

# Foundations of Calculus

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November 5, 2022

# Preface

the main objectives the audience the structure of the book how to use this book acknowledgements  
references

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**Part I**

**Sequences**

# Chapter 1

## Metric spaces

### 1.1 Metric spaces

**1.1** (Definition of metric spaces). Let  $X$  be a set. A *metric* is a function  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  such that

- (i)  $d(x, y) = 0$  if and only if  $x = y$ , (nondegeneracy)
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ , (symmetry)
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ . (triangle inequality)

A pair  $(X, d)$  of a set  $X$  and a metric on  $X$  is called a *metric space*. We often write it simply  $X$ .

- (a) A normed space  $X$  is a metric space with a metric defined by  $d(x, y) := \|x - y\|$ .
- (b) A subset of a metric space is a metric space with a metric given by restriction.

**1.2** (System of open balls). A metric is often misunderstood as something that measures a distance between two points and belongs to the study of geometry. The main function of a metric is to make a system of small balls, sets of points whose distance from specified center points is less than fixed numbers. The balls centered at each point provide a concrete images of “system of neighborhoods at a point” in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly.

Note that taking either  $\varepsilon$  or  $\delta$  in analysis really means taking a ball of the very radius. Investigation of the distribution of open balls centered at a point is now an important problem.

Let  $X$  be a metric space. A set of the form

$$\{y \in X : d(x, y) < \varepsilon\}$$

for  $x \in X$  and  $\varepsilon > 0$  is called an *open ball centered at  $x$  with radius  $\varepsilon$*  and denoted by  $B(x, \varepsilon)$  or  $B_\varepsilon(x)$ .

**1.3** (Convergence and continuity in metric spaces). Let  $\{x_n\}_n$  be a sequence of points on a metric space  $(X, d)$ . We say that a point  $x$  is a *limit* of the sequence or the sequence *converges to  $x$*  if for arbitrarily small ball  $B(x, \varepsilon)$ , we can find  $n_0$  such that  $x_n \in B(x, \varepsilon)$  for all  $n > n_0$ . If it is satisfied, then we write

$$\lim_{n \rightarrow \infty} x_n = x,$$

or simply  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

A function  $f : X \rightarrow Y$  between metric spaces is called *continuous at  $x \in X$*  if for any ball  $B(f(x), \varepsilon) \subset Y$ , there is a ball  $B(x, \delta) \subset X$  such that  $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ . The function  $f$  is called *continuous* if it is continuous at every point on  $X$ .

- (a) A sequence  $x_n$  in a metric space  $X$  converges to  $x \in X$  if and only if  $d(x_n, x)$  converges to zero.
- (b) Let  $f : X \rightarrow Y$  be a function between two metric spaces. If there is a constant  $C$  such that  $d(x, y) \leq C d(f(x), f(y))$  for all  $x$  and  $y$  in  $X$ , then  $f$  is continuous. In this case,  $f$  is particularly called *Lipschitz continuous* with the *Lipschitz constant*  $C$ .

1.4 (Separable metric spaces). separable iff second countable iff lindelof

## 1.2 Normed spaces

banach space

## 1.3 Open sets and closed sets

convergence, limit point

## 1.4 Compact sets

Bolzano-Weierstrass

## 1.5 Connected sets

## Exercises

## Problems

## Chapter 2

# Real sequences

### 2.1 Monotone sequences

preserving inequalities limsup and liminf monotone convergence

### 2.2 Extended real numbers

2.1 (Operations in the extended real numbers). We can extend addition (except  $\infty + (-\infty)$ ), subtraction, multiplication (except  $\infty \times 0$ ), division (except dividing by zero).

2.2 (Limits in the extended real numbers).

### 2.3 Asymptotic analysis

sufficiently large asymptotic expressions growth and decay

Approximate sequences( $\varepsilon/3$ )

2.3 (Change of limits).

$$|a_n - a| \leq |a_n - b_{mn}| + |b_{mn} - b_m| + |b_m - a|$$

$$\limsup_m \sup_n |a_n - b_{mn}| = 0$$

$$\lim_n |b_{mn} - b_m| = 0$$

$$a_n = b_{mn} + c_{mn} \leq b_{mn} + \varepsilon$$

## Exercises

2.4.

2.5 (Newton method).

## Problems

1. Show that every real sequence  $(a_n)_{n=1}^{\infty}$  has a subsequence  $(a_{n_k})_{k=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n$ .



# Chapter 3

## Series

### 3.1 Absolute convergence

3.1 (Unconditional convergence).

### 3.2 Convergence tests

comparison limit comparison cauchy condensation integral....

ratio root

3.2 (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_k B_k - A_{k-1} B_{k-1}$$
$$\sum_{m < k \leq n} A_k b_k = A_n B_n - A_m B_m - \sum_{m < k \leq n} a_k B_{k-1}.$$

abel test

3.3 (Dirichlet test).

3.4 (Mertens' theorem). If  $\sum_{k=0}^{\infty} a_k$  converges to  $A$  absolutely and  $\sum_{k=0}^{\infty} b_k$  converges to  $B$ , then their Cauchy product  $\sum_{k=0}^{\infty} c_k$  with  $c_k := \sum_{l=0}^k a_l b_{k-l}$  converges to  $AB$ .

(a) We have

$$\lim_{m \rightarrow \infty} \sup_n \sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l = 0.$$

(b) We have for each  $m$  that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^m \sum_{l=n-k+1}^n a_k b_l = 0$$

*Proof.* Let

$$A_n := \sum_{k=0}^n a_k, \quad B_n := \sum_{k=0}^n b_k, \quad \text{and} \quad C_n := \sum_{k=0}^n c_k.$$

As  $m \rightarrow \infty$ .

$$\left| \sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l \right| \leq \sum_{k=m+1}^n |a_k| \left| \sum_{l=n-k+1}^n b_l \right| = \sum_{k=m+1}^n |a_k| |B_n - B_{n-k}| \lesssim \sum_{k=m+1}^{\infty} |a_k| \rightarrow 0.$$

For fixed  $m$ , as  $n \rightarrow \infty$ ,

$$\left| \sum_{k=0}^m \sum_{l=n-k+1}^n a_k b_l \right| \leq \sum_{k=0}^m |a_k| \left| \sum_{l=n-k+1}^n b_l \right| = \sum_{k=0}^m |a_k| |B_n - B_{n-k}| \rightarrow \sum_{k=0}^m |a_k| |B - B| = 0.$$

We will prove

$$A_n B_n - C_n = \sum_{k=0}^n \sum_{l=n-k+1}^n a_k b_l \rightarrow 0$$

as  $n \rightarrow \infty$ . For  $\varepsilon > 0$ , take  $m$  such that

$$\left| \sup_n \sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l \right| < \varepsilon.$$

Then for every  $n$  we have

$$\left| \sum_{k=0}^n \sum_{l=n-k+1}^n a_k b_l \right| \leq \varepsilon + \left| \sum_{k=0}^m \sum_{l=n-k+1}^n a_k b_l \right|.$$

Taking limits  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  in order, we are done. □

## Exercises

3.5 (Cesàro mean).

3.6 (Recursive sine sequence). Let  $a_{n+1} = \sin a_n$  and  $a_n = 1$ . We can use  $\sin x = x - \frac{x^3}{6} + O(x^5)$ .

$$a_n = \sqrt{3}n^{-\frac{1}{2}} - \frac{3\sqrt{3}}{20}n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}).$$

3.7 (Convergence rates of recursive sequences). If  $a_{n+1} = a_n - f(a_n)$ ,  $f(0) = 0$ ,  $f(x) > 0$  for  $0 < x < \varepsilon$ ,  $f \in C^2$ ? then

$$f'(a_n) \sim \lim_{x \rightarrow 0+} \frac{f'(x)^2}{f''(x)f(x)} \frac{1}{n}.$$

## Problems

1. If  $a_n \rightarrow 0$ , then  $\frac{1}{n} \sum_{k=1}^n a_k \rightarrow 0$ .
2. If  $a_n \geq 0$  and  $\sum a_n$  diverges, then  $\sum \frac{a_n}{1+a_n}$  also diverges.

# **Part II**

# **Functions**

## Chapter 4

# Continuity

### 4.1 Intermediate and extreme value theorems

left and right limits semicontinuous

### 4.2 Various continuities

#### Exercises

#### Problems

1. The set of local minima of a convex real function is connected.
2. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. The equation  $f(x) = c$  cannot have exactly two solutions for every constant  $c \in \mathbb{R}$ .
3. A continuous function that takes on no value more than twice takes on some value exactly once.
4. Let  $f$  be a function that has the intermediate value property. If the preimage of every singleton is closed, then  $f$  is continuous.

## Chapter 5

# Differentiation

### 5.1 Differentiability

5.1 (L'hospital theorem).

### 5.2 Monotonicity and convexity

### 5.3 Mean value theorem

5.2 (Taylor theorem).

### 5.4 Smooth functions

### Exercises

5.3 (Variations on the mean value theorem). Let  $f$  be a differentiable function on the unit closed interval.

- (a) If  $f(0) = 0$  there is  $c$  such that  $cf'(c) = f(c)$ . (Flett)
- (b) If  $f(0) = 0$  there is  $c$  such that  $cf(c) = (1 - c)f'(c)$ .

5.4 (Dini derivatives).

5.5 (Darboux theorem).

### Problems

1. If  $\lim_{x \rightarrow \infty} f(x) = a$  and  $\lim_{x \rightarrow \infty} f'(x) = b$ , then  $a = 0$ .
2. Let  $f$  be a real  $C^2$  function with  $f(0) = 0$  and  $f''(0) \neq 0$ . Defined a function  $\xi$  such that  $f(x) = xf'(\xi(x))$  with  $|\xi| \leq |x|$ , we have  $\xi'(0) = 1/2$ .
3. Let  $f$  be a  $C^2$  function such that  $f(0) = f(1) = 0$ . We have  $\|f\| \leq \frac{1}{8}\|f''\|$ .
4. A smooth function such that for each  $x$  there is  $n$  having the  $n$ th derivative vanish is a polynomial.
5. If a real  $C^1$  function  $f$  satisfies  $f(x) \neq 0$  for  $x$  such that  $f'(x) = 0$ , then in a bounded set there are only finite points at which  $f$  vanishes.

6. Let a real function  $f$  be differentiable. For  $a < a' < b < b'$  there exist  $a < c < b$  and  $a' < c' < b'$  such that  $f(b) - f(a) = f'(c)(b - a)$  and  $f(b') - f(a') = f'(c')(b' - a')$ .

## Chapter 6

# Integration

### 6.1 Riemann integral

tagged partition

### 6.2 Henstock-Kurzweil intergral

bounded compact support  $\leftrightarrow$  lebesgue

### 6.3 Improper integral

### 6.4 Fundamental theorem of calculus for continuous functions

### Exercises

### Problems

1. Find the value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x) dx \right)$ .
2. Find all  $a > 0$  and  $b > 0$  such that  $\int_0^\infty x^{-b} |\tan x|^a dx$  converges.
- \*3. If  $xf'(x)$  is bounded and  $x^{-1} \int_0^x f \rightarrow L$  then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .

## **Part III**

# **Function spaces**



## Chapter 7

# Continuous functions

### 7.1 Uniform convergence

*Proof.* Divide the error

$$|f(x_n) - f(x)| \leq |f(x_n) - f_m(x_n)| + |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|.$$

Using the uniform convergence, we can take  $m$  such that  $\|f_m - f\| < \varepsilon$ , so we have

$$|f(x_n) - f(x)| < \varepsilon + |f_m(x_n) - f_m(x)| + \varepsilon.$$

Then, taking  $\limsup_{n \rightarrow \infty}$  on the both-hand sides, we get

$$\limsup_{n \rightarrow \infty} |f(x_n) - f(x)| \leq \varepsilon + 0 + \varepsilon = 2\varepsilon.$$

Since  $\varepsilon > 0$  has been arbitrarily taken,

$$\lim_{n \rightarrow \infty} |f(x_n) - f(x)| = 0.$$

□

### 7.2 Arzela-Ascoli theorem

### 7.3 Stone-Weierstrass theorem

### Exercises

7.1 (Weierstrass' nowhere differentiable function).

### Problems

- \*1. If a sequence of real functions  $f_n : [0, 1] \rightarrow [0, 1]$  satisfies  $|f(x) - f(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$ , then it has a uniformly convergent subsequence.

## Chapter 8

# Differentiable functions

### 8.1 Differentiable class

completeness

### 8.2 Hölder spaces

### 8.3 Analytic functions

Power series uniform convergence and absolute convergence, abel theorem? differentiation convergence of radius, complex domain sum, product, composition, reciprocal? closed under uniform convergence identity theorem

## Chapter 9

# Integrable functions

### 9.1

9.1 (Lebesgue criterion of Riemann integrability).

**Part IV**

**Multivariable Calculus**

## Chapter 10

# Fréchet derivatives

### 10.1 Tangent spaces

10.1 (Vector fields).

### 10.2 Inverse function theorem

# Chapter 11

## Differential forms

### 11.1 Multilinear algebra

11.1 (Tensor product).

11.2 (Wedge product).

11.3 (One-forms).

11.4 (Multiple integral). volume forms, stone weierstrass and fubini

### 11.2 Vector calculus

11.5 (Exterior derivative).

11.6 (Musical isomorphisms).

11.7 (Inner product of differential forms). ONB

11.8 (Hodge star operator). Identification of 2-forms and vector fields

11.9 (Gradient, curl, and divergence).

11.10 (Potentials).

11.11 (Vector calculus identities).

### Exercises

11.12 (Multivariable Taylor's theorem). Symmetric product

11.13 (Vector analysis in two dimension).

11.14 (Geometric algebra).

## Chapter 12

# Stokes theorems

### 12.1 Local coordinates

12.1 (Spherical coordinates). Let  $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = 0, y \geq 0\}$ .

$$(x, y, z) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$$

for  $(r, \theta, \varphi) \in (0, \infty) \times (0, \pi) \times (0, 2\pi)$ . Orthonormal bases are

$$\left( \partial_r, \frac{1}{r} \partial_\theta, \frac{1}{r \sin \theta} \partial_\varphi \right),$$

$$(dr, r d\theta, r \sin \theta d\varphi),$$

$$(r^2 \sin \theta d\theta \wedge d\varphi, r \sin \theta d\varphi \wedge dr, r dr \wedge d\theta).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}.$$

*Proof.* Write  $df$  in the orthonormal basis

$$\begin{aligned} df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi \\ &= \left( \frac{\partial f}{\partial r} \right) dr + \left( \frac{1}{r} \frac{\partial f}{\partial \theta} \right) r d\theta + \left( \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) r \sin \theta d\varphi. \end{aligned}$$

After taking the Hodge star operator

$$\begin{aligned} *df &= \left( \frac{\partial f}{\partial r} \right) r^2 \sin \theta d\theta \wedge d\varphi + \left( \frac{1}{r} \frac{\partial f}{\partial \theta} \right) r \sin \theta d\varphi \wedge dr + \left( \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) r dr \wedge d\theta \\ &= r^2 \sin \theta \frac{\partial f}{\partial r} d\theta \wedge d\varphi + \sin \theta \frac{\partial f}{\partial \theta} d\varphi \wedge dr + \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} dr \wedge d\theta, \end{aligned}$$

the differential is computed as

$$\begin{aligned} d * df &= d \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right) d\theta \wedge d\varphi + d \left( \sin \theta \frac{\partial f}{\partial \theta} \right) d\varphi \wedge dr + d \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right) dr \wedge d\theta \\ &= \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right] dr \wedge d\theta \wedge d\varphi, \end{aligned}$$

so that we have

$$\begin{aligned}\Delta f &= *d*df = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}\end{aligned}$$

□

## 12.2 Integration on curves and surfaces

12.2 (Line integral).

12.3 (Surface integral).

## 12.3 Stokes theorems

12.4 (Bump functions).

12.5 (Partition of unity).

12.6.