Homological Algebra

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1. Modules

References: Atsushi Shiho, Yukiyoshi Kawada

1.1. R-modules

Definition 1.1. Let R be a ring with 1. A (left) R-module is an abelian group M with a map $R \times M \to M : (a, x) \mapsto ax$ satisfying $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ab)x = a(bx)$, $1x = x$.

Example 1.2. (a) Every abelian group is a $\mathbb{Z}$-module. The R-module structures on an abelian group M has 1-1 correspondence with the ring homomorphisms $R \to \mathrm{End}_{\mathbb{Z}}(M)$.

(b) $M = C^{\infty}(\mathbb{R})$, $R = \mathbb{R}[T]$ a polynomial ring, $R \times M \to M : (P(T), f(x)) \mapsto P(\frac{d}{dx})f(x)$.

Definition 1.3. A (left) R-submodule of M is a subgroup $N \subset M$ such that $ax \in N$ for $a \in R$, $x \in N$. A (left) R-homomorphism is a group homomorphism $M \to N$ which preserves the action of R.

Example 1.4. (a) $M = C^{\infty}(\mathbb{R})$, $R = \mathbb{R}[T]$, then $\varphi : M \to M : f(x) \mapsto f(x+1)$ is an R-homomorphism.

Definition 1.5. Let $f : M \to N$ be an R-homomorphism. The kernel of f is $\ker f := \{x \in M : f(x) = 0\} \xrightarrow{i} M$, and the cokernel of f is $N \xrightarrow{p} \operatorname{coker} f := N/\operatorname{im} f$, where the image is $\operatorname{im} f := \{f(x) \in N : x \in M\} \xrightarrow{j} N$.

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & M \arrow[rr, "f"] \arrow[dr, "f"'] & & N \arrow[r, "p"] & \operatorname{coker} f \\
& & \operatorname{im} f \arrow[ur, "j"'] & &
\end{tikzcd}$$

On each of them, there is a unique R-module structure such that the each map i, j, p becomes an R-homomorphism respectively.

Theorem 1.6 (Universal property)**.** *For the above setting, note that $fi = 0$ and $pf = 0$. If an R-homomorphism $g : M' \to M$ satisfies $fg = 0$, then there is a unique R-homomorphism $h : M' \to \ker f$ such that $g = ih$. If an R-homomorphism $g : N \to N'$ satisfies $gf = 0$, then there is a unique R-homomorphism $h : \operatorname{coker} f \to N'$ such that $g = hp$.*

1.1 Commutative diagrams and exact sequences

Definition 1.7 (Diagram)**.** Among some R-modules suppose we have R-homomorphisms as the following diagram:

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "f_3"'] & M_2 \arrow[d, "g_1"] \arrow[dl, "f_2"'] \\
M_3 \arrow[r, "g_2"] & M_4
\end{tikzcd}.$$

Then, if the compositions sharing each source and target coincide, then we say the diagram is commutative. For example, we say the triangle formed by M_2, M_3, M_4 is commutative iff $g_1 = g_2 f_2$.

Definition 1.8 (Sequence)**.** A sequence is a diagram of R-modules placed linearly as

$$\cdots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \longrightarrow \cdots.$$

If $\operatorname{im} f_n = \ker f_{n+1}$ for all n, then we say the sequence is exact.

Example 1.9. (a) $f : M \to N$ is injective iff $0 \to M \xrightarrow{f} N$ is exact. $f : M \to N$ is surjective iff $M \xrightarrow{f} N \to 0$ is exact.

(b)

$$0 \longrightarrow \ker f \xrightarrow{\ i\ } M \xrightarrow{\ f\ } N \xrightarrow{\ p\ } \operatorname{coker} f \longrightarrow 0$$

is exact.

(c)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ n\ } \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is exact.

(d)

$$0 \longrightarrow \mathbb{R}\cos x \oplus \mathbb{R}\sin x \xrightarrow{\ n\ } C^\infty(\mathbb{R}) \xrightarrow{\frac{d^2}{dx^2}+1} C^\infty(\mathbb{R}) \longrightarrow 0$$

is exact.

Proposition 1.10 (Five lemma)**.** *Suppose each row is exact in the folloing commutative diagram:*

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & M_5 \arrow[d, "h_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5.
\end{tikzcd}$$

Then,

(a)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, hook] & {} \arrow[d, hook, dashed] & {} \arrow[d, hook] \\
{} & {} & {} & {}
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, two heads, dashed] & {} \arrow[d, two heads] & {} \arrow[d, hook] \\
{} & {} & {} & {}
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, "\sim"] & {} \arrow[d, dashed, "\sim"] & {} \arrow[d, "\sim"] & {} \arrow[d, hook] \\
{} & {} & {} & {} & {}
\end{tikzcd}$$

Proof. (a) We will show $x \in \ker h_3$ is in the image of $f_2 f_1$: $h_3(x) = 0 \implies f_3(x) = 0 \implies x = f_2(y) \implies g_2 h_2(y) = 0 \implies h_2(y) = g_1(z) \implies z = h_1(u) \implies f_1(u) = y$. Then, $x = f_2(y) = f_2 f_1 = 0$.

(b) Similar.

(c) Clear. □

Proposition 1.11 (Snake lemma)**.** *Suppose the second and the third rows are exact in the following commutative diagram:*

$$\begin{array}{ccccc}
 & \ker h_1 & \ker h_2 & \ker h_3 & \\
 & M_1 & M_2 & M_3 & 0 \\
0 & N_1 & N_2 & N_3 & \\
 & \operatorname{coker} h_1 & \operatorname{coker} h_2 & \operatorname{coker} 3 &
\end{array}$$

(a) *There is* $\delta : \ker h_3 \to \operatorname{coker} h_1$ *such that*

$$\ker h_1 \xrightarrow{k_1} \ker h_2 \xrightarrow{k_2} \ker h_3 \xrightarrow{\delta} \operatorname{coker} h_1 \xrightarrow{l_1} \operatorname{coker} h_2 \xrightarrow{l_2} \operatorname{coker} 3$$

is exact. Here k_1, k_2, l_1, l_2 *are induced from* f_1, f_2, g_1, g_2*, respectively. The element* $\delta(x)$ *is determined by* u *such that* $x = f_2(y)$*,* $z = h_2(y)$*,* $z = g_1(u)$*, and we can check that* u *does not depend on the choice of* y*.*

(b)

Proof. (a) We have to show the well-definedness of δ, $\ker \subset \operatorname{im}$, and $\operatorname{im} \subset \ker$. Skip. □

In the general abelian cateogies, the five lemma and the snake lemma hold but the proofs become more complicated.

1.2 Direct sum, direct product, inductive limit, direct limit

Definition 1.12. Let M_λ be a family of R-modules. The direct product is

$$\prod_\lambda M_\lambda := \{(x_\lambda) : x_\lambda \in M_\lambda\} \twoheadrightarrow M_\lambda,$$

and the direct sum is the submodule of the direct product such that

$$\bigoplus_\lambda M_\lambda := \{(x_\lambda) : x_\lambda = 0 \text{ but finitely many}\} \hookleftarrow M_\lambda$$

Proposition 1.13 (Universal property)**.** (a) *For* $f_\mu : M_\mu \to N$ *there is unique* $f : \bigoplus_\lambda M_\lambda \to N$ *such that* $f i_\mu = f_\mu$*.*

(b) *For* $g_\mu : N \to M_\mu$ *there is unique* $g : N \to \prod_\lambda M_\lambda$ *such that* $p_\mu g = g_\mu$*.*

Remark 1.14. (a) The direct sum and direct product is unique up to isomorphism by the universal property.

(b) For R-homomorphisms $f_\lambda : M_\lambda \to N_\lambda$ we can induce $\prod_\lambda f_\lambda : \prod_\lambda M_\lambda \to \prod_\lambda N_\lambda$ and $\bigoplus_\lambda f_\lambda : \bigoplus_\lambda M_\lambda \to \bigoplus_\lambda N_\lambda$.

(c) In the category of modules, even for infinite indices, direct product and sum commute with the kernel, cokernel, and image. In an abelian category, we may not have infinie direct product/sum.

(d) exactness also preserved under products and sums

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Let $(\Lambda, \prec)$ be a totally ordered set. By a direct system, we refer the family of R-modules M_λ for each $\lambda \in \Lambda$ and the family of R-homomorphisms $\tau_{\mu\lambda} : M_\lambda \to M_\mu$ for $\lambda \prec \mu$ such that $\tau_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ and $\tau_{\kappa\lambda} = \tau_{\kappa\mu}\tau_{\mu\lambda}$ for $\lambda \prec \mu \prec \kappa$.

Example.1.3.3.

(a) Let $\Lambda = \mathbb{N}$ and $n \prec m \Leftrightarrow n \mid m$, $M_n = \mathbb{Z}$ and $\tau_{mn}(z) : M_n \to M_m : z \mapsto (m/n)z$.

(b) Let M be a R-module, $\{M_\lambda\}$ are finitely generated R-submodules of M, and $\lambda \prec \mu \Leftrightarrow M_\lambda \subset M_\mu$, with $\tau_{\mu\lambda}$ inclusions.

Definition.

$$\varinjlim M_\lambda = \varinjlim(M_\lambda, \tau_{\mu\lambda}) := \operatorname{coker}\Big(\bigoplus_{\substack{(\lambda,\mu)\in\Lambda \\ \lambda\prec\mu}} M_\lambda \xrightarrow{\Phi} \bigoplus_{\lambda\in\Lambda} M_\lambda\Big),$$

where $\Phi((x_{\lambda\mu})) = \sum_{\lambda\prec\mu} \iota_\mu \tau_{\mu\lambda}(x_{\lambda\mu}) - \iota_\lambda(x_{\lambda\mu})$, and $\iota_\lambda : M_\lambda \to \bigoplus_\lambda M_\lambda$ is a componentwise embedding. That is, we want to identify $x \in M_\lambda$ and $\tau_{\mu\lambda}(x) \in M_\mu$ with the map Φ.

Proposition.1.3.4. *Let* $\tau_\mu : M_\mu \xrightarrow{\iota_\mu} \bigoplus_\lambda M_\lambda \twoheadrightarrow \lim_{\longrightarrow} M_\lambda$.

(a) $\tau_\mu = \tau_\kappa \tau_{\kappa\mu}$.

(b) $M_\mu \xrightarrow{f_\mu} N$ *for* $\mu \in \Lambda$ *are R-homomorphisms, and they satisfy* $f_\mu = f_\kappa \tau_{\kappa\mu}$. *Then, there is a unique* $f : \lim_{\longrightarrow} M_\lambda \to N$ *such that* $f_\mu = f\tau_\mu$

For each example in 1.3.3, $\mathbb{Q}$ and M are the direct limits because it satisfies the universal property (1.3.4(b))

Remark. (1) The direct limit is unique by the universal property up to isomorphism.

(2) If $f_\lambda : M_\lambda \to M'_\lambda$ are R-homomorphism such that

$$\begin{array}{ccc} M_\lambda & \xrightarrow{f_\lambda} & M\lambda' \\ \downarrow & & \downarrow \\ M_\mu & \xrightarrow{f_\mu} & M'_\mu \end{array}$$

commutes for all $\lambda \prec \mu$, then there is a unique f such that

$$\begin{array}{ccccccc} \bigoplus_{\lambda\prec\mu} M_\lambda & \longrightarrow & \bigoplus_\lambda M_\lambda & \longrightarrow & \lim_{\longrightarrow} M_\lambda & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \exists! f} & & \downarrow \\ \bigoplus_{\lambda\prec\mu} M'_\lambda & \longrightarrow & \bigoplus_\lambda M'_\lambda & \longrightarrow & \lim_{\longrightarrow} M'_\lambda & \longrightarrow & 0 \end{array}$$

commutes, and f is denoted by $\lim_{\longrightarrow} f_\lambda$. It is by the universal property of cokernel.

Definition.1.3.6. A preordered set Λ is a directed set if $\forall \lambda, \lambda' \in \Lambda$, there is $\mu \in \Lambda$ such that $\lambda, \lambda' \prec \mu$.

Proposition. *If Λ is a directed set, then there is a 1-1 correspondence*

$$\Big(\coprod_\lambda M_\lambda\Big)/\sim \longrightarrow \varinjlim M_\lambda : [x_\lambda] \mapsto \tau_\lambda(x_\lambda),$$

where $x_\lambda \sim y_{\lambda'}$ *iff there is* $\mu \succ \lambda, \lambda'$ *such that* $\tau_{\mu\lambda}(x_\lambda) = \tau_{\mu\lambda'}(y_{\lambda'})$.

Proposition. *If*

$$L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \longrightarrow 0$$

is exact, then

$$\operatorname{colim} L_\lambda \longrightarrow \operatorname{colim} M_\lambda \longrightarrow \operatorname{colim} N_\lambda \longrightarrow 0$$

is exact.

Proof. The only non-trivial part is the exactness at $\operatorname{colim} M_\lambda$. We can prove it by diagram chasing. □

Example. Examples of inverse limit

(a) projection $\mathbb{Z}/p^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ for $m > n$.

(b) restriction $C^\infty((-r,r)) \to C^\infty((-r',r')$ for $r' > r$.

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Example. Limit preserves injectivity, but not surjectivity: although the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\ \mathbb{Z}_p \arrow[r] & \cdots \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \end{tikzcd}$$

commutes, but the induced map $\mathbb{Z} \to \mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ is not surjective.

Lemma (Mittag-Leffler condition). *Let*

$$0 \longrightarrow M_n \longrightarrow N_n \longrightarrow L_n \longrightarrow 0$$

be a sequence of exact sequences. Suppose (M_n) satisfies that for each n we have a eventually constant monotonically decreasing sequence of submodules

$$M_n \supset \pi_{n,n+1}(M_{n+1}) \supset \pi_{n,n+2}(M_{n+2}) \supset \cdots .$$

Then,

$$0 \longrightarrow \lim M_n \longrightarrow \lim N_n \longrightarrow \lim L_n \longrightarrow 0.$$

When we consider the seuqence of kernels $p^n\mathbb{Z}$ of $\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$, we can check it does not satisfy the Mittag-Leffler condition.

1.4. Properties of Hom

Let R be a commutative ring and M, N be a R-modules. Define

$$\mathrm{Hom}_R(M,N) := \{f : M \to N \ R\text{-homomorphism}\}.$$

It is an R-module, which is not the case if R is not commutative. If $\varphi : N_1 \to N_2$ is an R-homomorphism, then

$$\mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2) : f \mapsto \varphi \circ f$$

is an R-homomorphism. If $\psi : M_1 \to M_2$ is an R-homomorphism, then

$$\mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N) : f \mapsto f \circ \psi$$

is an R-homomorphism.

Proposition.1.4.1.

(a) *If*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3$$

is exact, then

$$0 \longrightarrow \mathrm{Hom}_R(M,N_1) \longrightarrow \mathrm{Hom}_R(M,N_2) \longrightarrow \mathrm{Hom}_R(M,N_3)$$

is exact.

(b) *If*

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \mathrm{Hom}_R(M_3,N) \longrightarrow \mathrm{Hom}_R(M_2,N) \longrightarrow \mathrm{Hom}_R(M_1,N)$$

is exact.

Proof. (a) If $f \in \mathrm{Hom}_R(M, N_2)$ satisfies $\varphi_2 \circ f = 0$ where $\varphi : N_2 \to N_3$, then by the universal property

$$\begin{tikzcd}
 & & M \arrow[dl, dashed, "\exists !"'] \arrow[d] \arrow[dr] & \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3
\end{tikzcd}$$

□

Example. For

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

The maps

$$0 \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

and

$$\mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{n} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

are not surjective.

1.5. Projective modules

Definition.1.5.1. An R-module is said to be projective if for every surjective $\varphi : N_1 \twoheadrightarrow N_2$ and for every $f : M \to N_2$, there is a map $\widetilde{f} : M \to N_1$ such that

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\widetilde{f}"'] \arrow[d, "f"] \\
N_1 \arrow[r, two heads] & N_2
\end{tikzcd}$$

commutes, equivalently,

$$\mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to 0$$

is exact for every exact $N_1 \to N_2 \to 0$.

Proposition.1.5.2. *If M is a projective module, then* $\mathrm{Hom}_R(M, -)$ *is an exact functor.*

Proposition.1.5.3. *A direct sum of R-modules is projective iff its summands are all projective. In particular, a free R-module is projective.*

*Remark.*1.5.4. A module M is projective if and only if there is another module N such that $M \oplus N$ is free.

Proof. (⇒) Take generators of $\{e_\lambda\}_\lambda$ of M. Then, for

$$f : \bigoplus_\lambda R \twoheadrightarrow M : (a_\lambda) \mapsto \sum_\lambda a_\lambda e_\lambda,$$

we have a exact sequence

$$0 \to \ker f \to \bigoplus_\lambda R \to M \to 0,$$

which is right split by applying the definition of projective modules to extend the codomain of $\mathrm{id}_M : M \to M$.

(⇐) Clear from Proposition 1.5.3. □

*Remark.*1.5.5. Let R be a PID. Then, since a submodule of a free module is free, so a module is projective if and only if it is free.

1.6. Injective modules

Definition.1.6.1. An R-module is said to be injective if for every injective $\varphi : N_1 \hookrightarrow N_2$ and for every $g : N_1 \to M$, there is a map $\widetilde{g} : N_2 \to M$ such that

$$\begin{tikzcd} N_1 \arrow[r, hook, "\varphi"] \arrow[d, "g"] & N_2 \arrow[dl, dashed, "\widetilde{g}"] \\ M & \end{tikzcd}$$

commutes, equivalently,

$$\mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to 0$$

is exact for every exact $N_1 \to N_2 \to 0$.

Proposition. *1.6.3. An R-module M is injective iff the restriction* $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M)$ *is surjective for every ideal* I *of R.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) Suppose there is $x \in N_2$ such that $N_2 = N_1 + Rx$. By letting I be the kernel of a ring homomorphism $R \to (N_1 + Rx)/N_1 : a \mapsto ax + N_1$, we have an exact sequence

$$0 \to I \to N_1 \oplus R \to N_1 + Rx = N_2 \to 0$$

in which the first map sends b to $(-bx, b)$ and the second map sends (y, a) to $y + ax$.

$\square$

Remark. 1.6.4. If R is a PID, then an R-module M is injective iff for all $0 \neq a \in R$ the map $M \xrightarrow{\cdot a} M$ is surjective.

Proof. $\square$

Example. If $R = \mathbb{Z}$, then $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, and $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are not injective.

Ab-enriched: preadditive binary biproduct: semiadditive existence of ker/coker normality of mono/epi constructions: universals(products and equalizers, pullbacks, limits, representables)