Complex Analysis

Ikhan Choi

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Part I

Holomorphic functions

Chapter 1

Cauchy theory

1.1 Complex differentiability

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1.3 Power series

Analyticity, Laurent series,

1.4 Cauchy estimates

1.1. Let $p \in \mathbb{C}[z]$ with $p(z) = \sum_{k=0}^{n} a_k z^k$.

(a) $|p(z)| \lesssim |z|^n$.

(b) There is $R > 0$ such that $|p(z)| \gtrsim |z|^n$ for $|z| \geq R$.

Proof. If we take $R > 0$ such that

$$\sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2},$$

then $|z| \geq R$ implies

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \\ &\geq |a_n||z|^n - \sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} |z|^n \\ &\geq \frac{|a_n|}{2} |z|^n. \end{aligned}$$

□

1.2. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain. Then, $\overline{f(z)} = f(\overline{z})$ if and only if $f(z) \in \mathbb{R}$ for $z \in \Omega \cap \mathbb{R}$.

Exercises

1.3. If a holomorphic function has positive real parts on the open unit disk then $|f'(0)| < 2 \operatorname{Re} f(0)$.

1.4. If at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.

1.5. If a holomorphic function on a domain containing the closed unit disk is injective on the unit circle, then so is on the disk.

1.6. For a holomorphic function f and every z_0 in the domain, there are $z_1 \neq z_2$ such that $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$.

1.7. For two linearly independent entire functions, one cannot dominate the other.

1.8. The uniform limit of injective holomorphic function is either constant or injective.

1.9. If the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions converges has a limit point, then it compactly converges.

1.10. Find all entire functions f satisfying $f(z)^2 = f(z^2)$.

1.11. An entire function maps every unbounded sequence to an unbounded sequence is a polynomial.

1.12. Let f be a holomorphic function on the open unit disk such that $f(0) = 1$ and $f'(0) > 2$. Then, there is z such that $|z| < 1$ and $f(z)$ is pure imaginary.

Chapter 2

Singularities

2.1 Classification of singularities

Riemann removable singularity theorem, Casorati-Weierstrass theorem, Picard's theorem

2.2 Residue theorem

$$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \quad -1 < a < 1$$

2.1 (Semicircles). (a)

$$\int_0^\infty \frac{1}{1+x^2}\,dx =$$

(b)

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

(c)

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx =$$

(d)

$$\int_0^\infty \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1$$

2.2 (Computation of Fourier transforms).

2.3 (Laplace transforms).

2.4 (Gamma function).

2.3 Zeros and poles

2.5 (Argument principle). (a)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = i\text{winding number}.$$

(b)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(\text{number of zeros} - \text{number of poles}).$$

2.6 (Rouché theorem). Let f be a meromorphic function on Ω. Let γ be a curve...

(a) If $h : [0,1]\times\Omega \to \mathbb{C}$ is continuous and

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

(b) In particular, if $|g(z)| < |f(z)|$ on $z \in \gamma$, then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz.$$

2.7. Fundamental theorem of algebra, proof by the Liouville theorem, and proof by the Rouché theorem.

open mapping theorem

Chapter 3

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Mergelyan

Part II

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4.2 Open unit disk

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