Functional Analysis

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Contents

Part I

Topological vector spaces

Chapter 1

Locally convex spaces

1.1 The Hahn-Banach theorem

Chapter 2

Barreled spaces

2.1 (The Baire category theorem).

2.2 (Barreled spaces). A barrel.

If a closed convex cone contains a dense subset of absorbing at a point, then it is entire?

2.3 (Uniform boundedness principle). Let $f : S \subset X \to \mathbb{R}_{\geq 0}$. Suppose $\|T_\alpha x\| \leq f(x)$ on S.

(a) $S \subset \bigcup_{n=1}^\infty \bigcap_\alpha T_\alpha^{-1} B_n$.

(b) If X is the closed linear span of S, then $\bigcap_\alpha T_\alpha^{-1} B_1$ is a barrel of X.

2.4 (Open mapping theorem). Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Suppose T is surjective.

(a) There is $r > 0$ such that $B_r \subset \overline{TB_1}$.

(b) There is $r > 0$ such that $B_r \subset TB_1$.

(c) T is open.

(d) T is open even for complete locally convex X and barreled Y.

Proof. (a) The set $\overline{TB_1}$ is clearly closed and absolutely convex. The surjectivity of T implies $\overline{TB_1}$ is absorbing. Since Y is barreled, $\overline{TB_1}$ contains an open ball B_r.

(b) Let $r > 0$ such that $B_r \subset \overline{TB_{1/2}}$. For $y \in B_r$, we are going to construct $x \in B_1 \subset X$ such that $y = Tx$. We claim for n that

$$(y + TB_{1-1/2^n}) \cap B_{r/2^n} \neq \varnothing.$$

We have

$$(y + B_{r/2}) \cap TB_{1/2} \neq \varnothing,$$

and

$$(y + TB_{1/2}) \cap B_{r/2} \neq \varnothing,$$

and

$$(y + TB_{1/2}) \cap \overline{TB_{1/4}} \neq \varnothing,$$

and

$$(y + TB_{1/2} + B_{r/4}) \cap TB_{1/4} \neq \varnothing,$$

and

$$(y + TB_{3/4}) \cap B_{r/4} \neq \varnothing.$$

□

2.5. Let (T_n) be a sequence in $B(X, Y)$. If T_n coverges then $\|T_n\|$ is bounded by the uniform boundedness principle.

2.6. We show that there is no projection from ℓ^∞ onto c_0.

(a) Show that a Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

2.7 (Bounded below maps in Banach spaces)**.** Let $T : X \to Y$ be a bounded linear map between Banach spaces. Show that the following statements are equivalent:

(a) It is bounded below.

(b) It is injective and has closed range.

(c) It is a isometric isomorphism onto its image.

2.8 (Bounded below maps in Hilbert spaces)**.** Let $T : H \to K$ be a bounded linear operator between Hilbert spaces. Show that the following statements are equivalent:

(a) It is bounded below.

(b) It has a left inverse.

(c) Its adjoint has right inverse.

(d) The product T^*T is invertible.

In particular, a normal operator in $B(H)$ is bounded below if and only if it is invertible.

2.9 (Injectivity and surjectivity of dual map)**.** Let $T : X \to Y$ be a bounded linear operator between Banach spaces and $T^* : Y^* \to X^*$ be its dual.

(a) Show that T^* is injective if and only if T has dense range.

(b) Show that T^* is surjective if and only if T is bounded below.

2.10. For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. If T is normal, then the kernel of T and T^* are equal.

(a) Show that if T is surjective bounded operator, then T is invertible.

2.11 (Schur's property of ℓ^1)**.** .

2.12. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

Chapter 3

Fréchet, Banach, and Hilbert spaces

Part II

Weak topologies

Chapter 4

Weak* topologies

4.1 The Banach-Alaoglu theorem

4.2 The Krein-Milman theorem

4.1 (Predual correspondence)**.** Let X be a Banach space and Z be a linear subspace of X^*. Define $\varphi : X \to Z^*$ as the restriction of the dual map of inclusion $Z \subset X^*$.

(a) Show that if φ is an isometric isomorphism, then closed ball of X is compact Hausdorff in $\sigma(X, Z)$.

(b) Show that the converse holds by using Goldstine's theorem.

4.2. Let X be a closed subspace of a Banach space Y and

$$i : X \to Y$$

the inclusion. Suppose X and Y have preduals X_* and Y_* respectively. Let

$$j := i^*|_{Y_*} : Y_* \to Z \subset X^*,$$

where $Z := i^*(Y_*)^-$. Then we can show

$$j^* : Z^* \subset X^{**} \to Y$$

coincides with i on $X \cap Z^*$. From the existence of X_* we have $X^{**} \to X$, which is restricted to define a map $k : Z^* \to X$.

$$\begin{tikzcd}
& X \arrow[r, "i"] & Y \\
X^{**} \arrow[ur] \arrow[r] & Z^* \arrow[u, "k"] \arrow[ur, "j"] &
\end{tikzcd}$$

We can show k is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

Chapter 5

Distribution theory

Chapter 6

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6.1 (Compact left multiplications and SOT). Let T_n be a sequence of bounded linear operators on a Hilbert space that converges in SOT. For compact K, T_nK converges in norm, but KT_n generally does not unless T is self-adjoint.

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11.1 (Operator monotonicity of square and commitativity)**.** Let $\mathcal{A}$ be a C*-algebra in which the square function is operator monotone, that is, $0 \le a \le b$ implies $a^2 \le b^2$ for any positive elements a and b in $\mathcal{A}$. We are going to show that $\mathcal{A}$ is necessarily commutative. Let a and b denote arbitrary positive elements of $\mathcal{A}$.

(a) Show that $ab + ba \ge 0$.

(b) Let $ab = c + id$ where c and d are self adjoints. Show that $d^2 \le c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \le c^2$. Show that $c^2d^2 + d^2c^2 - 2\lambda d^4 \ge 0$.

(d) Show that $\lambda(cd + dc)^2 \le (c^2 - d^2)^2$.

(e) Show that $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \le c^2$ and deduce $d = 0$.

(f) Extend the result for general exponent: $\mathcal{A}$ is commitative if $f(x) = x^\beta$ is operator monotone for $\beta > 1$.

11.2 (Injective *-homomorphism is an isometry)**.**

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