# Low Dimensional Topology

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# Contents

# Part I

# Topology of 3-manifolds

# Chapter 1

# Chapter 2

# Chapter 3

# Part II

# Geometry of 3-manifolds

# Chapter 4

# Hyperbolization

## 4.1 Geometric structures

I need to check more carefully the followings... All statements in here may not be true, anyway.

**4.1** (Geometric structure)**.** Let $M$ be a connected smooth manifold. We are concerned with geometric structures on $M$. We restrict our interests on geometries that having length, area, volume, and angle measurements. For example, affine, projective, or conformal geometries are not considered to be candidates of geometric structures. Precisely, we suggest to define a *geometric structure* as a metric $d$ on $M$ such that

(i) $(M,d)$ is geodesically connected,

(ii) $(M,d)$ is geodesically complete,

(iii) $(M,d)$ is a Riemannian manifold,

(iv) $(M,d)$ is locally homogeneous.

In other words, a geometric structure on $M$ is a Riemannian metric satisfying (i), (ii), and (iv). Each condition has been obtained by modifying the first four postulates of Euclid's Elements.

(a) A homogeneous Riemannian metric is a geometric structure.

(b) If $M$ is simply connected, then a geometric structure on $M$ is homogeneous.

(c) If $M$ is compact, then a geometric structure on $M$ is homogeneous.

(d) If $p : C \to M$ is covering map, then there is a unique geometric structure on $C$ which makes $p$ a local isometry.

(e) If $p : M \to B$ is covering map, then there is a unique geometric structure on $B$ which makes $p$ a local isometry.

*Proof.* (a) Let $g$ be a homogeneous Riemannian metric on $M$. We will prove $g$ satisfies (ii) and (i).

(b) Ambrose-Singer theorem.

(c) □

**4.2** (Homogeneous Riemannian metrics)**.** Let $M$ be a connected smooth manifold. We want to establish the following correspondence.

$$\left\{ \begin{array}{c} \text{Homogeneous} \\ \text{Riemannian metrics} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Homogeneous} \\ \text{maximal smooth group actions} \\ \text{with compact stabilizers} \end{array} \right\}.$$

(a) If $g$ is a homogeneous Riemannian metric on $M$, the group action on $M$ by $\mathrm{Isom}(M,g)$ is maximal among smooth group actions with compact stabilizers.

(b) If a smooth group action on $M$ by $G$ is maximal among smooth group actions with compact stabilizers, then there is a homogeneous Riemannian metric on $M$ such that $G \cong \mathrm{Isom}(M,g)$.

*Proof.* □

**4.3** (Pseudogroup structure). Let $X$ be a topological space. A *pseudogroup* $\Gamma$ on $X$ is a wide subgroupoid of $\mathrm{Homeo}(X)$ such that $U \mapsto \{g \in \Gamma : \operatorname{dom} g = U\}$ is a sheaf on the topology of $X$. Let $\Gamma$ be a pseudogroup on $X$, and $M$ be a topological space. A $\Gamma$-*atlas* on $M$ is an atlas whose charts have $X$ as the codomain and transition maps belong to $\Gamma$. A $\Gamma$-*structure* on $M$ is defined as an equivalence class of $\Gamma$-atlases on $M$.

(a) For $G \le \mathrm{Homeo}(X)$, $\{g|_U : g \in G,\ U \in \mathcal{T}\}$ is a pseudogroup on $X$. We write this pseudogroup as $(G,X)$.

(b) ... Note that $G$ does not act on $(G,X)$-manifold $M$.

**4.4** (Complete $(G,X)$-structure). Let $(G,X)$... Let $M$ be a connected smooth manifold.
Developing map and holonomy.
We will show the equivalence of the following statements.

(i) $M$ admits a geometric structure such that the universal covering is $X$.

(ii) $M$ admits a complete $(G,X)$-structure.

Therefore, for a model geometry $(G,X)$, a complete $(G,X)$-structure on $M$ be called a *geometric structure* on $M$.

(a) (i) implies (ii).

(b) (ii) implies (i).

analyticity of isometries of homogeneous Riemannian manifolds?
examples.

**4.5** (Thurston's eight geometries). We define a *model geometry* or a *Thurston geometry* as a simply connected geometric manifold $X$ such that there exists at least one space form of finite volume.

## 4.2 Poincaré polygon and polyhedron theorems

## 4.3 Mostow rigidity

Kleinian groups Several topological invariants: volume, trace fields, etc.

## 4.4 Hyperbolization Dehn surgery

**4.6** (Ideal triangulation of knot complement). Cusped hyperbolic 3-manifolds

**4.7** (Cusp and horoball).

**4.8** (Thick-thin decomposition). Margulis constant.

**4.9** (Thurston's Hyperbolic Dehn surgery).

## 4.5 Orbifolds

Chapter 5

# Teichmüller theory

## 5.1

# Chapter 6

# Geometric group theory

## 6.1

# Part III

# Topology of 4-manifolds

# Chapter 7

# Surgery theory

# Chapter 8

# Intersection forms

## Chapter 9

# Kirby calculus

# Part IV

# Geometry of 4-manifolds

# Chapter 10

# Chapter 11

# Chapter 12