Lie group representation theory

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Contents

1 Day 1: April 11

- local properties: curvature, local homogeneous structure,
- global properties: compactness, finiteness of diameter, hausdorff

Theorem 1.1 (Bonnet-Meyers). *Let (X,g) be an n-dimensional complete Riemannian manifold whose Ricci curvature satisfies* $\mathrm{Ric}(X,g) \geq (n-1)k$ *for some $k>0$. Then, X is compact with diameter $\leq \pi/\sqrt{k}$.*

Definition 1.2 (Pseudo-Riemannian manifolds).

Example 1.3. The signature of a pseudo-Riemannian structure is locally constant. A pesudo-Riemannian manifold is called a Riemannian manifold if $q=0$ and a Lorentzian manifold if $q=1$, where q is the negtive component of the signature.

Example 1.4. S^2 does not, but S^3 admits a Lorentzian structure.

On a pseudo-Riemannian manifold, one can define sectional curvature, geodesics, and the Levi-Civita connection.

Theorem 1.5. *The group of isometries* $\mathrm{Isom}(X,g)$ *is a Lie group for any pseudo-Riemannian manifold (X,g).*

Remark 1.6. The group of biholomorphic maps of a complex manifold is not necessarily a Lie group.

Example 1.7. For example, biholomorphic maps for $\mathbb{C}^2$ gives rise to an infinite dimensional group.

Example 1.8 (Examples of Lorentzian manifolds). (1) (Minkowski space)

$$\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2).$$

(2) (De Sitter space)

$$dS^{n,1} := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}$$

has a Lorentzian structure induced from $\mathbb{R}^{n,1}$. Its sectional curvature is identically equal to one.

Definition 1.9. A (geodesically) complete Lorentzian manifold is called a de Sitter manifold if its sectional curvature is constantly one.

Example 1.10. The de Sitter space is an example of a de Sitter manifold. The de Sitter space is a model space in a sense of what we will explain later.

Theorem 1.11 (Calabi-Marlcus phenomenon). *Any de Sitter manifold is non-compact.*

Two key lemmas to prove Theorem 1.11. For simplicity, we consider the case that the dimension is ≥ 3.

Lemma 1.12. *Any de Sitter manifold is obtained as the quotient of the Sitter space by an isometric discontinuous group.*

Lemma 1.13. *Any such a discontinuous group is finite.*

The uniformization theorem states that a connected Riemann surface is classified into three classes. classification of group actions: which groups act on which spaces?

Basic notions for transformation on groups

Let L be a locally compact group, and X a locally compact topological space. For $S \subset X$, let $L_S := (S|S) = \{g \in L : gS \cap S \neq \varnothing\}$.

Definition 2.1. The L-action on X is called *free* if $\#L_{\{x\}} = 1$ whenever $\#S = 1$, *properly discontinuous* if $\#L_S < \infty$ whenever S is compact, and *proper* if L_S is compact whenever S is compact.

Example 2.2. Let M be a manifold, and X the universal covering. Then, the fundamental group $\Gamma := \pi_1(M)$ acts on X as the covering transformation also called the Deck transformation, and the action is free and properly discontinuous. The quotient space $\Gamma \setminus X$ is naturally diffeomorphic to X.

Exercise 2.3. Let X be a smooth manifold and Γ a discrete group acting on X freely and properly discontinuously. Show that the quotient space $\Gamma \setminus X$ is Hausdorff in the quotient topology. Show that $\Gamma \setminus X$ carries a smooth structure such that the quotient map is a smooth covering map.

Example 2.4. $X = \mathbb{R}$, $\Gamma = \mathbb{Z}$, $\Gamma \setminus X \cong S^1$.

Example 2.5. Let M be a compact Riemann surface with genus ≥ 2. Then, M is biholomorphic to $\Gamma \setminus \mathbb{H}$, where Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ acting on $\mathbb{H}$ by linear fractional transformations, and is isomorphic to $\pi_1(M)$.

Γ is called a discontinuous group for X if the Γ-action on X is free and properly discontinuous.

2 Day 2: April 18

Proper actions

Let L be a locally compact (Hausdorff) group continuously acting on X a locally compact Hausdorff space.

Theorem 2.3. *Suppose L properly act on X.*

(a) *$L \setminus X$ is Hausdorff.*

(b) *Every orbit is closed in X.*

(c) *Every isotropy group is compact.*

Remark 2.4. (b) and (c) are easily verified for actual cases.

Theorem 2.5. *Suppose (X, g) is a Riemannian manifold and G be the group of isometries. Let Γ be a subgroup of G. Then, Γ acts on G properly discontinuously if and only if Γ is a discrete subgroup. This equivalence may fail if X is pseudo-Riemannian.*

Proof. ($\Leftarrow$) Suppose Γ acts on X not properly discontinuous. Then, there eixst a compact subset S of X such that for some infinite sequences γ_k in Γ and s_k in S such that $\gamma_k s_k \in S$. We shall prove γ_k is not discrete in G. Let d denote the distance induced from the Riemannian manifold X. For $x \in X$, we set $M(x) := M_S(x) := \max_{a \in S} d(x, a) < \infty$. Then, $d(x, \gamma_k x) \leq d(x, \gamma_k s_k) + d(\gamma_k s_k, \gamma_k, x) \leq 2M(x)$. This shows that $\{\gamma_k x\}$ is a bounded set, so it contains a convergent subsequence for each x in X.

Take a countable dense subset x_j of X. By Cantor's diagonal argument, we may assume that $\gamma_k x_j$ converges as $k \to \infty$ for each j. For every compact C of X, we show $\gamma_k : X \to X$ converges uniformly as $k \to \infty$ using the idea of the Arzela-Ascoli. (Fix $\varepsilon > 0$ and take N such that for any $x \in X$ there is i satisfying $d(x, x_j) < \varepsilon/3$ for $j \leq N$. Take $R > 0$ such that $d(\gamma_k x_j, \gamma_{k'} x_j) < \varepsilon/3$ for $k, k' \geq R$.)

Take an increasing sequence $C_1 \subset C_2 \subset \cdots$ of compact subsets in X such that $C_j \uparrow X$. The above argument gives a family of maps $\gamma_{C_j} : C_j \to X$ such that $\gamma_{C_j}|_{C_i} = \gamma_{C_i}$ for $i < j$ by the uniqueness of convergence. This defines $\gamma : X \to X$ such that γ_k converges to γ compactly. We can check the resulting map γ is an isometry. Moreover, if we do the same thing with γ_k^{-1}, then we can see that γ is surjective. So γ_k is not discrete in G. $\square$

3 Day 3: April 25

Problem 4.1. Find a criterion for the triple (L, G, H) with locally compact groups $L \subset G \supset H$ such that the natural action on G/H by L is proper.

Example 4.2. $L := \{\mathrm{diag}(a, a^{-1}) : a > 0\}$, $G := \mathrm{SL}(2, \mathbb{R})$, $H := \{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{R}\}$. Then, G/H is identified with $\mathbb{R}^2 \setminus \{0\}$, via $gH \mapsto g(1,0)$, since H is the stabilizer group of an action of G on $\mathbb{R}^2 \setminus \{0\}$. The induced action of L on $\mathbb{R}^2 \setminus$ is not proper.

Lemma 4.3. *Let S be a compact subset of G. Then, $L_{\overline{S}} = L \cap SHS^{-1}$.*

Proposition 4.4. *If at least one of L on H is compact, then the L-action on G/H is proper.*

Proof. $L_{\overline{S}} = L \cap SHS^{-1}$ is compact for any compact subset S in G. □

Theorem 4.5 (von Neumann-Cartan)**.** *Suppose G is a Lie group, and H is a closed subgroup. Then, H and G/H carry a unique smooth structure such that the natural maps $H \hookrightarrow G \to G/H$ are smooth.*

Example 4.6.

Definition 4.7. The triple (L, G, H) is of *compact isotropy property* or *CI property* if the isotropy subgroup L_x is compact for any $x \in G/H$.

Proposition 4.8. *(L, G, H) is of CI iff (H, G, L) is of CI. If the L-action on G/H is proper, then (L, G, H) is of CI.*

Conjecture 4.9.(Lipsman's conjecture(1995)) Suppose G is a simply connected nilpotent Lie group, and H, L are connected closed subgroups. Then, L acts on G/H properly if and only if (L, G, H) is of CI.

Definition 4.10. Let $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$. $\mathfrak{g}$ is nilpotent iff $g^{(k)} = \{0\}$ for some k. $\mathfrak{g}$ is k-step nilpotent iff $g^{(k)} = \{0\}$. $\mathfrak{g}$ is one-step nilpotent iff it is abelian.

Example 4.11 (Heisenberg Lie algbera)**.** The $(2n+1)$-dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is two-step nilpotent.

$$\mathfrak{h}_7 \cong \{ \begin{bmatrix} 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \} \cong \mathrm{span}_{\mathbb{R}}\{x_1, x_2, x_3, \partial_1, \partial_2, \partial_3, 1\}.$$

It is contained in the Weyl algebra $\mathbb{R}[x, \partial]$.

Affirmative results to Lipsman's conjecture. Nasrin: 2-step, Yoshino: 3-step, Baklout et al: 3-step. But a counterexample was found by Yoshino (07?) for a 4-step nilpotent Lie group G.

Definition 4.14. For two subsets L and H in a locally compact group G, we write $L \pitchfork H$ if and only if $L \cap SHS^{-1}$ or equivalently $L \cap SHS$ is compact for any compact subset S in G, and we write $L \sim H$ if and only if there is a compact $S \subset G$ such that $L \subset SHS$ and $H \subset SLS$.

Lemma 4.15. *$L \pitchfork H$ iff $H \pitchfork L$.*

Proposition 4.16. *Assume that H and L are closed subgroups of G. Then, the L-action on G/H is proper iff $L \pitchfork H$.*

Corollary 4.17. *The L-action on G/H is proper iff the H-action on G/L is proper.*

In 4.2, the H-action on G/L, where G/L is the 2-dimensional de Sitter space dS^2, is not proper.

4 Day 4: May 2

Properness criterion - reductive case

Cartan decomposition (polar decomposition). We can view $\mathbb{R}^n$ as $(0,\infty) \times S^{n-1}$ with origin.

Theorem (Cartan decomposition). *Let $G := \mathrm{GL}(n,\mathbb{R})$ and $K := \mathrm{O}(n)$. Let $\mathfrak{p}$ be the set of symmetric matrices, $\mathfrak{a}$ be the set of diagonal matrices, and $\overline{\mathfrak{a}}_+$ be the diagonal matrices whose terms are non-increasing, in $M(n,\mathbb{R})$. Then, one has a diffeomorphism $K \times \mathfrak{p} G : (k,X) \mapsto ke^X$. Furthermore, one has a surjective map $K \times \mathfrak{a} \times K \to G : (k_1, X, k_2) \mapsto k_1 e^X k_2$, whose kernel includes the symmetric group S_n. Thus we have the Cartan projection $\mu : G \to \mathfrak{a}/S_n \cong \overline{\mathfrak{a}}_+$.*

Theorem. *For closed subsets L, H of $G = \mathrm{GL}(n,\mathbb{R})$, $L \sim H$ iff $\mu(L) \sim \mu(H)$, and $L \pitchfork H$ iff $\mu(L) \pitchfork \mu(H)$.*