Algebra II

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Contents

Part I

Modules

Chapter 1

Modules

1.1 Modules

1.1 (Definition of modules)**.** Let R be a ring, which is possibly neither commutative nor unital. A *left R-module* is an abelian group $(M,+,0)$ together with a binary operation $\cdot : R\times M\to M$ satisfying

(i) for all $r,s\in R$ and $m\in M$ we have $(rs)m=r(sm)$, (associativity)

(ii) for all $r,s\in R$ and $m\in M$ we have $(r+s)m=rm+sm$. (distributivity)

When R is unital, a left R-module M is called *unital* if

(iii) for all $m\in M$ we have $1m=m$. (identity)

Throughout the entire book, we will always assume modules are unital over commutative unital rings.

(a)

submodules quotient modules isomorphism theorems

1.2 Free modules

generators, cyclic direct sum free modules

1.3 Tensor product modules

1.2 (Tensor product of algebras)**.** Let R be a commutative unital ring. Let M and N are R-modules. A *bilinear form* or a *pairing* is a function $M\times N\to R$ such that...

1.3 (Base change of modules)**.** Given a ring homomorphism $R\to A$, we can write $A\in\mathrm{Mod}_R$, and the induced tensoring functor $-\otimes_R A:\mathrm{Mod}_R\to\mathrm{Mod}_A$ is left adjoint to the forgetful functor, that is,

$$\mathrm{Hom}_A(M\otimes_R A,N)\cong\mathrm{Hom}_R(M,N),\qquad M\in\mathrm{Mod}_R,\ N\in\mathrm{Mod}_A.$$

1.4 Homomorphism modules

Chapter 2

Exact sequences

2.1 Chain complexes

Let R be a commutative unital ring. Let $C_\bullet \in \mathrm{Ch}_{\geq 0}(R)$ be a non-negatively graded chain complex of R-modules. Let M be an R-module.

Define the homology group with coefficients in M by

$$(C \otimes_R M)_\bullet := C_\bullet \otimes_R M \in \mathrm{Ch}_{\geq 0}(R), \qquad H_n(C, M) := H_n((C \otimes_R M)_\bullet) \in \mathrm{Mod}_R\,.$$

Define the cohomology group with coefficients in M by

$$\mathrm{Hom}_R(C, M)^\bullet := \mathrm{Hom}_R(C_\bullet, M) \in \mathrm{Ch}^{\geq 0}(R), \qquad H^n(C, M) := H^n(\mathrm{Hom}_R(C, M)^\bullet) \in \mathrm{Mod}_R\,.$$

If M is a commutative unital R-algebra, then the resulting homology groups are M-modules.

When do we have $H^n(C, M) \otimes_R N \cong H^n(C, M \otimes_R N)$?

2.2 Projective and injective modules

2.1 (Projective modules)**.** Let R be a commutative unital ring. An R-module P is called *projective* if the zero map $0 \to P$ has the left lifting property with respect to surjective module maps. That is, for every surjective module map $M_1 \to M_0$ and a module map $P \to M_0$ there exists a module map $P \to M_1$ such that we have a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ M_1 \arrow[r] & M_0 \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & M_1 \arrow[d, two heads] \\ P \arrow[ur, dashed, "\exists"] \arrow[r] & M_0 \end{tikzcd}$$

Let P be an R-module.

free implies projective, every module is a quotient of a free module....

(a) P is projective if and only if it is a direct summand of a free module.

(b) P is projective if and only if the left exact functor $\mathrm{Hom}_R(P, -)$ preserves surjectivity.

(c) P is projective if and only if every short exact sequence $0 \to M_1 \to M_0 \to P \to 0$ is split.

(d) The direct sum $\bigoplus_i P_i$ is projective iff P_i are projective.

PID: projective iff free (note sub of free is free in PID)

2.2 (Injective modules)**.** Let R be a commutative unital ring. An R-module I is called *injective* if the zero map $I \to 0$ has the right lifting property with respect to injective module maps. That is, for every injective module map $M^0 \to M^1$ and a module map $M^0 \to I$ there exists a module map $M^1 \to I$ such that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M^0 \arrow[r] \arrow[d] & M^1 \arrow[dl, dashed, "\exists"'] \\ & I & \end{tikzcd}$$

$$\begin{tikzcd} M^0 \arrow[r] \arrow[d, hook] & I \\ M^1 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

(a)

(b) Every module is embedded in an injective module.

(c) I is injective if and only if the left exact contravariant functor $\mathrm{Hom}_R(-, I)$ preserves the surjectivity.

(d) direct product of injectives is injective

PID: injective iff divisible ($r\cdot : M \to M$ surj) (lem: $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is injective if M is injective $\mathbb{Z}$-module)

2.3 (Flat modules)**.** (a) PID: flat iff ($\cdot a : M \to M$ inj)

(b) M flat iff $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is injective

(c) M flat iff $I \otimes M \to R \otimes M$ inj

(d) if projective, then flat

2.4 (Projective resolutions)**.** Let R be a commutative unital ring, and M be an R-module. A *projective resolution* of M is a chain complex $P_\bullet \in \mathrm{Ch}_{\geq 0}(R)$ together with an R-homomorphism $q : P_0 \to M$ such that each module in $P_\bullet$ is projective and we have an exact sequence of R-modules

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{q} M \to 0.$$

(a)

2.3 Tor and Ext

2.5 (Tor functor)**.** Let R be a commutative unital ring, and let M and N be R-modules. We define the *Tor functor* as either

$$\mathrm{Tor}_n^R(M, N) := H_n(P_\bullet \otimes_R N) \quad \text{or} \quad \mathrm{Tor}_n^R(M, N) := H_n(M \otimes_R Q_\bullet),$$

where $P_\bullet$ and $Q_\bullet$ are projective resolutions of M and N respectively. It is the left derived functor of a right exact functor. It is symmetric by definition.

(a) Two definitions coincide.

(b) It does not depend on the choice of resolutions.

(c) It has a long exact sequence.

(d) It preserves possibly infinite direct sums and filtered colimits in each variable.

(e) We may only assume $P_\bullet$ is a flat resolution. (Flat resolution lemma)

2.6 (Ext functor). Let R be a commutative unital ring, and let M and N be R-modules. We define the *Ext functor* as wither

$$\operatorname{Ext}^n_R(M,N) := H^n(\operatorname{Hom}_R(P_\bullet, N)) \quad \text{or} \quad \operatorname{Ext}^n_R(M,N) := H^n(\operatorname{Hom}_R(M, I^\bullet)),$$

where $P_\bullet$ and $I^\bullet$ are projective and injective resolutions of M and N respectively. It is the right derived functor of a left exact functor.

(a) Two definitions coincide.

(b) It does not depend on the choice of resolutions.

(c) It has a long exact sequence.

(d) It preserves...

2.7 (Universal coefficient theorem). Let R be a commutative unital ring. Let $C_\bullet \in \mathrm{Ch}_{\geq 0}(R)$ be a chain complex of flat right R-modules and M be a left R-module.

$$0 \to H_n(C) \otimes_R M \to H_n(C, M) \to \operatorname{Tor}^R_1(H_{n-1}(C), M) \to 0.$$

(a) If R is a principal ideal domain, then the Künneth formula splits non-canonically.

Proof. We first prove the Künneth formula. Note that modules in $Z_\bullet$ and $B_\bullet$ are also flat. We start from that we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0.$$

Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since $H_n(B_{\bullet-1}) = H_{n-1}(B_\bullet)$ for any chain complex $C_\bullet$, we have a long exact sequence

$$H_n(B_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(C_\bullet \otimes_R M) \to H_{n-1}(B_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet \otimes_R M).$$

Since every module map inside $B_\bullet$ and $Z_\bullet$ is zero, we have an exact sequence

$$B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet \otimes_R M) \to B_{n-1} \otimes_R M \xrightarrow{f_{n-1}} Z_{n-1} \otimes_R M.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{coker} f_n \to H_n(C_\bullet \otimes_R M) \to \ker f_{n-1} \to 0.$$

Now we want to compute the cokernel and kernel of f_n.

Since

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

is a flat resolution of $H_n(C_\bullet)$, by the flat resolution lemma, we have a long exact sequence

$$\operatorname{Tor}^R_1(Z_n, M) \to \operatorname{Tor}^R_1(H_n(C_\bullet), M) \to B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet) \otimes_R M \to 0.$$

Since Z_n is flat so that $\mathrm{Tor}_1^R(Z_n, M) = 0$, we have

$$\operatorname{coker} f_n = H_n(C_\bullet) \otimes_R M, \quad \ker f_n = \mathrm{Tor}_1^R(H_n(C_\bullet), M).$$

Therefore, we have an exact sequence

$$0 \to H_n(C_\bullet) \otimes_R M \to H_n(C_\bullet \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), M) \to 0.$$

$\square$

$$\begin{array}{ccccccc} K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \twoheaddownarrow & & \downarrow & & \\ K' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 0 \end{array}$$

(a) If $A \to A'$ is monic, then $K \to K'$ is monic.

(b) If $B \to B'$ is monic, then $K \to K'$ is epic.

hom functor and tensor functor commtues...? no

Chapter 3

Linear algebra

3.1 Modules over principal ideal domains

Over a principal ideal, a finitely generated module is also finitely presented, a projective module is free.

3.1 (Torsion modules)**.** Let R be a commutative unital ring. An element of an R-module is called a *torsion element* if there is $r \in R$ annihilating the element. An R-module is called a *torsion-free module* if every non-zero element is not a torsion element, and called a *torsion module* if every element is a torsion element.

(a) A finitely generated torsion-free module embeds in a free module, over an integral domain.

(b) A submodule of a free module is a free module, over a principal ideal ring.

(c) A finitely generated module is the direct sum of a free module and a torsion module, over a principal ideal domain.

Proof. (a) Let M be a finitely generated torsion-free module over an integral domain R. We may assume M is non-zero. Since M is finitely generated, there is a finite set $X \subset M$ that generates M. Take a maximal subset $Y \subset X$ that is R-linearly independent. If we denote by $N := RY \subset M$ the submodule of M generated by Y, then N is free by the linear independence of Y. For each $x \in X \setminus Y$, since $Y \cup \{x\}$ is R-linearly dependent by the maximality assumption, there is a non-zero $r_x \in R$ such that $r_x x \in RY = N$. If we define $r := \prod_{x \in X \setminus Y} r_x$, which is valid since X is finite, then $r(X \setminus Y) \subset N$ implies $rM \subset N$. Since M is torsion-free and since r is non-zero because R is an integral domain, the multiplication $r\cdot : M \to M$ is injective, so M embeds to a free module N. Note that N can be assumed finitely generated.

(b) (Converse also holds)

(c) Let M be a finitely generated module over a principal ideal domain R. Let $\mathrm{Tor}(M)$ be the set of all torsion elements of M. Then, $\mathrm{Tor}(M)$ is a torsion module, and $M/\mathrm{Tor}(M)$ is a torsion-free module. (proof?)

The quotient module $M/\mathrm{Tor}(M)$ is finitely generated and torsion-free, so it is free by the parts (a) and (b), and is projective. The projectivity of $M/\mathrm{Tor}(M)$ concludes that M is the direct sum of $M/\mathrm{Tor}(M)$ and $\mathrm{Tor}(M)$.

□

3.2 (Primary modules)**.** Let R be a commutative unital ring.

We will decompose torsion modules into primary modules.

elementary divisors

3.3 (Cyclic modules)**.** Let R be a commutative unital ring. An R-module M is said to be *cyclic* if it is generated by one element.

invariant factors

(a) A cyclic R-module is isomorphic to a quotient of R.

(b) A cyclic R-module is torsion-free if and only if it is isomorphic to R.

$$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/48\mathbb{Z}) \Longleftrightarrow \begin{array}{c|cccc} & 2 & 4 & 12 & 48 \\ \hline 2 & 2^1 & 2^2 & 2^2 & 2^4 \\ 3 & 0 & 0 & 3^1 & 3^1 \end{array}$$

$$(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2^2\mathbb{Z})^2 \oplus (\mathbb{Z}/2^4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^2 \Longleftrightarrow \begin{array}{c|cccc} p \setminus e & 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 2 & 0 & 1 \\ 3 & 2 & 0 & 0 & 0 \end{array}$$

3.2 Normal forms

3.4 (Frobenius normal form)**.** Let F be a field. Each element $a \in M_n(F) := \operatorname{End}(F^n)$ gives rise to a finitely generated $F[x]$-module F^n.

Let M be a finitely generated $F[x]$-module without free component? Let $e_i \in M$ be generators of the $F[x]$-module. We can define a matrix $a_{ij} \in F$ such thtat $xe_j = \sum_j a_{ij} e_i$.

$a_{ij} = \langle ae_j, e_i \rangle$, $v = \sum_j v_j e_j$, $av = \sum_{i,j} a_{ij} v_j e_i$

$av = \sum_{i,j} \langle a v_j e_j, e_i \rangle e_i = \sum_{i,j} \langle ae_j, e_i \rangle v_j e_i$

Frobenius normal form or the *rational canonical form*

have the same normal form iff they generate isomorphic $F[x]$-modules...

Invariant factor form

(a) There is a one-to-one correspondence between the similarity classes of square matrices over F and the isomorphism classes of finitely generated $F[x]$-modules.

(b) Every finitely generated $F[x]$-module is a direct sum of cylic torsion $F[x]$-modules, i.e. no free submodules.

(c) Every cyclic torsion $F[x]$-module $V \cong R/(a)$ can be represented by the associated companion matrix C_a, constructed by the coefficients of a.

For $A \in M_n(F)$, the minimal polynomial $m_A(x)$ can be defined by the generator of the annihilator of the associated $F[x]$-module (V,A). The minimal polynomial is the largest invariant factor of (V,A). For each invariant factor a_i, we can construct a companion matrix with its coefficients.

Proof. □

3.5 (Jordan normal form)**.**

3.6 (Commuting matrices)**.**

3.3 Vector spaces

3.7 (Fields)**.** homomorphisms

3.8 (Dual spaces)**.** Double dual

3.9 (Polarization identity)**.** (a) Let F be a field of characteristic not 2. If $\langle -,- \rangle$ is a symmetric bilinear form, then

$$\langle x, y \rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2).$$

(b) Let $F = \mathbb{C}$. If $\langle -,- \rangle$ is a sesquilinear form, then

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

(c) isometry check

3.10 (Cauchy-Schwarz inequality). (a) Let $F = \mathbb{R}$. If $\langle -,- \rangle$ is a positive semi-definite symmetric bilinear form, then

(b) Let $F = \mathbb{C}$. If $\langle -,- \rangle$ is a positive semi-definite Hermitian form, then

3.11 (Dual space identification). Let $\langle -,- \rangle$ be a non-degenerate bilinear form

3.12 (Adjoint linear transforms).

spectral theorems

Exercises

3.13 (Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p)$). The conjugacy classes are classified by normal forms. There are four cases: for some a and b in $\mathbb{F}_p$,

(a) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$: $\binom{p-1}{2}$ classes of size $\frac{|G|}{(p-1)^2} = p(p+1)$.

(b) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$: $p-1$ classes of size 1.

(c) $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$: $p-1$ classes of size $\frac{|G|}{p(p-1)} = p^2 - 1$.

(d) otherwise, the eigenvalues are in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. In this case, the number of conjugacy classes is same as the number of monic irreducible qudratic polynomials over $\mathbb{F}_p$; $\frac{|\mathbb{F}_{p^2}|-|\mathbb{F}_p|}{2} = \frac{p(p-1)}{2}$ classes. Their size is $\frac{p(p-1)}{2}$.

3.14 (Conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_p)$). There are eight types of invariant factors:

$$(x-a)(x-b)(x-c),\ (x-a)^2(x-b),\ (x-a)^3,\ (x^2+ax+b)(x-c),\ (x^3+ax^2+bx+c),$$

$$(x-a) \mid (x-a)(x-b),\ (x-a) \mid (x-a)^2,\ (x-a) \mid (x-a) \mid (x-a)$$

Show that a square matrix A over $\mathbb{F}_p$ satisfying $A^p = A$ is diagonalizable.

Part II

Algebras

Chapter 4

Tensor algebras

4.1 Algebras

4.1 (Definition of algebras)**.** Let R be a commutative ring. An *associative algebra* or simply an *algebra* over R, or more simply *R-algebra*, is a ring A that is also an R-module satisfying

(i) for all $r \in R$ and $a, b \in A$ we have $r(ab) = (ra)b = a(rb)$.

Unital?

Although there are some important examples of *non-associative* algebras in which the associativity of multiplication is dropped, we will assume that an R-algebra is associative if no mention.

(a) The set of matrices $M_n(R)$ over a ring R is a unital R-algebra.

(b) The set of quaternions $\mathbb{H}$ is an $\mathbb{R}$-algebra.

4.2 Graded and filtered algebras

All of them are possible for R-modules?

4.2. Let V be a vector space over a field F. As vector spaces, define $T(V) := \bigoplus_{k=0}^{\infty} T^k(V)$, where $T^k(V) := V^{\otimes_R k}$. Then, it has a canonical algebra structure. This tensor algebra has the universal property. For any linear map $f : V \to A$ to an F-algebra A, there is a unique algebra homomorphism $\varphi : T(V) \to A$ such that

For any linear map $f : V \to A$ such that $f(v)^2 = 0$ for all $v \in V$, there is a unique algebra homomorphism $\varphi : \Lambda(v) \to A$ such that

4.3 (Multilinear forms)**.** A *multilinear form* is an element of $T^k(V)^*$. We have a canonical isomorphism $T^k(V)^* \cong T^k(V^*)$ defined such that

$$T^k(V^*) \to T^k(V)^* : v_1^* \otimes \cdots \otimes v_k^* \mapsto (v_1 \otimes \cdots \otimes v_k \mapsto v_1^*(v_1) \cdots v_k^*(v_k)),$$

The *alternatization* or the *anti-symmetrization* is an idempotent linear map $\mathrm{Alt} : T(V)^* \to T(V)^*$ defined degree-wise such that

$$\mathrm{Alt}(\omega)(v_1 \otimes \cdots \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}), \qquad \omega \in T^k(V)^*,\ v_j \in V,\ 1 \le j \le k.$$

An *alternating multilinear form* is an element of the image $\mathrm{Alt}(T(V)^*)$ of the alternatization.

For each $k \geq 0$ we canonically have a commutative diagram of linear maps

$$\begin{array}{ccccccc} \mathrm{Alt}(T^k(V)^*) & \subset & T^k(V)^* & \cong & T^k(V^*) & \twoheadrightarrow & \Lambda^k(V^*) \\ \cap & & \cap & & \cap & & \cap \\ \mathrm{Alt}(T(V)^*) & \subset & T(V)^* & \cong & T(V^*) & \longrightarrow\!\!\!\!\!\rightarrow & \Lambda(V^*) \end{array}$$

such that the horizontal composition $\mathrm{Alt}(T^k(V)^*) \to \Lambda^k(V^*)$ is a linear isomorphism for each degree $k \geq 0$. Then, we can describe the wedge product in terms of alternating forms by $\omega \wedge \eta := \mathrm{Alt}(\omega \otimes \eta)$, where the tensor product is induced from the identification $T(V)^* \cong T(V^*)$. Concretely,

$$(\omega \wedge \eta)(v_1 \otimes \cdots \otimes v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)})\eta(v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(k+l)}).$$

4.4 (Geometric convention). In geometry, we often differently choose the canonical isomorphism

$$T^k(V^*) \to T^k(V)^* : v_1^* \otimes \cdots \otimes v_k^* \mapsto (v_1 \otimes \cdots \otimes v_k \mapsto k!\ v_1^*(v_1) \cdots v_k^*(v_k)),$$

which makes $T^k(V)^*$ an algebra such that the geometric area of the unit hypercube $[0,1]^k$ is one, not $n!$. Then, to make the linear isomorphism $\mathrm{Alt}(T(V)^*) \to \Lambda(V^*)$ an algebra isomorphism, we have no choice but to define

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta), \qquad \omega \in \mathrm{Alt}(T^k(V)^*),\ \eta \in \mathrm{Alt}(T^l(V)^*),$$

or equivalently,

$$(\omega \wedge \eta)(v_1 \otimes \cdots \otimes v_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)})\eta(v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(k+l)}).$$

In this convention, we have

$$dx \wedge dy = dx \otimes dy - dy \otimes dx.$$

(geometric: Kobayashi-Nomizu convention, algebraic: Spivak convention)

4.3 Exterior algebras

4.5 (Determinants).

4.4 Symmetric algebras

Chapter 5

5.1 Clifford algebras

Let V be a quadratic vector space over a field k with a quadratic form Q, usually assumed to be non-degenerate. The *Clifford algebra* of V is defined as the universal map $V \to \mathrm{Cl}(V,Q)$ among linear maps $f : V \to A$ to a unital k-algebra such that $f(v)^2 = Q(v)$. We have a construction $T(V)/(v^2 - Q(v) : v \in V)$. Note that it is the exterior algebra if $Q = 0$. It has a natural $\mathbb{Z}/2\mathbb{Z}$-grading.

5.1 (Real Clifford algebras)**.** If $V = \mathbb{R}^n$, then the grading automorphism is represented by the Clifford multiplication of the complexified volume element $\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n$ of the complexified Clifford algebra, and the direct sum decomposition into even and odd parts is the eigenspace decomposition with respect to $\omega_{\mathbb{C}}$.

5.2. $\mathrm{Cl}(V,Q)$

Chapter 6

Semi-simple algebras

6.1 Artin-Wedderburn theorem

6.2 Character theory

6.3 Central simple algebras