# Homological Algebra

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# Part I

# Chapter 1

A left $R$-module $P$ is projective if and only if the left exact functor $\mathrm{Hom}_R(P,-)$ is exact.

A left $R$-module $I$ is injective if and only if the left exact contravariant functor $\mathrm{Hom}_R(-,I)$ is exact.

**1.1** (Tor functor). Let $R$ be a ring and $M$ be a left $R$-module. We define the *Tor functor* as the left derived functor of the right exact functor $-\otimes_R M : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$

$$\mathrm{Tor}_n^R(N,M) := H_n(P_\bullet \otimes_R M),$$

where $P_\bullet$ is a projective resolution of a right $R$-module $N$.

(a) In fact, the Tor functor may be defined by the left derived functor of the right exact functor $M \otimes_R - : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ for a right $R$-module $M$.

(b) In fact, only for Tor functors, we may only assume $P_\bullet$ is a flat resolution. (Flat resolution lemma)

**1.2** (Ext functor). Let $R$ be a ring and $M$ be a left $R$-module. We define the *Ext functor* as the right derived functor of left exact functor $\mathrm{Hom}_R(M,-)$

$$\mathrm{Ext}_R^n(M,N) := H^n(M,I^\bullet),$$

where $I^\bullet$ is an injective resolution of $N$.

(a) In fact, the Ext functor may be defined by the right derived functor of the left exact contravariant functor $\mathrm{Hom}(-,M)$.

long exact seuqence

**1.3** (Universal coefficient theorem). Let $R$ be a ring. Let $C_\bullet$ be a chain complex of flat right $R$-modules and $M$ be a left $R$-module.

*Proof.* We first prove the Künneth formula. Note that modules in $Z_\bullet$ and $B_\bullet$ are also flat. We start from that we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0.$$

We have a short exact sequence of chain complexes

$$\mathrm{Tor}_1^R(B_{\bullet-1},M) \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since modules in $B_{\bullet-1}$ are flat so that $\mathrm{Tor}_1^R(B_{\bullet-1},M) = 0$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since $H_n(C_{\bullet-1}) = H_{n-1}(C_\bullet)$ for any chain complex $C$, we have a long exact sequence

$$H_n(B_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(C_\bullet \otimes_R M) \to H_{n-1}(B_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet \otimes_R M).$$

Since every morphism in $B_\bullet$ and $Z_\bullet$ is zero, we have an exact sequence

$$B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet \otimes_R M) \to B_{n-1} \otimes_R M \xrightarrow{f_{n-1}} Z_{n-1} \otimes_R M.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{coker} f_n \to H_n(C_\bullet \otimes_R M) \to \ker f_{n-1} \to 0.$$

Since

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

is a flat resolution of $H_n(C_\bullet)$, by the flat resolution lemma, we have a long exact sequence

$$\operatorname{Tor}_1^R(Z_n, M) \to \operatorname{Tor}_1^R(H_n(C_\bullet), M) \to B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet) \otimes_R M \to 0.$$

Since $Z_n$ is flat so that $\operatorname{Tor}_1^R(Z_n, M) = 0$, we have

$$\operatorname{coker} f_n = H_n(C_\bullet) \otimes_R M, \quad \ker f_n = \operatorname{Tor}_1^R(H_n(C_\bullet), M).$$

Therefore, we have an exact sequence

$$0 \to H_n(C_\bullet) \otimes_R M \to H_n(C_\bullet \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M) \to 0.$$

Universal coefficient theorem states that if $R$ is a PID, then the Künneth formula splits non-canonically.

□