Von Neumann Algebras

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Contents

Part I

Fundamentals

Chapter 1

1.1

1.1 (Support projections of operators)**.** Let x be an element of a von Neumenna algebra M. The *left support projection* of x is the minimal projection $p \in M$ such that $x = px$, denoted by $s_l(x)$. The *right support projection* of x is defined as the left support projection of x^*. The projections $s_l(x)$ and $1-s_r(x)$ are also called the *range* and *kernel* projections of x, respectively.

(a) Support projections of x uniquely exist.

(b) $x^*yx = 0$ if and only if $s_l(x)ys_l(x) = 0$ for every $y \in M$.

(c) We have $s_r(x) = s_r(x^*x) = s_r(|x|)$. In particular, $s_l(x) = s_r(x)$ if x is normal.

(d) If $x^*x \le y^*y$, then there is a unique $v \in M$ such that $x = vy$ and $s_r(v) \le s_l(y)$.

(e) There is unique $v \in M$ such that the polar decomposition $x = v|x|$ holds and that $s_r(x) = v^*v$. Moreover, $x^* = v^*|x^*|$ and $s_l(x) = vv^*$. In particular, $s_l(x)$ and $s_r(x)$ are Murray-von Neumann equivalent.

Proof. (a) Let $x \in M$. Since $\operatorname{im} x = \operatorname{im}(xx^*)^{\frac{1}{2}}$, we may assume $0 \le x \le 1$. Then, $x^{2^{-n}}$ is an increasing sequence in M bounded by one, so it converges strongly to some $p \in M_+$. We can check $p^2 = p$ by... We can check p is the range projection of x by...

(e) Since $x^*x \le |x|^*|x|$, there is a unique $v \in M$ such that $x = v|x|$ and $v = vs_l(|x|) = vs_r(x)$. Then, $s_r(x) - v^*v = s_r(x)(1-v^*v)s_r(x) = 0$ from $|x|(1-v^*v)|x| = |x|^2 - |x|^2 = 0$, and $s_l(x) - vv^* = s_l(x)(1-vv^*)s_l(x) = 0$ from $x^*(1-vv^*)x = |x|^2 - |x|^2 = 0$. The partial isometry v is unique since $s_r(x) = v^*v$ implies $s_r(v) = s_r(v^*v) = s_r(s_r(x)) = s_r(x)$. Similarly, $s_l(v) = s_l(x)$. The equality $xv^* = |x^*|$ follows from $xv^* = v|x|v^* \ge 0$ and $|xv^*|^2 = vx^*xv^* = v|x|^2v^* = xx^* = |x^*|^2$. □

1.2 (Support projections of states)**.**

1.3 (Cyclic and separating vectors)**.** A vector state is separating iff it is faithful.

If $M \subset B(H)$ admits a separating vector, then every normal state is a vector state. (T:V.1.12, J:7.1.4?)

1.4 (Countable decomposable von Neumann algebras)**.** Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal nonzero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is. The followings are all equivalent.

(a) M is countably decomposable.

(b) M admits a faithful normal state.

(c) M admits a module with a cyclic and separating vector.

(d) The unit ball of M is metrizable in the σ-strong topology.

Proof. □

1.5 (Separable von Neumann algebras)**.** Let M be a von Neumann algebra. The followings are all equivalent.

(a) M has the separable predual.

(b) M admits a faithful separable module.

(c) M is countably decomposable and countably generated.

(d) The unit ball of M is metrizable in the σ-weak topology.

Proof. □

Chapter 2

Modular theory

2.1 Weights

2.1 (Ideals associated to weights)**.** left ideal, definition ideal

2.2 (Semi-cyclic representations)**.** Let A be a C*-algebra. A *semi-cyclic representation* is a representation $\pi : A \to B(H)$ together with a linear map $\psi : \mathfrak{n} \to H$ from a left ideal $\mathfrak{n}$ of A into H with dense range, such that $\pi(x)\psi(y) = \psi(xy)$ for $x \in A$ and $y \in \mathfrak{n}$.

For a semi-cyclic representation, if we denote $\mathfrak{m} := \mathfrak{n}^*\mathfrak{n}$, then we have a bilinear form

$$\Theta : \mathfrak{m} \times \pi(A)' \to \mathbb{C} : (y^*x, z) \mapsto \langle z\psi(x), \psi(y)\rangle.$$

With this, we can construct a linear map $\theta : \mathfrak{m} \to (\pi(A)')_*$ and its transpose $\theta^* : \pi(A)' \to \mathfrak{m}^\#$.

Consider a weight φ.

(a) A (it might require some condition here if A is not W*) weight on A defines a semi-cyclic representation and vice versa?

(b) If $A = M$ is a von Neumann algebra, then we can let $\theta_* : \pi(M)' \to M_*$ to have $\theta^{**} = \theta$.

(c) θ^* is bijective onto the space of linear functionals on $\mathfrak{m}$ absolutely continuous with respect to φ. (bounded Radon-Nikodym)

2.3 (Normal weights)**.** Let M be a von Neumann algebra. Let ω be a weight of M.

(a) ω is normal.

(b) ω is σ-weakly lower semi-continuous.

(c) ω is the supremum of a set of normal positive linear functionals.

Proof. (c)⇒(b)⇒(a) are clear.

(a)⇒(b)

Suppose first M is countably decomposable so that B is metrizable.

□

2.2 Hilbert algebras

2.4. A *left Hilbert algebra* is a $*$-algebra A together with an inner product such that the left multiplication defines a nondegenerate $*$-homomorphism $\lambda : A \to B(H)$, where $H := \overline{A}$, and the involution is a closable antilinear operator whose domain contains A.

If an involution is an isometry, then it is also a right Hilbert algebra, which is the unimodular case.

2.3 Traces

2.5 (Semi-finite and tracial von Neumann algebras)**.** Let M be a von Neumann algebra. We say M is *semi-finite* if it admits a faithful normal semi-finite trace, and *tracial* if it admits a faithful normal tracial state.

(a) regular representation and antilinear isometric involution J. $L(G) = \rho(G)'$

(b) M is semi-finite if and only if type III does not occur in the direct sum.

(c) A factor M has at most one tracial state, which is normal and faithful.

(d) A factor is tracial if and only if it is type II_1.

2.6 (Semi-finite traces)**.** Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x \in M^+$ has a net $x_\alpha \in L^1(M,\tau)^+$ such that $x_\alpha \uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x) = \sup\{\, \tau(y) : y \le x,\ y \in L^1(M,\tau)^+ \,\} \quad \text{for} \quad x \in M^+.$$

2.7 (Uniformly hyperfinite algebras)**.** Let A be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique tracial state.

(b) Every UHF algebra admits a unique tracial state.

(c) Every hyperfinite

measurable operators, unbounded operators affilated with M, noncommutative L^p spaces for semi-finite con Neumann algebras, noncommutative L^p space for general von Neumann algebras: by Haagerup(crossed product), and by Kosaki-Terp(complex interpolation).

On semi-finite von Neumann algebras, measurable operators are affiliated. On a finite von Neumann algebras, affiliated operators are measurable.

- density of $C(X)$ in $L^p(X,\mu)$
- Hölder inequality
- Radon-Nikodym
- Riesz representation
- Fubini
- maximality of L^∞ in $B(L^2)$

2.4 Modular automorphisms

2.8 (Unitary group)**.** (a) $U(H)$ is strongly* complete.

(b) $U(H)$ is not strongly complete.

(c) $U(H)$ is weakly relatively compact.

Let A be a C*-algebra. Then, $\overline{U(A) \cap B(1,r)}^{s*} = U(A'') \cap B(1,r)$. In particular, $U(A)$ is strongly* dense in $U(A'')$. (Kaplansky?)

Exercises

2.9 (Lower semi-continuous weights)**.** Let φ be a weight on a C*-algebra A. The semi-cyclic representation of φ is non-degenerate if either A is unital or φ is lower semi-continuous. On a von Neumann algebra, there exists a weight that is not lower semi-continuous.

2.10 (Completely additive weights)**.** Let φ be a *completely additive* weight on a von Neumann algebra in the sense that for every orthogonal family $\{p_\alpha\}$ of projections we have $\varphi(\sum_\alpha p_\alpha) = \sum_\alpha \varphi(p_\alpha)$.

(a) A completely additive state on a von Neumann algebra is normal.

(b) A completely additive and lower semi-continuous weight on a commutative von Neumann algebra is normal.

Chapter 3

Direct integral

3.1 Commutative von Neumann algebras

σ-field is a unital σ-ring. σ-ideal is an ideal of a σ-ring which is a σ-ring. σ-ideal is sometimes called the measure class because it corresponds to an equivalence class of measures up to absolute continuity.

3.1 (Enhanced measurable spaces)**.** An *enhanced measurable space* is a measurable space (X, M) together with a σ-ideal N of M. A morphism between enhanced measurable spaces is a partial function $f : X_1 \to X_2$ on a conegligible set such that f^* induces a ring homomorphism $M_2/N_2 \to M_1/N_1$.

(a) Maharam's theoem: every enhanced measurable space is isomorphic to the disjoint union of $\{0,1\}^I$, where I is an aribitrary cardinality...?

(b) A σ-finite enhanced measurable space is isomorphic to a enhenced measurable space induced from a standard probability space...?

(c) For σ-finite enhanced measurable spaces, a $*$-homomorphism $L^\infty(X_2) \to L^\infty(X_1)$ induces a morphism $X_1 \to X_2$...?

3.2 (Maharam classification)**.**

3.3. Noncommutative L^p spaces for a general weight?

(a) For $1 \le p < \infty$, $C_0(X) \to L^p(X, \mu)$ is a bounded linear maps of dense range.

(b) $L^\infty(X, \mu)$ is a m.a.s.a. of $B(L^2(X, \mu))$.

Proof. We will show bounded linear maps $L^\infty(X, \mu)' \to M(X)$ and $L^\infty(X, \mu) \to M(X)$ have the same image. Let $y \in L^\infty(X, \mu)'$ and define $\mu_y \in M(X)$ by

$$\mu_y(a) := \langle \pi_\mu(a) y \psi_\mu, \psi_\mu \rangle.$$

We claim that μ_y factors through $L^1(X, \mu)$. □

Monotone convergence theorem states that a measure on a countably decomposable(?) enhanced measurable space X uniquely defines a 'countably' normal weight on the space of all measurable functions. Note that a 'countably' normal weight is normal on a countably decomposable von Neumann algebra.

3.4 (Maximal commutative subalgebras)**.** A commutative von Neumann algebra M is m.a.s.a. if and only if it admits a cyclic vector. In this case, M is spatially isomorphic to some L^∞ (if separable?).

Proof. □

separable commutative von Neumann algebra is generated by one self-adjoint element.

hyperstonean sapces

3.2 Tensor products

$L^2(X,\mu,H) = L^2(X,\mu) \otimes H$ vector or operator-valued integrals

3.3 Measurable fields

3.5 (Effros Borel structure)**.**

3.6 (Decomposition of states)**.**

3.4 Types

finite, infinite, purely infinite, properly infinite, abelian projections

Type I factors. It possess a minimal projection. It is isomorphic to the whole $B(H)$ for some Hilbert space. Therefore, it is classified by the cardinality of H.

Type II factors. No minimal projection, but there are non-zero finite projections so that every projection can be "halved" by two Murray-von Neumann equivalent projections.

In type II_1 factors, the identity is a finite projection Also, Murray and von Neumann showed there is a unique finite tracial state and the set of traces of projections is $[0,1]$. Examples of II_1 factors include crossed product, tensor product, free product, ultraproduct. Free probability theory attacks the free groups factors, which are type II_1.

In type II_∞ factors. There is a unique semifinite tracial state up to rescaling and the set of traces of projections is $[0,\infty]$.

In type III factors no non-zero finite projections exists. Classified the $\lambda \in [0,1]$ appeared in its Connes spectrum, they are denoted by III_λ. Tomita-Takesaki theory. It is represented as the crossed product of a type II_∞ factor and $\mathbb{R}$.

Amenability, equivalently hyperfiniteness is a very nice condition in von Neumann algebra theory. Group-measure space construction can construct them. There are unique hyperfinite type II_1 and II_∞ factors, and their property is well-known. Fundamental groups of type II factors, discrete group theory, Kazhdan's property (T) are used.

Tensor product factors such as Araki-Woods factors and Powers factors.

Part II

Factors

Chapter 4

Type II factors

4.1. Let M be a von Neumann algebra. Since every σ-weakly closed ideal of M admits a unit z so that we have $zM, Mz \subset I \subset zIz \subset zMz$, and it implies z is a central projection of M. A von Neumann algebra M on H is called a *factor* if $M \cap M' = \mathbb{C}\,\mathrm{id}_H$, which is equivalent to that there are only two σ-weakly closed ideals of M. In a factor, every ideal of M is σ-weakly dense in M

4.1

4.2 (Crossed products)**.** A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ gives

$$\alpha : \Gamma \to \mathrm{Aut}(L^\infty(X)),$$

which has the Koopman representation

$$\sigma : \Gamma \to B(L^2(X)).$$

Then, we have a injective $*$-homomorphism

$$C_c(\Gamma, L^\infty(X)) \to B(L^2(X) \otimes \ell^2(\Gamma)) = B(\ell^2(\Gamma, L^2(X))),$$

whose element $s \mapsto x_s$ is written in

$$\sum_{s\in\Gamma,\ fin} (x_s \otimes 1)(\sigma_s \otimes \lambda_s).$$

(a) $L(\Gamma)$ is a II_1 factor if and only if Γ is a i.c.c. group.

(b) $L^\infty(X)$ is a m.a.s.a. of $L^\infty(X) \rtimes \Gamma$ if and only if the p.m.p. action $\Gamma \curvearrowright X$ is free.

(c) $L^\infty(X) \rtimes \Gamma$ is a II_1 factor if and only if the p.m.p. action $\Gamma \curvearrowright X$ is ergodic.

4.2 Ergodic theory

4.3 Rigidity theory

4.4 Free probability

4.5

Existentially closed II_1 factors

Chapter 5

Type III factors

Part III

Subfactors

Chapter 6

Standard invariant

The way how quantum systems are decomposed. And has Galois analogy.

6.1 (Jones index theorem). A *subfactor* of a factor M is a factor N containing 1_M.

Tensor categories and topological invariants of 3-folds. Ergodic flows.
Ocneanu's paragroups Popa's λ-lattices Jones' planar algebras Quantum entropy

Part IV

Noncommutative probability