Geometry

Ikhan Choi

May 26, 2024

Contents

Part I

Classical geometry

Chapter 1

Euclidean geometry

1.1 Plane geometry

1.2 Solid geometry

1.3 Axiomatization

Chapter 2

Non-Euclidean geometry

2.1 Absolute geometry

axioms 1 to 4

2.2 Spherical and elliptic geometry

axioms 2 and 4

2.3 Hyperbolic geometry

axiomes 1 to 4

Models of hyperbolic geometry (metric description) Elementary figures Isometries Length, volume, angle

Chapter 3

Non-metric geometry

3.1 Ordered and incidence geometry

axioms 1 and 2

3.2 Affine and projective geometry

axioms 1,2,5

3.3 Conformal and inversive geometry

Part II

Smooth surfaces

Chapter 4

Smooth manifolds

4.1 Local coordinates

4.2 Space curves

4.3 Space surfaces

Reparametrizations

Theorem 4.3.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 4.3.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 4.3.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 4.3.4. *Metric coefficients*

$$\langle \alpha_x, \alpha_x \rangle =: g_{11} \qquad \langle \alpha_x, \alpha_y \rangle =: g_{12}$$
$$\langle \alpha_y, \alpha_x \rangle =: g_{21} \qquad \langle \alpha_y, \alpha_y \rangle =: g_{22}$$

Theorem 4.3.5 (Normal coordinates)**.** *...?*

Differentiation of tangent vectors

Definition 4.3.6. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x, y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x, y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 4.3.7 (Gauss formula, Γ_{ij}^k, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$ and $\{L_{ij}\}_{i,j=1}^2$ by the Gauss formula

$$\begin{aligned} \alpha_{xx} &=: \Gamma_{11}^1 \alpha_x + \Gamma_{11}^2 \alpha_y + L_{11}\,\nu, \qquad & \alpha_{xy} &=: \Gamma_{12}^1 \alpha_x + \Gamma_{12}^2 \alpha_y + L_{12}\,\nu, \\ \alpha_{yx} &=: \Gamma_{21}^1 \alpha_x + \Gamma_{21}^2 \alpha_y + L_{21}\,\nu, \qquad & \alpha_{yy} &=: \Gamma_{22}^1 \alpha_x + \Gamma_{22}^2 \alpha_y + L_{22}\,\nu. \end{aligned}$$

The *Christoffel symbols* refer to eight functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij}\,\nu. \end{aligned}$$

Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu \rangle = \frac{1}{2} \partial_X \langle \nu, \nu \rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 4.3.8. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 4.3.9. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y) \rangle = \langle X, -\partial_Y \nu \rangle = \langle \partial_Y X, \nu \rangle = \langle \partial_X Y, \nu \rangle = \langle \mathcal{S}(X), Y \rangle. \qquad \square$$

Theorem 4.3.10. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i \alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j \alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}_i^j X_i$. Then,

$$g_{ik} X^i \mathcal{S}_j^k Y^j = \langle X, \mathcal{S}(Y) \rangle = \langle \partial_X Y, \nu \rangle = X^i Y^j L_{ij}$$

implies $g_{ik}\, \mathcal{S}_j^k = L_{ij}$. $\square$

Chapter 5

Fundamental forms

5.1 Riemannian metrics

5.2 Gaussian curvatures

Theorema egregium surfaces of constant gaussian curvature

Definition 5.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$\begin{aligned} E &:= \langle \alpha_x, \alpha_x \rangle = g_{11}, \qquad & F &:= \langle \alpha_x, \alpha_y \rangle = g_{12}, \qquad & G &:= \langle \alpha_y, \alpha_y \rangle = g_{22}, \\ L &:= \langle \alpha_{xx}, \nu \rangle = L_{11}, \qquad & M &:= \langle \alpha_{xy}, \nu \rangle = L_{12}, \qquad & N &:= \langle \alpha_{yy}, \nu \rangle = L_{22}. \end{aligned}$$

Corollary 5.2.2. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\begin{aligned} \nu_x &= \frac{FM - GL}{EG - F^2} \alpha_x + \frac{FL - EM}{EG - F^2} \alpha_y, \\ \nu_y &= \frac{FN - GM}{EG - F^2} \alpha_x + \frac{FM - EN}{EG - F^2} \alpha_y. \end{aligned}$$

Theorem 5.2.3.

$$\Gamma_{ij}^l = \frac{1}{2} g^{kl} (g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2} (\log g)_x = \Gamma_{11}^1.$$

$$\nu_x \times \nu_y = K \sqrt{\det g} \ \nu.$$

$$\alpha_x \times \alpha_y = \sqrt{\det g} \ \nu$$

$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det \begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

5.1 (Gaussian curvature formula). (a) In general,

$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) For orthogonal coordinates such that $F \equiv 0$,

$$K = -\frac{1}{2\sqrt{\det g}} \left((\frac{1}{\sqrt{\det g}} E_y)_y + (\frac{1}{\sqrt{\det g}} G_x)_x \right).$$

(c) For $f(x,y,z)=0$,

$$K=-\frac{1}{|\nabla f|^4}\begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f)\end{vmatrix},$$

where ∇f denotes the gradient $\nabla f=(f_x,f_y,f_z)$.

(d) (Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then

$$K=-\tau^2.$$

(e) (Brioschi) E,F,G describes K.

Proof. (a) Clear.

(b) We have $GM=EM$ and

$$\nu_x=-\frac{L}{E}\alpha_x-\frac{M}{G}\alpha_y,\qquad \nu_y=-\frac{M}{E}\alpha_x-\frac{N}{G}\alpha_y.$$

$$\nu_x\times\nu_y=\frac{LN-M^2}{EG}\alpha_x\times\alpha_y$$

After curvature tensors...

□

5.2 (Computation of Gaussian curvatures). (a) (Monge's patch) For $(x,y,f(x,y))$,

$$K=\frac{f_{xx}f_{yy}-f_{xy}^2}{(1+f_x^2+f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t)=(r(t),z(t))$ be a plane curve with $r(t)>0$. If $t\mapsto(r(t),z(t))$ is a unit-speed curve, then

$$K=-\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

Proof. (b) Let

$$\alpha(\theta,t)=(r(t)\cos\theta,r(t)\sin\theta,z(t))$$

be a parametrization of a surface of revolution. Then,

$$\begin{aligned}\alpha_\theta&=(-r(t)\sin\theta,r(t)\cos\theta,0)\\ \alpha_t&=(r'(t)\cos\theta,r'(t)\sin\theta,z'(t))\\ \nu&=\frac{1}{\sqrt{r'(t)^2+z'(t)^2}}(z'(t)\cos\theta,z'(t)\sin\theta,-r'(t)),\end{aligned}$$

and

$$\begin{aligned}\alpha_{\theta\theta}&=(-r(t)\cos\theta,-r(t)\sin\theta,0)\\ \alpha_{\theta t}&=(-r'(t)\sin\theta,-r'(t)\cos\theta,0)\\ \alpha_{tt}&=(r''(t)\cos\theta,r''(t)\sin\theta,z''(t)).\end{aligned}$$

Thus we have

$$E=r(t)^2,\quad F=0,\quad G=r'(t)^2+z'(t)^2,$$

and

$$L=-\frac{r(t)z'(t)}{\sqrt{r'(t)^2+z'(t)^2}},\quad M=0,\quad N=\frac{r''(t)z'(t)-r'(t)z''(t)}{\sqrt{r'(t)^2+z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

□

5.3 (Local isomorphism)**.** Surfaces of the same constant Gaussian curvature are locally isomorphic.

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 6

Part III

Riemann surfaces

Chapter 7

Riemann-Roch theorem

Let X be a compact Riemann surface. Consider a vector space $\mathcal{M}^\times(X) \cup \{0\}$.

$$L(D) := H^0(X, \mathcal{O}(D)) = \{f \in \mathcal{M}^\times(X) : (f) + D \geq 0\} \cup \{0\}.$$

$$\mathrm{Div}(X) = H^0(X, \mathcal{M}^\times/\mathcal{O}^\times) = \Gamma(\mathcal{M}^\times/\mathcal{O}^\times).$$

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\times).$$

First Chern class $H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z})$.

7.1. Let X be a compact Riemann surface. A *Weil divisor* D on X is an element of the free abelian group $\mathrm{Div}(X)$ generated by points of X. By compactness of X, a meromorphic function $f \in \mathcal{M}(X)$ gives rise to a divisor $(f) := \sum_{p \in X} \mathrm{ord}_p(f) p$. Such a divisor is called a *principal divisor*.

Let $D = \sum n_i p_i$ on X be a Weil divisor on X. Each point $P \in X$ has a meromorphic function f on an open neighborhood U of P such that $(f) = D$ on U. It implies that there is a collection $\{f_\alpha\}$ of meromorphic functions f_α defined on U_α, where $\{U_\alpha\}$ is an open cover of X, such that f_α/f_β is a well-defined holomorphic functions on $U_\alpha \cap U_\beta$. The collection $\{f_\alpha\}$ is called a *Cartier divisor*.

A Cartier divisor defines a line bundle.

7.2. Given $\{p_i\}_{i=1}^n$ points and $\{f_i\}_{i=1}^n$ principal parts, there is a meromorphic function f with predescribed principal parts if and only if for every holomorphic 1-form ω we have $\sum_{i=1}^n \mathrm{Res}(f_i\omega, p_i) = 0$.

7.3.

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

The genus can be defined by $g = h^0(X, \Omega^1)$. For algebraic curves, it can be proved as follows: Assuming the Serre duality, we have $\chi(D) = h^0(D) - h^1(D) = l(D) - l(K - D)$ and $\chi(0) = h^0(0) - h^1(0) = 1 - g$. Then, the Riemann-Roch is boiled down to $\chi(D) = \deg(D) + \chi(0)$, which can be shown inductively.

However, we want to prove a compact Riemann surface is projective as an application of the Riemann-Roch theorem, we need to prove the Riemann-Roch theorem without theory of algebraic curves.

(a) If $\deg D < 0$, then $l(D) = 0$.

Proof. (a) Let $f \in L(D) \setminus \{0\}$. Then, $(f) + D \geq 0$ and $\deg(f) = 0$ imply $\deg D \geq 0$, which is a contradiction.

(b) Let $D = 0$. Then, it follows from $l(K) = g$ and $l(0) = 1$.

Let $D > 0$. We may assume $D = \sum_{i=1}^{n} n_i p_i$ with $n_i > 0$. (why?) Let

$$V_i := \Big\{ \sum_{k=-n_i}^{-1} c_k (z - p_i)^k : c_k \in \mathbb{C} \Big\}$$

and $V := \bigoplus_{i=1}^{n} V_i$. (how can we define the principal part of f on Riemann surface?) Then, $\dim V = \deg D$. Define $L(D) \to V$ by principal part at each point p_i.

□

7.4 (Embedding theorem)**.** Let X be a compact Riemann surface. The *complete linear system* of a divisor D on X is

$$|D| := \{(f) + D : f \in \mathcal{O}(X)\}.$$

Then, $|D|$ can be identified with the projective space $(L(D) \setminus \{0\})/\mathbb{C}^\times = \mathbb{CP}^{l(D)-1}$. Let $(f_i)_{i=0}^{l(D)-1}$ be a basis of $L(D)$.

For a linear system Δ of projective dimension $n - 1$, we can take (how?) a basis $(f_i)_{i=0}^{n-1}$ such that the following map is well-defined:

$$X \setminus \mathrm{Bl}(\Delta) \to \mathbb{CP}^{n-1} : p \mapsto (f_0 : \cdots : f_{n-1}).$$

Chapter 8

Algebraic curves

8.1

multiplicities, Bezout theorem

8.2

divisors, line bundles euler characteristic (tangent line bundle degree 2-2g, canonical line bundle 2g-2)
$L(D) := \Gamma(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D))$

Jacobian variety (moduli spaces....)

8.1 (Chow theorem)**.** A complex submanifold of a projective space is algebraic.

Chapter 9

Uniformization

The uniformization theorem provides one philosophy to classify compact Riemann surfaces. The universal covering is one of the three: the Riemann sphere, the complex plane, and the open unit disk. Each compact Riemann surface is realized as a quotient of these model space with a properly discontinuous action.

- $g = 0$: Riemann sphere (spherical) $\to$ Riemann sphere itself
- $g = 1$: complex plane (Euclidean) $\to$ elliptic curves
- $g \geq 2$: open unit disk (hyperbolic) $\to$ hyperbolic surfaces, classified by Fuchsian groups(with which properties?)

Part IV

Topological surfaces

Chapter 10

Fundamental groups

10.1 Homotopy

10.1. A *homotopy of paths* is a continuous map $h : I \times I \to X$ such that $h(0,) = x_0$ and

(a) linear homotopy

(b) reparametrization

10.2. The fundamental group is a group composition

10.3 (Van Kampen theorem)**.**

10.2 Covering spaces

10.4 (Unique path lifting property)**.** Let $p : Y \to X$ be a covering map. For a path $\gamma : [0, 1] \to X$ and a point $y_0 \in Y$ such that $p(y_0) = \gamma(0)$, there is a unique lift $\widetilde{\gamma} : I \to Y$ of γ such that $\widetilde{\gamma}(0) = y_0$.

As a corollary, if γ_0 and γ_1 are end-fixing homotopic and have lifts $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ such that $\widetilde{\gamma}_0(0) = \widetilde{\gamma}_1(0)$, then $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are basepoint-preserving homotopic.

As a corollary, for $p(y_0) = x_0$, the induced map $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective.

Proof. (Uniqueness) The interval $[0, 1]$ can be replaced to any connected set.

(Existence) By the compactness of $[0, 1]$, there is an increasing finite sequence $(t_i)_{i=0}^n$ such that

$$t_0 = 0, \quad t_n = 1, \quad [t_i, t_{i+1}] \subset \gamma^{-1}(U_i), \qquad 0 \le i < n,$$

where U_i is trivializing p.

□

10.5 (Universal covering)**.** connected, locally path connected, semi-locally simply connected

10.6 (Classification of covering spaces)**.** connected, locally path connected, semi-locally simply connected

We say p is *regular* if $p_*(\pi_1(Y, y_0))$ is normal in $\pi_1(X, x_0)$.

$\pi_1(X, x_0)/p_*(\pi_1(Y, y_0)) \to p^{-1}(x_0)$ is always injective, and bijective if Y is path connected.

Examples: S^1, $\mathbb{RP}^n$.

Chapter 11

Homology groups

11.1 Singular homology

11.1 (Eilenberg-Steenrod axioms).

11.2 Simplicial homology

Simplicial homology is defined for simplicial complex, which is purely combinatorial. The singular chain complex of a topological space is the most natural simplicial complex on it. The simplicial homology of this is the singular homology as it is just the definition.

One can associate some other simplicial complexes by *triangulations* to a topological space which are more convenient to compute the homology. We now have to investigate which conditions make a simplicial complex generate same homology groups with singular homology.

Let X be a simplicial complex. A *geometric realization* of X is a topological space $|X|$ defined by.... For a topological space, a *triangulation* is a homeomorphism from the geometric realization of a simplicial complex to the topological space.

11.3 Cellular homology

11.4 Cohomology

Chapter 12

Classification of surfaces

12.1 Combinatorial surfaces

triangulation orientability euler characteristic genus connected sum