

Harmonic Analysis

Ikhan Choi

April 3, 2022

Contents

I	Fourier analysis	2
1	Fourier transform	3
2	Fourier series and inversions	4
3		5
II	Singular integral operators	6
4	Caldéron-Zygmund theory	7
4.1	Hilbert transform	7
4.2	Calderón-Zygmund operator of convolution type	7
4.3	L^2 -boundedness of truncated integrals	9
4.4	Calderón-Zygmund operator of non-convolution type	9
5	Littlewood-Paley theory	10
6	Multiplier theorems	11
III	Pseudo-differential operators	12
IV	Oscillatory integral operators	13

Part I

Fourier analysis

Chapter 1

Fourier transform

Chapter 2

Fourier series and inversions

Chapter 3

Part II

Singular integral operators

Chapter 4

Caldéron-Zygmund theory

4.1 Hilbert transform

4.2 Calderón-Zygmund operator of convolution type

4.1 (Calderón-Zygmund decomposition of sets). Let $E_n f$ be the conditional expectation with respect to the σ -algebra generated by dyadic cubes with side length 2^{-n} . Let $Mf = \sup_n E_n |f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|} \int_{Q_x} |f| > \lambda.$$

- (a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω .
- (b) We have an weak type estimate $|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1}$.
- (c) $\|f\|_{L^\infty(\mathbb{R}^d \setminus \Omega)} \leq \lambda$.
- (d) For $x \in \Omega$

$$\frac{1}{|Q_x|} \int_{Q_x} |f| \leq 2^d \lambda.$$

4.2 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|} \int_{Q_x} |f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \leq 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof.

□

4.3 (Calderón-Zygmund operator of convolution type). Let $T : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that

$$Tf(x) = \int K(x-y)f(y)dy$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \text{supp } f$. If T is L^2 -bounded

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$$

and satisfies the *Hörmander condition*

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \lesssim 1,$$

then it is called a *Calderón-Zygmund operator*.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2 -boundedness implies

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2} \|Tg\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda} \int_{\mathbb{R}^d \setminus \Omega^*} |Tb(x)| dx \leq \frac{2}{\lambda} \sum_i \int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\text{supp } b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y)b_i(y) dy = \int_{Q_i} [K(x-y) - K(x)]b_i(y) dy,$$

and

$$\int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d \setminus Q_i^*} |K(x-y) - K(x)| dx dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.3 L^2 -boundedness of truncated integrals

4.4 Calderón-Zygmund operator of non-convolution type

standard kernels

Exercises

4.4 (Gradient size condition). Let $|\nabla K(x)| \lesssim \frac{1}{|x|^{d+1}}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

Chapter 5

Littlewood-Paley theory

Chapter 6

Multiplier theorems

Part III

Pseudo-differential operators

Part IV

Oscillatory integral operators