Complex Analysis

Ikhan Choi

July 20, 2022

Contents

Part I

Holomorphic functions

Chapter 1

Cauchy theory

1.1 Complex differentiability

1.2 Contour integral

Cauchy-Goursat theorem

1.3 Power series

Analyticity, Laurent series,

1.4 Cauchy estimates

1.1. Let $p \in \mathbb{C}[z]$ with $p(z) = \sum_{k=0}^{n} a_k z^k$.

(a) $|p(z)| \lesssim |z|^n$.

(b) There is $R > 0$ such that $|p(z)| \gtrsim |z|^n$ for $|z| \geq R$.

Proof. If we take $R > 0$ such that

$$\sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2},$$

then $|z| \geq R$ implies

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \\ &\geq |a_n||z|^n - \sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} |z|^n \\ &\geq \frac{|a_n|}{2}|z|^n. \end{aligned}$$

□

1.2.

1.3 (Open mapping theorem).

Problems

1. If a holomorphic function has positive real parts on the open unit disk then $|f'(0)| < 2\operatorname{Re} f(0)$.
2. If at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.
3. If a holomorphic function on a domain containing the closed unit disk is injective on the unit circle, then so is on the disk.
4. For a holomorphic function f and every z_0 in the domain, there are $z_1 \neq z_2$ such that $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$.
5. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain. Then, $\overline{f(z)} = f(\overline{z})$ if and only if $f(z) \in \mathbb{R}$ for $z \in \Omega \cap \mathbb{R}$.
6. For two linearly independent entire functions, one cannot dominate the other.
7. The uniform limit of injective holomorphic function is either constant or injective.
8. If the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions converges has a limit point, then it compactly converges.
9. Find all entire functions f satisfying $f(z)^2 = f(z^2)$.
10. An entire function maps every unbounded sequence to an unbounded sequence is a polynomial.
11. Let f be a holomorphic function on the open unit disk such that $f(0) = 1$ and $f'(0) > 2$. Then, there is z such that $|z| < 1$ and $f(z)$ is pure imaginary.

Chapter 2

Singularities

2.1 Classification of singularities

Riemann removable singularity theorem, Casorati-Weierstrass theorem, Picard's theorem

2.2 Residue theorem

2.1.

2.2 (Unit circle substitution).

$$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \quad -1 < a < 1$$

2.3 Contour integrals

2.3 (Semicircular contour). Jordan lemma

$$|\int_{C_R} e^{iz} f(z)\,dz| \le \pi \sup_{z\in C_R} |f(z)|$$

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx, \quad \int_0^\infty \frac{1}{1+x^4}\,dx, \quad \int_0^\infty \frac{\cos x}{1+x^2}\,dx,$$

2.4 (Indented contour). Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

2.5 (Sector contour). Fresnel integral

$$\int_0^\infty \cos x^2\,dx = \sqrt{\frac{\pi}{8}}$$

2.6 (Hankel contour).

$$\int_0^\infty \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1), \quad \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$\log z$ trick

$$\int_0^\infty \frac{dx}{1+x^3}$$

2.7 (Rectangular contour)**.** Fourier integral?

$$\int_0^\infty \frac{\sin x}{e^x - 1}\,dx, \quad \int_0^\infty \frac{\cos x}{\cosh x}\,dx$$

2.4 Zeros and poles

2.8 (Argument principle)**.** (a)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = i\text{winding number}.$$

(b)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(\text{number of zeros} - \text{number of poles}).$$

2.9 (Rouché theorem)**.** Let f be a meromorphic function on Ω. Let γ be a curve...

(a) If $h : [0,1] \times \Omega \to \mathbb{C}$ is continuous and

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

(b) In particular, if $|g(z)| < |f(z)|$ on $z \in \gamma$, then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{f'(z) + g'(z)}{f(z) + g(z)}\,dz.$$

Exercises

2.10 (Fundamental theorem of algebra)**.** proof by the Liouville theorem, and proof by the Rouché theorem.

2.11 (Computation of Fourier transforms)**.** sector and Gaussian integral, rectangular integral

2.12 (Laplace transforms)**.**

2.13 (Gamma function)**.** Hankel representation

2.14 (Abel-Plana formula)**.**

Sokhotski-Plemelj theorem, Kramers-Konig relations, Titchmarsh theorem for Hilbert transform, Phragmén-Lindelöf principle, Carlson's theorem

Chapter 3

Polynomial approximation

3.1 Mittag-Leffler theorem

3.2 Weierstrass factorization theorem

3.3 Runge's approximation

Mergelyan

Part II

Geometric function theory

Chapter 4

Conformal mappings

4.1 Riemann sphere and open unit disk

4.2 Riemann mapping theorem

Exercises

4.1 (Special solution of Laplace' equation).

Problems

1. Find a conformal mapping that maps the open unit disk onto $A := \{ z \in \mathbb{C} : \max\{|z|, |z-1|\} < 1 \}$.

Chapter 5

Univalent functions

5.1 Bierbach conjecture

5.2 Harmonic functions

Chapter 6

Maximum principle; Schwarz's lemma, Lindelöf principle,

6.1 Riemann-Hilbert problem

Hilbert transform

6.2 Quasi-conformal mappings

Beltrami equations and Teichmüler theory?

Part III

Riemann surfaces

Chapter 7

Analytic continuation

7.1 Monodromy

7.2 Covering surfaces

7.3 Algebraic functions

7.4 Elliptic curves

Chapter 8

Differential forms

Chapter 9

Uniformization theorem

Part IV

Several complex variables