Operator Algebra Seminar Note II

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Contents

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TODO

- lower semi-continuous weights, Thomsen's Combes theorem
- normal stinespring dilation
- cocycle, commutant, central, spatial derivatives
- approximation techniques: E_n, R_n for actions

1 October 18

1.1 Countably decomposable von Neumann algebras

Definition 1.1 (Countably decomposable von Neumann algebras). Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal non-zero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is.

Proposition 1.2. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M is countably decomposable.*

(b) *M admits a faithful normal state.*

(c) *M admits a faithful normal non-degenerate representation with a cyclic and separating vector.*

(d) *The unit ball of M is metrizable in one of the following topologies: σ-strong*, σ-strong, strong*, strong.*

Proof. (a)$\Longleftrightarrow$(b) Suppose M is countably decomposable. Let $\{\xi_i\} \subset H$ be a maximal family of unit vectors such that $\overline{M'\xi_i}$ are mutually orthogonal subspaces, taken by Zorn's lemma. If we let p_i be the projection on $\overline{M'\xi_1}$, then $p_i z p_i = z p_i$ for $z \in M'$ implies $p_i \in M'' = M$. By the assumption, the family $\{\xi_i\}$ is countable. Define a state ω of M such that

$$\omega(x) := \sum_{i=1}^{\infty} \omega_{2^{-i}\xi_i}(x), \qquad x \in M.$$

It converges due to $\|\omega_{2^{-i}\xi_i}\| = 2^{-i+1}$. It is normal since the sequence $(2^{-i}\xi_i)$ belongs to $\ell(\mathbb{N}, H)$, and it is faithful because $\omega(x^*x) = 0$ implies $x\xi_i = 0$ for all i, which deduces that $x = \sum_i xp_i = 0$.

Conversely, if ω is a faithful normal state, then for a mutually orthogonal family of non-zero projections $\{p_i\} \subset M$, we have

$$\{p_i\} = \bigcup_{n=1}^{\infty} \{p_i : \varphi(p_i) > n^{-1}\}$$

the countable union of finite sets. Thus M is countable decomposable.

(b)$\Longleftrightarrow$(c) Let ω be a faithful normal state of M. Consider any faithful normal nondegenerate representation in which ω is a vector state so that the corresponding vector is a separating vector by the faithfullness of ω. Examples include the GNS representation of ω, and the composition with the diagonal map $B(H) \to B(\ell^2(\mathbb{N}, H))$. Then, $\overline{M\Omega}$ admits a cyclic and separating vector Ω of M. The converse is immediate, i.e. the vector state ω_Ω is a faithful normal state of M.

(a)$\Longleftrightarrow$(d) Suppose M is countably decomposable and take $\{\xi_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ as we did. Define

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \|(x - y)\xi_i\|.$$

Clearly it generates a topology coarser than strong topology. It is also finer because if a bounded net x_α in M converges to zero in the metric d so that $x\xi_i \to 0$ for all i, then $H = \bigoplus_i M'\xi_i$ implies that for every $\xi \in H$ and $\varepsilon > 0$ we have $\|\xi - \sum_{k=1}^{n} z_k \xi_{i_k}\| < \varepsilon$ for some $z_k \in M'$ so that

$$\|x_\alpha \xi\| \le \|x_\alpha(\xi - \sum_{k=1}^{n} z_k \xi_{i_k})\| + \sum_{k=1}^{n} \|x_\alpha z_k \xi_{i_k}\| < \varepsilon + \sum_{k=1}^{n} \|z_k\| \|x_\alpha \xi_{i_k}\| \to \varepsilon.$$

Since on the bounded part the strong and σ-strong topologies coincide, the two topologies on the unit ball are metrizable. We can do similar for the strong* and the σ-strong* topologies.

Conversely, for a mutually orthogonal family of non-zero projections $\{p_i\}_{i \in I} \subset M$, since the net of finite partial sums $p_F := \sum_{i \in F} p_i$ is an non-decreasing net in the closed unit ball whose supremum is the identity of M, there is a convergent subsequence $p_{F_n} \uparrow 1$ by the metrizability, which implies $I = \bigcup_{n=1}^{\infty} F_n$, the countable union of finite sets. $\square$

Proposition 1.3. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M has the separable predual.*

(b) *M admits a faithful normal non-degenerate representation on a separable Hilbert space.*

(c) *M is countably decomposable and countably generated.*

(d) *The unit ball of M is metrizable in one of the following topologies: σ-weak, weak.*

1.2 Weights and semi-cyclic representations

Definition 1.4 (Weights)**.** Let M be a von Neumann algebra. A *weight* is a function $\varphi : M^+ \to [0,\infty]$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\lambda x) = \lambda\varphi(x), \qquad x,y \in M^+,\ \lambda \geq 0,$$

where we use the convention $0 \cdot \infty = 0$.

Definition 1.5. Let φ be a weight on a von Neumann algebra M. Define

$$\mathfrak{n} := \{x \in M : \varphi(x^*x) < \infty\}, \qquad \mathfrak{a} := \mathfrak{n}^* \cap \mathfrak{n}, \qquad \mathfrak{m} := \mathfrak{n}^*\mathfrak{n}.$$

It easily follows that $\mathfrak{n}$ is a left ideal of M with a sesquilinear form $\langle,\rangle_\varphi : \mathfrak{n} \times \mathfrak{n} \to \mathbb{C}$ such that

$$\langle x, y\rangle_\varphi := \varphi(y^*x), \qquad x,y \in \mathfrak{n},$$

$\mathfrak{a}$ is a $*$-subalgebra of M, and $\mathfrak{m}$ is a hereditary $*$-subalgebra of M with a positive linear functional $\varphi : \mathfrak{m} \to \mathbb{C}$ such that

$$\varphi(y^*x) := \sum_{k=0}^{3} i^k \varphi((x+i^k y)^*(x+i^k y)), \qquad x,y \in \mathfrak{n},$$

which extends the original φ.

Proposition 1.6. *Let φ be a weight on a von Neumann algebra M.*

(a) *Every element of $\mathfrak{m}^+$ can be written to be x^*x for some $x \in \mathfrak{n}$.*

(b) *Every element of $\mathfrak{m}$ can be written to be y^*x for some $x, y \in \mathfrak{n}$.*

Proof. (a) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}^+$ for some $x_i, y_i \in \mathfrak{n}$. The polarization writes

$$a = \frac{1}{4}\sum_{i=1}^{n}\sum_{k=0}^{3} i^k |x_i + i^k y_i|^2$$

and $a^* = a$ implies

$$a = \frac{1}{2}\sum_{i=1}^{n}(|x_i + y_i|^2 - |x_i - y_i|^2) \leq \frac{1}{2}\sum_{i=1}^{n}|x_i + y_i|^2$$

implies

$$\varphi(a) \leq \frac{1}{2}\sum_{i=1}^{n}\varphi(|x_i + y_i|^2) < \infty.$$

Therefore, if $x := a^{\frac{1}{2}} \in \mathfrak{n}$, then $a = x^*x$.

(b) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}$ for some $x_i, y_i \in \mathfrak{n}$. Let $x := (\sum_{i=1}^n x_i^* x_i)^{\frac{1}{2}} \in \mathfrak{n}$. Since $x_i^* x_i \leq x^2$, we have $s_i \in M$ such that $x_i = s_i x$. If we let $y := \sum_{i=1}^n s_i^* y_i \in \mathfrak{n}$, then

$$a = \sum_{i=1}^{n} y_i^* x_i = \sum_{i=1}^{n} y_i^* s_i x = (\sum_{i=1}^{n} s_i^* y_i)x = y^*x. \qquad \square$$

Definition 1.7 (Semi-cyclic representations). Let A be a C*-algebra. A *semi-cyclic representation* of A is a pair (π, Λ) of a representation $\pi : A \to B(H)$ and a linear map $\Lambda : \operatorname{dom}\Lambda \to H$ of dense image such that $\pi(a)\Lambda(b) = \Lambda(ab)$ for $x \in A$ and $b \in \operatorname{dom}\Lambda$, where $\operatorname{dom}\Lambda$ is a left ideal of A.

Proposition 1.8. *Let φ be a weight on a von Neumann algebra and (π, Λ) be the associated semi-cyclic representation to φ. For $h \in \pi(M)'$, the linear functional $\mathfrak{m} \to \mathbb{C} : y^*x \mapsto \langle h\Lambda(x), \Lambda(y)\rangle$ is well-defined.*

Proof. We first check the well-definedness on $\mathfrak{m}^+$. Let $x^*x = y^*y \in \mathfrak{m}^+$ for $x, y \in \mathfrak{n}$. Then, there is $s \in M$ such that $y = sx$ and $s = sp$, where p is the range projection of x, so $x^*(1 - s^*s)x = x^*x - y^*y = 0$ implies $0 = p(1 - s^*s)p = p - s^*s$ and $x = px = s^*sx = s^*y$. The well-definedness follows from

$$\langle z\Lambda(x), \Lambda(x)\rangle = \langle \pi(s)z\pi(s^*)\Lambda(y), \Lambda(y)\rangle = \langle z\Lambda(ss^*y), \Lambda(y)\rangle = \langle z\Lambda(y), \Lambda(y)\rangle.$$

The homogeneity is clear, so now we prove the addivitiy. Let $x^*x, y^*y \in \mathfrak{m}^+$ for some $x, y \in \mathfrak{n}$. Let $a := (x^*x + y^*y)^{\frac{1}{2}}$ and take $s, t \in M$ such that $x = sa$, $y = ta$, $s = sa$, and $t = ta$, where p is the range projection of a. Then, $a(1 - s^*s - t^*t)a = a^*a - x^*x - y^*y = 0$ implies $p(1 - s^*s - t^*t)p = p - s^*s - t^*t$. It follows that

$$\begin{aligned}\langle z\Lambda(a), \Lambda(a)\rangle &= \langle z\pi(p)\Lambda(a), \Lambda(a)\rangle \\ &= \langle z\pi(s^*s)\Lambda(a), \Lambda(a)\rangle + \langle z\pi(t^*t)\Lambda(a), \Lambda(a)\rangle \\ &= \langle z\Lambda(x), \Lambda(x)\rangle + \langle z\Lambda(y), \Lambda(y)\rangle.\end{aligned}$$

Now the $\Theta(\cdot, z)$ is linearly extendable to $\mathfrak{m}$. □

Proposition 1.9 (Radon-Nikodym affiliated with commutant). *Let φ be a weight on a von Neumann algebra and (π, Λ) be the associated semi-cyclic representation to φ. Let ψ be a ... There is h such that*

$$\psi(y^*x) = \langle h\Lambda(x), \Lambda(y)\rangle$$

In particular, if $l \in \mathfrak{m}^\#$ satisfies $|l| \le C\varphi$ for some $C > 0$, then there is $h \in \pi(M)'$ such that $\|h\| \le C$ and $l(y^*x) = \langle h\Lambda(x), \Lambda(y)\rangle$ for $x, y \in \mathfrak{n}$.

Proof. (a)

(b) The linear map θ^* is injective since Λ has dense range. Take $z \in \pi(M)'$ and consider $\theta^*(z)$, which maps x^*x to $\langle z\Lambda(x), \Lambda(x)\rangle$ for $x \in \mathfrak{n}$. The image is majorized by φ as

$$|\langle z\Lambda(x), \Lambda(x)\rangle| \le \|z\|\|\Lambda(x)\|^2 = \|z\|\varphi(x^*x).$$

Conversely, let $l \in \mathfrak{m}^\#$ is a linear functional majorized by φ, i.e. there is a constant $C > 0$ such that

$$|l(x^*x)| \le C\varphi(x^*x), \qquad x \in \mathfrak{n}.$$

Define a sesquilinear form $\sigma : \mathfrak{n} \times \mathfrak{n} \to \mathbb{C}$ such that $\sigma(x, y) := l(y^*x)$. It is well-defined after separation of $\mathfrak{n}$ and is bounded by the Cauhy-Schwartz inequality

$$|\sigma(x, y)|^2 = |l(y^*x)|^2 \le \|l(x^*x)\|\|l(y^*y)\| \le \varphi(x^*x)\varphi(y^*y) = \|\Lambda(x)\|^2\|\Lambda(y)\|^2.$$

Therefore, σ defines a bounded linear operator $z \in \pi(M)'$ such that

$$\sigma(x, y) = \langle z\Lambda(x), \Lambda(y)\rangle,$$

exactly meaning $\theta^*(z)(y^*x) = l(y^*x)$ for $x, y \in \mathfrak{n}$. □

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, "\Lambda"] \arrow[dd, "|\cdot|^2"'] & H \arrow[d, "\omega"] \\
 & B(H)_* \arrow[d, "\mathrm{res}"] \\
\mathfrak{m}^+ \arrow[r, "\theta"] & \pi(M)'_*.
\end{tikzcd}$$

In particular, for $x \in \mathfrak{n}^+$ we have

$$\|\theta(x^2)\| = \|\omega_{\Lambda(x)}\| = \|\Lambda(x)\|^2 = \varphi(x^2).$$

1.3 Normal weights and normal semi-cyclic representations

Definition 1.10 (Normal semi-cyclic representations)**.** Let M be a von Neumann algebra. We say a weight φ on M is *normal* if it is the supremum of normal positive linear functionals. We say a semi-cyclic representation (π, Λ) of M is *normal* if π is normal and Λ is closed with respect to σ-weak topology of M and weak topology of H.

Proposition 1.11. *Let φ be a weight on M. Let H_φ be the Hilbert space defined by the separation and completion of a sesquilinear form $\mathfrak{n}_\varphi \times \mathfrak{n}_\varphi \to \mathbb{C} : (x, y) \mapsto \varphi_\varphi(y^*x)$, and let $\Lambda_\varphi : \mathfrak{n}_\varphi \to H_\varphi$ be the canonical map.*

Let (π, Λ) be a semi-cyclic representation of M. Let

$$\mathcal{F}_\Lambda := \{\omega \in M_*^+ : \omega(x^*x) \le \|\Lambda(x)\|^2,\ x \in \operatorname{dom}\Lambda\}$$

and $\varphi_\Lambda(x^*x) := \sup_{\omega\in\mathcal{F}_\Lambda} \omega(x^*x)$ for $x \in M$. Then, it is clear that φ_Λ is a weight.

(a) *If φ is normal, then $(\pi_\varphi, \Lambda_\varphi)$ is a normal semi-cyclic representation such that $\varphi = \varphi_{(\pi_\varphi,\Lambda_\varphi)}$.*

(b) *If (π, Λ) is normal, then φ_Λ is a normal weight such that there is a unitary $u : H \to H_{\varphi_\Lambda}$ satisfying $\pi_{\varphi_\Lambda} = (\operatorname{Ad} u)\pi$ and $\Lambda_{\varphi_\Lambda} = u\Lambda$.*

(c) *For a normal φ, φ is faithful if and only if Λ is injective.*

(d) *For a normal φ, φ is semi-finite if and only if Λ is σ-weakly densely defined.*

Proof. (a) We show π_φ is normal. The proof is almost same as the normality of cyclic representation associated to normal states. Consider $\pi_\varphi^* : B(H)_* \to M^*$, which is bounded. Since

$$\pi_\varphi^*(\omega_{\Lambda_\varphi(y)})(x) = \langle \pi_\varphi(x)\Lambda_\varphi(y), \Lambda_\varphi(y)\rangle = \varphi(y^*xy), \qquad x \in M, y \in \mathfrak{n}_\varphi,$$

and φ is order continuous, we can see that $\pi_\varphi^*(\omega_{\Lambda_\varphi(y)})$ is also order continuous, so it is contained in M_*. Because the image of Λ_φ is dense, the linear span of states of the form $\omega_{\Lambda_\varphi(y)}$ for $y \in \mathfrak{n}_\varphi$ is norm-dense in $B(H)_*$ by the inequality

$$\|\omega_\xi - \omega_\eta\| \le \|\xi - \eta\|(\|\xi\| + \|\eta\|), \qquad \xi, \eta \in H.$$

Since M_* is norm-closed M^*, so $\pi_\varphi^*(B(H)_*) \to M_*$ and π_φ is normal.

For closedness of Λ,

(b) First we show $\operatorname{dom}\Lambda = \mathfrak{n}_{\varphi_{(\pi,\Lambda)}}$. One direction $\operatorname{dom}\Lambda \subset \mathfrak{n}_{\varphi_{(\pi,\Lambda)}}$ is clear because $x \in \operatorname{dom}\Lambda$ implies $\varphi_{(\pi,\Lambda)}(x^*x) \le \|\Lambda(x)\|^2$ by definition of $\mathcal{F}_{(\pi,\Lambda)}$ and $\varphi_{(\pi,\Lambda)}$. Conversely, we let $x \in \mathfrak{n}_{\varphi(\pi,\Lambda)}$ and claim $x \in \operatorname{dom}\Lambda$. We may assume $x \ge 0$ and $\varphi_{(\pi,\Lambda)}(x^2) = 1$. If we consider the σ-weak and weak topologies on $\operatorname{dom}\Lambda$ and H respectively, then since the graph of Λ is closed and the projection $\operatorname{dom}\Lambda \times H_1 \to \operatorname{dom}\Lambda$

is a closed map due to the tube lemma, the set $\{y \in \operatorname{dom}\Lambda : \|\Lambda(y)\| \leq 1\}$ and its positive part is σ-weakly closed. Since the square root is strongly continuous, if we temporarily consider a sufficiently large representation of M in which every normal state is a vector state so that a strong and σ-strong topology coincide on M, we can conclude that $C := \{y^2 : \|\Lambda(y)\|^2 \leq 1,\ y \in (\operatorname{dom}\Lambda)^+\}$ is σ-weakly closed with its convexity. If $x^2 \notin C$, then there is $\omega \in M_*^{sa}$ such that

$$\sup_{y^2 \in C} \omega(y^2) \leq 1 < \omega(x^2)$$

by the Hahn-Banach separation. Since all functional arguments in the above inequality are all positive, we may assume ω is positive. Then, for every $y \in \operatorname{dom}\Lambda$ we have

$$\omega(y^*y) = \|\Lambda(y)\|^2 \omega(\frac{y^*y}{\|\Lambda(y)\|^2}) \leq \|\Lambda(y)\|^2$$

because $y^*y/\|\Lambda(y)\|^2 \in C$, which means $\omega \in \mathcal{F}_{(\pi,\Lambda)}$. Thus, $\omega(x^2) \leq \varphi_{(\pi,\Lambda)}(x^2) = 1$ by definition of $\varphi_{(\pi,\Lambda)}$, which leads a contradiction, so $x^2 \in C$ and $x \in \operatorname{dom}\Lambda$.

Next, fixing $x \in \operatorname{dom}\Lambda = \mathfrak{n}_{\varphi_{(\pi,\Lambda)}}$, we can check $\|\Lambda(x)\|^2 = \varphi(x^*x) = \|\Lambda_{\varphi_{(\pi,\Lambda)}}(x)\|^2$. The rest is routine.

(c) Suppose φ is faithful. If $x \in \mathfrak{n}$ satisfies $\Lambda(x) = 0$, then $\varphi(x^*x) = \|\Lambda(x)\|^2 = 0$ implies $x = 0$, so Λ is injective.

Suppose Λ is injective. Take a non-zero $x \in \mathfrak{n}$ so that $\|\Lambda(x)\|^2 > 0$. We claim $\varphi(x^*x) \neq 0$.

(d) Also clear. $\square$

Theorem 1.12. *Let φ is a weight on a von Neumann algebra M. Then, φ is normal if and only if φ is σ-weakly lower semi-continuous.*

Proof. ($\Rightarrow$) Endow a partial order on the set of all weights. Then, every set of monotonically increasing subadditive homogeneous functions $\varphi : M^+ \to [0,\infty]$ always have its supremum given by its pointwise supremum. Since if φ is the supremum of σ-weakly lower semi-continuous φ_i, then

$$\varphi^{-1}([0,1]) = \bigcap_i \varphi_i^{-1}([0,1])$$

implies the σ-weak lower semi-continuity of φ. Conversly, the following theorem holds.

($\Leftarrow$) Let $F := \varphi^{-1}([0,1])$. It is a hereditary closed convex subset of the real locally convex space $(M^{sa}, \sigma w)$. Denote by the superscript circle the real polar set. Since

$$\mathcal{F}_\varphi = F^{\circ+} = \{\omega \in M_*^+ : \omega \leq \varphi\}, \qquad F^{\circ+\circ+} = \{x \in M^+ : \sup_{\omega \in \mathcal{F}_\varphi} \omega(x) \leq 1\},$$

it is enough to show $F^{\circ+\circ+} = F$. The positive part of the real polar of F is generally written as

$$F^{\circ+} = F^\circ \cap M_*^+ = F^\circ \cap (-M^+)^\circ = (F \cup -M^+)^\circ = (F - M^+)^\circ.$$

Consider a sequence of inclusions

$$F \subset \overline{F} \subset \overline{(F-M^+)^+} \subset \overline{(F-M^+)}^+ \subset (F-M^+)^{\circ\circ+} = F^{\circ+\circ+}.$$

The first, second, and forth inclusions are in fact full because F is closed, hereditary, and convex. The forth one uses the bipolar theorem. So we claim that the reverse of the third inclusion $\overline{(F-M^+)}^+ \subset \overline{(F-M^+)^+}$.

Let $x \in \overline{(F-M^+)}^+$. For arbitrary $\varepsilon > 0$, it is enough to show $f_\varepsilon(x) \in F - M^+$ because $x \geq 0$ implies $f_\varepsilon(x) \geq 0$ and $f_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. Let x_α be a net in $F - M^+$ that converges to x σ-strongly, which can be done by the convexity of $F - M^+$. Let y_α be a net in F such that $f_{\varepsilon/2}(x_\alpha) \leq y_\alpha$. Since $f_{\varepsilon/2}(y_\alpha)$

is a bounded net, we may assume it is σ-weakly convergent. By the σ-strong continuity of f_ε, the net $f_\varepsilon(x_\alpha)$ converges to $f_\varepsilon(x)$ σ-strongly, hence σ-weakly. Therefore, by the closedness of F,

$$f_\varepsilon(x) = \lim_\alpha f_\varepsilon(x_\alpha) \le \lim_\alpha f_{\varepsilon/2}(y_\alpha) \in F,$$

so we conclude $f_\varepsilon(x) \in F - M^+$. □

Lemma 1.13. *Let For $z \in \mathfrak{m}^{sa}$, we have*

$$\inf\{\varphi(a) : z \le a \in \mathfrak{m}^+\} \le \|\theta(z)\|.$$

In particular, for $x, y \in \mathfrak{n}^+$ and for any $\varepsilon > 0$ there is $a \in \mathfrak{m}^+$ such that $x^2 - y^2 \le a$ and

$$\varphi(a) \le \|\theta(x^2 - y^2)\| + \varepsilon = \|\omega_{\Lambda(x)} - \omega_{\Lambda(y)}\| + \varepsilon.$$

Proof. Denote by $p(z)$ the left-hand side of the inequality. Then, we can check $p : \mathfrak{m}^{sa} \to \mathbb{R}_{\ge 0}$ is a semi-norm such that $p(z) = \varphi(z)$ for $z \ge 0$. (If we take $p(z) := \varphi(z^+)$, then it seems to be dangerous when checking the sublinearity. I could not find the counterexample for $(z_1 + z_2)^+ \le z_1^+ + z_2^+$.)

Fix any non-zero $z_0 \in \mathfrak{m}^{sa}$. By the Hahn-Banach extension, there is an algebraic real linear functional $l : \mathfrak{m}^{sa} \to \mathbb{R}$ such that

$$l(z_0) = p(z_0), \qquad |l(z)| \le p(z), \qquad z \in \mathfrak{m}^{sa}.$$

Extend linearly l to be $l : \mathfrak{m} \to \mathbb{C}$. Since $|l(z)| \le \varphi(z)$ for $z \in \mathfrak{m}^+$, by the bounded Radon-Nikodym theorem, we have a corresponding operator $a \in \pi(M)'_1$ such that $\theta^*(a) = l$, hence

$$p(z_0) = l(z_0) = \theta^*(a)(z_0) = \theta(z_0)(a) \le \|\theta(z_0)\|.$$

Since $z_0 \in \mathfrak{m}^{sa}$ is aribtrary, we are done. □

Theorem 1.14. *Let φ is a weight on a von Neumann algebra M. Then, φ is σ-weakly lower semi-continuous if and only if φ is order continuous.*

Proof. ($\Rightarrow$) Easy

($\Leftarrow$) Let φ be an order continuous weight on M. We first claim that the associated semi-cyclic representation (π, Λ) to φ is normal if M is countably decomposable.

Suppose a sequence $x_n \in \mathfrak{n}_1$ satisfies $x_n \to x$ σ-strongly in M_1 and $\Lambda(x_n) \to \xi$ in H_1. Since $\Lambda(x_n)$ is Cauchy and bounded, $\omega_{\Lambda(x_n)}$ is also Cauchy in the norm topology of $B(H)_*$, so we may assume $\|\omega_{\Lambda(x_{n+1})} - \omega_{\Lambda(x_n)}\| < \varepsilon 2^{-n}$, for arbitrarily taken $\varepsilon > 0$. In order to dominate x_n with an monotone sequence, we take $a_n \in \mathfrak{m}^+$ such that $|x_{n+1}|^2 - |x_n|^2 \le a_n$ and $\varphi(a_n) < \varepsilon 2^{-n}$ using the previous lemma. Since the limit of the increasing sequence $\sum_{k=1}^n a_k$ in $n \to \infty$ may not exist, we introduce the cutoff $f_\varepsilon(t) := t(1 + \varepsilon t)^{-1}$. By taking the limit $\varepsilon \to 0$ on the inequality

$$\varphi(f_\varepsilon(|x|^2)) = \varphi(\lim_{n\to\infty} f_\varepsilon(|x_n|^2)) \le \varphi(\sup_n f_\varepsilon(|x_1|^2 + \sum_{k=1}^n a_k)) = \sup_n \varphi(f_\varepsilon(|x_1|^2 + \sum_{k=1}^n a_k)) < 1 + \varepsilon,$$

we have $x \in (\mathfrak{n}_\varphi)_1$ and $\Lambda(x) \in H_1$. Next, since $\Lambda(x_n - x)$ is Cauchy, we may assume $\|\omega_{\Lambda(x_n - x)} - \omega_{\Lambda(x_{n+1} - x)}\| < 2^{-n}$. Take $b_n \in \mathfrak{m}^+$ such that $|x_n - x|^2 - |x_{n+1} - x|^2 \le b_n$ and $\varphi(b_n) < 2^{-n}$. As we did previously, by taking $\varepsilon \to 0$ on the inequality

$$\varphi(f_\varepsilon(|x_m - x|^2)) = \varphi(\lim_{n\to\infty} f_\varepsilon(|x_m - x|^2 - |x_n - x|^2)) \le \varphi(\sup_n f_\varepsilon(\sum_{k=m}^n b_k)) = \sup_n \varphi(f_\varepsilon(\sum_{k=m}^n b_k)) < 2^{-(m-1)},$$

we have $\|\Lambda(x_n) - \Lambda(x)\|^2 = \varphi(|x_n - x|^2) \to 0$ and $\xi = \lim_{n\to\infty} \Lambda(x_n) = \Lambda(x)$. Thus (π, Λ) is normal.

In the spirit of the Krein-Šmulian theorem, the σ-weak lower semi-continuity is equivalent to the σ-weak closedness of the bounded part of the inverse image of the closed interval

$$\begin{aligned}\varphi^{-1}([0,1])_1 &= \{x \in M^+ : \varphi(x) \le 1,\ \|x\| \le 1\} \\ &= \{x^*x \in \mathfrak{m}^+ : \|\Lambda(x)\| \le 1,\ \|x\| \le 1\}.\end{aligned}$$

Since the σ-weak and strong closedness of a bounded convex set are equivalent and that the square root operation is strongly continuous, we are enough to show the square root

$$\varphi^{-1}([0,1])_1^{\frac{1}{2}} = \{x \in \mathfrak{n}^+ : \|\Lambda(x)\| \le 1,\ \|x\| \le 1\}$$

is σ-weakly closed. This set, if we denote the graph of $\Lambda : \mathfrak{n} \to H$ by Γ_Λ, is exactly the image of the positive part of the unit ball

$$(\Gamma_\Lambda)_1^+ = \{(x, \Lambda(x)) \in \mathfrak{n}^+ \oplus_\infty H : \|\Lambda(x)\| \le 1,\ \|x\| \le 1\}$$

under the projection $M \oplus_\infty H \to M$. Observe (x_α, ξ_α) converges to (x, ξ) weakly* in $M \oplus_\infty H \cong (M_* \oplus_1 H)^*$ if and only if $x_\alpha \to x$ σ-weakly and $\xi_\alpha \to \xi$ weakly. Since the graph of Λ and the closed ball in $M \oplus_\infty H$ is a closed with respect to the σ-weak topology of $\mathfrak{n}$ and the weak topology of H, their intersection $(\Gamma_\Lambda)_1^+$ is weakly* closed. Therefore, by its compactness, φ is σ-wealy lower semi-continuous done provided M is countably decomposable.

Now, let M be an arbitrary von Neumann algebra, and let φ be a order continuous weight on M. Let Σ be the set of all countably decomposable projections of M and let $M_0 := \bigcup_{p \in \Sigma} pMp$. The equivalent condition for $x \in M$ to belong to M_0 is that the left and right support projections of x are countably decomposable. Since then the left support projection p and the right support projection q of x are Murray-von Neumann equivalent so that there is a $*$-isomorphism between pMp and qMq, the countable decomposability is equivalent for p and q. It implies that M_0 is an algebraic ideal of M. (Moreover, M_0 is σ-weakly sequentially closed in M since if a sequence $x_n \in M_0$ converges to $x \in M$ σ-weakly, then for $p_n \in \Sigma$ such that $x_n = p_n x_n p_n$, we have $p \in \Sigma$ with $p_n \le p$ so that $x_n = p x_n p$ converges to $x = pxp$ σ-weakly. This fact is not needed in the proof.)

We first claim that $\varphi^{-1}([0,1])_1$ is relatively σ-weakly closed in M_0. Let $y \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0$ so that there is a net $y_\alpha \in \varphi^{-1}([0,1])_1$ converges σ-weakly to y, and there is $p \in \Sigma$ such that $pyp = y$. Note that the previous theorem states that $\varphi^{-1}([0,1]) \cap pMp$ is σ-weakly closed. Since $py_\alpha p$ is a net in $\varphi^{-1}([0,1])_1 \cap pMp$ that also converges σ-weakly to $pyp = y$, we have $y \in \varphi^{-1}([0,1])$. The claim proved.

We now claim that $\varphi^{-1}([0,1])_1$ is σ-weakly closed in M. Suppose a net $x_\alpha \in \varphi^{-1}([0,1])_1$ converges to $x \in M$ σ-weakly. Clearly $x \in M_1^+$. Let $\{p_i\}_{i \in I}$ be a maximal mutually orthogonal projections in Σ, and let $p_J := \sum_{i \in J} p_i$ for finite sets $J \subset I$ so that $\sup_J p_J = 1$. It clearly follows that for each α we have

$$x_\alpha^{\frac{1}{2}} p_J x_\alpha^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

Then, we can show easily with boundedness of x_α that

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w}.$$

Because $p_J \in M_0$ and M_0 is an ideal,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0.$$

By the above claim,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

By the complete additivity of φ, we finally obtain

$$x \in \varphi^{-1}([0,1])_1.$$

Therefore, $\varphi^{-1}([0,1])_1$ is σ-weakly closed. $\square$

2 November 10

2.1 Hilbert algebras

Definition 2.1 (Left Hilbert algebra). A *left Hilbert algebra* is a $*$-algebra A together with an inner product such that the involution is closable on H and the square A^2 is dense in H, where $H := \overline{A}$. A left Hilbert algebra A has the following additional devices:

(i) a closable densely defined anti-linear operator $S : A \to H$, defined by the involution,

(ii) a faithful non-degenerate $*$-homomorphism $\lambda : A \to B(H)$, defined by the left multiplication.

The *associated von Neumann algebra* of a left Hilbert algebra A is defined as $M := \lambda(A)''$.

Definition 2.2 (Right Hilbert algebra). Let A be a left Hilbert algebra. For $\eta \in H$, define:

(i) a linear functional $F\eta : A \to \mathbb{C}$ such that $F\eta(\xi) := \langle \eta, S\xi \rangle$ for $\xi \in A$,

(ii) a linear operator $\rho(\eta) : A \to H$ such that $\rho(\eta)\xi := \lambda(\xi)\eta$ for $\xi \in A$.

Define also:

$$D' := \{\eta \in H \mid F\eta \text{ is bounded}\}, \qquad B' := \{\eta \in H \mid \rho(\eta) \text{ is bounded}\}, \qquad A' := B' \cap D'.$$

Then, for $\eta \in D'$, we can identify $F\eta$ with a vector in H by the Riesz representation theorem, and for $\eta \in B'$, we can identify $\rho(\eta)$ with an element of $B(H)$.

Proposition 2.3. *Let A be a left Hilbert algebra.*

(a) *A' is a $*$-algebra such that $\eta^* := F\eta$ and $\eta\zeta := \rho(\zeta)\eta$.*

(b) *$\rho(A')A'$ is dense in H.*

(c) *A' is a right Hilbert algebra such that $\overline{A'} = H$.*

Proof. (a) Combining from (i) to (iv) in the below, the claim follows clearly:

(i) For $\eta \in D'$, we have $FF\eta = \eta$ in H by

$$FF\eta(\xi) = \langle F\eta, S\xi \rangle = \langle SS\xi, \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi \in A.$$

Therefore, if $\eta \in D'$, then $F\eta \in D'$.

(ii) For $\eta \in D'$, we have $\rho(F\eta) = \rho(\eta)^*$ on A by

$$\begin{aligned}\langle \rho(F\eta)\xi, \xi \rangle &= \langle \lambda(\xi)F\eta, \xi \rangle = \langle F\eta, \lambda(\xi)^*\xi \rangle = \langle S\lambda(\xi)^*\xi, \eta \rangle \\ &= \langle \lambda(\xi)^*\xi, \eta \rangle = \langle \xi, \lambda(\xi)\eta \rangle = \langle \xi, \rho(\eta)\xi \rangle = \langle \rho(\eta)^*\xi, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta \in A'$, then $F\eta \in B'$.

(iii) For $\eta, \zeta \in B'$, we have $F(\rho(\eta)^*\zeta) = \rho(\zeta)^*\eta$ in H by

$$\begin{aligned}\langle F(\rho(\eta)^*\zeta), \xi \rangle &= \langle S\xi, \rho(\eta)^*\zeta \rangle = \langle \rho(\eta)S\xi, \zeta \rangle = \langle \lambda(\xi)^*\eta, \zeta \rangle \\ &= \langle \eta, \lambda(\xi)\zeta \rangle = \langle \eta, \rho(\zeta)\xi \rangle = \langle \rho(\zeta)^*\eta, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta, \zeta \in B'$, then $\rho(\eta)^*\zeta \in D'$.

(iv) For $\eta \in B'$ and $\zeta \in H$, we have $\rho(\rho(\eta)^*\zeta) = \rho(\eta)^*\rho(\zeta)$ on A by

$$\begin{aligned}\langle \rho(\rho(\eta)^*\zeta)\xi, \xi \rangle &= \langle \lambda(\xi)\rho(\eta)^*\zeta, \xi \rangle = \langle \zeta, \rho(\eta)\lambda(\xi)^*\xi \rangle = \langle \zeta, \lambda(\lambda(\xi)^*\xi)\eta \rangle \\ &= \langle \zeta, \lambda((S\xi)\xi)\eta \rangle = \langle \zeta, \lambda(\xi)^*\lambda(\xi)\eta \rangle = \langle \lambda(\xi)\zeta, \lambda(\xi)\eta \rangle \\ &= \langle \rho(\zeta)\xi, \rho(\eta)\xi \rangle = \langle \rho(\eta)^*\rho(\zeta)\xi, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta, \zeta \in B'$, then $\rho(\eta)\zeta \in B'$.

(b) Since D' is dense in H by the closability of S, it suffices to verify the inclusion $D' \subset \overline{\rho(A')A'}$. Let $\eta \in D'$. Since $\rho(\eta)$ has densely defined adjoint $\rho(F\eta)$, we may assume $\rho(\eta)$ to be closed and densely defined by taking closure, so we can write down the polar decomposition

$$\rho(\eta) = vh = kv, \qquad h := |\rho(\eta)|, \quad k := |\rho(\eta)^*|.$$

To control the unboundedness of $\rho(\eta)$, we introduce $f \in C_c((0,\infty))^+$ to cutoff $\rho(\eta)$. Let $\acute{f}(t) := tf(t)$ and $\grave{f}(t) := t^{-1}f(t)$. Now we have $f(k) \in \rho(B')$ since $f(k)$ is bounded and

$$f(k) = f(vhv^*) = vf(h)v^* = v\grave{f}(h)\rho(\eta)^* = \rho\left(v\grave{f}(h)F\eta\right).$$

We also have $f(k)\eta \in B'$ since

$$\rho(f(k)\eta) = f(k)\rho(\eta) = \acute{f}(k)v$$

is bounded. Applying the above arguments for $f^{\frac{1}{3}} \in C_c((0,\infty))$,

$$f(k)\eta = (f(k)^{\frac{1}{3}})^3\eta \in \rho(B')^*\rho(B')\rho(B')^*B'.$$

Because $\rho(B')^*B' \subset A'$ and $\rho(B')^*\rho(B) \subset \rho(A')$ by (iii) and (iv) in the part (a), we have $f(k)\eta \in \rho(A')A'$.

If we construct a non-decreasing net $f_\alpha \in C_c((0,\infty))$ such that $\sup_\alpha f_\alpha = 1_{(0,\infty)}$, then the strong limit implies

$$\lim_\alpha f_\alpha(k)\eta = 1_{(0,\infty)}(k)\eta = s(k)\eta = s_l(\rho(\eta))\eta.$$

Here we use the non-degeneracy of λ to verify η belongs to the closure of the range of $\rho(\eta)$, i.e. since M contains the identity operator on H, we have a net $\xi_\alpha \in A$ such that $\lambda(\xi_\alpha)$ converges to the identity strongly so that $\lambda(\xi_\alpha)\eta \to \eta$. It implies that $\eta \in \overline{\lambda(A)\eta} = \overline{\rho(\eta)A}$ and $s_l(\rho(\eta))\eta = \eta$. Therefore, $\eta = s_l(\rho(\eta))\eta \in \overline{\rho(A')A'}$.

(c) The involution $F : A' \to H$ is a closable densely defined anti-linear operator because A' is dense in H by (b) and the closability follows from the dense domain of its adjoint S. The right multiplication $\rho : A'^{\mathrm{op}} \to B(H)$ is a faithful non-degenerate $*$-homomorphism because $\rho(A')H$ is dense in H by (b) and the faithfulness follows from the non-degeneracy of λ. Therefore, A' is a right Hilbert algebra with $\overline{A}' = H$. $\square$

Corollary 2.4. $\rho(A')' = M$.

Proof. One direction is clear, i.e. $\rho(A') \subset M'$ implies $\rho(A')'' \subset M'$. Conversely, let $y \in M'^+$. Since $\rho : A'^{\mathrm{op}} \to B(H)$ is non-degenerate, there is a net $\eta_\alpha \in A'$ such that $\rho(\eta_\alpha)$ converges to the identity σ-weakly. Then,

$$\rho(\eta_\alpha)^*y\rho(\eta_\alpha) = \rho(y^{\frac{1}{2}}\eta_\alpha)^*\rho(y^{\frac{1}{2}}\eta_\alpha) \in \rho(B')^*\rho(B') \subset \rho(A')$$

converges to y σ-weakly, hence $y \in \rho(A')''$. $\square$

Definition 2.5 (Full Hilbert algebra)**.** Let A be a left Hilbert algebra. Symmetrically as above, starting from the right Hilbert algebra A', we can construct a left Hilbert algebra A''. We say A is *full* if $A = A''$.

Definition 2.6 (Modular operator and conjugation)**.** Let A be a left Hilbert algebra. Denote the polar decomposition of S by $S = J\Delta^{\frac{1}{2}}$. The unbounded operators Δ and J are called the *modular operator* and the *modular conjugation*.

Corollary 2.7. *From the polar decomposition theorem for unbounded (anti-)linear operators, we have*

(a) *S is injective with $S = S^{-1}$ and $D = \operatorname{dom} S = \operatorname{dom} \Delta^{\frac{1}{2}}$.*

(b) *F is injective with $F = F^{-1}$ and $D' = \operatorname{dom} F = \operatorname{dom} \Delta^{-\frac{1}{2}}$.*

(c) *Δ is an injective positive self-adjoint operator.*

(d) *J is a conjugation, i.e. an anti-linear isometric involution.*

(e) *$S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$, $F = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J$, and $J\Delta J = \Delta^{-1}$.*

2.2 Faithful semi-finite normal weights

Definition 2.8. Let φ be a weight on a von Neumann algebra M. We say φ is *faithful* if $\varphi(x^*x) = 0$ implies $x = 0$ for $x \in \mathfrak{n}$. We say φ is *semi-finite* if $\mathfrak{m}$ is σ-weakly dense in M. Recall that a weight φ on a von Neumann algebra M is normal if and only if it is obtained by the pointwise supremum of a set of normal positive linear functionals.

In the proofs of theorems of this section, the following diagram might be helpful:

$$\begin{array}{ccccccccc} \mathfrak{m} := \mathfrak{n}^*\mathfrak{n} & \subset & \mathfrak{a} := \mathfrak{n} \cap \mathfrak{n}^* & \subset & \mathfrak{n} & \subset & \pi(M) & \subset & B(H) \\ & & {\scriptstyle\lambda}\uparrow\downarrow{\scriptstyle\Lambda} & & {\scriptstyle\lambda}\uparrow\downarrow{\scriptstyle\Lambda} & & & & \\ & & A & \subset & B & \subset & H. & & \end{array}$$

Recall that for a weight φ on a von Neumann algebra M and its semi-cyclic representation (π, Λ) of M we have $\varphi(x^*x) = \|\Lambda(x)\|^2$ for $x \in \mathfrak{n}$.

Theorem 2.9. *Let M be a von Neumann algebra. If A is a full left Hilbert algebra together with a faithful normal non-degenerate representation $\pi : M \to B(H)$ such that $\lambda(A)'' = \pi(M)$, then*

$$\varphi(x^*x) := \begin{cases} \|\xi\|^2 & \text{if } \pi(x) = \lambda(\xi) \in \lambda(B), \\ \infty & \text{otherwise,} \end{cases}$$

is a faithful semi-finite normal weight on M.

Proof. We use the notation $x = \pi(x)$. We first check that the weight φ is well-defined. Let $x_1 = \lambda(\xi_1), x_2 = \lambda(\xi_2) \in \lambda(B)$ such that $x_1^*x_1 = x_2^*x_2$. Since $x_1, x_2 \in M$, we have a partial isometry $v \in M$ such that $x_2 = vx_1$ and $v^*v = s_l(x_1)$, and it is not diffcult to see $\xi_2 = v\xi_1$. As we know $s_l(x)\xi_1 = \xi_1$,

$$\|\xi_2\|^2 = \langle \xi_2, \xi_2\rangle = \langle v\xi_1, v\xi_1\rangle = \langle v^*v\xi_1, \xi_1\rangle = \langle \xi_1, \xi_1\rangle = \|\xi_1\|^2,$$

which proves the well-definedness.

With this weight φ, we can see

$$\mathfrak{n} = \lambda(B), \qquad \mathfrak{a} = \lambda(A), \qquad \mathfrak{m} = \lambda(B)^*\lambda(B).$$

The first one is by definition of φ, and the third one is by definition of $\mathfrak{m}$. Since A is full so that $A = B \cap D$, λ is injective, $\lambda(A)^* = \lambda(A)$, and $\lambda(D)^* = \lambda(D)$, we have $\lambda(A) = \lambda(B) \cap \lambda(D) = \lambda(B)^* \cap \lambda(D)$, which implies $\lambda(A) = \lambda(B) \cap \lambda(B)^* \cap \lambda(D)$. If $\xi_1, \xi_2 \in B$ satisfy $\lambda(\xi_1) = \lambda(\xi_2)^*$, then

$$\begin{aligned} S\xi_1(\rho(\eta)^*\zeta) &= \langle F\rho(\eta)^*\zeta, \xi_1\rangle = \langle \rho(\zeta)^*\eta, \xi_1\rangle = \langle \eta, \rho(\zeta)\xi_1\rangle = \langle \eta, \lambda(\xi_1)\zeta\rangle \\ &= \langle \lambda(\xi_2)\eta, \zeta\rangle = \langle \rho(\eta)\xi_2, \zeta\rangle = \langle \xi_2, \rho(\eta)^*\zeta\rangle, \qquad \eta, \zeta \in A'. \end{aligned}$$

We have $\xi_1 \in D$ by the density of A'^2 in H, so $\lambda(B) \cap \lambda(B)^* \subset \lambda(D)$, hence the second equality follows.

From now in the rest of proof, we will always denote $y = \rho(\eta)$ and $z = \rho(\zeta)$ for $y, z \in \mathfrak{n}'$. The weight φ is clearly faithful, and semi-finiteness follows from the assumption $\mathfrak{a}'' = \lambda(A)'' = M$ that a net e_α in $\mathfrak{a}$ convergent σ-strongly/weakly to the identity has a σ-weak limit $x = \lim_\alpha e_\alpha x e_\alpha \in \mathfrak{m}''$ for $x \in M$. To verify the normality of φ, we will show

$$\varphi(x^*x) = \sup_{y \in \mathfrak{n}'_1} \omega_\eta(x^*x), \qquad x \in \mathfrak{n},$$

where $\mathfrak{n}' := \rho(B')$.

($\geq$) We may assume $x = \lambda(\xi) \in \mathfrak{n} = \lambda(B)$ so that $\varphi(x^*x) < \infty$. Since the unit ball $\mathfrak{n}'_1$ has a net y_α that converges to id_H strongly by the Kaplansky density theorem, we have an inequality

$$\omega_{\eta_\alpha}(x^*x) = \|x\eta_\alpha\|^2 = \|\lambda(\xi)\eta_\alpha\|^2 = \|\rho(\eta_\alpha)\xi\|^2 = \|y_\alpha\xi\|^2 \leq \|\xi\|^2 = \varphi(x^*x),$$

in which the equality condition is attained at its limit.

($\le$) Suppose $x \in M$ is taken such that the right-hand side $\sup_{y\in\mathfrak{n}'_1}\omega_\eta(x^*x)$ is finite. If we show $x \in \mathfrak{n}$, then we are done from $\varphi(x^*x) < \infty$ by the previous argument. To motivate the strategy, consider the opposite weight

$$\varphi'(y^*y) := \begin{cases} \|\eta\|^2 & \text{if } y \in \rho(B'), \\ \infty & \text{otherwise,} \end{cases}$$

and the associated linear map

$$\theta'^* : M \to \mathfrak{m}'^{\#} : x^*x \mapsto (z^*y \mapsto \langle x^*x\eta, \zeta\rangle), \qquad y, z \in \mathfrak{n}',$$

where we can check $\mathfrak{m}' = \rho(B')^*\rho(B')$. The idea is to show a well-designed linear functional $l \in \mathfrak{m}'^{\#}$ such that $l = \theta'^*(x^*x)$ is contained in the image $\theta'^*(\mathfrak{m})$ using the assumption that the right-hand side is finite to verify $x \in \mathfrak{n}$.

Define a linear functional

$$l : \mathfrak{m}' \to \mathbb{C} : z^*y \mapsto \langle x^*x\eta, \zeta\rangle.$$

Then, by the assumption we have

$$\|l\| = \sup_{y\in\mathfrak{n}'_1}\langle x^*x\eta, \eta\rangle = \sup_{y\in\mathfrak{n}'_1}\omega_\eta(x^*x) < \infty,$$

and

$$|l(y)| \le \|l\| l(y^*y)^{\frac{1}{2}} = \|l\|\|x\eta\|, \qquad y \in \mathfrak{n}'$$

implies the well-definedness as well as boundedness of the linear functional $\overline{xH} \to \mathbb{C} : x\eta \mapsto l(y)$ for any $\eta \in H$, and it follows the existence of $\xi \in \overline{xH}$ such that

$$l(y) = \langle x\eta, \xi\rangle, \qquad y \in \mathfrak{n}'$$

by the Riesz representation theorem on $\overline{xH}$. We have $\lambda(\xi)\zeta \in \overline{xH}$ and

$$\begin{aligned}\langle x\eta, x\zeta\rangle &= l(z^*y) = \langle x\rho^{-1}(z^*y), \xi\rangle = \langle xz^*\eta, \xi\rangle \\ &= \langle z^*x\eta, \xi\rangle = \langle x\eta, z\xi\rangle = \langle x\eta, \rho(\zeta)\xi\rangle = \langle x\eta, \lambda(\xi)\zeta\rangle, \qquad y, z \in \mathfrak{n}',\end{aligned}$$

hence $x = \lambda(\xi)$. The vector ξ is left bounded by definition and $x = \lambda(\xi) \in \lambda(B) = \mathfrak{n}$. □

Theorem 2.10. *Let M be a von Neumann algebra. If φ is a faithful semi-finite normal weight on M and (π, Λ, H) is the associated semi-cyclic representation of M, then $A := \Lambda(\mathfrak{a})$ is a full left Hilbert algebra with*

$$\Lambda(x_1)\Lambda(x_2) := \Lambda(x_1x_2), \qquad \Lambda(x)^* := \Lambda(x^*),$$

such that $\lambda(A)'' = \pi(M)$.

Proof. We use the notation $x = \pi(x)$. It does not make any confusion because the semi-cyclic representation $\pi : M \to B(H)$ is always unital and is faithful due to the assumption that φ is faithful. We clearly see that A is a $*$-algebra and the left multiplication provides a $*$-homomorphism $\lambda : A \to B(H)$. By the construction of the semi-cyclic representation associated to φ, A is dense in H. We need to show the non-degeneracy of λ, the closability of the involution, and the fullness of A.

(non-degeneracy) Since φ is semi-finite, there is a net x_α in $\mathfrak{a}_1$ converges strongly to the identity of M by the Kaplansky density theorem. Then, it follows that λ is non-degenerate from

$$\lambda(\Lambda(x_\alpha))\Lambda(x) = \Lambda(x_\alpha x) = x_\alpha\Lambda(x) \to \Lambda(x), \qquad x \in \mathfrak{a}.$$

(closability) We need to prove the domain of the adjoint

$$D' := \{\eta \in H \mid A \to \mathbb{C} : \Lambda(x) \mapsto \langle \eta, \Lambda(x^*)\rangle \text{ is bounded}\}$$

is dense in H. Let

$$\mathcal{G} := \{\omega \in M_*^+ : (1+\varepsilon)\omega \le \varphi \text{ for some } \varepsilon > 0\}.$$

Note that the normality of φ says that $\varphi(x^*x) = \sup_{\omega\in\mathcal{G}} \omega(x^*x)$ for any $x \in M$. For each $\omega \in \mathcal{G}$, by the bounded Radon-Nikodym theorem, there is $h_\omega \in M'^+$ such that $\|h_\omega\| < 1$ and

$$\omega(x^*x) = \langle h_\omega \Lambda(x), \Lambda(x)\rangle, \qquad x \in \mathfrak{n}.$$

Also, if we take a net $x_\alpha \in \mathfrak{n}_1$ that converges σ-strongly to the identity of M using the strong density of $\mathfrak{n}$ in M, the Kaplansky density, and the coincidence of strong and the σ-strong topologies on the bounded part, then we have a σ-weak limit $\lim_{\alpha,\beta} |x_\alpha - x_\beta|^2 = 0$ so that by the normality of ω we obtain

$$\lim_{\alpha,\beta} \|h_\omega^{\frac{1}{2}}\Lambda(x_\alpha) - h_\omega^{\frac{1}{2}}\Lambda(x_\beta)\|^2 = \lim_{\alpha,\beta} \omega(|x_\alpha - x_\beta|^2) = 0.$$

Thus, the vector $\Lambda_\omega := \lim_\alpha h_\omega^{\frac{1}{2}}\Lambda(x_\alpha)$ can be defined, and in particular, we have $h_\omega^{\frac{1}{2}}\Lambda(x) = x\Lambda_\omega$ for $x \in \mathfrak{n}$ and $\omega = \omega_{\Lambda_\omega}$.

If $\eta := h_{\omega_2}^{\frac{1}{2}} y \Lambda_{\omega_1}$ for some $y \in M'$ and $\omega_1, \omega_2 \in \mathcal{G}$, then

$$\begin{aligned}
|\langle \eta, \Lambda(x^*)\rangle| &= |\langle h_{\omega_2}^{\frac{1}{2}} y\Lambda_{\omega_1}, \Lambda(x^*)\rangle| = |\langle y\Lambda_{\omega_1}, h_{\omega_2}^{\frac{1}{2}}\Lambda(x^*)\rangle| = |\langle y\Lambda_{\omega_1}, x^*\Lambda_{\omega_2}\rangle| \\
&= |\langle yx\Lambda_\omega, \Lambda_{\omega_2}\rangle| = |\langle y h_{\omega_1}^{\frac{1}{2}}\Lambda(x), \Lambda_{\omega_2}\rangle| = |\langle \Lambda(x), h_{\omega_1}^{\frac{1}{2}} y^*\Lambda_{\omega_2}\rangle| \\
&\le \|\Lambda(x)\| \|h_{\omega_1}^{\frac{1}{2}} y^*\Lambda_{\omega_2}\|, \qquad x \in \mathfrak{a},
\end{aligned}$$

which deduces that $\eta \in D'$. Therefore, it suffices to show the following space is dense in H:

$$\{h_{\omega_2}^{\frac{1}{2}} y\Lambda_{\omega_1} : \omega_1, \omega_2 \in \mathcal{G},\ y \in M'\}.$$

Thanks to the normality of φ, we can write

$$\begin{aligned}
\langle \Lambda(x), \Lambda(x)\rangle &= \|\Lambda(x)\|^2 = \varphi(x^*x) = \sup_{\omega\in\mathcal{G}} \omega(x^*x) \\
&= \sup_{\omega\in\mathcal{G}} \|x\Lambda_\omega\|^2 = \sup_{\omega\in\mathcal{G}} \|h_\omega^{\frac{1}{2}}\Lambda(x)\|^2 = \sup_{\omega\in\mathcal{G}} \langle h_\omega\Lambda(x), \Lambda(x)\rangle, \qquad x \in \mathfrak{a}.
\end{aligned}$$

Because A in H, for any $\xi \in H$ and $\varepsilon > 0$ there is $x \in \mathfrak{n} \cap \mathfrak{n}^*$ such that $\|\xi - \Lambda(x)\| < \varepsilon$, so the inequality

$$\langle (1-h_\omega)\xi, \xi\rangle \le \varepsilon(\|\xi\| + \|\Lambda(x)\|) + \langle (1-h_\omega)\Lambda(x), \Lambda(x)\rangle$$

deduces $\inf_{\omega\in\Phi} \langle (1-h_\omega)\xi, \xi\rangle = 0$ by limiting $\varepsilon \to 0$ and taking infinimum on $\omega \in \mathcal{G}$. In other words, for each $\xi \in H$ and $\varepsilon > 0$, we can find $\omega \in \mathcal{G}$ such that $\langle (1-h_\omega)\xi, \xi\rangle < \varepsilon$. At this point, we claim the set $\{h_\omega : \omega \in \mathcal{G}\}$ is upward directed. If the claim is true, then we can construct an increasing net ω_α in $\mathcal{G}$ such that h_{ω_α} converges weakly to the identity of M, and also strongly by the nature of increasing nets. To prove the claim, take $h_1 = h_{\omega_1}$ and $h_2 = h_{\omega_2}$ with $\omega_1, \omega_2 \in \mathcal{G}$. Introduce a operator monotone function $f(t) := t/(1+t)$ and its inverse $f^{-1}(t) = t/(1-t)$ to define

$$h_0 := f(f^{-1}(h_1) + f^{-1}(h_2)).$$

Then, we have $h_0 \ge h_1$, $h_0 \ge h_2$, and $\|h_0\| < 1$. Consider a linear functional

$$\omega_0 : \mathfrak{n} \to \mathbb{C} : x \mapsto \langle h_0\Lambda(x), \Lambda(x)\rangle.$$

Write

$$\begin{aligned}
\omega_0(x^*x) &\le \langle f^{-1}(h_1)\Lambda(x), \Lambda(x)\rangle + \langle f^{-1}(h_2)\Lambda(x), \Lambda(x)\rangle \\
&\le (1-\|h_1\|)^{-1}\langle h_1\Lambda(x), \Lambda(x)\rangle + (1-\|h_2\|)^{-1}\langle h_2\Lambda(x), \Lambda(x)\rangle \\
&= (1-\|h_1\|)^{-1}\omega_1(x^*x) + (1-\|h_2\|)^{-1}\omega_2(x^*x), \qquad x \in \mathfrak{n}.
\end{aligned}$$

Then, since ω_1 and ω_2 are normal, we can define $\Lambda_0 := \lim_\alpha h_0^{\frac{1}{2}}\Lambda(x_\alpha) \in H$ for a σ-strongly convergent net $x_\alpha \in \mathfrak{n}_1$ to the identity of M as we have taken above, and we have the vector functional $\omega_0 = \omega_{\Lambda_0}$. Henceforth, ω_0 is extended to a normal positive linear functional on the whole M, and finally the norm condition $\|h_0\| < 1$ tells us that $\omega_0 \in \mathcal{G}$, so the claim is true.

Now the problem is reduced to the density of $\{y\Lambda_\omega : \omega \in \mathcal{G},\ y \in M'\}$ in H. Let $p \in B(H)$ be the projection to the closure of this space. Then, $p\Lambda_\omega = \Lambda_\omega$ for every $\omega \in \mathcal{G}$. Since the space is left invariant under the action of the self-adjoint set M', we have $p \in M$. Then,

$$\varphi(1-p) = \sup_{\omega\in\mathcal{G}} \omega(1-p) = \sup_{\omega\in\mathcal{G}} \langle (1-p)\Lambda_\omega, \Lambda_\omega\rangle = 0$$

implies $p = 1$, hence the density.

(fullness) We have $\lambda(\Lambda(x)) = x$ for $x \in \mathfrak{a}$ since $\Lambda(\mathfrak{a}) = A$ is dense in H and

$$x_1\Lambda(x_2) = \Lambda(x_1x_2) = \Lambda(x_1)\Lambda(x_2) = \lambda(\Lambda(x_1))\Lambda(x_2), \qquad x_1, x_2 \in \mathfrak{n}\cap\mathfrak{n}^*.$$

Also we have for $\xi = \Lambda(x) \in A$ that

$$\Lambda(\lambda(\xi)) = \Lambda(\lambda(\Lambda(\xi))) = \Lambda(x) = \xi.$$

For $\xi \in B$ so that $\lambda(\xi) \in M$, since

$$\varphi(\lambda(\xi)^*\lambda(\xi)) = \|\Lambda(\lambda(\xi))\|^2 = \|\xi\|^2 < \infty,$$

we get $\lambda(B) \subset \mathfrak{n}$. Therefore, A is full by

$$\lambda(A'') = \lambda(B) \cap \lambda(B)^* \subset \mathfrak{a} = \lambda(A).$$

□

Corollary 2.11. *The operations giving a faithful semi-finite normal weight and a full left Hilbert algebra in the above two theorems are mutually inverses of each other.*

Proposition 2.12. *Every von Neumann algebra admits a faithful semi-finite normal weight.*

Proof. Let M be a von Neumann algebra and let $\{\omega_i\}_{i\in I}$ be a maximal family of normal states on M with orthogonal support projections $p_i := s(\omega_i)$. Here, the support projection $s(\omega)$ of a normal state ω is the minimal projection p such that $\omega(px) = \omega(x) = \omega(xp)$ for all $x \in M$. Since every countably decomposable projection p is a support of a normal state, a faithful normal state on pMp, we have $\sum_i p_i = 1$. Define a weight φ by

$$\varphi(x) := \sum_{i\in I} \omega_i(x) = \sup_{J\Subset I} \sum_{i\in J} \omega_i(x).$$

It is faithful because $\varphi(x) = 0$ with $x \geq 0$ means $\omega_i(x) = 0$ and $p_ixsp_i = 0$ for all i, and it implies

$$x^{\frac{1}{2}} = x^{\frac{1}{2}}\sum_i p_i = \sum_i x^{\frac{1}{2}}p_i = 0.$$

It is normal because it is the supremum of normal positive linear functionals $\omega_J = \sum_{i\in J}\omega_i$. It is semi-finite because $p_J \uparrow 1$ with $\varphi(p_J) < \infty$ as $J \to I$, where $p_J := \sum_{i\in J} p_i$ and J runs through finite subsets of I. □

2.3 Examples

Example 2.13 (Locally compact groups)**.** For a locally compact group G, the set $A := C_c(G)$ together with a left Haar measure on G has the following left Hilbert algebra structure

$$\langle \xi_1, \xi_2\rangle := \int \overline{\xi_2(s)}\xi_1(s)\,ds, \qquad (\xi_1\xi_2)(s) := \int_G \xi_1(t)\xi_2(t^{-1}s)\,dt, \qquad \xi^*(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}.$$

We have S, F, Δ, and J given by

$$S\xi(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}, \qquad F\xi(s) = \overline{\xi(s^{-1})},$$

$$\Delta\xi(s) = \Delta(s)\xi(s), \qquad J\xi(s) = \Delta(s)^{-\frac{1}{2}}\overline{\xi(s^{-1})},$$

and they have the following norm formulas

$$\|S\xi\|_2 = \|\Delta^{\frac{1}{2}}\xi\|_2, \quad \|F\xi\|_2 = \|\Delta^{-\frac{1}{2}}\xi\|_2, \quad \|S\xi\|_1 = \|\xi\|_1, \quad \|F\xi\|_1 = \|\Delta^{-1}\xi\|_1.$$

The left von Neumann algebra $\lambda(A)''$ is called the *group von Neumann algebra*.

For a locally compact abelian group G, the corresponding f.n.s. weight is a suitably normalized Haar measure on the Pontryagin dual group $\widehat{G}$, called the Plancherel measure, not the Haar measure on the original group G. For a locally compact non-abelian group G, there is no characterization of the corresponding f.n.s. weight as a measure because the left Hilbert algebra $(C_c(G), *)$ is not commutative.

Example 2.14 (Locally compact abelian groups)**.** If G is a locally compact abelian group, then $A = \mathcal{F}^{-1}(L^2(\widehat{G}) \cap L^\infty(\widehat{G}))$ is a full Hilbert algebra, where $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is the Fourier transform, such that $B = A$, $D = H = L^2(G)$.

Example 2.15 (Measure spaces)**.** If (X, μ) is a σ-finite measure space, then $L^2(X) \cap L^\infty(X)$ is a full Hilbert algebra.

Example 2.16 (Cyclic separating vector)**.** Let M be a countably decomposable von Neumann algebra and ω be a faithful normal state. If we consider the associated cyclic representation of ω, then we have an action of M on H together with a cyclic separating vector $\Omega \in H$. Then, $A := M\Omega$ has the following left Hilbert algebra structure:

$$\langle x\Omega, y\Omega\rangle \text{ is defined as it is,} \qquad (x\Omega)(y\Omega) := xy\Omega, \qquad (x\Omega)^* := x^*\Omega.$$

There is no specific description of Δ and J in general, but it is known that $\mathfrak{n} = \mathfrak{a} = \mathfrak{m} = M$ so that $A = B = M\Omega$ is full, and

$$D = \{c\Omega : c \in C(H) \text{ affiliated with } M \text{ such that } \Omega \in \operatorname{dom} c \cap \operatorname{dom} c^*\}.$$

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3.1 Pettis integral

Definition 3.1 (Properties of dual pairs). Let (X,F) be a dual pair. For example, if X is a topological vector space and F is a linear subspace of X^*, then (X,F) is a dual pair if and only if F is weakly* dense in X^*. Conversely, every dual pair (X,F) can be understood as (X,X^*) by endowing with the weak topology $\sigma(X,F)$ on X. Then, we say (X,F) has the *Krein property* if the closed balanced convex hull of a compact subset of X is compact in the topology $\sigma(X,F)$, and say (X,F) has the *Goldstine property* if X is $\beta(X,F_\beta)$-closed in the strong bidual $(F_\beta)^*_\beta$.

Remark. Let X a Banach space. The weak dual pair (X,X^*) satisfies the Krein property by the Krein-Šmulian theorem, and the Goldstine property by the closedness of X in X^{**}. Suppose there is a predual X_* of X, i.e. a norm closed subspace X_* of X^* such that the restriction of $(X^*)^* \to (X_*)^*$ on X gives rise to a isometric isomorphism. Then, the weak* dual pair (X,X_*) satisfies the Krein property by the fact that the closed convex hull of a bounded set is bounded, and the Golstine property because the norm topology and $\beta(X,(X_*)_\beta)$ coincide by the Goldstine theorem. In particular, a dual pair (X,F) with $F \subset X^*$ has the Goldstine property if and only if the closed unit ball $F_1 = F \cap X_1^*$ is weakly* dense in the closed ball X_1^*.

Proposition 3.2 (Well-definedness of Pettis integral). *Let $x : \Omega \to X$ be a $\sigma(X,F)$-bounded $\sigma(X,F)$-measurable function, where (Ω,μ) is a σ-finite measure space and (X,F) is a dual pair. Then, it determines a linear operator $F \to L^\infty(\mu)$ by definition. By the adjoint and restriction, we have a linear operator $\phi_x : L^1(\mu) \to F^\#$, which satisfies*

$$\langle \phi_x(f), x^* \rangle := \int_\Omega f(s)\langle x(s), x^*\rangle \, d\mu(s), \qquad f \in L^1(\mu),\ x^* \in F.$$

We usually write as

$$\phi_x(f) = \int_\Omega f(s)x(s)\, d\mu(s).$$

(a) *$\phi_x(L^1(\mu)) \subset (F_\beta)^*$ and ϕ_x is always weak-$\sigma((F_\beta)^*,F)$-continuous.*

(b) *Suppose (X,F) has the Krein property. If x is $\sigma(X,F)$-compactly valued, then $\phi_x(L^1(\mu)) \subset X$.*

(c) *Suppose (X,F) has the Krein and Goldstine property. Suppose Ω is a locally compact Hausdorff space with a Radon measure μ. If x is $\sigma(X,F)$-continuous, then $\phi_x(L^1(\mu)) \subset X$.*

(d) *Suppose we have $\phi_x(L^1(\mu)) \subset X$. Let Y be another topological vector space and G is a weakly* dense subspace of Y^*. If $T : X \to Y$ is a $\sigma(X,F)$-$\sigma(Y,G)$-continuous linear operator, then $T\phi_x = \phi_{T\circ x}$. In other words,*

$$T\int_\Omega f(s)x(s)\, d\mu(s) = \int_\Omega f(s)Tx(s)\, d\mu(s), \qquad f \in L^1(\mu).$$

(e) *Suppose we have $\phi_x(L^1(\mu)) \subset X$, (X,F) has the Goldstine property, and X is a Banach space. Then,*

$$\|\int f(s)x(s)\, d\mu(s)\| \le \int \|f(s)x(s)\|\, d\mu(s), \qquad f \in L^1(\mu).$$

Proof. (a) Let $B^* \subset F$ be a $\beta(F,X_\sigma)$-bounded set. For $x^* \in F$ we have an inequality

$$|\langle \phi_x(f), x^*\rangle| \le \int_\Omega |f(s)\langle x(s), x^*\rangle| \, d\mu(s) \le \|f\|_{L^1} \sup_{y\in x(\Omega)} |\langle y, x^*\rangle|,$$

and a bound

$$\sup_{x_*\in B^*}\sup_{y\in x(\Omega)}|\langle y,x^*\rangle|<\infty$$

due to the $\sigma(X,F)$-boundedness of $x(\Omega)$, so $\phi_x(f)\in(F_\beta)^*$. If $f_\alpha\in L^1(\mu)$ converges weakly to zero, then

$$\langle\phi_x(f_\alpha),x^*\rangle=\int_\Omega f(s)\langle x(s),x^*\rangle\,d\mu(s)\to 0,\qquad x^*\in F$$

because x is $\sigma(X,F)$-integrable so that $(s\mapsto\langle x(s),x^*\rangle)\in L^\infty(\mu)$, so the continuity of ϕ_x.

(b) Fix $p\in L^\infty(\mu)$ and let C be the $\sigma(X,F)$-closed balanced convex hull of $x(\Omega)\subset X$. Then C is $\sigma(X,F)$-compact by the Krein property. Since for every $x^*\in F$ we have

$$|\langle\phi_x(f),x^*\rangle|\le\int_\Omega|f(s)\langle x(s),x^*\rangle|\,d\mu(s)\le\|f\|_{L^1}\sup_{y\in x(\Omega)}|\langle y,x^*\rangle|\le\|f\|_{L^1}\sup_{y\in C}|\langle y,x^*\rangle|,$$

the linear functional $\phi_x(f)$ on F is continuous with respect to the Mackey topology $\tau(F,X)$, which is a dual topology so that $\phi_x(f)$ can be naturally identified with a vector in $(F_\tau)^*=X$.

(c) Fix $f\in L^1(\mu)$. By the tightness of μ, there is a sequence of compact sets $K_n\subset\Omega$ such that $\int_{\Omega\setminus K_n}|f(s)|\,d\mu(s)<n^{-1}$. Since for each $x^*\in F$ we have

$$|\langle\phi_x(f)-\phi_{x|_{K_n}}(f),x^*\rangle|\le\int_{\Omega\setminus K_n}|f(s)|\,d\mu(s)\cdot\sup_{s\in\Omega}|\langle x(s),x^*\rangle|<n^{-1}\sup_{y\in x(\Omega)}|\langle y,x^*\rangle|$$

so that

$$\sup_{x^*\in B^*}|\langle\phi_x(f)-\phi_{x|_{K_n}}(f),x^*\rangle|\le n^{-1}\sup_{x_*\in B^*}\sup_{y\in x(\Omega)}|\langle y,x^*\rangle|\to 0,\qquad n\to\infty,$$

which means that $\phi_{x|_{K_n}}(f)$ converges to $\phi_x(f)$ in $\beta((F_\beta)^*,F_\beta)$. Since $\phi_{x|_{K_n}}(f)\in X$ by the part (b) and X is closed in $\beta((F_\beta)^*,F_\beta)$ by the Goldstine property, we have $\phi_x(f)\in X$.

(d) By the continuity of T, the adjoint $T^*:G\to F$ is well-defined. The measurability of T and the existence of the adjoint T^* imply that the composition $T\circ x:\Omega\to Y$ is $\sigma(Y,G)$-bounded and $\sigma(Y,G)$-measurable, so the operator $\phi_{T\circ x}:L^1(\mu)\to G^\#$ is well-defined. Then,

$$\begin{aligned}\langle T\phi_x(f),y^*\rangle=\langle\phi_x(f),T^*y^*\rangle&=\int_\Omega f(s)\langle x(x),T^*y^*\rangle\,d\mu(s)\\&=\int_\Omega f(s)\langle Tx(s),y^*\rangle\,d\mu(s)=\langle\phi_{T\circ x}(f),y^*\rangle,\qquad f\in L^1(\mu),\ y^*\in G.\end{aligned}$$

In particular, $\phi_{T\circ x}:L^1(\mu)\to Y$.

(e) By the Goldstine property,

$$\begin{aligned}\|\int f(s)x(s)\,d\mu(s)\|=\sup_{x^*\in F_1}|\int f(s)x(s)\,d\mu(s)|&\le\sup_{x^*\in F_1}\int|f(s)x(s)|\,d\mu(s)\\&\le\int\sup_{x^*\in F_1}|f(s)x(s)|\,d\mu(s)\le\int\|f(s)x(s)\|\,d\mu(s).\end{aligned}$$ □

Proposition 3.3. *Let X be a Banach space together with a weakly* dense subspace F of X^* such that (X,F) satisfies the Krein and Goldstine property. A $\sigma(X,F)$-holomorphic function $x:\Omega\subset\mathbb{C}\to X$ is holomorphic.*

Proof. It may be false. □

3.2 Isometric actions

From now on, we always let G be a locally compact group, and let X be a Banach space together with a weakly* dense subspace F of X^* such that (X,F) satisfies the Krein and Goldstine property.

Definition 3.4 (Isometric group actions). By an *isometric action* of G on (X,F), we mean a $\sigma(X,F)$-continuous group homomorphism $\alpha : G \to \mathrm{Isom}(X) \subset B(X)$ of $\sigma(X,F)$-continuous linear isometries. Let A be a C*-algebra. Then, we always consider an *action* of G on A as an isometric action $\alpha : G \to \mathrm{Aut}(A)$ in the above sense, where $(X,F) = (A,A^*)$. Let M be a von Neumann algebra. Then, we always consider an *action* of G on M as an isometric action $\alpha : G \to \mathrm{Aut}(M)$ in the above sense, where $(X,F) = (M,M_*)$.

Remark. Suppose, for an isometric action $\alpha : G \to \mathrm{Isom}(X)$ on (X,F), we want to justify the following integral:

$$\int_G \alpha_s(x)\,d\mu(s), \qquad x \in X,\ \mu \in M(G).$$

For actions on C*-algebras, the point-weak continuity is in fact equivalent to the point-norm continuity because a weakly continuous semi-group on a Banach space is strongly continuous. (Not that easy.) It means that the function $G \to X : s \mapsto \alpha_s(a)$ is norm continuous for each $a \in A$ so that it is strongly measurable, which allows to use the Bochner integral to justify the above integral. However, for von Neumann algebras, the σ-weak continuity cannot imply the strong measurability in general, so we need to develop the Pettis integral.

Proposition 3.5 (Representation of groups and measure algebras). *Let $\alpha : G \to \mathrm{Isom}(X)$ be an isometric action of G on (X,F). There is a (faithful non-degenerate?) homomorphism $\pi_\alpha : M(G) \to B(X)$ defined by*

$$\pi_\alpha(\mu)x := \int_G \alpha_s(x)\,d\mu(s),$$

which is justified by the Pettis integral.

Proof. For each $x \in X$, since $G \to X : s \mapsto \alpha_s(x)$ is bounded and continuous with respect to $\sigma(X,F)$, by (c) of the previous proposition, we can define the Pettis integral

$$\pi_\alpha(\mu)x := \phi_{s\mapsto\alpha_s(x)}(1) = \int_G \alpha_s(x)\,d\mu(s), \qquad x \in X,\ \mu \in M(G).$$

For μ, $\nu \in M(G)$,

$$\begin{aligned}\pi_\alpha(\mu * \nu)x &= \iint \alpha_{st}(x)\,d\mu(s)\,d\nu(t) = \iint \alpha_s(\alpha_t(x))\,d\nu(t)\,d\mu(s)\\ &= \int \alpha_s\Big(\int \alpha_t(x)\,d\nu(t)\Big)\,d\mu(s) = \pi_\alpha(\mu)\pi_\alpha(\nu)x, \qquad x \in M.\end{aligned}$$

□

Proposition 3.6 (Dual actions). *Let $\alpha : G \to \mathrm{Isom}(X)$ be an isometric action of G on (X,F). For $\mu \in M(G)$, the linear map*

$$X \to X : x \mapsto \int \alpha_s(x)\,d\mu(s)$$

is $\sigma(X,F)$-$\sigma(X,F)$-continuous.

Proof. Consider the dual one-parameter group $\alpha^* : G \to \mathrm{Isom}(F)$, which is $\sigma(F,X)$-continuous group of $\sigma(F,X)$-$\sigma(F,X)$-continuous linear isometries. Since it satisfies the conditions in (c) of the proposition at the first with the dual pair (F,X), the Pettis integral

$$\int \alpha_s^*(x^*)\,d\mu(s)$$

is well-defined in F. Therefore, if a net x_i converges to zero in $\sigma(X,F)$, then for $x^* \in F$

$$\langle \int \alpha_s(x_i)\,d\mu(s), x^*\rangle = \int \langle \alpha_s(x_i), x^*\rangle\,d\mu(s) = \int \langle x_i, \alpha_s^*(x^*)\rangle\,d\mu(s) = \langle x_i, \int \alpha_s^*(x^*)\,d\mu(s)\rangle \to 0. \quad \square$$

3.3 One-parameter group of isometries

If $G = \mathbb{R}$, then an isometric action is somtimes called a *flow*.

Proposition 3.7 (Smoothing operators). *Let $\alpha : \mathbb{R} \to \mathrm{Isom}(X)$ be a flow on (X,F). For each $n > 0$, define a linear operator $R_n : X \to X$ such that*

$$R_n(x) := \sqrt{\frac{n}{\pi}} \int e^{-ns^2} \alpha_s(x)\,ds, \qquad x \in X.$$

(a) *R_n is contractive.*

(b) *$R_n(x) \to x$ in $\sigma(X,F)$.*

(c) *$R_n(x) \to x$ in norm if $F = X^*$.*

(d) *$R_n(x) \to x$ σ-strongly* if $(X,F) = (M, M_*)$ for a von Neumann algebra M.*

Proof. Relatively obvious. $\square$

Theorem 3.8 (Analytic continuations). *Let $\alpha : \mathbb{R} \to \mathrm{Isom}(X)$ be a flow on (X,F). We have a family of densely defined closed operators $\{\alpha_z : z \in \mathbb{C}\}$ on X which extends the original α, such that*

(i) $\alpha_z\alpha_t = \alpha_{z+s} = \alpha_s\alpha_z$ *and* $\alpha_z\alpha_w \subset \alpha_{z+w}$ *for $s \in \mathbb{R}$ and $z, w \in \mathbb{C}$,*

(ii) $\alpha_z^{-1} = \alpha_{-z}$,

(iii) $\mathrm{dom}\,\alpha_z \subset \mathrm{dom}\,\alpha_w$ *if* $\mathrm{Im}\,z \geq \mathrm{Im}\,w \geq 0$,

(iv) $\bigcap_{z\in\mathbb{C}} \mathrm{dom}\,\alpha_z$ *is dense in X.*

Proof. Consider the set of regularized vectors

$$\{R_n(x) : n \in \mathbb{N},\ x \in X\}.$$

Now we define $\alpha_z : X_0 \to X$ for $z \in \mathbb{C}$ such that

$$\alpha_z\Big(\int_{\mathbb{R}} f(s)\alpha_s(x)\,ds\Big) := \int_{\mathbb{R}} f(s-z)\alpha_s(x)\,ds.$$

It satisfies some properties:

(a) It extends the original $\{\alpha_s : s \in \mathbb{R}\}$.

(b) For fixed $x \in X_0$, $z \mapsto \alpha_z(x)$ is $\sigma(X,F)$-entire.

(c) X_0 is $\sigma(X,F)$-dense in E, so α_z is densely defined for each $z \in \mathbb{C}$.

(d) α_z is closable for each $z \in \mathbb{C}$.

(a) is clear by coordinate change, and (b) follows from the Fubini and the Morera after taking arbitrary elements of E^*. (c) is by an approximate identity e_n of $L^1(\mathbb{R})$ has $x = \lim_{n\to\infty} \int_{\mathbb{R}} e_n(s)\alpha_s(x)\,ds$. For (d), we have the adjoint $(\alpha_z)_0^* \supset (\alpha_{-\bar{z}})_0$, which is densely defined. Now we have a family of closed densely defined operators $\{\alpha_z : z \in \mathbb{C}\}$ on E such that $\alpha_z\alpha_w \subset \alpha_{z+w}$ for all $z, w \in \mathbb{C}$. $\square$

Proposition 3.9. (a) $R_n(x) \to x$

Definition 3.10 (Entire elements). The set of entire elements is $\bigcup_{z\in\mathbb{C}} \mathrm{dom}\,\alpha_z = \bigcup_{n\in\mathbb{Z}} \mathrm{dom}\,\alpha_{ni}$, which is dense. If X is a von Neumann algebra, then it is a $*$-subalgebra of M.

3.4 Tomita-Takesaki commutation theorem

In this section, we let A be a left Hilbert algebra. We will use the following notations freely:

$$H, M, S, \lambda, F, \rho, B, D, A', B', D', \Delta, J.$$

Also note that

$$\begin{tikzcd}
\mathfrak{m} := \mathfrak{n}^*\mathfrak{n} & \subset & \mathfrak{a} := \mathfrak{n}^* \cap \mathfrak{n} \arrow[d, shift left, "\Lambda"] & \subset & \mathfrak{n} \arrow[d, shift left, "\Lambda"] & \subset & M \\
 & & A \arrow[u, shift left, "\lambda"] & \subset & B \arrow[u, shift left, "\lambda"] & \subset & H.
\end{tikzcd}$$

The goal of this section is to prove that there exists the following commutative "cube" diagram:

$$\begin{tikzcd}
 & A \arrow[rr, "\Delta^{it}"] & & A \arrow[dd, "\lambda"] \\
A' \arrow[ur, leftrightarrow, "J"] \arrow[rr, "\Delta^{it}"'] \arrow[ddd, "\rho"'] & & A' \arrow[ur, leftrightarrow, "J"'] \arrow[ddd, "\rho"] & \\
 & & & \lambda(A) \arrow[dl, leftrightarrow, "\mathrm{Ad}\,J"] \\
 & & \, & \\
\rho(A') \arrow[rr, "\mathrm{Ad}\,\Delta^{it}"'] & & \rho(A') & .
\end{tikzcd}$$

Lemma. *For every $t \in \mathbb{R}$, the unitary operator Δ^{it} commutes with J, S, and F.*

Proof. It is enough to show $\Delta^{it}J = J\Delta^{it}$. By the relation $J\Delta J = \Delta^{-1}$, the anti-linearity of J, and the uniqueness of the bounded Borel functional calculus, we have the commutation. More precisely, if we let $f(s) := e^{it\log s}$ on $(0, \infty)$, then

$$\Delta^{-it} = f(\Delta^{-1}) = f(J\Delta J) = J\overline{f(\Delta)}J = J(\Delta^{it})^* J = J\Delta^{-it}J.$$

(Here we omit the detailed proof of $f(J\Delta J) = J\overline{f(\Delta^{-1})}J$.) □

Lemma. *$J : D' \to D$ and $\Delta^{it} : D \to D$.*

Proof. We have $J : D' \to D$ since $\eta \in D'$ implies that $SJ\eta = JF\eta$ is well-defined in H. We have $\Delta^{it} : D \to D$ for real t since $\xi \in D$ implies that $S\Delta^{it}\xi = \Delta^{it}S\xi$ is well-defined in H because $S\xi \in D$. □

We need two critical lemmas.

Lemma 3.11 (Fourier inversion of sech)**.** *Let α be a flow. Let H be a Hilbert space.*

(a) *We have a Pettis integral*

$$\int_{\mathbb{R}} (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}}) \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \alpha_t(x)\, dt = x, \qquad x \in \operatorname{dom}\alpha_{-\frac{i}{2}} \cap \operatorname{dom}\alpha_{\frac{i}{2}}.$$

(b) *If $\alpha = \sigma : \mathbb{R} \to \mathrm{Aut}(B(H))$ such that $\sigma_t = \mathrm{Ad}_{\Delta^{it}}$ for a injective positive self-adjoint operator Δ on H, then we have a σ-weak Pettis integral*

$$\int_{\mathbb{R}} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t(x)\, dt = (e^{-\frac{s}{2}}\sigma_{-\frac{i}{2}} + e^{\frac{s}{2}}\sigma_{\frac{i}{2}})^{-1} x, \qquad x \in B(H).$$

(c) *If $\alpha = u^* : \mathbb{R} \to U(H)$ such that $u_t = \Delta^{it}$ for a injective positive self-adjoint operator Δ on H, then we have a Pettis integral*

$$\int_{\mathbb{R}} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} u_t^*(\xi)\, dt = (e^{-\frac{s}{2}}u_{\frac{i}{2}} + e^{\frac{s}{2}}u_{-\frac{i}{2}})^{-1}\xi, \qquad \xi \in H.$$

The adjoint u_t^* is called the propagator.

Remark. If we let $f(t) := (e^{\frac{t}{2}} + e^{-\frac{t}{2}})^{-1}$ and write $\alpha_t = e^{t\delta}$, then the equation in the lemma can be rewritten formally as the Fourier inversion

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it(-i\delta - s)} \widehat{f}(t)\, dt = f(-i\delta - s), \qquad s \in \mathbb{R}.$$

However, this Fourier calculus in general setting using an unbounded holomorphic functional calculus for unbounded operators acting on Banach spaces is impossible, because even for a fairly normal example (e.g. $\sigma_t = \mathrm{Ad}\, u_t$, u_t is given by the translation on $L^2(\mathbb{R})$) we have a counterexample having the entire spectrum of the analytic generator $\sigma(\sigma_{-i}) = \mathbb{C}$.

Proof. (a) We use the special fact that the function $\widehat{f}(t) := \sqrt{2\pi}(e^{\pi t} + e^{-\pi t})^{-1}$ has imaginary period i. Fix $s \in \mathbb{R}$ and $x \in \mathrm{dom}\, \alpha_{-\frac{i}{2}} \cap \mathrm{dom}\, \alpha_{\frac{i}{2}}$. Define a weakly* meromorphic vector function $g : \mathbb{C} \setminus i\mathbb{Z} \to X$ such that

$$g(z) := -i\sqrt{2\pi} \frac{e^{-isz}}{e^{\pi z} - e^{-\pi z}} \alpha_z(x).$$

It satisfies relations

$$g(t - \frac{i}{2}) = e^{-\frac{s}{2}} \alpha_{-\frac{i}{2}}(e^{-ist} \widehat{f}(t) \alpha_t(x)), \qquad g(t + \frac{i}{2}) = -e^{\frac{s}{2}} \alpha_{\frac{i}{2}}(e^{-ist} \widehat{f}(t) \alpha_t(x))$$

and enjoys an estimate

$$\sup_{|r| \le \frac{1}{2}} \|g(t + ir)\| \le \sup_{|r| \le \frac{1}{2}} \sqrt{2\pi} \frac{e^{sr}}{|e^{\pi(t+ir)} - e^{-\pi(t+ir)}|} \|x\| = O(e^{-\pi|t|}), \qquad |t| \to \infty.$$

Then, by the residue theorem

$$\begin{aligned} \sqrt{2\pi} x = 2\pi \lim_{z \to 0} z g(z) &= \int_{-\infty}^{\infty} g(t - \frac{i}{2})\, dt - \int_{-\infty}^{\infty} g(t + \frac{i}{2})\, dt \\ &= \int_{-\infty}^{\infty} (e^{-\frac{s}{2}} \alpha_{-\frac{i}{2}} + e^{\frac{s}{2}} \alpha_{\frac{i}{2}}) e^{-ist} \widehat{f}(t) \alpha_t(x)\, dt. \end{aligned}$$

Extend for $x \in X$ using the boundedness.

(b) Let $A = (e^{-\frac{s}{2}} \sigma_{-\frac{i}{2}} + e^{\frac{s}{2}} \sigma_{\frac{i}{2}})$ be a densely defined operator on $B(H)$, and let R be a bounded linear operator on $B(H)$ such that

$$R(x) := \int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t(x)\, dt, \qquad x \in B(H).$$

Fix $x \in B(H)$. If we let $e_n := 1_{[n^{-1},n]}(\Delta)$ for some $n > 1$, then $\sigma_{\pm \frac{i}{2}}$ acts on the compression $e_n B(H) e_n$ as bounded invertible operators $e_n B(H) e_n \to e_n B(H) e_n$, which is continuous between σ-weak topologies. Thus, the part (a) and the commutation with integral imply

$$e_n x e_n = RA(e_n x e_n) = AR(e_n x e_n) = e^{-\frac{s}{2}} \Delta^{\frac{1}{2}} R(e_n x e_n) \Delta^{-\frac{1}{2}} + e^{\frac{s}{2}} \Delta^{-\frac{1}{2}} R(e_n x e_n) \Delta^{\frac{1}{2}}.$$

Since R is continuous in σ-weak topologies, we obtain by letting $n \to \infty$ an equation $x = AR(x)$ as sesquilinear forms on a dense subspace $\mathrm{dom}\, \Delta^{\frac{1}{2}} \cap \mathrm{dom}\, \Delta^{-\frac{1}{2}}$ of H, and hence as bounded operators because linear functionals given by vectors in a dense subspace of H separate points of $B(H)$. In particular, A is surjective. Since the injectivity of A follows from the part (a), we have $A^{-1} = R$.

(c) Similar to (b), but cut ξ off into $e_n \xi$. □

Lemma 3.12. *Let A be a left Hilbert algebra. For $s \in \mathbb{R}$, we have $(e^{-s} + \Delta)^{-1} : A' \to A \cap D'$. In particular, $A \cap D'$ is dense in H.*

Proof. Let $\eta \in A'$ and $\xi := (e^{-s} + \Delta)^{-1}\eta$. Then, $\Delta\xi = \eta - e^{-s}\xi \in H$ implies $\xi \in \operatorname{dom}\Delta \subset \operatorname{dom}\Delta^{\frac{1}{2}} = D$, and $F\xi = e^s(F\eta - S\xi) \in H$ implies $\xi \in D'$. The only non-trivial fact is $\xi \in B$. Since $\xi \in D$, by the polar decomposition, we have

$$\lambda(\xi) = vh = kv, \qquad h := |\lambda(\xi)|, \quad k := |\lambda(\xi)^*|.$$

Let $f \in C_c((0,\infty))^+$. Since

$$\begin{aligned}\langle f(h)S\xi, \zeta\rangle = \langle S\xi, f(h)\zeta\rangle &= \langle Ff(h)\zeta, \xi\rangle = \langle Fv^*\grave{f}(k)\lambda(\xi)\zeta, \xi\rangle = \langle F\lambda(v^*\grave{f}(k)\xi)\zeta, \xi\rangle \\ &= \langle F\rho(\zeta)v^*\grave{f}(k)\xi, \xi\rangle = \langle \rho(v^*\grave{f}(k)\xi)^*F\zeta, \xi\rangle = \langle F\zeta, \rho(v^*\grave{f}(k)\xi)\xi\rangle \\ &= \langle F\zeta, \lambda(\xi)v^*\grave{f}(k)\xi\rangle = \langle F\zeta, f(k)\xi\rangle = \langle Sf(k)\xi, \zeta\rangle, \qquad \xi \in D,\ \zeta \in D',\end{aligned}$$

we have

$$\begin{aligned}\|f(k)\eta\|^2 &= \|f(k)(e^{-s} + \Delta)\xi\|^2 \\ &= e^{-2s}\|f(k)\xi\|^2 + \|f(k)\Delta\xi\|^2 + 2e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\ &\geq 2e^{-s}\|f(k)\xi\|\|f(k)\Delta\xi\| + 2e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\ &\geq 4e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\ &= 4e^{-s}\operatorname{Re}\langle f(k)^2\xi, FS\xi\rangle \\ &= 4e^{-s}\operatorname{Re}\langle Sf(k)^2\xi, S\xi\rangle \\ &= 4e^{-s}\operatorname{Re}\langle f(h)^2S\xi, S\xi\rangle \\ &= 4e^{-s}\|f(h)S\xi\|^2,\end{aligned}$$

and

$$\begin{aligned}\|f(k)\eta\|^2 = \|\grave{f}(k)k\eta\|^2 &= \|\grave{f}(k)v\lambda(\xi)^*\eta\|^2 = \|v\grave{f}(h)\rho(\eta)S\xi\|^2 \\ &= \|\rho(\eta)v\grave{f}(h)S\xi\|^2 \leq \|\rho(\eta)\|^2\|\grave{f}(h)S\xi\|^2, \qquad \eta \in A',\ f \in C_c((0,\infty))^+,\end{aligned}$$

which imply that $c := \frac{1}{2}e^{\frac{s}{2}}\|\rho(\eta)\|$ satisfies

$$\|f(h)S\xi\| \leq c\|\grave{f}(h)S\xi\|.$$

For arbitrary $\varepsilon > 0$, by considering a net $f_\alpha \uparrow 1_{(c+\varepsilon,\infty)}$ and defining $p_\varepsilon := 1_{[0,c+\varepsilon]}(h)$, we have $\grave{f} \leq (c+\varepsilon)^{-1}f$ and that

$$\|(1-p_\varepsilon)S\xi\| \leq \frac{c}{c+\varepsilon}\|(1-p_\varepsilon)S\xi\|,$$

which implies $p_\varepsilon S\xi = S\xi$ for all $\varepsilon > 0$, so $p_0S\xi = S\xi$. Then,

$$\|\lambda(\xi)^*\zeta\| = \|p_0\lambda(\xi)^*\zeta\| = \|1_{[0,c]}(h)hv^*\zeta\| \leq c\|v^*\zeta\| \leq c\|\zeta\|, \qquad \zeta \in A'.$$

Therefore, $S\xi \in B$, which implies $S\xi \in A$ and $\xi \in A$.

For the density of $A \cap D'$, we approximate $\zeta \in \operatorname{dom}\Delta$. Define a sequence $\eta_n \in A'$ such that $\eta_n \to (1+\Delta)\zeta$. Then, since $(1+\Delta)^{-1}$ is bounded, we have $(1+\Delta)^{-1}\eta_n \to \zeta \in \overline{A \cap D'}$. Since $\operatorname{dom}\Delta$ is dense in H, we are done. □

Theorem 3.13 (Tomita-Takesaki commutation theorem). *Let A be a left Hilbert algebra. Then, for every $t \in \mathbb{R}$, the following diagram commutes:*

$$\begin{tikzcd} & D \arrow[r, "u_t = \Delta^{it}"] & D \arrow[d, "\lambda"] \\ A' \arrow[ur, "J"] \arrow[d, "\rho"'] & & \\ & & C(H). \\ \rho(A') \arrow[r, "\sigma_t = \operatorname{Ad}\Delta^{it}"'] & B(H) \arrow[ur, "\operatorname{Ad}J"'] & \end{tikzcd}$$

Proof. Fix $\eta \in A'$ and define

$$\begin{aligned}\xi &:= (e^{-\frac{s}{2}}u_{\frac{i}{2}} + e^{\frac{s}{2}}u_{-\frac{i}{2}})^{-1}J\eta = (e^{-\frac{s}{2}}\Delta^{-\frac{1}{2}} + e^{\frac{s}{2}}\Delta^{\frac{1}{2}})^{-1}J\eta \\ &= (e^{-\frac{s}{2}}\Delta^{-\frac{1}{2}} + e^{\frac{s}{2}}\Delta^{\frac{1}{2}})^{-1}\Delta^{-\frac{1}{2}}F\eta = e^{-\frac{s}{2}}(e^{-s}+\Delta)^{-1}F\eta \in A\cap D'.\end{aligned}$$

By the computations

$$\begin{aligned}\langle \rho(F\xi)\zeta_1, \zeta_2\rangle &= \langle \lambda(\zeta_1)F\xi, \zeta_2\rangle = \langle F\xi, \lambda(\zeta_1)^*\zeta_2\rangle = \langle S\lambda(\zeta_1)^*\zeta_2, \xi\rangle \\ &= \langle \lambda(\zeta_2)^*\zeta_1, \xi\rangle = \langle \rho(\zeta_1)S\zeta_2, \xi\rangle = \langle S\zeta_2, \rho(\zeta_1)^*\xi\rangle \\ &= \langle S\zeta_2, \lambda(\xi)F\zeta_1\rangle = \langle F\lambda(\xi)F\zeta_1, \zeta_2\rangle, \qquad \zeta_1 \in A\cap D',\ \zeta_2 \in A, \\ \langle \rho(S\xi)\zeta_1, \zeta_2\rangle &= \langle \lambda(\zeta_1)S\xi, \zeta_2\rangle = \langle S\xi, \lambda(\zeta_1)^*\zeta_2\rangle = \langle S\xi, \rho(\zeta_2)S\zeta_1\rangle \\ &= \langle \rho(\zeta_2)^*S\xi, S\zeta_1\rangle = \langle \lambda(\xi)^*F\zeta_2, S\zeta_1\rangle = \langle F\zeta_2, \lambda(\xi)S\zeta_1\rangle \\ &= \langle S\lambda(\xi)S\zeta_1, \zeta_2\rangle, \qquad \zeta_1 \in A\cap D',\ \zeta_2 \in D',\end{aligned}$$

the domains of $\rho(F\xi)$ and $\rho(S\xi)$ contain $A\cap D'$ and we have

$$\begin{aligned}\rho(\eta) &= \rho(J(e^{-\frac{s}{2}}\Delta^{-\frac{1}{2}} + e^{\frac{s}{2}}\Delta^{\frac{1}{2}})\xi) \\ &= e^{-\frac{s}{2}}\rho(F\xi) + e^{\frac{s}{2}}\rho(S\xi) \\ &= e^{-\frac{s}{2}}F\lambda(\xi)F + e^{\frac{s}{2}}S\lambda(\xi)S \\ &= e^{-\frac{s}{2}}\Delta^{\frac{1}{2}}J\lambda(\xi)J\Delta^{-\frac{1}{2}} + e^{\frac{s}{2}}\Delta^{-\frac{1}{2}}J\lambda(\xi)J\Delta^{\frac{1}{2}} \\ &= (e^{-\frac{s}{2}}\sigma_{-\frac{i}{2}} + e^{\frac{s}{2}}\sigma_{\frac{i}{2}})(\mathrm{Ad}J)\lambda(\xi)\end{aligned}$$

as sesquilinear forms on $A\cap D'$. The conditions for ξ and ζ_1 to belong to $A\cap D'$ are necessary in the above computation. By the density of $A\cap D'$ in H, we have the bounded operators

$$(\mathrm{Ad}J)(e^{-\frac{s}{2}}\sigma_{-\frac{i}{2}} + e^{\frac{s}{2}}\sigma_{\frac{i}{2}})^{-1}\rho(\eta) = \lambda((e^{-\frac{s}{2}}u_{\frac{i}{2}} + e^{\frac{s}{2}}u_{-\frac{i}{2}})^{-1}J\eta).$$

Then, we get the equation of bounded linear operators

$$(\mathrm{Ad}J)\Big(\int \frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\sigma_t(\rho(\eta))\,dt\Big) = \lambda\Big(\int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}u_t(J\eta)\,dt\Big), \qquad s\in\mathbb{R},\ \eta\in A',$$

changing the variable using that the hyperbolic secant functions is even. For every $\zeta \in B'$, since $\mathrm{Ad}J : B(H)\to B(H)$, $\cdot\zeta : B(H)\to H$, and $\rho(\zeta) : H \to H$ are all continuous between weak* topologies, we have

$$\int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}(\mathrm{Ad}J)\sigma_t(\rho(\eta))\zeta\,dt = (\mathrm{Ad}J)\Big(\int \frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\sigma_t(\rho(\eta))\,dt\Big)\zeta,$$

which is equal to

$$\begin{aligned}\lambda\Big(\int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}u_t(J\eta)\,dt\Big)\zeta &= \rho(\zeta)\int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}u_t(J\eta)\,dt \\ &= \int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}\rho(\zeta)u_t(J\eta)\,dt \\ &= \int \frac{e^{ist}}{e^{\pi t}+e^{-\pi t}}\lambda(u_t(J\eta))\zeta\,dt.\end{aligned}$$

Then, by taking arbitrary bounded linear functionals of H on the above integral, and by the injectivity of the Fourier transform, we finally obtain $\mathrm{Ad}_J\circ\sigma_t\circ\rho = \lambda\circ u_t\circ J$ on A'. □

Corollary 3.14. *Let A be a full left Hilbert algebra. Then, for $t \in \mathbb{R}$, the following diagram is well-defined and commutes:*

$$\begin{tikzcd}
 & A \arrow[rr, "\Delta^{it}"] \arrow[dl, leftrightarrow, "J"'] & & A \arrow[dd, "\lambda"] \arrow[dl, leftrightarrow, "J"] \\
A' \arrow[rr, "\Delta^{it}"'] \arrow[dd, "\rho"'] & & A' \arrow[dd, "\rho"] & \\
 & & & \lambda(A) \arrow[dl, leftrightarrow, "\mathrm{Ad}_J"] \\
\rho(A') \arrow[rr, "\mathrm{Ad}_{\Delta^{it}}"'] & & \rho(A') & .
\end{tikzcd}$$

In particular, we have

$$JA = A', \qquad \Delta^{it}A = A, \qquad JMJ = M', \qquad \Delta^{it}M\Delta^{-it} = M,$$

and J is an anti-homomorphism, Δ^{it} is a $*$-homomorphism

Proof. □

Corollary 3.15 (Flow on a Hilbert space)**.**

$$R_n(\xi) := \sqrt{\frac{n}{\pi}} \int e^{-ns^2} \Delta^{is} \xi \, ds.$$

$R_n(\xi) - \xi \to 0$ *in norm,* $\Delta^{\frac{1}{2}}(R_n(\xi) - \xi) \to 0$ *in norm if* $\xi \in D$, $\|\lambda(R_n(\xi))\| \le \|\lambda(\xi)\|$ *if* $\xi \in B$, $\lambda(R_n(\xi)) \to \lambda(\xi)$ σ*-strongly**. *Stone's theorem, Spectral truncation technique*

Example 3.16 (Flow on a von Neumann algebra)**.**

$$R_n(x) := \sqrt{\frac{n}{\pi}} \int e^{-ns^2} \sigma_s(x) \, ds.$$

A *Tomita algebra* is a left Hilbert algebra A_0 such that every element of A_0 is entire with respect to the associated modular automorphism group.

Spectral truncation technique

4 January 17

4.1 Cocycle conjugacy

Definition (Actions and flows)**.** Let M be a von Neumann algebra and G be a locally compact group. An *action* of G on M is a σ-weakly continuous group homomorphism $\alpha : G \to \mathrm{Aut}(M)$. The triple (M,G,α) is called a *W*-dynamical system*. If $G = \mathbb{R}$, then an action is also called a *flow*. A *covariant representation* of a W*-dynamical system (M,G,α) is a pair (π,u,H) of a normal representation $\pi : M \to B(H)$ and a strongly continuous unitary representation $u : G \to U(H)$ such that $\alpha_s(x) = u_s x u_s^*$.

Definition 4.1 (Cocycle conjugacy)**.** Let (M,G,α) be a W*-dynamical system. A *α-(one)-cocycle* of a strongly continuous map $u : G \to U(M)$ such that $u_{st} = u_s\alpha_s(u_t)$, and we denote by $Z^1_\alpha(G,U(M))$ the set of all α-cocycles.

Theorem 4.2 (Connes cocycle derivative)**.** *Let φ and ψ be faithful semi-finite normal weights on a von Neumann algebra M.*

(a) *The representations π_φ and ψ_ψ are unitarily equivalent. In particular, every normal state is a vector state in a semi-cyclic reprsentation of a faithful semi-finite normal weight.*

(b) *The modular automorphism groups σ_t^φ and σ_t^ψ are conjugate up to cocycles. In other words, there is a canonical continuous group homomorphism $\mathbb{R} \to \mathrm{Out}(M)$ for M.*

(c)

Proof. Consider the *balanced weight* $\varphi\oplus\psi$ on $M\otimes M_2(\mathbb{C}) = M_2(M)$. Then, it is also faithful, semi-finite, and normal.***

We investigate the semi-cyclic representation and the left Hilbert algebra structure corresponding to $\varphi\oplus\psi$. First, we have

$$\mathfrak{n}_{\varphi\oplus\psi} = \begin{pmatrix} \mathfrak{n}_\varphi & \mathfrak{n}_\psi \\ \mathfrak{n}_\varphi & \mathfrak{n}_\psi \end{pmatrix}, \qquad \mathfrak{a}_{\varphi\oplus\psi} = \begin{pmatrix} \mathfrak{a}_\varphi & \mathfrak{n}_\varphi^*\cap\mathfrak{n}_\psi \\ \mathfrak{n}_\psi^*\cap\mathfrak{n}_\varphi & \mathfrak{a}_\psi \end{pmatrix}, \qquad \mathfrak{m}_{\varphi\oplus\psi} = \begin{pmatrix} \mathfrak{m}_\varphi & \mathfrak{n}_\varphi^*\mathfrak{n}_\psi \\ \mathfrak{n}_\psi^*\mathfrak{n}_\varphi & \mathfrak{m}_\psi \end{pmatrix}.$$

The semi-cyclic representation of $\varphi\oplus\psi$ can be realized on the identification with the direct sum

$$H_{\varphi\oplus\psi} = H_\varphi\oplus H_\varphi\oplus H_\psi\oplus H_\psi$$

such that $\Lambda_{\varphi\oplus\psi} : \mathfrak{n}_{\varphi\oplus\psi} \to H_{\varphi\oplus\psi}$ and $\pi_{\varphi\oplus\psi} : M_2(M) \to B(H_{\varphi\oplus\psi})$ given by

$$\Lambda_{\varphi\oplus\psi}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{bmatrix} \Lambda_\varphi(x_{11}) \\ \Lambda_\varphi(x_{21}) \\ \Lambda_\psi(x_{12}) \\ \Lambda_\psi(x_{22}) \end{bmatrix}, \quad \pi_{\varphi\oplus\psi}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{bmatrix} \pi_\varphi(x_{11}) & \pi_\varphi(x_{12}) & 0 & 0 \\ \pi_\varphi(x_{21}) & \pi_\varphi(x_{22}) & 0 & 0 \\ 0 & 0 & \pi_\psi(x_{11}) & \pi_\psi(x_{12}) \\ 0 & 0 & \pi_\psi(x_{21}) & \pi_\psi(x_{22}) \end{bmatrix}.$$

The Hilbert algebra structure $S_{\varphi\oplus\psi}$, $\Delta_{\varphi\oplus\psi}$, $J_{\varphi\oplus\psi} : A_{\varphi\oplus\psi} \to H_{\varphi\oplus\psi}$ on $A_{\varphi\oplus\psi} = \Lambda_{\varphi\oplus\psi}(\mathfrak{a}_{\varphi\oplus\psi})$ are computed as

$$S_{\varphi\oplus\psi} = \begin{bmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi,\psi} & 0 \\ 0 & S_{\psi,\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{bmatrix}, \quad J_{\varphi\oplus\psi} = \begin{bmatrix} J_\varphi & 0 & 0 & 0 \\ 0 & 0 & J_{\varphi,\psi} & 0 \\ 0 & J_{\psi,\varphi} & 0 & 0 \\ 0 & 0 & 0 & J_\psi \end{bmatrix}, \quad \Delta_{\varphi\oplus\psi} = \begin{bmatrix} \Delta_\varphi & 0 & 0 & 0 \\ 0 & \Delta_{\varphi,\varphi} & 0 & 0 \\ 0 & 0 & \Delta_{\psi,\psi} & 0 \\ 0 & 0 & 0 & \Delta_\psi \end{bmatrix}.$$

(a) Since

$$\pi_{\varphi\oplus\psi}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{bmatrix} \pi_\varphi(x) & 0 & 0 & 0 \\ 0 & \pi_\varphi(x) & 0 & 0 \\ 0 & 0 & \pi_\psi(x) & 0 \\ 0 & 0 & 0 & \pi_\psi(x) \end{bmatrix}, \quad J\pi_{\varphi\oplus\psi}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ J_{\psi,\varphi}J_\varphi & 0 & 0 & 0 \\ 0 & J_\psi J_{\psi,\varphi} & 0 & 0 \end{bmatrix}$$

are commuting by the Tomita-Takesaki commutation theorem, we obtain

$$\begin{bmatrix} \pi_\psi(x)J_{\psi,\varphi}J_\varphi & 0 \\ 0 & \pi_\psi(x)J_\psi J_{\psi,\varphi} \end{bmatrix} = \begin{bmatrix} J_{\psi,\varphi}J_\varphi\pi_\varphi(x) & 0 \\ 0 & J_\psi J_{\psi,\varphi}\pi_\varphi(x) \end{bmatrix},$$

so if we define $u_{\psi,\varphi} := J_{\psi,\varphi}J_\varphi = J_\psi J_{\psi,\varphi} : H_\varphi \to H_\psi$, then it is unitary such that $\pi_\psi(x) = u_{\psi,\varphi}\pi_\varphi(x)u_{\psi,\varphi}^*$ for all $x \in M$. Be cautious that $\Lambda_\psi(x) \neq u\Lambda_\varphi(x)$ unless $\varphi = \psi$ in general, so the two semi-cyclic representations are not unitarily equivalent in the full sense.

(b) Define $\sigma_t^{\varphi,\psi}$ and $\sigma_t^{\psi,\varphi}$ such that

$$\sigma_t^{\varphi\oplus\psi}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} =: \begin{pmatrix} \sigma_t^\varphi(x_{11}) & \sigma_t^{\varphi,\psi}(x_{12}) \\ \sigma_t^{\psi,\varphi}(x_{21}) & \sigma_t^\psi(x_{22}) \end{pmatrix}.$$

□

4.2 Commuting weights

Definition 4.3 (Kubo-Martin-Schwinger weights)**.** Let M be a von Neumann algebra. Let α be a flow on M, and φ be a faithful semi-finite normal weight on M. For $x, y \in M$, their *two-point function* at inverse temperature $\beta \in \mathbb{R}$ is a bounded continuous function $f : \mathrm{Im}^{-1}([\beta,0] \cup [0,\beta]) \to \mathbb{C}$ which is holomorphic on its interior such that

$$f(t) = \varphi(y\sigma_t(x)), \qquad f(t+i\beta) = \varphi(\sigma_t(x)y), \qquad t \in \mathbb{R}.$$

If φ is invariant under σ and every pair $x, y \in \mathfrak{a}$ admits a two-point function at β, then we say φ is a *Kubo-Martin-Schwinger weight* or *KMS weight* for α at β. From now on, we always assume $\beta = -1$.

Remark. If φ is a state, then for φ to be a KMS state the invariance condition is superfluous: since $\mathfrak{m} = \mathfrak{a} = M$, we can put $y = 1$ to show the invariance using the Liouville theorem and the Schwarz reflection principle.

Lemma 4.4 (Action of entire elements)**.** *Let φ be a faithful semi-finite normal weight on a von Neumann algebra M. If x is entire with respect to σ, then $x\mathfrak{m} \cup \mathfrak{m}x \subset \mathfrak{m}$.*

Proof. We first show $x\mathfrak{a} \cup \mathfrak{a}x \subset \mathfrak{a}$. Let $y \in \mathfrak{a}$. By symmetry, it suffices to show $xy \in \mathfrak{a}$. Since $\mathfrak{n}$ is a left ideal of M, $xy \in \mathfrak{n}$, which implies $\Lambda(xy) \in B$. Consider

$$\xi(t) := \Delta^{it}\Lambda(xy) = \Delta^{it}x\Lambda(y) = \sigma_t(x)\Delta^{it}\Lambda(y).$$

Since $\Lambda(y) \in D = \mathrm{dom}\,\Delta^{\frac{1}{2}}$, the function f is holomorphically extended to the strip $\mathrm{Im}^{-1}([-\frac{1}{2},0])$. It means that $\xi(0) = \Lambda(xy) \in \mathrm{dom}\,\Delta^{\frac{1}{2}}$, so $\Lambda(xy) \in D$. Thus we have $\Lambda(xy) \in A$ and $xy \in \mathfrak{a}$.

For the original claim, if $y \in \mathfrak{m}^+$, since $y^{\frac{1}{2}} \in \mathfrak{a}$ and $xy^{\frac{1}{2}} \in \mathfrak{a}$ as above, we have $xy = (xy^{\frac{1}{2}})y^{\frac{1}{2}} \in \mathfrak{a}^2 = \mathfrak{m}$. The linear span and symmetry show $x\mathfrak{m} \cup \mathfrak{m}x \subset \mathfrak{m}$. □

Lemma 4.5 (Existence of two-point functions)**.** *Let M be a von Neumann algebra. For σ the associated modular automorphism group for a faithful semi-finite normal weight φ on M, the followings hold.*

(a) *If $x, y \in \mathfrak{a}$, then they admit a two-point function.*

(b) *If $x \in M$ is entire for σ and $y \in \mathfrak{m}$, then they admit an entire two-point function.*

(c) *If $x \in M$ satisfies $x\mathfrak{m} \cup \mathfrak{m}x \subset \mathfrak{m}$ and $y \in \mathfrak{a}_0^*\mathfrak{a}_0$, then they admit an entire two-point function.*

Proof. We may assume $x, y \geq 0$. In this case, $\Delta^{\frac{1}{2}}\Lambda(y) = JS\Lambda(y) = J\Lambda(y)$ for $y \in \mathfrak{n}^+$.

(a) Define

$$f(z) := \langle \Delta^{i\frac{z}{2}}\Lambda(x), \Delta^{-i\frac{\bar{z}}{2}}\Lambda(y)\rangle.$$

Since $\Lambda(x),\Lambda(y)\in A\subset\operatorname{dom}\Delta^{\frac{1}{2}}$, the function f is bounded and continuous on $\operatorname{Im}^{-1}([-1,0])$, and holomorphic on its interior. Also, for $t\in\mathbb{R}$

$$\begin{aligned}\varphi(y\sigma_t(x))&=\langle\Delta^{it}\Lambda(x),\Lambda(y)\rangle=\langle\Delta^{i\frac{t}{2}}\Lambda(x),\Delta^{-i\frac{t}{2}}\Lambda(y)\rangle=f(t),\\ \varphi(\sigma_t(x)y)&=\langle\Lambda(y),\Delta^{it}\Lambda(x)\rangle=\langle\Lambda(y),J\Delta^{it}J\Lambda(x)\rangle=\langle\Delta^{it}J\Lambda(x),J\Lambda(y)\rangle\\ &=\langle\Delta^{it}\Delta^{\frac{1}{2}}\Lambda(x),\Delta^{\frac{1}{2}}\Lambda(y)\rangle=\langle\Delta^{i\frac{t-i}{2}}\Lambda(x),\Delta^{-i\frac{t+i}{2}}\Lambda(y)\rangle=f(t-i).\end{aligned}$$

(b) Since σ_t sends entire elements to entire elements, we have $y\sigma_t(x),\sigma_t(x)y\in\mathfrak{m}$ by the previous lemma. Define

$$f(z):=\langle\sigma_{z+\frac{i}{2}}(x)\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle.$$

Since x is entire, the function f is entire and bounded on the strip $\operatorname{Im}^{-1}([-1,0])$. Then, for $t\in\mathbb{R}$,

$$\begin{aligned}\varphi(y\sigma_t(x))&=\langle\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\sigma_t(x)\Lambda(y^{\frac{1}{2}})\rangle=\langle\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\sigma_{t-\frac{i}{2}}(x)\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle\\ &=\langle\sigma_{t+\frac{i}{2}}(x)\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle=f(t),\\ \varphi(\sigma_t(x)y)&=\langle\Delta^{\frac{1}{2}}\sigma_t(x)\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle=\langle\sigma_{t-\frac{i}{2}}\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle=f(t-i).\end{aligned}$$

(c) We may assume $y^{\frac{1}{2}}\in\varphi_0$. Define

$$f(z):=\langle x\Delta^{-i(z+i)}\Lambda(y^{\frac{1}{2}}),\Delta^{-i\overline{z}}\Lambda(y^{\frac{1}{2}})\rangle.$$

Since $\Lambda(y^{\frac{1}{2}})\in A_0\subset\operatorname{dom}\Delta$, the function f is bounded and continuous on $\operatorname{Im}^{-1}([-1,0])$, and holomorphic on its interior, and in fact it is entire. Then, for $t\in\mathbb{R}$,

$$\begin{aligned}\varphi(y\sigma_t(x))&=\langle\Lambda(y^{\frac{1}{2}}\sigma_t(x)),\Lambda(y^{\frac{1}{2}})\rangle=\langle J\Lambda(y^{\frac{1}{2}}),J\Lambda(y^{\frac{1}{2}}\sigma_t(x))\rangle\\ &=\langle\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(\sigma_t(x)y^{\frac{1}{2}})\rangle=\langle\Delta\Lambda(y^{\frac{1}{2}}),\sigma_t(x)\Lambda(y^{\frac{1}{2}})\rangle\\ &=\langle\Delta\Lambda(y^{\frac{1}{2}}),\Delta^{it}x\Delta^{-it}\Lambda(y^{\frac{1}{2}})\rangle=\langle x\Delta^{-it+1}\Lambda(y^{\frac{1}{2}}),\Delta^{-it}\Lambda(y^{\frac{1}{2}})\rangle=f(t),\\ \varphi(\sigma_t(y))&=\langle\Lambda(y^{\frac{1}{2}}),\Lambda(y^{\frac{1}{2}}\sigma_t(x))\rangle=\langle J\Lambda(y^{\frac{1}{2}}\sigma_t(x)),J\Lambda(y^{\frac{1}{2}})\rangle\\ &=\langle\Delta^{\frac{1}{2}}\Lambda(\sigma_t(x)y^{\frac{1}{2}}),\Delta^{\frac{1}{2}}\Lambda(y^{\frac{1}{2}})\rangle=\langle\sigma_t(x)\Lambda(y^{\frac{1}{2}}),\Delta\Lambda(y^{\frac{1}{2}})\rangle\\ &=\langle\Delta^{it}x\Delta^{-it}\Lambda(y^{\frac{1}{2}}),\Delta\Lambda(y^{\frac{1}{2}})\rangle=\langle x\Delta^{-it}\Lambda(y^{\frac{1}{2}}),\Delta^{it+1}\Lambda(y^{\frac{1}{2}})\rangle=f(t-i).\end{aligned}$$

□

Proposition 4.6 (Centralizers). *Let φ be a faithful semi-finite normal weight on a von Neumann algebra M. For $x\in M$, the followings are all equivalent:*

(a) $\sigma_t(x)=x$ *for all* $t\in\mathbb{R}$,

(b) $x\mathfrak{m}\cup\mathfrak{m}x\subset\mathfrak{m}$ *and* $\varphi(xy)=\varphi(yx)$ *for all* $y\in\mathfrak{m}$.

The set of all x satisfying one of the above conditions is called the centralizer *or the* fixed point algebra *of φ, and denoted by M^φ.*

Proof. (a)⇒(b) Let $\sigma_t(x)=x$ and $y\in\mathfrak{m}$, then since the constant function is entire, we have $x\in M_0$. By (b) of the previous lemma, observing the two-point function f is constant on the real line so that it is entirely constant by the identity principle, we can check that the KMS condition gives

$$\varphi(yx)=\varphi(y\sigma_t(x))=f(t)=f(t-i)=\varphi(\sigma_t(x)y)=\varphi(xy).$$

(b)⇒(a) Let $x\mathfrak{m}\cup\mathfrak{m}x\subset\mathfrak{m}$ and $y\in(\mathfrak{a}_0^*\mathfrak{a}_0)^+$. Then, $\sigma_t(y)\in\mathfrak{a}_0^*\mathfrak{a}_0$. By (c) of the previous lemma, we have a two-point function such that

$$f(t)=\varphi(y\sigma_t(x))=\varphi(\sigma_{-t}(y)x)=\varphi(x\sigma_{-t}(y))=\varphi(\sigma_t(x)y)=f(t-i).$$

By the Liouville theorem f is constant, so we have

$$0=\varphi(y\sigma_t(x))-\varphi(yx)=\langle(\sigma_t(x)-x)\Lambda(y^{\frac{1}{2}}),\Lambda(y^{\frac{1}{2}})\rangle.$$

Since $\Lambda(\mathfrak{a}_0^*\mathfrak{a}_0)=A_0^2$ is dense in H, we have $\sigma_t(x)=x$. □

Proposition 4.7 (Centrally perturbed weights)**.** *Let φ be a faithful semi-finite normal weight on a von Neumann algebra M. Let h be a non-negative self-adjoint operator affiliated with the centralizer M^φ. Then,*

$$\varphi_h(x) := \lim_{\varepsilon \to 0} \varphi(h_\varepsilon^{\frac{1}{2}} x h_\varepsilon^{\frac{1}{2}}), \qquad x \in M^+$$

is a semi-finite normal weight, where $h_\varepsilon = h(1+\varepsilon h)^{-1}$. The weight φ is faithful if and only if h is non-singular.

Proof. 2.7, 2.8 First assume $h, k \in M_*^+$.

□

Theorem 4.8 (Commuting weights)**.** *Let φ, ψ be a faithful semi-finite normal weights on a von Neumann algebra M. TFAE*

(a) $\psi = \varphi_h$.

(b) $\psi = \psi \circ \sigma_t^\varphi$.

Proof. (a)⇒(b) First we claim that: Let h be a positive non-singular self-adjoint operator affiliated with M^φ. Then, the modular automorphism group of $\psi := \varphi_h$ is given by

$$\sigma_t^\psi(x) = h^{it} \sigma_t^\varphi(x) h^{-it}.$$

Since $h^{it} \in M^\varphi$, we can show the invariance.

First suppose $h \in M^{\varphi+}$ invertible. Since h is entire, $\mathfrak{m}_\varphi = \mathfrak{m}_\psi$. By the uniqueness theorem, it is enough to show that $t \mapsto h^{it}\sigma_t^\varphi(x)h^{-it}$ satisfies the KMS condition. We will construct f. Take a sequence

For the general unbounded case, we skip.

(b)⇒(a)

$$\sigma_s^\varphi((D\psi : D\varphi)_t) = (D\psi \circ \sigma_{-s}^\varphi : D\varphi \circ \sigma_{-s}^\varphi)_t = (D\psi : D\varphi)_t.$$

Stone's theorem, construct h.

□

Theorem 4.9 (Semi-finiteness)**.** *Existence of trace vs inner modular automorphism*

Proof. □

4.3 Standard form

If (H, P, J) is a standard form of M, then there is a unique covariant representation of (M, G, α), called the *standard covariant representation*.

By the Friedrichs extension (a non-negative densely defined symmetric operator admits a canonical non-negative self-adjoint extension)

4.4 Noncommutative integration

5 March 8

5.1 Takesaki duality

abelian group

$T(M)$, $S(M)$

Type III begin

Appendix

Proposition 5.1. *For positive elements*

$$x(1+\varepsilon x)^{-1}$$

At first, they are operator monotone. Next, they are σ-strongly continuous on a closed subset of its domain due to the boundedness of f_ε, as we can see in the proof of the Kaplansky density theorem. Finally, for each $x \in M_+$, the increasing limit $f_\varepsilon(x) \uparrow x$ in norm as $\varepsilon \to 0$ implies that $\sup_\varepsilon f_\varepsilon(x) = x$.

Consider for a while, a family of functions

$$f_\varepsilon(t) := t(1+\varepsilon t)^{-1}, \qquad t \in (-\varepsilon^{-1}, \infty),$$

parametrized by $\varepsilon > 0$. They have several properties. At first, they are operator monotone. Next, they are σ-strongly continuous on a closed subset of its domain due to the boundedness of f_ε, as we can see in the proof of the Kaplansky density theorem. Finally, for each $x \in M_+$, the increasing limit $f_\varepsilon(x) \uparrow x$ in norm as $\varepsilon \to 0$ implies that $\sup_\varepsilon f_\varepsilon(x) = x$.

Proposition 5.2.

$$E_n(x) := 1_{[n^{-1},n]}(\Delta)x1_{[n^{-1},n]}(\Delta), \qquad E_\varepsilon(\xi) := 1_{[n^{-1},n]}(\Delta)\xi$$

Proposition 5.3.

$$R_n(x) := \sqrt{\frac{n}{\pi}} \int e^{-ns^2} \alpha_s(x)\, ds, \qquad R_n(\xi) := \sqrt{\frac{n}{\pi}} \int e^{-ns^2} \Delta^{is}(x)\, ds$$