Measure Theory

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Contents

Part I

Measures

Chapter 1

Measure spaces

1.1 Measurable spaces

1.1 (Measurable spaces).

1.2 Measure spaces

1.2 (Definition of measures). Let $(\Omega, \mathcal{M})$ be a measurable space. A *measure* on $\mathcal{M}$ is a set function $\mu : \mathcal{M} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i)$$

for $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$. Here the squared cup notation reads the disjoint union.

1.3 (Continuity of measures).

1.4 (Pushforward measures).

1.5 (Complete measures).

1.3 Carathéodory extension

1.6 (Outer measures). Let Ω be a set. An *outer measure* on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty] : \varnothing \mapsto 0$ such that

(i) μ^* is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \leq \mu^*(S_2)$$

for $S_1, S_2 \in \mathcal{P}(\Omega)$,

(ii) μ^* is *countably subadditive*: we have

$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \leq \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

Compairing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(\Omega) \to [0,\infty] : \varnothing \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $S \in \mathcal{P}(\Omega)$ and $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

(b) For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^{\infty} \rho(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{A} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

Proof. □

1.7 (Carathéodory measurability)**.** Let μ^* be an outer measure on a set Ω. We want to construct a measure by restriction of μ^* on a properly defined σ-algebra. A subset $E \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(\Omega)$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is complete.

Proof. □

1.8 (Carathéodory extension theorem)**.** The Carathéodory extension is a construction method for a measure extending a given set function ρ. The idea is to restrict the outer measure μ^* associated to ρ in order to obtain a measure μ. We want to find a sufficient condition for μ to be a measure on a σ-algebra containing $\mathcal{A}$.

For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. Let $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0,\infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) We have $\mu^*|_{\mathcal{A}} = \rho$ if ρ satisfies the monotone countable subadditivity:

$$A \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \rho(A) \le \sum_{i=1}^{\infty} \rho(B_i)$$

for $A \in \mathcal{A}$ and $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$.

(b) We have $\mathcal{A} \subset \mathcal{M}$ if ρ satisfies the following property: for every $B, A \in \mathcal{A}$, and for any $\varepsilon > 0$, there are $\{C_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty} \subset \mathcal{A}$ such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} C_j \quad \text{and} \quad B \setminus A \subset \bigcup_{j=1}^{\infty} D_j,$$

and

$$\rho(B) + \varepsilon > \sum_{j=1}^{\infty} \rho(C_j) + \sum_{j=1}^{\infty} \rho(D_j).$$

Proof. (a) Clearly $\mu^*(A) \le \rho(A)$ for $A \in \mathcal{A}$. We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{B_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(B_i) \ge \rho(A).$$

Limiting $\varepsilon \to 0$, we get $\mu^*(A) \ge \rho(A)$.

(b) Let $S \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. It is enough to check the inequality $\mu^*(S) \ge \mu^*(S \cap A) + \mu^*(S \setminus A)$ for S with $\mu^*(S) < \infty$, so we may assume there is a countable family $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. Then, we have $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$ satisfying

$$\mu^*(S) + \varepsilon > \sum_{i=1}^\infty (\rho(B_i) + \frac{\varepsilon}{2^{i+1}}) > \sum_{i,j=1}^\infty \rho(C_{i,j}) + \sum_{i,j=1}^\infty \rho(D_{i,j}) \ge \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Therefore, A is Carathéodory measurable relative to μ^*. $\square$

1.9 (Uniqueness of extension of measures)**.** The existence of the Carathéodory extension provides a uniqueness theorem for the extension of measures. The important property here is *σ-finiteness*: for $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. Then, we say ρ is σ-finite if there is a countable cover $(B_i)_{i=1}^\infty \subset \mathcal{A}$ of Ω such that $\rho(B_i) < \infty$ for each i.

Let μ^* be the outer measure associated to ρ. Let $\mathcal{M}$ be a σ-algebra such that the restriction $\mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is a measure, and $\mu : \mathcal{M} \to [0,\infty]$ be any measure. Suppose further that $\mu^*(A) = \rho(A) = \mu(A)$ for all $A \in \mathcal{A}$. Let $E \in \mathcal{M}$.

(a) $\mu(E) \le \mu^*(E)$.

(b) If $E_1, E_2 \in \mathcal{M}$ satisfy $\mu(E_1) = \mu^*(E_1)$ and $\mu(E_2) = \mu^*(E_2)$, then $\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2)$.

(c) $\mu(E) = \mu^*(E)$ if $\mu^*(E) < \infty$.

(d) If ρ is σ-finite, then $\mu(E) = \mu^*(E)$ for $\mu^*(E) = \infty$.

Proof. (a) If $\mu^*(E) = \infty$, then $\mu(E) \le \mu^*(E)$ trivially. Suppose $\mu^*(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Also, we have

$$\mu(E) \le \mu\Big(\bigcup_{i=1}^\infty B_i\Big) \le \sum_{i=1}^\infty \mu(B_i) = \sum_{i=1}^\infty \rho(B_i)$$

whenever $E \subset \bigcup_{i=1}^\infty B_i$, so $\mu(E) \le \mu^*(E)$.

(b) In the light of the inclusion-exclusion principle,

$$\begin{aligned}\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) \le \mu^*(E_1 \cup E_2) + \mu^*(E_1 \cap E_2) &= \mu^*(E_1) + \mu^*(E_2)\\ &= \mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2)\end{aligned}$$

proves the identity we want.

(c) Because $\mu^*(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(E) + \varepsilon > \sum_{i=1}^\infty \rho(B_i).$$

Applying the part (b) inductively, we have for every n that

$$\mu\Big(\bigcup_{i=1}^n B_i\Big) = \mu^*\Big(\bigcup_{i=1}^n B_i\Big),$$

and by limiting $n \to \infty$ the continuity from below gives

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \mu^*\Big(\bigcup_{i=1}^{\infty} B_i\Big).$$

Then, we have

$$\mu^*(E) \le \mu^*\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) + \mu(E)$$

and

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) \le \mu^*\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) = \mu^*\Big(\bigcup_{i=1}^{\infty} B_i\Big) - \mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(B_i) - \mu^*(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu^*(E) < \varepsilon,$$

we get $\mu^*(E) < \mu(E) + \varepsilon$ and $\mu^*(E) \le \mu(E)$ by limiting $\varepsilon \to 0$.

(d) Let $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$ be such that $\rho(B_i) < \infty$ and $\Omega = \bigcup_{i=1}^{\infty} B_i$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \ge 2$. Then, $(E_i)_{i=1}^{\infty}$ is a pairwise disjoint cover of Ω with

$$\mu^*(E \cap E_i) \le \mu^*(E_i) \le \mu^*(B_i) = \rho(B_i) < \infty$$

for each i, so we have by the part (c) that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \sum_{i=1}^{\infty} \mu^*(E \cap E_i) = \mu^*(E). \qquad \square$$

Exercises

1.10 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A}$ be a collection of subsets of a set Ω such that $\varnothing \in \mathcal{A}$. We say $\mathcal{A}$ is a semi-ring if it is closed under finite intersection, and the complement is a finite union of elements of $\mathcal{A}$. We say $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over Ω. Suppose a set function $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} \rho(A_i)$$

for $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

1.11 (Monotone class lemma). A collection $\mathcal{C} \subset \mathcal{P}(\Omega)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let H be a vector space closed under bounded monotone convergence. If $\operatorname{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^{\infty}(\sigma(\mathcal{A})) \subset H$.

Chapter 2

Measures on the real line

2.1 (Distribution functions).

2.2 (Helly selection theorem).

2.3 (Non-Lebesgue measurable set).

Exercises

2.4 (Steinhaus theorem). Let $\mathbb{E} \subset \mathbb{R}$ be Lebesgue measurable with $\lambda(E) > 0$.

(a) For any $\alpha < 1$, there is an interval $I = [a, b]$ such that $\lambda(E \cap I)/\lambda(I) > \alpha$.

(b) $E - E$ contains an open interval containing zero.

Proof. (a) □

Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

Chapter 3

Measurable functions

3.1 Extended real numbers

3.2 Simple functions

3.1 (Measurability of pointwise limits)**.** Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

$\square$

3.3 Almost everywhere convergence

3.2 (Almost everywhere convergence)**.** Let (Ω, μ) be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence f_n is defined as the set

$$\{ x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x) \},$$

and the set of divergence is defined as its complement. We say f_n converges to f *alomst everywhere* with respect to μ if the set of divergence is a null set in μ. We simply write

$$f_n \to f \text{ a.e.}$$

if f_n converges to f almost everywhere, and we frequently omit the measure μ if it has no confusion.

(a) If μ is complete and, if $f_n \to f$ a.e., then f is measurable.

3.3 (Tail events)**.** Let (Ω, μ) be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon>0} \bigcap_{n>0} \bigcup_{i\geq n} T_i^\varepsilon,$$

where

$$T_n^\varepsilon := \{ x : |f_n(x) - f(x)| \geq \varepsilon \},$$

which is called the *tail event*. The term is originated from probability theory.

(a) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n\to\infty} T_n^\varepsilon) = 0.$$

3.4 (Borel-Cantelli lemma)**.**

3.5 (Convergence in measure)**.** Let (Ω,μ) be a measure space and let $f_n:\Omega\to\overline{\mathbb{R}}$ be a sequence of measurable functions. We say f_n converges to a measurable function $f:\Omega\to\overline{\mathbb{R}}$ *in measure* if for each $\varepsilon>0$ we have

$$\lim_{n\to\infty}\mu(\{x:|f_n(x)-f(x)|>\varepsilon\})=\lim_{n\to\infty}\mu(T_n^\varepsilon)=0.$$

(a) If $f_n\to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k}\to f$ a.e.

(b) If every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}}\to f$ a.e., then $f_n\to f$ in measure.

Proof. (a) Since $\mu(T_n^{1/k})\to 0$ for each k as $n\to\infty$, there is n_k such that

$$\mu(T_{n_k}^{1/k})<\frac{1}{2^k}.$$

We claim that $f_{n_k}\to f$ a.e. Since

$$\sum_{k=1}^\infty \mu(T_{n_k}^{1/k})<\infty,$$

by the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k\to\infty}T_{n_k}^{1/k})=0.$$

For each $\varepsilon>0$,

$$\limsup_{k\to\infty}T_{n_k}^\varepsilon=\bigcap_{k>\varepsilon^{-1}}\bigcup_{j\ge k}T_{n_j}^\varepsilon\subset\bigcap_{k>\varepsilon^{-1}}\bigcup_{j\ge k}T_{n_j}^{1/k}=\limsup_{k\to\infty}T_{n_k}^{1/k}$$

implies $f_{n_k}\to f$ a.e.

(b) □

3.6 (Egorov theorem)**.** Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient "$\varepsilon/2^m$ argument", occasionally used throughout measure theory to construct a measurable set having a special property.

Let (Ω,μ) be a measure space and let $f_n:\Omega\to\overline{\mathbb{R}}$ be a sequence of measurable functions. Our idea is to consider a family of sequences of increasing measurable subsets which converge to full sets. Let

$$E_n^m:=\bigcap_{i\ge n}\{x:|f_i(x)-f(x)|<\tfrac{1}{m}\}.$$

Note that $\Omega\setminus E_n^m=\bigcup_{i\ge n}T_n^{1/m}$.

(a) Suppose $\mu(\Omega\setminus E_n^m)\to 0$ as $n\to\infty$ for each m. Then, for every $\varepsilon>0$ there is a measurable $K\subset\Omega$ such that $\mu(\Omega\setminus K)<\varepsilon$ and for each m there is n satisfying $K\subset E_n^m$.

(b) Let $\mu(\Omega)<\infty$. Then, $f_n\to f$ a.e. if and only if $\mu(\Omega\setminus E_n^m)\to 0$ as $n\to\infty$ for each m.

(c) Let $\mu(\Omega)<\infty$. If $f_n\to f$ a.e., then for every $\varepsilon>0$ there is a measurable $K\subset\Omega$ such that $\mu(\Omega\setminus K)<\varepsilon$ and $f_n\to f$ uniformly on K.

Proof. (a) For each m, we can find n_m such that

$$\mu(\Omega\setminus E_{n_m}^m)<\frac{\varepsilon}{2^m}.$$

If we define

$$K:=\bigcap_{m=1}^\infty E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(\Omega\setminus K)=\mu\Big(\bigcup_{m=1}^{\infty}(\Omega\setminus E^m_{n_m})\Big)\le\sum_{m=1}^{\infty}\mu(\Omega\setminus E^m_{n_m})<\sum_{m=1}^{\infty}\frac{\varepsilon}{2^m}=\varepsilon.$$

(b) The set of divergence of the sequence f_n is given by

$$\bigcup_{m>0}\bigcap_{n>0}\bigcup_{i\ge n}\{x:|f_i(x)-f(x)|\ge\tfrac{1}{m}\}=\bigcup_{m>0}\bigcap_{n>0}(\Omega\setminus E^m_n).$$

Then, the convergence $f_n\to f$ a.e. means that for every fixed m the intersection

$$\bigcap_{n>0}(\Omega\setminus E^m_n)=\limsup_n T^m_n$$

is a null set. Since $\mu(\Omega)<\infty$ and we have $\Omega\setminus E^m_n\supset\Omega\setminus E^m_{n+1}$ clearly by definition, we are done by the continuity from above.

(c) Fix $m>0$. Since $n\ge n_m$ implies $K\subset E^m_{n_m}\subset E^m_n$, we have

$$n\ge n_m\quad\Rightarrow\quad\sup_{x\in K}|f_n(x)-f(x)|<\frac{1}{m}.\qquad\square$$

Exercises

3.7 (Cauchy's functional equation)**.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x+y)=f(x)+f(y)$, satisfied for all $x,y\in\mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x)=ax$ for all $x\in\mathbb{R}$, where $a:=f(1)$.

(a) $f(x)=ax$ for all $x\in\mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If f is conitnuous at a point, then f is linear.

(c) If f is Lebesgue measurable, then f is linear.

Part II

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The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i} u v + u \partial_{x_i} v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i} u v + u \partial_{x_i} v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega_x \times \Omega_y \to \mathbb{R}$ be such that $\partial_{x_i} f$ is well-defined. Suppose f and $\partial_{x_i} f$ are locally integrable in x and integrable y.

Then,

$$\partial_{x_i} \int f(x,y)\,dy = \int \partial_{x_i} f(x,y)\,dy.$$

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}
(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞
(b) f is AC iff f' is L^1
(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

Lebesgue differentiation theorem