# Measure Theory

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# Contents

# Part I

# Chapter 1

# Measures

## 1.1 Measures

**1.1** ($\sigma$-algebras)**.** Let $\Omega$ be a set. A *$\sigma$-algebra* of sets on $\Omega$ is a collection $\mathcal{M} \subset \mathcal{P}(\Omega)$ which is closed under countable unions and complements.

(a) generated by a set.

(b) countable and cocountable sets

(c) Borel

**1.2** (Measures)**.** A *measurable space* is a pair $(\Omega, \mathcal{M})$ of a set $\Omega$ and a $\sigma$-algebra $\mathcal{M}$ on $\Omega$. Each element of $\mathcal{M}$ is called *measurable*. We often omit $\mathcal{M}$ to just write $\Omega$ for $(\Omega, \mathcal{M})$ if there is no confusion.

Let $(\Omega, \mathcal{M})$ be a measurable space. A *measure* on $(\Omega, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i)$$

for $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$. Here the squared cup notation reads the disjoint union. A *measure space* is a triple $(\Omega, \mathcal{M}, \mu)$, where $\mu$ is a measure on $(\Omega, \mathcal{M})$. Let $\mu$ be a measure on $\Omega$.

(a) $\mu$ is monotone: for $E, F \in \mathcal{M}$ if $E \subset F$ then $\mu(E) \le \mu(F)$.

(b) $\mu$ is countably subadditive: for

(c) $\mu$ is continuous from below:

(d) $\mu$ is continuous from above:

**1.3** (Complete measures)**.** Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. A *null set* is a measurable set $N$ satisfying $\mu(N) = 0$, and a *full set* is a measurable set whose complement is a null set.

A *complete measure* is a measure such that every subset of a null set is measurable.

For a predicate $P$ of points $x \in \Omega$, we say $P$ is true *almost everywhere* or *a.e.* on $\Omega$ if there is a full set $F \subset \Omega$ such that $P(x)$ is true for all $x \in F$.

## 1.2 Carathéodory extension

**1.4** (Outer measures)**.** Let $\Omega$ be a set. An *outer measure* on $\Omega$ is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty] :$ $\varnothing \mapsto 0$ such that

(i) $\mu^*$ is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \le \mu^*(S_2)$$

for $S_1, S_2 \in \mathcal{P}(\Omega)$,

(ii) $\mu^*$ is *countably subadditive*: we have

$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

Compairing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(\Omega) \to [0,\infty] : \varnothing \mapsto 0$ is an outer measure if and only if $\mu^*$ is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $S \in \mathcal{P}(\Omega)$ and $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

(b) For $\varnothing \in \mathcal{B} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{B} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^{\infty} \rho(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{B} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

*Proof.* □

**1.5** (Carathéodory measurable sets)**.** Let $\mu^*$ be an outer measure on a set $\Omega$. We want to construct a measure by restriction of $\mu^*$ on a properly defined $\sigma$-algebra. A subset $E \subset \Omega$ is called *Carathéodory measurable* relative to $\mu^*$ if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(\Omega)$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to $\mu^*$.

(a) $\mathcal{M}$ is an algebra and $\mu^*$ is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a $\sigma$-algebra and $\mu^*$ is countably additive on $\mathcal{M}$. That is, $\mu := \mu^*|_{\mathcal{M}}$ is a measure.

(c) The measure $\mu$ is complete.

*Proof.* □

**1.6** (Carathéodory extension theorem)**.** The Carathéodory extension is a construction method for a measure extending a given set function $\rho$ on $\mathcal{B} \subset \mathcal{P}(\Omega)$ for a set $\Omega$. The idea is to restrict the outer measure $\mu^*$ associated to $\rho$ in order to obtain a measure $\mu$. We want to find a sufficient condition for $\mu$ to be a measure on a $\sigma$-algebra containing $\mathcal{B}$.

For $\varnothing \in \mathcal{B} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{B} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. Let $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ be the associated outer measure of $\rho$, and $\mu : \mathcal{M} \to [0,\infty]$ the measure defined by the restriction of $\mu^*$ on Carathéodory measurable subsets.

(a) $\mu^*$ extends $\rho$ if $\rho$ satisfies the monotone countable subadditivity: for $B \in \mathcal{B}$ and $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$, we have

$$B \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \rho(B) \le \sum_{i=1}^{\infty} \rho(B_i).$$

(b) $\mu$ extends $\rho$ if $\rho$ satisfies the following property in addition: for $B, A \in \mathcal{B}$ and any $\varepsilon > 0$, there are $\{C_j\}_{j=1}^{\infty}, \{D_j\}_{j=1}^{\infty} \subset \mathcal{B}$ such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} C_j, \quad B \setminus A \subset \bigcup_{j=1}^{\infty} D_j, \quad \sum_{j=1}^{\infty} \rho(C_j) + \sum_{j=1}^{\infty} \rho(D_j) < \rho(B) + \varepsilon.$$

*Proof.* (a) Clearly $\mu^*(A) \leq \rho(A)$ for $A \in \mathcal{B}$. For the opposite direction, we may assume $\mu^*(A) < \infty$. For any $\varepsilon > 0$ we have $\{B_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} B_i$ and

$$\rho(A) \leq \sum_{i=1}^{\infty} \rho(B_i) < \mu^*(A) + \varepsilon.$$

Limiting $\varepsilon \to 0$, we get $\rho(A) \leq \mu^*(A)$.

(b) Let $A \in \mathcal{B}$. It is enough to check the inequality $\mu^*(S \cap A) + \mu^*(S \setminus A) \leq \mu^*(S)$ for $S \in \mathcal{P}(\Omega)$ with $\mu^*(S) < \infty$. By the finiteness of $\mu^*(S)$, we may assume there is $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ such that $S \subset \bigcup_{i=1}^{\infty} B_i$. From the condition, we have $B_i \cap A \subset \bigcup_{j=1}^{\infty} C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^{\infty} D_{i,j}$ satisfying

$$\begin{aligned}
\mu^*(S \cap A) + \mu^*(S \setminus A) &\leq \mu^*\Big(\bigcup_{j=1}^{\infty}(B_i \cap A)\Big) + \mu^*\Big(\bigcup_{j=1}^{\infty}(B_i \setminus A)\Big) \\
&\leq \sum_{i,j=1}^{\infty} \rho(C_{i,j}) + \sum_{i,j=1}^{\infty} \rho(D_{i,j}) \\
&\leq \sum_{i=1}^{\infty} (\rho(B_i) + 2^{-i}\varepsilon) \\
&< \mu^*(S) + \varepsilon.
\end{aligned}$$

Therefore, $A$ is Carathéodory measurable relative to $\mu^*$, so the domain of $\mu$ contains the domain of $\rho$. The values coincide by the part (a). □

**1.7** (Uniqueness of extension of measures)**.** The Carathéodory extension also provides a uniqueness result for measure extensions. Let $\rho : \mathcal{B} \to [0, \infty] : \varnothing \mapsto 0$ be a set function, where $\varnothing \in \mathcal{B} \subset \mathcal{P}(\Omega)$ for a set $\Omega$. We say $\rho$ is *$\sigma$-finite* if there is a cover $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ of $\Omega$ such that $\rho(B_i) < \infty$ for each $i$.

Let $\mathcal{M}$ be a $\sigma$-algebra containing $\mathcal{B}$. Let $\mu$ be a measure on $\mathcal{M}$, which extends $\rho$, given by the restriction of the outer measure $\mu^*$ associated to $\rho$. Let $\nu$ be another measure on $\mathcal{M}$ which extends $\rho$. Let $E \in \mathcal{M}$ and $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$.

(a) $\nu(E) \leq \mu(E)$.

(b) $\nu(E_i) = \mu(E_i)$ implies $\nu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big)$.

(c) $\nu(E) = \mu(E)$ for $\mu(E) < \infty$.

(d) $\nu(E) = \mu(E)$ for $\mu(E) = \infty$, if $\rho$ is $\sigma$-finite

*Proof.* (a) We may assume $\mu(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^{\infty} B_i$. Also, whenever $E \subset \bigcup_{i=1}^{\infty} B_i$ we have

$$\nu(E) \leq \nu\Big(\bigcup_{i=1}^{\infty} B_i\Big) \leq \sum_{i=1}^{\infty} \nu(B_i) = \sum_{i=1}^{\infty} \rho(B_i) = \sum_{i=1}^{\infty} \mu(B_i),$$

hence $\nu(E) \leq \mu(E)$.

(b) In the light of the inclusion-exclusion principle, we have

$$\mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j) - \mu(E_i \cap E_j) \leq \nu(E_i) + \nu(E_j) - \nu(E_i \cap E_j) = \nu(E_i \cup E_j),$$

so that $\mu(E_i \cup E_j) = \nu(E_i \cap E_j)$. Applying it inductively, we have for every $n$ that

$$\mu\Big(\bigcup_{i=1}^{n} B_i\Big) = \nu\Big(\bigcup_{i=1}^{n} B_i\Big),$$

and by limiting $n \to \infty$ the continuity from below gives

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i\Big).$$

(c) Because $\mu(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^{\infty} \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^{\infty} B_i$ and

$$\sum_{i=1}^{\infty} \rho(B_i) < \mu(E) + \varepsilon.$$

Applying the part (b) Then, we have

$$\mu(E) \leq \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) + \nu(E)$$

and

$$\nu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) \leq \mu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) - \mu(E) \leq \sum_{i=1}^{\infty} \mu(B_i) - \mu(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu(E) < \varepsilon,$$

we get $\mu(E) < \nu(E) + \varepsilon$ and $\mu(E) \leq \nu(E)$ by limiting $\varepsilon \to 0$.

(d) Let $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ be a cover of $\Omega$ such that $\rho(B_i) < \infty$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \geq 2$ so that $\{E_i\}_{i=1}^{\infty}$ is a pairwise disjoint cover of $\Omega$ with

$$\mu(E \cap E_i) \leq \mu(E_i) \leq \mu(B_i) = \rho(B_i) < \infty$$

for each $i$, so we have by the part (c) that

$$\nu(E) = \sum_{i=1}^{\infty} \nu(E \cap E_i) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \mu(E). \qquad \square$$

## 1.3 Measures on the real line

**1.8** (Borel $\sigma$-algebra).

**1.9** (Distribution functions).

**1.10** (Helly selection theorem).

**1.11** (Vitali set).

## Exercises

**1.12** (Boolean algebras and rings).

**1.13** (Cardinalities). infinite $\sigma$-algebra is $\geq \mathfrak{c}$.

**1.14** (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ such that $\varnothing \in \mathcal{A}$. We say that $\mathcal{A}$ is a semi-ring if it is closed under finite intersections, and each relative complement is a finite union of elements of $\mathcal{A}$. We say that $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over $\Omega$. Suppose a set function $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ satisfies

(i) $\rho$ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \rho(A_i)$$

for $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$,

(ii) $\rho$ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

**1.15** (Monotone class lemma)**.** A collection $\mathcal{C} \subset \mathcal{P}(\Omega)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let $H$ be a vector space closed under bounded monotone convergence. If $\operatorname{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^\infty(\sigma(\mathcal{A})) \subset H$.

**1.16** (Steinhaus theorem)**.** Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$ and let $\mathbb{E} \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.

(a) For any $0 < \alpha < 1$, there is an interval $I = (a, b)$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.

(b) $E - E = \{x - y : x, y \in E\}$ contains an open interval containing zero.

*Proof.* (a) We may assum $\lambda(E) < \infty$. Since $\lambda$ is outer measure and $\lambda(E) \neq 0$, we have an open subset $U$ of $\mathbb{R}$ such that $\lambda(U) < \alpha^{-1}\lambda(E)$. Because $U$ is a countable disjoint union of open intervals $U = \bigsqcup_{i=1}^{\infty}(a_i, b_i)$, we have

$$\sum_{i=1}^{\infty} \lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1}\sum_{i=1}^{n} \lambda(E \cap (a_i, b_i)).$$

Therefore, there is $i$ such that $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$. □

## Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

# Chapter 2

# Measurable functions

## 2.1 Simple functions

**2.1** (Measurability of pointwise limits)**.** Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

*Proof.* Let $f(x) = \lim_{n\to\infty} s_n(x)$.

□

## 2.2 Almost everywhere convergence

**2.2** (Almost everywhere convergence)**.** Let $(\Omega, \mu)$ be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence $f_n$ is defined as the set

$$\{x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say $f_n$ converges to $f$ *alomst everywhere* with respect to $\mu$ if the set of divergence is a null set in $\mu$. We simply write

$$f_n \to f \text{ a.e.}$$

if $f_n$ converges to $f$ almost everywhere, and we frequently omit the measure $\mu$ if it has no confusion.

(a) If $\mu$ is complete and, if $f_n \to f$ a.e., then $f$ is measurable.

**2.3** (Borel-Cantelli lemma)**.** Let $(\Omega, \mu)$ be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\{x : \limsup_{n\to\infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(c) $f_n \to f$ a.e. if for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \infty.$$

*Proof.* (b) The set of divergence of the sequence $f_n$ is given by

$$\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \tfrac{1}{m}\} = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}(\Omega \setminus E_n^m).$$

(c) Since

$$\mu\Big(\bigcup_{i=1}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \leq \sum_{i=1}^{\infty}\mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned}
\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}) &= \mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&= \lim_{n\to\infty}\mu\Big(\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&\leq \lim_{n\to\infty}\sum_{i=n}^{\infty}\mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0.
\end{aligned}$$

□

**2.4** (Convergence in measure). Let $(\Omega, \mu)$ be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions. We say $f_n$ converges to a measurable function $f : \Omega \to \overline{\mathbb{R}}$ *in measure* if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(a) If $f_n \to f$ in measure, then there is a subsequence $f_{n_k}$ such that $f_{n_k} \to f$ a.e.

(b) If every subsequence $f_{n_k}$ of $f_n$ has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}} \to f$ a.e., then $f_n \to f$ in measure.

*Proof.* (a) Since for each positive integer $k$ we have $\mu(\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}) \to 0$ as $n \to \infty$, there exists $n_k$ such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) = 0.$$

Then, for each $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \varepsilon\} &= \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \geq \varepsilon\} \\
&\subset \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \geq \tfrac{1}{k}\} \\
&= \limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}
\end{aligned}$$

implies the limit superior of the tail events is a null set, hence $f_{n_k} \to f$ a.e.

(b) □

**2.5** (Egorov theorem). Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient "$\varepsilon/2^m$ argument", occasionally used throughout measure theory to construct a measurable set having a special property.

Let $(\Omega, \mu)$ be a finite measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that $f_n \to f$ a.e. For each positive integer $m$, which indexes the tolerance $1/m$, consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty} \{x : |f_i(x) - f(x)| < \tfrac{1}{m}\}.$$

(a) $E_n^m$ converges to a full set for each $m$.

(b) For every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and for each $m$ there is finite $n$ satisfying $K \subset E_n^m$.

(c) For every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on $K$.

*Proof.* (a) Recall that the a.e. convergence $f_n \to f$ means that for every fixed $m$ the intersection

$$\bigcap_{n=1}^{\infty} (\Omega \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \tfrac{1}{m}\}$$

is a null set. Since $\mu(\Omega) < \infty$, it is equivalent to $E_n^m$ converges to a full set for each $m$ by the continuity from above.

(b) For each $m$, we can find $n_m$ such that

$$\mu(\Omega \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(\Omega \setminus K) = \mu\Big(\bigcup_{m=1}^{\infty} (\Omega \setminus E_{n_m}^m)\Big) \leq \sum_{m=1}^{\infty} \mu(\Omega \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix $m > 0$. Since $n \geq n_m$ implies $K \subset E_{n_m}^m \subset E_n^m$, we have

$$n \geq n_m \quad \Rightarrow \quad \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \qquad \square$$

# Exercises

**2.6** (Cauchy's functional equation). Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x+y) = f(x) + f(y)$, satisfied for all $x, y \in \mathbb{R}$. Suppose $f$ satisfies the Cauchy functional equation. We ask if $f$ is linear, that is $f(x) = ax$ for all $x \in \mathbb{R}$, where $a := f(1)$.

(a) $f(x) = ax$ for all $x \in \mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If $f$ is conitnuous at a point, then $f$ is linear.

(c) If $f$ is Lebesgue measurable, then $f$ is linear.

**2.7** (Pointwise approximation by simple functions). Let $(\Omega, \mu)$ be a measure space and $X$ a metric space with Borel measurable structure. By a *simple function* we mean a measurable function $s : \Omega \to X$ of finite image.

(a) For each open set $U \subset X$ there is a sequence of open sets $U_i$ such that $U = \bigcup_i U_i$ and $\overline{U}_i \subset U$. Let $f : \Omega \to X$ be any function.

(b) If $f$ is the pointwise limit of a sequence of measurable functions, then $f$ is measurable.

(c) If $f$ is measurable, then $f$ is the pointwise limit of a sequence of simple functions, if $X$ is separable.

*(d) The pointwise limit of a net of simple functions may not be measurable.

*Proof.* (b) Suppose a sequence $(f_n)_n$ of measurable functions converges pointwisely to a function $f$. For fixed open $U \subset X$ we claim

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} \liminf_{n\to\infty} f_n^{-1}(U_i).$$

If it is true, then $f^{-1}(U)$ is the countable set operation of measurable sets $f_n^{-1}(U_i)$. Let $U_i$ be the sequence associated to $U$ taken by the part (a).

($\subset$) If $\omega \in f^{-1}(U)$, then for some $i$ we have $f(\omega) \in U_i$, so $f_n(\omega)$ is eventually in $U_i$, thus we have $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$.

($\supset$) If $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$ for some $i$, then $f_n(\omega)$ is eventually in $U_i$, so $f(\omega) \in \overline{U}_i \subset U$, thus we have $\omega \in f^{-1}(U)$.

(c) Suppose there is a increasing sequence of finite tagged partitions $\mathcal{P}_n \subset \mathcal{B}$ satisfying the following property: for each open-neighborhood pair $(x, U)$ there is $n$ and $i$ such that $P_{n,i} \in \mathcal{P}_n$ and $x \in P_{n,i} \subset U$. We denote the tags by $t_{n,i} \in P_{n,i}$ for each $P_{n,i} \in \mathcal{P}_n$. Define

$$s_n(\omega) := t_{n,i} \quad \text{for} \quad f(\omega) \in P_{n,i}.$$

To show $s_n(\omega) \to f(\omega)$, fix an open $f(\omega) \in U \subset X$. Then, there is $n_0$ such that there is a sequence $(P_{n,i_n})_{n=n_0}^{\infty}$ satisfying $P_{n,i_n} \in \mathcal{P}_n$ and $f(\omega) \in P_{n,i_n} \subset U$. Then, for all $n \geq n_0$, we have for $f(\omega) \in P_{n,i_n}$ that $s_n(\omega) = t_{n,i_n} \in P_{n,i_n} \subset U$.

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem. □

# Chapter 3

# Lebesgue integral

## 3.1 Monotone convergence theorem

**3.1** (Lebesgue integral of non-negative functions)**.** Let $(\Omega, \mu)$ be a measure space. Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be a measurable function. The *Lebesgue integral* of $f$ is defined by

$$\int f\, d\mu := \sup\left\{\int s\, d\mu : 0 \leq s \leq f,\ s \text{ simple}\right\}$$

**3.2** (Monotone convergence theorem)**.** Let $(\Omega, \mu)$ be a measure space. Let $f_n$ and $f$ be measurable functions $\Omega \to \mathbb{R}_{\geq 0}$.

(a) $E \mapsto \int_E f\, d\mu$ is a measure if $f$ is simple.

(b) $E \mapsto \int_E f\, d\mu$ is a measure even if $f$ is not simple.

(c) If $f_n \uparrow f$ a.e., then $\int f_n \to \int f$.

*Proof.* (a) Clear from the linearity of the integral for simple functions.

(b) For $E_n \uparrow E$, we want to show the continuity from below, $\int_{E_n} f \to \int_E f$. Take $\varepsilon > 0$. We introduce a continuous bijection $\beta : [0, \infty] \to [0, 1] : t \mapsto t/(1+t)$ to avoid dividing the cases for infinity. By the definition of the Lebesgue integral, we have a simple function $s$ such that $0 \leq s \leq f$ and

$$\beta(\textstyle\int_E f) - \beta(\int_E s) < \varepsilon,$$

whether or not $\int_E f$ diverges. Then,

$$\begin{aligned}\beta(\textstyle\int_E f) - \beta(\int_{E_n} f) &= [\beta(\textstyle\int_E f) - \beta(\int_E s)] + [\beta(\int_E s) - \beta(\int_{E_n} s)] + [\beta(\int_{E_n} s) - \beta(\int_{E_n} f)] \\ &< \qquad \varepsilon \qquad + [\beta(\textstyle\int_E s) - \beta(\int_{E_n} s)] + \qquad 0 \\ &\xrightarrow{n\to\infty} \varepsilon\end{aligned}$$

by the part (a). We are done by letting $\varepsilon \to 0$.

(c) Define $E_n := \{x : f(x) < (1+\varepsilon) f_n(x)\}$, which converges to a full set because $f_n \to f$ a.e. Since $f$ is a measure, we can choose $N$ such that

$$\beta(\textstyle\int_E f) - \beta(\int_{E_N} f) < \varepsilon.$$

With this $N$, we have

$$\beta(\textstyle\int_{E_N} f) - \beta(\int_{E_N} f_n) \leq \dfrac{\int_{E_N} f - \int_{E_N} f_n}{(1 + \int_{E_N} f)(1 + \int_{E_N} f_n)} < \varepsilon, \qquad n \geq N.$$

Then, we have for $n \geq N$ that

$$\begin{aligned}\beta(\textstyle\int_E f) - \beta(\int_E f_n) &= [\beta(\textstyle\int_E f) - \beta(\int_{E_N} f)] + [\beta(\int_{E_N} f) - \beta(\int_{E_N} f_n)] + [\beta(\int_{E_N} f_n) - \beta(\int_E f_n)] \\ &< 0 + \varepsilon + \varepsilon,\end{aligned}$$

so we are done by letting $n \to \infty$ and $\varepsilon \to 0$. □

**3.3** (Corollaries of monotone convergence theorem)**.** Fatou's lemma, linearity of the integral, $f \geq 0$ and $\int f = 0$ imply $f = 0$ a.e.

## 3.2 Dominated convergence theorem

**3.4** (Lebesgue integral of complex-valued functions)**.**

**3.5** (Bounded convergence theorem)**.** Semifinite measures

(a)

$$\sup_{g \leq f} \int g \, d\mu = \int f \, d\mu$$

where $g$ runs through bounded measurable functions.

(b)

## 3.3 Product measures

**3.6** (Fubini-Tonelli theorem)**.** Lebesgue measure on Euclidean spaces

## Exercises

**3.7** (Convergence of one-parameter family)**.**

# Part II

# Chapter 4

# Signed measures

## 4.1 Radon-Nikodym theorem

An integrable function as a measure $\sigma$-finite measures

# Chapter 5

# Borel measures

## 5.1 Continuous functions on metric spaces

Urysohn and Tietze.

**5.1** (Regular Borel measures on metric spaces)**.** Let $\mu$ be a Borel measure on a metric space $\Omega$. We say $\mu$ is *outer regular* if

$$\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\},$$

and say $\mu$ is *inner regular* if

$$\mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ closed}\},$$

for every Borel subset $E \subset \Omega$. If $\mu$ is both outer and inner regular, we say $\mu$ is *regular*.

(a) Let $E$ be $\sigma$-finite. Then, $E$ is $\mu$-regular if and only if for any $\varepsilon > 0$ there are open $U$ and closed $F$ such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.

(b) If $\mu$ is $\sigma$-finite, then the set of $\mu$-regular subsets is a $\sigma$-algebra. (may be extended?)

(c) Every closed set is $G_\delta$.

(d) Every finite Borel measure on $\Omega$ is regular.

*Proof.* □

**5.2** (Luzin's theorem)**.** Let $\mu$ be a regular Borel measure on a metric space $\Omega$. Let $f : \Omega \to \mathbb{R}$ be a Borel measurable function. Two proofs: direct and Egoroff.

(a) If $E \subset \Omega$ is $\sigma$-finite, then there is a continuous $g$ blabla

(b) If $f$ vanishes outside a $\sigma$-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset \Omega$ such that $f|_F : F \to \mathbb{R}$ is continuous and $\mu(\Omega \setminus F) < \varepsilon$.

(c) If $f$ vanishes outside a $\sigma$-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset \Omega$ and continuous $g : \Omega \to \mathbb{R}$ such that $f|_F = g|_F$ and $\mu(\Omega \setminus F) < \varepsilon$.

(d) If $f$ is further bounded, then $g$ also can be taken to be bounded.

*Proof.* (a) Let $\varepsilon > 0$ and suppose $E \subset \Omega$ is measurable with $\mu(E) < \infty$. Since $E$ is $\sigma$-finite, we have open $U$ and closed $F$ such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon/2$. By the Urysohn lemma, there is a continuous function $g : \Omega \to [0,1]$ such that $g|_{U^c} = 0$ and $g|_F = 1$. Then,

$$\int |\mathbf{1}_E - g|\, d\mu = \int_{U\setminus F} |\mathbf{1}_E - g|\, d\mu \le 2\mu(U \setminus F) < \varepsilon.$$

(b) Since $\mathbb{R}$ is second countable, we have a base $(V_n)_{n=1}^\infty$ of $\mathbb{R}$. Since $\mu$ is $\sigma$-finite, for each $n$ we can take open $U_n$ and closed $F_n$ such that

$$F_n \subset f^{-1}(V_n) \subset U_n$$

and $\mu(U_n \setminus F_n) < \varepsilon/2^n$. Define $F := \left(\bigcup_{n=1}^\infty (U_n \setminus F_n)\right)^c$ so that $\mu(\Omega \setminus F) < \varepsilon$ and $F$ is closed. Then,

$$\begin{aligned} U_n \cap F &= U_n \cap ((U_n^c \cup F_n) \cap F) \\ &= (U_n \cap (U_n^c \cup F_n)) \cap F \\ &= (\varnothing \cup (U_n \cap F_n)) \cap F \\ &\subset F_n \cap F \end{aligned}$$

proves $f^{-1}(V_n)$ is open in $F$ for every $n$, hence the continuity of $f|_F$. (In fact, we require that $X$ to be just a topological space.)

(b') We can alternatively use the part (a) and the Egoroff theorem. By the part (a), we can construct a sequence $(f_n)$ of continuous functions $X \to \mathbb{R}$ such that $f_n \to f$ in $L^1$. By taking a subsequence, we may assume $f_n \to f$ pointwise. Assuming $\mu$ is finite, by the Egorov theorem, there is a measurable $A \subset X$ such that $f_n \to f$ uniformly on $A$ and $\mu(X \setminus A) < \varepsilon/2$. Since $\mu$ is inner regular, we have closed $F \subset A$ such that $\mu(A \setminus F) < \varepsilon/2$, so that we have $\mu(X \setminus F) < \varepsilon$. Then, $f$ is continuous on $A$, and of course on $F$.

□

## 5.2 Locally compact spaces

compact closed set not containing infty
open open not containing infty
closed closed set containing infty

for a measure that "vanishes at infty" = tight
two definitions of inner regularity is equivalent.

inner regular on compact sets -> inner regular on closed sets
inner regular on compact sets + sigma finite -> tight

**5.3** (One-point compactification).

**5.4** (Regular Borel measures on locally compact metric spaces). sss

(a) $C_c(\Omega)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

(b) If $\mu$ is $\sigma$-finite, then for any $\varepsilon > 0$ there is compact $K \subset \Omega$ and continuous $g : \Omega \to \mathbb{R}$ such that $f|_K = g|_K$ and $\mu(\Omega \setminus K) < \varepsilon$.

**5.5** (Tightness and inner regularity). We have a similar but confusing concept called tightness; we say a Borel measure $\mu$ on a topological space $X$ is *tight* if for any $\varepsilon > 0$ there is a compact $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$.

History of Bourbaki's text.

(a)

## 5.3 Riesz-Markov-Kakutani representation theorem

**5.6** (Riesz-Markov-Kakutani representation theorem for $C_0$)**.** Let $\Omega$ be a locally compact metric space. We want to establish the following one-to-one correspondence:

$$\begin{array}{ccc} \{\text{finite Borel measures on } \Omega\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_0(\Omega)\} \\ \mu & \mapsto & (f \mapsto \int f\, d\mu). \end{array}$$

Let $I$ a positive linear functional on $C_0(\Omega)$. Let $\mathcal{T}$ be the set of all open subsets of $\Omega$ and $\rho : \mathcal{T} \to [0, \infty]$ a set function such that

$$\rho(U) := \sup\{ I(f) : f \in C_c(U, [0, 1]) \}$$

for open $U$. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the associated outer measure defined from $\rho$, and $\mu := \mu^*|_{\mathcal{M}}$ the Carathéodory measure, where $\mathcal{M}$ is the $\sigma$-algebra of Carathéodory measurable subsets relative to $\mu^*$, and $\mathcal{B}$ is the Borel $\sigma$-algebra of $\Omega$.

(a) $\mu^*|_{\mathcal{T}} = \rho$.

(b) $\mathcal{B} \subset \mathcal{M}$.

(c) $I(f) = \int f\, d\mu$ for $f \in C(\Omega)$, i.e. the map given above is surjective.

(d) The map given above is injective.

*Proof.* (a) It suffices to show that $\rho$ satisfies monotonically countably subadditive. Take an open set $U$ and a countable open cover $\{U_i\}_{i=1}^\infty$ of $U$. Take any $f \in C_c(U, [0, 1])$ and let $K := \operatorname{supp} f$. Since $K$ is compact, there is a finite subcover $\{U_j\}_{j=1}^n$ of $K$, and since $K$ is paracompact Hausdorff, there is a partition of unitiy $\{\chi_j\}_j$ on $K$ subordinate to the open cover $\{U_j \cap K\}_j$. Note that $\operatorname{supp} \chi_j \subset U_j \cap K$ for each $j$.

The set $\operatorname{supp}(f\chi_j)$ is closed in $K$ so the compactness, and we also have the inclusion $\operatorname{supp}(f\chi_j) \subset \operatorname{supp} \chi_j \subset U_j$. For every $0 < a \le 1$, since $(f\chi_j)^{-1}((a, 1])$ is open in the interior of $K$ and $(f\chi_j)^{-1}([a, 1])$ is closed in $K$, $f\chi_j$ is continuous on $U_j$. Now we have checked $f\chi_j \in C_c(U_j, [0, 1])$.

Then, because $I$ is linear so that it preserves finite sum, we have

$$I(f) = I\left(\sum_{j=1}^n f\chi_j\right) = \sum_{j=1}^n I(f\chi_j) \le \sum_{j=1}^n \rho(U_j) \le \sum_{i=1}^\infty \rho(U_i).$$

Since $f$ is arbitrary, we get $\rho(U) \le \sum_{i=1}^\infty \rho(U_i)$.

(b) It suffices to show $\mathcal{T} \subset \mathcal{M}$. Clearly $\mu^*(E) \le \mu^*(E \cap U) + \mu^*(E \setminus U)$ for any measurable $E$ and open $U$. For the opposite direction, take $\varepsilon > 0$. Note that we may assume $\mu^*(E) < \infty$. There are open $U_i$ such that $E \subset \bigcup_{i=1}^\infty U_i$ and

$$\mu^*(E) + \frac{\varepsilon}{3} > \sum_{i=1}^\infty \rho(U_i).$$

Take $f_i \in C_c(U_i \cap U, [0, 1])$ such that

$$\rho(U_i \cap U) - \frac{1}{3} \cdot \frac{\varepsilon}{2^i} < I(f_i),$$

and take $g_i \in C_c(U_i \setminus \operatorname{supp} f_i, [0, 1])$ such that

$$\rho(U_i \setminus \operatorname{supp} f_i) - \frac{1}{3} \cdot \frac{\varepsilon}{2^i} < I(g_i).$$

Then, since $f_i + g_i \in C_c(U_i, [0, 1])$, we have

$$\begin{aligned} \rho(U_i) \ge I(f_i + g_i) &> \rho(U_i \cap U) + \rho(U_i \setminus \operatorname{supp} f_i) - \frac{2}{3} \cdot \frac{\varepsilon}{2^i} \\ &\ge \rho(U_i \cap U) + \rho(U_i \setminus U) - \frac{2}{3} \cdot \frac{\varepsilon}{2^i}. \end{aligned}$$

It implies

$$\mu^*(E)+\varepsilon > \sum_{i=1}^\infty \rho(U_i\cap U)+\sum_{i=1}^\infty \rho(U_i\setminus U)) \ge \mu^*(E\cap U)+\mu^*(E\setminus U)$$

because $E\cap U\subset \bigcup_{i=1}^\infty U_i\cap U$ and $E\setminus U\subset\bigcup_{i=1}^\infty U_i\setminus U$.

(c) Note that we have

$$\rho(U)=\sup_{f\in C_c(U,[0,1])} I(f), \qquad \mu(E)=\inf_{\substack{E\subset U\\ U \text{ open}}}\rho(U).$$

We first claim that for $g\in C_c(\Omega,[0,1])$, if $K$ and $K'$ are compact sets such that $g|_K=1$ and $g|_{K'^c}=0$ respectively, then we have

$$\mu(K)\le I(g)\le \mu(K').$$

The one inequality directly follows from

$$I(g)\le \inf_{K'\subset U}\rho(U)=\mu(K').$$

For the other, take sufficiently small $\varepsilon>0$ such that $U:=g^{-1}((1-\varepsilon,1])$ satisfies $K\subset U\subset \operatorname{supp} g$. For any $h\in C_c(U,[0,1])$, the inequality $(1-\varepsilon)h\le g$ implies $I(h)\le(1-\varepsilon)^{-1}I(g)$, so

$$\mu(K)\le\rho(U)\le I(h)\le(1-\varepsilon)^{-1}I(g).$$

By limiting $\varepsilon\to 0$, we get $\mu(K)\le I(g)$, the claim proved.

Since $C_c(\Omega)$ is the linear span of $C_c(\Omega,[0,1])$, it is enough to show $I(f)=\int f\,d\mu$ for $f\in C_c(X,[0,1])$. For a fixed positive integer $n$ and for each index $1\le i\le n$, let $K_i:=f^{-1}([i/n,1])$ and define

$$f_i(x):=\begin{cases}0 & \text{if } x\in K_{i-1}^c,\\ f(x)-\frac{i-1}{n} & \text{if } x\in K_{i-1}\setminus K_i,\\ \frac{1}{n} & \text{if } x\in K_i,\end{cases}$$

where $K_0:=\operatorname{supp} f$. Note that $nf_i\in C_c(X,[0,1])$ and $f=\sum_{i=1}^n f_i$. For $1\le i\le n$ we have $\mu(K_i)<\infty$ because $K_i$ is compact subsets contained in a locally compact Hausdorff space $U:=f^{-1}((0,1])$, but $\mu(K_0)$ is possibly infinite. By the previous claim and the property of integral, we have

$$\frac{\mu(K_i)}{n}\le I(f_i),\qquad \frac{\mu(K_i)}{n}\le\int f_i\,d\mu$$

for $1\le i\le n$ and

$$I(f_i)\le\frac{\mu(K_{i-1})}{n},\qquad \int f_i\,d\mu\le\frac{\mu(K_{i-1})}{n}$$

for $2\le i\le n$. Then, using the above inequalities and $\mu(K_n)\ge 0$, we have

$$I(f)\le I(f_1)+\int f\,d\mu \quad\text{and}\quad \int f\,d\mu\le\int f_1\,d\mu+I(f).$$

Note that $f_1=\min\{f,1/n\}$ is a sequence of functions indexed by $n$. By the monotone convergence theorem, $\int f_1\,d\mu\to 0$ as $n\to\infty$. We now show $I(f_1)$ converges to zero.

(d) Let $\mu$ and $\nu$ be finite Borel measures on $\Omega$ such that

$$\int g\,d\mu=\int g\,d\nu$$

for all $g \in C(\Omega)$. Let $E$ be any measurable set. Since $\mu + \nu$ is a finite Borel measure, it is regular, and by the Luzin theorem, we have a closed set $F$ and $g \in C(\Omega)$ with $0 \le g \le 1$ such that $\mathbf{1}_E|_F = g|_F$ and $(\mu + \nu)(\Omega \setminus F) < \varepsilon/2$. Then,

$$\begin{aligned}
|\mu(E) - \nu(E)| &= |\int \mathbf{1}_E \, d\mu - \int \mathbf{1}_E \, d\nu| \\
&\le \int_{\Omega \setminus F} |\mathbf{1}_E - g| \, d\mu + \int_{\Omega \setminus F} |g - \mathbf{1}_E| \, d\nu \\
&\le 2\mu(\Omega \setminus F) + 2\nu(\Omega \setminus F) < \varepsilon.
\end{aligned}$$

By limiting $\varepsilon \to 0$, we have $\mu(E) = \nu(E)$. □

**5.7** (Dual of continuous function spaces).

## 5.4 Hausdorff measures

## Exercises

# Chapter 6

# Lebesgue spaces

## 6.1 $L^p$ spaces

**6.1** (Hölder inequality).

*Proof.*

$$\int fg \leq C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take $C$ such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \Big(\int |f|^p\Big)^{\frac{1}{p}} \Big(\int |g|^p\Big)^{\frac{1}{q}}.$$

Note that we can show that $1 \leq 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \leq 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. □

## 6.2 $L^1$ spaces

**6.2** (Convolution?).

**6.3** (Approximate identity?).

**6.4** (Continuity of translation?).

## 6.3 $L^2$ spaces

## 6.4 $L^\infty$ spaces

# Part III

# Distribution theory

# Chapter 7

# Test functions

## Chapter 8

# Distributioins

# Chapter 9

# Linear operators

## 9.1 Boundedness

Translation and multiplication operators

**9.1** (Bitranspose extension).

## 9.2 Kernels

**9.2** (Schur test).

**9.3** (Young's inequality of integral operators).

## 9.3 Convolution

**9.4** (Approximation of identity). Fejér, Poisson, box?

**9.5** (Summability methods).

# Part IV

# Fundamental theorem of calculus

# Chapter 10

# Absolute continuity

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\mathrm{loc}}$.

**10.1** (Product rule for weakly differentiable functions). We want to show that if $u$, $v$, and $uv$ are weakly differentiable with respect to $x_i$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

(a) If $u$ is weakly differentiable with respect to $x_i$ and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

**10.2** (Interchange of differentiation and integration). Let $f : \Omega_x \times \Omega_y \to \mathbb{R}$ be such that $\partial_{x_i}f$ is well-defined. Suppose $f$ and $\partial_{x_i}f$ are locally integrable in $x$ and integrable $y$.

Then,

$$\partial_{x_i}\int f(x,y)\,dy = \int \partial_{x_i}f(x,y)\,dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if $F$ is $C^2$ then the *classical* partial differentiation commute, and if $F$ is not $C^2$ then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.

## 10.1 Absolutely continuity

(a) $f$ is $\mathrm{Lip}_{\mathrm{loc}}$ iff $f'$ is $L^\infty_{\mathrm{loc}}$
(b) $f$ is $\mathrm{AC}_{\mathrm{loc}}$ iff $f'$ is $L^1_{\mathrm{loc}}$

(a) $f$ is Lip iff $f'$ is $L^\infty$
(b) $f$ is AC iff $f'$ is $L^1$
(c) $f$ is BV iff $f'$ is a finite regular Borel measure

**10.3** (Absolute continuous measures).

**10.4** (Absolute continuous functions).

**10.5** (Functions of bounded variation).

## 10.2 Interpolations

weak Lp, marcinkiewicz

**Definition 10.2.1.** Let $f$ be a measurable function on a measure space $(X, \mu)$. The *distribution function* $\lambda_f : [0,\infty) \to [0,\infty)$ is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}).$$

Do not use $\mu(\{x : |f(x)| \geq \alpha\})$. The strict inequality implies the *lower semi-continuity* of $\lambda_f$.

(a) For $p > 0$, we have

$$\|f\|_{L^p}^p = p\int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^p \frac{d\alpha}{\alpha}.$$

**Definition 10.2.2.**

$$\|f\|_{L^{p,q}}^q := p\int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0<\alpha<\infty} \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right].$$

**Theorem 10.2.3.** *For $p \geq 1$ we have $\|f\|_{p,\infty} \leq \|f\|_p$.*

*Proof.* By the Chebyshev inequality,

$$\sup_{0<\alpha<\infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha)\, d\alpha = \|f\|_{L^p}^p.$$

□

**10.6** (Marcinkiewicz interpolation)**.** Let $X$ be a $\sigma$-finite measure space and $Y$ be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator $T : L^{p_0}(X) + L^{p_1}(X) \to M(Y)$ has two weak-type estimates

$$\|T\|_{L^{p_0}(X)\to L^{p_0,\infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X)\to L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X)\to L^p(X)} < \infty.$$

*Proof.* Let $f \in L^p(X)$ and denote $f_h = \chi_{|f|>\alpha} f$ and $f_l = \chi_{|f|\leq\alpha} f$. It is easy to show $f_h \in L^{p_0}$ and $f_l \in L^{p_1}$. Then,

$$\begin{aligned}
\|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha)\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^p \cdot \mu(|T(f\cdot \mathbf{1}_{|f|>\alpha})| > \alpha)\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha)\frac{d\alpha}{\alpha} \\
&\leq \int \alpha^p \cdot \frac{1}{\alpha^{p_0}}\|Tf_h\|_{L^{p_0,\infty}}^{p_0}\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}}\|Tf_l\|_{L^{p_1,\infty}}^{p_1}\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^{p-p_0}\|f_h\|_{p_0}^{p_0}\frac{d\alpha}{\alpha} + \int \alpha^{p-p_1}\|f_l\|_{p_1}^{p_1}\frac{d\alpha}{\alpha} \\
&\sim \|f\|_p^p.
\end{aligned}$$

by (1) Fubini, (2) Sublinearlity, (3) Chebyshev, (4) Boundedness, (5) Fubini. □

**10.7** (Hadamard's three line lemma)**.** Let $f$ be a bounded holomorphic function on the vertical unit strip $\{z : 0 < \operatorname{Re} z < 1\}$. Then, for $0 < \theta < 1$,

$$\|f\|_{L^\infty(\mathrm{Re}=\theta)} \leq \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta}\|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta}.$$

*Proof.* Fix $n$ and define

$$g_n(z) := \frac{f(z)}{\|f\|_{L^\infty(\mathrm{Re}=0)}^{1-z}\|f\|_{L^\infty(\mathrm{Re}=1)}^{z}} e^{-\frac{z(1-z)}{n}}.$$

Then,

$$|g_n(z)| \le e^{-\frac{(\mathrm{Im}\, z)^2}{n}}$$

for $z$ in the strip. By the maximum principle,

$$|f(z)| \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta}\|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta} e^{\frac{y^2}{n}}.$$

Letting $n \to \infty$, we are done. $\square$

**10.8** (Riesz-Thorin interpolation)**.** Let $X, Y$ be $\sigma$-finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \qquad \frac{1}{q_\theta} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \to q_\theta} \le \|T\|_{p_0 \to q_0}^{1-\theta}\|T\|_{p_1 \to q_1}^{\theta}.$$

*Proof.* Note that

$$\|T\|_{p_\theta \to q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g\rangle|}{\|f\|_{p_\theta}\|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z\rangle = \int \overline{g_z(y)}\, Tf_z(y)\, dy,$$

where $f_z$ and $g_z$ are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z)+\frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have $f_\theta = f$ and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for $\mathrm{Re}\, z = x$.

Then,

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0}\|f_z\|_{p_0}\|g_z\|_{q'_0} = \|T\|_{p_0 \to q_0}\|f\|_{p_\theta}^{p_\theta/p_0}\|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for $\mathrm{Re}\, z = 0$, and

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_1 \to q_1}\|f_z\|_{p_1}\|g_z\|_{q'_1} = \|T\|_{p_1 \to q_1}\|f\|_{p_\theta}^{p_\theta/p_1}\|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for $\mathrm{Re}\, z = 1$. By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0}^{1-\theta}\|T\|_{p_1 \to q_1}^{\theta}\|f\|_{p_\theta}\|g\|_{q'_\theta}$$

for $\mathrm{Re}\, z = \theta$. Putting $z = \theta$ in the last inequality, we get the desired result. $\square$

# Chapter 11

# Lebesgue differentiation theorem

## 11.1 Hardy-Littlewood maximal function

Let $T_m$ be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m,\ \varepsilon:|\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

**11.1** (Hardy-Littlewood maximal function)**.** The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

**11.2** (Weak type estimate)**.**

$$\|Mf\|_{1,\infty} \le 3^d \|f\|_{L^1(\Omega)}.$$

(a) Proof by covering lemma.

*Proof.* (a) By the inner regularity of $\mu$, there is a compact subset $K$ of $\{|Mf| > \lambda\}$ such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every $x \in K$, since $|Mf(x)| > \lambda$, we can choose an open ball $B_x$ such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover $\{B_i\}_i$ of $K$. Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection $\{B_k\}_k$ such that

$$K \subset \bigcup_i Bi \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \le \sum_k 3^d \mu(B_k) \le \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \le \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of $B_k$'s. $\square$

**11.3** (Radially bounded approximate identity)**.** If an approximate identity $K_n$ is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every $n$ and $x$, hence has a weak type estimate.

**11.4** (Almost everywhere convergence of operators)**.** Suppose is $T_m$ is a sequence of linear operators such that the maximal function $T^*f$ is dominated by $Mf$. If $f \in L^1(\Omega)$ and $T_m g \to g$ pointwise for $g \in C(\Omega)$, then $T_m f \to f$ a.e.

*Proof.* Take $\varepsilon > 0$ and $g \in C(\Omega)$ such that $\|f - g\|_{L^1(\Omega)} < \varepsilon$. Since $T_m g(x) \to g(x)$ pointwise, we have

$$\begin{aligned}
&\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\
&\quad \le \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \tfrac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \tfrac{\lambda}{2}\}) \\
&\quad \le \mu(\{x : M(f-g)(x) > \tfrac{\lambda}{2}\}) + \frac{2}{\lambda}\|f - g\|_{L^1(\Omega)} \\
&\quad \lesssim \frac{1}{\lambda}\varepsilon
\end{aligned}$$

for every $\lambda > 0$. Limiting $\varepsilon \to 0$, we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every $\lambda > 0$, hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

**Definition 11.1.1.**

$$f^*(x) := \lim_{r\to 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\,dy.$$

**Theorem 11.1.2** (Lebesgue differentiation)**.** $f^* = 0$ *a.e.*

*Proof.* Note that $f^* \le Mf + |f|$ implies

$$\|f^*\|_{1,\infty} \le \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that $g^* = 0$ for $g \in C_c$. Approximate using $f^* = (f - g)^*$. □

# Exercises

**11.5** (Doubling measure)**.**