Harmonic Functions Vanishing at Infinity on the Punctured Domain

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1 Kelvin transform and decay rates

2 Potential field from a source

Theorem. *Let $d \geq 3$. A distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity if and only if there is a distribution $\rho \in \mathcal{D}'(\mathbb{R}^d)$ such that $u = \Phi * \rho$ and $\operatorname{supp}(\rho) \subset \{0\}$, where Φ denotes the fundamental solution of Laplace's equation.*

Proof. ($\Rightarrow$) Define a distribution ρ by

$$\langle \rho, \varphi \rangle := -\langle u, \Delta\varphi \rangle$$

for $\varphi \in C_c^\infty(\mathbb{R}^d)$. In other words, $\rho = -\Delta u$ in distributional sense. Then, ρ has the support contained in $\{0\}$ because if $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ then

$$\langle \rho, \varphi \rangle = -\langle u, \Delta\varphi \rangle = -\int u(x)\Delta\varphi(x)\,dx = -\int \Delta u(x)\varphi(x)\,dx = 0.$$

Therefore, we only need to verify $u = \Phi * \rho$ to complete the proof.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Be cautious that the argument

$$\langle \Phi * \rho, \varphi \rangle = \langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle = \langle u, \varphi \rangle$$

fails to provide a proof because the function $\Phi * \rho$ is not compactly supported so that we cannot deduce $\langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$, and here we use the condition that

u vanishes at infinity to justify the equality. Define a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \le \frac{5}{4} \\ 0 & \text{if } |x| \ge \frac{7}{4} \end{cases}.$$

If we denote $\chi_r(x) := \chi(\frac{x}{r})$, then we have

$$\langle \rho, (\Phi\chi_r) * \varphi \rangle = -\langle u, \Delta((\Phi\chi_r) * \varphi)\rangle$$

by the definition of ρ. We have the limit of the left-hand side

$$\lim_{r\to\infty} \langle \rho, (\Phi\chi_r) * \varphi \rangle = \langle \rho, \Phi * \varphi\rangle$$

because

$$\begin{aligned} \operatorname{supp}((\Phi(1-\chi_r) * \varphi) &\subset \operatorname{supp}(\Phi(1-\chi_r)) + \operatorname{supp}(\varphi) \\ &\subset \mathbb{R}^d \setminus B(0,2R) + \overline{B}(0,R) = \mathbb{R}^d \setminus B(0,R) \end{aligned}$$

for all $r > 2R$ so that the supports of $\Phi(1-\chi_r) * \varphi$ and ρ are disjoint, where we define $R := \sup_{x\in\operatorname{supp}(\varphi)} |x|$. However, the right-hand limit

$$-\lim_{r\to\infty} \langle u, \Delta((\Phi\chi_r) * \varphi)\rangle = -\langle u, \Delta(\Phi * \varphi)\rangle$$

is not a trivial result.

Assuming $\chi(x) = \chi(-x)$ without loss of generality, we have

$$\langle u, \Delta(\Phi(1-\chi_r) * \varphi)\rangle = \langle u * \Delta(\Phi(1-\chi_r)), \varphi\rangle.$$

Because

$$\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big] = 0$$

for $|y| < R$ and $x \in \operatorname{supp}(\varphi)$ if $r > 2R$, we can write

$$\langle u * \Delta(\Phi(1-\chi_r)), \varphi\rangle = \int \varphi(x) \int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\,dy\,dx.$$

We compute

$$\begin{aligned} \Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big] &= 2\nabla\Phi(x-y)\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r}) - \Phi(x-y)\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r}) \\ &= -\frac{2}{\omega_d}\frac{x-y}{|x-y|^d}\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r}) - \frac{1}{(d-2)\omega_d}\frac{1}{|x-y|^{d-2}}\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r}). \end{aligned}$$

Then, since $\frac{5}{4}r \le |x-y| \le \frac{7}{4}r$ if $\nabla\chi(\frac{x-y}{r}) \neq 0$ and $\Delta\chi(\frac{x-y}{r}) \neq 0$, we obtain

$$\left|\Delta_y\left[\Phi(x-y)\left(1-\chi(\tfrac{x-y}{r})\right)\right]\right| \le C\frac{1}{r^d}\psi(\frac{x-y}{r})$$

for some constant $C > 0$, where

$$\psi(y) := |\nabla\chi(y)| + |\Delta\chi(y)|.$$

For each $x \in \operatorname{supp}(\varphi)$, since we have $\frac{5}{4}r \le |x-y| \le \frac{7}{4}r$ implies $r \le |y| \le 2r$ if $r > 4R$, it follows that

$$\begin{aligned}|\int u(y)\Delta_y\left[\Phi(x-y)\left(1-\chi(\tfrac{x-y}{r})\right)\right]dy| &\le C\int |u(y)\frac{1}{r^d}\psi(\frac{x-y}{r})|\,dy \\ &\le C\max_{r\le|y|\le 2r} u(y)\end{aligned}$$

converges to zero as $r \to \infty$. By the bounded convergence theorem, we can deduce

$$\lim_{r\to\infty}\int \varphi(x)\int u(y)\Delta_y\left[\Phi(x-y)\left(1-\chi(\tfrac{x-y}{r})\right)\right]dy\,dx = 0,$$

so we are done.

($\Leftarrow$) Let $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Since

$$\langle \Phi * \rho, \Delta\varphi\rangle = \langle \rho, \Phi * (\Delta\varphi)\rangle = \langle \rho, \varphi\rangle = 0,$$

the distribution $\Phi * \rho$ on $\mathbb{R}^d \setminus \{0\}$ is weakly harmonic, and by Weyl's lemma for distributions, it is a smooth harmonic function on $\mathbb{R}^d \setminus \{0\}$.

Since ρ is supported at zero, we have a positive integer k and constants a_α such that

$$|\langle \rho, \varphi\rangle| \le \sum_{|a|\le k} |a_\alpha D^\alpha \varphi(0)|$$

for $\varphi \in C^\infty(\mathbb{R}^d)$. Then, for non-zero $x \in \mathbb{R}^d$, by taking a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(y) = \begin{cases} 1 & \text{if } |y-x| \le \frac{1}{3}|x| \\ 0 & \text{if } |y| \le \frac{1}{3}|x| \end{cases},$$

we have

$$|\Phi * \rho(x)| = |(\Phi\chi) * \rho(x)| = |\langle \rho(x-y), \Phi(y)\chi(y)\rangle_y| \le \sum_{|a|\le k} |a_\alpha D^\alpha \Phi(x)| = O(r^{2-d})$$

as $r \to \infty$. Therefore, $\Phi * \rho$ vanishes at infinity. $\square$

Lemma. *Let ρ be a distribution on $\mathbb{R}^d$ such that* $\operatorname{supp}(\rho) \subset \{0\}$. *Then, there is a constant coefficient partial differential operator $P(D)$ such that $\rho = P(D)\delta$.*

Corollary. *Let $d \geq 3$. If a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity, then there are an integer $k \geq 0$ and constants a_α such that*

$$u(x) = \sum_{|a| \leq k} a_\alpha D^\alpha \Phi(x)$$

for $x \neq 0$, where Φ denotes the fundamental solution of Laplace's equation.