Real Reductive Groups

Ikhan Choi
Lectured by Yoshiki Oshima
University of Tokyo, Spring 2023

April 19, 2023

Contents

1 Day 1: April 5 **2**

2 Day 2: April 19 **3**

1 Day 1: April 5

We know the finite dimensional representations of complex reductive Lie groups, which has a 1-1 correspondence with finite dimensional(unitary) reps of compact Lie groups via unitarian trick. For example, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$ belong to former, and $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are in the latter.

For the construction and classification of irreducible reps (highest weight theory) of complex reductive Lie groups, we have several methods:

- as quotients of a Verma module,
- as holomorphic sections of line bundles on a flag varieity (Borel-Weil theory).

For infinite dim reps of a real reductive Lie groups such as

$$\mathrm{SL}(n,\mathbb{R}),\quad \mathrm{GL}(n,\mathbb{R}),\quad \mathrm{O}(p,q)=\{g\in M_{p+q}(\mathbb{R}):{}^t g\mathrm{I}_{p,q}g=\mathrm{I}_{p,q}\}\ (\mathrm{I}_{p,q}:=I_p\oplus(-I_q)),$$

- asymptotic behaviors of matrix elements, quotients of principal series representations (Langlands)
- D-modules over flag variety (Beilinson-Bernstein, Brylinski-Kashiwara)
- minimal K-type (Vogar)

Classification of infinite-dimensional unitary reps is still unsolved.

Definition 1.1. A Lie group is informally both a manifold and a group. A C^∞ (complex) manifold is a Hausdorff second countable space that is locally homeomorphic to open sets in $\mathbb{R}^n$ ($\mathbb{C}^n$), such that the transition maps are C^∞ (holomorphic).

A Lie group is a group with a structure of C^∞ manifolds such that maps from the group structures $G\times G\to G:(g,g')\mapsto gg'$ and $G\to G:g\mapsto g^{-1}$ are C^∞. We can do same for complex Lie groups.

Example 1.2 (Lie groups)**.** $(\mathbb{R},+)$, $(\mathbb{R}^\times,\times)$, $\mathrm{GL}(n,\mathbb{R})$ (C^∞ structure is induced from $\mathbb{R}^{n^2}$ as an open subset), $\mathrm{SL}(n,\mathbb{R})$ (preimage theorem from) are Lie groups.

Example 1.3 (Complex Lie groups)**.** $(\mathbb{C}^n,+)$, $(\mathbb{C}^\times,\times)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$ are complex Lie groups. $U(n)$ is not complex.

Exercise. Check that the above examples.

The definitions of representations differ in references. In this lecture, we follow:

Definition 1.4 ((Finite dimensional) Representation)**.** Let G be a Lie group, V a finite-dimensional vector space over $\mathbb{C}$. A (finite-dimensional) representation is a Lie group homomoprphism $\pi:G\to GL_{\mathbb{C}}(V)$. We can do same for holomorphic representations.

Remark 1.5. For a group homomorphism $\pi:G\to\mathrm{GL}(V)$ from a Lie group G, TFAE:

(a) π is C^∞

(b) π is continuous

(c) $G\times V\to V$ is continuous.

Example 1.6. $(\det,\mathbb{C})$ and $(\mathrm{id}_{\mathrm{GL}(n,\mathbb{C})},\mathbb{C}^n)$ are holomorphic representations of $\mathrm{GL}(n,\mathbb{C})$. If we define $\mu^m:\mathbb{C}^\times\to\mathbb{C}^\times:z\mapsto z^m$, then $(\mu^m,\mathbb{C})$ is a holomorphic representation of both $\mathbb{C}^\times$ and $\mathrm{U}(1)$.

Definition 1.7. For two reps (π, V), (π', V') of G, we say thery are equivalent if there is a linear isomorphism $i : V \to V'$ such that $\pi(g)i = i\pi'(g)$ for all $g \in G$. For a subspace $W \subset V$, if $\pi(g)(W) \subset W$ for $g \in G$, then we say a representation (π_W, W) is a subrepresentation of (π, V). Irreducible representations are representations having only two subrepresentations. They are "minimal units" of representations.

For reps $(\pi_1, V_1), \cdots, (\pi_n, V_n)$ of G, we define the direct sum as a representation on $V_1 \oplus \cdots \oplus V_n$ with

$$(\pi_1 \oplus \cdots \oplus \pi_n)(g)(v_1, \cdots, v_n) := (\pi_1(g)v_1, \cdots, \pi_n(g)v_n).$$

Proposition 1.8 (Holomorphic representations of $\mathbb{C}^\times$ and $U(1)$)**.**

(a) *If (π, V) is a holomorphic representation of $\mathbb{C}^\times$, then there is $m_1, \cdots, m_n \in \mathbb{Z}$ such that*

$$\pi \sim \mu^{m_1} \oplus \cdots \oplus \mu^{m_n}.$$

(b) *If (π, V) is a holomorphic representation of* $\mathrm{U}(1)$*, then there is $m_1, \cdots, m_n \in \mathbb{Z}$ such that*

$$\pi \sim \mu^{m_1} \oplus \cdots \oplus \mu^{m_n}.$$

Proof. We first show the following lemma: If $(\pi, \mathbb{C}^n)$ is a representation of a Lie group $(\mathbb{R}, +)$, then there is $X \in M_n(\mathbb{C})$ such that $\pi(t) = \exp(tX)$ for $t \in \mathbb{R}$, i.e. π factors through $\mathbb{R} \to M_n(\mathbb{C}) : t \mapsto tX$.

Proof of the lemma: If we take a small open ball U of $M_n(\mathbb{C})$ centered at the origin, then $\exp : U \to \mathrm{GL}(n, \mathbb{C})$ is injective, so we can take t_0 small enough so that $\pi([-t_0, t_0]) \subset \exp(\frac{1}{2}U)$. Let $Y \in U, Z \in \frac{1}{2}U$ such that $\pi(t_0) = \exp(Y)$, $\pi(\frac{t_0}{2}) = \exp(Z)$. Then, $\pi(t_0) = \exp(2Z)$, so $Y = 2Z$. Repeating this, $\pi(\frac{t_0}{2^N}) = \exp(\frac{Y}{2^N})$ for all N. Since $\{\frac{M}{2^N}t_0\}$ is dense in $\mathbb{R}$ and π is continuous, $\pi(at_0) = \exp(aY)\ \forall a \in \mathbb{R}$. Thus we have $X = t_0^{-1}Y$ which satisfies the lemma. (Remark: we only have used the continuity of π, not the smoothness) Then we back to the proof of the proposition.

(b) By composition of $e : \mathbb{R} \to U(1) : t \mapsto e^{2\pi i t}$, we have a representation $(\pi \circ e, V)$ of $\mathbb{R}$. By the lemma, $\pi \circ e(t) = \exp(tX)$ for some $X \in M_n(\mathbb{C})$, and it satisfies $\exp(X) = \pi \circ e(1) = \pi(1) = \mathrm{I}_n$. Since X is diagonalizable, we have

$$X \sim 2\pi i \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix} \quad \Rightarrow \quad \pi(z) = \begin{pmatrix} z^{m_1} & & 0 \\ & \ddots & \\ 0 & & z^{m_n} \end{pmatrix}.$$

(a) $U(1) \to \mathbb{C}^\times \to \mathrm{GL}(V)$. By the identity theorem from complex analysis, we have

$$\pi(z) = \begin{pmatrix} z^{m_1} & & 0 \\ & \ddots & \\ 0 & & z^{m_n} \end{pmatrix}.$$

□

2 Day 2: April 19

Reference: Kobayashi-Oshima, Carter-Segal-Macdonald, Warner

Remark 2.1. From the above proposition, we have

$$\left\{ \text{ representations of } U(1) \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{holomorphic} \\ \text{representations of } \mathrm{GL}(1, \mathbb{C}) \end{array} \right\}.$$

More generally, Weyl's unitarian trick states

$$\left\{\text{ representations of } U(n) \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{holomorphic} \\ \text{representations of } \mathrm{GL}(n,\mathbb{C}) \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{representations of} \\ \text{a compact Lie group} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{holomorphic representations of} \\ \text{a complex reductive Lie group} \end{array} \right\}.$$

Remark 2.2. In particular, a holomorphic representation of $\mathbb{C}^\times$ is the direct sum of irreducible representations. However, a holomorphic representation of $\mathbb{C}$ may not be the direct sum of irreducible representations, i.e. not completely reducible. We have a counterexample $\mathbb{C} \to \mathrm{GL}(2,\mathbb{C}) : t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Every finite-dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ is completely reducible.

Let $G = \mathrm{GL}(n,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{C})$. Let X be For each $\lambda = (\lambda_i) \in \mathbb{Z}^n$, a holomorphic line bundle L_λ over X is determined.

Theorem 2.3. $\Gamma(X, L_\lambda) \neq 0$ *if and only if* $\lambda_1 \geq \cdots \geq \lambda_n$.

[ccc] $\Gamma(\mathrm{U}_1, L_{\mathcal{O}(k)}) \cong$ {holomorphic functions on $\mathbb{C}$}
z$_1$: z_2
$\mapsto ([z_1 : z_2], f_1(z_1/z_2)) \mapsto f$.

every integral weight corresponds to a holomorphic line bundle