

# Functional Analysis

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## **Part I**

# **Topological vector spaces**

# Chapter 1

## Locally convex spaces

### 1.1 General vector topologies

canonical uniformity, canonical bornology, metrizability(Birkhoff-Kakutani), boundedness and continuity

### 1.2 Seminorms and convex sets

boundedness by seminorms, normability

### 1.3 Continuous linear functionals

1.1. Let  $\{x_i^*\}_{i=1}^n \subset X^*$ . If  $x^* \in X^*$  vanishes on  $\bigcap_{i=1}^n \ker x_i^*$ , then  $x^*$  is a linear combination of  $\{x_i^*\}$ .

1.2 (Dual space).

1.3 (Adjoint operator).

### 1.4 Hahn-Banach theorem

1.4 (Hahn-Banach theorem).

## Chapter 2

# Barreled spaces

### 2.1 Uniform boundedness principle

**2.1** (Barreled spaces). A barrel. A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

If a closed convex cone contains a dense subset of absorbing at a point, then it is entire?

**2.2** (Uniform boundedness principle). Let  $f : S \subset X \rightarrow \mathbb{R}_{\geq 0}$ . Suppose  $\|T_\alpha x\| \leq f(x)$  on  $S$ .

(a)  $S \subset \bigcup_{n=1}^{\infty} \bigcap_{\alpha} T_\alpha^{-1} B_n$ .

(b) If  $X$  is the closed linear span of  $S$ , then  $\bigcap_{\alpha} T_\alpha^{-1} B_1$  is a barrel of  $X$ .

### 2.2 Baire spaces

**2.3** (The Baire category theorem).

### 2.3 Open mapping theorem

**2.4** (Open mapping theorem). Let  $X$  be a F-space and  $Y$  a topological vector space. Suppose  $T : X \rightarrow Y$  is continuous and surjective.

(a) There is  $r > 0$  such that  $B_r \subset \overline{TB_1}$ .

(b) There is  $r > 0$  such that  $B_r \subset TB_1$ .

(c)  $T$  is open.

*Proof.* (a) The set  $\overline{TB_1}$  is clearly closed and absolutely convex. The surjectivity of  $T$  implies  $\overline{TB_1}$  is absorbing. Since  $Y$  is barreled,  $\overline{TB_1}$  contains an open ball  $B_r$ .

(b) Let  $r > 0$  such that  $B_r \subset \overline{TB_{1/2}}$ . For  $y \in B_r$ , we are going to construct  $x \in B_1 \subset X$  such that  $y = Tx$ . We claim for  $n$  that

$$(y + TB_{1-1/2^n}) \cap B_{r/2^n} \neq \emptyset.$$

We have

$$(y + B_{r/2}) \cap TB_{1/2} \neq \emptyset,$$

and

$$(y + TB_{1/2}) \cap B_{r/2} \neq \emptyset,$$

and

$$(y + TB_{1/2}) \cap \overline{TB_{1/4}} \neq \emptyset,$$

and

$$(y + TB_{1/2} + B_{r/4}) \cap TB_{1/4} \neq \emptyset,$$

and

$$(y + TB_{3/4}) \cap B_{r/4} \neq \emptyset.$$

□

## Exercises

**2.5.** Let  $(T_n)$  be a sequence in  $B(X, Y)$ . If  $T_n$  converges then  $\|T_n\|$  is bounded by the uniform boundedness principle.

## Chapter 3

# Fréchet, Banach, and Hilbert spaces

### 3.1 Freéchet spaces

dual is not Freéchet.

### 3.2 Banach spaces

dual is Banach. Basis problem, Mazur' duck.

### 3.3 Hilbert spaces

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

**3.1.** (a) A Banach space  $X$  is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of  $X$ .

**3.2** (Riesz representation theorem). Let  $H$  be a Hilbert space over a field  $\mathbb{F}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ .

We use the bilinear form  $\langle -, - \rangle : X \times X^* \rightarrow \mathbb{F}$  of canonical duality. *Dirac* notation  $\langle - | - \rangle$  for the inner product of a complex Hilbert spaces such that  $\langle x, y \rangle = \langle y | x \rangle$ . The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each  $x^* \in H^*$ , there is a unique  $x \in H$  such that  $\langle y, x^* \rangle = \langle y, x \rangle$  for every  $y \in H$ .

(b)  $H \rightarrow H^* : x \mapsto \langle -, x \rangle$  is a natural linear and anti-linear isomorphism if  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{C}$ , respectively.

### 3.4 Bounded linear operators

**3.3** (Bounded belowness in Banach spaces). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ . The following statements are equivalent:

- (a)  $T$  is bounded below.
- (b)  $T$  is injective and has closed range.
- (c)  $T$  is a topological isomorphism onto its image.

**3.4** (Bounded belowness in Hilbert spaces). Let  $T \in B(H, K)$  for Hilbert spaces  $H$  and  $K$ . The following statements are equivalent:

- (a)  $T$  is bounded below.



- (b)  $T$  is left invertible.
- (c)  $T^*$  is right invertible.
- (d)  $T^*T$  is invertible.

**3.5** (Injectivity and surjectivity of adjoint). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ .

- (a)  $T^*$  is injective if and only if  $T$  has dense range.
- (b)  $T^*$  is surjective if and only if  $T$  is bounded below.

**3.6** (Normal operators). For  $T \in B(H)$ , we have an obvious fact  $(\text{im } T)^\perp = \ker T^*$ . Suppose  $T$  is normal.

- (a)  $\ker T = \ker T^*$ .
- (b)  $T$  is bounded below if and only if  $T$  is invertible.
- (c) If  $T$  is surjective, then  $T$  is invertible.

## Exercises

**3.7.** There is no metric  $d$  on  $C([0, 1])$  such that  $d(f_n, f) \rightarrow 0$  if and only if  $f_n \rightarrow f$  pointwise as  $n \rightarrow \infty$  for every sequence  $f_n$ . Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

**3.8.** Let  $T$  be an invertible linear operator on a normed space. Then,  $T^{-2} + \|T\|^{-2}$  is injective if it is surjective.

**3.9.** We show that there is no projection from  $\ell^\infty$  onto  $c_0$ .

**3.10** (Schur's property of  $\ell^1$ ).

**3.11.** Let  $\varphi : L^\infty([0, 1]) \rightarrow \ell^\infty(\mathbb{N})$  be an isometric isomorphism. Suppose  $\varphi$  is realised as a sequence of bounded linear functionals on  $L^\infty$ .

- (a) Show that  $\varphi^*(\ell^1) \subset L^1$  where  $\ell^1$  and  $L^1$  are considered as closed linear subspaces of  $(\ell^\infty)^*$  and  $(L^\infty)^*$  respectively.
- (b) Show that  $\varphi^*$  is indeed an isometric isomorphism, and deduce  $\varphi$  cannot be realised as bounded linear functionals on  $L^\infty$ .

## **Part II**

# **Weak topologies**

## Chapter 4

# Dual space of Banach spaces

### 4.1 Weak and weak\* topologies

boundedness, incompleteness

**4.1** (Weak convergence by dense set). Let  $X$  be a Banach space,  $D$  a subset of  $X^*$ , and  $\overline{D}$  the norm closure of  $D$ . For example, if  $X$  has a predual  $X_* \subset X^*$  and  $D$  is dense in  $X_*$ , then  $\sigma(X, \overline{D})$  is the weak\* topology.

- (a) There is a sequence  $x_n \in X$  converges to zero in  $\sigma(X, D)$  but not in  $\sigma(X, \overline{D})$ .
- (b) A sequence  $x_n \in X$  converges to zero in  $\sigma(X, \overline{D})$  if in  $\sigma(X, D)$ , if  $\|x_n\| \leq 1$ .

*Proof.* (b) Let  $x^* \in \overline{D}$  and choose  $y^* \in D$  such that  $\|x^* - y^*\| < \varepsilon$ . Then,

$$|\langle x_n, x^* \rangle| \leq \|x_n\| \|x^* - y^*\| + |\langle x_n, y^* \rangle|.$$

□

### 4.2 Weak compactness

**4.2** (Banach-Alaoglu theorem).

**4.3** (Eberlein-Šmulian theorem).

**4.4** (James' theorem).

### 4.3 Weak density

Bishop-Phelps theorem

**4.5** (Goldstine's theorem). Let  $X$  be a Banach space and  $J : X \rightarrow X^{**}$  the canonical embedding. Let  $\{x_i^*\}_{i=1}^m \subset X^*$  and  $x^{**} \in X^{**}$ .

- (a) There is  $x \in X$  such that  $\langle x_i^*, J(x) \rangle = \langle x_i^*, x^{**} \rangle$  for all  $i$ .
- (b) If  $\|x^{**}\| \leq 1$ , then there is  $x \in X$  such that  $\|x\| \leq 1 + \varepsilon$  and  $\langle x_i^*, J(x) \rangle = \langle x_i^*, x^{**} \rangle$  for all  $i$ , for any  $\varepsilon > 0$ .
- (c)  $J(\overline{B}_X)$  is weak\*-dense in  $\overline{B}_{X^{**}}$

*Proof.* (b) Let  $z \in X$  such that  $\langle x_i^*, J(x) \rangle = \langle x_i^*, x^{**} \rangle$  for all  $i$ . Let  $Y$  be the set of all  $y \in X$  such that  $\langle x_i^*, J(y) \rangle = 0$  for all  $i$ . Then,  $z + Y$  is the closed affine subspace of  $X$  containing all  $y \in X$  such that  $\langle x_i^*, J(y) \rangle = \langle x_i^*, x^{**} \rangle$  for all  $i$ . If we assume  $z + Y$  does not contain any  $x \in X$  such that  $\|x\| \leq 1 + \varepsilon$ , then  $\text{dist}(z, Y) = \text{dist}(0, z + Y) > 1 + \varepsilon$ . By the Hahn-Banach theorem, there is  $y^* \in X^*$  such that  $\|y^*\| = 1$ ,  $y^*|_Y = 0$ , and  $\langle z, y^* \rangle > 1 + \varepsilon$ . Then,  $y^*$  is a linear combination of  $\{x_i^*\}_{i=1}^m$ , so

$$1 + \varepsilon < \langle z, y^* \rangle = \langle y^*, J(z) \rangle = \langle y^*, x^{**} \rangle \leq \|x^{**}\| \|y^*\| \leq 1.$$

(c) Fix  $x^{**} \in X^{**}$  such that  $\|x^{**}\| \leq 1$  and let

$$U = \bigcap_{i=1}^m \{y^{**} \in X^{**} : |\langle x_i^*, y^{**} - x^{**} \rangle| < 1\}$$

be an open weak\*-neighborhood of  $x^{**}$ . Choose  $\varepsilon > 0$  such that

$$\varepsilon \max_{1 \leq i \leq m} \|x_i^*\| < 1.$$

By the part (b), there is  $x \in X$  such that  $\|x\| \leq 1 + \varepsilon$  and  $\langle x_i^*, x^{**} \rangle = \langle x_i^*, J(x) \rangle$ . If we let  $y := (1 + \varepsilon)^{-1}x$ , then  $\|y\| \leq 1$  so that

$$|\langle x_i^*, J(y) - x^{**} \rangle| = |\langle x_i^*, J(y) - J(x) \rangle| = |\langle x_i^*, \varepsilon J(y) \rangle| \leq \varepsilon \|x_i^*\| \|y\| < 1$$

for all  $i$  implies  $J(y) \in U$ , hence we get  $J(\overline{B_X}) \cap U \neq \emptyset$ . □

## 4.4 Krein-Milman theorem

Choquet theory

### Exercises

**4.6** (James' space). not reflexive but isometrically isomorphic to bidual

**4.7** (Predual correspondence). Let  $X$  be a Banach space. Let

$$\{(Y, \varphi) \mid \varphi : X \rightarrow Y^* \text{ is an isometric isomorphism}\}$$

and

$$\{Z \leq X^* \mid \overline{B_X} \text{ is compact Hausdorff in } (X, \sigma(X, Z))\}.$$

$$(Y, \varphi) \mapsto \text{im } \varphi^*|_{J(Y)}$$

(a) The map is well-defined.

(b) The map is surjective. (by Goldstein)

(c) The map is injective up to isomorphism for  $Y$ .

**4.8.** Let  $X$  be a closed subspace of a Banach space  $Y$  and

$$i : X \rightarrow Y$$

the inclusion. Suppose  $X$  and  $Y$  have preduals  $X_*$  and  $Y_*$  respectively. Let

$$j := i^*|_{Y_*} : Y_* \rightarrow Z \subset X^*,$$

where  $Z := i^*(Y_*)^\perp$ . Then we can show

$$j^* : Z^* \subset X^{**} \rightarrow Y$$

coincides with  $i$  on  $X \cap Z^*$ . From the existence of  $X_*$  we have  $X^{**} \rightarrow X$ , which is restricted to define a map  $k : Z^* \rightarrow X$ .

$$\begin{array}{ccccc} & & X & \xrightarrow{i} & Y \\ & \nearrow & \uparrow k & \nearrow j & \\ X^{**} & \longrightarrow & Z^* & & \end{array}$$

We can show  $k$  is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

**4.9** (Mazur's lemma).

**4.10** (Dunford-Pettis property).

## Chapter 5

# Polar topologies

### 5.1 Dual pair

### 5.2 Strong topologies

Mackey-Arens

## Chapter 6

# Operator topologies

**6.1** (Compact left multiplications and SOT). Let  $T_n$  be a sequence of bounded linear operators on a Hilbert space that converges in SOT. For compact  $K$ ,  $T_n K$  converges in norm, but  $K T_n$  generally does not unless  $T$  is self-adjoint.

**6.2.** Let  $f$  be a linear functional on  $B(H)$  for a Hilbert space  $H$ . Then, TFAE:

- (a)  $f$  is WOT-continuous,
- (b)  $f$  is SOT-continuous,
- (c)  $f(T) = \sum_{i=1}^n \langle T x_i, y_i \rangle$  for some  $x_i, y_i$ .

*Proof.* (2)  $\Rightarrow$  (3) is the only nontrivial implication. By the definition of SOT, there exists  $v \in \mathcal{H}^n$  such that

$$|f(T)| \leq \|T^{\oplus n} v\|.$$

The functional  $f : \mathcal{A} \rightarrow \mathbb{C}$  factors through  $\mathcal{H}^n$  such that

$$\mathcal{A} \rightarrow v\mathcal{H}^n \rightarrow \mathbb{C}.$$

□

## **Part III**

# **Spectral theory**



# Chapter 7

## Compact operators

$K(X, Y)$  is closed in  $B(X, Y)$ .  $K(X)$  is an ideal of  $B(X)$ . adjoint is  $K(X, Y) \rightarrow K(Y^*, X^*)$ . integral operators are compact. riesz operator, quasi-nilpotent operator.

### 7.1 Finite-rank operators

### 7.2 Fredholm operators

**7.1.** A bounded linear operator  $T : X \rightarrow Y$  between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

(a) A Fredholm operator  $T$  has closed range.

*Proof.* (a) Let  $C$  be a finite dimensional subspace of  $Y$  such that  $\text{im } T \oplus C = Y$ . Let  $\tilde{T} : X/\ker T \rightarrow Y$  be the induced operator of  $T$ . Define  $S : (X/\ker T) \oplus C \rightarrow Y$  such that  $S(x + \ker T, c) := \tilde{T}(x + \ker T) + c$ . Then,  $S$  is an topological isomorphism between Banach spaces by the open mapping theorem, so  $S(X/\ker T \oplus \{0\}) = \text{im } \tilde{T} = \text{im } T$  is closed.  $\square$

**7.2** (Atkinson's theorem). An operator  $T \in B(X, Y)$  is Fredholm if and only if there is  $S \in B(Y, X)$  such that  $TS - I$  and  $ST - I$  is finite rank.

**7.3** (Fredholm index). locally constant, in particular, continuous. composition makes the addition of indices.

### 7.3 Nuclear operators

tensor products

### Exercises

**7.4.** If  $T : L^2([0, 1]) \rightarrow L^2([0, 1])$  is a compact operator, then for any  $\varepsilon > 0$  there is a constant  $C_\varepsilon > 0$  such that

$$\|Tf\|_{L^2} \lesssim \varepsilon \|f\|_{L^2} + C_\varepsilon \|f\|_{L^1}.$$

*Proof.* Suppose there is  $\varepsilon > 0$  such that we have sequence  $f_n \in L^2$  satisfying  $\|f_n\|_2 = 1$  and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of  $T$ , there is a subsequence  $Tf_{n_k}$  converges to  $g \neq 0$  in  $L^2$ . Then,  $\|f_{n_k}\|_1 \rightarrow 0$  implies  $f_{n_k} \rightarrow 0$  weakly in  $L^2$ , hence also for  $Tf_{n_k}$ . It means  $g = 0$ , which contradicts to the assumption.  $\square$

## Chapter 8

# Normal operators

### 8.1 Spectral theorem for compact normal operators

There is an orthonormal basis  $E \subset H$  such that

$$T = \sum_{e \in E} \lambda_e |e\rangle \langle e|.$$

### 8.2 Spectral theorem for bounded normal operators

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

## Chapter 9

# Unbounded operators

Kato-Rellich theorem

## **Part IV**

# **Operator algebras**

# Chapter 10

## Banach algebras

### 10.1 Spectral theory of unital Banach algebras

**10.1** (Unital Banach algebras). (a) If  $\|a\| < 1$ , then  $1 - a$  is invertible. So  $\mathcal{A}^\times$  is open.

(b)  $\mathcal{A}^\times \rightarrow \mathcal{A} : a \mapsto a^{-1}$  is differentiable.

(c)  $\mathbb{C} \setminus \sigma(a) \rightarrow \mathcal{A} : \lambda \mapsto (\lambda - a)^{-1}$  is differentiable.

**10.2** (Vector-valued complex function theory). Let  $\Omega$  be an open subset of  $\mathbb{C}$  and  $X$  a Banach space. For a vector-valued function  $f : \Omega \rightarrow X$ , we say  $f$  is *differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} \mu^{-1}(f(\lambda) - f(\lambda_0))$$

exists in  $X$ , and *weakly differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} \mu^{-1}\langle f(\lambda) - f(\lambda_0), x^* \rangle$$

exists in  $\mathbb{C}$  for each  $x^* \in X^*$ . Then, the followings are all equivalent.

(a)  $f$  is differentiable.

(b)  $f$  is weakly differentiable.

(c) For each  $\lambda_0 \in \Omega$ , there is a sequence  $(x_k)_{k=0}^\infty$  such that the power series

$$\sum_{k=0}^{\infty} (\lambda - \lambda_0)^k x_k$$

converges to  $f(\lambda)$  absolutely and uniformly on any closed ball  $\overline{B(\lambda_0, r)} \subset \Omega$ .

**10.3** (Gelfand-Mazur).  $\sigma(a)$  is non-empty. In particular, if  $\mathcal{A}^\times = \mathcal{A} \setminus \{0\}$ , then  $\mathcal{A} \cong \mathbb{C}$ .

**10.4** (Beurling).

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} \leq \|a\|.$$

*Proof.* Let  $\lambda \in \mathbb{C}$  such that  $|\lambda| < r(a)^{-1}$ . Then we have  $\lambda^{-1} \notin \sigma(a)$  so that  $1 - \lambda a = \lambda(\lambda^{-1} - a)$  is invertible.

Then,  $1 - \lambda a = \sum_{i=0}^{\infty} (\lambda a)^i$ .

If  $|\lambda| < \|a\|^{-1} \leq r(a)^{-1}$ , then the inverse of  $1 - \lambda a$  is given by the power series. If  $|\lambda| < r(a)^{-1}$ , then we can only deduce the invertibility of  $1 - \lambda a$ . Complex function theory let us to write the inverse even if we have only  $|\lambda| < r(a)^{-1}$ . Also, the radius of convergence is exactly  $r(a)^{-1}$ .  $\square$

**10.5** (Spectrum in closed subalgebras). For fixed element, smaller the ambient algebra, less “holes” in the spectrum. Let  $\mathcal{B} \subset \mathcal{A}$  be a closed subalgebra containing  $1_{\mathcal{A}}$ . Note that  $\mathcal{B}$  may be unital even for  $1_{\mathcal{A}} \notin \mathcal{B}$ .

- (a)  $\mathcal{B}^\times$  is clopen in  $\mathcal{A}^\times \cap \mathcal{B}$ .

## 10.2 Ideals

**10.6** (Ideals). (a) If  $I$  is a left ideal, then  $\mathcal{A}/I$  is a left  $\mathcal{A}$ -module.

**10.7** (Modular left ideals). A left ideal  $I$  is called *modular* if there is  $e \in \mathcal{A}$  such that  $a - ae \in I$  for all  $a \in \mathcal{A}$ . The element  $e$  is called a *right modular unit* for  $I$ .

- (a)  $I$  is modular if and only if  $\mathcal{A}/I$  is unital(?).  
(b) A proper modular left ideal is contained in a maximal left ideal.  
(c)  $I$  is a maximal modular left ideal if and only if  $I$  is a modular maximal left ideal.  
(d) There is a non-modular maximal ideal in the disk algebra.

**10.8** (Closed ideals). (a) closure of proper left ideal is proper left.

- (b) maximal modular left ideal is closed.

**10.9** (Unitization). Let  $\mathcal{A}$  be an algebra. Recall that we always assume algebras are associative. Consider an embedding  $\mathcal{A} \rightarrow B(\mathcal{A}) : a \mapsto L_a$ , where  $L_a(b) = ab$ . Define

$$\tilde{\mathcal{A}} := \{ L_a + \lambda \text{id}_{B(\mathcal{A})} : a \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital  $\mathcal{A}$ .

- (a) If  $\mathcal{A}$  is normed, then  $\tilde{\mathcal{A}}$  is a normed algebra such that there is an isometric embedding  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ .  
(b) If  $\mathcal{A}$  is Banach, then  $\tilde{\mathcal{A}}$  is a Banach algebra.  
(c)  $\mathcal{A} \oplus \mathbb{C}$  is topologically isomorphic to  $\tilde{\mathcal{A}}$  as normed spaces.

*Proof.* (a) The space of bounded operators  $B(\mathcal{A})$  is a normed algebra. Then,  $\tilde{\mathcal{A}}$  is a normed  $*$ -algebra with induced norm

$$\|L_a + \lambda \text{id}_{B(\mathcal{A})}\| = \sup_{b \in \mathcal{A}} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then,  $\mathcal{A}$  is a normed  $*$ -subalgebra of  $\tilde{\mathcal{A}}$  because the norm and involution of  $\mathcal{A}$  agree with  $\tilde{\mathcal{A}}$ .

(b) Suppose  $(x_n, \lambda_n)$  is Cauchy in  $\tilde{\mathcal{A}}$ . Since  $\mathcal{A}$  is complete so that it is closed in  $\tilde{\mathcal{A}}$ , we can induce a norm on the quotient  $\tilde{\mathcal{A}}/\mathcal{A}$  so that the canonical projection is (uniformly) continuous so that  $\lambda_n$  is Cauchy. Also, the inequality  $\|x\| \leq \|(x, \lambda)\| + |\lambda|$  shows that  $x_n$  is Cauchy in  $\mathcal{A}$ .

Since a finite dimensional normed space is always Banach and  $\mathcal{A}$  is Banach,  $\lambda_n$  and  $x_n$  converge. Finally, the inequality  $\|(x, \lambda)\| \leq \|x\| + |\lambda|$  implies that  $(x_n, \lambda_n)$  converges.

- (c) Check the topology on  $\mathcal{A} \oplus \mathbb{C}$  in detail... □

unitization, homomorphisms, category(direct sum, product, etc.)

$B(\mathbb{C}^n)$  is simple, but  $B(X)$  is not simple.

### 10.3 Gelfand theory of commutative Banach algebras

also important spectrum for non-unital banach algebras Banach algebra of single generator semisimplicity and symmetricity

**10.10** (Character space). Let  $\mathcal{A}$  be a commutative Banach algebra. A *character* of  $\mathcal{A}$  is a non-zero homomorphism  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ . Denote by  $\sigma(\mathcal{A})$  the set of all characters of  $\mathcal{A}$ . We will show that all characters are bounded. Then, endow with the weak\* topology on  $\sigma(\mathcal{A})$  from the inclusion  $\sigma(\mathcal{A}) \subset \mathcal{A}^*$ . We call this space as the *character space* or the *spectrum* of  $\mathcal{A}$ . Let  $\varphi \in \sigma(\mathcal{A})$ .

- (a)  $\|\varphi\| = 1$ .
- (b) If  $\mathcal{A}$  is unital, then  $\sigma(\mathcal{A})$  is compact and Hausdorff.
- (c) Even if  $\mathcal{A}$  is non-unital,  $\sigma(\mathcal{A})$  is locally compact and Hausdorff.

**10.11** (Gelfand-Naimark representation). Let  $\mathcal{A}$  be a commutative Banach algebra.

$$\Gamma : \mathcal{A} \rightarrow C_0(\sigma(\mathcal{A})).$$

- (a)  $\Gamma(\mathcal{A})$  separates points.
- (b)  $\Gamma$  has closed range if
- (c)  $\Gamma$  is injective if
- (d)  $\Gamma$  is isometric if  $r(a) = \|a\|$  for all  $a \in \mathcal{A}$ .

### 10.4 Holomorphic functional calculus

Dunford-Reisz functional calculus

#### Exercises

**10.12.** Let  $\mathcal{A}$  be a unital algebra.

- (a)  $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ .
- (b) If  $\sigma(a)$  is non-empty, then  $\sigma(p(a)) = p(\sigma(a))$ .

*Proof.* (a) Intuitively, the inverse of  $1 - ab$  is  $c = 1 + ab + abab + \dots$ . Then,  $1 + bca = 1 + ba + baba + \dots$  is the inverse of  $1 - ba$ . □

$$C_b(\Omega) \ell^\infty(S) L^\infty(\Omega) B_b(\Omega) A(\mathbb{D}) B(X)$$

**10.13.** In  $C(\mathbb{R})$ , the modular ideals correspond to compact sets.

**10.14** (Disk algebra). (a) Every continuous homomorphism is an evaluation.

**10.15** (Polynomial convexity). (conway)

**10.16** (Inclusion relation on spectra). (a)  $\sigma(a + b) \subset \sigma(a) + \sigma(b)$  and  $\sigma(ab) \subset \sigma(a)\sigma(b)$  for unital cases.

- (b)  $\sigma(a^{-1}) = \sigma(a)^{-1}$  for unital cases.
- (c)  $r(a)^n = r(a^n)$ .

spectral radius is upper semi-continuous



# Chapter 11

## C\*-algebras

### 11.1 C\* identity

**11.1 (C\* identity).** A normed \*-algebra  $\mathcal{A}$  is called a C\*-algebra if

- (a)  $\mathcal{A}$  is Banach,
- (b)  $\mathcal{A}$  satisfies the C\*-identity:  $\|x^*x\| = \|x\|^2$ .

**11.2 (Unitization of C\*-algebras).**

$$(L_a + \lambda \text{id}_{B(\mathcal{A})})^* = L_{a^*} + \bar{\lambda} \text{id}_{B(\mathcal{A})}.$$

*Proof.* The C\*-identity easily follows from the following inequality:

$$\begin{aligned} \|(x, \lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\ &= \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\| \\ &= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y)\| \\ &\leq \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y\| \\ &= \|(x, \lambda)^*(x, \lambda)\|. \end{aligned}$$

□

**11.3 (Spectra of normal elements).** Let  $\mathcal{A}$  be a C\*-algebra.

- (a) If  $a \in \mathcal{A}$  is unitary, then  $\sigma(a) \subset \mathbb{T}$ .
- (b) If  $a \in \mathcal{A}$  is self-adjoint, then  $\sigma(a) \subset \mathbb{R}$ .

*Proof.* (a) (b) By the holomorphic functional calculus,

$$e^{itx} = \sum_{n=1}^{\infty} \frac{(itx)^n}{n!}.$$

Since the involution is continuous,

$$(e^{itx})^* = \sum_{n=1}^{\infty} \frac{(-itx)^n}{n!} = e^{-itx},$$

so we have  $\|e^{itx}\|^2 = \|e^{itx}e^{-itx}\| = 1$ . Then, the inequality

$$1 = \|e^{itx}\| \geq |h(e^{itx})| = |e^{ith(x)}| = e^{-t \text{Im} h(x)}$$

proves  $h(x) \in \mathbb{R}$ .

□

## 11.2 Continuous functional calculus

**11.4** (Gelfand-Naimark representation for  $C^*$ -algebras). For a commutative unital  $C^*$ -algebra  $\mathcal{A}$ , consider the Gelfand transform  $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ .

- (a)  $\Gamma$  is a  $*$ -homomorphism.
- (b)  $\Gamma$  is an isometry.
- (c)  $\Gamma$  is a  $*$ -isomorphism.

*Proof.* (a)

(b) Note that we have

$$\|\hat{x}\| = \sup_{h \in \sigma(\mathcal{A})} |\hat{x}(h)| = \sup_{h \in \sigma(\mathcal{A})} |h(x)| = r(x).$$

For self adjoint  $x \in \mathcal{A}$ , since we have  $\|x\|^2 = \|x^*x\| = \|x^2\|$ , the spectral radius coincides with the norm by the Gelfand formula for spectral radius in Banach algebras:

$$r(x) = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

Hence

$$\|x\|^2 = \|x^*x\| = \|\widehat{x^*x}\| = \|\hat{x}^*\hat{x}\| = \|\hat{x}\|^2$$

for arbitrary  $x \in \mathcal{A}$ .

$\Gamma(\mathcal{A})$  is a unital  $*$ -subalgebra of  $C(\sigma(\mathcal{A}))$ , and it separates points by definition. By the Stone-Weierstrass theorem,  $\Gamma(\mathcal{A})$  is dense in  $C(\sigma(\mathcal{A}))$ . The step 2 shows that  $\Gamma(\mathcal{A})$  is complete and hence closed so that  $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A}))$ .  $\square$

**11.5** (Finitely generated  $C^*$ -algebras). joint spectrum.

**11.6** (Continuous functional calculus). 1.  $\text{id} \mapsto a$ , 2.  $(f+g)(a) = f(a) + g(a)$ ,  $(fg)(a) = f(a)g(a)$ , 3.  $(f \circ g)(a) = f(g(a))$ .

We have shown unitary element has spectrum in the circle, and self-adjoint element has spectrum in real line. The converses of these two statements also hold if we assume  $a$  is normal.

## 11.3 Positive linear functionals

**11.7.** (a) If  $a, b \geq 0$ , then  $a + b \geq 0$ .

(b) If  $a^*a \leq 0$ , then  $a^*a = 0$ .

(c)  $a^*a \geq 0$  for all  $a \in \mathcal{A}$ .

**11.8** (Operator monotone functions). (a) inverse

(b) conjugation

**11.9** (Operator monotonicity of square and commutativity). Let  $\mathcal{A}$  be a  $C^*$ -algebra in which the square function is operator monotone, that is,  $0 \leq a \leq b$  implies  $a^2 \leq b^2$  for any positive elements  $a$  and  $b$  in  $\mathcal{A}$ . We are going to show that  $\mathcal{A}$  is necessarily commutative. Let  $a$  and  $b$  denote arbitrary positive elements of  $\mathcal{A}$ .

(a) Show that  $ab + ba \geq 0$ .

(b) Let  $ab = c + id$  where  $c$  and  $d$  are self adjoints. Show that  $d^2 \leq c^2$ .

(c) Suppose  $\lambda > 0$  satisfies  $\lambda d^2 \leq c^2$ . Show that  $c^2 d^2 + d^2 c^2 - 2\lambda d^4 \geq 0$ .

- (d) Show that  $\lambda(cd + dc)^2 \leq (c^2 - d^2)^2$ .
- (e) Show that  $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \leq c^2$  and deduce  $d = 0$ .
- (f) Extend the result for general exponent:  $\mathcal{A}$  is commutative if  $f(x) = x^\beta$  is operator monotone for  $\beta > 1$ .

11.10 (Injective  $*$ -homomorphism is an isometry). SS

## 11.4 Representation theory

## 11.5 Gelfand-Naimark-Siegel representation

11.11 (States on unitization). Let  $\mathcal{A}$  and  $\tilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$  be a  $C^*$ -algebra and its unitization respectively. Let  $\tilde{\rho} = \rho \oplus \lambda$  be a bounded linear functional on  $\tilde{\mathcal{A}}$ , where  $\rho \in \mathcal{A}^*$  and  $\lambda \in \mathbb{C}^* = \mathbb{C}$ .

- (a)  $\tilde{\rho}$  is positive if and only if  $\lambda \geq 0$  and  $0 \leq \rho \leq \lambda$ .
- (b)  $\tilde{\rho}$  is a state if and only if  $\lambda = 1$  and  $\rho$  is positive with  $\|\rho\| \leq 1$ .
- (c)  $\tilde{\rho}$  is a pure state if and only if  $\lambda = 1$  and  $\rho$  is either a pure state or zero.

## Exercises

11.12. A  $C^*$ -algebra is commutative if and only if a function  $f(x) = \frac{x}{1+x}$  is operator subadditive.

## Chapter 12

# Von Neumann algebras

### 12.1 The double commutant theorem

**Theorem 12.1.1** (Double commutant theorem). *Let  $\mathcal{A}$  be a  $C^*$ -subalgebra of  $B(H)$  containing  $\text{id}_H$ .*

- (a)  $\mathcal{A}'$  is strongly closed.
- (b) For  $b \in \mathcal{A}''$  and  $\xi \in H$ , there is a sequence  $a_n \in \mathcal{A}$  such that  $a_n(\xi) \rightarrow b(\xi)$ .
- (c) For  $b \in \mathcal{A}''$  and  $\xi_1, \dots, \xi_m \in H$ , there is a sequence  $a_n \in \mathcal{A}$  such that  $a_n(\xi_i) \rightarrow b(\xi_i)$  for all  $i$ .
- (d)  $\mathcal{A}$  is strongly dense in  $\mathcal{A}''$ .
- (e)  $\mathcal{A}$  is von Neumann algebra if and only if  $\mathcal{A} = \mathcal{A}''$ .

*Proof.* (b) Let  $K$  be the cyclic subspace  $\overline{\mathcal{A}\xi}$  of  $\xi$  in  $H$  and  $p$  its orthogonal projection. We will show  $b\xi \in K$ . Note that  $\xi \in K$  due to the assumption  $\text{id}_H \in \mathcal{A}$ .

For every  $a \in \mathcal{A}$ , we have  $aK \subset K$  because the multiplication by  $a$  is continuous on  $H$ , and  $a^*K \subset K$  because  $\mathcal{A}$  is self-adjoint. It means that  $K$  reduces all  $a \in \mathcal{A}$ , and then  $ap = pa$  implies  $bp = pb$ , so  $K$  also reduces  $b$ . Therefore,  $bK \subset K$  proves  $b\xi \in K$ .

- (e) Since  $\overline{\mathcal{A}}^{\text{wot}}$  is closed convex,  $\overline{\mathcal{A}}^{\text{sot}} = \overline{\mathcal{A}}^{\text{wot}}$ . Also,  $\mathcal{A}''$  is weakly closed,  $\overline{\mathcal{A}}^{\text{wot}} \subset \mathcal{A}''$ . □

### 12.2 The Kaplansky density theorem

### 12.3 Borel functional calculus

resolution of identity

normal operator theories: multiplicity, invariant subspaces

### 12.4 Traces

Every trace of factor is faithful

**12.1.** Normal states is a state in which the monotone convergence theorem holds. Precisely, a state  $\rho$  is *normal* if a monotone net  $a_\alpha$  strongly converges to  $a$  then  $\rho(a_\alpha) \rightarrow \rho(a)$ .