \end{array}$$

It implies $x_\delta \in F$, and if $x_\delta \to x$, then $x \in F$.

Definition

For $\delta > 0$, we define $f_\delta : (-\delta^{-1}, \infty) \to \mathbb{R}$ such that $f_\delta(t) := t(\delta t + 1)^{-1}$ for $t > -\delta^{-1}$.

It has many interesting properties such as operator monotonicity, semi-group property, increasing strong convergence to the identity, etc. Haagerup used the σ-strong topology to have $f_\delta(x_i) \to f_\delta(x)$ in the proof of (1).

Suppression by the one-parameter family of functional calculi

Since $F^{r+r+} = (F - E^+)^{rr+} = (\overline{F - E^+})^+$ by the usual real bipolar theorem, where r denotes the real polar, each statement is equivalent to $(\overline{F - E^+})^+ \subset F$. For example in (1), fixing $\delta > 0$, we want to suppress y_i to get $y_{i,\delta}$ with $\|y_{i,\delta}\| \le \delta^{-1}$ so that $y_{i,\delta} \to y_\delta$ σ-weakly.

$$\begin{array}{ccccccc} F - M^+ & \ni & x_i & \le & y_i & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x & \le & ? & \in & F \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} F - M^+ & \ni & x_{i,\delta} & \le & y_{i,\delta} & \in & F \\ & & \downarrow & & \downarrow & & \\ 0 & \le & x_\delta & \le & y_\delta & \in & F \end{array}$$

It implies $x_\delta \in F$, and if $x_\delta \to x$, then $x \in F$.

Definition

For $\delta > 0$, we define $f_\delta : (-\delta^{-1}, \infty) \to \mathbb{R}$ such that $f_\delta(t) := t(\delta t + 1)^{-1}$ for $t > -\delta^{-1}$.

It has many interesting properties such as operator monotonicity, semi-group property, increasing strong convergence to the identity, etc. Haagerup used the σ-strong topology to have $f_\delta(x_i) \to f_\delta(x)$ in the proof of (1). Since A^* has no analogue of the σ-strong topology, we use an inequality like $t - \varepsilon \le f_\delta(t) \le t$ on a suitable interval.

$$\begin{array}{ccccccccc} F - M^+ & \ni & x_i - \varepsilon & \le & f_\delta(x_i) & \le & f_\delta(y_i) & \in & F \\ & & \downarrow & & & & \downarrow & & \\ 0 & \le & x & & \le & & y_\delta & \in & F \end{array}$$

Bounded commutant Radon-Nikodym derivatives

To take functional calculi on linear functionals:

Bounded commutant Radon-Nikodym derivatives

To take functional calculi on linear functionals:

Definition

Let M be a von Neumann algebra, and let $\psi \in M_*^+$. Consider the Gelfand-Naimark-Segal representation $\pi : M \to B(H)$ associated to ψ with the canonical cyclic vector $\Omega \in H$. Then, we have a positive bounded linear map $\theta : \pi(M)' \to M_*$ defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega\rangle, \qquad h \in \pi(M)',\ x \in M.$$

It has the image

$$\operatorname{im}\theta = \{\omega \in M_* : \text{there is } C > 0 \text{ such that } |\omega(x)| \le C\psi(x) \text{ for all } x \in M^+\}.$$

We will call $\theta^{-1}(\omega)$ the *commutant Radon-Nikodym derivative* of ω with respect to ψ.

Bounded commutant Radon-Nikodym derivatives

To take functional calculi on linear functionals:

Definition

Let M be a von Neumann algebra, and let $\psi \in M_*^+$. Consider the Gelfand-Naimark-Segal representation $\pi : M \to B(H)$ associated to ψ with the canonical cyclic vector $\Omega \in H$. Then, we have a positive bounded linear map $\theta : \pi(M)' \to M_*$ defined such that

$$\theta(h)(x) := \langle h\pi(x)\Omega, \Omega\rangle, \qquad h \in \pi(M)',\ x \in M.$$

It has the image

$$\operatorname{im}\theta = \{\omega \in M_* : \text{there is } C > 0 \text{ such that } |\omega(x)| \le C\psi(x) \text{ for all } x \in M^+\}.$$

We will call $\theta^{-1}(\omega)$ the *commutant Radon-Nikodym derivative* of ω with respect to ψ.

For example in (2), when $\omega_n \in F_* - M_*^+$ converges to $\omega \in M_*^+$ in norm, we can find a suitable $\psi \in M_*^+$ such that

$$\begin{array}{ccccccc} F_* - M_*^+ & \ni & \theta(f_\delta(\theta^{-1}(\omega_n)) & \le & \theta(f_\delta(\theta^{-1}(\varphi_n)) & \in & F_* \\ & & \downarrow & & \downarrow & & \\ 0 & \le & \theta(f_\delta(\theta^{-1}(\omega)) & \le & \varphi_\delta & \in & F_* \end{array}$$

Contents

Proof of (1)

We prove (1) in a different way to motivate the proof methods of (4). Recall that we need to prove $(\overline{F - M^+})^+ \subset F$. To use the Krein-Šmulian theorem, we define a subset G satisfying $F - M^+ \subset G$ and $G^+ \subset F$ and $\overline{G} \subset G$.

Proof of (1)

We prove (1) in a different way to motivate the proof methods of (4). Recall that we need to prove $(\overline{F-M^+})^+ \subset F$. To use the Krein-Šmulian theorem, we define a subset G satisfying $F-M^+ \subset G$ and $G^+ \subset F$ and $\overline{G} \subset G$. Haagerup originally defined and proved the σ-strong closedness of

$$G := \{x \in M^{sa} : f_\delta(x) \in F - M^+ \text{ for all } \delta < \|x_-\|^{-1}\}.$$

Instead, to avoid the use of σ-strong topology, we define

$$G := \left\{ x \in M^{sa} : \begin{array}{c} \text{for any } \varepsilon > 0, \text{ there is a net } y_\delta \in F \\ \text{indexed on } 0 < \delta \le (1+\|x\|)^{-1} \text{ such that} \\ \|y_\delta\| \le \delta^{-1} \text{ and } f_\delta(x) \le y_\delta + \varepsilon\delta^{\frac{1}{2}} \end{array} \right\}.$$

Proof of (1)

We prove (1) in a different way to motivate the proof methods of (4). Recall that we need to prove $(\overline{F-M^+})^+ \subset F$. To use the Krein-Šmulian theorem, we define a subset G satisfying $F-M^+ \subset G$ and $G^+ \subset F$ and $\overline{G} \subset G$. Haagerup originally defined and proved the σ-strong closedness of

$$G := \{x \in M^{sa} : f_\delta(x) \in F - M^+ \text{ for all } \delta < \|x_-\|^{-1}\}.$$

Instead, to avoid the use of σ-strong topology, we define

$$G := \left\{ x \in M^{sa} : \begin{array}{c} \text{for any } \varepsilon > 0, \text{ there is a net } y_\delta \in F \\ \text{indexed on } 0 < \delta \le (1+\|x\|)^{-1} \text{ such that} \\ \|y_\delta\| \le \delta^{-1} \text{ and } f_\delta(x) \le y_\delta + \varepsilon\delta^{\frac{1}{2}} \end{array} \right\}.$$

- $F - M^+ \subset G$: Easy.
- $G^+ \subset F$: Relatively easy. Fix $\delta' > 0$ and obtain $(1+\delta'\|x\|)^{-1} f_\delta(x) \in F$ by limiting
$$0 \le (1+\delta'\|x\|)^{-1} f_\delta(x) \le f_{\delta'}(f_\delta(x)) \le f_{\delta'}(y_\delta + \delta^{\frac{1}{2}}) \le f_{\delta'}(y_\delta) + \delta^{\frac{1}{2}}.$$
- $\overline{G} \subset G$: If $x_i \in G$ is bounded and $x_i \to x$ σ-weakly, then we can construct $y_\delta \in F$ such that $y_{i,\delta} \to y_\delta$ for $\delta \le \delta_0$ and $y_\delta := f_{\delta-\delta_0}(y_{\delta_0})$ for $\delta > \delta_0$ for small $\delta_0 > 0$. The convexity follows from $F - M^+ \subset G$ and $\overline{G \cap M_r} \subset G$, so the Krein-Šmulian theorem completes the proof.

Proof of (4)

For (4), we define

Proof of (4)

For (4), we define

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{for any } \varepsilon > 0, \text{ there are nets } \psi_\delta \in A^{*+} \text{ and } \varphi_\delta \in F^* \\ \text{indexed on } 0 < \delta \le (1 + 4\|\omega\|)^{-6} \text{ such that} \\ \text{the following five conditions are satisfied:} \\ |\omega(a)| \le \delta^{-\frac{1}{6}} \psi_\delta(a) \text{ for all } a \in A^+,\ \|\psi_\delta\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \\ \omega_\delta \le \varphi_\delta + \varepsilon \delta^{\frac{1}{2}} \psi_\delta, \text{ and } \omega_\delta \to \omega \text{ weakly}^* \text{ in } A^* \text{ as } \delta \to 0 \end{array} \right\},$$

where $\omega_\delta := \theta_\delta(f_\delta(\theta_\delta^{-1}(\omega)))$, and here θ_δ is associated to ψ_δ.

Proof of (4)

For (4), we define

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{for any } \varepsilon > 0 \text{, there are nets } \psi_\delta \in A^{*+} \text{ and } \varphi_\delta \in F^* \\ \text{indexed on } 0 < \delta \le (1 + 4\|\omega\|)^{-6} \text{ such that} \\ \text{the following five conditions are satisfied:} \\ |\omega(a)| \le \delta^{-\frac{1}{6}} \psi_\delta(a) \text{ for all } a \in A^+,\ \|\psi_\delta\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \\ \omega_\delta \le \varphi_\delta + \varepsilon \delta^{\frac{1}{2}} \psi_\delta, \text{ and } \omega_\delta \to \omega \text{ weakly}^* \text{ in } A^* \text{ as } \delta \to 0 \end{array} \right\},$$

where $\omega_\delta := \theta_\delta(f_\delta(\theta_\delta^{-1}(\omega)))$, and here θ_δ is associated to ψ_δ.

- $F^* - A^{*+} \subset G^*$: Take $\psi_\delta := (1 + \|\omega\|)^{-1}([\omega] + (1 + \|\varphi\|)^{-1}\varphi)$ and $\varphi_\delta := \theta(f_\delta(\theta(\varphi)))$.
- $G^{*+} \subset F^*$: Take the Radon-Nikodym for $\omega + \delta\varphi_\delta + \psi_\delta$ and do the same thing as (1).
- $\overline{G^*} \subset G^*$: we can prove in a similar way to (1), but long computations.

Questions

- Simpler proof? (in conversation with N. Ozawa)
  Yes, and we succeeded in proving with

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{there is } \psi \in A^{*+} \text{ and there is a net } \varphi_\delta \in F^* \text{ such that} \\ \|\psi\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \text{ and } \omega \le \varphi_\delta + \delta^{\frac{1}{2}}\psi \\ \text{for any sufficiently small } \delta > 0 \end{array} \right\}.$$

- Weight theory on C*-algebras?
- Convex hereditary subsets instead of convex balanced subsets?
- Non-commutative L^p spaces?

References I

[Haa75] Uffe Haagerup. Normal weights on W^*-algebras. *J. Functional Analysis*, 19:302–317, 1975.

[Tak02] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.