Quantum Field Theory II

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1. The BRST operator Q_B is an integral of the BRST current $j_B(z)$:

$$Q_B := \oint \frac{dz}{2\pi i} j_B(z)$$

with

$$j_B(z) := cT^{(m)}(z) + :bc\partial c:(z) + \frac{3}{2}\partial^2 c(z),$$

where $T^{(m)}(z)$ is the energy-momentum tensor in the matter sector whose central charge is given by $c^{(m)} = 26$.

(1) Show that $Q_B \cdot cV^{(m)} = 0$ for any primary field in the matter sector $V^{(m)}$ of weight 1.

(2) Show that $j_B(z)$ is a primary field of weight 1.

(3) Show that $Q_B^2 = 0$.

Solution of (1). Since

$$Q_B \cdot cV^{(m)}(0) = \oint \frac{dz}{2\pi i} j_B(z) cV^{(m)}(0),$$

we are going to compute the OPE of $j_B(z)cV^{(m)}(0)$ at zero up to regular terms. First,

$$\begin{aligned}
cT^{(m)}(z)cV^{(m)}(0) &= [T^{(m)}(z)V^{(m)}(0)]c(z)c(0) \\
&= \left[\frac{1}{z^2}V^{(m)}(0) + \frac{1}{z}\partial V^{(m)}(0)\right]c(z)c(0) \\
&\sim \left[\frac{1}{z^2}V^{(m)}(0) + \frac{1}{z}\partial V^{(m)}(0)\right]\Big(c(0) + z\partial c(0)\Big)c(0) \\
&\sim \boxed{-\frac{1}{z}c\partial cV^{(m)}(0)}.
\end{aligned}$$

Next,

$$:bc\partial c:(z)cV^{(m)}(0) \sim [b(z)c(0)]c\partial c(z)V^{(m)}(0) \sim \frac{1}{z}c\partial c(z)V^{(m)}(0) \sim \boxed{\frac{1}{z}c\partial cV^{(m)}(0)}.$$

Finally, since matter and ghost sectors do not generate singular terms,

$$\frac{3}{2}\partial^2 c(z)cV^{(m)}(0) \sim \boxed{0}.$$

By combining the above three boxed terms, we obtain the desired conclusion. □

Solution of (2). It suffices to show

$$T(z)j_B(0) \sim \frac{h}{z^2}j_B(0) + \frac{1}{z}\partial j_B(0),$$

where $h = 1$. Recall that

$$T(z) = T^{(m)}(z) + :(\partial b)c:(z) - 2\partial(:bc:)(z),$$
$$j_B(0) = cT^{(m)}(0) + :bc\partial c:(0) + \frac{3}{2}\partial^2 c(0).$$

We compute nine OPEs:

(i) By the TT OPE,

$$\begin{aligned} T^{(m)}(z)cT^{(m)}(0) &= [T^{(m)}(z)T^{(m)}(0)]c(0) \\ &\sim \left[\frac{13}{z^4} + \frac{2}{z^2}T^{(m)}(0) + \frac{1}{z}\partial T^{(m)}(0)\right]c(0) \\ &= \boxed{\frac{13}{z^4}c(0) + \frac{2}{z^2}cT^{(m)}(0)}. \end{aligned}$$

(ii) Since the product of the matter sector and the ghost sector does not generate singular terms, we have

$$T^{(m)}(z):bc\partial c:(0) \sim \boxed{0}.$$

(iii) Similarly,

$$T^{(m)}(z)\frac{3}{2}\partial^2 c(0) \sim \boxed{0}.$$

(iv)

$$\begin{aligned} :(\partial b)c:(z)cT^{(m)}(0) &\sim [\partial b(z)c(0)]c(z)T^{(m)}(0) \\ &\sim \frac{1}{z^2}c(z)T^{(m)}(0) \\ &\sim \frac{1}{z^2}\Big(c(0) + z\partial c(0)\Big)T^{(m)}(0) \\ &= \boxed{\frac{1}{z^2}cT^{(m)}(0) + \frac{1}{z}(\partial c)T^{(m)}(0)}. \end{aligned}$$

(v)

$$\begin{aligned} &:(\partial b)c:(z):bc\partial c:(0) \\ &\quad \sim [\partial b(z)c(0)]:c(z)b\partial c(0): - [\partial b(z)\partial c(0)]:c(z)bc(0): + [c(z)b(0)]:\partial b(z)c\partial c(0): \\ &\qquad + [\partial b(z)c(0)][c(z)b(0)]\partial c(0) + [\partial b(z)\partial c(0)][c(z)b(0)]c(0) \\ &\quad \sim -\frac{1}{z^2}:c(z)b\partial c(0): + \frac{2}{z^3}:c(z)bc(0): + \frac{1}{z}:\partial b(z)c\partial c(0): - \frac{1}{z^3}\partial c(0) + \frac{2}{z^4}c(0) \\ &\quad \sim -\frac{1}{z^2}\Big(-:bc\partial c:(0) - z:b(\partial c)^2:(0)\Big) + \frac{2}{z^3}\Big(z:bc\partial c:(0) + \frac{z^2}{2}:bc\partial^2 c:(0)\Big) \\ &\qquad + \frac{1}{z}:(\partial b)c\partial c:(0) - \frac{1}{z^3}\partial c(0) + \frac{2}{z^4}c(0) \\ &\quad = \boxed{\frac{2}{z^4}c(0) - \frac{1}{z^3}\partial c(0) + \frac{3}{z^2}:bc\partial c:(0) + \frac{1}{z}\partial(:bc\partial c:)(0)}. \end{aligned}$$

(vi)

$$\begin{aligned} :(\partial b)c:(z)\frac{3}{2}\partial^2 c(0) &\sim -\frac{3}{2}[\partial b(z)\partial^2 c(0)]c(z) \sim \frac{9}{z^4}c(z) \\ &\sim \frac{9}{z^4}\Big(c(0) + z\partial c(0) + \frac{z^2}{2}\partial^2 c(0) + \frac{z^3}{6}\partial^3 c(0)\Big) \\ &= \boxed{\frac{9}{z^4}c(0) + \frac{9}{z^3}\partial c(0) + \frac{9}{2z^2}\partial^2 c(0) + \frac{3}{2z}\partial^3 c(0)}. \end{aligned}$$

(vii) The OPE

$$:bc:(z)cT^{(m)}(0) \sim -[b(z)c(0)]c(z)T^{(m)}(0) \sim -\frac{1}{z}c(z)T^{(m)}(0) \sim -\frac{1}{z}cT^{(m)}(0)$$

implies

$$-2\partial(:bc:)(z)cT^{(m)}(0) \sim \boxed{-\frac{2}{z^2}cT^{(m)}(0)}.$$

(viii) The OPE

$$\begin{aligned}
&:bc:(z):bc\partial c:(0)\\
&\quad\sim [b(z)c(0)]:c(z)b\partial c(0): -[b(z)\partial c(0)]:c(z)bc(0): +[c(z)b(0)]:b(z)c\partial c(0):\\
&\qquad +[b(z)c(0)][c(z)b(0)]\partial c(0) - [b(z)\partial c(0)][c(z)b(0)]c(0)\\
&\quad\sim \frac{1}{z}:c(z)b\partial c(0): -\frac{1}{z^2}:c(z)bc(0): +\frac{1}{z}:b(z)c\partial c(0): +\frac{1}{z^2}\partial c(0) - \frac{1}{z^3}c(0)\\
&\quad\sim -\frac{1}{z^2}(z:bc\partial c:)(0) + \frac{1}{z^2}\partial c(0) - \frac{1}{z^3}c(0)\\
&\quad\sim -\frac{1}{z^3}c(0) + \frac{1}{z^2}\partial c(0) - \frac{1}{z}:bc\partial c:(0)
\end{aligned}$$

implies

$$-2\partial(:bc:)(z):bc\partial c:(0) \sim \boxed{-\frac{6}{z^4}c(0) + \frac{4}{z^3}\partial c(0) - \frac{2}{z^2}:bc\partial c:(0)}.$$

(ix) The OPE

$$\begin{aligned}
:bc:(z)\partial^2 c(0) &\sim -[b(z)\partial^2 c(0)]c(z) \sim -\frac{2}{z^3}c(z)\\
&\sim -\frac{2}{z^3}(c(0) + z\partial c(0) + \frac{z^2}{2}\partial^2 c(0))\\
&= -\frac{2}{z^3}c(0) - \frac{2}{z^2}\partial c(0) - \frac{1}{z}\partial^2 c(0)
\end{aligned}$$

implies

$$-2\partial(:bc:)(z)\frac{3}{2}\partial^2 c(0) \sim \boxed{-\frac{18}{z^4}c(0) - \frac{12}{z^3}\partial c(0) - \frac{3}{z^2}\partial^2 c(0)}.$$

By summing up the above boxed nine terms, we obtain

$$\begin{aligned}
T(z)j_B(0) \sim\ & \frac{1}{z^4}(13+2+9-6-18)c(0)\\
&+\frac{1}{z^3}(-1+9+4-12)\partial c(0)\\
&+\frac{1}{z^2}\Big((2+1-2)cT^{(m)}(0) + (3-2):bc\partial c:(0) + (\frac{9}{2}-3)\partial^2 c(0)\Big)\\
&+\frac{1}{z}\Big(c\partial T(0) + (\partial c)T(0) + \partial:bc\partial c:(0) + \frac{3}{2}\partial^3 c(0)\Big)\\
=\ & \frac{1}{z^2}j_B(0) + \frac{1}{z}\partial j_B(0).
\end{aligned}$$

□

Solution of (3). Write

$$\begin{aligned}
2Q_B^2 = \{Q_B, Q_B\} &= \oint_{|w|=1} \frac{dw}{2\pi i} j_B(w) \oint_{|z|=\frac{1}{2}} \frac{dz}{2\pi i} j_B(z) + \oint_{|z|=\frac{3}{2}} \frac{dz}{2\pi i} j_B(z) \oint_{|w|=1} \frac{dw}{2\pi i} j_B(w) \\
&= \oint_{|w|=1} \frac{dw}{2\pi i} \oint_{|z-w|=\frac{1}{2}} \frac{dz}{2\pi i} j_B(z) j_B(w) \\
&= \operatorname*{Res}_{w=0} \operatorname*{Res}_{z=w} j_B(z) j_B(w).
\end{aligned}$$

Since the compensated term $\frac{3}{2}\partial^2 c$ in the BRST current j_B is a total derivative, to determine the OPE of the integrand $j_B(z)j_B(w)$ at w, it is enough to show

$$\operatorname*{Res}_{w=0} \operatorname*{Res}_{z=w} \Big(cT^{(m)}(z) + :bc\partial c:(z) \Big)\Big(cT^{(m)}(w) + :bc\partial c:(w) \Big) = 0. \tag{†}$$

Temporarily fix w and consider the following residue computations at $z = w$:

(i) The TT OPE and the Taylor expansion

$$\begin{aligned}
cT^{(m)}(z)cT^{(m)}(w) = &\Big[\frac{13}{(z-w)^4} + \frac{2}{(z-w)^2} T^{(m)}(w) + \frac{1}{z-w} \partial T^{(m)} + :T^{(m)}(z)T^{(m)}(w): \Big] \\
&\cdot \Big[-(z-w)c\partial c(w) - \frac{(z-w)^2}{2} c\partial^2 c(w) - \frac{(z-w)^3}{6} c\partial^3 c(w) - \cdots \Big]
\end{aligned}$$

implies

$$\operatorname*{Res}_{z=w} cT^{(m)}(z)cT^{(m)}(w) = \boxed{-2c\partial cT^{(m)}(w) - \frac{13}{6} c\partial^3 c(w)}.$$

(ii) We have

$$\operatorname*{Res}_{z=w} cT^{(m)}(z) :bc\partial c:(w) = \boxed{c\partial cT^{(m)}(w)}$$

from

$$cT^{(m)}(z) :bc\partial c:(w) \sim [c(z)b(w)]T^{(m)}(z)c\partial c(w) \sim \frac{1}{z-w} c\partial cT^{(m)}(w).$$

(iii) We have

$$\operatorname*{Res}_{z=w} :bc\partial c:(z)cT^{(m)}(w) = \boxed{c\partial cT^{(m)}(w)}$$

from

$$:bc\partial c:(z)cT^{(m)}(w) \sim [b(z)c(w)]c\partial c(z)T^{(m)}(w) \sim \frac{1}{z-w} c\partial cT^{(m)}(w).$$

(iv) Observe

$$\begin{aligned}
&:bc\partial c:(z) :bc\partial c:(w) \\
&\sim -[b(z)c(w)] :c\partial c(z)b\partial c(w): + [b(z)\partial c(w)] :c\partial c(z)bc(w): \\
&\quad -[c(z)b(w)] :b\partial c(z)c\partial c(w): + [\partial c(z)b(w)] :bc(z)c\partial c(w): \\
&\quad +[b(z)c(w)][c(z)b(w)]\partial c(z)\partial c(w) - [b(z)c(w)][\partial c(z)b(w)]c(z)\partial c(w) \\
&\quad -[b(z)\partial c(w)][c(z)b(w)]\partial c(z)c(w) + [b(z)\partial c(w)][\partial c(z)b(w)]c(z)c(w).
\end{aligned}$$

If we only see the coefficient of $1/(z-w)$ to take the residue at $z = w$, then the computation for the sum of these eight terms can be summarized as

$$\begin{aligned}
&\operatorname*{Res}_{z=w} :bc\partial c:(z) :bc\partial c:(w) \\
&= (-1-1+1+1) :bc(\partial c)^2:(w) + (-1-\frac{1}{2})(\partial c)(\partial^2 c)(w) + (\frac{1}{2}+\frac{1}{6})c\partial^3 c(w) \\
&= \boxed{-\frac{3}{2}(\partial c)(\partial c^2)(w) + \frac{2}{3}c\partial^3 c(w)}.
\end{aligned}$$

Summing up the boxed four terms, the left-hand side in (†) at $w=0$ becomes

$$\operatorname*{Res}_{w=0}\Big(-\frac{3}{2}(\partial c)(\partial^2 c)(w)+(-\frac{13}{6}+\frac{2}{3})c\partial^3 c(w)\Big)=-\frac{3}{2}\operatorname*{Res}_{w=0}\partial(c\partial^2 c)(w)=0.$$

Therefore, $Q_B^2=0$. □

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2. For the string field Ψ given by

$$\Psi = tc_1|0\rangle + uc_{-1}|0\rangle + vL_{-2}^{(m)}c_1|0\rangle,$$

calculate the following quantity:

$$\frac{V(t,u,v)}{T_{25}} = 2\pi^2\left[\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi,\Psi*\Psi\rangle\right]_{density}.$$

Solution. We will only compute the given function assuming $u=v=0$. Under the state-field correspondence, since the identity operator correponds to the vacuum state, we have the correspondences

$$\Psi = tc_1|0\rangle = t\oint\frac{dz}{2\pi i}\frac{1}{z}c(z)|0\rangle \rightsquigarrow t\oint\frac{dz}{2\pi i}\frac{1}{z}c(z) = tc(0)$$

and

$$Q_B\Psi = t\oint\frac{dz}{2\pi i}j_B(z)c_1|0\rangle \rightsquigarrow t\oint\frac{dz}{2\pi i}j_B(z)c(0) = tc\partial c(0).$$

The last equality follows from the OPEs

$$cT^{(m)}(z)c(0)\sim 0, \qquad :bc\partial c:(z)c(0)\sim\frac{1}{z}c\partial c(0), \qquad \frac{3}{2}\partial^2 c(z)c(0)\sim 0.$$

Now we compute the BPZ inner product $\langle\Psi, Q_B\Psi\rangle$ using the conformal transformation of c and $c\partial c$, which are primary fields of conformal weights -1. Introduce conformal transformations

$$f_1(z)=\frac{z-1}{z+1}, \qquad f_2(z)=-\frac{z+1}{z-1},$$

whose derivatives are

$$f_1'(z)=\frac{2}{(z+1)^2}, \qquad f_2'(z)=\frac{2}{(z-1)^2}.$$

Then, the conformal transformations of the fields are

$$\begin{aligned} c(0)\to f_1\circ c(0) &= f_1'(0)^{-1}c(f_1(0)) = \frac{1}{2}c(-1),\\ c\partial c(0)\to f_2\partial c(0) &= f_2'(0)^{-1}c\partial c(f_2(0)) = \frac{1}{2}c\partial c(1). \end{aligned}$$

Thus, the first BPZ product can be given by

$$\begin{aligned} \langle\Psi, Q_B\Psi\rangle &= t^2\langle f_1\circ c(0) \quad f_2\circ c\partial c(0)\rangle_{\text{UHP}}\\ &= \frac{t^2}{4}\langle c(-1) \quad c(1) \quad \partial c(1)\rangle_{\text{UHP}}\\ &= \frac{t^2}{4}\partial_{z_3}\langle c(z_1) \quad c(z_2) \quad c(z_3)\rangle|_{z_1=-1,\ z_2=z_3=1} \end{aligned}$$

By normalizing with the space-time volume factor, we can compute the value of the uniquely determined correlation function

$$\langle \Psi, Q_B \Psi \rangle_{density} = -\frac{t^2}{4} \partial_{z_3} (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|_{z_1=-1,\ z_2=z_3=1} = -t^2.$$

For the second BPZ inner product $\langle \Psi, \Psi * \Psi \rangle$, introduce

$$f_1(z) = \tan\Big(\frac{2}{3}(\arctan z - \frac{\pi}{2})\Big), \qquad f_2(z) = \tan(\frac{2}{3}\arctan z), \qquad f_3(z) = \tan\Big(\frac{2}{3}(\arctan z + \frac{\pi}{2})\Big)$$

such that

$$f_1'(0) = \frac{8}{3}, \qquad f_2'(0) = \frac{2}{3}, \qquad f_3'(0) = \frac{8}{3},$$

which can be computed by assuming $|z| \ll 1$. For example, using $\arctan z \approx z$, we can write

$$f_1(z) \approx \tan(\frac{2}{3}(z - \frac{\pi}{2})) = \tan(-\frac{\pi}{3} + \frac{2}{3}z) \approx \tan(-\frac{\pi}{3}) + \tan'(-\frac{\pi}{3})\frac{2}{3}z = -\sqrt{3} + \frac{8}{3}z.$$

Then the conformal transformations are

$$\begin{aligned}
c(0) &\to f_1 \circ c(0) = f_1'(0)^{-1} c(f_1(0)) = \frac{3}{8} c(-\sqrt{3}), \\
c(0) &\to f_2 \circ c(0) = f_2'(0)^{-1} c(f_2(0)) = \frac{3}{2} c(0), \\
c(0) &\to f_3 \circ c(0) = f_3'(0)^{-1} c(f_3(0)) = \frac{3}{8} c(\sqrt{3}),
\end{aligned}$$

and hence we have

$$\begin{aligned}
\langle \Psi, \Psi * \Psi \rangle &= t^3 \langle f_1 \circ c(0) \quad f_2 \circ c(0) \quad f_3 \circ c(0) \rangle_{\mathrm{UHP}} \\
&= \frac{27}{128} t^3 \langle c(-\sqrt{3}) \quad c(0) \quad c(\sqrt{3}) \rangle_{\mathrm{UHP}}
\end{aligned}$$

and the density

$$\langle \Psi, \Psi * \Psi \rangle_{density} = -\frac{27}{128} t^3 (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|_{z_1=-\sqrt{3},\ z_2=0,\ z_3=\sqrt{3}} = -\frac{81\sqrt{3}}{64} t^3.$$

Therefore,

$$\frac{V(t)}{T_{25}} = 2\pi^2 \left[-\frac{1}{2} t^2 - \frac{27\sqrt{3}}{64} t^3 \right]. \qquad \square$$