Abstract Harmonic Analysis

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Part I

Fourier analysis on groups

Chapter 1

Locally compact groups

1.1 Topological groups

1.2 Haar measures

1.1 (Non-σ-finite measures)**.** Following technical issues are important

(a) Positive linear functionals on C_c

(b) The Fubini theorem

(c) The Radon-Nikodym theorem

(d) The dual space of L^1 space

1.2 (Radon measures)**.** Let Ω be a locally compact Hausdorff space. A *Radon measure* is a Borel measure μ on Ω such that

(i) μ is outer regular for every Borel set: $\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\}$ for Borel E,

(ii) μ is inner regular for every open set: $\mu(U) = \inf\{\mu(K) : K \subset U,\, K \text{ compact}\}$ for open U,

(iii) μ is locally finite.

(a) A σ-finite Radon measure is regular.

(b) If every open subset of Ω is σ-compact, then a locally finite Borel measure is Radon.

(c) $C_c(\Omega)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

1.3 (Riesz-Markov-Kakutani representation theorem for C_c)**.** Let Ω be a locally compact Hausdorff space and consider the following map:

$$\begin{array}{ccc} \{\text{Radon measures on } \Omega\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_c(\Omega, \mathbb{R})\}, \\ \mu & \mapsto & (f \mapsto \int f\, d\mu). \end{array}$$

(a) a

1.4 (Existence of the Haar measure)**.**

1.3 Group algebras

1.5 (Modular functions)**.**

1.6 (Convolution)**.**

1.7 (Positive definite functions)**.** Bochner theorem

1.8 (Fourier-Stieltjes algebra)**.**

1.9 (GNS construction for locally compact groups)**.** Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

$$\begin{tikzcd}
G \arrow[r] & M(G) \\
L_1(G) \arrow[ur, hook] \arrow[r, hook] \arrow[d, dashed, "*"] & C^*(G) \arrow[r, two heads] \arrow[d, dashed, "*"] & C_r^*(G) \arrow[r, hook] \arrow[d, dashed, "*"] & L(G) \arrow[d, dashed, "* \text{ with } \sigma w"] \\
L^\infty(G) & B(G) \arrow[l, hook'] & C_r^*(G)^* \arrow[l, hook'] & A(G) \arrow[l, hook'] \\
& C_0(G) \arrow[ul, hook']
\end{tikzcd}$$

1.10 (Uniformly continuous functions)**.** G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$.

1.4 Pontryagin duality

1.11 (Dual group)**.**

1.12 (Fourier inversion theorem)**.**

1.13 (Plancherel's theorem)**.**

1.5 Structure theorems

Exercises

1.14.

Problems

1. Let Ω be a topological space. For every positive linear functional I on $C_c(\Omega, \mathbb{R})$, show that there exists a Borel measure μ on Ω such that $I(f) = \int f\, d\mu$ for all $f \in C_c(\Omega, \mathbb{R})$. (Hint: Consider the uncountable wedge sum of circles as an example.)

Solution. 1. The constructed Carathéodory measure μ on Ω is outer regular Borel measure, but we do not have local finiteness. Everything is same to when Ω is locally compact Hausdorff except that $\mu(\operatorname{supp} f)$ may be infinite. Now it is enough to show $I(\min\{f, \frac{1}{n}\})$ converges to zero as $n \to \infty$ for $f \in C_c(\Omega, [0, 1])$.

Let $U := f^{-1}((0, 1])$. For $g \in C_0(U, [0, 1])$, it clearly has compact support, and and it is also continuous because $g^{-1}((a, 1])$ is open in U and $g^{-1}([a, 1])$ is closed in K for any $0 < a \leq 1$, so that we have $C_0(U) \subset C_c(X)$. We also have $f_1 \in C_0(U)$ since $f_1^{-1}([\varepsilon, 1])$ is a compact set in U for every $\varepsilon > 0$. Therefore, I is a positive linear functional on $C_0(U)$. Since a positive linear map between C*-algebras is bounded, there is a constant C such that $I(g) \leq C$ for all $g \in C_0(U, [0, 1])$, and it proves $I(f_1) \leq C/n \to 0$ as $n \to \infty$. Therefore, $I(f) = \int f \, d\mu$. $\square$

1.6 Spectral synthesis

Chapter 2

Representation theory

2.1 (Schur's lemma).

2.2 (Operator-valued Fourier transform).

2.1 Group C*-algerbas

- How can we describe L^1-norm intrinsically?
- How can we show the equivalences between representations of G, $C_c(G)$, $L^1(G)$, and $C^*(G)$? Note that $\|\cdot\|_r \leq \|\cdot\| \leq \|\cdot\|_1$ and $L^1(G) \hookrightarrow C^*(G) \twoheadrightarrow C_r^*(G)$.
- How to show and interpret the inclusion $L^1(G) \hookrightarrow C_r^*(G)$.

Since it is not easy to introduce the quantum dual of G for now, we cannot discuss $L^1(G)$ as the Fourier algebra, the predual of the quantum group von Neumann algebra. ($A(G) = L(G)_* = L^1(\widehat{G})$ and also is the closed linear span of matrix coefficients of the left regular representation.)

Chapter 3

Compact groups

3.1 Peter-Weyl theorem

3.2 Tannaka-Krein duality

3.3 Example of compact Lie groups

Chapter 4

Mackey machine

4.1 Example of non-compact Lie groups

Wigner classification

Chapter 5

Kac algebras

Part II

Topological quantum groups

Chapter 6

Compact quantum groups

Chapter 7

Locally compact quantum groups

7.1 Multiplicative unitaries

7.1.1 Measures on locally compact Hausdorff spaces

compact closed set not containing infty open open not containing infty closed closed set containing infty

for a measure that "vanishes at infty" = tight two definitions of inner regularity is equivalent.

IRK -> IRF IRK + sigma finite -> tight

Thm. The measure contructed by RMK is lf and regular(cpt version). 1. open set is approx by cpt sets (by def of rho, if X is LCH) 2. meas set is approx by opn sets (by def of outer meas) 3. sigma finite set is approx by cpt sets (by thm)

Consider

$$
\begin{tikzcd}
\mathrm{regBorel}_{fin} \arrow[r, hook] \arrow[d, hook] & \mathrm{Borel}_{fin} \arrow[r] \arrow[d, hook] & \mathrm{Baire}_{fin} \arrow[r] \arrow[d, hook] & C_b^{*+} \arrow[d, two heads] & \\
\mathrm{regBorel}_{locfin} \arrow[r, hook] & \mathrm{Borel}_{locfin} \arrow[r] & \mathrm{Baire}_{locfin} \arrow[r] & C_c^{*+} \arrow[r, hook] & \text{pos lin on } C_c.
\end{tikzcd}
$$

for locally compact Hausdorff X.

$\mathrm{Borel}_{locfin} \to$ pos lin on C_c is surjective for all topological spaces.

$\mathrm{regBorel}_{fin} \to C_b^{*+}$ is injective for normal spaces.

$\mathrm{regBorel}_{locfin} \to C_c^{*+}$ is injective for locally compact Hausdorff spaces.(maybe)

Lemma 7.1.1. *Let μ be a Borel measure on a LCH X. Then, μ is inner regular on open sets iff*

$$\mu(U) = \|\mu\|_{C_c(U)^*}$$

for every open U in X.

Proof. ($\Leftarrow$) ($\geq$) For $f \in C_c(U)$, we have

$$\left|\int f\,d\mu\right| = \left|\int_U f\,d\mu\right| \leq \mu(U)\,\|f\|.$$

($\leq$) Since μ is inner regular on U, there is a compact set $K \subset U$ such that $\mu(U) - \mu(K) < \varepsilon$ (for the case $\mu(U) = \infty$, we can deal with separately). We can find a nonnegative function $f \in C_c(U)$ with $f|_K \equiv 1$ and $f \leq 1$ by the construction of Urysohn. Then, for all $\varepsilon > 0$ we have

$$\mu(U) < \mu(K) + \varepsilon \leq \int f\,d\mu + \varepsilon \leq \|\mu\|_{C_c^*(U)} + \varepsilon.$$

($\Rightarrow$) Let $f \in C_c(U)$ be a function such that $\|f\| = 1$ and

$$\mu(U) - \varepsilon < \int f\,d\mu.$$

Let $K = \mathrm{supp}(f)$. Then

$$\mu(K) \geq \int f > \mu(U) - \varepsilon.$$

$\square$

Proposition 7.1.2. *A Radon measure is inner regular on all σ-finite Borel sets.(Folland's)*

Proof. First we approximate Borel sets of finite measure, with compact sets. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U-E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned}\mu(K-V) &\geq \mu(K) - \mu(V)\\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U-E) + \frac{\varepsilon}{2}\right)\\ &\geq \mu(E) - \varepsilon.\end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^{\infty} K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^{\infty} \mu(K_n) > \sum_{n=1}^{\infty} \left(\mu(E_n) - \frac{\varepsilon}{2^n}\right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. □

Theorem 7.1.3. *If every open set in X is σ-compact(i.e. Borel sets and Baire sets coincide), then every locally finite Borel measure is regular.*

Proposition 7.1.4. *In a second countable space, every open set is σ-compact(i.e. Borel sets and Baire sets coincide).*

Two corollaries are presented as follows:

$$\begin{tikzcd}
\text{locally finite Borel regular} \arrow[r] & \text{Radon} \arrow[l, dashed, bend right, "X \text{ is } \sigma\text{-compact}"'] \arrow[r] & \text{locally finite Borel} \arrow[ll, dashed, bend left, "X \text{ is second countable}"]
\end{tikzcd}$$

7.1. Let X be compact. A positive linear functional ρ on $C(X)$ is bounded with norm $\rho(1)$.

Proof. Since $0 \leq \rho(\|f\| \pm f) = \|f\|\rho(1) \pm \rho(f)$, we have $|\rho(f)| \leq \rho(1)\|f\|$. □

7.2. Let X be a locally compact Hausdorff space.

(a) The Baire σ-algebra is generated by compact G_δ sets.

(b) If X is second countable, then every Baire set is Borel.

Solution. (b) (A second countable locally compact space is σ-compact.

Since X is σ-compact and Hausdorff, every closed set is a countable union of compact sets, so the Borel σ-algebra on X is generated by compact sets.)

Since locally compact Hausdorff space is regular, the Urysohn metrization implies X is metrizable, and every closed sets in metrizable space is G_δ set. □

7.3. Let X be compact. There is a map from the set of finite Baire measures to the set of positive linear functionals on $C(X)$.

Solution. A function in $C(X)$ is Baire measurable and bounded. Thus the integration is well-defined. □

7.4. Let X be compact. There is a map from the set of positive linear functionals on $C(X)$ to the set of finite regular Borel measures.

Solution. i. and ii. and iii. of Theorem 7.2. □

7.5. Let X be compact. Let ρ be a positive linear functional on $C(X)$. Let ν be the regular Borel measure associated to ρ. Then, $\rho(f) = \int f\,d\nu$.

Solution. iv. of Theorem 7.2. □

7.6. Let X be compact. Let ν be a finite regular Borel measure. Let ν' be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\,d\nu$. Then, $\nu = \nu'$ on Borel sets.

Solution. Theorem 7.8. □

The two results above establish the correspondence between positive linear functionals and regular Borel measures. The following is an additional topic: Borel extension of Baire measures.

7.7. Let X be compact. Let μ be a finite Baire measure. Let ν be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\,d\mu$. Then, $\mu = \nu$ on Baire sets.

Solution. Let μ, ν be finite Baire measures. Enough to show if $\int f\,d\mu = \int f\,d\nu$ then $\mu = \nu$ according to the preceding two results.

Enough to show the regularity of Baire measures. □