

## Contents

<b>1</b>	<b>Nets of measurable functions</b>	<b>2</b>
<b>2</b>	<b>Potential from a source</b>	<b>3</b>
<b>3</b>	<b>Kinetic theory</b>	<b>5</b>
3.1	Velocity averaging lemmas . . . . .	5
<b>4</b>	<b>Sturm-Liouville theory</b>	<b>6</b>
4.1	Self-adjointness . . . . .	6
4.2	Regular Sturm-Liouville problem . . . . .	7
4.3	Legendre's equation . . . . .	8
4.4	Bessel's equation . . . . .	8
<b>5</b>	<b>Peetre's theorem</b>	<b>9</b>
<b>6</b>	<b>Characteristic curve</b>	<b>10</b>
6.1	Wave equation . . . . .	11
6.2	Burgers' equation . . . . .	12
<b>7</b>	<b>Statements in functional analysis and general topology</b>	<b>13</b>
<b>8</b>	<b>Ultrafilter</b>	<b>13</b>
<b>9</b>	<b>Selected analysis problems</b>	<b>14</b>
<b>10</b>	<b>Physics problem</b>	<b>17</b>
10.1	Resonance . . . . .	17

## 1 Nets of measurable functions

### 1.1. (a)

If  $f_\alpha$  is continuous, then  $f$  is lower semi-continuous. We use the inner regularity of the measure on the open set  $f^{-1}(j2^{-n}, \infty)$ .

## 2 Potential from a source

**Theorem.** Let  $d \geq 3$ . A distribution  $u \in \mathcal{D}'(\mathbb{R}^d)$  is a harmonic function on  $\mathbb{R}^d \setminus \{0\}$  and vanishes at infinity if and only if there is a distribution  $\rho \in \mathcal{D}'(\mathbb{R}^d)$  such that  $u = \Phi * \rho$  and  $\text{supp}(\rho) \subset \{0\}$ , where  $\Phi$  denotes the fundamental solution of Laplace's equation.

*Proof.* ( $\Rightarrow$ ) Define a distribution  $\rho$  by

$$\langle \rho, \varphi \rangle := -\langle u, \Delta \varphi \rangle$$

for  $\varphi \in C_c^\infty(\mathbb{R}^d)$ . In other words,  $\rho = -\Delta u$  in distributional sense. Then,  $\rho$  has the support contained in  $\{0\}$  because if  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$  then

$$\langle \rho, \varphi \rangle = -\langle u, \Delta \varphi \rangle = -\int u(x) \Delta \varphi(x) dx = -\int \Delta u(x) \varphi(x) dx = 0.$$

Therefore, we only need to verify  $u = \Phi * \rho$  to complete the proof.

Let  $\varphi \in C_c^\infty(\mathbb{R}^d)$ . Be cautious that the argument

$$\langle \Phi * \rho, \varphi \rangle = \langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle = \langle u, \varphi \rangle$$

fails to provide a proof because the function  $\Phi * \rho$  is not compactly supported so that we cannot deduce  $\langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$ , and here we use the condition that  $u$  vanishes at infinity to justify the equality. Define a cutoff function  $\chi \in C_c^\infty(\mathbb{R}^d)$  such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{5}{4} \\ 0 & \text{if } |x| \geq \frac{7}{4} \end{cases}.$$

If we denote  $\chi_r(x) := \chi(\frac{x}{r})$ , then we have

$$\langle \rho, (\Phi \chi_r) * \varphi \rangle = -\langle u, \Delta((\Phi \chi_r) * \varphi) \rangle$$

by the definition of  $\rho$ . We have the limit of the left-hand side

$$\lim_{r \rightarrow \infty} \langle \rho, (\Phi \chi_r) * \varphi \rangle = \langle \rho, \Phi * \varphi \rangle$$

because

$$\begin{aligned} \text{supp}((\Phi(1 - \chi_r) * \varphi)) &\subset \text{supp}(\Phi(1 - \chi_r)) + \text{supp}(\varphi) \\ &\subset \mathbb{R}^d \setminus B(0, 2R) + B(0, R) = \mathbb{R}^d \setminus B(0, R) \end{aligned}$$

for all  $r > 2R$  so that the supports of  $\Phi(1 - \chi_r) * \varphi$  and  $\rho$  are disjoint, where we define  $R := \sup_{x \in \text{supp}(\varphi)} |x|$ . However, the right-hand limit

$$-\lim_{r \rightarrow \infty} \langle u, \Delta((\Phi \chi_r) * \varphi) \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$$

is not a trivial result.

Assuming  $\chi(x) = \chi(-x)$  without loss of generality, we have

$$\langle u, \Delta(\Phi(1 - \chi_r) * \varphi) \rangle = \langle u * \Delta(\Phi(1 - \chi_r)), \varphi \rangle.$$

Because

$$\Delta_y \left[ \Phi(x - y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] = 0$$

for  $|y| < R$  and  $x \in \text{supp}(\varphi)$  if  $r > 2R$ , we can write

$$\langle u * \Delta(\Phi(1 - \chi_r)), \varphi \rangle = \int \varphi(x) \int u(y) \Delta_y \left[ \Phi(x - y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy dx.$$

We compute

$$\begin{aligned}\Delta_y \left[ \Phi(x-y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] &= 2\nabla\Phi(x-y) \cdot \frac{1}{r} \nabla\chi\left(\frac{x-y}{r}\right) - \Phi(x-y) \frac{1}{r^2} \Delta\chi\left(\frac{x-y}{r}\right) \\ &= -\frac{2}{\omega_d} \frac{x-y}{|x-y|^d} \cdot \frac{1}{r} \nabla\chi\left(\frac{x-y}{r}\right) - \frac{1}{(d-2)\omega_d} \frac{1}{|x-y|^{d-2}} \frac{1}{r^2} \Delta\chi\left(\frac{x-y}{r}\right).\end{aligned}$$

Then, since  $\frac{5}{4}r \leq |x-y| \leq \frac{7}{4}r$  if  $\nabla\chi\left(\frac{x-y}{r}\right) \neq 0$  and  $\Delta\chi\left(\frac{x-y}{r}\right) \neq 0$ , we obtain

$$\left| \Delta_y \left[ \Phi(x-y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] \right| \leq C \frac{1}{r^d} \psi\left(\frac{x-y}{r}\right)$$

for some constant  $C > 0$ , where

$$\psi(y) := |\nabla\chi(y)| + |\Delta\chi(y)|.$$

For each  $x \in \text{supp}(\varphi)$ , since we have  $\frac{5}{4}r \leq |x-y| \leq \frac{7}{4}r$  implies  $r \leq |y| \leq 2r$  if  $r > 4R$ , it follows that

$$\begin{aligned}\left| \int u(y) \Delta_y \left[ \Phi(x-y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy \right| &\leq C \int |u(y)| \frac{1}{r^d} \psi\left(\frac{x-y}{r}\right) dy \\ &\leq C \max_{r \leq |y| \leq 2r} u(y)\end{aligned}$$

converges to zero as  $r \rightarrow \infty$ . By the bounded convergence theorem, we can deduce

$$\lim_{r \rightarrow \infty} \int \varphi(x) \int u(y) \Delta_y \left[ \Phi(x-y) \left( 1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy dx = 0,$$

so we are done.

( $\Leftarrow$ ) Let  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ . Since

$$\langle \Phi * \rho, \Delta\varphi \rangle = \langle \rho, \Phi * (\Delta\varphi) \rangle = \langle \rho, \varphi \rangle = 0,$$

the distribution  $\Phi * \rho$  on  $\mathbb{R}^d \setminus \{0\}$  is weakly harmonic, and by Weyl's lemma for distributions, it is a smooth harmonic function on  $\mathbb{R}^d \setminus \{0\}$ .

Since  $\rho$  is supported at zero, we have a positive integer  $k$  and constants  $a_\alpha$  such that

$$|\langle \rho, \varphi \rangle| \leq \sum_{|\alpha| \leq k} |a_\alpha D^\alpha \varphi(0)|$$

for  $\varphi \in C^\infty(\mathbb{R}^d)$ . Then, for non-zero  $x \in \mathbb{R}^d$ , by taking a cutoff function  $\chi \in C_c^\infty(\mathbb{R}^d)$  such that

$$\chi(y) = \begin{cases} 1 & \text{if } |y-x| \leq \frac{1}{3}|x| \\ 0 & \text{if } |y| \leq \frac{1}{3}|x| \end{cases},$$

we have

$$|\Phi * \rho(x)| = |(\Phi\chi) * \rho(x)| = |\langle \rho(x-y), \Phi(y)\chi(y) \rangle_y| \leq \sum_{|\alpha| \leq k} |a_\alpha D^\alpha \Phi(x)| = O(r^{2-d})$$

as  $r \rightarrow \infty$ . Therefore,  $\Phi * \rho$  vanishes at infinity.  $\square$

**Lemma.** Let  $\rho$  be a distribution on  $\mathbb{R}^d$  such that  $\text{supp}(\rho) \subset \{0\}$ . Then, there is a constant coefficient partial differential operator  $P(D)$  such that  $\rho = P(D)\delta$ .

**Corollary.** Let  $d \geq 3$ . If a distribution  $u \in \mathcal{D}'(\mathbb{R}^d)$  is a harmonic function on  $\mathbb{R}^d \setminus \{0\}$  and vanishes at infinity, then there are an integer  $k \geq 0$  and constants  $a_\alpha$  such that

$$u(x) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha \Phi(x)$$

for  $x \neq 0$ , where  $\Phi$  denotes the fundamental solution of Laplace's equation.

### 3 Kinetic theory

#### 3.1 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

**Theorem 3.1** (Velocity averaging). *Let  $L$  be a free transport operator  $\partial_t + v \cdot \nabla_x$  on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . Then,*

$$\left\| \int u \varphi dv \right\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}$$

for  $\varphi \in C_c^\infty(\mathbb{R}_v^n)$ ,

*Proof.* Let  $m(t, x) = \int u \varphi dv$ . By Fourier transform with respect to  $t$  and  $x$ , we have

$$\widehat{u}(\tau, \xi, v) = \frac{1}{i} \frac{\widehat{Lu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v) \varphi(v) dv.$$

Fixing  $\tau, \xi$ , decompose the integral and use Hölder's inequality to get

$$\begin{aligned} |\widehat{m}(\tau, \xi)| &\leq \int_{|\tau + v \cdot \xi| < \alpha} |\widehat{u} \varphi| dv + \int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\widehat{Lu} \varphi|}{|\tau + v \cdot \xi|} dv \\ &\leq \|\widehat{u}\|_{L_v^2}^{1/2} \left( \int_{|\tau + v \cdot \xi| < \alpha} |\varphi|^2 dv \right)^{1/2} + \|\widehat{Lu}\|_{L_v^2}^{1/2} \left( \int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2} dv \right)^{1/2}, \end{aligned}$$

where  $\alpha > 0$  is an arbitrary constant that will be determined later. Let

$$I_s(\tau, \xi, \alpha) := \int_{|\tau + v \cdot \xi| < \alpha} |\varphi|^2 dv, \quad I_n(\tau, \xi, \alpha) := \int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|} dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \quad I_n \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

Define coordinates  $(v_1, v_2)$  on  $\mathbb{R}_v$  as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}, \quad v_2 := v - \frac{v \cdot \xi}{|\xi|^2} \xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = \left(v_1 - \frac{\tau}{|\xi|}\right)^2 + |v_2|^2 \quad \text{and} \quad \int dv = \iint dv_2 dv_1.$$

For the first integral, suppose that  $\varphi$  is supported on a ball  $|v| \leq R$ . If  $\frac{|\tau| - \alpha}{|\xi|} > R$ , then the region of integration vanishes so that  $I_s = 0$ . If  $|\tau| \leq \alpha + R|\xi|$ , then

$$\begin{aligned} I_s &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 dv_1 \\ &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}, |v_1| \leq R} \int_{|v_2| \leq R} dv_2 dv_1 \\ &\lesssim \min\left\{\frac{2\alpha}{|\xi|}, R\right\} \cdot R^{n-1} \\ &\approx \frac{1}{\sqrt{1 + \left(\frac{|\xi|}{\alpha}\right)^2}} \\ &\lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}. \end{aligned}$$

For the second integral, suppose that  $\varphi$  is supported on  $|\nu| < R$  so that  $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < R$ . Then,

$$\begin{aligned} I_n &\lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|}| < R} \int_{|v_2| < R} \frac{1}{v_1^2 |\xi|^2} dv_2 dv_1 \\ &\simeq \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - R\} \leq v_1 < \frac{|\tau|}{|\xi|} + R} \frac{1}{v_1^2 |\xi|^2} dv_1 \\ &\simeq \frac{1}{|\xi|^2} \left( \frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - R\}} - \frac{1}{\frac{|\tau|}{|\xi|} + R} \right). \end{aligned}$$

If  $\frac{|\tau|}{|\xi|} - R > \frac{\alpha}{|\xi|}$ , then

$$I_n \lesssim \frac{2R}{\tau^2 - (R|\xi|)^2} < \frac{2R}{\alpha(|\tau| + R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

If  $|\tau| \leq \alpha + R|\xi|$ , then

$$I_n \lesssim \frac{1}{|\xi|} \frac{(|\tau| + R|\xi|) - \alpha}{\alpha(|\tau| + R|\xi|)} \leq \frac{2R}{\alpha(|\tau| + R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

To sum up, we have

$$|\widehat{m}(\tau, \xi)| \lesssim \frac{1}{(\tau^2 + |\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Lu}\|_{L_v^2}^{1/2}).$$

Letting  $\alpha = \sqrt{\|\widehat{Lu}\|_{L_v^2} / \|\widehat{u}\|_{L_v^2}}$  and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2} \|\widehat{Lu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on  $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$  and Plancherel's theorem gives

$$\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}.$$

□

**Corollary 3.2.** *Let  $\mathcal{F}$  be a family of functions on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  and  $L\mathcal{F}$  are bounded in  $L_{t,x,v}^2$ , then  $\int \mathcal{F} \varphi dv$  is bounded in  $H_{t,x}^{1/2}$ .*

**Theorem 3.3.** *Let  $\mathcal{F}$  be a family of functions on  $I_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  is weakly relatively compact and  $L\mathcal{F}$  is bounded in  $L_{t,x,v}^1$ , then  $\int \mathcal{F} \varphi dv$  is relatively compact in  $L_{t,x}^1$ .*

## 4 Sturm-Liouville theory

### 4.1 Self-adjointness

Let  $I = [a, b]$  and

$$\begin{aligned} L &= -\frac{1}{w(x)} \left[ \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \right], \\ 0 &\leq p(x) \in C^\infty(I), \quad q(x) \in C^\infty(I), \quad 0 < w(x) \in C^\infty(I). \end{aligned}$$

We expect  $L$  to be self-adjoint. In this regard, our interest is elimination of the difference term

$$\langle f, Lg \rangle - \langle Lf, g \rangle = p(f'g - fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[ \frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right) - \frac{m^2}{1-x^2} \right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2} \left[ \frac{d}{dx} \left( e^{-x^2} \frac{d}{dx} \right) \right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

## 4.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that  $p$  does not vanish on the boundary of  $I$  that we should cancel  $f'g - fg'|_a^b$ . View the Sturm-Liouville operator  $L$  as a non-densely defined operator on the space  $C^\infty(I)$  with inner product  $\langle f, g \rangle = \int_I f g w$  with domain

$$V = \{u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0, \beta_0 u(b) + \beta_1 u'(b) = 0\},$$

the subspace for the *separated* Robin boundary condition.

**Proposition 4.1.** *The operator  $L : V \rightarrow C^\infty(I)$  is self-adjoint when  $C^\infty(I)$  has the inner product  $\langle f, g \rangle = \int_I f g w$ .*

We are interested in the eigenvalue problem of  $L : V \rightarrow C^\infty(I)$  on  $V$ . Fortunately, if we choose a constant  $z \in \mathbb{C} \setminus \mathbb{R}$ , then  $(L - z)^{-1} : C^\infty(I) \rightarrow V$  is well-defined.

**Proposition 4.2.** *If  $z$  is not an eigenvalue of  $L$ , then  $L - z : V \rightarrow C^\infty(I)$  is bijective.*

*Proof.* The injectivity follows from the definition of eigenvalues. We may assume that  $L$  is injective by translation  $q \mapsto q - \lambda$ .

Suppose  $f \in C^\infty(I)$ . The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$\begin{aligned} -pu'' - p'u' - qu &= f w, \\ \alpha_0 u(a) + \alpha_1 u'(a) &= 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0. \end{aligned}$$

Let  $u_a, u_b$  be the unique solutions of the corresponding homogeneous equation with initial conditions

$$u_a(a) = -\alpha_1, \quad u'_a(a) = \alpha_0, \quad u_b(b) = -\beta_1, \quad u'_b(b) = \beta_0.$$

Then we can define  $L^{-1} : C^\infty([0, 1]) \rightarrow D(L)$  by

$$L^{-1}f(x) := u_a(x) \int_x^b \frac{u_b}{W[u_a, u_b]} \frac{f}{(-p)} w + u_b(x) \int_a^x \frac{u_a}{W[u_a, u_b]} \frac{f}{(-p)} w,$$

where  $W[u_a, u_b] := u_a u'_b - u_b u'_a$  denotes the Wronskian. This formula is derived from variation of parameters: we can compute  $c_a$  and  $c_b$  from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u'_a & u'_b \end{pmatrix} \begin{pmatrix} c'_a \\ c'_b \end{pmatrix} \implies L(c_a u_a + c_b u_b) = f.$$

Then, we can check that

$$L^{-1}Lu = u$$

for  $u \in D(L)$  by computation, which implies  $L$  is surjective. □

### 4.3 Legendre's equation

The Legendre equation is

$$(1-x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right).$$

Since  $p(\pm 1) = 0$ , the operator  $L : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1])$  is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

$l$	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x \log \frac{1+x}{1-x}, \quad \dots \in L^2(-1, 1) \setminus C^\infty([-1, 1])$$

as eigenvectors of  $L$ , then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x) \left( Q'_0(x)f(x) - Q_0(x)f'(x) \right) \Big|_{-1}^1 \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for  $f \in C^\infty([-1, 1])$ .

### 4.4 Bessel's equation

The Bessel equation is

$$x^2 u'' + x u' + (k^2 x^2 - \nu^2) u = 0, \quad \text{on } (0, \infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x} \left[ \frac{d}{dx} \left( x \frac{d}{dx} \right) - \nu^2 \frac{1}{x} \right].$$



## 5 Peetre's theorem

**Lemma 5.1.** Suppose a linear operator  $L : C_c^\infty(M) \rightarrow C_c^\infty(M)$  satisfies

$$\text{supp}(Lu) \subset \text{supp}(u) \quad \text{for } u \in C_c^\infty(X).$$

For each point  $x \in M$ , there is a bounded neighborhood  $U$  together with a nonnegative integer  $m$  such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for  $u \in C_c^\infty(U \setminus \{x\})$ .

*Proof.* Suppose not. There is a point  $x$  at which the inequality fails; for every bounded neighborhood  $U$  and for every nonnegative  $m$ , we can find  $u \in C_c^\infty(U \setminus \{x\})$  such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large  $C$ . We want to construct a function  $u \in C_c^\infty(U)$  such that  $Lu$  has a singularity at  $x$ .

(Induction step) Take a bounded neighborhood  $U_m$  of  $x$  such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U_i}.$$

There is  $u_m \in C_c^\infty(U_m \setminus \{x\})$  such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that  $u \in C_c^\infty(U)$  since the series converges in the inductive topology of the LF space  $C_c^\infty(U)$ : it converges absolutely with respect to the seminorms  $\|\cdot\|_{C^m}$  for all  $m$ :

$$\begin{aligned} \sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\ &\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\ &< \infty. \end{aligned}$$

Also, since the supports of each term are disjoint and  $L$  is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction. □

## 6 Characteristic curve

Algorithm:

- (a) Establish the associated vector field by substituting  $u \mapsto y$ .
- (b) Find the integral curve.
- (c) Eliminate the auxiliary variables to get an algebraic equation.
- (d) Verify the computed solution is in fact the real solution.

**Proposition 6.1.** *Suppose that there exists a smooth solution  $u : \Omega \rightarrow \mathbb{R}_y$  of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, (t, x) \in \Omega \subset \mathbb{R}_{t \geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, \text{ at } x \in \mathbb{R}, \end{cases}$$

and let  $M$  be the embedded surface defined by  $y = u(t, x)$ .

Let  $\gamma : I \rightarrow \Omega \times \mathbb{R}_y$  be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that  $\gamma(0) \in M$ . Then,  $\gamma(\theta) \in M$  for all  $\theta \in I$ .

*Proof.* We may assume  $\gamma$  is maximal. Define  $\tilde{\gamma} : \tilde{I} \rightarrow M$  as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that  $\tilde{\gamma}(0) = \gamma(0)$ . Since  $X$  and  $\tilde{X}$  coincide on  $M$ , the curve  $\tilde{\gamma}$  is also an integral curve of  $X$  with  $\tilde{\gamma}(0) = \gamma(0)$ . By the uniqueness of the integral curve, we get  $\tilde{I} \subset I$  and  $\gamma(\theta) = \tilde{\gamma}(\theta)$  for all  $\theta \in \tilde{I}$ .

Since  $M$  is closed in  $E$ , the open interval  $\tilde{I} = \gamma^{-1}(M)$  is closed in  $I$ , hence  $\tilde{I} = I$  by the connectedness of  $I$ . □

**Definition 6.2.** The projection of the integral curve  $\gamma$  onto  $\Omega$  is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface  $M$  explicitly by finding the integral curves of the vector field  $X$ . Once we find a necessary condition of the form of algebraic equation, we can demonstrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since  $X$  does not depend on  $u$ , we can solve the ODE: let  $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$  be the integral curve of  $X$  such that  $\gamma(0) = (0, \xi, \xi)$ . Then, the system of ODEs

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi \end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad y(\theta) = \xi, \quad x(\theta) = \xi^2 \theta + \xi.$$

Therefore,

$$u(t, x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain  $\Omega$  as

$$\Omega = \{(t, x) : tx > -\frac{1}{4}\}.$$

## 6.1 Wave equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } t, x > 0, \\ u(0, x) &= g(x), \quad u_t(0, x) = h(x), \quad u_x(t, 0) = \alpha(t). \end{aligned}$$

Define  $v := u_t - cu_x$ . Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0, x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t, x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0, x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t, 0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter  $\xi > 0$ ,

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t, x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi) \end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad x(\theta) = -c\theta + \xi, \quad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta') d\theta',$$

hence for  $x > ct > 0$ ,

$$\begin{aligned} u(t, x) &= g(\xi) - \int_0^\theta v(s, \xi - cs) ds \\ &= g(x + ct) \\ &= \frac{3g(x + ct) - g(x - ct)}{2} - \int_0^t h(x + c(t - 2s)) ds \end{aligned}$$

## 6.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

- (a) Suppose  $u(0, x) = \tanh(x)$ . For what values of  $t > 0$  does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (b) Suppose  $u(0, x) = -\tanh(x)$ . For what values of  $t > 0$  does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (c) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and “paste” the solution together.

## 7 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator  $T$  induces a Hilbert space structure on its domain. If the inclusion is bounded, then  $T$  has the bounded inverse. If the inclusion is compact, then  $T$  has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting  $L^2$  inner product on  $C([0, 1])$ , define  $\phi(f) = \int_0^{\frac{1}{2}} f$ .
- Every separable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem  $\rightarrow$  continuous embedding is really an embedding.
- $D(\Omega)$  is defined by a countable strict inductive limit of  $D_K(\Omega)$ ,  $K \subset \Omega$  compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever  $\alpha < \beta$  the induced topology by  $\mathcal{T}_\beta$  coincides with  $\mathcal{T}_\alpha$ )
- A net  $(\phi_d)_d$  in  $D(\Omega)$  converges if and only if there is a compact  $K$  such that  $\phi_d \in D_K(\Omega)$  for all  $d$  and  $\phi_d$  converges uniformly.
- The integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as  $\delta$  is called *singular*.
- $D'$  is equipped with the weak\* topology.
- $\frac{\partial}{\partial x} : D' \rightarrow D'$  is continuous. They commute (Schwarz theorem holds).
- $D \rightarrow S \rightarrow L^p$  are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$  and  $H \subset \widehat{\mathbb{C}}$  have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

## 8 Ultrafilter

**Definition 8.1.** An *ultrafilter* is a synonym for maximal filter. If we say  $\mathcal{U}$  is an *ultrafilter* on a set  $A$ , then it means  $\mathcal{U}$  is a maximal filter as a directed subset of  $\mathcal{P}(A)$ .

existence of ultrafilter.

**Theorem 8.2.** Let  $\mathcal{U}$  be an ultrafilter on a set  $A$  and  $X$  be a compact space. For a function  $f : A \rightarrow X$ , the limit  $\mathcal{U}\text{-}\lim f$  always exists.

**Theorem 8.3.** Let  $X = \prod_{\alpha \in A} X_\alpha$  be a product space of compact spaces  $X_\alpha$ . A net  $f : \mathcal{D} \rightarrow X$  has a convergent subnet.

1. Use Tychonoff. Compactness and net compactness are equivalent. □
2. It is a proof without Tychonoff. Let  $\mathcal{U}$  be an ultrafilter on a set  $\mathcal{D}$  containing all  $\uparrow d$ . Define a directed set  $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$  as  $(d, U) \succ (d', U')$  for  $U \subset U'$ . Let  $f : \mathcal{E} \rightarrow X$  be a subnet of  $f : \mathcal{D} \rightarrow X$  defined by  $f_{(d, U)} = f_d$ .

By the previous theorem,  $\mathcal{U}\text{-}\lim \pi_\alpha f_d \in X_\alpha$  exists for each  $\alpha$ . Define  $f \in X$  such that  $\pi_\alpha f = \mathcal{U}\text{-}\lim \pi_\alpha f_d$ . Let  $G = \prod_\alpha G_\alpha \subset X$  be any open neighborhood of  $f$ . Then,  $\pi_\alpha f \in G_\alpha$  and we have  $G_\alpha = X_\alpha$  except finite. For  $\alpha$ , we can take  $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$  by definition of convergence with ultrafilter. Since  $U_\alpha = \mathcal{D}$  except finites, we can take an upper bound  $U_0 \in \mathcal{U}$  of  $\{U_\alpha\}_\alpha$ . Then, by taking any  $d_0 \in U_0$ , we have  $f_{(d, U)} \in G$  for every  $(d, U) \succ (d_0, U_0)$ . This means  $f = \lim_{\mathcal{E}} f_{(d, U)}$ , so we can say  $\lim_{\mathcal{E}} f_{(d, U)}$  exists. □

## 9 Selected analysis problems

9.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

*Solution.* Let  $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$ . Divide the unit circle  $\mathbb{R}/2\pi\mathbb{Z}$  by  $7k$  uniform arcs. There are at least  $2^k/7k$  integers that do not exceed  $2^k$  and are in a same arc. Let  $S$  be the integers and  $x_0$  be the smallest element. Since,  $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$  for  $x \in S$ ,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also,  $1 \leq x - x_0 \leq x \leq 2^k$ ,  $x - x_0 \in A_k$ .

$$|A_k| \geq \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\ &\geq \sum_{k=1}^N (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^N \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &> \sum_{k=1}^N \frac{2^k}{2^{k+2}} \frac{1}{7k} \\ &= \frac{1}{28} \sum_{k=1}^N \frac{1}{k} \\ &\rightarrow \infty. \end{aligned}$$

□

**9.2.** If  $|xf'(x)| \leq M$  and  $\frac{1}{x} \int_0^x f(y) dy \rightarrow L$ , then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .

*Solution.* It is a kind of Tauberian theorems. Since for each fixed  $\varepsilon > 0$  we have

$$\begin{aligned} |f(x) - \frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x f(y) dy| &\leq \frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x |f(x) - f(y)| dy \\ &\leq \frac{M}{\varepsilon x} \int_{(1-\varepsilon)x}^x \frac{x-y}{y} dy \\ &= M \left( \frac{1}{\varepsilon} \log \frac{1}{1-\varepsilon} - 1 \right) = O(\varepsilon) \end{aligned}$$

by the mean value theorem and

$$\frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x f(y) dy = \frac{1}{\varepsilon x} \int_0^x f(y) dy - \frac{1}{\varepsilon x} \int_0^{(1-\varepsilon)x} f(y) dy \rightarrow \frac{1}{\varepsilon} L - \frac{1-\varepsilon}{\varepsilon} L = L$$

as  $x \rightarrow \infty$ , we get

$$\limsup_{x \rightarrow \infty} |f(x) - L| = O(\varepsilon),$$

so we are done. □

**9.3.** Let  $f_n : [0, 1] \rightarrow [0, 1]$  be a sequence of functions such that  $|f_n(x) - f_n(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$  for each  $n \geq 1$ . Then, it has a uniformly convergent subsequence.

*Solution.* By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that  $f_n$  converges to a function  $f : \mathbb{Q} \cap [0, 1] \rightarrow [0, 1]$  pointwisely.

Let  $n \geq 4$ . Then, for  $x \in [0, 1]$  there is  $z \in [0, 1]$  such that  $|x - z| = \frac{2}{n}$  so that

$$|f_n(x) - f_n(z)| \leq |x - z| = \frac{2}{n}.$$

Whenever  $y \in [0, 1]$  satisfies  $|x - y| \leq \frac{1}{n}$ , then we have  $|y - z| \geq |x - z| - |x - y| \geq \frac{1}{n}$ , so we get

$$|f_n(y) - f_n(z)| \leq |y - z| \leq |y - x| + |x - z| \leq \frac{3}{n}.$$

Combining the two inequalities, we obtain

$$|x - y| \leq \frac{1}{n} \implies |f_n(x) - f_n(y)| \leq \frac{5}{n} \quad (1)$$

for  $n \geq 4$ .

Let  $\varepsilon > 0$  and suppose  $|x - y| \leq \frac{\varepsilon}{5}$ . For every  $n \geq \max\{\frac{10}{\varepsilon}, 4\}$ , since  $|x - y| \leq \frac{1}{n}$  implies by the inequality (1) that

$$|f_n(x) - f_n(y)| \leq \frac{5}{n} \leq \frac{\varepsilon}{2},$$

and since  $|x - y| > \frac{1}{n}$  implies by the condition in the problem that

$$|f_n(x) - f_n(y)| \leq |x - y| \leq \frac{\varepsilon}{5} < \frac{\varepsilon}{2},$$

we have

$$|x - y| \leq \frac{\varepsilon}{5} \implies |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{2} \quad (2)$$

for all  $n \geq \max\{\frac{10}{\varepsilon}, 4\}$ .

For  $\varepsilon > 0$ , take  $\delta := \varepsilon/5$  and fix  $x$  and  $y$  in  $\mathbb{Q} \cap [0, 1]$  satisfying  $|x - y| < \delta$ . Then, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + \frac{\varepsilon}{2} + |f_n(y) - f(y)| \end{aligned}$$

for all  $n \geq \max\{\frac{10}{\varepsilon}, 4\}$ , and by limiting  $n \rightarrow \infty$ ,

$$|f(x) - f(y)| \leq 0 + \frac{\varepsilon}{2} + 0 < \varepsilon.$$

Therefore,  $f$  is uniformly continuous on  $\mathbb{Q} \cap [0, 1]$  so that it has a unique continuous extension on the whole  $[0, 1]$ . Let it denoted by the same notation  $f$ .

Finally, we are going to show  $f_n \rightarrow f$  uniformly on  $[0, 1]$ . By the uniform continuity of  $f$ , for each  $\varepsilon > 0$  we have  $\delta > 0$  such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}. \quad (3)$$

Take a finite subset  $F \in \mathbb{Q} \cap [0, 1]$ , such that for every  $x$  there is  $y$  satisfying  $|x - y| < \min\{\frac{\varepsilon}{5}, \delta\}$ . Then, by (2) and (3), we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(y)| + |f_n(y) - f(y)| + |f(y) - f(x)| \\ &< \frac{\varepsilon}{2} + \max_{z \in F} |f_n(z) - f(z)| + \frac{\varepsilon}{2} \end{aligned}$$

for all  $n \geq \max\{\frac{10}{\varepsilon}, 4\}$ . Therefore, by taking supremum for  $x$  and limiting  $n \rightarrow \infty$  on it we have

$$\limsup_{n \rightarrow \infty} \|f_n - f\| \leq \varepsilon,$$

so we are done because  $\varepsilon$  is arbitrary. □



## 10 Physics problem

### 10.1 Resonance

Let  $m, b, k, A, \omega_d$  be positive real constants. Consider an underdamped oscillator with sinusoidal driving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0, \quad x'(0) = 0.$$

There are some observations:

- (a) The underdamping condition means  $b^2 - 4mk < 0$  so that the roots of characteristic equation are imaginary.
- (b) The positivity of  $m, b$  implies the real part of solution that will be denoted by  $-\beta = -\frac{b}{2m}$  is negative; it shows exponential decay of solutions.
- (c) Introducing the natural frequency  $\omega_n = \sqrt{k/m}$ , we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega t.$$

- (d) The complementary solution is computed as

$$x_c(t) = x_0 e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \rightarrow \infty} x_c(t) = 0.$$

- (e) The condition  $\beta > \omega_n$  is equivalent to that the oscillator is underdamped.
- (f) Let  $m, k$  be fixed. Then, the solution  $x_c$  decays most fastly when  $b$  satisfied  $b^2 = 4mk$ , equivalently,  $\beta = \omega_n$ .
- (g) When  $\omega_d = \omega_n$  such that the amplitude of particular solution diverges.