

# Functional Analysis

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# Contents

<b>I</b>	<b>Topological vector spaces</b>	<b>3</b>
<b>1</b>	<b>Locally convex spaces</b>	<b>4</b>
1.1	Vector topologies . . . . .	4
1.2	Seminorms and convex sets . . . . .	4
1.3	Continuous linear functionals . . . . .	4
<b>2</b>	<b>Barreled spaces</b>	<b>6</b>
2.1	Uniform boundedness principle . . . . .	6
2.2	Baire category theorem . . . . .	6
2.3	Open mapping theorem . . . . .	7
<b>3</b>	<b>Weak topologies</b>	<b>9</b>
3.1	Dual spaces . . . . .	9
3.2	Weak compactness . . . . .	10
3.3	Weak density . . . . .	10
3.4	Krein-Milman theorem . . . . .	10
3.5	Polar topologies . . . . .	10
<b>II</b>	<b>Banach spaces</b>	<b>12</b>
<b>4</b>	<b>Operators on Banach spaces</b>	<b>13</b>
4.1	Bounded operators . . . . .	13
4.2	Compact operators . . . . .	13
4.3	Fredholm operators . . . . .	13
4.4	Nuclear operators . . . . .	14
<b>5</b>	<b>Geometry of Banach spaces</b>	<b>16</b>
5.1	Tensor products . . . . .	16
5.2	Approximation property . . . . .	16
<b>6</b>		<b>17</b>
<b>III</b>	<b>Spectral theory</b>	<b>18</b>
<b>7</b>	<b>Operators on Hilbert spaces</b>	<b>19</b>
7.1	Operator topologies . . . . .	19
7.2	Closed operators . . . . .	20
7.3	Spectral theorems . . . . .	21
7.4	Decomposition of spectrum . . . . .	23

<b>8</b>	<b>Operator theory</b>	<b>24</b>
8.1	Toeplitz operators . . . . .	24
<b>9</b>		<b>25</b>
<b>IV</b>	<b>Operator algebras</b>	<b>26</b>
<b>10</b>	<b>Banach algebras</b>	<b>27</b>
10.1	Spectra of elements . . . . .	27
10.2	Ideals . . . . .	29
10.3	Holomorphic functional calculus . . . . .	30
10.4	Gelfand theory . . . . .	30
<b>11</b>	<b><math>C^*</math>-algebras</b>	<b>32</b>
11.1	$C^*$ identity . . . . .	32
11.2	Continuous functional calculus . . . . .	32
11.3	Positive elements . . . . .	34
11.4	Representations of $C^*$ -algebras . . . . .	35
<b>12</b>	<b>Von Neumann algebras</b>	<b>37</b>
12.1	Density theorems . . . . .	37
12.2	Borel functional calculus . . . . .	38
12.3	Predual . . . . .	39

## **Part I**

# **Topological vector spaces**

# Chapter 1

## Locally convex spaces

### 1.1 Vector topologies

1.1 (Canonical uniformity and bornology).

1.2 (Metrizability). Birkhoff-Kakutani

1.3 (Boundedness of linear operators).

### 1.2 Seminorms and convex sets

1.4 (Seminorms).

$$\bigcap_{i=1}^m \{p_i < 1\}$$

Equivalent conditions on the continuity of seminorms

*Proof.*

□

boundedness by seminorms, normability

### 1.3 Continuous linear functionals

1.5. Let  $\overline{x^*} = (x_1^*, \dots, x_n^*) \in X^{*n}$ .  $\overline{x^*} : X \rightarrow \mathbb{F}^n$ . If  $x^* \in X^*$  vanishes on  $\bigcap_{i=1}^n \ker x_i^*$ , then  $x^*$  is a linear combination of  $\{x_i^*\}$ .

1.6 (Hahn-Banach extension). Let  $X$  be a real vector space. Suppose  $V$  is a linear subspace of  $X$  and  $l : V \rightarrow \mathbb{R}$  is a linear functional dominated by a sublinear functional  $q : X \rightarrow \mathbb{R}$ , that is,  $l(v) \leq q(v)$  for all  $v \in V$ .

- (a) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$ .
- (b) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$  and is dominated by  $q$  if  $\dim X/V = 1$ .
- (c) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$  and is dominated by  $q$ .

*Proof.* (a) It can be done by the Hamel basis.

(b) Let  $e \in X \setminus V$  so that every vector  $x \in X$  can be uniquely written as  $x = v + te$  with  $v \in V$  and  $t \in \mathbb{R}$ . For  $v_1, v_2 \in V$ ,

$$l(v_1) + l(v_2) = l(v_1 + v_2) \leq q(v_1 + v_2) \leq q(v_1 - e) + q(v_2 + e)$$

implies

$$l(v_1) - q(v_1 - e) \leq -l(v_2) + q(v_2 + e).$$

Define a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  such that  $\tilde{l}(v) = v$  and

$$l(v) - q(v - e) \leq \tilde{l}(e) \leq -l(v) + q(v + e)$$

for all  $v \in V$ . Since

$$\tilde{l}(v + te) = l(v) + t\tilde{l}(e) \leq l(v) + t(-l(t^{-1}v) + q(t^{-1}v + e)) = q(v + te)$$

if  $t \geq 0$  and

$$\tilde{l}(v + te) = l(v) + t\tilde{l}(e) \leq l(v) + t(l(-t^{-1}v) - q(-t^{-1}v - e)) = q(v + te)$$

if  $t \leq 0$ , we have  $\tilde{l}(x) \in q(x)$  for all  $x \in X$ .

(c) With  $X$  and  $q$ , Consider a partially ordered set

$$\{(\tilde{V}, \tilde{l}) \mid V \leq \tilde{V} \leq X, \tilde{l} : \tilde{V} \rightarrow \mathbb{R} \text{ is a linear extension of } l \text{ dominated by } q\}$$

such that  $(V_1, l_1) \prec (V_2, l_2)$  if and only if  $V_1 \leq V_2$  and  $l_2|_{V_1} = l_1$ . The nonemptiness and the chain condition is easily satisfied, so it has a maximal element  $(\tilde{V}, \tilde{l})$  by the Zorn lemma. By the part (b), we have  $\tilde{V} = X$ .  $\square$

**1.7 (Complex linear functionals).** Let  $X$  be a complex vector space. Consider a map

$$\begin{array}{ccc} \{\mathbb{C}\text{-linear functionals on } X\} & \rightarrow & \{\mathbb{R}\text{-linear functionals on } X\} \\ l & \mapsto & \operatorname{Re} l. \end{array}$$

Let  $p$  be a seminorm on  $X$  and  $l$  a complex linear functional on  $X$ .

(a) The above map is bijective.

(b)  $|l(x)| \leq p(x)$  if and only if  $|\operatorname{Re} l(x)| \leq p(x)$ .

*Proof.* (b) There is  $\lambda$  such that  $|\lambda| = 1$  and  $l(\lambda x) \geq 0$ . Then,

$$|l(x)| = |\lambda^{-1}l(\lambda x)| = l(\lambda x) = \operatorname{Re} l(\lambda x) \leq p(\lambda x) = p(x).$$

$\square$

**1.8 (Hahn-Banach separation).**

## Exercises

**1.9 (Topology of compact convergence).**

## Chapter 2

# Barreled spaces

### 2.1 Uniform boundedness principle

**2.1** (Barreled spaces). Let  $X$  be a topological vector space. A *barrel* is an absorbing, balanced, convex, and closed subset of  $X$ . A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

**2.2** (Uniform boundedness principle). Let  $X$  and  $Y$  be topological vector spaces. Let  $\mathcal{F}$  be a family of continuous linear operator from  $X$  to  $Y$ . Suppose  $\bigcup_{T \in \mathcal{F}} Tx$  is bounded for each  $x \in D$ , where  $D \subset X$ .

- (a) If  $D$  is dense in  $X$ , then  $\bigcap_{T \in \mathcal{F}} T^{-1}\overline{U}$  is absorbing.
- (b) If  $X$  is barreled, then  $\mathcal{F}$  is equicontinuous.

### 2.2 Baire category theorem

**2.3** (Baire spaces). A topological space is called a *Baire space* if the countable intersection of open dense subsets is always dense.

- (a) If a topological vector space is Baire, then it is barreled.
- (b) A Baire space is second category in itself.
- (c) A topological group that is second category in itself is Baire.

**2.4** (Absorbing sets). Let  $X$  be a topological vector space that is Baire. A subset  $U \subset X$  is said to be *absorbing* if for every  $x \in X$  there is a sufficiently large  $t > 0$  such that  $x \in tU$ . Let  $U \subset X$ .

- (a) If  $U$  is closed and absorbing, then  $U$  has a non-empty open subset.
- (b) If  $U$  is closed and absorbing, then  $U - U$  is a neighborhood of zero.
- (c) If  $U$  is closed, convex, and absorbing, then  $U$  is a neighborhood of zero.

**2.5** (Baire category theorem). The Baire category theorem proves many examples of topological vector space are Baire, in particular barreled.

- (a) A complete metric space is Baire.
- (b) A locally compact Hausdorff space is Baire.

## 2.3 Open mapping theorem

**2.6** (Open mapping theorem). Let  $X$  be a  $F$ -space and  $Y$  a barreled space. Suppose  $T : X \rightarrow Y$  is a continuous and surjective linear operator.

(a)  $\overline{TU}$  is a neighborhood of zero.

(b)  $TU$  is a neighborhood of zero.

*Proof.* (a) Let  $U'$  be a neighborhood of zero such that  $U \supset U' - U'$ . Because  $T$  is surjective, the set  $\overline{TU'}$  is a closed absorbing set, so it contains a non-empty open subset, since  $Y$  is barreled. Thus,  $\overline{TU} \supset \overline{TU'} - \overline{TU'}$  is a neighborhood of zero.

(b) We claim  $\overline{TU_{2^{-1}}} \subset TU_1$ . Take  $y_1 \in \overline{TU_{2^{-1}}}$ .

Assume  $y_n \in \overline{TU_{2^{-n}}}$ . Since  $\overline{TU_{2^{-(n+1)}}}$  is a neighborhood of zero, we have

$$(y_n + \overline{TU_{2^{-(n+1)}}}) \cap TU_{2^{-n}} \neq \emptyset.$$

Then, there is  $x_n \in U_{2^{-n}}$  such that  $Tx_n \in y_n + \overline{TU_{2^{-(n+1)}}}$ . Define

$$y_{n+1} := y_n - Tx_n.$$

Then,  $\sum_{n=1}^{\infty} x_n$  clearly converges to  $x \in U_1$ . Therefore,

$$Tx = \sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} (y_n - y_{n+1}) = y_1. \quad \square$$

## Exercises

**2.7.** Let  $(T_n)$  be a sequence in  $B(X, Y)$ . If  $T_n$  converges strongly then  $\|T_n\|$  is bounded by the uniform boundedness principle.

**2.8.** There is a closed absorbing set in  $\ell^2(\mathbb{Z}_{\geq 0})$  that is not a neighborhood of zero;

$$\overline{B}(0, 1) \setminus \bigcup_{i=2}^{\infty} B(i^{-1}e_i, i^{-2})$$

is a counterexample.

**2.9.** There is no metric  $d$  on  $C([0, 1])$  such that  $d(f_n, f) \rightarrow 0$  if and only if  $f_n \rightarrow f$  pointwise as  $n \rightarrow \infty$  for every sequence  $f_n$ . Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

**2.10.** We show that there is no projection from  $\ell^\infty$  onto  $c_0$ .

**2.11** (Schur property).  $\ell^1$

**2.12.** Let  $\varphi : L^\infty([0, 1]) \rightarrow \ell^\infty(\mathbb{N})$  be an isometric isomorphism. Suppose  $\varphi$  is realised as a sequence of bounded linear functionals on  $L^\infty$ .

(a) Show that  $\varphi^*(\ell^1) \subset L^1$  where  $\ell^1$  and  $L^1$  are considered as closed linear subspaces of  $(\ell^\infty)^*$  and  $(L^\infty)^*$  respectively.

(b) Show that  $\varphi^*$  is indeed an isometric isomorphism, and deduce  $\varphi$  cannot be realised as bounded linear functionals on  $L^\infty$ .

**2.13** (Daugavet property). (a) The real Banach space  $C([0, 1])$  satisfies the Daugavet property.



*Proof.* Let  $T$  be a finite rank operator on  $C([0, 1])$ , and  $e_i$  be a basis of  $\text{im } T$ . Then, for some measures  $\mu_i$ ,

$$Tf(t) = \sum_{i=1}^n \int_0^1 f d\mu_i e_i(t).$$

Let  $M := \max \|e_i\|$ .

Take  $f_0$  such that  $\|f_0\| = 1$  and  $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$ . Reversing the sign of  $f_0$  if necessary, take an open interval  $\Delta$  such that  $Tf_0(t) \geq \|T\| - \frac{\varepsilon}{2}$  and  $|\mu_i|(\Delta) \leq \frac{\varepsilon}{4nM}$  for all  $i$ . Define  $f_1$  such that  $f_0 = f_1$  on  $\Delta^c$ ,  $f_1(t_0) = 1$  for some  $t_0 \in \Delta$ , and  $\|f_1\| = 1$ . Then,  $\|Tf_1 - Tf_0\| \leq \frac{\varepsilon}{2}$  shows  $Tf_1 \geq \|T\| - \varepsilon$  on  $\Delta$ . Therefore,

$$\|1 + T\| \geq \|f_1 + Tf_1\| \geq f_1(t_0) + Tf_1(t_0) \leq 1 + \|T\| - \varepsilon.$$

□

**2.14** (Bartle-Graves theorem). Let  $E$  be a Banach space and  $N$  a closed subspace. For  $\varepsilon > 0$ , there is a continuous homogeneous map  $\rho : E/N \rightarrow E$  such that  $\pi\rho(y) = y$  and  $\|\rho(y)\| \leq (1 + \varepsilon)\|y\|$  for all  $y \in E/N$ .

*Proof.* We want to construct a continuous map  $\psi : S_{E/N} \rightarrow E$  with  $\|\psi(y)\| \leq 1 + \varepsilon$  for all  $y \in S_{E/N}$ . If then,  $\rho$  can be made from  $\psi$ .

For each  $y_0 \in S_{E/N}$ , choose  $x_0 \in \pi^{-1}(y_0) \cap B_{1+\varepsilon}$ . There is a neighborhood  $V_{y_0} \subset S_{E/N}$  of  $y_0$  such that  $y \in V_{y_0}$  implies  $x_0$  belongs to  $(\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$ , which is convex. With a locally finite subcover  $V_{y_\alpha}$  and a partition of unity  $\eta_\alpha(y)$ , define  $\psi_1(y) = \sum_\alpha \eta_\alpha(y)x_\alpha$ . Then,  $\psi_1(y) \in (\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$ .

For  $i \leq 2$ , choose for each  $y_0$  the element  $x_0$  in  $\pi^{-1}(y_0) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}})$ . Then, we obtain

$$\psi_i(y) \in \left( \pi^{-1}(y) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}}) \right) + U_{2^{-i}}.$$

Therefore,  $\|\psi_i(y) - \psi_{i-1}(y)\| < 2^{-i-2}$ , so it converges uniformly to  $\psi$  such that  $\psi(y) \in \pi^{-1}(y) \cap B_{1+\varepsilon}$ . □

## Problems

**2.15.** Let  $T$  be an invertible linear operator on a normed space. Then,  $T^{-2} + \|T\|^{-2}$  is injective if it is surjective.

## Chapter 3

# Weak topologies

### 3.1 Dual spaces

3.1 (Bidual).

3.2. Let  $X$  be a locally convex space. The *weak topology* is the topology  $w$  on  $X$  defined by the family of seminorms  $\{x \mapsto |\langle x, \xi \rangle|\}_{\xi \in X^*}$ . The *weak\* topology* is the topology  $w^*$  on  $X^*$  defined by the family of seminorms  $\{\xi \mapsto |\langle x, \xi \rangle|\}_{x \in X}$ . Let  $J : X \rightarrow X^{**}$  be the canonical embedding.

- (a)  $(X, w)$  and  $(X^*, w^*)$  are locally convex.
- (b)  $(X, w)^* = X^*$ .
- (c)  $(X^*, w^*)^* = X$ . Every locally convex space is a dual of a locally convex space.

*Proof.* (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show  $(X^*, w^*)^* \subset X$ . If  $u \in (X^*, w^*)^*$ , then there are  $x_1, \dots, x_m \in X$  such that

$$|\langle u, \xi \rangle| \leq \sum_{i=1}^m |\langle x_i, \xi \rangle|$$

for all  $\xi \in X^*$ . If we let  $\ker \vec{x} := \bigcap_{i=1}^m \ker x_i$ , then it is a closed subspace of  $X^*$  such that  $\ker \vec{x} \subset \ker u$ , so we have  $u \in \text{span } \vec{x} \subset X$ . □

### 3.3. closure and weak closure of convex subsets

*Proof.* Hahn-Banach □

### 3.4 (Polar).

boundedness, incompleteness

3.5 (Weak convergence by dense set). Let  $X$  be a Banach space,  $D^*$  a subset of  $X^*$ , and  $\overline{D^*}$  the norm closure of  $D^*$ . For example, if  $X$  has a predual  $X_* \subset X^*$  and  $D^*$  is dense in  $X_*$ , then  $\sigma(X, \overline{D^*})$  is the weak\* topology.

- (a) There is a sequence  $x_n \in X$  converges to zero in  $\sigma(X, D^*)$  but not in  $\sigma(X, \overline{D^*})$ .
- (b) A bounded sequence  $x_n \in X$  converges to zero in  $\sigma(X, \overline{D^*})$  if in  $\sigma(X, D^*)$ .

*Proof.* (b) Let  $\xi \in \overline{D^*}$  and choose  $\eta \in D^*$  such that  $\|\xi - \eta\| < \varepsilon$ . Then,

$$|\langle x_n, \xi \rangle| \leq \|x_n\| \|\xi - \eta\| + |\langle x_n, \eta \rangle| \lesssim \varepsilon + |\langle x_n, \eta \rangle| \rightarrow \varepsilon.$$

□

## 3.2 Weak compactness

3.6 (Banach-Alaoglu theorem).

*Proof.* Consider

$$B_{X^*} \rightarrow \prod_{x \in X} \|x\|B : l \mapsto (l(x))_{x \in X}.$$

Since it is an embedding into a compact space, it suffices to show the closedness of image: for  $l(x) := \lim_{\alpha} l_{\alpha}(x)$ , we have

$$\|l(x)\| \leq \|l(x) - l_{\alpha}(x)\| + \|x\| \xrightarrow{\alpha \rightarrow \infty} \|x\|,$$

so  $l$  is contained in the range. □

3.7 (Eberlein-Šmulian theorem).

3.8 (James' theorem).

## 3.3 Weak density

Bishop-Phelps theorem

3.9 (Goldstine theorem). Let  $X$  be a Banach space. Then,  $B_X$  is weakly\* dense in  $B_{X^{**}}$ .

*Proof.* Take  $x^{**} \in B_{X^{**}} \setminus \overline{B_X}^{w*}$ . By the Hahn-Banach separation, there are  $x^* \in X^*$  and  $r \in \mathbb{R}$  such that

$$\operatorname{Re}\langle x, x^* \rangle \leq r < \operatorname{Re}\langle x^{**}, x^* \rangle$$

for every  $x \in B_X$ . Since the left hand side can approximate  $\|x^*\|$ , we have  $\|x^*\| \leq r$  and obtain a contradiction

$$r < \operatorname{Re}\langle x^{**}, x^* \rangle \leq \|x^*\| \leq r. \quad \square$$

## 3.4 Krein-Milman theorem

Choquet theory

## 3.5 Polar topologies

Mackey-Arens

## Exercises

3.10 (James' space). not reflexive but isometrically isomorphic to bidual

3.11 (Preduals). Let  $X$  be a Banach space. A *predual* of  $X$  is a Banach space  $F$  together with an isometric isomorphism  $\varphi : X \rightarrow F^*$ . Two preduals  $\varphi_1 : X \rightarrow F_1^*$  and  $\varphi_2 : X \rightarrow F_2^*$  are said to be equivalent if there is an isometric isomorphism  $\theta : F_1 \rightarrow F_2$  such that  $\theta^* = \varphi_1 \varphi_2^{-1}$ .

- (a) There is a one-to-one correspondence between the equivalence class of preduals of  $X$  and the set of closed subspaces  $X_*$  of  $X^*$  such that  $B_X$  is compact and Hausdorff in  $(X, \sigma(X, X_*))$ . Such a subspace  $X_*$  is also called a predual of  $X$ .
- (b) If  $X$  admits a predual  $X_* \subset X^*$ , then a  $\sigma(X, X_*)$ -closed subspace  $V$  of  $X$  also admits a predual  $X_*|_V$ .

*Proof.* (a) Goldstine theorem for surjectivity.

(b) It is easy if we apply the part (a). We can show more directly. If we let  $V_* := X_*|_V$  the image of  $X_*$  under the map  $X^* \rightarrow V^*$ , then we have isometric injections  $V \rightarrow (V_*)^* \rightarrow X$ . We can show  $V$  is  $\sigma(X, X_*)$  dense in  $(V_*)^*$ , hence the closedness proves the bijectivity of  $V \rightarrow (V_*)^*$ .  $\square$

**3.12** (Mazur's lemma).

## **Part II**

# **Banach spaces**

## Chapter 4

# Operators on Banach spaces

### 4.1 Bounded operators

**4.1** (Bounded belowness in Banach spaces). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ . The following statements are equivalent:

- (a)  $T$  is bounded below.
- (b)  $T$  is injective and has closed range.
- (c)  $T$  is a topological isomorphism onto its image.

**4.2** (Bounded belowness in Hilbert spaces). Let  $T \in B(H, K)$  for Hilbert spaces  $H$  and  $K$ . The following statements are equivalent:

- (a)  $T$  is bounded below.
- (b)  $T$  is left invertible.
- (c)  $T^*$  is right invertible.
- (d)  $T^*T$  is invertible.

**4.3** (Injectivity and surjectivity of adjoint). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ .

- (a)  $T^*$  is injective if and only if  $T$  has dense range.
- (b)  $T^*$  is surjective if and only if  $T$  is bounded below.

### 4.2 Compact operators

$K(X, Y)$  is closed in  $B(X, Y)$ .  $K(X)$  is an ideal of  $B(X)$ . adjoint is  $K(X, Y) \rightarrow K(Y^*, X^*)$ . integral operators are compact. riesz operator, quasi-nilpotent operator.

### 4.3 Fredholm operators

**4.4.** A bounded linear operator  $T : X \rightarrow Y$  between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

- (a) A Fredholm operator  $T$  has closed range.

*Proof.* (a) Let  $C$  be a finite dimensional subspace of  $Y$  such that  $\text{im } T \oplus C = Y$ . Let  $\tilde{T} : X/\ker T \rightarrow Y$  be the induced operator of  $T$ . Define  $S : (X/\ker T) \oplus C \rightarrow Y$  such that  $S(x + \ker T, c) := \tilde{T}(x + \ker T) + c$ . Then,  $S$  is an topological isomorphism between Banach spaces by the open mapping theorem, so  $S(X/\ker T \oplus \{0\}) = \text{im } \tilde{T} = \text{im } T$  is closed.  $\square$

**4.5** (Atkinson's theorem). An operator  $T \in B(X, Y)$  is Fredholm if and only if there is  $S \in B(Y, X)$  such that  $TS - I$  and  $ST - I$  is finite rank.

**4.6** (Fredholm index). locally constant, in particular, continuous. composition makes the addition of indices.

## 4.4 Nuclear operators

tensor products

### Exercises

**4.7** (Completely continuous operators). On reflexive spaces, completely continuous operators are same with compact operators.

**4.8** (Dunford-Pettis property). A Banach space  $X$  is said to have the *Dunford-Pettis property* if all weakly compact operators  $T : X \rightarrow Y$  to any Banach space  $Y$  is completely continuous.

- (a)  $X$  has the Dunford-Pettis property if and only if for every sequences  $x_n \in X$  and  $x_n^* \in X^*$  that converge to  $x$  and  $x^*$  weakly we have  $x_n^*(x_n) \rightarrow x^*(x)$ .
- (b)  $C(\Omega)$  for a compact Hausdorff space  $\Omega$  has the Dunford-Pettis property.
- (c)  $L^1(\Omega)$  for a probability space  $\Omega$  has the Dunford-Pettis property.
- (d) Infinite dimensional reflexive Banach space does not have the Dunford-Pettis property.

**4.9.**

- (a) (Mazur-Ulam, 1932) A surjective isometry  $T : X \rightarrow Y$  between normed spaces is affine.
- (b) (Mankiewicz, 1972) Let  $U, V$  be open sets in  $X, Y$ , normed spaces. A surjective isometry  $U \rightarrow V$  is uniquely extended to a surjective isometry  $X \rightarrow Y$ .
- (c) (Mori) A surjective local isometry  $T : X \rightarrow Y$  between Banach spaces is an isometry, if  $X$  is separable. (Use the Baire category)

*Solution.* (a)  $T$  is continuous. It is easy to see for continuous map  $T$  that it is affine if and only if  $T$  preserves the midpoint. For  $x_1 \neq x_2 \in X$  let  $x_0$  be the midpoint. Define inductively

$$C_1 := \{x \in X : \|x - x_1\| = \|x - x_2\| = \frac{1}{2}\|x_1 - x_2\|\}, \quad C_k := \{x \in C_{k-1} : \sup_{x' \in C_{k-1}} \|x - x'\| \leq \frac{1}{2} \text{diam } C_{k-1}\}.$$

Since  $x_0 \in C_{k-1}$  and  $x' \in C_{k-1}$  imply  $x_0 \in C_k$  by  $\|x_0 - x'\| = \frac{1}{2}\|(2x_0 - x') - x'\| \leq \frac{1}{2} \text{diam } C_{k-1}$ , and since  $\text{diam } C_k \leq \frac{1}{2} \text{diam } C_{k-1}$ , we have  $\{x_0\} = \bigcup_{k=1}^{\infty} C_k$ . It follows that the midpoint can be detected from the metric structure of  $X$ , not depending on the linear structure of  $X$ .  $\square$

## Problems

1. If  $T \in B(L^2([0, 1]))$  is a compact operator, then for any  $\varepsilon > 0$  there is a constant  $C_\varepsilon > 0$  such that

$$\|Tf\|_{L^2} \leq \varepsilon \|f\|_{L^2} + C_\varepsilon \|f\|_{L^1}.$$

*Proof.* 1. Suppose there is  $\varepsilon > 0$  such that we have sequence  $f_n \in L^2$  satisfying  $\|f_n\|_2 = 1$  and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of  $T$ , there is a subsequence  $Tf_{n_k}$  converges to  $g \neq 0$  in  $L^2$ . Then,  $\|f_{n_k}\|_1 \rightarrow 0$  implies  $f_{n_k} \rightarrow 0$  weakly in  $L^2$ , hence also for  $Tf_{n_k}$ . It means  $g = 0$ , which contradicts to the assumption.  $\square$



## Chapter 5

# Geometry of Banach spaces

### 5.1 Tensor products

### 5.2 Approximation property

dual is Banach. Basis problem, Mazur' duck.

**5.1 (Approximation property).** Every compact operator is a limit of finite-rank operators.

(a) An Hilbert space has the AP

(b)

*Proof.* (a) Let  $H$  be a Hilbert space and  $K \in K(H)$ . Since  $\overline{KB_H}$  is a compact metric space, it is separable, which means  $\overline{KH}$  is separable. Let  $(e_i)_{i=1}^\infty$  be an orthonormal basis of  $\overline{KH}$ , and let  $P_n$  be the orthogonal projection on the space spanned by  $(e_i)_{i=1}^n$ . If we let  $K_n := P_n K$ , then  $K_n \rightarrow K$  strongly and  $K_n$  has finite rank. Take any  $\varepsilon > 0$  and find, using the totally boundedness of  $KB_H$ , a finite subset  $\{x_j\} \subset B_H$  such that for any  $x \in B_H$  there is  $x_j$  satisfying  $\|Kx - Kx_j\| < \frac{\varepsilon}{2}$ . Then,

$$\begin{aligned} \|Kx - K_n x\| &\leq \|Kx - Kx_j\| + \|Kx_j - K_n x_j\| + \|P_n(Kx_j - Kx)\| \\ &\leq \frac{\varepsilon}{2} + \|Kx_j - K_n x_j\| + \frac{\varepsilon}{2}. \end{aligned}$$

By taking the supremum on  $x \in B_H$ , we have

$$\|K - K_n\| \leq \max_j \|Kx_j - K_n x_j\| + \varepsilon,$$

which deduces  $K_n \rightarrow K$  in norm.

□

### Exercises

Tingley problem

## Chapter 6

## **Part III**

# **Spectral theory**

## Chapter 7

# Operators on Hilbert spaces

### 7.1 Operator topologies

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

7.1. (a) A Banach space  $X$  is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of  $X$ .

7.2 (Riesz representation theorem). Let  $H$  be a Hilbert space over a field  $\mathbb{K}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ .

We use the bilinear form  $\langle -, - \rangle : X \times X^* \rightarrow \mathbb{K}$  of canonical duality. The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

- (a) For each  $x^* \in H^*$ , there is a unique  $x \in H$  such that  $\langle y, x^* \rangle = \langle y, x \rangle$  for every  $y \in H$ .
- (b)  $H \rightarrow H^* : x \mapsto \langle -, x \rangle$  is a natural linear and anti-linear isomorphism if  $\mathbb{K} = \mathbb{R}$  and  $\mathbb{C}$ , respectively.

Let  $H$  be a separable Hilbert space. Find a positive sequence  $a_n$  such that every sequence  $x_n$  of unit vectors of  $H$  satisfying  $|\langle x_i, x_j \rangle| \leq a_j$  for all  $i < j$  converges weakly to zero.

7.3 (Normal operators). For  $T \in B(H)$ , we have an obvious fact  $(\text{im } T)^\perp = \ker T^*$ . Suppose  $T$  is normal.

- (a)  $\ker T = \ker T^*$ .
- (b)  $T$  is bounded below if and only if  $T$  is invertible.
- (c) If  $T$  is surjective, then  $T$  is invertible.

7.4 (Invariant and Reducing subspaces). Let  $K$  be a closed subspace of  $H$ .

- (a)  $K$  is reducing for  $T$  if and only if  $K$  is invariant for  $T$  and  $T^*$ .
- (b)  $K$  is reducing for  $T$  if and only if  $TP = PT$ , where  $P$  is the orthogonal projection on  $K$ .

7.5 (Trace class operators). Let  $K \in B(H)$  The *trace* of  $K$  is

$$\text{Tr}(K) := \sum_i \langle Ke_i, e_i \rangle,$$

where  $(e_i) \subset H$  is an orthonormal basis. The operator  $K$  is said to be in the *trace-class* if  $\text{Tr}(|K|) < \infty$ .

- (a) The trace does not depend on the choice of  $(e_i)$ .
- (b)  $K$  is a trace class if and only if  $K = \sum_{i=1}^{\infty} \lambda_i \theta_{x_i, y_i}$  for some  $(\lambda_i)_{i=1}^{\infty} \subset \ell^1(\mathbb{N})$  and orthogonal sequences  $(x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty} \subset H$ .

(c)  $B(H) \rightarrow L^1(H)^* : T \mapsto \text{Tr}(T \cdot)$  is an isometric isomorphism.

*Proof.* (b) Conversely, we can check the diagonalization  $K^*K = \sum_{i=1}^{\infty} |\lambda_i|^2 \theta_{y_i}$ , which implies  $|K| = \sum_{i=1}^{\infty} |\lambda_i| \theta_{y_i}$ . Thus,

$$\text{Tr}(|K|) = \sum_{j=1}^{\infty} \langle |K| y_j, y_j \rangle = \sum_{i=1}^{\infty} |\lambda_i| < \infty.$$

□

**7.6.** (a) A net  $T_\alpha$  converges to  $T$  strongly in  $B(H)$  if and only if  $\|(T_\alpha - T)^{\oplus n} \bar{\xi}\| \rightarrow 0$  for all  $\bar{\xi} \in H^{\oplus n}$ .

(b) A net  $T_\alpha$  converges to  $T$   $\sigma$ -strongly in  $B(H)$  if and only if  $\|(T_\alpha - T)^{\oplus \infty} \bar{\xi}\| \rightarrow 0$  for all  $\bar{\xi} \in H^{\oplus \infty}$ .

**7.7** (Strong\* operator topology). Let  $H$  be a Hilbert space. We provides some conditions for a strongly convergent sequence to converge strongly\*. Let  $(T_\alpha) \subset B(H)$  and suppose  $T_\alpha \rightarrow T$  strongly.

**7.8** (Continuity of linear functionals). Let  $f$  be a linear functional on  $B(H)$  for a Hilbert space  $H$ .

(a)  $f$  is weakly continuous if and only if it is strongly\* continuous, and in this case we have  $f = \sum_i \omega_{x_i, y_i}$  for some  $(x_i), (y_i) \in c_c(\mathbb{N}, H)$ .

(b)  $f$  is  $\sigma$ -weakly continuous if and only if it is  $\sigma$ -strongly\* continuous, and in this case we have  $f = \sum_i \omega_{x_i, y_i}$  for some  $(x_i), (y_i) \in \ell^2(\mathbb{N}, H)$ .

*Proof.* Suppose  $f$  is strongly continuous. There exists  $\bar{x} \in H^{\oplus n}$  such that

$$|f(T)| \leq \|T^{\oplus n} \bar{x}\|.$$

The functional  $f : A \rightarrow \mathbb{C}$  factors through  $H^{\oplus n}$  such that

$$A \xrightarrow{\bar{x}} H^{\oplus n} \rightarrow \mathbb{C}.$$

□

For  $\bar{x} = (x_i) \in \ell^2(\mathbb{N}, H)$ ,

$$p_{\bar{x}}^{\sigma s*}(T) = \left( \sum_i \|Tx_i\|^2 + \|T^*x_i\|^2 \right)^{\frac{1}{2}} \quad p_{\bar{x}}^{\sigma s}(T) = \left( \sum_i \|Tx_i\|^2 \right)^{\frac{1}{2}} \quad p_{\bar{x}}^{\sigma w}(T) = \left| \sum_i \langle Tx_i, x_i \rangle \right|$$

## 7.2 Closed operators

**7.9** (Closed operators). (a) a

**7.10** (Adjoint operators). Let  $T : \text{dom } T \subset X \rightarrow Y$  be a densely defined linear operator between Banach spaces. Define an unbounded operator  $T^* : \text{dom } T^* \subset Y^* \rightarrow X^*$  such that  $\langle x, T^*y^* \rangle := \langle Tx, y^* \rangle$  for all  $x \in \text{dom } T$  and  $y^* \in \text{dom } T^*$ , where

$$\text{dom } T^* := \{y^* \in Y^* \mid \text{dom } T \rightarrow \mathbb{C} : x \mapsto \langle Tx, y^* \rangle \text{ is bounded}\}.$$

(a) If  $T \subset S$ , then  $S^* \subset T^*$ .

(b)  $T^*$  is always closed.

(c)  $T$  is closable if and only if  $T^*$  is densely defined. If it is, then  $T^{**}$  is the closure of  $T$ . (Only on reflexive spaces?)

(d)  $T^*$  is injective if and only if  $T$  has dense range, and surjective if and only if  $T$  is bounded below.

*Proof.* (d) Suppose  $T$  is bounded below. Fix  $x^* \in X^*$ . Since  $T$  is bounded below,  $x^*$  defines a bounded linear functional on  $\text{dom } T$  with respect to  $\|x\| + \|Tx\|$ , which is embedded in  $Y$  as a closed subspace. By the Hahn-Banach extension, we have an element  $y^* \in Y^*$  such that  $\langle Tx, y^* \rangle = \langle x, x^* \rangle$  for all  $x \in X$ , which is contained in  $\text{dom } T^*$  by the definition of  $\text{dom } T^*$ . This implies that  $T$  is surjective because  $T^*y^* = x^*$ .

Conversely, suppose  $T^*$  is surjective. Let  $F := \{x \in \text{dom } T : \|Tx\| \leq 1\}$ . Since for every  $x^* \in X^*$  we have for some  $y^* \in \text{dom } T^*$  such that

$$\sup_{x \in F} |\langle x, x^* \rangle| = \sup_{x \in F} |\langle x, T^*y^* \rangle| = \sup_{x \in F} |\langle Tx, y^* \rangle| \leq \|y^*\|.$$

By the uniform boundedness principle, we have  $\sup_{x \in F} (\|x\| + \|Tx\|)$  is bounded, we are done.  $\square$

**7.11** (Operations of unbounded operators). inverse, composition, addition

**7.12** (Symmetric operators). A densely defined operator  $T : \text{dom } T \rightarrow H$  is called *symmetric* if

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad x, y \in \text{dom } T.$$

Let  $T$  be a densely defined symmetric operator. If the closure of  $T$  is self-adjoint, then it is called *essentially self-adjoint*.

- (a)  $T$  has the closed and densely defined closure.
- (b) Every symmetric extension of  $T$  is a restriction of  $T^*$ , which is not symmetric in general. In particular,  $T$  has a maximal symmetric extension.
- (c) A maximal symmetric operator is closed since the closure of a .
- (d) A self-adjoint operator is maximal.
- (e) A densely defined closed symmetric operator is essentially self-adjoint if and only if it is indeed the unique self-adjoint extension if and only if the adjoint is symmetric.

**7.13** (Cayley transform). There is a one-to-one correspondence between the unitary operators from  $K_+$  to  $K_-$ , the deficiency subspaces.

Let  $T$  be a symmetric operator on a Hilbert space  $H$ . We will always assume that  $T$  is densely defined and closed. We want to ask the following questions: Is  $T$  self-adjoint? If not, does  $T$  admit self-adjoint extensions? Which self-adjoint extension generate the appropriate quantum dynamics?

**Example.** Let  $T := i d/dx$  on  $L^2([0, 1])$  with

$$\text{dom } T = H_0^1((0, 1)).$$

It is densely defined and closed. Then,

$$\text{dom } T^* = H^1((0, 1)) \subset C([0, 1])$$

and  $T^*$  is not self-adjoint since... The set of self-adjoint extensions is  $\{T_\alpha : \alpha \in \mathbb{T}\}$ , where

$$\text{dom } T_\alpha = \{f \in H^1((0, 1)) : \alpha f(0) = f(1)\}.$$

## 7.3 Spectral theorems

**7.14** (Spectral measure). Let  $(\Omega, \mathcal{A})$  be a measurable space and  $H$  a Hilbert space. A *projection-valued measure* on  $\Omega$  for  $H$  is a map  $E : \mathcal{A} \rightarrow B(H)$  with  $E(\emptyset) = 0$  such that  $E(A)$  is a projection for every  $A \in \mathcal{A}$ , and one of the following equivalent conditions hold:

- (i) the set function  $E_{x,y} : \mathcal{A} \rightarrow \mathbb{C} : A \mapsto \langle E(A)x, y \rangle$  is a complex measure on  $\Omega$  for each  $x, y \in H$ .
- (ii) the countable additivity holds, i.e.  $E(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} E(A_i)$  in the strong operator topology of  $B(H)$  for  $(A_i)_{i=1}^{\infty} \subset \mathcal{M}$ .
- (a)  $E(A \cap B) = E(A)E(B)$  for  $A, B \in \mathcal{M}$ .

**7.15.** Let  $T \in B(H)$  be a normal operator. Then, there exists a projection-valued measure  $E$  on  $\sigma(T)$  for  $H$  such that

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

This spectral measure  $E$  is also called the *resolution of the identity*.

A multiplication operator by any Borel measurable function  $\Omega \rightarrow \mathbb{C}$  always defines a densely defined closed normal operator.

Let  $E$  be the spectral measure of a normal operator  $T \in B(H)$ . If we choose  $\xi \in E(B(\lambda, n^{-1}))H$ , then since  $E(B(\lambda, n^{-1})^c)\xi = 0$ , or since  $E(B(\lambda, n^{-1}))\xi = \xi$ , we have

$$\begin{aligned} \|(\lambda - T)\xi\|^2 &= \int |\lambda - z|^2 d\langle E(z)\xi, \xi \rangle \\ &= \int_{B(\lambda, n^{-1})} |\lambda - z|^2 d\langle E(z)\xi, \xi \rangle \\ &\leq n^{-2} \int_{B(\lambda, n^{-1})} d\langle E(z)\xi, \xi \rangle \\ &\leq n^{-2} \int d\langle E(z)\xi, \xi \rangle \\ &= n^{-2} \|\xi\|^2. \end{aligned}$$

**7.16** (Spectral representation). Let  $T$  be a bounded normal operator on a Hilbert space  $H$ . Then, the unital  $C^*$ -algebra  $C^*(T)$  generated by  $T$  is  $*$ -isomorphic to  $C(\sigma(T))$ , and it has a canonical faithful representation  $\pi : C(\sigma(T)) \rightarrow B(H)$ . This representation exactly corresponds to the object called spectral measure. We now decompose  $\pi = \bigoplus_{\alpha} \pi_{\alpha}$  to cyclic representations  $\pi_{\alpha} : C(\sigma(T)) \rightarrow B(H_{\alpha})$  with cyclic unit vectors  $\psi_{\alpha}$ . Each vector state  $\psi_{\alpha}$  induces a probability measure  $\mu_{\alpha}$  on  $\sigma(T)$ . It is called the spectral measure associated to the cyclic vector  $\psi_{\alpha}$ . We can check that the GNS-representation of  $\mu_{\alpha}$  is unitarily equivalent to  $\pi_{\alpha}$ . The direct sum  $C(\sigma(T)) \rightarrow \bigoplus_{\alpha} B(L^2(\mu_{\alpha}))$  can be defined. Then, we can show the bounded normal operator  $T$  is always unitarily equivalent to the multiplication operator on a finite measure space. However, in order for  $T$  to be unitarily equivalent to the multiplication operator by the identity function of  $\mathbb{C}$ ,  $T$  must be multiplicity free, equivalently,  $T$  must have a cyclic vector of  $H$ .

Two bounded normal operators are unitarily equivalent if and only if the sequence of multiplicity measure classes are identical.

Two spectral theorems: Multiplication operator form(unitary equivalence), Projection-valued measure form(functional calculus).

Kato-Rellich theorem

For a densely defined closed operator  $T : H \rightarrow H$ ,  $\sigma(T^*) = \overline{\sigma(T)}$ .

**7.17** (Polar decomposition). polar decomposition polar decomposition of symmetric operator? polar decomposition changes spectrum or domains?

support projection

**7.18** (Stone theorem).

**7.19** (Analytic vectors). (a) If  $T$  is symmetric and  $D_0$  is dense, then  $T|_{D_0}$  is essentially self-adjoint.

**7.20** (Resolvent convergence).

## 7.4 Decomposition of spectrum

$$\begin{aligned}\sigma &= \sigma_p \cup \sigma_c \cup \sigma_r \\ &= \sigma_{ess} \cup \sigma_d \\ &= \overline{\sigma_{pp}} \cup \sigma_{ac} \cup \sigma_{sc}.\end{aligned}$$

$$\sigma = \sigma_p \sqcup \sigma_c \sqcup \sigma_r = \overline{\sigma_{pp}} \cup \sigma_{ac} \cup \sigma_{sc} = \sigma_d \sqcup \sigma_{ess,5}.$$

### Exercises

**7.21** (Strict topology). Let  $H$  be a Hilbert space. Let  $(T_\alpha) \subset B(H)$  and  $K \in K(H)$ .

- (a) The strong\* topology and the strict topology agree on bounded sets of  $B(H)$ .

**7.22** (Unitary group). Let  $H$  be a Hilbert space.

- (a) The weak topology and the strict topology agree on  $U(H)$ .

**7.23** (Bounded increasing nets). Let  $T_\alpha$  be a bounded increasing net of bounded self-adjoint operators on  $H$ .

- (a)  $T_\alpha$  converges strictly. In particular,  $T_\alpha \rightarrow T$  strictly iff  $T_\alpha \rightarrow T$  weakly.

*Proof.* Define  $T$  such that

$$\langle Tx, y \rangle := \lim_\alpha \sum_{k=0}^3 i^k \langle T_\alpha(x + i^k y), x + i^k y \rangle.$$

The convergence is due to the monotone convergence in  $\mathbb{R}$ . We can check it is a well-defined bounded linear operator by considering the bounded sesquilinear form. Then,  $T_\alpha \rightarrow T$  weakly by definition, and  $\sigma$ -strongly because the net is increasing.  $\square$

**7.24** (Distributional operators). (a) Every continuous linear operator  $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$  naturally defines a closable densely defined operator  $T : \text{dom } T \rightarrow L^2(\mathbb{R})$  with  $\text{dom } T := \mathcal{D}(\mathbb{R})$ .

**7.25** (Hydrogen atom). For  $V \in L^\infty(\mathbb{R}^d)$ , the operator

$$H\psi(x) := -\frac{\hbar^2}{2m}\Delta\psi(x) - V(x)\psi(x), \quad x \in \mathbb{R}^d$$

is called the *Schrödinger operator*, and simply we write  $H = -\Delta + V$ . The eigenvectors associated to the discrete spectrum is called *bound eigenstates*.

Consider the Schrödinger operator  $H := -\Delta - |x|^{-1}$  on  $L^2(\mathbb{R}^3)$ . We want to investigate the spectral decomposition of  $H$  by diagonalization.

- (a)  $H$  is self-adjoint.  
(b)  $\sigma_d(H) = \{\}$

The orbital comes from the diagonalization of the Laplace-Beltrami operator on the unit sphere.

The periodic Schrödinger operator is diagonalized to the direct integral of elliptic operators defined on the Brillouin torus.



## Chapter 8

# Operator theory

### 8.1 Toeplitz operators

invariant subspace problem Beurling theorem Hardy and Bergman and Bloch spaces  $JB^*$  triple

## Chapter 9

## **Part IV**

# **Operator algebras**

# Chapter 10

## Banach algebras

### 10.1 Spectra of elements

**10.1 (Banach algebras).** For a Banach algebra  $A$  with multiplicative unit, there is a complete renorming such that  $\|1\| = 1$ , i.e. we can always assume  $\|1\| = 1$ . It provides a definition of *unital Banach algebra*.

Let  $A$  be a unital Banach algebra.

(a) If  $\|a\| < 1$ , then  $1 - a$  is invertible. So  $A^\times$  is open.

(b)  $A^\times \rightarrow A^\times : a \mapsto a^{-1}$  is continuous.

(c)  $A^\times \rightarrow A^\times : a \mapsto a^{-1}$  is differentiable.

*Proof.* (a) We can show

$$(1 - a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

If  $a$  is invertible, then  $a + h = a(1 + a^{-1}h)$  has the inverse  $(1 + a^{-1}h)^{-1}a^{-1}$  if  $h$  is sufficiently small such that  $\|a^{-1}h\| < 1$ .

(b) Clear from

$$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}.$$

(c)

$$\begin{aligned} \frac{\|b^{-1} - a^{-1} - (-a^{-1}(b - a)a^{-1})\|}{\|b - a\|} &= \frac{\|(a^{-1} - b^{-1})(b - a)a^{-1}\|}{\|b - a\|} \\ &\leq \|a^{-1} - b^{-1}\| \|a^{-1}\| \xrightarrow{b \rightarrow a} 0. \end{aligned}$$

□

**10.2 (Spectrum and resolvent).** Let  $a$  be an element of a unital Banach algebra  $A$ . The *spectrum* of  $a$  in  $A$  is defined to be the set

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A\},$$

and the *resolvent* of  $a$  in  $A$  is defined to be its complement  $\rho_A(a) := \mathbb{C} \setminus \sigma_A(a)$ . We can now define the *resolvent map*  $R : \rho_A(a) \rightarrow A$  such that

$$R(\lambda) = R(\lambda; a) := (\lambda - a)^{-1}.$$

If  $A$  is clear in its context, we omit it to just write  $\sigma(a)$  and  $\rho(a)$ .

- (a)  $\sigma(a)$  is compact.
- (b)  $\sigma(a)$  is non-empty.
- (c) If  $A$  is a division ring, then  $A \cong \mathbb{C}$ . This result is called the *Gelfand-Mazur theorem*.

*Proof.* (a) If  $|\lambda| > \|a\|$ , then  $\lambda - a$  is always invertible, so the spectrum is bounded. Closedness follows from the fact that the set of invertibles is open.

(b) Suppose the spectrum  $\sigma(a) = \emptyset$  so that the resolvent function  $R : \mathbb{C} \rightarrow A$  is well-defined on the entire  $\mathbb{C}$ . Note that  $a \neq 0$ . Since  $R$  is continuous and since

$$\|(\lambda - a)^{-1}\| = \|\lambda^{-1}(1 - \lambda^{-1}a)^{-1}\| = \left\| \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1}a)^k \right\| < (2\|a\|)^{-1} \sum_{k=0}^{\infty} 2^{-k} = \|a\|^{-1}$$

on  $\{\lambda \in \mathbb{C} : |\lambda| > 2\|a\|\}$ , the function  $R$  is bounded. Also, for every  $l \in A^*$  we have that the function  $\mathbb{C} \rightarrow \mathbb{C} : \lambda \mapsto \langle R(\lambda), l \rangle$  is holomorphic since  $a \mapsto a^{-1}$  is differentiable.

Therefore, by the Liouville theorem, the bounded entire function  $\lambda \mapsto \langle R(\lambda), l \rangle$  is constant for all  $l \in A^*$ . Because  $A^*$  separates points of  $A$ , the function  $R$  is constant, which implies  $a \in \mathbb{C}$  and contradicts to  $\sigma(a) = \emptyset$ .

(c) For any  $a \in A$ , by the part (b), there must be  $\lambda$  such that  $\lambda - a$  is not invertible. In a division ring, zero is the only non-invertible element, so  $\lambda = a$ .  $\square$

**10.3 (Spectral radius).** Let  $a$  be an element of a unital Banach algebra  $A$ . The *spectral radius* of  $a$  in  $A$  is defined to be

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$

- (a)  $r(a) \leq \inf_n \|a^n\|^{\frac{1}{n}}$ .
- (b)  $\limsup_n \|a^n\|^{\frac{1}{n}} \leq r(a)$ , i.e.  $r(a) = \lim_n \|a^n\|^{\frac{1}{n}}$ .

*Proof.* (a) Since  $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1}$  exists if  $|\lambda| > \|a\|$ , we have  $r(a) \leq \|a\|$  for all  $a \in \mathcal{A}$ . For every  $\lambda \in \sigma(a)$  and every integer  $n \geq 1$  we have

$$|\lambda|^n = |\lambda^n| \leq r(a^n) \leq \|a^n\|,$$

and it proves  $r(a) \leq \inf_n \|a^n\|^{\frac{1}{n}}$ .

(b) Consider a holomorphic function

$$f : \{\lambda \in \mathbb{C} : |\lambda| > r(a)\} \rightarrow \mathbb{C} : \lambda \mapsto \langle R(\lambda), l \rangle$$

for each  $l \in A^*$ . Since on a smaller domain  $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$ , the function  $f$  can be given by

$$f(\lambda) = \left\langle \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1}a)^k, l \right\rangle,$$

we can determine the coefficients of the Laurent series of  $f$  at infinity as

$$f(\lambda) = \sum_{k=0}^{\infty} \langle a^k, l \rangle \lambda^{-k-1}$$

on  $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$ .

Take  $\lambda$  such that  $|\lambda| > r(a)$ . Then, the sequence  $(a^k \lambda^{-k-1})_k \in A$  is weakly bounded, hence is normly bounded by the uniform boundedness principle. Let  $\|a^n\| \leq C_\lambda |\lambda|^{n+1}$  for all  $n \geq 1$ . Then,

$$\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} C_\lambda^{\frac{1}{n}} |\lambda|^{n+1} = |\lambda|.$$

If we limit  $|\lambda| \downarrow r(a)$ , we are done.  $\square$

**10.4** (Spectrum in closed subalgebras). For fixed element, smaller the ambient algebra, less “holes” in the spectrum. Let  $A \subset B$  be a closed subalgebra containing  $1_A$ . Note that  $A$  may be unital even for  $1_B \notin A$ .

- (a)  $B^\times$  is clopen in  $A^\times \cap B$ .

## 10.2 Ideals

**10.5** (Ideals). (a) If  $I$  is a left ideal, then  $A/I$  is a left  $A$ -module.

**10.6** (Modular left ideals). A left ideal  $I$  is called *modular* if there is  $e \in A$  such that  $a - ae \in I$  for all  $a \in A$ . The element  $e$  is called a *right modular unit* for  $I$ .

- (a)  $I$  is modular if and only if  $A/I$  is unital(?).  
(b) A proper modular left ideal is contained in a maximal left ideal.  
(c)  $I$  is a maximal modular left ideal if and only if  $I$  is a modular maximal left ideal.  
(d) There is a non-modular maximal ideal in the disk algebra.

**10.7** (Closed ideals). (a) closure of proper left ideal is proper left.

- (b) maximal modular left ideal is closed.

**10.8** (Unitization). Let  $A$  be an algebra. Recall that we always assume algebras are associative. Consider an embedding  $A \rightarrow B(A) : a \mapsto L_a$ , where  $L_a(b) = ab$ . Define

$$\tilde{A} := \{ L_a + \lambda \text{id}_{B(A)} : a \in A, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital  $A$ .

- (a) If  $A$  is normed, then  $\tilde{A}$  is a normed algebra such that there is an isometric embedding  $A \rightarrow \tilde{A}$ .  
(b) If  $A$  is Banach, then  $\tilde{A}$  is a Banach algebra.  
(c)  $A \oplus \mathbb{C}$  is topologically isomorphic to  $\tilde{A}$  as normed spaces.

*Proof.* (a) The space of bounded operators  $B(A)$  is a normed algebra. Then,  $\tilde{A}$  is a normed  $*$ -algebra with induced norm

$$\|L_a + \lambda \text{id}_{B(A)}\| = \sup_{b \in A} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then,  $A$  is a normed  $*$ -subalgebra of  $\tilde{A}$  because the norm and involution of  $A$  agree with  $\tilde{A}$ .

(b) Suppose  $(x_n, \lambda_n)$  is Cauchy in  $\tilde{A}$ . Since  $A$  is complete so that it is closed in  $\tilde{A}$ , we can induce a norm on the quotient  $\tilde{A}/A$  so that the canonical projection is (uniformly) continuous so that  $\lambda_n$  is Cauchy. Also, the inequality  $\|x\| \leq \|(x, \lambda)\| + |\lambda|$  shows that  $x_n$  is Cauchy in  $A$ .

Since a finite dimensional normed space is always Banach and  $A$  is Banach,  $\lambda_n$  and  $x_n$  converge. Finally, the inequality  $\|(x, \lambda)\| \leq \|x\| + |\lambda|$  implies that  $(x_n, \lambda_n)$  converges.

- (c) Check the topology on  $A \oplus \mathbb{C}$  in detail... □

unitization, homomorphisms, category(direct sum, product, etc.)

$B(\mathbb{C}^n) = M_n(\mathbb{C})$  is simple, but  $B(H)$  is not simple.

## 10.3 Holomorphic functional calculus

Fréchet space valued

**10.9** (Holomorphic functional calculus). Let  $a$  be an element of a unital Banach algebra  $A$ . Let  $f$  be a holomorphic function on a neighborhood  $U$  of  $\sigma(a)$ . Let  $C$  be a positively oriented smooth simple closed curve in  $U$  enclosing  $\sigma(a)$ . Define  $f(a) \in A^{**}$  as the Dunford integral

$$\langle f(a), l \rangle := \int_C f(\lambda) \langle (\lambda - a)^{-1}, l \rangle d\lambda, \quad l \in A^*.$$

Let  $\mathcal{O}(\sigma(a))$  be the space of all holomorphic functions on a neighborhood of  $\sigma(a)$  endowed with the topology of compact convergence. Note that  $\mathcal{O}(\sigma(a))$  is a Fréchet algebra, but not Banach. We define the *holomorphic functional calculus* by the map

$$\mathcal{O}(\sigma(a)) \rightarrow A : f \mapsto f(a).$$

It is also called the Riesz or the *Riesz-Dunford functional calculus*.

- (a)  $f(a) \in A$ , i.e.  $f(a)$  is in fact given by the Pettis integral.
- (b)  $f(a)$  is independent of the choice of  $C$ .
- (c) The functional calculus is an algebra homomorphism.
- (d) The functional calculus is bounded.
- (e) injective.
- (f) unital and  $\text{id}_C \mapsto a$ .
- (g) spectral mapping.
- (h) power series.

*Proof.* (a)

□

## 10.4 Gelfand theory

Banach algebra of single generator semisimplicity and symmetricity

**10.10** (Spectrum of a Banach algebra). Let  $A$  be a commutative Banach algebra. A *character* of  $A$  is a non-trivial algebra homomorphism  $\pi : A \rightarrow \mathbb{C}$ . Denote by  $\sigma(A)$  the set of all characters of  $A$  and endow with the weak\* topology on  $\sigma(A) \subset A^*$ . We call this space as the *spectrum* of  $A$ .

- (a) If  $A$  is unital,  $\sigma(A)$  is contained in the unit sphere of  $A^*$ .
- (b)  $\sigma(A)$  is locally compact and Hausdorff.

*Proof.*

□

**10.11** (Gelfand transform). Let  $A$  be a commutative Banach algebra. The *Gelfand transform* or the *Gelfand representation* is the following algebra homomorphism

$$\Gamma : A \rightarrow C_0(\sigma(A)) : a \mapsto (\pi \mapsto \pi(a)).$$

- (a)  $\Gamma$  has the image separating points by definition.
- (b)  $\Gamma$  has closed range if  $A$  is a symmetric Banach \*-algebra.
- (c)  $\Gamma$  is injective if and only if  $A$  is semisimple.
- (d)  $\Gamma$  is isometric if and only if  $r(a) = \|a\|$  for all  $a \in A$ .

## Exercises

**10.12** (Basic properties of spectrum). Let  $A$  be a unital algebra.

- (a)  $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ .
- (b) If  $\sigma(a)$  is non-empty, then  $\sigma(p(a)) = p(\sigma(a))$ .

*Proof.* (a) Intuitively, the inverse of  $1-ab$  is  $c = 1+ab+abab+\dots$ . Then,  $1+bca = 1+ba+baba+\dots$  is the inverse of  $1-ba$ .  $\square$

$$C_b(\Omega) \ell^\infty(S) L^\infty(\Omega) B_b(\Omega) A(\mathbb{D}) B(X)$$

**10.13.** In  $C(\mathbb{R})$ , the modular ideals correspond to compact sets.

**10.14** (Disk algebra). (a) Every continuous homomorphism is an evaluation.

**10.15** (Polynomial convexity). (See Conway)

**10.16** (Inclusion relation on spectra). (a)  $\sigma(a+b) \subset \sigma(a) + \sigma(b)$  and  $\sigma(ab) \subset \sigma(a)\sigma(b)$  for unital cases.

- (b)  $\sigma(a^{-1}) = \sigma(a)^{-1}$  for unital cases.
- (c)  $r(a)^n = r(a^n)$ .

**10.17** (Spectral radius function). (a) upper semi-continuous

**10.18** (Vector-valued complex function theory). Let  $\Omega$  be an open subset of  $\mathbb{C}$  and  $X$  a Banach space. For a vector-valued function  $f : \Omega \rightarrow X$ , we say  $f$  is *differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{-1} (f(\lambda) - f(\lambda_0))$$

exists in  $X$  for every  $\lambda \in \Omega$ , and *weakly differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{-1} \langle f(\lambda) - f(\lambda_0), x^* \rangle$$

exists in  $\mathbb{C}$  for each  $x^* \in X^*$  and every  $\lambda \in \Omega$ . Then, the followings are all equivalent.

- (a)  $f$  is differentiable.
- (b)  $f$  is weakly differentiable.
- (c) For each  $\lambda_0 \in \Omega$ , there is a sequence  $(x_k)_{k=0}^\infty$  such that we have the power series expansion

$$f(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k x_k,$$

where the series on the right hand side converges absolutely and uniformly on any closed ball in  $\Omega$  centered at  $\lambda_0$ .

**10.19** (Exponential of an operator).



# Chapter 11

## C\*-algebras

### 11.1 C\* identity

11.1 (\*-algebras). normed?

11.2 (C\*-identity). A C\*-algebra is a Banach \*-algebra  $A$  satisfying the C\*-identity  $\|a^*a\| = \|a\|^2$  for all  $a \in A$ .

11.3 (Unitization).

$$(L_a + \lambda \text{id}_{B(A)})^* = L_{a^*} + \bar{\lambda} \text{id}_{B(A)}.$$

*Proof.* The C\*-identity easily follows from the following inequality:

$$\begin{aligned} \|(a, \lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\ &= \sup_{\|b\|=1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\ &= \sup_{\|b\|=1} \|b^*((a^*a + \lambda a^* + \bar{\lambda}a)b + |\lambda|^2 y)\| \\ &\leq \sup_{\|b\|=1} \|(a^*a + \lambda a^* + \bar{\lambda}a)b + |\lambda|^2 b\| \\ &= \|(a, \lambda)^*(a, \lambda)\|. \end{aligned}$$

□

### 11.2 Continuous functional calculus

11.4 (Gelfand-Naimark representation for C\*-algebras). For a commutative C\*-algebra  $A$ , consider the Gelfand transform  $\Gamma : A \rightarrow C_0(\sigma(A))$ .

- (a)  $\Gamma$  is a \*-homomorphism.
- (b)  $\Gamma$  is an isometry.
- (c)  $\Gamma$  is a \*-isomorphism.

*Proof.* (a)

(b) Note that we have

$$\|\Gamma a\| = \sup_{\varphi \in \sigma(A)} |\Gamma a(\varphi)| = \sup_{\varphi \in \sigma(A)} |\varphi(a)| = r(a)$$

for all  $a \in A$ . If we assume  $a$  is self-adjoint, then since  $\|a\|^2 = \|a^*a\| = \|a^2\|$ , the spectral radius coincides with the norm by the Beurling formula for spectral radius in Banach algebras:

$$\|\Gamma a\| = r(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Hence we have for all  $a \in A$  that

$$\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|(\Gamma a)^*(\Gamma a)\| = \|\Gamma a\|^2.$$

(c) By the part (a) and (b), the image  $\Gamma(A)$  is a closed unital  $*$ -subalgebra of  $C(\sigma(A))$ , and it separates points by definition. Then,  $\Gamma(A)$  is dense in  $C(\sigma(A))$  by the Stone-Weierstrass theorem, which implies  $\Gamma(A) = C(\sigma(A))$ .  $\square$

**11.5** (Generators of a  $C^*$ -algebra). joint spectrum.

**11.6** (Continuous functional calculus). Let  $A$  be a unital  $C^*$ -algebra, and  $a \in A$  a normal element. Then, we have a  $*$ -isomorphism

$$C(\sigma(a)) \rightarrow \tilde{C}^*(1, a) : \text{id}_{\sigma(a)} \mapsto a$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

(a) spectral mapping:  $\lambda \in \sigma_p(a)$  implies  $f(\lambda) \in \sigma_p(f(a))$ ,  $\lambda \in \sigma(a)$  iff  $f(\lambda) \in \sigma(f(a))$ , composition, ...

**11.7** (Normal elements). Let  $a$  be an element of a unital  $C^*$ -algebra  $A$ . We say  $a$  is *normal*, *unitary*, and *self-adjoint* if  $a^*a = aa^*$ ,  $a^*a = aa^* = e$ , and  $a^* = a$  respectively. For normality and self-adjointness, the definitions can be extended to non-unital  $C^*$ -algebras.

(a) If  $a$  is normal, then  $a$  is unitary if and only if  $\sigma(a) \subset \mathbb{T}$ .

(b) If  $a$  is normal, then  $a$  is self-adjoint if and only if  $\sigma(a) \subset \mathbb{R}$ .

*Proof.* (a)

(b) We may assume  $A$  is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \in A,$$

and the inverse of  $e^{ia}$  is  $e^{-ia}$ . Since the involution on  $A$  is continuous, we can check  $e^{ia}$  is unitary by

$$(e^{ia})^* = \sum_{n=1}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia}.$$

For every  $\varphi \in \sigma(A)$ , then by the part (a) the equality

$$e^{-\text{Im } \varphi(a)} = |e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$$

proves  $\varphi(a) \in \mathbb{R}$ , hence  $\sigma(a) \subset \mathbb{R}$ .  $\square$

**11.8** ( $*$ -homomorphism). Let  $\varphi : A \rightarrow B$  be a  $*$ -homomorphism between  $C^*$ -algebras.

(a)  $\varphi$  is determined by self-adjoint elements.

(b)  $\|\varphi\| = 1$  if  $\varphi$  is non-trivial.

(c) The image of  $\varphi$  is closed.

(d) The induced map  $A/\ker \varphi \rightarrow B$  is an isometry.

### 11.3 Positive elements

**11.9** (Positive elements). Let  $a, b$  be elements of a  $C^*$ -algebra  $A$ . We say  $a$  is *positive* and write  $a \geq 0$  if it is normal and  $\sigma(a) \subset \mathbb{R}_{\geq 0}$ . If we define a relation  $a \leq b$  as  $b - a \geq 0$ , then we can see that it is a partial order on  $A$ .

- (a)  $a \geq 0$  if and only if  $\|\lambda - a\| \leq \lambda$  for some  $\lambda \geq \|a\|$ .
- (b) If  $a \geq 0$  and  $\sigma(b) \subset \mathbb{R}_{\geq 0}$ , then  $\sigma(a + b) \subset \mathbb{R}_{\geq 0}$ .
- (c)  $a \geq 0$  if and only if  $a = b^*b$  for some  $b \in A$ .

*Proof.* Let  $a := b^*b$ . Let  $a = a_+ - a_-$ . Then we have  $(ba_-)^*(ba_-) = a_-aa_- = -a_-^3 \leq 0$ , which also implies  $(ba_-)(ba_-)^* \leq 0$  and

$$0 \leq (ba_-)^*(ba_-) + (ba_-)(ba_-)^* \leq 0.$$

Thus we have  $ba_- = 0$  and  $a_-^3 = 0$ .

□

**11.10** (Operator monotone operations). (a) If  $0 \leq a \leq b$ , then  $a^{-1} \geq b^{-1}$ .

- (b) If  $a \leq b$ , then  $cac^* \leq cbc^*$ .

**11.11** (Positive linear functionals). Let  $A$  be a  $C^*$ -algebra. A *state* of  $A$  is a positive linear functional  $\omega$  such that  $\|\omega\| = 1$ .

- (a) For a normal element  $a \in A$  there is a state  $\omega$  such that  $|\omega(a)| = \|a\|$ .
- (b) A self-adjoint linear functional is the difference of two positive linear functional. It is called the *Jordan decomposition*.

*Proof.* (b) We first show the real dual  $(A^{sa})^*$  can be identified with the self adjoint part  $(A^*)^{sa}$  of the complex dual. By this identification, we can describe the weak\* topology on  $(A^*)^{sa}$  as  $\sigma((A^*)^{sa}, A^{sa})$ .

We may assume  $A$  is unital. The closed unit ball of the real Banach space  $(A^*)^{sa}$  is weakly\* compact. We are enough to show

$$(A^*)_1^{sa} = \overline{\text{conv}}(S(A) \cup -S(A)),$$

where the closure is taken in the weak\* topology, because  $S(A)$  and  $-S(A)$  are weakly\* compact and convex due to the unit of  $A$ , the closure on the right-hand side is not necessary. Suppose not and take  $l \in (A^*)_1^{sa}$  which is not approximated weakly\* by  $\text{conv}(S(A) \cup -S(A))$ . By the Hahn-Banach separation, there is  $a \in A^{sa}$  such that

$$\sup_{\omega \in S(A) \cup -S(A)} \omega(a) < l(a).$$

If we take  $\omega \in S(A) \cup -S(A)$  such that  $\omega(a) = \|a\|$  using the part (a), then we get a contradiction to the bound  $\|l\| \leq 1$ .

□

**11.12** (Approximate identity). Let  $e_\alpha$  be an approximate identity of  $A$ .

- (a) Exists.
- (b) For a positive linear functional  $\omega$ , we have  $\lim_\alpha \omega(e_\alpha) = \|\omega\|$ .
- (c)
- (d) separable.

## 11.4 Representations of $C^*$ -algebras

**11.13** (Non-degenerate representations). Let  $A$  be a  $C^*$ -algebra. A *representation* of  $A$  on a Hilbert space  $H$  is a  $*$ -homomorphism  $\pi : A \rightarrow B(H)$ . We say a representation  $\pi : A \rightarrow B(H)$  is *non-degenerate* if  $\pi(A)H$  is dense in  $H$ .

- (a) Every representation has a unique non-degenerate subrepresentation.
- (b) The following statements are equivalent:
  - (i)  $\pi$  is non-degenerate.
  - (ii) For each  $\xi \in H$  there is  $a \in A$  such that  $\pi(a)\xi \neq 0$ .
  - (iii)  $\pi(e_\alpha) \rightarrow \text{id}_H$  strongly for an approximate identity  $e_\alpha$  of  $A$ .

**11.14** (Cyclic representations). *cyclic* if there is a vector  $\psi \in H$  such that  $A\psi$  is dense in  $H$ . Cyclic decomposition

**11.15** (Irreducible representations). *irreducible* if there is no proper closed subspace  $K \subset H$  such that  $\pi(A)K \subset K$ . The following statements are equivalent:

- (i)  $\pi$  is irreducible.
- (ii)  $\pi(A)' = \mathbb{C} \text{id}_H$ .
- (iii)  $\pi(A)$  is strongly dense in  $B(H)$ .
- (iv) Every non-zero vector in  $H$  is cyclic.

**11.16** (Gelfand-Naimark-Segal representation). Let  $A$  be a  $C^*$ -algebra, and  $\omega$  be a state on  $A$ . The *left kernel* of  $\omega$  is defined to be

$$N_\omega := \{a \in A : \omega(a^*a) = 0\}.$$

- (a)  $N_\omega$  is a left ideal of  $A$ .
- (b)  $\langle a + N_\omega, b + N_\omega \rangle := \omega(b^*a)$  is an inner product on  $A/N_\omega$ .
- (c) There is a unique representation  $\pi_\omega : A \rightarrow B(H_\omega)$  such that  $\pi_\omega(a)(b + N_\omega) := ab + N_\omega$  for  $a, b \in A$ .
- (d)  $\pi_\omega : A \rightarrow B(H_\omega)$  is a cyclic representation.

## Exercises

**11.17** (Projections in  $M_2(\mathbb{C})$ ). The space of self-adjoint elements in  $M_2(\mathbb{C})$  is a real vector space spanned by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

- (a)  $(p - q)^2 = \frac{1}{2}$ .
- (b) If we let  $\lambda_\pm$  be the eigenvalues of  $ap + bq$ , then  $\lambda_+ + \lambda_- = a + b$  and  $\lambda_+ - \lambda_- = \sqrt{a^2 + b^2}$ .
- (c) Every functional calculus  $f(x)$  of self-adjoint  $x$  is a linear combination of  $x$  and  $1$ .
- (d)  $ap + bq + c \geq 0$  if and only if  $a + b + 2c \geq \sqrt{a^2 + b^2}$ .
- (e) Every projection of rank one is given by  $ap + bq + (1 - a - b)/2$  for  $a^2 + b^2 = 1$ .

**11.18** (Operator monotone square). Let  $A$  be a  $C^*$ -algebra in which the square function is operator monotone, that is,  $0 \leq a \leq b$  implies  $a^2 \leq b^2$  for any positive elements  $a$  and  $b$  in  $A$ . We are going to show that  $A$  is necessarily commutative. Let  $a$  and  $b$  denote arbitrary positive elements of  $A$ .

- (a) Show that  $ab + ba \geq 0$ .
- (b) Let  $ab = c + id$  where  $c$  and  $d$  are self adjoints. Show that  $d^2 \leq c^2$ .
- (c) Suppose  $\lambda > 0$  satisfies  $\lambda d^2 \leq c^2$ . Show that  $c^2 d^2 + d^2 c^2 - 2\lambda d^4 \geq 0$ .
- (d) Show that  $\lambda(cd + dc)^2 \leq (c^2 - d^2)^2$ .
- (e) Show that  $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \leq c^2$  and deduce  $d = 0$ .
- (f) Extend the result for general exponent:  $A$  is commutative if  $f(x) = x^\beta$  is operator monotone for  $\beta > 1$ .

**11.19** (States on unitization). Let  $A$  be a non-unital  $C^*$ -algebra and  $\tilde{A}$  be its unitization. Let  $\tilde{\omega} = \omega \oplus \lambda$  be a bounded linear functional on  $\tilde{A}$ , where  $\omega \in A^*$  and  $\lambda \in \mathbb{C}^* = \mathbb{C}$ .

Since  $A$  is hereditary in  $\tilde{A}$ , the extension defines a well-defined injective map  $S(A) \rightarrow S(\tilde{A})$ . We can identify  $PS(A)$  as a subset of  $PS(\tilde{A})$  whose complement is a singleton.

- (a)  $\tilde{\rho}$  is positive if and only if  $\lambda \geq 0$  and  $0 \leq \rho \leq \lambda$ .
- (b)  $\tilde{\omega}$  is a state if and only if  $\lambda = 1$  and  $0 \leq \omega \leq 1$ .
- (c)  $\tilde{\omega}$  is a pure state if and only if  $\lambda = 1$  and  $\omega$  is either a pure state or zero.

**11.20** (Representations of  $C_0(X)$ ). Let  $A = C_0(X)$  and  $\mu$  be a state on  $A$ , a regular Borel probability measure on a locally compact Hausdorff space  $X$ .

- (a) The left kernel of  $\mu$  is  $N_\mu = \{f \in A : f|_{\text{supp } \mu} = 0\}$ .
- (b)  $H_\mu = L^2(X, \mu)$ .
- (c) The canonical cyclic vector is the unity function on  $X$ .

**11.21** (Representations of  $K(H)$ ).

**11.22** (Automorphism group of  $K(H)$  and  $B(H)$ ).

**11.23** (Approximate eigenvectors).

**11.24** (Kadison transitivity theorem).

**11.25** (Hereditary  $C^*$ -algebras).

**11.26** (Extreme points of the ball). Let  $A$  be a  $C^*$ -algebra and let  $B_A$  be the closed unit ball of  $A$ .

- (a) Extreme points of  $A_+ \cap B_A$  is the projections in  $A$ .
- (b) Extreme points of  $A_{sa} \cap B_A$  is the self-adjoint unitaries in  $A$ .
- (c) Every extreme point of  $B_A$  is a partial isometry.

## Problems

- \*1. A  $C^*$ -algebra is commutative if and only if a function  $f(x) = x(1+x)^{-1}$  is operator subadditive.

# Chapter 12

## Von Neumann algebras

### 12.1 Density theorems

**12.1** (Von Neumann algebras). A *von Neumann algebra* on a Hilbert space  $H$  is a  $\sigma$ -weakly closed  $*$ -subalgebra of  $B(H)$  including  $\text{id}_H$ . A positive linear map  $\varphi$  between von Neumann algebras is said to be *normal* if  $\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha)$  for any bounded increasing net  $x_\alpha$  of positive elements.

(a) A positive map  $\varphi$  is normal if and only if it is continuous between  $\sigma$ -weak topologies.

**12.2** (Normal states). Let  $N \subset M \subset B(H)$  be von Neumann algebras. The space of  $\sigma$ -weakly continuous linear functionals on  $M$  is denoted by  $M_*$ .

(a)  $M_*$  is a predual of  $M$ .

(b) The restriction of a normal state of  $M$  on  $N$  is normal.

(c) A normal state of  $N$  is extended to a normal state of  $M$ .

(d) A state  $\omega$  of  $M$  is normal if and only if  $\omega(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle$  for some  $(\xi_i) \in \ell^2(\mathbb{N}, H)$ .

(e) The GNS representation of a normal state is normal.

**12.3** (Double commutant theorem). The *commutant* of a subset  $A \subset B(H)$ , denoted by  $A'$ , is the set of all elements of  $B(H)$  that commute every  $a \in A$ . Suppose  $A$  is a non-degenerate  $*$ -subalgebra of  $B(H)$ . One can describe the von Neumann algebra generated by  $A$  in  $B(H)$  purely algebraically in terms of commutants.

(a)  $A''$  is weakly closed  $*$ -algebra.

(b) If  $x \in A''$ , for any  $\varepsilon > 0$  and  $\xi \in H$  there is  $a \in A$  such that  $\|(x - a)\xi\| < \varepsilon$ .

(c)  $A$  is  $\sigma$ -strongly\* dense in  $A''$ .

*Proof.* (a) Suppose a net  $x_\alpha \in A''$  weakly converges to  $x \in B(H)$ . For any  $y \in A'$ ,

$$\langle xy\xi, \eta \rangle = \lim_\alpha \langle x_\alpha y\xi, \eta \rangle = \lim_\alpha \langle yx_\alpha\xi, \eta \rangle = \langle yx\xi, \eta \rangle, \quad \xi, \eta \in H.$$

Hence  $x \in A''$ .

(b) We claim  $x\xi \in \overline{A\xi}$  for each  $\xi \in H$ . Let  $p$  be the projection onto  $\overline{A\xi}$ . For any  $a \in A$ , the operator  $ap$  ranges into  $\overline{A\xi}$  so that  $pap = ap$ , and we also have  $pa^*p = a^*p$  by the self-adjointness of  $A$ . It implies  $ap = pa$ , which deduces  $p \in A'$ . Thus  $xp = px$  for  $x \in A''$ . On the other hand, observe that  $a(1-p)\xi = (1-p)a\xi = 0$  for all  $a \in A$ . Then,  $\langle (1-p)\xi, \eta \rangle = 0$  for any  $\eta \in H = \overline{AH}$  by the non-degeneracy, so  $p\xi = \xi$ . Combining  $xp = px$  and  $p\xi = \xi$ , we obtain  $x\xi = xp\xi = px\xi$  so that  $x\xi \in \overline{A\xi}$ .

(c) It suffices to show  $A$  is  $\sigma$ -strongly dense in  $A''$  because  $A$  is self-adjoint. Consider  $A$  as the non-degenerate  $*$ -subalgebra of  $B(\ell^2(\mathbb{N}, H))$  via the diagonal map  $B(H) \rightarrow B(\ell^2(\mathbb{N}, H))$ , which is a injective normal unital  $*$ -homomorphism. We can check that  $A''$  does not change if we replace  $B(H)$  to  $B(\ell^2(\mathbb{N}, H))$ . By applying the part (b) for arbitrary  $\xi \in \ell^2(\mathbb{N}, H)$ , we deduce the desired result.  $\square$

**12.4** (Kaplansky density theorem).

## 12.2 Borel functional calculus

**12.5** (Sherman-Takeda theorem). Let  $A$  be a  $C^*$ -algebra. Define  $M(\pi) := \pi(A)''$  for  $\pi : A \rightarrow B(H)$  a representation. Let  $\pi_u : A \rightarrow B(H_u)$  be the universal representation of  $A$ , the direct sum of all the GNS-representations of states of  $A$ . Consider the following three maps

$$\pi_u : A \rightarrow (M(\pi_u), \sigma_w), \quad \pi_u^* : M(\pi_u)_* \rightarrow A^*, \quad \pi_u^{**} : A^{**} \rightarrow M(\pi_u),$$

constructed by adjoints.

- (a)  $\pi_u^*$  is isometric.
- (b)  $\pi_u^*$  is surjective. In particular,  $\pi_u^{**}$  is a normal  $*$ -isomorphism.
- (c)  $A^{**}$  enjoys a universal property in the sense that every  $*$ -homomorphism  $\varphi : A \rightarrow M$  to a von Neumann algebra  $M$  has a unique normal extension  $\tilde{\varphi} : A^{**} \rightarrow M$  of  $\varphi$ .

*Proof.* (a) It holds for any representation of  $\pi : A \rightarrow B(H)$ . For each  $l \in M(\pi)_*$  we have

$$\|\pi^*(l)\| = \sup_{\substack{\|a\| \leq 1 \\ a \in A}} |l(\pi(a))| = \sup_{\substack{\|x\| \leq 1 \\ x \in M(\pi)}} |l(x)| = \|l\|$$

by the Kaplansky density theorem and the  $\sigma$ -weak continuity of  $l$ .

(b) Let  $\omega$  be a state of  $A$ . Since the universal representation  $\pi_u$  has the GNS representation of  $\omega$  as a subrepresentation,  $\omega$  is given by a vector state in  $\pi_u$ . By restriction of this vector state, we have a normal state of  $M(\pi_u)$ , which extends  $\omega$ . Now the Jordan decomposition can be applied to verify that every bounded linear functional of  $A$  has a  $\sigma$ -weakly continuous extension on  $M(\pi_u)$ .

(c) We can define  $\tilde{\varphi}$  as the bitranspose of  $\varphi : A \rightarrow (M, \sigma_w)$ , and it is a unique extension because  $A$  is  $\sigma$ -weakly dense in  $A^{**}$ .  $\square$

*Remark 12.2.1.* The bidual  $A^{**}$  is frequently viewed as a von Neumann algebra, and we call it the *enveloping von Neumann algebra* of a  $C^*$ -algebra  $A$ . By the universal property, we have a normal  $*$ -homomorphism  $M(\pi_u) \rightarrow M(\pi)$  that is in fact surjective for every representation  $\pi$  of  $A$ , and it fails to be injective even if  $\pi$  is faithful.

**12.6** (Bounded Borel functions). Let  $X$  be a compact Hausdorff space and denote by  $B^\infty(X)$  the space of bounded Borel functions on  $X$ . The linear combinations of projections in  $B^\infty(X)$  are called *simple functions*.

- (a) There are natural inclusions  $C(X) \subset B^\infty(X) \subset C(X)^{**}$  among  $C^*$ -algebras.
- (b)  $B^\infty(X)$  is the norm closure of simple functions.
- (c)  $B^\infty(X)$  factors through all  $L^\infty(X, \mu) := M(\pi_\mu)$  for GNS-representations  $\pi_\mu$  of  $C(X)$ .

**12.7** (Borel functional calculus). Let  $x \in B(H)$  be a normal operator. Consider

$$B^\infty(\sigma(x)) \subset C(\sigma(x))^{**} \rightarrow W^*(x) \subset B(H).$$

- (a) If we endow the topology of pointwise convergence on  $B^\infty(\sigma(a))$  and the strong operator topology on  $M$ , then the Borel functional calculus is continuous.
- (b) Every von Neumann algebra is the norm closed span of projections.

*Proof.* (a) By the bounded convergence theorem.

(b) This is because  $\sigma(a) \subset \mathbb{C}$  is compact so that it is separable and metrizable; every bounded measurable function is a pointwise limit of simple functions. □

For normal  $a \in B(H)$ , the continuous functional calculus for  $a$  is just a non-degenerate representation

$$C(\sigma(a)) \rightarrow B(H)$$

which maps  $\text{id}_{\sigma(a)}$  to  $a$ . Also, a projection valued-measure on a compact Hausdorff space  $X$  is just a non-degenerate representation

$$C(X) \rightarrow B(H).$$

To show this, note that a projection-valued measure defines a “normal” unital  $*$ -homomorphism

$$\text{span } P(B^\infty(X)) \rightarrow B(H).$$

Then, mimic the definition of Lebesgue integral to construct a unital  $*$ -homomorphism  $C(X) \rightarrow B(H)$ .

## 12.3 Predual

**12.8** (Conditional expectations). Let  $A$  be a closed subalgebra of a  $C^*$ -algebra  $B$ . Let  $\varphi : B \rightarrow A$  be a contractive idempotent surjective linear map. Such a map is called a *conditional expectation*.

- (a)  $\varphi$  is an  $A$ -bimodule map.
- (b)  $\varphi$  is completely positive.

*Proof.* Since each conclusion of (a) and (b) still holds for restriction, we may assume  $A$  and  $B$  are von Neumann algebras by thinking of the bitranspose  $\varphi^{**} : B^{**} \rightarrow A^{**}$ .

(a) Since the linear span of projections is  $\sigma$ -weakly dense in a von Neumann algebra, we are enough to show  $p\varphi(b) = \varphi(pb)$  and  $\varphi(bp) = \varphi(b)p$  for any projection  $p \in A$ .

Let  $p \in A$  be a projection and let  $b \in B$ . Note that the surjectivity of  $\varphi$  implies that  $p\varphi$  is also idempotent. Then, where  $1 = 1_B$ ,

$$\begin{aligned} (1+t)^2 \|p\varphi((1-p)b)\|^2 &= \|p\varphi((1-p)b) + tp\varphi(p\varphi((1-p)b))\|^2 \\ &\leq \|(1-p)b + tp\varphi((1-p)b)\|^2 \\ &= \|(1-p)b\|^2 + t^2 \|p\varphi((1-p)b)\|^2 \end{aligned}$$

implies  $p\varphi((1-p)b) = 0$  by letting  $t \rightarrow \infty$ . Putting  $1_A - p$  and  $1_A$  instead of  $p$ , we obtain

$$(1-p)\varphi((1-1_A+p)b) = 0, \quad \varphi((1-1_A)b) = 0$$

respectively, which imply  $(1-p)\varphi(pb) = 0$ . Hence for any  $b \in B$  we have

$$p\varphi(b) = p\varphi(pb) = \varphi(pb).$$

Similarly we can show  $\varphi(b(1-p))p = 0$  and  $\varphi(bp)(1-p) = 0$  for  $b \in B$ , we are done.



(b) Let  $[b_{ij}] \in M_n(B)_+$ . Let  $\pi : A \rightarrow B(H)$  be a cyclic representation with a cyclic vector  $\psi$ . Then,  $[\xi_i] \in H^n$  can be replaced to  $[\pi(a_i)\psi]$ , so we can check the positivity of inflations  $\varphi_n$  as

$$\sum_{i,j} \langle \pi(\varphi(b_{ij}))\pi(a_j)\psi, \pi(a_i)\psi \rangle = \langle \pi(\varphi(\sum_{i,j} a_i^* b_{ij} a_j))\psi, \psi \rangle \geq 0,$$

because it follows  $\sum_{i,j} a_i^* b_{ij} a_j \geq 0$  by the positivity of  $b_{ij}$  from

$$\langle \pi_B(\sum_{i,j} a_i^* b_{ij} a_j)\xi, \xi \rangle = \sum_{i,j} \langle \pi_B(b_{ij})\pi_B(a_j)\xi, \pi_B(a_i)\xi \rangle \geq 0,$$

where  $\pi_B$  is any representation of  $B$ . □

**12.9 (Sakai theorem).** Suppose  $A$  is a  $C^*$ -algebra which admits a predual  $F$ .

- (a) There is an injective  $*$ -homomorphism  $\pi : A \rightarrow A^{**}$  with weakly $*$  closed image.
- (b)  $\pi$  is a topological embedding with respect to  $\sigma(A, F)$  and  $\sigma(A^{**}, A^*)$ .
- (c) The predual  $F$  is unique in  $A^*$ .

In particular, since  $A^{**}$  admits a faithful normal representation, so does  $A$ .

*Proof.* (a) By taking the adjoint for the inclusion  $i : F \hookrightarrow A^*$ , we have a conditional expectation  $\varepsilon : A^{**} \rightarrow A$ . Its kernel is a  $A$ -bimodule, and by the  $\sigma$ -weak density of  $A$  in  $A^{**}$  and the continuity of  $\varepsilon$  between weak $*$  topologies, so it is in fact a  $A^{**}$ -bimodule, which means it is a  $\sigma$ -weakly closed ideal of  $A^{**}$ . Thus we have a central projection  $z \in A^{**}$  such that  $\ker \varepsilon = (1 - z)A^{**}$ .

Define  $\pi : A \rightarrow A^{**}$  such that  $\pi(a) := za$ . It is clearly a  $*$ -homomorphism. The injectivity follows from  $a = \varepsilon(a) = \varepsilon(za)$  for  $a \in A$ . The image is weakly $*$  closed because  $\varepsilon(x - \varepsilon(x)) = 0$  implies  $z(x - \varepsilon(x)) = 0$  for  $x \in A^{**}$  so that  $zA^{**} = zA$ .

(b) Since  $\langle a, f \rangle = \langle \varepsilon(za), f \rangle = \langle za, f \rangle$  for  $a \in A$  and  $f \in F$ , in which the second equality holds by the definition of  $\varepsilon$ , it is enough to show  $\sigma(zA, A^*) = \sigma(zA, F)$ .

For  $l \in A^*$ , we claim there exists  $f$  such that  $\langle za, l \rangle = \langle za, f \rangle$ . Define  $\tilde{l} \in A^*$  such that  $\langle x, \tilde{l} \rangle := \langle zx, l \rangle$  for  $x \in A^{**}$ . Then,  $\langle zx, l \rangle = \langle z^2x, l \rangle = \langle zx, \tilde{l} \rangle$  for  $x \in A^{**}$ . Suppose  $\tilde{l} \notin F$ . Because  $F$  is closed in  $A^*$ , there is  $x \in A^{**}$  such that  $\langle x, \tilde{l} \rangle \neq 0$  and  $\langle x, f \rangle = 0$  for all  $f \in F$  by the Hahn-Banach separation. Then,  $0 = \langle x, f \rangle = \langle x, i(f) \rangle = \langle \varepsilon(x), f \rangle$  implies  $\varepsilon(x) = 0$  so that  $zx = 0$ , which leads a contradiction  $\langle x, \tilde{l} \rangle = \langle zx, l \rangle = 0$ , so we have  $\tilde{l} \in F$ .

(c) If closed subspaces  $F_1$  and  $F_2$  of  $A^*$  are preduals of  $A$ , then  $\sigma(A, F_1) = \sigma(A, F_2)$  by the part (b). If  $l \in F_1$ , which is obviously continuous on  $\sigma(A, F_1)$ , and the continuity in  $\sigma(A, F_2)$  implies that  $l$  is contained in a linear span of some finitely many elements of  $F_2$ , hence  $F_1 \subset F_2$ . □

## Exercises

**12.10** (Extremally disconnected space).  $\sigma(B^\infty(\Omega))$  is extremally disconnected.

resolution of identity normal operator theories: multiplicity, invariant subspaces  $L^\infty$  representation  $\sigma$ -weakly closed left ideal has the form  $Mp$ . II.3.12

Let  $\mathfrak{m}$  be an algebraic ideal of a von Neumann algebra  $M$ , and  $\overline{\mathfrak{m}}$  be its  $\sigma$ -weak closure. If  $x \in (\overline{\mathfrak{m}})_+$ , then there is an increasing net  $(x_i) \subset \mathfrak{m}$  converges to  $x$  strongly. II.3.13

binary expansion and hereditary subalgebras