C^*-Algebras

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Contents

Part I

Constructions

Chapter 1

Completely positive maps

1.1 Operator systems and spaces

1.1 (Choi-Effros characterization)**.**

1.2 (Von Neumann inequality)**.**

1.3 (n-positive maps)**.** Let S be an operator system. Let A and B be C*-algebras.

(a) (Cauchy-Schwarz inequality) Let $\varphi : A \to B$ be a 2-positive map. Then,

$$\varphi(a)^*\varphi(a) \le \lim_\alpha \|\varphi(e_\alpha)\|\varphi(a^*a)$$

for all $a \in A$, where e_α be an approximate unit of A. In particular, $\lim_\alpha \|\varphi(e_\alpha)\| = \|\varphi\|$.

(b) (Multiplicative domain) Let $\varphi : A \to B$ be a 4-positive map with $\|\varphi\| = 1$. If $a \in A$ satisfies $\varphi(a)^*\varphi(a) = \varphi(a^*a)$, then $\varphi(b)\varphi(a) = \varphi(ba)$ for all $b \in A$. In particular, if $\varphi : B \to C$ is an extension of a $*$-homomorphism $\pi : A \to C$, then $\varphi(ab) = \pi(a)\varphi(b)$ and $\varphi(ba) = \varphi(b)\pi(a)$ for $a \in A$ and $b \in B$.

Proof. (a) Consider B to act on a Hilbert space H non-degenerately and faithfully. The 2-positivity of φ and

$$\begin{pmatrix} e_\alpha^2 & e_\alpha a \\ a^* e_\alpha & a^* a \end{pmatrix} = \begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix} \ge 0$$

implies

$$\begin{pmatrix} \varphi(e_\alpha^2) & \varphi(e_\alpha a) \\ \varphi(a^* e_\alpha) & \varphi(a^* a) \end{pmatrix} \ge 0,$$

which is equivalent to have

$$\langle \varphi(e_\alpha^2)\xi, \xi\rangle + 2\operatorname{Re}\langle \varphi(e_\alpha a)\eta, \xi\rangle + \langle \varphi(a^*a)\eta, \eta\rangle \ge 0$$

for any $\xi, \eta \in H$. We put $\xi := -(\|\varphi(e_\alpha)\| + \varepsilon)^{-1}\varphi(e_\alpha a)\eta$ for $\varepsilon > 0$ to get

$$\begin{aligned}(\|\varphi(e_\alpha)\| + \varepsilon)\varphi(a^*a) &\ge \varphi(e_\alpha a)^*(2 - (\|\varphi(e_\alpha)\| + \varepsilon)^{-1}\varphi(e_\alpha^2))\varphi(e_\alpha a) \\ &\ge \varphi(e_\alpha a)^*\varphi(e_\alpha a).\end{aligned}$$

We have the desired inequality by taking limits for α and ε.

(b) The 2-positivity of φ_2 gives

$$\varphi_2\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right)^* \varphi_2\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \le \varphi_2\left(\begin{pmatrix} a^*a & a^*b \\ b^*a & b^*b \end{pmatrix}\right),$$

so

$$\begin{pmatrix} 0 & \varphi(a^*b)-\varphi(a^*)\varphi(b) \\ \varphi(b^*a)-\varphi(b^*)\varphi(a) & \varphi(b^*b)-\varphi(b^*)\varphi(b) \end{pmatrix} \geq 0,$$

which implies $\varphi(b^*a)-\varphi(b^*)\varphi(a)=0$ for any $b\in A$.

Note that $\|\pi\|=1$ if π is not trivial. Using the above argument for a and a^*, we are done. □

1.4 (Russo-Dye theorem)**.** If $C(X)\to B$ is positive, then it is completely positive.

1.5 (Completely positive maps for matrix algebras)**.** Let A be a C*-algebra.

(a) Choi matrix

(b) There is a one-to-one correspondence

$$\mathrm{CP}(M_n(\mathbb{C}),A)\to M_n(A)_+ : \varphi\mapsto[\varphi(e_{ij})].$$

(c) Let A be unital. There is a one-to-one correspondence

$$\mathrm{CP}(A,M_n(\mathbb{C}))\to M_n(A)^*_+ : \varphi\mapsto (s_\varphi : [a_{ij}]\mapsto \sum_{i,j}\langle\varphi(a_{ij})e_j,e_i\rangle).$$

(d) The above correspondences are (maybe?) isometric if we endow the complete norm on CP.

Proof. (b)

□

1.2 Dilations and Extensions

1.6 (Stinespring dilation)**.** Let A be a C*-algebra and $\varphi: A\to B(H)$ is a completely positive map. There exist a representation $\pi : A\to B(K)$ and a bounded linear operator $V : H\to K$ such that $\varphi(a)=V^*\pi(a)V$ for $a\in A$.

$$\begin{tikzcd} B(K) \arrow[dr, "V^*\cdot V"] & \\ A \arrow[u, "\pi"] \arrow[r, "\varphi"'] & B(H) \end{tikzcd}$$

(a) If $\|\varphi\|=1$, then V is an isometry.

(b)

(c) We can take π to be minimal in the sense that $\overline{\pi(A)VH}=K$.

Proof. □

1.7 (Arveson extension)**.** Let $A\subset B$ be C*-algebras. Let $\varphi: A\to B(H)$ be a completely positive map and consider the following diagram:

$$\begin{tikzcd} B \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ A \arrow[u] \arrow[r, "\varphi"'] & B(H). \end{tikzcd}$$

(a) The norm preserving completely positive extension $\widetilde{\varphi}$ of φ exists if B is unital and $1_B\in A$.

(b) The norm preserving completely positive extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is unital and $B=A\oplus\mathbb{C}$.

(c) The norm preserving completely positive extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is non-unital and $B=\widetilde{\mathcal{A}}$.

(d) The norm preserving completely positive extension $\widetilde{\varphi}$ of φ always exists.

extension of representations for ideals
unique extension of c.p. maps for hereditary subalgebras.

1.3 Completely bounded maps

1.4 Tensor products

1.8 (Maximal tensor products)**.** Let A and B be C*-algebras.

(a) A commuting pair of $*$-homomorphisms $\pi : A \to B(H)$ and $\pi' : B \to B(H)$ corresponds to a $*$-homomorphism $\Pi : A \odot B \to B(H)$ via the relation $\Pi(a \otimes b) = \pi(a)\pi'(b)$.

(b) $A \odot B$ admits a $*$-representation and every norms induced from these $*$-representations are uniformly bounded. So, we can define a maximal tensor norm on $A \odot B$.

(c) $a \otimes - : B \to A \odot B$ is a bounded linear map for each $a \in A$ with respect to any C*-norm on $A \odot B$. [BO, 3.2.5]

1.9 (Minimal tensor product)**.** spatiality

1.10 (Takesaki theorem)**.**

Tensors with $M_n(\mathbb{C})$, $C_0(X)$.

1.11 (Haagerup tensor product)**.**

Trick

Exercises

1.12. Let A be a hereditary C*-subalgebra of a C*-algebra B and let $b \in B_+$. If for any $\varepsilon > 0$ there is $a \in A_+$ such that $b - a \leq \varepsilon$, then $b \in A$.

Proof. For $a \in A_+$ satisfying $b \leq a + \varepsilon \leq (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^2$, define

$$a_\varepsilon := a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} \in A.$$

Then,

$$\|b^{\frac{1}{2}} - b^{\frac{1}{2}} a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\|^2 = \varepsilon \|(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\| \leq \varepsilon.$$

Thus $a_\varepsilon \to b$ in norm as $\varepsilon \to 0$. □

Chapter 2

Hilbert modules

2.1 Hilbert modules

2.1 (Banach modules)**.** Let A be a Banach algebra. A *Banach A-module* is a Banach space $\mathcal{E}$ which is a A-module such that the action is bounded.

(a) (Cohen factorization theorem) If A has a left approximate unit, then $A\mathcal{E}$ is closed in $\mathcal{E}$.

Proof. Suppose ξ belongs to the closure of $A\mathcal{E}$ and take $\varepsilon > 0$. We will construct a decreasing sequence a_n in the unitization $\widetilde{A}$ such that $a_n^{-1}\xi$ and a_n are both Cauchy. In order to do this, we first need to check $a_n^{-1} \in \widetilde{A} \setminus A$ can act on $\mathcal{E}$, which is easy anyway.

Let $a_0 = 1$ and suppose we have defined $a_n \geq 2^{-n}$. Take $b \in A$ and η such that $\|\xi - b\eta\| < \varepsilon 2^{-(2n+1)}$. Take $e \in A$ such that $\|a_n^{-1}b - ea_n^{-1}b\|\|\eta\| < \varepsilon 2^{-(n+1)}$. Now inductively define

$$a_{n+1} := a_n - 2^{-(n+1)}(1-e) \in \widetilde{A}$$

so that $a_{n+1} \geq 2^{-(n+1)}$ is invertible.

Then, we can check a_n is Cauchy whose limit point belongs to A, and $a_n^{-1}\xi$ is Cauchy because by the identity

$$a_{n+1}^{-1} - a_n^{-1} = a_{n+1}^{-1}(a_n - a_{n+1})a_n^{-1} = a_{n+1}^{-1}2^{-(n+1)}(1-e)a_n^{-1}$$

we get

$$\begin{aligned}\|a_{n+1}^{-1}\xi - a_n^{-1}\xi\| &\leq \|a_{n+1}^{-1} - a_n^{-1}\|\|\xi - b\eta\| + \|(a_{n+1}^{-1} - a_n^{-1})b\|\|\eta\| \\ &\leq 2^n \cdot \varepsilon 2^{-(2n+1)} + \varepsilon 2^{-(n+1)} = \varepsilon 2^{-n}.\end{aligned}$$

□

2.2 (Finsler modules)**.** Let A be a C*-algebra.

2.3 (Hilbert modules)**.** Let A be a C*-algebra. A *Hilbert A-module* is a complex linear space $\mathcal{E}$ which is a right A-module together with a

(i) a ring homomorphism $A^{op} \to \mathrm{End}_{\mathbb{C}}(\mathcal{E})$,

(ii) an A-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to A$ which is A-linear in second argument,

which is complete with respect to the norm $\|\xi\| := \|\langle\xi,\xi\rangle\|^{\frac{1}{2}}$.

(a)

constructions: direct sum, tensor product, localization
examples: A itself

2.2 Multiplier algebras

2.4 (Double centralizer characterization)**.** Let A be a C*-algebra. A *double centralizer* of A is a pair (L,R) of bounded linear maps on A such that $aL(b)=R(a)b$ for all $a,b\in A$. The *multiplier algebra* $M(A)$ of A is defined to be the set of all double centralizers of A. There is another characterization $M(A):=L_A(A)$, the set of adjointable operators to itself.

2.5 (Cohen factorization theorem)**.** Let A be a non-unital C*-algebra and $\mathcal{E}$ be a left Banach A-module, i.e. a Then,

This theorem is generalized to a non-unital Banach algebra A with a bounded left approximate unit.

2.6 (Strict topology)**.** (a) $\|\pi(a-e_\alpha a)\xi\|^2$

2.7 (Essential ideals)**.** (a) Hilbert C*-module description

2.8 (Examples of multiplier algebras)**.** (a) $M(K(H))\cong B(H)$.

(b) $M(C_0(\Omega))\cong C_b(\Omega)$.

Proof. (a)

(b) First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega)\cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega)\cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective $*$-homomorphism $C_b(\Omega)\to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g\in M(C_0(\Omega))_+$. □

2.3 Pimsner algebras

2.9 (C*-correspondences)**.** Let A be a C*-algebra. A *C*-correspondence* over A is a right Hilbert A-module $\mathcal{E}$ together with a $*$-homomorphism $\varphi:A\to B(\mathcal{E})$, called the *left action*. We say $\mathcal{E}$ is *faithful* or *non-degenerate* if φ is faithful or non-degenerate, respectively.

(a) If $\varphi:A\to M(B)$ is a unital completely positive map, then we can construct a natural A-B-correspondence $\mathcal{E}$ by mimicking the GNS construction on $A\odot B$.

(b) If $\varphi:A\to M(B)$ is a non-degenerate $*$-homomorphism, $\varphi\in\mathrm{Mor}(A,B)$ in other words, then we can associate a canonical A-B-correspondence B such that the left action is realized with φ. More precisely, $\iota:\mathcal{E}\to B: a\otimes b\mapsto\varphi(a)b$ provides a well-defined linear isomorphism (surjectivity follows from the density of $\varphi(A)B$ in B and the Cohen factorization theorem) and the two actions on $\mathcal{E}$ is described by $\iota(a\xi b)=\varphi(a)\iota(\xi)b$.

2.10. Let $\mathcal{E}$ be a C*-correspondence over a C*-algebra A. Let B be a C*-algebra and see it as a trivial C*-correspondence over B. A *representation* of $\mathcal{E}$ on B is a pair (π,τ) of a $*$-homomorphism $\pi:A\to B$ and a linear map $\tau:\mathcal{E}\to B$ such that

$$\pi(\langle\xi,\eta\rangle)=\tau(\xi)^*\tau(\eta),\qquad \tau(\varphi(a)\xi)=\pi(a)\tau(\xi).$$

We define the *Katsura ideal*

$$J(\mathcal{E}):=\varphi^{-1}(K(\mathcal{E}))\cap\varphi^{-1}(0)^\perp.$$

A *covariant representation* is a representation of $\mathcal{E}$ such that

$$\psi(\varphi(a))=\pi(a),\qquad a\in J(\mathcal{E}).$$

(a) Let $(A, \mathbb{Z}, \alpha)$ be a C*-dynamical system and consider the canonical C*-correspondence A over A with the left action $\varphi := \alpha_1 \in \mathrm{Aut}(A) \subset \mathrm{Mor}(A)$. This correspondence is full, faithful, and non-degenerate. Note that also we have $J(A) = \varphi^{-1}(A) \cap A = A$. If (π, τ) is an any representation of this C*-correspondence A on B, then

How can we decribe representations of C*-correspondence A with left action $\varphi \in \mathrm{Aut}(A)$ in terms of covariant representations of the C*-dynamical system $(A, \mathbb{Z}, \alpha)$ with $\alpha_n = \varphi^n$?

as a morphism sub and quotient, direct sum, tensor product,

Toeplitz-Cuntz Toeplitz-Pimsner Cuntz-Pimsner Cuntz-Krieger

Coactions and Fell bundles

2.4 Mortia equivalence

Induced representations?

Chapter 3

Examples

3.1 Crossed products

3.1 (Group algebras)**.**

type I, subhomogeneous
crystallographic discrete heisenberg free groups projectionless of $C_r^*(F_2)$

3.2 (Enveloping C*-algeberas)**.** Let A be a $*$-algebra. A *C*-norm* is an submultiplicative norm satisfying the C*-identity. Does A have enough $*$-representations?

(a) A complete C*-norm is unique if it exists.

(b) For every C*-norm α on A, there is a $*$-isometry $\pi : A \to B(H)$.

(c) For maximal C*-norm, there is a universal property. The maximal C*-norm can be obtained by running through cyclic representations.

3.3 (C*-dynamical system)**.** Let G be a locally compact group. A *C*-dynamical system* or a *G-C*-algebra* is a C*-algebra A together with a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ that is continuous in the point-norm topology. We will often write a triple (A, G, α) instead of A to refer a C*-dynamical system.

(a) There is an equivalence between categories of locally compact transformation groups and C*-dynamical system on abelian C*-algebras.

On $U(H)$, the strict topology and the strong operator topology are equal. Therefore, we have three topologies to consider: strong, weak, and σ-weak.

3.4 (Covariant representation)**.** Let G be a locally compact group.

A *covariant representation* of a C*-dynamical system (A, G, α) is a G-equivariant $*$-homomorphism $\pi : (A, G, \alpha) \to (B(H), G, \beta)$ for a C*-dynamical system $(B(H), G, \beta)$, where a Hilbert space H.

(a) There exists a unitary representation $u : G \to B(H)$ such that $\pi(\alpha_s a) = u_s \pi(a) u_s^*$.

(b) (Integrated form) There is a one-to-one correspondence between covariant representations of (A, G, α) and $*$-representations of $L^1(G, A)$. (non-degenerate)

Note that we have a homeomorphism $\mathrm{Aut}(K(H)) \cong PU(H)$ between the point-norm topology and the strong operator topology.

$\mathbb{Z}$-action, Homeo-action, left multiplication of subgroup induced representation regular representation $(C_0(G), G, \lambda) \to (B(L^2(G)), G, \lambda)$.

commutative case

3.2 Graph algebras

3.3 Groupoid algebras

3.4 Free products

Part II

Properties

Chapter 4

Approximation properties

4.1 Nuclearity and exactness

finite dimensional[BO, 3.3.2], abelian, AF permanence properties

4.1 (Completely positive approximation property)**.** Let A be a C*-algebra.

(a) If A has the CPAP, then A is nuclear.

(b) If A is nuclear, then A has the CPAP.

Proof. (b)

Let $E \subset A$ and $F \subset A^*$ be finite subsets and fix $\varepsilon > 0$. We want to find completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon$$

for $a \in E$ and $l \in F$. To implement the approximation, we would like to regard a bounded linear operator on A as a state of a tensor product of C*-algebras, which maps $\theta \in B(A)$ to the linear functional characterized by $a \otimes l \mapsto l(\theta(a))$. However, since A^* is not a C*-algebra, we embed A^* locally in $B(H)$ through the Radon-Nikodym type result. Let $\pi : A \to B(H)$ be the cyclic representation obtained from a positive linear functional that dominates F and Ω the cyclic vector such that there is a linear map $\pi' : F \to \pi(A)'$ satisfying

$$l(a) = \omega_\Omega(\pi(a)\pi'(l)) = \langle \pi(a)\pi'(l)\Omega, \Omega\rangle$$

for $a \in E$ and $l \in F$.

Since A is nuclear, we have a well-defined representation

$$\pi \times i : A \otimes_{\min} \pi(A)' \to B(H).$$

If we take any faithful representation $\rho : A \to B(K)$, then we obtain a fathful representation

$$\rho \otimes i : A \otimes_{\min} \pi(A)' \to B(K \otimes H).$$

By the Hahn-Banach separation, the state $\omega_\Omega \circ (\pi \times i)$ on $A \otimes_{\min} \pi(A)'$ can be approximated weakly* by convex combinations of vector states in $B(K \otimes H)$. In particular, by the density of $\pi(A)\Omega$ in H, we have algebraic tensors $(\tau_k)_{k=1}^m \subset K \odot \pi(A)\Omega$ such that

$$\left|\omega_\Omega((\pi \times i)(a \otimes \pi'(l))) - \sum_{k=1}^m \lambda_k \omega_{\tau_k}((\rho \otimes i)(a \otimes \pi'(l)))\right| < \varepsilon \qquad (\dagger)$$

for all $a \in E$ and $l \in F$, where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda = 1$.

If we write each element $\tau \in K \odot \pi(\mathcal{A})\Omega$ as

$$\tau = \sum_{i=1}^{n} \eta_i \otimes \pi(b_i)\Omega,$$

then

$$\begin{aligned}\omega_\tau((\rho \otimes i)(a \otimes \pi'(l))) &= \left\langle (\rho(a) \otimes \pi'(l))\Big(\sum_{j=1}^{n} \eta_j \otimes \pi(b_j)\Omega\Big), \Big(\sum_{i=1}^{n} \eta_i \otimes \pi(b_i)\Omega\Big) \right\rangle \\ &= \sum_{i,j=1}^{n} \langle \rho(a)\eta_j, \eta_i\rangle \langle \pi'(l)\pi(b_i^* b_j)\Omega, \Omega\rangle \\ &= l\Big(\sum_{i,j=1}^{n} \langle \rho(a)\eta_j, \eta_i\rangle b_i^* b_j\Big).\end{aligned}$$

If we define completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ for each τ such that

$$\varphi(a) := [\langle \rho(a)\eta_j, \eta_i\rangle], \quad \psi([e_{ij}]) := b_i^* b_j,$$

then we have $\omega_\tau(a \otimes \pi'(l)) = l(\psi \circ \varphi(a))$.

Since $\mu(a \otimes \pi'(l)) = l(a)$ and since the completely positive contractions which factor through a matrix algebra form a convex set, we have completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that the inequality (†) is rewritten as

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon,$$

so we are done. □

quotients of nuclear local reflexivity

a separable C*-algebra is nuclear if and only if every factor representation is hyperfinite.

Extension properties weak expectation property relatively weakly injective maximal tensor product inclusion problem

4.2 Quasi-diagonality

4.2 (Weyl-von Neumann theorem)**.** A self-adjoint bounded operator is quasi-diagonal.

4.3 (Glimm lemma)**.**

4.4 (Voiculescu theorem)**.**

4.5 (Quasi-diagonal algebras)**.** An operator $a \in B(H)$ is called *quasi-diagonal* if there is a net of projections $p_i \in B(H)$ such that $[p_i, a] \to 0$ in norm and $p_i \uparrow \mathrm{id}_H$ strongly. A C*-algebra is called *quasi-diagonal* if it admits a faithful representation whose image is quasi-diagonal.

faithful non-degenerate essential representations of a quasi-diagonal C*-algebra are all quasi-diagonal locally quasi-diagonal

4.3 AF-embeddability

Chapter 5

Amenability

5.1 Amenable groups

5.2 Amenable actions

crossed products Z_2-grading Connes-Feldman-Weiss Anantharaman-Delaroche Gromov boundaries approximately central structure? dynamical Kirchberg-Phillips

stably finite dynamical Elliott program

Ornstein-Weiss-Rokhlin lemma

5.3 Exact groups

Exact groups

5.4 Other properties

Kazdahn property (T) factorization property Haagerrup property

Kaplansky conjecture

Chapter 6

Simplicity

Furstenburg boundary

Part III

Invariants

Chapter 7

Operator K-theory

7.1 Homotopy of C*-algebras

7.1 (Homotopy of $*$-homomorphisms)**.** Let A, B be C*-algebras. Two $*$-homomorphisms in $\mathrm{Mor}(A, B)$ are said to be *homotopic* if they are connected by a path in $\mathrm{Mor}(A, B)$ that is continuous with the point-norm topology.

(a) For pointed compact Hausdorff spaces $(X, x_0), (Y, y_0)$, two pointed maps $\varphi_0, \varphi_1 : X \to Y$ are homotopic if and only if $\varphi_0^*, \varphi_1^* : C_0(Y \setminus \{y_0\}) \to C_0(X \setminus \{x_0\})$ are homotopic.

Proof. (a) Suppose φ_0 and φ_1 are connected by a homotopy φ_t. Fixing $g \in C_0(Y)$ and $t_0 \in I$, we want to show

$$\lim_{t \to t_0} \sup_{x \in X} |g(\varphi_t(x)) - g(\varphi_{t_0}(x))| = 0.$$

Since the function g is uniformly continuous, with respect to an arbitrarily chosen uniformity on Y, so that there is an entourage $E \subset Y \times Y$ such that $(y, y') \in E \circ E$ implies $|g(y) - g(y')| < \varepsilon$. Using compactness we have a finite sequence $(y_i)_{i=1}^n \subset Y$ such that for every y there is y_i satisfying $(y, y') \in E$. Then, $\varphi^{-1}(E[y_i])$ is a finite open cover of $X \times I$, so we have δ such that $|t - t_0| < \delta$ implies for any $x \in X$ the existence of i satisfying $(\varphi_t(x), y_i) \in E$ and $(\varphi_{t_0}(x), y_i) \in E$, which deduces the desired inequality.

Conversely, suppose φ_0^* and φ_1^* are connected by a homotopy φ_t^*. By taking dual, we can induce $\varphi_t : X \to Y$ such that $g(\varphi_t(x)) = (\varphi_t^* g)(x)$ for each $g \in C(Y)$ from φ_t^* via the embedding $X \to M(X)$ by Dirac measures. Let V be an open neighborhood of $\varphi_{t_0}(x_0)$ and take $g \in C(Y)$ such that $g(\varphi_{t_0}(x_0)) = 1$ and $g(y) = 0$ for $y \notin V$. Now we have an open neighborhood U of x_0 such that $x \in U$ implies $|(\varphi_{t_0}^* g)(x) - (\varphi_{t_0}^* g)(x_0)| < \frac{1}{2}$. Also we have $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|\varphi_t^* g - \varphi_{t_0}^* g\| < \frac{1}{2}$. Therefore, $(x, t) \in U \times (t_0 - \delta, t_0 + \delta)$ implies $g(\varphi_t(x)) > 0$, hence $\varphi_t(x) \in V$, which means $X \times I \to Y : (x, t) \mapsto \varphi_t(x)$ is continuous. $\square$

We have $\widetilde{K}^n(X, x_0) = K_n(C_0(X \setminus \{x_0\}))$ for a pointed compact Hausdorff space X. Now then since the inclusion $\{x_0\} \to X$ induces the section so that

$$0 \to K_0(C_0(X \setminus \{x_0\})) \to K_0(C(X)) \to K_0(\{x_0\}) \to 0$$

splits, we have

$$K^0(X) = \widetilde{K}^0(X, x_0) \oplus \mathbb{Z} = K_0(C_0(X \setminus \{x_0\})) \oplus K_0(\{x_0\}) = K_0(C(X))$$

for a compact connected Hausdorff space X. The additivity of K_0 and K^0 removes the connectedness condition.

$$K_0(\mathbb{C}) = \mathbb{Z}, \quad K_0(C_0(\mathbb{R})) = 0, \quad K_1(C_0(\mathbb{R})) = K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$$
$$K^0(*) = \mathbb{Z}, \quad K^0(S^1) = \mathbb{Z}, \quad K^1(S^1) = K^0(S^2) = \mathbb{Z}[x]/(x-1)^2$$

7.2 Brown-Douglas-Fillmore theory

7.2 (Haagerup property).

Baum-Connes conjecture Non-commutative geometry Elliott theorem

Chapter 8

KK-theory

8.1 Cuntz pairs

8.2 Kasparov modules

Chapter 9

Cuntz semigroup

Part IV

Classification

Chapter 10

Simple nuclear algebras

10.1 AF-algebras

10.2 Elliott invariant

10.3 Kirchberg-Phillips theorem

10.4 Classifiability

Jiang-Su stability Universal coefficient theorem

Toms-Winter conjecture strongly self-absorbing nuclear dimension

successful in Kirchberg algebras

https://arxiv.org/pdf/2307.06480.pdf

Elliott classification problem Kirchberg-Phillipes theorem

operator K-theory and its pairing with traces

$\mathcal{Z}$-stability, Rosenberg-Schochet universal coefficient theorem

Connes-Haagerup classification of injective factors

Kirchberg: unital simple separable $\mathcal{Z}$-stable algebra is either purely infinte or stably finite. Haagerup, Blackadar, Handelman: unital simple stably finite algebra has a trace.

Glimm: uniformly hyperfinite algebras Murray-von Neumann: hyperfinite II_1 factors

Chapter 11

Continuous fields

11.1 Banach bundles

11.1 (Banach bundles)**.** A *Banach bundle*, introduced by Fell, is a continuous open surjection $\pi : E \to X$ between topological spaces together with Banach space structure on each fiber $\pi^{-1}(x)$ such that:

(i) the addition $\{(e, e') : \pi(e) = \pi(e')\} \subset E \times E \to E : (e, e') \mapsto e + e'$ is continuous,

(ii) the scalar multiplication $\mathbb{C} \times E \to E : (\lambda, e) \mapsto \lambda e$ is continuous,

(iii) the norm $E \to \mathbb{R}_{\geq 0} : e \mapsto \|e\|$ is continuous,

(iv) the family of subsets
$$\{e \in B : \pi(e) \in U,\ \|e\| < r\}_{U \in N(x), r>0}$$
forms a neighborhood basis of $0 \in \pi^{-1}(x)$ in E.

The forth condition is equivalent to that if $\|e_i\| \to 0$ and $\pi(e_i) \to x$ then $e_i \to 0_x \in \pi^{-1}(x)$.

(a) For a Banach bundle $E \to X$, if X is locally compact Hausdorff and every fiber E_x shares a same finite dimension, then the bundle is locally trivial.

11.2 (Continuous fields of Banach spaces)**.**

11.3 (Hilbert bundles)**.** A *Hilbert bundle* is a Banach bundle whose norm function satisfies the parallelogram law.

(a) On a compact X, there is an equivalence between the category of Hilbert $C(X)$-modules and the category of Hilbert bundles over X.

(b) On a compact X, there is an equivalence between the category of algebraically finitely generated Hilbert $C(X)$-modules and the category of classical locally trivial finite-rank complex vector bundle over X. It is due to that finitely generatedness implies the projectivity and the Serre-Swan theorem.

11.2 Dixmier-Douady theory

Fell's condition

A C*-algebra A is called *continuous trace* if the set of all $a \in \mathcal{A}$ such that $\widehat{A} \to \mathbb{R}_{\geq 0} : \pi \mapsto \mathrm{tr}(\pi(a^*a))$ is continuous is dense in A.

Dadarlat-Pennig theory