Operator Algebra Seminar Note II

Ikhan Choi

November 12, 2023

Contents

1 October 18 **2**
- 1.1 Countably decomposable von Neumann algebras 2
- 1.2 Semi-cyclic representations 3
- 1.3 σ-weak lower semi-continuity 6
- 1.4 Pointwise supremum of normal positive linear functionals 8

2 November 10 **9**
- 2.1 Hilbert algebras 9
- 2.2 Modular operator and conjugation 11
- 2.3 Faithful normal semi-finite weights 12

3 December 20 **17**
- 3.1 Tomita-Takesaki commutation theorem 17

4 January 17 **18**

5 February 9 **19**

Acknowledgement

This note has been written based on the first-year graduate seminar presented at the University of Tokyo in the 2023 Autumn semester. Each seminar was delivered for 105 minutes. I gratefully acknowledge advice of Prof. Yasuyuki Kawahigashi and support of my colleagues Futaba Sato, Yusuke Suzuki.

1 October 18

1.1 Countably decomposable von Neumann algebras

Definition 1.1 (Countably decomposable von Neumann algebras)**.** Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal non-zero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is.

Proposition 1.2. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M is countably decomposable.*

(b) *M admits a faithful normal state.*

(c) *M admits a faithful normal non-degenerate representation with a cyclic and separating vector.*

(d) *The unit ball of M is metrizable in the four strong topologies.*

Proof. (a)$\Longleftrightarrow$(b) Suppose M is countably decomposable. Let $\{\xi_i\} \subset H$ be a maximal family of unit vectors such that $\overline{M'\xi_i}$ are mutually orthogonal subspaces, taken by Zorn's lemma. If we let p_i be the projection on $\overline{M'\xi_1}$, then $p_i z p_i = z p_i$ for $z \in M'$ implies $p_i \in M'' = M$. By the assumption, the family $\{\xi_i\}$ is countable. Define a state ω of M such that

$$\omega(x) := \sum_{i=1}^{\infty} \omega_{2^{-i}\xi_i}(x), \qquad x \in M.$$

It converges due to $\|\omega_{2^{-i}\xi_i}\| = 2^{-i+1}$. It is normal since the sequence $(2^{-i}\xi_i)$ belongs to $\ell(\mathbb{N}, H)$, and it is faithful because $\omega(x^*x) = 0$ implies $x\xi_i = 0$ for all i, which deduces that $x = \sum_i x p_i = 0$.

Conversely, if ω is a faithful normal state, then for a mutually orthogonal family of non-zero projections $\{p_i\} \subset M$, we have

$$\{p_i\} = \bigcup_{n=1}^{\infty} \{p_i : \varphi(p_i) > n^{-1}\}$$

the countable union of finite sets. Thus M is countable decomposable.

(b)$\Longleftrightarrow$(c) Let ω be a faithful normal state of M. Consider any faithful normal nondegenerate representation in which ω is a vector state so that the corresponding vector is a separating vector. Examples include the GNS representation of ω, and the composition with the diagonal map $B(H) \to B(\ell^2(\mathbb{N}, H))$. Then, $\overline{M\Omega}$ admits a cyclic and separating vector Ω of M. The converse is immediate, i.e. the vector state ω_Ω is a faithful normal state of M.

(a)$\Longleftrightarrow$(d) Suppose M is countably decomposable and take $\{\xi_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ as we did. Define

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \|(x - y)\xi_i\|.$$

Clearly it generates a topology coarser than strong topology. It is also finer because if a bounded net x_α in M converges to zero in the metric d so that $x\xi_i \to 0$ for all i, then $H = \bigoplus_i M'\xi_i$ implies that for every $\xi \in H$ and $\varepsilon > 0$ we have $\|\xi - \sum_{k=1}^{n} z_k \xi_{i_k}\| < \varepsilon$ for some $z_k \in M'$ so that

$$\|x_\alpha \xi\| \le \|x_\alpha(\xi - \sum_{k=1}^{n} z_k \xi_{i_k})\| + \sum_{k=1}^{n} \|x_\alpha z_k \xi_{i_k}\| < \varepsilon + \sum_{k=1}^{n} \|z_k\| \|x_\alpha \xi_{i_k}\| \to \varepsilon.$$

Since on the bounded part the strong and σ-strong topologies coincide, the two topologies on the unit ball are metrizable. We can do similar for the strong* and the σ-strong* topologies.

Conversely, for a mutually orthogonal family of non-zero projections $\{p_i\}_{i\in I} \subset M$, since the net of finite partial sums $p_F := \sum_{i\in F} p_i$ is an non-decreasing net in the closed unit ball whose supremum is the identity of M, there is a convergent subsequence $p_{F_n} \uparrow 1$ by the metrizability, which implies $I = \bigcup_{n=1}^{\infty} F_n$, the countable union of finite sets. $\square$

1.2 Semi-cyclic representations

Definition 1.3 (Weights)**.** Let M be a von Neumann algebra. A *weight* is a function $\varphi : M^+ \to [0,\infty]$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\lambda x) = \lambda\varphi(x), \qquad x, y \in M^+,\ \lambda \in \mathbb{R}^{\geq 0},$$

where we use $0 \cdot \infty = 0$. A weight φ is said to be *normal* if

$$\varphi(\sup_\alpha x_\alpha) = \sup_\alpha \varphi(x_\alpha)$$

for any bounded non-decreasing net (x_α) in M^+.

Definition 1.4. Let φ be a weight on a von Neumann algebra M. Define a left ideal of M

$$\mathfrak{n} := \{x \in M : \varphi(x^*x) < \infty\},$$

and a hereditary $*$-subalgebra of M

$$\mathfrak{m} := \mathfrak{n}^*\mathfrak{n} = \{\sum_{i=1}^n y_i^* x_i : (x_i), (y_i) \in \mathfrak{n}^n\}.$$

Lemma 1.5. *If $x, y \in M$ satisfies $y^*y \leq x^*x$, then there is a unique $s \in B(H)$ such that $y = sx$ and $s = sp$, where p is the range projection of x, and $s \in M$.*

Proof. Suppose $\mathrm{id}_H \in M \subset B(H)$. The operator $s_0 : \overline{xH} \to \overline{yH} : x\xi \mapsto y\xi$ is well defined because

$$\|y\xi\|^2 = \langle y^*y\xi, \xi\rangle \leq \langle x^*x\xi, \xi\rangle = \|x\xi\|^2.$$

Let p be the range projection of x and let $s := s_0 p$. Then, $y\xi = sx\xi$ for all $\xi \in H$. If $y = s'x$ and $s' = s'p$, then

$$x^*(s-s')^*(s-s')x = (y-y)^*(y-y) = 0$$

implies

$$0 = p(s-s')^*(s-s')p = (s-s')^*(s-s').$$

Therefore, s is unique in $B(H)$. If $u \in M'$ is unitary, then usu^* satisfies the same property $y = usu^*x$ and $usu^* = usu^*p$, so $us = su$. Since the unitary span the whole C*-algebra, we have $s \in M'' = M$. $\square$

Proposition 1.6. *Let φ be a weight on a von Neumann algebra M.*

(a) *Every element of $\mathfrak{m}^+$ can be written to be x^*x for some $x \in \mathfrak{n}$.*

(b) *Every element of $\mathfrak{m}$ can be written to be y^*x for some $x, y \in \mathfrak{n}$.*

Proof. (a) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}^+$ for some $x_i, y_i \in \mathfrak{n}$. The polarization writes

$$a = \frac{1}{4}\sum_{i=1}^n \sum_{k=0}^3 i^k |x_i + i^k y_i|^2$$

and $a^* = a$ implies

$$a = \frac{1}{2}\sum_{i=1}^n (|x_i + y_i|^2 - |x_i - y_i|^2) \leq \frac{1}{2}\sum_{i=1}^n |x_i + y_i|^2$$

implies

$$\varphi(a) \leq \frac{1}{2}\sum_{i=1}^n \varphi(|x_i + y_i|^2) < \infty.$$

Therefore, if $x := a^{\frac{1}{2}} \in \mathfrak{n}$, then $a = x^*x$.

(b) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}$ for some $x_i, y_i \in \mathfrak{n}$. Let $x := (\sum_{i=1}^n x_i^* x_i)^{\frac{1}{2}} \in \mathfrak{n}$. Since $x_i^* x_i \leq x^2$, we have $s_i \in M$ such that $x_i = s_i x$. If we let $y := \sum_{i=1}^n s_i^* y_i \in \mathfrak{n}$, then

$$a = \sum_{i=1}^n y_i^* x_i = \sum_{i=1}^n y_i^* s_i x = (\sum_{i=1}^n s_i^* y_i)x = y^*x. \qquad \square$$

Definition 1.7 (Semi-cyclic representations)**.** Let φ be a weight on a von Neumann algebra. Let H be the Hilbert space defined by the separation and completion of a sesquilinear form

$$\mathfrak{n} \times \mathfrak{n} \to \mathbb{C} : (x, y) \mapsto \varphi(y^* x)$$

and let $\psi : \mathfrak{n} \to H$ be the canonical image map. The pair (π, ψ) is called the *semi-cyclic representation* associated to φ.

Proposition 1.8. *Let φ be a weight on a von Neumann algebra and (π, ψ) be the associated semi-cyclic representation to φ. Consider a map*

$$\Theta : \mathfrak{m} \times \pi(M)' \to \mathbb{C} : (y^* x, z) \mapsto \langle z\psi(x), \psi(y)\rangle$$

and define

$$\theta : \mathfrak{m} \to (\pi(M)')_*, \qquad \theta^* : \pi(M)' \to \mathfrak{m}^\#$$

such that $\Theta(x, z) = \theta(x)(z) = \theta^*(z)(x)$ for $x \in \mathfrak{m}$ and $z \in \pi(M)'$.

(a) *Θ is a well-defined bilinear form.*

(b) *θ^* is bijective onto the space of linear functionals on $\mathfrak{m}$ whose absolute value is majorized by φ. (bounded Radon-Nikodym theorem)*

Proof. (a) The linearity in the second argument is obvious. Fix $z \in \pi(M)'$. We first check the well-definedness on $\mathfrak{m}^+$. Let $x^* x = y^* y \in \mathfrak{m}^+$ for $x, y \in \mathfrak{n}$. Then, there is $s \in M$ such that $y = sx$ and $s = sp$, where p is the range projection of x, so

$$x^*(1 - s^* s)x = x^* x - y^* y = 0$$

implies

$$0 = p(1 - s^* s)p = p - s^* s$$

and $x = px = s^* s x = s^* y$. The well-definedness follows from

$$\Theta(x^* x, z) = \langle z\psi(x), \psi(x)\rangle = \langle \pi(s) z \pi(s^*)\psi(y), \psi(y)\rangle = \langle z\psi(ss^* y), \psi(y)\rangle = \Theta(y^* y, z).$$

The homogeneity is clear, so now we prove the additivity. Let $x^* x, y^* y \in \mathfrak{m}^+$ for some $x, y \in \mathfrak{n}$. Let $a := (x^* x + y^* y)^{\frac{1}{2}}$ and take $s, t \in M$ such that $x = sa$, $y = ta$, $s = sa$, and $t = ta$, where p is the range projection of a. Then,

$$a(1 - s^* s - t^* t)a = a^* a - x^* x - y^* y = 0$$

implies

$$p(1 - s^* s - t^* t)p = p - s^* s - t^* t.$$

It follows that

$$\begin{aligned}
\Theta(x^* x + y^* y, z) &= \langle z\psi(a), \psi(a)\rangle = \langle z\pi(p)\psi(a), \psi(a)\rangle \\
&= \langle z\pi(s^* s)\psi(a), \psi(a)\rangle + \langle z\pi(t^* t)\psi(a), \psi(a)\rangle \\
&= \langle z\psi(x), \psi(x)\rangle + \langle z\psi(y), \psi(y)\rangle \\
&= \Theta(x^* x, z) + \Theta(y^* y, z).
\end{aligned}$$

Now the $\Theta(\cdot, z)$ is linearly extendable to $\mathfrak{m}$.

(b) The linear map θ^* is injective since ψ has dense range. Take $z \in \pi(M)'$ and consider $\theta^*(z)$, which maps $x^* x$ to $\langle z\psi(x), \psi(x)\rangle$ for $x \in \mathfrak{n}$. The image is majorized by φ as

$$|\langle z\psi(x), \psi(x)\rangle| \le \|z\| \|\psi(x)\|^2 = \|z\| \varphi(x^* x).$$

Conversely, let $l \in \mathfrak{m}^{\#}$ is a linear functional majorized by φ, i.e. there is a constant $C > 0$ such that

$$|l(x^*x)| \leq C\varphi(x^*x), \qquad x \in \mathfrak{n}.$$

Define a sesquilinear form $\sigma : \mathfrak{n} \times \mathfrak{n} \to \mathbb{C}$ such that $\sigma(x,y) := l(y^*x)$. It is well-defined after separation of $\mathfrak{n}$ and is bounded by the Cauhy-Schwartz inequality

$$|\sigma(x,y)|^2 = |l(y^*x)|^2 \leq \|l(x^*x)\|\|l(y^*y)\| \leq \varphi(x^*x)\varphi(y^*y) = \|\psi(x)\|^2\|\psi(y)\|^2.$$

Therefore, σ defines a bounded linear operator $z \in \pi(M)'$ such that

$$\sigma(x,y) = \langle z\psi(x), \psi(y)\rangle,$$

exactly meaning $\theta^*(z)(y^*x) = l(y^*x)$ for $x, y \in \mathfrak{n}$. $\square$

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, "\psi"] \arrow[dd, "|\cdot|^2"'] & H \arrow[d, "\omega"] \\
 & B(H)_* \arrow[d, "\mathrm{res}"] \\
\mathfrak{m}^+ \arrow[r, "\theta"] & (\pi(M)')_*.
\end{tikzcd}$$

In particular, for $x \in \mathfrak{n}^+$ we have

$$\|\theta(x^2)\| = \|\omega_{\psi(x)}\| = \|\psi(x)\|^2 = \varphi(x^2).$$

Lemma 1.9. *Let For $z \in \mathfrak{m}^{sa}$, we have*

$$\inf\{\varphi(a) : z \leq a \in \mathfrak{m}^+\} \leq \|\theta(z)\|.$$

In particular, for $x, y \in \mathfrak{n}^+$ and for any $\varepsilon > 0$ there is $a \in \mathfrak{m}^+$ such that $x^2 - y^2 \leq a$ and

$$\varphi(a) \leq \|\theta(x^2 - y^2)\| + \varepsilon = \|\omega_{\psi(x)} - \omega_{\psi(y)}\| + \varepsilon.$$

Proof. Denote by $p(z)$ the left-hand side of the inequality. Then, we can check $p : \mathfrak{m}^{sa} \to \mathbb{R}_{\geq 0}$ is a semi-norm such that $p(z) = \varphi(z)$ for $z \geq 0$. (If we take $p(z) := \varphi(z^+)$, then it seems to be dangerous when checking the sublinearity. I could not find the counterexample.)

Fix any non-zero $z_0 \in \mathfrak{m}^{sa}$. By the Hahn-Banach extension, there is an algebraic real linear functional $l : \mathfrak{m}^{sa} \to \mathbb{R}$ such that

$$l(z_0) = p(z_0), \qquad |l(z)| \leq p(z), \qquad z \in \mathfrak{m}^{sa}.$$

Extend linearly l to be $l : \mathfrak{m} \to \mathbb{C}$. Since $|l(z)| \leq \varphi(z)$ for $z \in \mathfrak{m}^+$, the linear functional l is contained in the image of the closed unit ball under the injective map

$$\theta^* : \pi(M)' \to \mathfrak{m}^{\#}.$$

If we let $a \in (\pi(M)')_1$ be the corresponding operator such that $\theta^*(a) = l$, then we get

$$p(z_0) = l(z_0) = \theta^*(a)(z_0) = \theta(z_0)(a) \leq \|\theta(z_0)\|.$$

Since $z_0 \in \mathfrak{m}^{sa}$ is aribtrary, we are done. $\square$

1.3 σ-weak lower semi-continuity

Theorem 1.10. *Let M be a countably decomposable von Neumann algebra. Then, normal weight on M is σ-weakly lower semi-continuous.*

Proof. Let φ be a normal weight on M and let (π, ψ) be the associated semi-cyclic representation.

In the spirit of the Krein-Šmulian theorem, the σ-weak lower semi-continuity is equivalent to the σ-weak closedness of the intersection with the ball

$$\begin{aligned}\varphi^{-1}([0,1])_1 &= \{x \in M^+ : \varphi(x) \le 1,\ \|x\| \le 1\} \\ &= \{x \in M^+ : \|\psi(x^{\frac{1}{2}})\| \le 1,\ \|x^{\frac{1}{2}}\| \le 1\}.\end{aligned}$$

Since that the σ-weak and σ-strong closedness of a convex set are equivalent and that the square root operation on M_1^+ is σ-strongly continuous, we are enough to show the set

$$(\varphi^{-1}([0,1])_1)^{\frac{1}{2}} = \{x \in M^+ : \|\psi(x)\| \le 1,\ \|x\| \le 1\}$$

is σ-weakly closed. This set, if we denote the graph of $\psi : \mathfrak{n} \to H$ by Γ_ψ, is the image of the positive part of the unit ball

$$(\Gamma_\psi)_1^+ = \{(x, \psi(x)) \in M^+ \oplus_\infty H : \|\psi(x)\| \le 1,\ \|x\| \le 1\}$$

under the projection $M \oplus_\infty H \to M$. Observing $M \oplus_\infty H \cong (M_* \oplus_1 H)^*$, if we prove $(\Gamma_\psi)_1^+$ is weakly* closed, then we are done by its compactness.

Consider a linear functional $l : M \oplus_\infty H \to \mathbb{C}$ that is continuous with respect to $(\sigma s, \|\cdot\|)$. If we define $l_1 : M \to \mathbb{C}$ and $l_2 : H \to \mathbb{C}$ such that $l_1(x) := l(x, 0)$ and $l_2(\xi) = (0, \xi)$, then they satisfy $l(x, \xi) = l_1(x) + l_2(\xi)$, and are continuous in σ-strong and norm topologies, hence to σ-weak and weak topologies, respectively. Since a net (x_α, ξ_α) converges to (x, ξ) weakly* if and only if $x_\alpha \to x$ σ-weakly and $\xi_\alpha \to \xi$ weakly, l is weakly* continuous. Because $(\Gamma_\psi)_1^+$ is convex, we will now show that $(\Gamma_\psi)_1^+$ is closed in $(M, \sigma s) \times (H, \|\cdot\|)$.

Note that the unit ball M_1 is metrizable in σ-strong topology since M is countably decomposable. Suppose a sequence $x_n \in \mathfrak{n}_1^+$ satisfies $x_n \to x$ σ-strongly and $\psi(x_n) \to \xi$ in H. Then, it suffices to show the following two statements: $x \in \mathfrak{n}_1^+$ and $\psi(x) = \xi$. We first observe that since $\psi(x_n)$ is Cauchy, so is $\omega_{\psi(x_n)}$ in $(\pi(M)')_*$.

Consider for a while, a family of functions

$$f_a(t) := \frac{t}{1 + at}, \qquad t \in (-a^{-1}, \infty),$$

parametrized by $a > 0$. They have several properties. At first, they are operator monotone. Next, they are σ-strongly continuous on a closed subset of its domain due to the boundedness of f_a, as we can see in the proof of the Kaplansky density theorem. Finally, for each $x \in M_+$, the increasing limit $f_a(x) \uparrow x$ in norm as $a \to 0$ implies that $\sup_a f_a(x) = x$.

First we show $x \in \mathfrak{n}_1^+$. It is clear that $x \in M_1^+$, so it is enough to show $\varphi(x^2) < \infty$. By taking a subsequence, we may assume $\|\omega_{\psi(x_{n+1})} - \omega_{\psi(x_n)}\| < \frac{1}{2^n}$. In order to dominate x_n with an monotone sequence, find $a_n \in \mathfrak{m}^+$ such that

$$x_{n+1}^2 - x_n^2 \le a_n, \qquad \varphi(a_n) < \frac{1}{2^n},$$

using the previous lemma. Then, we can write

$$x_{n+1}^2 \le x_1^2 + \sum_{k=1}^{n} (x_{k+1}^2 - x_k^2) \le x_1 + \sum_{k=1}^{n} a_k.$$

Here the right-hand side is non-decreasing but not a bounded sequence so we take f_a to get the σ-strong limit

$$f_a(x^2) \le \sup_n f_a(x_1^2 + \sum_{k=1}^n a_k).$$

Then, by the normality of φ, we have

$$\begin{aligned}\varphi(f_a(x^2)) &\le \sup_n \varphi(f_a(x_1^2 + \sum_{k=1}^n a_k)) \\ &\le \sup_n \varphi(x_1^2 + \sum_{k=1}^n a_k) \\ &= \varphi(x_1^2) + \sum_{k=1}^\infty \varphi(a_k) \\ &< \varphi(x_1^2) + 1 < \infty\end{aligned}$$

which implies by sending $a \to 0$ that $\varphi(x^2) < \infty$, whence $x \in \mathfrak{n}$.

Next we show $\psi(x) = \xi$. If we prove $\varphi((x_n - x)^2) \to 0$, then

$$\|\xi - \psi(x)\| \le \|\xi - \psi(x_n)\| + \|\psi(x_n) - \psi(x)\| = \|\xi - \psi(x_n)\| + \varphi((x_n - x)^2)^{\frac{1}{2}} \to 0$$

deduces the desired result. By taking a subsequence, since $\psi(x_n - x)$ is Cauchy, we may assume

$$\|\omega_{\psi(x_n - x)} - \omega_{\psi(x_{n+1} - x)}\| < \frac{1}{2^n}.$$

Let $b_n \in \mathfrak{m}^+$ such that

$$(x_n - x)^2 - (x_{n+1} - x)^2 \le b_n, \qquad \varphi(b_n) < \frac{1}{2^n}$$

As we did previously, we have

$$f_a((x_n - x)^2) \le f_a((x_{m+1} - x)^2) + f_a(\sum_{k=n}^m b_k) \to \sup_m f_a(\sum_{k=n}^m b_k)$$

as $m \to \infty$ and

$$\varphi(f_a((x_n - x)^2)) \le \sup_m \varphi(f_a(\sum_{k=n}^m b_k)) \le \sup_m \varphi(\sum_{k=n}^m b_k) < \frac{1}{2^{n-1}}.$$

Therefore,

$$\varphi((x_n - x)^2) \le \frac{1}{2^{n-1}} \to 0.$$

□

Theorem 1.11. *Let M be an arbitrary von Neumann algebra. Then, a normal weight on M is σ-weakly lower semi-continuous.*

Proof. Let φ be a normal weight of M. Let Σ be the set of all countably decomposable projections of M and let $M_0 := \bigcup_{p \in \Sigma} pMp$. The equivalent condition for $x \in M$ to belong to M_0 is that the left and right support projections of x are countably decomposable. Since then the left support projection p and the right support projection q of x are Murray-von Neumann equivalent so that there is a $*$-isomorphism between pMp and qMq, the countable decomposability is equivalent for p and q. It implies that M_0 is an algebraic ideal of M. (Moreover, M_0 is σ-weakly sequentially closed in M since if a sequence $x_n \in M_0$ converges to $x \in M$ σ-weakly, then for $p_n \in \Sigma$ such that $x_n = p_n x_n p_n$, we have $p \in \Sigma$ with $p_n \le p$ so that $x_n = p x_n p$ converges to $x = pxp$ σ-weakly. This fact is not needed in the proof.)

We first claim that $\varphi^{-1}([0,1])_1$ is relatively σ-weakly closed in M_0. Let $y \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0$ so that there is a net $y_\alpha \in \varphi^{-1}([0,1])_1$ converges σ-weakly to y, and there is $p \in \Sigma$ such that $pyp = y$. Note that the previous theorem states that $\varphi^{-1}([0,1]) \cap pMp$ is σ-weakly closed. Since $py_\alpha p$ is a net in $\varphi^{-1}([0,1])_1 \cap pMp$ that also converges σ-weakly to $pyp = y$, we have $y \in \varphi^{-1}([0,1])$. The claim proved.

We now claim that $\varphi^{-1}([0,1])_1$ is σ-weakly closed in M. Suppose a net $x_\alpha \in \varphi^{-1}([0,1])_1$ converges to $x \in M$ σ-weakly. Clearly $x \in M_1^+$. Let $\{p_i\}_{i\in I}$ be a maximal mutually orthogonal projections in Σ, and let $p_J := \sum_{i\in J} p_i$ for finite sets $J \subset I$ so that $\sup_J p_J = 1$. It clearly follows that for each α we have

$$x_\alpha^{\frac{1}{2}} p_J x_\alpha^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

Then, we can show easily with boundedness of x_α that

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w}.$$

Because $p_J \in M_0$ and M_0 is an ideal,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0.$$

By the above claim,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

By the normality of φ, we finally obtain

$$x \in \varphi^{-1}([0,1])_1.$$

Therefore, $\varphi^{-1}([0,1])_1$ is σ-weakly closed. $\square$

1.4 Pointwise supremum of normal positive linear functionals

Endow a partial order on the set of all weights. Then, every set of monotonically increasing subadditive homogeneous functions $\varphi : M^+ \to [0,\infty]$ always have its supremum given by its pointwise supremum. Since if φ is the supremum of σ-weakly lower semi-continuous φ_i, then

$$\varphi^{-1}([0,1]) = \bigcap_i \varphi_i^{-1}([0,1])$$

implies the σ-weak lower semi-continuity of φ. Conversly, the following theorem holds.

Theorem 1.12. *Let M be a von Neumann algebra. Then, a σ-weakly lower semi-continuous monotonically increasing additivie homogeneous function $\varphi : M^+ \to [0,\infty]$ is given by the pointwise supremum of a set of normal positive linear functionals.*

Proof. Let $F := \varphi^{-1}([0,1])$. It is a hereditary closed convex subset of the real locally convex space $(M^{sa}, \sigma w)$. Denote by the superscript circle the real polar set. Since

$$F^{\circ+} = \{\omega \in M_*^+ : \omega \le \varphi\}, \qquad F^{\circ+\circ+} = \{x \in M^+ : \sup_{\omega\le\varphi,\ \omega\in M_*^+} \omega(x) \le 1\},$$

it is enough to show $F^{\circ+\circ+} = F$. The positive part of the real polar of F is generally written as

$$F^{\circ+} = F^\circ \cap M_*^+ = F^\circ \cap (-M^+)^\circ = (F \cup -M^+)^\circ = (F - M^+)^\circ.$$

Consider a sequence of inclusions

$$F \subset \overline{F} \subset \overline{(F-M^+)^+} \subset \overline{(F-M^+)}^+ \subset (F-M^+)^{\circ\circ+} = F^{\circ+\circ+}.$$

The first, second, and forth inclusions are in fact surjective becuase F is closed, hereditary, and convex. So we claim that the reverse of the third inclusion $\overline{(F-M^+)}^+ \subset \overline{(F-M^+)^+}$... $\square$

2 November 10

2.1 Hilbert algebras

Let A be a $*$-algebra together with an inner product, and H be its closure. Then, we have a $*$-homomorphism $\lambda : A \to B(H)$, and a densely defined anti-linear operator $S : A \to H$. We say A is a *left Hilbert algebra* if λ is non-degenerate and S is closable...? (closable and closure notations....)

The faithfulness of λ is deduced after the density of A'^2 in H:

$$0 = \langle \lambda(\xi)\eta, \zeta \rangle = \langle \xi, \zeta F \eta \rangle, \qquad \eta\zeta \in A'.$$

Definition 2.1 (Left Hilbert algebra)**.** A *left Hilbert algebra* is a $*$-algebra A together with an inner product such that A^2 is dense in A and the involution is closable. Then, we can induce the following additional devices:

(i) the left multiplication defines a non-degenerate $*$-homomorphism $\lambda : A \to B(H)$, where $H := \overline{A}$,

(ii) the involution defines by its closure a closed and densely defined anti-linear operator S on H.

The *left von Neumann algebra* of a left Hilbert algebra A is defined as $\lambda(A)''$.

Definition 2.2 (Right Hilbert algebra)**.** Let A be a left Hilbert algebra. For $\eta \in H$, let $\rho(\eta) : A \to H$ be a densely defined operator such that $\rho(\eta)\xi := \lambda(\xi)\eta$ for $\xi \in A$, and let F be the adjoint of S. Define

$$\begin{aligned} B' &:= \{\eta \in H \mid A \to A : \xi \mapsto \lambda(\xi)\eta \text{ is bounded}\} = \{\eta \in H \mid \rho(\eta) \text{ is bounded}\}, \\ D' &:= \{\eta \in H \mid A \to \mathbb{C} : \xi \mapsto \langle \eta, S\xi \rangle \text{ is bounded}\} = \operatorname{dom} F, \\ A' &:= B' \cap D'. \end{aligned}$$

Proposition 2.3. *Let A be a left Hilbert algebra.*

(a) $\rho(B') \subset \lambda(A)'$.

(b) $\rho : B' \to B(H)$ *preserves multiplication.*

(c) $\rho : D' \to \{$*closed densely defined operators*$\}$ *preserves involution.*

(d) *The multiplication and the involution are compatible on A'.*

(e) *B' is a module over $\lambda(A)'$ and $\rho(B')$ is a left ideal of $\lambda(A)'$.*

(f) $\rho(B')^*\rho(B') \subset \rho(A')$.

(g) *A'^2 is dense in H.*

(h) $\lambda(A)'' = \rho(A')'$.

In particular, (d) means A' is a $*$-algebra, (b), (c) mean that $\rho : A'^{\text{op}} \to B(H)$ is a $*$-homomorphism, and (g) implies that $\rho : A'^{\text{op}} \to B(H)$ is non-degenerate and F is densely defined, i.e. A' is indeed a right Hilbert algebra.

Proof. (a) It follows from

$$\lambda(\xi)\rho(\eta)\xi_0 = \lambda(\xi)\lambda(\xi_0)\eta = \lambda(\xi\xi_0)\eta = \rho(\eta)\xi\xi_0 = \rho(\eta)\lambda(\xi)\xi_0, \qquad \xi_0 \in A.$$

(b) By the part (a),

$$\rho(\eta\zeta)\xi = \rho(\rho(\zeta)\eta)\xi = \lambda(\xi)\rho(\zeta)\eta = \rho(\zeta)\lambda(\xi)\eta = \rho(\zeta)\rho(\eta)\xi, \qquad \eta, \zeta \in B',\ \xi \in A.$$

(c) It follows from

$$\begin{aligned}\langle \rho(F\eta)\xi,\xi\rangle &= \langle \lambda(\xi)F\eta,\xi\rangle = \langle F\eta,\lambda(S\xi)\xi\rangle = \langle F\eta,(S\xi)\xi\rangle \\ &= \langle S((S\xi)\xi),\eta\rangle = \langle (S\xi)\xi,\eta\rangle = \langle \lambda(S\xi)\xi,\eta\rangle \\ &= \langle \xi,\lambda(\xi)\eta\rangle = \langle \xi,\rho(\eta)\xi\rangle = \langle \rho(\eta)^*\xi,\xi\rangle, \qquad \eta\in D',\ \xi\in A.\end{aligned}$$

(d) By the part (c),

$$\begin{aligned}\langle F(\eta\zeta),\xi\rangle &= \langle F(\rho(\zeta)\eta),\xi\rangle = \langle S\xi,\rho(\zeta)\eta\rangle = \langle \rho(F\zeta)S\xi,\eta\rangle \\ &= \langle \lambda(S\xi)F\zeta,\eta\rangle = \langle F\zeta,\lambda(\xi)\eta\rangle = \langle F\zeta,\rho(\eta)\xi\rangle \\ &= \langle \rho(F\eta)F\zeta,\xi\rangle = \langle (F\zeta)(F\eta),\xi\rangle, \qquad \eta,\zeta\in A',\ \xi\in A.\end{aligned}$$

(e) Note that

$$\rho(y\eta)\xi = \lambda(\xi)y\eta = y\lambda(\xi)\eta = y\rho(\eta)\xi, \qquad y\in\lambda(A)',\ \eta\in B',\ \xi\in A.$$

Hence B' is a module over $\lambda(A)'$ because

$$\|\rho(y\eta)\xi\| = \|y\rho(\eta)\xi\| \le \|y\|\|\rho(\eta)\|\|\xi\|$$

implies $y\eta\in B'$ for $y\in\lambda(A)'$ and $\eta\in B'$, and $\rho(B')$ is a left ideal of $\lambda(A)'$ because $y\rho(\eta)=\rho(y\eta)\in\rho(B')$.

(f) We first claim $\rho(B')^*B'\subset A'$. For fixed $\eta,\zeta\in B'$, we have $\rho(\zeta)^*\eta\in D'$ since

$$\begin{aligned}|\langle \rho(\zeta)^*\eta,S\xi\rangle| &= |\langle \eta,\rho(\zeta)S\xi\rangle| = |\langle \eta,\lambda(S\xi)\zeta\rangle| = |\langle \lambda(\xi)\eta,\zeta\rangle| \\ &= |\langle \rho(\eta)\xi,\zeta\rangle| = |\langle \xi,\rho(\eta)^*\zeta\rangle| \le \|\rho(\eta)^*\zeta\|\|\xi\|, \qquad \xi\in A,\end{aligned}$$

and we have $\rho(\zeta)^*\eta\in B'$ since B' is a module over $\lambda(A)'$, hence $\rho(\zeta)^*\eta\in A'$. Now we have $\rho(B')^*\rho(B')\subset\rho(A')$ since

$$\rho(\zeta)^*\rho(\eta)\xi = \rho(\zeta)^*\lambda(\xi)\eta = \lambda(\xi)\rho(\zeta)^*\eta = \rho(\rho(\zeta)^*\eta)\xi, \qquad \xi\in A.$$

(g) Since D' is dense in H, it suffices to verify $D'\subset\overline{A'^2}$. Let $\eta\in D'$. Then, by (c), $\rho(\eta)$ has a densely defined adjoint $\rho(F\eta)$, it is closable. Denoting its closure by the same notation $\rho(\eta)$, we can write down the polar decomposition $\rho(\eta)=vh=kv$. To control the unboundedness of $\rho(\eta)$, we introduce $f\in C_c((0,\infty))^+$ to cutoff $\rho(\eta)$.

First, we have $f(h)F\eta\in B'$ since

$$\begin{aligned}\|\lambda(\xi)f(h)F\eta\| &= \|f(h)\lambda(\xi)F\eta\| = \|f(h)\rho(F\eta)\xi\| \\ &= \|f(h)\rho(\eta)^*\xi\| = \|f(h)hv^*\xi\| \le \sup_{t\ge 0} tf(t)\|\xi\|, \qquad \xi\in A.\end{aligned}$$

We have $vf(h)F\eta\in B'$ since B' is a module over $\lambda(A)'$. Since

$$\rho(vf(h)F\eta) = v\rho(f(h)F\eta) = vf(h)(vh)^* = vf(h)hv^* = f(k)k,$$

we have $kf(k)\in\rho(B')$, and in fact we also have $f(k)\in\rho(B')$ because $\rho(B')$ is a left ideal of $\lambda(A)'$ and $f(t)=g(t)(tf(t))$ if we take $g\in C_c((0,\infty))$ such that $g(t)=t^{-1}$ on the support of f. Applying the above arguments for $f^{\frac14}\in C_c((0,\infty))$, we may assume

$$f(k) = (f(k)^{\frac14})^4 \in \rho(B)^*\rho(B)\rho(B)^*\rho(B)\subset\rho(A'^2).$$

Take $\zeta_1,\zeta_2\in A'$ such that $f(k)=\rho(\zeta_1\zeta_2)$. For $\eta,\zeta_1\in D'$, we have $\eta\zeta_1\in D'$ because D' is multiplicatively closed, and $\eta\zeta_1\in B'$ because B' is a left module of $\lambda(A)'$, so $\eta\zeta_1\in A'$ by definition of A'. Then,

$$f(k)\eta = \rho(\zeta_1\zeta_2)\eta = (\eta\zeta_1)\zeta_2\in A'^2.$$

If we construct a non-decreasing net $f_\alpha \in C_c((0,\infty))$ such that $\sup_\alpha f_\alpha = \mathbf{1}_{(0,\infty)}$, then the strong limit implies

$$\lim_\alpha f_\alpha(k)\eta = \mathbf{1}_{(0,\infty)}(k)\eta = s(k)\eta = s_r(\rho(F\eta))\eta = s_l(\rho(\eta))\eta.$$

Here we use the non-degeneracy of λ to verify η belongs to the closure of the range of $\rho(\eta)$, i.e. since $\mathrm{id}_H \in \lambda(A)''$, we have a net $\xi_\alpha \in A$ such that $\lambda(\xi_\alpha) \to \mathrm{id}_H$ strongly so that $\lambda(\xi_\alpha)\eta \to \eta$. It implies that $\eta \in \overline{\lambda(A)\eta} = \overline{\rho(\eta)A}$ and $s_l(\rho(\eta))\eta = \eta$. Therefore, $\eta = s_l(\rho(\eta))\eta \in \overline{A'^2}$.

(h) One direction is clear, i.e. $\rho(A') \subset \rho(B') \subset \lambda(A)'$ implies $\rho(A')'' \subset \lambda(A)'$. Conversely, let $y \in \lambda(A)'$. By the part (g), $\rho : A'^{\mathrm{op}} \to B(H)$ is non-degenerate and the σ-weak closure of $\rho(A')$ contains id_H, so there is a net $\eta_\alpha \in B'$ such that $y = \lim_\alpha \rho(\eta_\alpha)^* y \rho(\eta_\alpha) \in (\rho(B')^*\rho(B'))'' \subset \rho(A')''$, so we are done. □

Definition 2.4 (Full Hilbert algebra)**.** Let A be a left Hilbert algebra. Symmetrically as above, starting from the right Hilbert algebra A', we can construct a left Hilbert algebra A''. We say A is *full* if $A = A''$.

Example 2.5 (Commutative full Hilbert algebras)**.** If (X,μ) is a σ-finite measure space, then $L^2(X) \cap L^\infty(X)$ is a full Hilbert algebra. If G is a locally compact abelian group, then $A := \mathcal{F}^{-1}(L^2(\widehat{G}) \cap L^\infty(\widehat{G}))$ is a full Hilbert algebra, where $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is the Fourier transform.

2.2 Modular operator and conjugation

Definition 2.6 (Modular operator and conjugation)**.** Let A be a left Hilbert algebra. Denote the polar decomposition of S by $S = J\Delta^{\frac{1}{2}}$. The unbounded operators Δ and J are called the *modular operator* and the *modular conjugation*.

Corollary 2.7. *From the polar decomposition theorem for unbounded (anti-)linear operators, we have*

(a) *S is injective with $S = S^{-1}$ and $D = \mathrm{dom}\, S = \mathrm{dom}\,\Delta^{\frac{1}{2}}$.*

(b) *F is injective with $F = F^{-1}$ and $D' = \mathrm{dom}\, F = \mathrm{dom}\,\Delta^{-\frac{1}{2}}$.*

(c) *Δ is an injective positive self-adjoint operator.*

(d) *J is a conjugation, i.e. an anti-linear isometric involution.*

(e) *$S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$, $F = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J$, and $J\Delta J = \Delta^{-1}$.*

Example 2.8 (Group von Neumann algebra)**.** For a locally compact group G, the set $A := C_c(G)$ together with a left Haar measure on G has the following left Hilbert algebra structure

$$\langle \xi_1, \xi_2 \rangle := \int \overline{\xi_2(s)}\xi_1(s)\,ds, \qquad (\xi_1\xi_2)(s) := \int_G \xi_1(t)\xi_2(t^{-1}s)\,dt, \qquad \xi^*(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}.$$

We have S, F, Δ, and J given by

$$S\xi(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}, \qquad F\xi(s) = \overline{\xi(s^{-1})},$$
$$\Delta\xi(s) = \Delta(s)\xi(s), \qquad J\xi(s) = \Delta(s)^{-\frac{1}{2}}\overline{\xi(s^{-1})},$$

and they have the following norm formulas

$$\|S\xi\|_2 = \|\Delta^{\frac{1}{2}}\xi\|_2, \quad \|F\xi\|_2 = \|\Delta^{-\frac{1}{2}}\xi\|_2, \quad \|S\xi\|_1 = \|\xi\|_1, \quad \|F\xi\|_1 = \|\Delta^{-1}\xi\|_1.$$

The left von Neumann algebra $\lambda(A)''$ is called the *group von Neumann algebra*.

Example 2.9 (Cyclic separating vector)**.** Let M be a von Neumann algebra on H together with a cyclic separating vector $\Omega \in H$. Then, $A := M\Omega$ has the following left Hilbert algebra structure:

$$\langle x\Omega, y\Omega \rangle \text{ is defined as it is}, \qquad (x\Omega)(y\Omega) := xy\Omega, \qquad (x\Omega)^* := x^*\Omega.$$

There is no specific description of Δ and J in general, but it is known that

$$D = \{c\Omega : c \text{ closed densely defined on } H \text{ and affiliated with } M,\ \Omega \in \mathrm{dom}\, c \cap \mathrm{dom}\, c*\}.$$

2.3 Faithful normal semi-finite weights

Definition 2.10. Let φ be a weight on a von Neumann algebra M. We say φ is *faithful* if $\varphi(x^*x) = 0$ implies $x = 0$ for $x \in \mathfrak{n}$. We say φ is *semi-finite* if $\mathfrak{m}$ is σ-weakly dense in M. Recall that a weight φ on a von Neumann algebra M is normal if and only if it is obtained by the pointwise supremum of a set of normal positive linear functionals.

In the proofs of theorems of this section, the following diagram might be helpful:

$$\begin{tikzcd}
\mathfrak{m} = \mathfrak{n}^*\mathfrak{n} & \subset & \mathfrak{n} \cap \mathfrak{n}^* & \subset & \mathfrak{n} & \subset & \pi(M) & \subset & B(H) \\
 & & A \arrow[u, "\lambda"] & \subset & B \arrow[u, shift left, "\lambda"] & \subset & H. \arrow[from=1-5, to=2-5, shift left, "\psi"]
\end{tikzcd}$$

Recall that for a weight φ on a von Neumann algebra M and its semi-cyclic representation (π, ψ) of M we have $\varphi(x^*x) = \|\psi(x)\|^2$ for $x \in \mathfrak{n}$.

Theorem 2.11. *Let M be a von Neumann algebra. If A is a full left Hilbert algebra together with a faithful normal non-degenerate representation $\pi : M \to B(H)$ such that $\lambda(A)'' = \pi(M)$, then*

$$\varphi(x^*x) := \begin{cases} \|\xi\|^2 & \text{if } \pi(x) = \lambda(\xi) \in \lambda(B), \\ \infty & \text{otherwise,} \end{cases}$$

is a faithful normal semi-finite weight on M.

Proof. We use the notation $\pi(x) = x$. We first check that the weight φ is well-defined. Let $x_1 = \lambda(\xi_1), x_2 = \lambda(\xi_2) \in \lambda(B)$ such that $x_1^*x_1 = x_2^*x_2$. Since $x_1, x_2 \in M$, we have a partial isometry $v \in M$ such that $x_2 = vx_1$ and $v^*v = s_l(x_1)$, and it is not diffcult to see $\xi_2 = v\xi_1$. As we have seen in the proof of the part (g) of Proposition 2.3, we know $s_l(x)\xi_1 = \xi_1$, so

$$\|\xi_2\|^2 = \langle \xi_2, \xi_2 \rangle = \langle v\xi_1, v\xi_1 \rangle = \langle v^*v\xi_1, \xi_1 \rangle = \langle \xi_1, \xi_1 \rangle = \|\xi_1\|^2,$$

which proves the well-definedness.

With this weight φ, we can see

$$\mathfrak{n} = \lambda(B), \qquad \mathfrak{n} \cap \mathfrak{n}^* = \lambda(A), \qquad \mathfrak{m} = \lambda(B)^*\lambda(B).$$

The first one is by definition of φ, and the third one is by definition of $\mathfrak{m}$. Since A is full so that $A = B \cap D$, λ is injective, $\lambda(A)^* = \lambda(A)$, and $\lambda(D)^* = \lambda(D)$, we have $\lambda(A) = \lambda(B) \cap \lambda(D) = \lambda(B)^* \cap \lambda(D)$, which implies $\lambda(A) = \lambda(B) \cap \lambda(B)^* \cap \lambda(D)$. Because $\lambda(\xi_1) = \lambda(\xi_2)^*$ implies

$$\begin{aligned}
|\langle \xi_1, F(\eta\zeta) \rangle| &= |\langle \xi_1, \rho(F\eta)F\zeta \rangle| = \langle \rho(\eta)\xi_1, F\zeta \rangle| = |\langle \lambda(\xi_1)\eta, F\zeta \rangle| \\
&= |\langle \eta, \lambda(\xi_2)F\zeta \rangle| = |\langle \eta, \rho(F\zeta)\xi_2 \rangle| = |\langle \rho(\zeta)\eta, \xi_2 \rangle| \\
&= |\langle \eta\zeta, \xi_2 \rangle| \le \|\eta\zeta\|\|\xi_2\|, \qquad \eta, \zeta \in A',
\end{aligned}$$

we have $\xi_1 \in D$ due to the density of A'^2 in H, so $\lambda(B) \cap \lambda(B)^* \subset \lambda(D)$, hence the second equality follows. From now in the rest of proof, we will always denote $y = \rho(\eta)$ and $z = \rho(\zeta)$ for $y, z \in \mathfrak{n}'$.

The weight φ is clearly faithful, and semi-finiteness is because $x \in M$ is approximated by a net $\lambda(\xi_\alpha)x\lambda(\xi_\alpha) \in \lambda(B)^*\lambda(B) = \mathfrak{m}$, where $\lambda(\xi_\alpha) \in \lambda(B)$ converges σ-weakly to id_H. To verify the normality of φ, we will show

$$\varphi(x^*x) = \sup_{y \in \mathfrak{n}_1'} \omega_\eta(x^*x), \qquad x \in \mathfrak{n},$$

where $\mathfrak{n}' := \rho(B')$.

($\geq$) We may assume $x = \lambda(\xi) \in \mathfrak{n} = \lambda(B)$ so that $\varphi(x^*x) < \infty$. Since the unit ball $\mathfrak{n}_1'$ has a net y_α that converges to id_H strongly by the Kaplansky density theorem, we have an inequality

$$\omega_{\eta_\alpha}(x^*x) = \|x\eta_\alpha\|^2 = \|\lambda(\xi)\eta_\alpha\|^2 = \|\rho(\eta_\alpha)\xi\|^2 = \|y_\alpha\xi\|^2 \leq \|\xi\|^2 = \varphi(x^*x),$$

in which the equality condition is attained at its limit.

($\leq$) Suppose $x \in M$ is taken such that the right-hand side $\sup_{y\in\mathfrak{n}_1'} \omega_\eta(x^*x)$ is finite. If we show $x \in \mathfrak{n}$, then we are done from $\varphi(x^*x) < \infty$ by the previous argument. To motivate the strategy, consider the opposite weight

$$\varphi'(y^*y) := \begin{cases} \|\eta\|^2 & \text{if } y \in \rho(B'), \\ \infty & \text{otherwise,} \end{cases}$$

and the associated linear map

$$\theta'^* : M \to \mathfrak{m}'^\# : x^*x \mapsto (z^*y \mapsto \langle x^*x\eta, \zeta\rangle), \qquad y, z \in \mathfrak{n}',$$

where we can check $\mathfrak{m}' = \rho(B')^*\rho(B')$. The idea is to show a well-designed linear functional $l \in \mathfrak{m}'^\#$ such that $l = \theta'^*(x^*x)$ is contained in the image $\theta'^*(\mathfrak{m})$ using the assumption that the right-hand side is finite to verify $x \in \mathfrak{n}$.

Define a linear functional

$$l : \mathfrak{m}' \to \mathbb{C} : z^*y \mapsto \langle x^*x\eta, \zeta\rangle.$$

Then, by the assumption we have

$$\|l\| = \sup_{y\in\mathfrak{n}_1'} \langle x^*x\eta, \eta\rangle = \sup_{y\in\mathfrak{n}_1'} \omega_\eta(x^*x) < \infty,$$

and

$$|l(y)| \leq \|l\| l(y^*y)^{\frac{1}{2}} = \|l\|\|x\eta\|, \qquad y \in \mathfrak{n}'$$

implies the well-definedness as well as boundedness of the linear functional $\overline{xH} \to \mathbb{C} : x\eta \mapsto l(y)$ for any $\eta \in H$, and it follows the existence of $\xi \in \overline{xH}$ such that

$$l(y) = \langle x\eta, \xi\rangle, \qquad y \in \mathfrak{n}'$$

by the Riesz representation theorem on $\overline{xH}$. We have $\lambda(\xi)\zeta \in \overline{xH}$ and

$$\begin{aligned} \langle x\eta, x\zeta\rangle &= l(z^*y) = \langle x\rho^{-1}(z^*y), \xi\rangle = \langle xz^*\eta, \xi\rangle \\ &= \langle z^*x\eta, \xi\rangle = \langle x\eta, z\xi\rangle = \langle x\eta, \rho(\zeta)\xi\rangle = \langle x\eta, \lambda(\xi)\zeta\rangle, \qquad y, z \in \mathfrak{n}', \end{aligned}$$

hence $x = \lambda(\xi)$. The vector ξ is left bounded by definition and $x = \lambda(\xi) \in \lambda(B) = \mathfrak{n}$. □

Theorem 2.12. *Let M be a von Neumann algebra. If φ is a faithful normal semi-finite weight on M and (π, ψ) is the associated semi-cyclic representation of M, then $A := \psi(\mathfrak{n} \cap \mathfrak{n}^*)$ is a full left Hilbert algebra with*

$$\langle\psi(x_1), \psi(x_2)\rangle := \varphi(x_2^*x_1), \qquad \psi(x_1)\psi(x_2) := \psi(x_1x_2), \qquad \psi(x)^* := \psi(x^*),$$

such that $\lambda(A)'' = \pi(M)$.

Proof. We use the notation $\pi(x) = x$. It does not make any confusion because the semi-cyclic representation $\pi : M \to B(H)$ is always unital and is faithful due to the assumption that φ is faithful. We clearly see that A is a $*$-algebra and the left multiplication provides a $*$-homomorphism $\lambda : A \to B(H)$. By the construction of the semi-cyclic representation associated to φ, A is dense in H. We need to show the non-degeneracy of λ, the closability of the involution, and the fullness of A.

(non-degeneracy) Since φ is semi-finite, there is a net x_α in $(\mathfrak{n}\cap\mathfrak{n}^*)_1$ converges strongly to the identity of M by the Kaplansky density theorem. Then, it follows that λ is non-degenerate from

$$\lambda(\psi(x_\alpha))\psi(x) = \psi(x_\alpha)\psi(x) = \psi(x_\alpha x) = x_\alpha\psi(x) \to \psi(x), \qquad x \in \mathfrak{n}\cap\mathfrak{n}^*.$$

(closability) We need to prove the domain of the adjoint

$$D' := \{\eta \in H \mid A \to \mathbb{C} : \psi(x) \mapsto \langle \eta, \psi(x^*)\rangle \text{ is bounded}\}$$

is dense in H. Let

$$\Phi := \{\omega \in M_*^+ : (1+\varepsilon)\omega \le \varphi \text{ for some } \varepsilon > 0\}.$$

Note that the normality of φ says that $\varphi(x^*x) = \sup_{\omega\in\Phi}\omega(x^*x)$ for any $x \in M$. For each $\omega \in \Phi$, by the bounded Radon-Nikodym theorem, there is $h_\omega \in M'^+$ such that $\|h_\omega\| < 1$ and

$$\omega(x^*x) = \langle h_\omega\psi(x), \psi(x)\rangle, \qquad x \in \mathfrak{n}.$$

Also, if we take a net $x_\alpha \in \mathfrak{n}_1$ that converges σ-strongly to the identity of M using the strong density of $\mathfrak{n}$ in M, the Kaplansky density, and the coincidence of strong and the σ-strong topologies on the bounded part, then we have a σ-weak limit $\lim_{\alpha,\beta}|x_\alpha - x_\beta|^2 = 0$ so that by the normality of ω we obtain

$$\lim_{\alpha,\beta}\|h_\omega^{\frac{1}{2}}\psi(x_\alpha) - h_\omega^{\frac{1}{2}}\psi(x_\beta)\|^2 = \lim_{\alpha,\beta}\omega(|x_\alpha - x_\beta|^2) = 0.$$

Thus, the vector $\psi_\omega := \lim_\alpha h_\omega^{\frac{1}{2}}\psi(x_\alpha)$ can be defined, and in particular, we have $h_\omega^{\frac{1}{2}}\psi(x) = x\psi_\omega$ for $x \in \mathfrak{n}$ and $\omega = \omega_{\psi_\omega}$.

If $\eta = h_{\omega_2}^{\frac{1}{2}}y\psi_{\omega_1}$ for some $y \in M'$ and $\omega_1, \omega_2 \in \Phi$, then

$$\begin{aligned}
|\langle \eta, \psi(x^*)\rangle| &= |\langle h_{\omega_2}^{\frac{1}{2}}y\psi_{\omega_1}, \psi(x^*)\rangle| = |\langle y\psi_{\omega_1}, h_{\omega_2}^{\frac{1}{2}}\psi(x^*)\rangle| = |\langle y\psi_{\omega_1}, x^*\psi_{\omega_2}\rangle| \\
&= |\langle yx\psi_\omega, \psi_{\omega_2}\rangle| = |\langle yh_{\omega_1}^{\frac{1}{2}}\psi(x), \psi_{\omega_2}\rangle| = |\langle \psi(x), h_{\omega_1}^{\frac{1}{2}}y^*\psi_{\omega_2}\rangle| \\
&\le \|\psi(x)\|\|h_{\omega_1}^{\frac{1}{2}}y^*\psi_{\omega_2}\|, \qquad x \in \mathfrak{n}\cap\mathfrak{n}^*,
\end{aligned}$$

which deduces that $\eta \in D'$. Therefore, it suffices to show the following space is dense in H:

$$\{h_{\omega_2}^{\frac{1}{2}}y\psi_{\omega_1} : \omega_1, \omega_2 \in \Phi,\ y \in M'\}.$$

Thanks to the normality of φ, we can write

$$\begin{aligned}
\langle \psi(x), \psi(x)\rangle &= \|\psi(x)\|^2 = \varphi(x^*x) = \sup_{\omega\in\Phi}\omega(x^*x) \\
&= \sup_{\omega\in\Phi}\|x\psi_\omega\|^2 = \sup_{\omega\in\Phi}\|h_\omega^{\frac{1}{2}}\psi(x)\|^2 = \sup_{\omega\in\Phi}\langle h_\omega\psi(x), \psi(x)\rangle, \qquad x \in \mathfrak{n}\cap\mathfrak{n}^*.
\end{aligned}$$

Because A in H, for any $\xi \in H$ and $\varepsilon > 0$ there is $x \in \mathfrak{n}\cap\mathfrak{n}^*$ such that $\|\xi - \psi(x)\| < \varepsilon$, so the inequality

$$\langle (1-h_\omega)\xi, \xi\rangle \le \varepsilon(\|\xi\| + \|\psi(x)\|) + \langle (1-h_\omega)\psi(x), \psi(x)\rangle$$

deduces $\inf_{\omega\in\Phi}\langle (1-h_\omega)\xi, \xi\rangle = 0$ by limiting $\varepsilon \to 0$ and taking infinimum on $\omega \in \Phi$. In other words, for each $\xi \in H$ and $\varepsilon > 0$, we can find $\omega \in \Phi$ such that $\langle (1-h_\omega)\xi, \xi\rangle < \varepsilon$. At this point, we claim the set $\{h_\omega : \omega \in \Phi\}$ is upward directed. If the claim is true, then we can construct an increasing net ω_α in Φ such that h_{ω_α} converges weakly to the identity of M, and also strongly by the nature of increasing nets. To prove the claim, take $h_1 = h_{\omega_1}$ and $h_2 = h_{\omega_2}$ with $\omega_1, \omega_2 \in \Phi$. Introduce a operator monotone function $f(t) := t/(1+t)$ and its inverse $f^{-1}(t) = t/(1-t)$ to define

$$h_0 := f(f^{-1}(h_1) + f^{-1}(h_2)).$$

Then, we have $h_0 \geq h_1$, $h_0 \geq h_2$, and $\|h_0\| < 1$. Consider a linear functional

$$\omega_0 : \mathfrak{n} \to \mathbb{C} : x \mapsto \langle h_0\psi(x), \psi(x)\rangle.$$

Write

$$\begin{aligned}\omega_0(x^*x) &\leq \langle f^{-1}(h_1)\psi(x), \psi(x)\rangle + \langle f^{-1}(h_2)\psi(x), \psi(x)\rangle \\ &\leq (1-\|h_1\|)^{-1}\langle h_1\psi(x), \psi(x)\rangle + (1-\|h_2\|)^{-1}\langle h_2\psi(x), \psi(x)\rangle \\ &= (1-\|h_1\|)^{-1}\omega_1(x^*x) + (1-\|h_2\|)^{-1}\omega_2(x^*x), \qquad x \in \mathfrak{n}.\end{aligned}$$

Then, since ω_1 and ω_2 are normal, we can define $\psi_0 := \lim_\alpha h_0^{\frac{1}{2}}\psi(x_\alpha) \in H$ for a σ-strongly convergent net $x_\alpha \in \mathfrak{n}_1$ to the identity of M as we have taken above, and we have the vector functional $\omega_0 = \omega_{\psi_0}$. Henceforth, ω_0 is extended to a normal positive linear functional on the whole M, and finally the norm condition $\|h_0\| < 1$ tells us that $\omega_0 \in \Phi$, so the claim is true.

Now the problem is reduced to the density of $\{y\psi_\omega : \omega \in \Phi,\ y \in M'\}$ in H. Let $p \in B(H)$ be the projection to the closure of this space. Then, $p\psi_\omega = \psi_\omega$ for every $\omega \in \Phi$. Since the space is left invariant under the action of the self-adjoint set M', we have $p \in M$. Then,

$$\varphi(1-p) = \sup_{\omega\in\Phi}\omega(1-p) = \sup_{\omega\in\Phi}\langle(1-p)\psi_\omega, \psi_\omega\rangle = 0$$

implies $p = 1$, hence the density.

(fullness) We have $\lambda(\psi(x)) = x$ for $x \in \mathfrak{n}\cap\mathfrak{n}^*$ since $\psi(\mathfrak{n}\cap\mathfrak{n}^*) = A$ is dense in H and

$$x_1\psi(x_2) = \psi(x_1x_2) = \psi(x_1)\psi(x_2) = \lambda(\psi(x_1))\psi(x_2), \qquad x_1, x_2 \in \mathfrak{n}\cap\mathfrak{n}^*.$$

Also we have for $\xi = \psi(x) \in A$ that

$$\psi(\lambda(\xi)) = \psi(\lambda(\psi(\xi))) = \psi(x) = \xi.$$

For $\xi \in B$ so that $\lambda(\xi) \in M$, since

$$\varphi(\lambda(\xi)^*\lambda(\xi)) = \|\psi(\lambda(\xi))\|^2 = \|\xi\|^2 < \infty,$$

we get $\lambda(B) \subset \mathfrak{n}$. Therefore, A is full by

$$\lambda(A'') = \lambda(B)\cap\lambda(B)^* \subset \mathfrak{n}\cap\mathfrak{n}^* = \lambda(A). \qquad \square$$

Corollary 2.13. *The operations giving a faithful normal semi-finite weight and a full left Hilbert algebra in the above two theorems are mutually inverses of each other.*

Proposition 2.14. *Every von Neumann algebra admits a faithful normal semi-finite weight.*

Proof. Let M be a von Neumann algebra and let $\{\omega_i\}_{i\in I}$ be a maximal family of normal states on M with orthogonal support projections $p_i := s(\omega_i)$. Here, the support projection $s(\omega)$ of a normal state ω is the minimal projection p such that $\omega(px) = \omega(x) = \omega(xp)$ for all $x \in M$. Since every countably decomposable projection p is a support of a normal state, a faithful normal state on pMp, we have $\sum_i p_i = 1$. Define a weight φ by

$$\varphi(x) := \sum_i \omega_i(x).$$

It is faithful because $\varphi(x) = 0$ with $x \geq 0$ means $\omega_i(x) = 0$ and $p_ixsp_i = 0$ for all i, and it implies

$$x^{\frac{1}{2}} = x^{\frac{1}{2}}\sum_i p_i = \sum_i x^{\frac{1}{2}}p_i = 0.$$

It is normal because it is completely additive. It is semi-finite because $p_J \uparrow 1$ with $\varphi(p_J) < \infty$ as $J \to I$, where $p_J := \sum_{i\in J} p_i$ and J runs through finite subsets of I. $\square$

Example 2.15. For a locally compact abelian group G, the corresponding f.n.s. weight is a suitably normalized Haar measure on the Pontryagin dual group $\widehat{G}$, called the Plancherel measure, not the Haar measure on the original group G. For a locally compact non-abelian group G, there is no characterization of the corresponding f.n.s. weight as a measure because the left Hilbert algebra $(C_c(G), *)$ is not commutative.

3 December 20

3.1 Tomita-Takesaki commutation theorem

1.17, 1.18, 1.20, 1.21, 1.22

Theorem 3.1 (Tomita-Takesaki commutation theorem). *Goal:* $\Delta^{it}R_l(A)\Delta^{-it} = R_l(A)$ *and* $JR_l(A)J = R_l(A)'$. *1.19*

modular automorphism groups and Tomita algebras centralizer, Connes cocyle

Chapter IX: standard form, unitary implementation Chapter X: crossed product duality, W*-dynamical system

4 January 17

abelian group

5 February 9

Type III