

Category Theory

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September 9, 2023

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Part I

Chapter 1

Categories

set theoretical issues duality morphisms monic

1.1 Functors

full, faithful natural transformations and equivalence 2-category

Chapter 2

Universal property

2.1 Construction

products, equalizers, pullbacks

2.2 Representable functors

2.1 (Yoneda lemma). Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor from a locally small category \mathcal{C} . Fix $c \in \text{Ob}(\mathcal{C})$. we can define a function

$$\text{Nat}(\text{Hom}(c, -), F) \rightarrow F(c).$$

A *representation* of F is a pair (c, η) of an object $c \in \mathcal{C}$ and a natural isomorphism $\eta : \text{Hom}(c, -) \rightarrow F$.

A *universal element* of F is a pair (c, x) with $x \in F(c)$ such that for any pair (d, y) with $y \in F(d)$ there is a unique morphism $f \in \text{Hom}(c, d)$ satisfying $F(f) : x \mapsto y$.

(a)

Proof. (a) Consider the diagram

$$\begin{array}{ccc}
 \text{Hom}(c, c) & \xrightarrow{\eta_c} & F(c) \\
 \downarrow & \cong & \downarrow \\
 \text{Hom}(c, d) & \xrightarrow{\eta_d} & F(d)
 \end{array}
 \quad
 \begin{array}{ccc}
 & \text{id}_c \mapsto x := \eta_c(\text{id}_c) & \\
 & \downarrow & \downarrow \\
 & f \mapsto \eta_d(f) := F(f)(x) & \\
 & \downarrow & \downarrow \\
 & &
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \downarrow F(f) \\
 & & F(d)
 \end{array}$$

For a natural transformation $\eta : \text{Hom}(c, -) \rightarrow F$, define $x := \eta_c(\text{id}_c)$ in $F(c)$. For $x \in F(c)$, conversely, define a $\eta_d : \text{Hom}(c, d) \rightarrow F(d)$ by $\eta_d(f) := F(f)(x)$ for $d \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}(c, d)$. Then, the collection $\eta = \{\eta_d : d \in \text{Ob}(\mathcal{C})\}$ provides a natural transformation because for each $g \in \text{Hom}(d, e)$ we can check the diagram

$$\begin{array}{ccc}
 \text{Hom}(c, d) & \xrightarrow{\eta_d} & F(d) \\
 g \circ - \downarrow & & \downarrow F(g) \\
 \text{Hom}(c, e) & \xrightarrow{\eta_e} & F(e)
 \end{array}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \quad f \in \text{Hom}(c, d).$$

The correspondences $\eta \mapsto x$ and $x \mapsto \eta$ are inverses of each other, hence the bijection.

□

Chapter 3

Limits

preservation, reflection, creation completeness functoriality

Part II

Chapter 4

4.1 Adjunctions

4.2 Monads

4.3 Kan extensions

Chapter 5

Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

5.1 Enriched categories

5.1 (Pointed category). A *pointed category* is a category with a zero object.

- (a) A category is \mathbf{Set}_* -enriched if and only if it admits a zero morphism.
- (b) Every pointed category is \mathbf{Set}_* -enriched.

Chapter 6

Abelian categories

6.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f) .

A category is called *regular* if every kernel pair admits a coequalizer.

6.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category **Grp** is regular but not coregular, since there is a monomorphism which is not regular.

6.2 Additive and abelian categories

6.2 (Pre-additive categories). A *pre-additive category* is another name of **Ab**-enriched category.

(a) a

6.3 (Semi-additive categories). A *semi-additive category* is a category with binary biproducts.

(a) A category is semiadditive if and only if it is pointed **CMon**-enriched.

6.4 (Additive categories). (a) additive completion by formally adjoining finite biproducts.

(b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a **Set**_{*}-enriched category.

6.5 (Pre-abelian categories). A *pre-abelian category* is an additive category in which every morphism has the kernel and cokernel. Equivalently, it is a finitely bicomplete pre-additive category.

(a)

6.6 (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen). A pointed, Baar-exact, proto-modular, with binary coproducts.

(a) short five lemma, 3×3 lemma, snake lemma, noether isomorphism hold.

(b) long exact homology sequence

(c) Every semi-abelian category is exact.

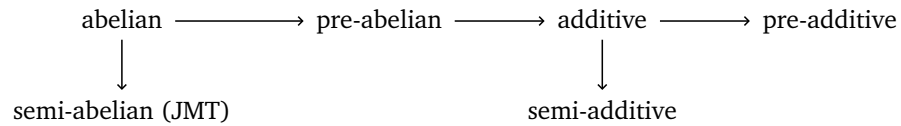
(d) Every semi-abelian category is finitely bicomplete.

(e) In general, a semi-abelian category is not pre-additive nor semi-additive.

6.7 (Abelian categories). We say \mathcal{C} is *abelian* if every morphism has the kernel and cokernel, and every mono and epi is normal.

(a) A category is abelian if and only if it is additive and exact.

6.8 (Freyd-Mitchell embedding).



- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C^* -algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

Chapter 7

site, topos $(\infty, 1)$ -category