Abstract Harmonic Analysis

Ikhan Choi

November 3, 2022

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Part I

Fourier analysis on groups

Chapter 1

Locally compact groups

1.1 Topological groups

1.2 Haar measures

- In $C_0(\Omega)$ theory for metrizable Ω, it was enough to consider **finite measures**: most applications including PDE, probability theory, spectral theory of single operators.
- In $C_0(\Omega)$ theory for non-metrizable Ω, it is enough to consider **regular finite measures** since a σ-finite Radon measure is regular: Choquet theory, spectral theory of general C*-algebras.
- In $C_c(\Omega)$ theory for non-metrizable Ω, we need general **Radon measures**, which may not be σ-finite: Haar measures on general locally compact groups.

1.1 (Non-σ-finite measures). Following technical issues are important

(a) Positive linear functionals on C_c

(b) The Fubini theorem

(c) The Radon-Nikodym theorem

(d) The dual space of L^1 space

1.2 (Radon measures). Let Ω be a locally compact Hausdorff space. A *Radon measure* is a Borel measure μ on Ω such that

(i) μ is outer regular for every Borel set: for every Borel set E we have

$$\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\},$$

(ii) μ is inner regular for every open set: for every open set U we have

$$\mu(U) = \inf\{\mu(K) : K \subset U,\, K \text{ compact}\},$$

(iii) μ is locally finite.

Radon measures generalize finite regular Borel measures which corresponds to positive linear functionals on $C_0(\Omega, \mathbb{R})$, but may be infinite. This infiniteness makes them define positive linear functionals on $C_c(\Omega, \mathbb{R})$, not $C_0(\Omega, \mathbb{R})$.

(a) A σ-finite Radon measure is regular.

(b) If every open subset of Ω is σ-compact, then a locally finite measure is Radon.

(c) $C_c(\Omega)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

1.3 (Riesz-Markov-Kakutani representation theorem for C_c)**.** Let Ω be a locally compact Hausdorff space and Consider the following map:

$$\begin{array}{ccc} \{\text{Radon measures on } \Omega\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_c(\Omega,\mathbb{R})\}, \\ \mu & \mapsto & (f \mapsto \int f\,d\mu). \end{array}$$

(a) a

1.4 (Existence of the Haar measure)**.**

1.2.1 Measures on locally compact Hausdorff spaces

compact closed set not containing infty open open not containing infty closed closed set containing infty

for a measure that "vanishes at infty" = tight two definitions of inner regularity is equivalent.

IRK -> IRF IRK + sigma finite -> tight

Thm. The measure contructed by RMK is lf and regular(cpt version). 1. open set is approx by cpt sets (by def of rho, if X is LCH) 2. meas set is approx by opn sets (by def of outer meas) 3. sigma finite set is approx by cpt sets (by thm)

Consider

$$\begin{tikzcd} \mathrm{regBorel}_{fin} \arrow[r, hook] \arrow[d, hook] & \mathrm{Borel}_{fin} \arrow[r] \arrow[d, hook] & \mathrm{Baire}_{fin} \arrow[r] \arrow[d, hook] & C_b^{*+} \arrow[d, two heads] & \\ \mathrm{regBorel}_{locfin} \arrow[r, hook] & \mathrm{Borel}_{locfin} \arrow[r] & \mathrm{Baire}_{locfin} \arrow[r] & C_c^{*+} \arrow[r, hook] & \text{pos lin on } C_c. \end{tikzcd}$$

for locally compact Hausdorff X.

$\mathrm{Borel}_{locfin} \to$ pos lin on C_c is surjective for all topological spaces.

$\mathrm{regBorel}_{fin} \to C_b^{*+}$ is injective for normal spaces.

$\mathrm{regBorel}_{locfin} \to C_c^{*+}$ is injective for locally compact Hausdorff spaces.(maybe)

Lemma 1.2.1. *Let μ be a Borel measure on a LCH X. Then, μ is inner regular on open sets iff*

$$\mu(U) = \|\mu\|_{C_c(U)^*}$$

for every open U in X.

Proof. ($\Leftarrow$) ($\ge$) For $f \in C_c(U)$, we have

$$|\int f\,d\mu| = |\int_U f\,d\mu| \le \mu(U)\,\|f\|.$$

($\le$) Since μ is inner regular on U, there is a compact set $K \subset U$ such that $\mu(U) - \mu(K) < \varepsilon$ (for the case $\mu(U) = \infty$, we can deal with separately). We can find a nonnegative function $f \in C_c(U)$ with $f|_K \equiv 1$ and $f \le 1$ by the construction of Urysohn. Then, for all $\varepsilon > 0$ we have

$$\mu(U) < \mu(K) + \varepsilon \le \int f\,d\mu + \varepsilon \le \|\mu\|_{C_c^*(U)} + \varepsilon.$$

($\Rightarrow$) Let $f \in C_c(U)$ be a function such that $\|f\| = 1$ and

$$\mu(U) - \varepsilon < \int f\,d\mu.$$

Let $K = \operatorname{supp}(f)$. Then

$$\mu(K) \geq \int f > \mu(U) - \varepsilon.$$

$\square$

Proposition 1.2.2. *A Radon measure is inner regular on all σ-finite Borel sets.(Folland's)*

Proof. First we approximate Borel sets of finite measure, with compact sets. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U - E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned} \mu(K - V) &\geq \mu(K) - \mu(V) \\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U - E) + \frac{\varepsilon}{2}\right) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^{\infty} K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^{\infty} \mu(K_n) > \sum_{n=1}^{\infty} \left(\mu(E_n) - \frac{\varepsilon}{2^n}\right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. $\square$

Theorem 1.2.3. *If every open set in X is σ-compact(i.e. Borel sets and Baire sets coincide), then every locally finite Borel measure is regular.*

Proposition 1.2.4. *In a second countable space, every open set is σ-compact(i.e. Borel sets and Baire sets coincide).*

Two corollaries are presented as follows:

$$\begin{tikzcd}
\begin{matrix}\text{locally finite}\\ \text{Borel regular}\end{matrix} \arrow[r] & \text{Radon} \arrow[l, bend right, dashed, "X \text{ is } \sigma\text{-compact}"'] \arrow[r] & \begin{matrix}\text{locally finite}\\ \text{Borel}\end{matrix} \arrow[ll, bend left, dashed, "X \text{ is second countable}"]
\end{tikzcd}$$

1.5. Let X be compact. A positive linear functional ρ on $C(X)$ is bounded with norm $\rho(1)$.

Proof. Since $0 \leq \rho(\|f\| \pm f) = \|f\|\rho(1) \pm \rho(f)$, we have $|\rho(f)| \leq \rho(1)\|f\|$. $\square$

1.6. Let X be a locally compact Hausdorff space.

(a) The Baire σ-algebra is generated by compact G_δ sets.

(b) If X is second countable, then every Baire set is Borel.

Solution. (b) (A second countable locally compact space is σ-compact.

Since X is σ-compact and Hausdorff, every closed set is a countable union of compact sets, so the Borel σ-algebra on X is generated by compact sets.)

Since locally compact Hausdorff space is regular, the Urysohn metrization implies X is metrizable, and every closed sets in metrizable space is G_δ set. □

1.7. Let X be compact. There is a map from the set of finite Baire measures to the set of positive linear functionals on $C(X)$.

Solution. A function in $C(X)$ is Baire measurable and bounded. Thus the integration is well-defined. □

1.8. Let X be compact. There is a map from the set of positive linear functionals on $C(X)$ to the set of finite regular Borel measures.

Solution. i. and ii. and iii. of Theorem 7.2. □

1.9. Let X be compact. Let ρ be a positive linear functional on $C(X)$. Let ν be the regular Borel measure associated to ρ. Then, $\rho(f) = \int f\,d\nu$.

Solution. iv. of Theorem 7.2. □

1.10. Let X be compact. Let ν be a finite regular Borel measure. Let ν' be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\,d\nu$. Then, $\nu = \nu'$ on Borel sets.

Solution. Theorem 7.8. □

The two results above establish the correspondence between positive linear functionals and regular Borel measures. The following is an additional topic: Borel extension of Baire measures.

1.11. Let X be compact. Let μ be a finite Baire measure. Let ν be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\,d\mu$. Then, $\mu = \nu$ on Baire sets.

Solution. Let μ, ν be finite Baire measures. Enough to show if $\int f\,d\mu = \int f\,d\nu$ then $\mu = \nu$ according to the preceding two results.

Enough to show the regularity of Baire measures. □

1.3 Group algebra

1.12 (Modular functions).

1.13 (Convolution).

1.4 Structure theorems

Exercises

1.14.

Problems

1. Let Ω be a topological space. For every positive linear functional I on $C_c(\Omega, \mathbb{R})$, show that there exists a Borel measure μ on Ω such that $I(f) = \int f\, d\mu$ for all $f \in C_c(\Omega, \mathbb{R})$. (Hint: Consider the uncountable wedge sum of circles as an example.)

Solution. 1. The constructed Carathéodory measure μ on Ω is outer regular Borel measure, but we do not have local finiteness. Everything is same to when Ω is locally compact Hausdorff except that $\mu(\operatorname{supp} f)$ may be infinite. Now it is enough to show $I(\min\{f, \frac{1}{n}\})$ converges to zero as $n \to \infty$ for $f \in C_c(\Omega, [0,1])$.

Let $U := f^{-1}((0,1])$. For $g \in C_0(U, [0,1])$, it clearly has compact support, and and it is also continuous because $g^{-1}((a,1])$ is open in U and $g^{-1}([a,1])$ is closed in K for any $0 < a \leq 1$, so that we have $C_0(U) \subset C_c(X)$. We also have $f_1 \in C_0(U)$ since $f_1^{-1}([\varepsilon,1])$ is a compact set in U for every $\varepsilon > 0$. Therefore, I is a positive linear functional on $C_0(U)$. Assume that I is not bounded; there is no constant C such that $g \in C_0(U, [0,1])$ implies $I(g) \leq C$. Construct a sequence $(h_k)_{k=1}^\infty$ in $C_0(U, [0,1])$ such that $I(h_k) \geq 2^k$, and define $h := \sum_{k=1}^\infty h_k/2^k$ so that $h \in C_0(U, [0,1])$. Then, $I(h) \geq \sum_{k=1}^m I(h_k)/2^k \geq m$ for every $m > 0$, it contradicts to the assumption, which means that there is a constant C such that $I(g) \leq C$ for all $g \in C_0(U, [0,1])$, and it proves $I(f_1) \leq C/n \to 0$ as $n \to \infty$. Therefore, $I(f) = \int f\, d\mu$. $\square$

Chapter 2

Pontryagin duality

2.1 Dual group

2.2

2.3 Fourier inversion

2.1 (Positive definite functions).

2.2 (Bochner's theorem).

2.3 (Fourier inversion theorem).

2.4 (Plancherel's theorem).

Chapter 3

Spectral synthesis

3.1 Closed ideals of the colvolution algebra

Part II

Representation theory

Chapter 4

Unitary representations

4.1

4.1 (Schur's lemma).

4.2 Group C*-algerbas

4.2 (Operator-value Fourier transform).

4.3 Functions of positive type

4.3 (Functions of positive type).

4.4 (Fourier-Stieltjes algebra).

4.5 (GNS construction for locally compact groups). Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

Chapter 5

Compact groups

5.1 Peter-Weyl theorem

5.2 Tannaka-Krein duality

5.3 Example of compact Lie groups

Chapter 6

Mackey machine

6.1 Example of non-compact Lie groups

Wigner classification

Part III

Kac algebras

Part IV

Topological quantum groups

Chapter 7

Compact quantum groups

Chapter 8

Locally compact quantum groups

8.1 Multiplicative unitaries