Probability Theory

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Part I

Probability distributions

Chapter 1

Random variables

1.1 Sample spaces and distributions

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joint distribution transformation of distributions distribution computations

1.2 Discrete probability distributions

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1.4 Independence

1.1 (Dynkin's π-λ lemma)**.** Let $\mathcal{P}$ be a π-system and $\mathcal{L}$ a λ-system respectively. Denote by $\ell(\mathcal{P})$ the smallest λ-system containing $\mathcal{P}$.

(a) If $A \in \ell(\mathcal{P})$, then $\mathcal{G}_A := \{B : A \cap B \in \ell(\mathcal{P})\}$ is a λ-system.

(b) $\ell(\mathcal{P})$ is a π-system.

(c) If a λ-system is a π-system, then it is a σ-algebra.

(d) If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

1.2 (Monotone class lemma)**.**

Chapter 2

Conditional probablity

2.1 (Monty Hall problem). Suppose you're on a game show, and you're given the choice of three doors A, B, and C. Behind one door is a car; behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say B, which has a goat. He then says to you, "Do you want to pick door C?" Is it to your advantage to switch your choice?

Proof. Let A, B, and C be the events that a car is behind the doors A, B, and C, respectively. Let X be the event that the challenger picked A, and Y the event that the game host opened B. Note $\{A,B,C\}$ is a partition of the sample space Ω, and X is independent to A, B, and C. Then, $P(A) = P(B) = P(C) = P(X) = 1/3$, and

$$P(Y|X,A) = \frac{1}{2}, \quad P(Y|X,B) = 0, \quad P(Y|X,C) = 1.$$

Therefore,

$$\begin{aligned} P(C|X,Y) &= \frac{P(X \cap Y \cap C)}{P(X \cap Y)} \\ &= \frac{P(Y|X,C)P(X \cap C)}{P(Y|X,A)P(X \cap A) + P(Y|X,B)P(X \cap B) + P(Y|X,C)P(X \cap C)} \\ &= \frac{1 \cdot \frac{1}{9}}{\frac{1}{2} \cdot \frac{1}{9} + 0 \cdot \frac{1}{9} + 1 \cdot \frac{1}{9}} = \frac{2}{3}. \end{aligned}$$

Similarly, $P(A|X,Y) = \frac{1}{3}$ and $P(B|X,Y) = 0$. $\square$

Chapter 3

Convergence of probability measures

3.1 Weak convergence in $\mathbb{R}$

3.1 (Portemanteau theorem)**.** Let F_n and F be distribution functions $\mathbb{R} \to [0,1]$. We will define the *weak convergence* as follows: F_n converges weakly to F if $F_n(x) \to F(x)$ for every continuity point x of $F(x)$.

(a) $F_n(x) \to F(x)$ for all continuity points x of F.

3.2 (Skorokhod representation theorem)**.**

3.3 (Continuous mapping theorem)**.**

3.4 (Slutsky's theorem)**.**

3.5 (Helly's selection theorem)**.** (a) Monotonically increasing functions $F_n : \mathbb{R} \to [0,1]$ has a pointwise convergent subsequence.

(b) If $(F_n)_n$ is tight, then

3.6 (Properties of probability Borel measures)**.** Let S be a topological space.

(a) Every single probability Borel measure is regular if S is perfectly normal.

(b) Every single probability Borel measure is tight if S is Polish.

3.2 Weak topology in the space of probability measures

3.7 (Local limit theorems)**.** Suppose f_n and f are density functions.

(a) If $f_n \to f$ a.s., then $f_n \to f$ in L^1. (Scheffé's theorem)

(b) $f_n \to f$ in L^1 if and only if in total variation.

(c) If $f_n \to f$ in total variation, then $f_n \to f$ weakly.

3.8 (Portmanteau theorem)**.** Let S be a normal space and, μ_α be a net in $\mathrm{Prob}(S)$. We define the *weak convergence* as follows: μ_α converges weakly to μ if

$$\int f \, d\mu_\alpha \to \int f \, d\mu$$

for every $f \in C_b(S)$. The following statements are all equivalent.

(a) $\mu_\alpha \Rightarrow \mu$

(b) $\mu_\alpha(g) \to \mu(g)$ for every uniformly continuous $g \in C_b(S)$.

(c) $\limsup_\alpha \mu_\alpha(F) \le \mu(F)$ for every closed F.

(d) $\liminf_\alpha \mu_\alpha(U) \ge \mu(U)$ for every open U.

(e) $\lim_\alpha \mu_\alpha(A) = \mu(A)$ for every Borel A such that $\mu(\partial A) = 0$.

Proof. (a)⇒(b) Clear.

(b)⇒(c) Let U be an open set such that $F \subset U$. There is uniformly continuous $g \in C_b(S)$ such that $\mathbf{1}_F \le g \le \mathbf{1}_U$. Therefore,

$$\limsup_\alpha \mu_\alpha(F) \le \limsup_\alpha \mu_\alpha(g) = \mu(g) \le \mu(U).$$

By the outer regularity of μ, we obtain $\limsup_\alpha \mu_\alpha(F) \le \mu(F)$.

(c)⇔(d) Clear.

(c)+(d)⇒(e) It easily follows from

$$\limsup_\alpha \mu_\alpha(\overline{A}) \le \mu(\overline{A}) = \mu(A) = \mu(A^\circ) \le \liminf_\alpha \mu_\alpha(A^\circ).$$

(e)⇒(a) Let $g \in C_b(S)$ and $\varepsilon > 0$. Since the pushforward measrue $g_*\mu$ has at most countably many mass points, there is a partition $(t_i)_{i=0}^n$ of an interval containing $[-\|g\|, \|g\|]$ such that $|t_{i+1} - t_i| < \varepsilon$ and $\mu(\{x : g(x) = t_i\}) = 0$ for each i. Let $(A_i)_{i=0}^{n-1}$ be a Borel decomposition of S given by $A_i := g^{-1}([t_i, t_{i+1}))$, and define $f_\varepsilon := \sum_{i=0}^{n-1} t_i \mathbf{1}_{A_i}$ so that we have $\sup_{x\in S} |g_\varepsilon(x) - g(x)| \le \varepsilon$. From

$$\begin{aligned} |\mu_\alpha(g) - \mu(g)| &\le |\mu_\alpha(g - g_\varepsilon)| + |\mu_\alpha(g_\varepsilon) - \mu(g_\varepsilon)| + |\mu(g_\varepsilon - g)| \\ &\le \varepsilon + \sum_{i=0}^{n-1} |t_i||\mu_\alpha(A_i) - \mu(A_i)| + \varepsilon, \end{aligned}$$

we get

$$\limsup_\alpha |\mu_\alpha(g) - \mu(g)| < 2\varepsilon.$$

Since ε is arbitrary, we are done. □

3.9 (Embedding by Dirac measures). Let S be a normal space.

(a) $S \to \mathrm{Prob}(S)$ is an embedding.

(b) $S \subset \mathrm{Prob}(S)$ is sequentially closed.

(c)

Proof. (a) It uses Urysohn.

(b) It uses (b)=>(c) of Portmanteau. □

3.10 (Lévy-Prokhorov metric). Let S be a metric space, and $\mathrm{Prob}(S)$ be the set of probability (regular) Borel measures on S. Define $\pi : \mathrm{Prob}(S) \times \mathrm{Prob}(S) \to [0, \infty)$ such that

$$\pi(\mu, \nu) := \inf\{\, \alpha > 0 : \mu(A) \le \nu(A^\alpha) + \alpha,\ \ \nu(A) \le \mu(A^\alpha) + \alpha,\ \ \forall A \in \mathcal{B}(S)\,\},$$

where A^α is the α-neighborhood of a.

(a) π is a metric.

(b) $\mu_n \to \mu$ in π implies $\mu_n \Rightarrow \mu$.

(c) $\mu_\alpha \Rightarrow \mu$ implies $\mu_\alpha \to \mu$ in π, if S is separable.

(d) (S,d) is separable if and only if $(\mathrm{Prob}(S),\pi)$ is separable.

(e) (S,d) is compact if and only if $(\mathrm{Prob}(S),\pi)$ is compact

(f) (S,d) is complete if and only if $(\mathrm{Prob}(S),\pi)$ is complete.

Proof. (c) □

3.11 (Prokhorov's theorem)**.** Let S be a normal space. Let $\mathrm{Prob}(S)$ be the space of probability measures on S endowed with the topology of weak convergence.

(a) The relative compactness implies the tightness if S is Polish.

(b) The tightness implies the relative compactness.

Proof. (a) Fix a complete metric d on S. For each entourage E, we want to find a finite union of closed E-neighborhood that is arbitrarily full. And we require the countable fundamental system of uniformity.

(b) □

3.3 Characteristic functions

3.12 (Characteristic functions)**.** Let μ be a probability measure on $\mathbb{R}$. Then, the *characteristic function* of μ is defined by

$$\varphi(t) := Ee^{itX} = \int e^{itx}\,d\mu(x).$$

Note that $\varphi(t) = \widehat{\mu}(-t)$ where $\widehat{\mu}$ is the Fourier transform of $\mu \in \mathcal{S}'(\mathbb{R})$.

(a) $\varphi \in C_b(\mathbb{R})$.

3.13 (Inversion formula)**.** Let μ be a probability measure on $\mathbb{R}$ and φ its characteristic function.

(a) For $a < b$, we have

$$\mu((a,b)) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\,dt.$$

(b) For $a \in \mathbb{R}$, we have

$$\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt$$

(c) If $\varphi \in L^1(\mathbb{R})$, then μ has density

$$f(x) = \frac{1}{2\pi}\int e^{-itx}\varphi(t)\,dt$$

in $C_0(\mathbb{R}) \cap L^1(\mathbb{R})$.

3.14 (Lévy's continuity theorem)**.** The continuity theorem provides with a tool to verify the weak convergence in terms of characteristic functions. Let μ_n and μ be probability distributions on $\mathbb{R}$ with characteristic functions φ_n and φ.

(a) If $\mu_n \to \mu$ weakly, then $\varphi_n \to \varphi$ pointwise.

(b) If $\varphi_n \to \varphi$ pointwise and φ is continuous at zero, then $(\mu_n)_n$ is tight and $\mu_n \to \mu$ weakly.

Proof. (a) For each t,

$$\varphi_n(t) = \int e^{itx}\, d\mu_n(x) \to \int e^{itx}\, d\mu(x) = \varphi(t)$$

because $e^{itx} \in C_b(\mathbb{R})$.

(b)

□

3.15 (Criteria for characteristic functions)**.** Bochner's theorem and Polya's criterion

There are two ways to represent a measure: A measure μ is absolutely continuous iff its distribution F is absolutely continuous iff its density f is integrable. So, the fourier transform of an absolutely continuous measure is just the fourier transform of L^1 functions.

3.4 Moments

moment problem

moment generating function defined on $|t| < \delta$

Exercises

3.16. Let φ_n be characteristic functions of probability measures μ_n on $\mathbb{R}$. If there is a continuous function φ such that $\varphi_n = \varphi$ on $n^{-1}\mathbb{Z}$, then μ_n converges weakly.

3.17 (Convergence determining class)**.**

3.18 (Vauge convergence)**.** Let S be a locally compact Hausdorff space.

(a) $\mu_\alpha \to \mu$ vaguely if and only if $\int g\, d\mu_\alpha \to \int g\, d\mu$ for all $g \in C_c(S)$.

(b) $\mu_\alpha \to \mu$ weakly if and only if vaguely.

(c) $\delta_n \to 0$ vaguely but not weakly. (escaping to infinity)

Proof. □

Part II

Discrete stochastic process

Chapter 4

Limit theorems

4.1 Laws of large numbers

Our purpose is to find appropriate a_n and slowly growing b_n such that $(S_n - a_n)/b_n \to 0$ in probability or almost surely.

4.1 (Kolmogorov-Feller theorem)**.** Let X_i be an uncorrelated sequence of random variables such that

$$\lim_{x\to\infty} \sup_i xP(|X_i| > x) = 0.$$

This condition is called the *Kolmogorov-Feller* condition. Let $Y_{n,i} := X_i \mathbf{1}_{|X_i|\le c_n}$.

(a) We have
$$\lim_{n\to\infty} P(S_n \neq T_n) = 0$$
if $n \lesssim c_n$.

(b) We have
$$\lim_{n\to\infty} P\left(\left|\frac{T_n - ET_n}{b_n}\right| > \varepsilon\right) = 0$$
if $nc_n \lesssim b_n^2$.

(c) We have
$$\frac{S_n - ET_n}{n} \to 0$$
in probability.

Proof. Write $g(x) := \sup_i xP(|X_i| > x)$ so that $g(x) \to 0$ as $x \to \infty$.

(a) It follows from

$$P(S_n \neq T_n) \le \sum_{i=1}^n P(|X_i| > c_n) \le \sum_{i=1}^n \frac{1}{c_n} g(c_n) \lesssim g(c_n).$$

(b) We write

$$
\begin{aligned}
P\left(\left|\frac{T_n - ET_n}{b_n}\right| > \varepsilon\right) &\le \frac{1}{\varepsilon^2 b_n^2} E|T_n - ET_n|^2 \\
&= \frac{1}{\varepsilon^2 b_n^2} \sum_{i=1}^n E|Y_{n,i} - EY_{n,i}|^2 \\
&\le \frac{1}{\varepsilon^2 b_n^2} \sum_{i=1}^n E|X_i \mathbf{1}_{|X_i| \le c_n}|^2 \\
&= \frac{1}{\varepsilon^2 b_n^2} \sum_{i=1}^n \int_0^{c_n} 2x P(|X_i| > x)\, dx \\
&\le \frac{2n}{\varepsilon^2 b_n^2} \int_0^{c_n} g(x)\, dx \\
&= \frac{2nc_n}{\varepsilon^2 b_n^2} \int_0^1 g(c_n x)\, dx \\
&\lesssim \int_0^1 g(c_n x)\, dx.
\end{aligned}
$$

Since $g(x) \le x$ and $g(x) \to 0$ as $x \to \infty$, the function g is bounded. By the bounded convergence theorem, we get $\int_0^1 g(c_n x)\, dx \to 0$ as $n \to \infty$. □

4.2 (St. Petersburg paradox)**.** We want see the asymptotic behavior of the partial sums S_n of i.i.d. random variables X_i such that $E|X_i| = \infty$. Let

$$P(X_n = 2^m) = 2^{-m} \quad \text{for } m \ge 1.$$

Let $Y_{n,i} := X_i \mathbf{1}_{|X_i| \le c_n}$.

(a) We have

$$\lim_{n\to\infty} P(S_n \ne T_n) = 0$$

if $n \ll c_n$.

(b) We have

$$\lim_{n\to\infty} P\left(\left|\frac{T_n - ET_n}{b_n}\right| > \varepsilon\right) = 0$$

if $nc_n \ll b_n^2$.

(c) We have

$$\frac{S_n - n\log_2 n}{n^{1+\varepsilon}} \to 0$$

in probability for every $\varepsilon > 0$.

Proof. (a) It follows from

$$P(S_n \ne T_n) \le \sum_{i=1}^n P(X_i \ne Y_{n,i}) = \sum_{i=1}^n P(|X_i| > c_n) \le \sum_{i=1}^n \frac{2}{c_n} = \frac{2n}{c_n}.$$

(b) It follows from

$$\begin{aligned}
P\left(\left|\frac{T_n - ET_n}{b_n}\right| > \varepsilon\right) &\le \frac{1}{\varepsilon^2 b_n^2} E|T_n - ET_n|^2 \\
&= \frac{1}{\varepsilon^2 b_n^2} \sum_{i=1}^n E|Y_{n,i} - EY_{n,i}|^2 \\
&\le \frac{1}{\varepsilon^2 b_n^2} \sum_{i=1}^n E|X_i \mathbf{1}_{|X_i| \le c_n}|^2 \\
&\le \frac{1}{\varepsilon^2 b_n^2} n \cdot 2c_n
\end{aligned}$$

□

4.3 (Borel-Cantelli lemmas).

4.4 (Head runs).

4.5 (Strong laws of large numbers for L^1). Proof by Etemadi

Random series proof

4.2 Renewal theory

4.3 Central limit theorems

4.6 (Central limit theorem for L^3). Replacement method by Lindeman and Lyapunov

4.7 (Lindeberg-Feller theorem). Let X_i be independent random variables such that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^n E|X_i - EX_i|^2 \mathbf{1}_{|X_i - EX_i| > \varepsilon s_n} = 0.$$

This condition is called the *Lindeberg-Feller* condition. Let $Y_{n,i} := \frac{X_i - EX_i}{s_n}$.

(a) We have

$$|Ee^{it(S_n - ES_n)/s_n} - e^{-\frac{1}{2}t^2}| \le \sum_{i=1}^n |Ee^{itY_{n,i}} - e^{-\frac{1}{2}E(tY_{n,i})^2}|.$$

(b) For any $\varepsilon > 0$, we have an estimate

$$\left|Ee^{itY} - \left(1 - \frac{1}{2}E(tY)^2\right)\right| \lesssim_t \varepsilon EY^2 + EY^2 \mathbf{1}_{|Y| > \varepsilon}$$

for all random variables Y such that $EY^2 < \infty$.

(c) For any $\varepsilon > 0$, we have an estimate

$$\left|e^{-\frac{1}{2}E(tY)^2} - \left(1 - \frac{1}{2}E(tY)^2\right)\right| \lesssim_t EY^2(\varepsilon^2 + EY^2 \mathbf{1}_{|Y| > \varepsilon}).$$

for all random variables Y such that $EY^2 < \infty$.

(d)

Proof. (a) Note

$$Ee^{it(S_n-ES_n)/s_n}=\prod_{i=1}^n Ee^{itY_{n,i}}\quad\text{and}\quad e^{-\frac12t^2}=\prod_{i=1}^n e^{-\frac12E(tY_{n,i})^2}.$$

(b) Since

$$\Big|e^{ix}-\big(1+ix-\frac12x^2\big)\Big|=\Big|\frac{i^3}{2}\int_0^x(x-y)^2e^{iy}\,dy\Big|\le\min\{\frac16|x|^3,x^2\}$$

for $x\in\mathbb{R}$, we have

$$\begin{aligned}\Big|Ee^{itY}-\big(1-\frac12E(tY)^2\big)\Big|&\le E\Big|e^{itY}-\big(1-\frac12(tY)^2\big)\Big|\\&\lesssim_t E\min\{|Y|^3,Y^2\}\\&\le E|Y|^3\mathbf{1}_{|Y|\le\varepsilon}+EY^2\mathbf{1}_{|Y|>\varepsilon}\\&\le\varepsilon EY^2+EY^2\mathbf{1}_{|Y|>\varepsilon}.\end{aligned}$$

(c) Since

$$|e^{-x}-(1-x)|=\Big|\int_0^x(x-y)e^{-y}\,dy\Big|\le\frac12x^2$$

for $x\ge0$, we have

$$\Big|e^{-\frac12E(tY)^2}-\big(1-\frac12E(tY)^2\big)\Big|\lesssim_t(EY^2)^2\le EY^2(\varepsilon^2+EY^2\mathbf{1}_{|Y|>\varepsilon}).$$

□

4.8. Let $X_n:\Omega\to\mathbb{R}$ be independent random variables. If there is $\delta>0$ such that the *Lyapunov condition*

$$\lim_{n\to\infty}\frac{1}{s_n^{2+\delta}}\sum_{i=1}^nE|X_i-EX_i|^{2+\delta}=0$$

is satisfied, then

$$\frac{S_n-ES_n}{s_n}\to N(0,1)$$

weakly, where $S_n:=\sum_{i=1}^nX_i$ and $s_n^2:=VS_n$.

Berry-Esseen ineaulity

Exercises

4.9 (Bernstein polynomial)**.** Let $X_n\sim\mathrm{Bern}(x)$ be i.i.d. random variables. Since $S_n\sim\mathrm{Binom}(n,x)$, $E(S_n/n)=x$, $V(S_n/n)=x(1-x)/n$. The L^2 law of large numbers implies $E(|S_n/n-x|^2)\to0$. Define $f_n(x):=E(f(S_n/n))$. Then, by the uniform continuity $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$,

$$|f_n(x)-f(x)|\le E(|f(S_n/n)-f(x)|)\le\varepsilon+2\|f\|P(|S_n/n-x|\ge\delta)\to\varepsilon.$$

4.10 (High-dimensional cube is almost a sphere)**.** Let $X_n\sim\mathrm{Unif}(-1,1)$ be i.i.d. random variables and $Y_n:=X_n^2$. Then, $E(Y_n)=\frac13$ and $V(Y_n)\le1$.

4.11 (Coupon collector's problem)**.** $T_n:=\inf\{\,t:|\{X_i\}_i|=n\,\}$ Since $X_{n,k}\sim\mathrm{Geo}(1-\frac{k-1}{n})$, $E(X_{n,k})=(1-\frac{k-1}{n})^{-1}$, $V(X_{n,k})\le(1-\frac{k-1}{n})^{-2}$. $E(T_n)\sim n\log n$

4.12 (An occupancy problem)**.**

4.13. Find the probability that arbitrarily chosen positive integers are coprime.

Poisson convergence, law of rare events, or weak law of small numbers (a single sample makes a significant attibution)

Chapter 5

Martingales

5.1 Submartingales

5.2 Martingale convergence theorem

5.1 (Doob's upcrossing inequality). (a)

5.2 (Martingale convergence theorems). (a)

5.3. (a)

5.3 Convergence in L^p and uniform integrability

5.4 Optional stopping theorem

Chapter 6

Markov chains

Part III

Continuous stochastic processes

Chapter 7

Brownian motion

7.1 Kolomogorov extension

7.1 (Kolmogorov extension theorem)**.** A *rectangle* is a finite product $\prod_{i=1}^n A_i \subset \mathbb{R}^n$ of measurable $A_i \subset \mathbb{R}$, and *cylinder* is a product $A^* \times \mathbb{R}^\mathbb{N}$ where A^* is a rectangle. Let $\mathcal{A}$ be the semi-algebra containing $\varnothing$ and all cylinders in $\mathbb{R}^\mathbb{N}$. Let $(\mu_n)_n$ be a sequence of probability measures on $\mathbb{R}^n$ that satisfies *consistency condition*

$$\mu_{n+1}(A^* \times \mathbb{R}) = \mu_n(A^*)$$

for any rectangles $A^* \subset \mathbb{R}^n$, and define a set function $\mu_0 : \mathcal{A} \to [0,\infty]$ by $\mu_0(A) = \mu_n(A^*)$ and $\mu_0(\varnothing) = 0$.

(a) μ_0 is well-defined.

(b) μ_0 is finitely additive.

(c) μ_0 is countably additive if $\mu_0(B_n) \to 0$ for cylinders $B_n \downarrow \varnothing$ as $n \to \infty$.

(d) If $\mu_0(B_n) \geq \delta$, then we can find decreasing $D_n \subset B_n$ such that $\mu_0(D_n) \geq \frac{\delta}{2}$ and $D_n = D_n^* \times \mathbb{R}^\mathbb{N}$ for a compact rectangle D_n^*.

(e) If $\mu_0(B_n) \geq \delta$, then $\bigcap_{i=1}^\infty B_i$ is non-empty.

Proof. (d) Let $B_n = B_n^* \times \mathbb{R}^\mathbb{N}$ for a rectangle $B_n^* \subset \mathbb{R}^{r(n)}$. By the inner regularity of $\mu_{r(n)}$, there is a compact rectangle $C_n^* \subset B_n^*$ such that

$$\mu_0(B_n \setminus C_n) = \mu_{r(n)}(B_n^* \setminus C_n^*) < \frac{\delta}{2^{n+1}}.$$

Let $C_n := C_n^* \times \mathbb{R}^\mathbb{N}$ and define $D_n := \bigcap_{i=1}^n C_i = D_n^* \times \mathbb{R}^\mathbb{N}$. Then,

$$\mu_0(B_n \setminus D_n) \leq \mu_0(\bigcup_{i=1}^n B_n \setminus C_i) \leq \mu_0(\bigcup_{i=1}^n B_i \setminus C_i) < \frac{\delta}{2},$$

which implies $\mu_0(D_n) \geq \frac{\delta}{2}$.

(e) Take any sequence $(\omega_n)_n$ in $\mathbb{R}^\mathbb{N}$ such that $\omega_n \in D_n$. Since each $D_n^* \subset \mathbb{R}^{r(n)}$ is compact and non-empty, by diagonal argument, we have a subsequence $(\omega_k)_k$ such that ω_k is pointwise convergent, and its limit is contained in $\bigcap_{i=1}^\infty D_i \subset \bigcap_{i=1}^\infty B_n = \varnothing$, which is a contradiction that leads $\mu_0(B_n) \to 0$. □

Part IV

Stochastic calculus