Partial Differential Equations

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May 9, 2022

Contents

Part I

Sobolev spaces

Chapter 1

Distribution theory

1.1 Space of test functions

1.1. (a) If a test function φ satisfies $\langle 1, \varphi \rangle = 0$, then there is $v \in \mathbb{R}^d$ and a test function ψ such that $\varphi = v \cdot \nabla \psi$.

(b) If a distribution has zero derivative, then it is a constant.

1.2 (Weak* convergence).

1.2 Space of distributions

1.3 (Rigged Hilbert space).

1.3 Well-posedness

1.4 (Extension of linear operators). Let $T : \mathcal{D} \to \mathcal{D}'$ be a contiuous linear operator. We can always define the adjoint $T^* : \mathcal{D} \subset \mathcal{D}'' \to \mathcal{D}'$. The most reasonable extension of T is $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$. For $f \in (T^*(\mathcal{D}))'$, we can define $\langle T(f), \varphi \rangle := \langle f, T^*\varphi \rangle$ for $\varphi \in \mathcal{D}$.

Suppose $T : (\mathcal{D}, \mathcal{T}) \to (T(\mathcal{D}), \mathcal{S})$ is proved to be continuous. If $(\mathcal{D}, \mathcal{T}) \to (T^*(\mathcal{D}))'$ and $(T(\mathcal{D}), \mathcal{S}) \to \mathcal{D}'$ are embeddings, then the extension of T to the completion of $(\mathcal{D}, \mathcal{T})$ agrees with $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$.

For example, if Φ is locally integrable, then since $(T_\Phi)^* = T_{\widetilde{\Phi}}$ and $\Phi * \varphi \in \mathcal{E} = C^\infty$ for $\varphi \in \mathcal{D}$, the convolution operator $T_\Phi : \mathcal{E}' \to \mathcal{D}'$ can be defined on the space of compactly supported distributions.

If $g * f$ is well-defined, is $f * g$ also well-defined? In other words, if $f \in (T_{\widetilde{g}}(\mathcal{D}))'$ so that $g * f \in \mathcal{D}'$, then $g \in (T_{\widetilde{f}}(\mathcal{D}))'$? Are they same?

$$\langle g, \widetilde{f} * \varphi \rangle =$$

Exercises

1.5. * Describe the range of the operator $T : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ defined by $Tf = \Phi * f$ for $d \geq 3$, where Φ is the fundamental solution of Laplace's equation.

Chapter 2

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2.2 (Difference quotient).

2.3 (Interior approximation).

2.4 (Myers-Serrin theorem).

2.2 Extensions and restrictions

2.5 (Lipschitz boundary).

2.6 (Extension theorem).

2.7 (Trace theorem).

2.8 (Vanishing at boundary). zero trace, whole domain

2.3 Sobolev embeddings

2.9 (Gagliardo-Nirenberg-Sobolev inequality).

2.10 (Hölder spaces).

2.11 (Morrey inequality).

2.12 (Poincaré inequality). BMO

2.13 (Rellich-Kondrachov theorem). Let Ω be bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary. Let $1 \le p < d$ and $1 \le q < p^*$ where $p^* := \frac{dp}{d-p}$ denotes the Sobolev conjugate. Let $(u_n)_n$ be a bounded sequence in $W^{1,p}(\Omega)$. We may assume it is also bounded in $W^{1,1}(\mathbb{R}^d)$ by the embedding $W^{1,p}(\Omega) \subset W^{1,1}(\Omega)$ and the extension theorem. Let η_ε be a standard mollifier.

(a) There is a subsequence of $(\eta_\varepsilon * u_n)_n$ that is Cauchy in $L^q(\Omega)$ for each $\varepsilon > 0$.

(b) $\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0$.

(c) $\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \to 0$ as $\varepsilon \to 0$.

(d) There is a subsequence of $(u_n)_n$ that is Cauchy in $L^q(\Omega)$.

(e) $W^{k,p}(\Omega) \to W^{l,q}(\Omega)$ is a compact embedding if

$$\frac{l}{d} - \frac{1}{q} < \frac{k}{d} - \frac{1}{p}.$$

Proof. (a) The sequence $(\eta_\varepsilon * u_n)_n$ is pointwise bounded from

$$\|\eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim_\varepsilon 1,$$

and equicontinuous from

$$\|\nabla \eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\nabla \eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim_\varepsilon 1.$$

By the Arzela-Ascoli theorem, since $\overline{\Omega}$ is compact, there is a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $C(\overline{\Omega})$, and hence in $L^q(\Omega)$.

(b) Write

$$\begin{aligned}
\eta_\varepsilon * u_n(x) - u_n(x) &= \frac{1}{\varepsilon^d} \int \eta\left(\frac{x-y}{\varepsilon}\right)(u_n(y) - u_n(x))\, dy \\
&= \int \eta(y)(u_n(x - \varepsilon y) - u_n(x))\, dy \\
&= \int \eta(y) \int_0^1 \frac{d}{dt}(u_n(x - t\varepsilon y))\, dt\, dy \\
&= \int \eta(y) \int_0^1 (-\varepsilon y) \cdot \nabla u_n(x - t\varepsilon y)\, dt\, dy.
\end{aligned}$$

Then, since $|y| \ge 1$ if $\eta(y) > 0$,

$$\|\eta_\varepsilon * u_n - u_n\|_{L^1(\mathbb{R}^d)} \le \varepsilon \int \eta(y) \int_0^1 \int |\nabla u_n(x - t\varepsilon y)|\, dx\, dt\, dy = \varepsilon \|\nabla u_n\|_{L^1(\mathbb{R}^d)}.$$

(c) The interpolation

$$\|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_n - u_n\|_{L^1(\Omega)}^{\theta} \|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)}^{1-\theta}$$

for $q = \frac{\theta}{1} + \frac{1-\theta}{p}$ with $0 < \theta \le 1$ and the Gagliardo-Nireberg-Sobolev inequality

$$\|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)} \lesssim \|\eta_\varepsilon * u_n - u_n\|_{W^{1,p}(\Omega)} \lesssim 1$$

give the L^q version of the part (b),

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \to 0$$

as $\varepsilon \to 0$.

(d) By the part (c), for any $\delta > 0$, there is $\varepsilon > 0$ such that

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} < \frac{\delta}{2},$$

so for a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $L^q(\Omega)$, we have

$$\|u_{n_k} - u_{n_{k'}}\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_{n_k} - \eta_\varepsilon * u_{n_{k'}}\|_{L^q(\Omega)} + \delta,$$

and by the diagonal argument reducing δ to zero, we can construct the desired subsequence.

(e) □

Chapter 3

Generalizations of Sobolev spaces

3.1 Fractional Sobolev spaces

3.2 Fourier transform methods

3.3 Almost everywhere differentiability

Lipschitz, Rademacher

3.4 Vector-valued functions

Part II

Elliptic equations

Chapter 4

Existence

4.1 Lax-Milgram theorem

4.1 (Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the problem

$$\begin{cases} -\Delta u(x) = f(x) & , \text{ in } x \in \Omega, \\ u(x) = 0 & , \text{ on } x \in \partial\Omega. \end{cases}$$

Define a bilinear form B on $H_0^1(\Omega)$ such that

$$B(u,v) := \int \nabla u(x) \cdot \nabla v(x)\,dx.$$

(a) If $u \in H_0^1(\Omega)$ and $f \in \mathcal{D}'(\Omega)$ satisfy $B(u,\varphi) = \langle f, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, then $-\Delta u = f$.

(b) B is another inner product equivalent to $\langle -, - \rangle_{H_0^1(\Omega)}$.

(c) For $f \in H^{-1}(\Omega)$, there is $u \in H_0^{-1}(\Omega)$ such that $-\Delta u = f$.

4.2 Fredholm alternative

4.3 Eigenvalue problems

4.4 Perron's method

Chapter 5

Regularity

5.1 L^p theory

5.1 (Interior regularity in H^2)**.** Let Ω be bounded open subset of $\mathbb{R}^d$ and $L : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ a uniformly elliptic operator given by

$$Lu := -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu$$

for $a^{ij} \in C^1(\Omega)$, $b^i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)$.

Fix an open subset $U \Subset \Omega$ and $\zeta \in C_c^\infty(\Omega)$ a cutoff function such that $\zeta = 1$ in U. Let $\varphi := -\partial_k^{-h}(\zeta^2\partial_k^h u)$ for $k = 1, \cdots, d$ and sufficiently small $h > 0$.

(a) We have

$$\|\nabla u\|_{L^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$

(b) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \int a^{ij}\partial_i u\partial_j\varphi + \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$.

(c) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}$$

for all u such that $Lu \in L^2(\Omega)$ and $u \in H^1(\Omega)$.

(d) We have

$$\|u\|_{H^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$.

Proof. (a) Since $\zeta^2 u \in H_0^1(\Omega)$,

$$
\begin{aligned}
\int \zeta^2 |\nabla u|^2 &\lesssim \int a^{ij} \zeta^2 \partial_i u \partial_j u \\
&= \int a^{ij}\ \partial_i u\ \partial_j(\zeta^2 u) - \int a^{ij}\ \partial_i u\ \partial_j(\zeta^2) u \\
&= \int (Lu - b^i \partial_i u - cu)\ \zeta^2 u - \int a^{ij}\ \partial_i u\ 2\zeta \partial_j \zeta\ u \\
&\lesssim \int (|Lu\ u| + |u\ \zeta \nabla u| + |u|^2 + |u\ \zeta \nabla u|) \\
&\lesssim \int (|Lu|^2 + |u|^2) + \frac{1}{\varepsilon} \int |u|^2 + \varepsilon \int \zeta^2 |\nabla u|^2.
\end{aligned}
$$

Taking small $\varepsilon > 0$, we are done.

(b) Write

$$
\begin{aligned}
\int a^{ij} \partial_i u \partial_j \varphi &= -\int a^{ij} \partial_i u \partial_j \partial_k^{-h} (\zeta^2 \partial_k^h u) \\
&= \int \partial_k^h (a^{ij} \partial_i u)\ \partial_j (\zeta^2 \partial_k^h u) \\
&= \int \partial_k^h a^{ij}\ \partial_i u\ \partial_j(\zeta^2)\ \partial_k^h u + \int \partial_k^h a^{ij}\ \partial_i u\ \zeta^2\ \partial_j \partial_k^h u \\
&\quad + \int a^{ij}\ \partial_k^h \partial_i u\ \partial_j(\zeta^2)\ \partial_k^h u + \int a^{ij}\ \partial_k^h \partial_i u\ \zeta^2\ \partial_j \partial_k^h u.
\end{aligned}
$$

The last term out of the four terms controls the difference quotient $|\partial_k^h \nabla u|$ as

$$
\int a^{ij}\ \partial_k^h \partial_i u\ \zeta^2\ \partial_j \partial_k^h u \gtrsim \int \zeta^2 |\partial_k^h \nabla u|^2,
$$

and the absolute values of other three terms are estimated up to constant by

$$
\begin{aligned}
&\int \zeta |\nabla u| |\partial_k^h u| + \int \zeta^2 |\nabla u| |\partial_k^h \nabla u| + \int \zeta |\partial_k^h \nabla u| |\partial_k^h u| \\
&\quad \lesssim \left(1 + \frac{1}{\varepsilon}\right) \int \zeta^2 |\nabla u|^2 + \left(1 + \frac{1}{\varepsilon}\right) \int |\partial_h^k u|^2 + \varepsilon \int \zeta^2 |\partial_k^h \nabla u|^2 \\
&\quad \lesssim \left(1 + \frac{1}{\varepsilon}\right) \int |\nabla u|^2 + \varepsilon \int \zeta^2 |\partial_k^h \nabla u|^2.
\end{aligned}
$$

Therefore,

$$
\int \zeta^2 |\partial_k^h \nabla u|^2 \lesssim \int a^{ij} \partial_i u \partial_j \varphi + \left(1 + \frac{1}{\varepsilon}\right) \int |\nabla u|^2 + \varepsilon \int \zeta^2 |\partial_k^h \nabla u|^2,
$$

and taking small $\varepsilon > 0$, we are done.

(c) Note that

$$
\int a^{ij} \partial_i u \partial_j \varphi = \int (Lu - b^i \partial_i u - cu) \varphi
$$

since $\varphi \in H_0^1(\Omega)$. Because

$$
\int (Lu - b^i \partial_i u - cu) \varphi \lesssim \frac{1}{\varepsilon} \int (|Lu|^2 + |\nabla u|^2 + |u|^2) + \varepsilon \int |\varphi|^2
$$

and

$$\begin{aligned}\int |\varphi|^2 &= \int |\partial_k^{-h}(\zeta^2 \partial_k^h u)|^2 \\ &\lesssim \int |\nabla(\zeta^2 \partial_k^h u)|^2 \\ &\lesssim \int |\partial_k^h u|^2 + \int \zeta^2 |\partial_k^h \nabla u|^2 \\ &\lesssim \int |\nabla u|^2 + \int \zeta^2 |\partial_k^h \nabla u|^2,\end{aligned}$$

we obtain

$$\int (Lu - b^i \partial_i u - cu)\varphi \lesssim \frac{1}{\varepsilon}\int (|Lu|^2 + |u|^2) + \left(\varepsilon + \frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon \int \zeta^2 |\partial_k^h \nabla u|^2.$$

Taking small $\varepsilon > 0$, we are done. □

5.2 Schauder theory

5.3 Weyl's lemma

Chapter 6

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Part III

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Chapter 7

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Chapter 8

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Chapter 9

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Part IV

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Chapter 11

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shocks NS