Functional Analysis

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Contents

Part I

Topological vector spaces

Chapter 1

Locally convex spaces

1.1 Vector topologies

1.1 (Topological vector spaces)**.** A vector space will always mean a real or complex vector space. We will define a *topological vector space* as a vector space together with a Hausdorff vector topology. For every topological vector space, there is a balanced neighborhood system at zero.

1.2 (Canonical uniformity and bornology)**.**

1.3 (Continuity and boundedness of linear operators)**.**

1.4 (Metrizable topological vector spaces)**.** Birkhoff-Kakutani

1.5 (Continuous linear functionals)**.** A linear functional $l : X \to \mathbb{F}$ is continuous if and only if $\ker l$ is closed, if and only if $|l|$ is continuous.

1.2 Seminorms and convex sets

1.6 (Locally convex spaces)**.** A *disk* is a convex balanced subset of a vector space. A topological vector space X is called a *locally convex space* if there is a neighborhood system of disks at zero.

1.7 (Seminorms)**.** Let X be a vector space. A *semi-norm* on X is a functional $p : X \to \mathbb{R}_{\geq 0}$ such that

$$p(x+y) \leq p(x)+p(y), \qquad p(\lambda x) = |\lambda| p(x), \qquad x, y \in X,\ \lambda \in \mathbb{C}.$$

1.8 (Absorbing convex balanced sets)**.** Let X be a vector space. We say a subset $D \subset X$ is *absorbing* if for every $x \in X$ there is a sufficiently large $r > 0$ such that $x \in rD$. For an absorbing disk D, a semi-norm on X defined by $p(x) := \inf\{r \geq 0 : x \in rD\}$ for $x \in X$ is the unique semi-norm satisfying $p^{-1}([0,1)) \subset D \subset p^{-1}([0,1])$. The semi-norm p is called the *gauge* or *Minkowski functional* of D.

In a given topological vector space, open convex sets correspond to continuous sublinear functionals, open convex balanced sets correspond to continuous semi-norms.

Equivalent conditions on the continuity of seminorms, boundedness by seminorms, normability

(a)

Proof. If X is a locally convex space generated by a family of semi-norms $\{p_i\}$, then a subset $B \subset X$ is bounded if and only if $\sup_{x \in B} p_i(x) < \infty$ for each semi-norm p_i. □

Let X be a locally convex space. Let U be an open absorbing convex balanced subset of X. Let p be the Minkowski functional of U. If x_i is a net in X such that $x_i \to 0$, then for every $\varepsilon > 0$ there is i_0 such that $x_i \in \varepsilon U$ for $i \succ i_0$, and it implies $p(x_i) \le \varepsilon$ so we have $p(x_i) \to 0$ by taking limit for i and $\varepsilon \to 0$. That is, p is continuous.

Let D be a disk in a vector space X. By separation and completion of a semi-normed space $(\operatorname{span} D, p_D)$, where p_D is a Minkowski functional of D, we obtain a Banach space $\widehat{X}_D$, called the *auxiliary Banach space* for D. Let X be a locally convex space, that is, a dual pair (X, X^*) together with a choice of polar topology between X_σ and X_τ. Then, D is a neighborhood if and only if there is a natural continuous linear map $X \to \widehat{X}_D$ of densen range, and D is *Banach* if and only if there is a natural continuous linear map $\widehat{X}_D \to X$ that is injective. A linear map $T : X \to Y$ between locally convex topologies is continuous if and only if for every continuous semi-norm q on Y there is a continuous semi-norm p on X such that $|p| \le |q|$.

1.3 Hahn-Banach theorems

1.9. Let $x_k^* \in X^*$ be a finite sequence. If $x^* \in X^*$ vanishes on $\bigcap_k \ker x_k^*$, then x^* is a linear combination of x_k^*.

1.10 (Hahn-Banach extension)**.** Let $X_0 \subset X$ be vector spaces over the real or complex field $\mathbb{F}$. A real functional $q : X \to \mathbb{R}$ is said to be *sublinear* if $q(x+y) \le q(x) + q(y)$ and $q(tx) = tq(x)$ for all $x, y \in X$ and $t \in \mathbb{R}$.

(a) For $q : X \to \mathbb{R}$ sublinear, any linear functional $l_0 : X_0 \to \mathbb{R}$ satisfying $l_0 \le q$ on X_0 admits a linear extension $l : X \to \mathbb{R}$ satisfying $l \le q$.

(b) For $p : X \to \mathbb{R}_{\ge 0}$ a semi-norm, any linear functional $l_0 : X_0 \to \mathbb{F}$ satisfying $|l_0| \le p$ on X_0 admits a linear extension $l : X \to \mathbb{F}$ satisfying $|l| \le p$.

(c) If X is locally convex, then a continuous linear functional $l_0 : X_0 \to \mathbb{F}$ admits a continuous linear extension $l : X \to \mathbb{F}$. If X is normed, then a bounded linear functional $l_0 : X_0 \to \mathbb{F}$ admits a norm-preserving linear extension $l : X \to \mathbb{F}$.

Proof. (a) Consider a partially ordered set of all linear extensions of l_0 dominated by q. Precisely, we consider the set

$$\{l : V \to \mathbb{R} \mid V \text{ is a linear space between } X_0 \subset X,\ l_0 = l|_{X_0},\ l \le q\},$$

on which the partial order is given by the restriction. The non-emptyness and the chain condition is easily satisfied, so the partially ordered set has a maximal element $l : V \to \mathbb{R}$ by the Zorn lemma.

Suppose $V \ne X$ and choose $e \in X \setminus V$. We want to assign an appropriate value to the vector e to extend our maximal extension l. The inequality

$$l(v) + l(v') = l(v+v') \le q(v+v') \le q(v-e) + q(v'+e), \qquad v, v' \in V$$

implies the existence of $r \in \mathbb{R}$ such that

$$\sup_{v \in V}(l(v) - q(v-e)) \le r \le \inf_{v \in V}(-l(v) + q(v+e)).$$

If we define $\widetilde{V} := V + \mathbb{R}e$ and $\widetilde{l} : \widetilde{V} \to \mathbb{R}$ such that

$$\widetilde{l}(v+te) := l(v) + tr, \qquad v \in V,\ t \in \mathbb{R},$$

then $\widetilde{l}$ extends l and is dominated by q as

$$l(v) + tr \le l(v) + t \cdot \begin{cases} -l\left(\frac{v}{t}\right) + q\left(\frac{v}{t} + e\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ l\left(-\frac{v}{t}\right) - q\left(-\frac{v}{t} - e\right) & \text{if } t < 0, \end{cases} = q(v+te), \qquad v \in V,\ t \in \mathbb{R},$$

which leads a contradiction to the maximality of l. Therefore, we conclude $V = X$.

(b)

Let $\mathbb{F} = \mathbb{C}$. Note that the real part map $\mathrm{Re} : \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(X, \mathbb{R})$ is bijective. Note also that $|l| \le p$ if and only if $\mathrm{Re}\, l \le p$ for any complex linear functional $l : V \to \mathbb{C}$ on a complex vector space V. It is because $|l| \le p$ implies $\mathrm{Re}\, l \le |l| \le p$ and conversely $\mathrm{Re}\, l \le p$ implies $|l| \le p$ by the inequality

$$|l(v)|^2 = l(v)\overline{l(v)} = l(\overline{l(v)}v) = \mathrm{Re}\, l(\overline{l(v)}v) \le p(\overline{l(v)}v) = |l(v)|p(v), \qquad v \in V.$$

Since $|l_0| \le p$, we have $\mathrm{Re}\, l_0 \le p$. Using the part (a), there is a linear functional $l : X \to \mathbb{C}$ such that $\mathrm{Re}\, l_0 = \mathrm{Re}\, l$ on X_0 and $\mathrm{Re}\, l \le p$. Then, we can deduce $l_0 = l$ on X_0 and $|l| \le p$.

(c) Since $l_0 : X_0 \to \mathbb{F}$ is continuous, there is a finite family $\{p_j\}$ of continuous semi-norms such that

$$|l_0(x)| \le \sum_j p_j(x), \qquad x \in X_0.$$

The sum of semi-norms is a semi-norm, so the part (b) implies that there is a linear extension $l : X \to \mathbb{F}$ such that

$$|l(x)| \le \sum_j p_j(x), \qquad x \in X.$$

This inequality shows the continuity of l.

□

1.11 (Hahn-Banach separation)**.** Let X be a locally convex space. Let C be a closed convex subset and K be a compact convex subset of X that are disjoint. Then, there is a continuous linear functional $x^* \in X^*$ such that

$$\sup_{x\in C} \mathrm{Re}\langle x, x^*\rangle < \inf_{x\in K} \mathrm{Re}\langle x, x^*\rangle.$$

Exercises

1.12 (Topology of compact convergence)**.**

Chapter 2

Barrelled spaces

2.1 Uniform boundedness principle

2.1 (Barreled spaces)**.** Let X be a topological vector space. A *barrel* is a closed absorbing convex balanced subset of X. A *barrelled space* is a topological vector space in which every barrel is a neighborhood of zero.

A locally convex space X is barrelled iff $X = X_\tau = X_\beta$.

2.2 (Uniform boundedness principle)**.** Let X and Y be topological vector spaces. We say that a family $\{T_i\}$ of continuous linear operators from X to Y is *pointwise bounded* if $\{T_i x\} \subset Y$ is bounded for each $x \in X$, and is *equicontinuous* if

The *uniform boundedness principle* states that a pointwise bounded family is equicontinuous. It is also frequently called the *Banach-Steinhaus theorem*.

(a) If X is barrelled and Y is locally convex, then the uniform boundedness principle holds.

(b) If X is complete and metrizable, then the uniform boundedness principle holds.

Proof. (a) Let V be a neighborhood of zero in Y. We may assume V is closed and balanced, and we may further assume that V is convex since Y is locally convex. If we define $U := \bigcap_i T_i^{-1} V$, then U is clearly a closed convex balanced set, and it is also absorbing because for any $x \in X$ the boundedness of $T_i x$ implies that there is $r \geq 0$ such that $T_i x \subset rV$ for all i, which is equivalent to $x \in rU$ by definition of U. Thuerefore, U is a barrel in a barrelled space X, so it is a neighborhood of zero. From $T_i U \subset V$ for every i, we have $\{T_i\}$ is equicontinuous. □

2.2 Baire category theorem

2.3 (Baire spaces)**.** A topological space is called a *Baire space* if the countable intersection of open dense subsets is always dense. Equivalently, if the union of a sequence of closed subsets is the whole set, then at least one of such closed subsets contains an open set.

(a) If a topological vector space is Baire, then it is barrelled.

(b) A Baire space is second category in itself.

(c) A topological group that is second category in itself is Baire.

2.4. Let $B \subset X$ be a closed absorbing subset of a topological vector space X that is Baire. Then, B has a non-empty open subset, and $B - B$ is a neighborhood of zero. If B is convex in addition, then B is a neighborhood of zero.

2.5 (Baire category theorem)**.** The Baire category theorem proves many exmples of topological vector space are Baire, in particular barrelled.

(a) A completely metrizable space is Baire.

(b) A locally compact Hausdorff space is Baire.

2.3 Open mapping theorem

2.6 (Open mapping theorem)**.** Let X and Y be topological vector spaces. The *open mapping theorem* states that continuous surjective linear operator $T : X \to Y$ is an open map.

(a) Suppose X is completely metrizable. If Y is Baire, or if X is locally convex and Y is barreled, the open mapping theorem holds.

Proof. (a) Let U be a balanced open neighborhood of zero in X. It is enough to prove TU is a neighborhood of zero. We first claim the closure $\overline{TU}$ is a neighborhood of zero. If we take a smaller balanced open neighborhood U' of zero in X such that $U' + U' \subset U$, then because U' is absorbing and T is surjective, the set $\overline{TU'}$ is closed and absorbing in Y. Since Y is Baire, $\overline{TU'}$ contains an open subset, so $\overline{TU'} + \overline{TU'}$ is a neighborhood of zero in Y. Thus, the claim follows from $\overline{TU'} + \overline{TU'} \subset \overline{TU}$.

(If X is locally convex (i.e. Fréchet) and Y is barreled...)

Since X is metrizable, we have a countable balanced open neighborhood system $\{U_n\}_{n=0}^\infty$ of zero in X such that $\overline{U_0 + U_0} \subset U$ and $U_{n+1} + U_{n+1} \subset U_n$ for all $n \geq 0$. Our goal is to prove $\overline{TU_0} \subset TU$. To prove this, we fix arbitrary $y_0 \in \overline{TU_0}$ and construct a sequence $y_n \in Y$ inductively such that

$$y_{n+1} := y_n - Tx_n, \qquad n \geq 0,$$

where $x_n \in X$ is taken such that

$$x_n \in U_n, \qquad Tx_n \in y_n + \overline{TU_{n+1}}, \qquad n \geq 0.$$

To verify that such x_n exists, it suffices to show $y_n \in \overline{TU_n}$ for all $n \geq 0$ because this gets TU_n to intersect the neighborhood $y_n + \overline{TU_{n+1}}$ of y_n. For $n = 0$ it is clear. By the induction hypothesis that there is $x_{n-1} \in X$ satisfying all the above conditions, we get $Tx_{n-1} \in y_{n-1} + \overline{TU_n}$ and $y_n = y_{n-1} - Tx_{n-1} \in \overline{TU_n}$, so the desired sequence x_n is well-defined.

Since X is complete, the partial sum $\sum_{k=0}^{n-1} x_k$ is convergent to $x \in X$ because it is Cauchy by

$$\sum_{k=n'}^{n-1} x_k \in U_{n'} + \cdots + U_{n-1} \subset U_{n'-1}, \qquad n > n' \geq 0,$$

and in fact we have $x \in \overline{U_0 + U_0} \subset U$ by

$$\sum_{k=0}^{n-1} x_k \in U_0 + \cdots + U_{n-1} \subset U_0 + U_0, \qquad n \geq 1.$$

Because $y_n \to 0$ as $n \to \infty$, we can check $y_0 = Tx \in TU$ by

$$y_0 = y_n + \sum_{k=0}^{n-1}(y_k - y_{k+1}) = y_n + \sum_{k=0}^{n-1} Tx_k = y_n + T\sum_{k=0}^{n-1} x_k \to Tx, \qquad n \to \infty.$$

This completes the proof. □

A first countable topological vector space is metrizable. A locally complete metrizable topological vector space is complete metrizable.

2.4 Vector bornologies

2.7 (Bornological spaces)**.** A *bornology* on a set X is an ideal $\mathcal{B}$ of $\mathcal{P}(X)$ which covers X. A bornology on a vector space is called a *vector bornology* if it is stable under translation, scaling, and the balanced hulls. For a topological vector space, there is a canonical bornology, sometimes called the *von Neumann bornology*, defined such that $B \subset X$ is bounded if and only if every open neighborhood U of zero in X has $r \geq 0$ such that $B \subset rU$.

Let X be a locally convex space. It is called *bornological* if one of the following holds.

(i) every bornivorous convex balanced set is a neighborhood of zero,

(ii) every bounded linear operator from X to any locally convex space is continuous,

(iii) it is an inductive limit of normable spaces

If X is metrizable, then X is bornological. If X is bornological, then X is Mackey. If X is bornological, then X_β is complete. If X is bornological and boundedly complete, then X is barrelled.

bornological space is Mackey barrelled space is Mackey: $X = X$

Exercises

2.8. Let (T_n) be a sequence in $B(X,Y)$. If T_n coverges strongly then $\|T_n\|$ is bounded by the uniform boundedness principle.

2.9. There is a closed absorbing set in $\ell^2(\mathbb{Z}_{\geq 0})$ that is not a neighborhood of zero;

$$\overline{B}(0,1) \setminus \bigcup_{i=2}^{\infty} B(i^{-1}e_i, i^{-2})$$

is a counterexample.

2.10. There is no metric d on $C([0,1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise as $n \to \infty$ for every sequence f_n. Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

2.11. We show that there is no projection from ℓ^∞ onto c_0.

2.12 (Schur property)**.** ℓ^1

2.13. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

2.14 (Daugavet property)**.** (a) The real Banach space $C([0,1])$ satisfies the Daugavet property.

Proof. Let T be a finite rank operator on $C([0,1])$, and e_i be a basis of $\operatorname{im} T$. Then, for some measures μ_i,

$$Tf(t) = \sum_{i=1}^{n} \int_0^1 f \, d\mu_i e_i(t).$$

Let $M := \max \|e_i\|$.

Take f_0 such that $\|f_0\| = 1$ and $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$. Reversing the sign of f_0 if necessary, take an open interval Δ such that $Tf_0(t) \geq \|T\| - \frac{\varepsilon}{2}$ and $|\mu_i|(\Delta) \leq \frac{\varepsilon}{4nM}$ for all i. Define f_1 such that $f_0 = f_1$ on Δ^c, $f_1(t_0) = 1$ for some $t_0 \in \Delta$, and $\|f_1\| = 1$. Then, $\|Tf_1 - Tf_0\| \leq \frac{\varepsilon}{2}$ shows $Tf_1 \geq \|T\| - \varepsilon$ on Δ. Therefore,

$$\|1 + T\| \geq \|f_1 + Tf_1\| \geq f_1(t_0) + Tf_1(t_0) \leq 1 + \|T\| - \varepsilon.$$

□

2.15 (Bartle-Graves theorem)**.** Let E be a Banach space and N a closed subspace. For $\varepsilon > 0$, there is a continuous homogeneous map $\rho : E/N \to E$ such that $\pi\rho(y) = y$ and $\|\rho(y)\| \leq (1 + \varepsilon)\|y\|$ for all $y \in E/N$.

Proof. We want to construct a continuous map $\psi : S_{E/N} \to E$ with $\|\psi(y)\| \leq 1 + \varepsilon$ for all $y \in S_{E/N}$. If then, ρ can be made from ψ.

For each $y_0 \in S_{E/N}$, choose $x_0 \in \pi^{-1}(y_0) \cap B_{1+\varepsilon}$. There is a neighborhood $V_{y_0} \subset S_{E/N}$ of y_0 such that $y \in V_{y_0}$ implies x_0 belongs to $(\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$, which is convex. With a locally finite subcover V_{y_α} and a partition of unity $\eta_\alpha(y)$, define $\psi_1(y) = \sum_\alpha \eta_\alpha(y) x_\alpha$. Then, $\psi_1(y) \in (\pi^{-1}(y) \cap B_{1+\varepsilon}) + U_{2^{-1}}$.

For $i \leq 2$, choose for each y_0 the element x_0 in $\pi^{-1}(y_0) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}})$. Then, we obtain

$$\psi_i(y) \in \Big(\pi^{-1}(y) \cap B_{1+\varepsilon} \cap (\psi_{i-1}(y_0) + U_{2^{-i-1}})\Big) + U_{2^{-i}}.$$

Therefore, $\|\psi_i(y) - \psi_{i-1}(y)\| < 2^{-i-2}$, so it converges uniformly to ψ such that $\psi(y) \in \pi^{-1}(y) \cap B_{1+\varepsilon}$. □

Problems

Let Y be a separable infinite-dimensional Banach space. Let $\{e_i\}_{i \in I}$ be a Hamel basis of Y. Since I is uncountable, embed $\mathbb{N} \subset I$. Let $X := c_c(I)$ with ℓ^1 norm. Define $T : X \to Y$ such that $Te_i := i^{-1}e_i$ for $i \in \mathbb{N}$ and $Te_i := e_i$ for $i \notin \mathbb{N}$. Then, T is surjective, but we can see $T(e_i) = i^{-1}e_i$ for $i \in \mathbb{N}$ has a limit point zero.

2.16. Let T be an invertible linear operator on a normed space. Then, $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

Chapter 3

Weak topologies

3.1 Dual pairs

3.1 (Dual pairs)**.** Let $\mathbb{F}$ be the real or complex field. A *dual pair* is a pair (X, X^*) of vector spaces over $\mathbb{F}$ together with a non-degenerate bilinear form $X \times X^* \to \mathbb{F}$. A pair (X, X^*) of a vector space X and a subspace X^* of $X^\#$ is a natural dual pair if and only if X^* separates points of X.

Let X be a topological vector space. We can canonically define X^* as the topological dual of X, and we consider (X, X^*) as a canonical dual pair associated to X. If F is a linear subspace of X^*, then (X, F) is another dual pair exactly when F is weakly* dense in X^* by the Hahn-Banach separation.

Note that if X is discrete, then $X^* = X^\#$. If (X, X^*) is a dual pair, then (X^*, X) is also a dual pair.

Proof. For a linear subspace V of a topological vector space X, $\overline{V} = V^{\perp\perp}$. If $x \in \overline{V}$, then for $x^* \in V^\perp$, we have $\langle x, x^* \rangle = 0$ by approximation, so $x \in V^{\perp\perp}$. Conversely, if $x \notin \overline{V}$, then the Hahn-Banach extension implies that there is x^* such that $\langle y, x^* \rangle = 0 < \langle x, x^* \rangle$ for all $y \in V$, which means $x^* \in V^\perp$ and $x \notin V^{\perp\perp}$.

□

3.2 (Weak topology of dual pairs)**.** Let (X, X^*) be a dual pair.

(a) X_σ and X^*_σ are locally convex.

(b) $(X_\sigma)^* = X^*$.

(c) $(X^*_\sigma)^* = X$. Every locally convex space is a dual of a locally convex space.

(d) A subset $B \subset X_\sigma$ is weakly bounded if and only if it is weakly totally bounded.

(e) If $X^* = X^\#$, then X^*_σ is complete.

Proof. (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show $(X^*_\sigma)^* \subset X$. If $x^{**} \in (X^*_\sigma)^*$, then there is a finite subset $\{x_i\}_{i \in J}$ of X such that

$$|\langle x^{**}, x^* \rangle| \le \sum_{i \in J} |\langle x_i, x^* \rangle|, \qquad x^* \in X^*.$$

Since $\bigcap_{i \in J} \ker x_i$ is a closed subspace of $\ker x^{**}$, we have $x^{**} \in \operatorname{span}\{x_i\}_{i \in J} \subset X$. □

3.3. closure and weak closure of convex subsets

Proof. Hahn-Banach □

3.4 (Polar topologies)**.** Let (X, X^*) be a dual pair. The *Mackey topology* on X is the topology τ of uniform convergence on compact convex balanced subsets of X^*_σ. The *strong topology* on X is the topology β of uniform convergence on bounded convex balanced subsets of X^*_σ. The space X together with the Mackey and strong topology is denoted by X_τ and X_β respectively. If X is a topological vector space and (X, X^*) is the canonical dual pair, then we always have continuous identity operators

$$X_\beta \to X_\tau \to X \to X_\sigma.$$

Let α is a polar topology on X generated by $\mathcal{G}^* \subset \mathcal{P}(X^*)$. If $x^* \in (X_\alpha)^*$, then there is $\sigma(X, X^*)$-closed convex balanced $C^* \in \mathcal{G}$ such that $\sup_{x \in (C^*)^\circ} |\langle x, x^* \rangle| \le 1$.

(a) If X is locally convex, then X is barrelled if and only if $X = X_\beta$.

(b) If X is locally covex and metrizable, then $X = X_\tau$.

(c) Mackey-Arens theorem

boundedness, incompleteness

3.5 (Weak convergence of bounded nets)**.** Let X be a Banach space, X_0^* a subset of X^*, and $\overline{D^*}$ the norm closure of X_0^*. For example, if X has a predual $X_* \subset X^*$ and X_0^* is dense in X_*, then $\sigma(X, \overline{X_0^*})$ is the weak* topology.

(a) There is a squence $x_n \in X$ converges to zero in $\sigma(X, X_0^*)$ but not in $\sigma(X, \overline{X_0^*})$.

(b) A bounded sequence $x_n \in X$ converges to zero in $\sigma(X, \overline{X_0^*})$ if in $\sigma(X, X_0^*)$.

Proof. (b) Let $x^* \in \overline{X_0^*}$ and choose $x_0^* \in X_0^*$ such that $\|x^* - x_0^*\| < \varepsilon$. Then,

$$|\langle x_n, x^* \rangle| \le \|x_n\| \|x^* - x_0^*\| + |\langle x_n, x_0^* \rangle| \lesssim \varepsilon + |\langle x_n, x_0^* \rangle| \to \varepsilon.$$

□

3.6 (Alaoglu theorem)**.** Let (X, X^*) be a dual pair. If U is a neighborhood of zero in X_τ, then the polar U° is compact in X^*_σ.

Proof. Note that X^*_σ is topologically embedded into $X^\#_\sigma$, which has the Heine-Borel property by the Tychonoff theorem. Since U is absorbing, U° is bounded in $X^\#_\sigma$. We can also see that U° is closed in $X^\#_\sigma$ because if $x_i^* \in U^\circ$ is a net such that $x_i^* \to x^*$ in $X^\#_\sigma$, then taking the limit for i and the supremum on $x \in U$ in the inequality

$$|\langle x, x^* \rangle| \le |\langle x, x^* - x_i^* \rangle| + |\langle x, x_i^* \rangle| \le |\langle x, x^* - x_i^* \rangle| + 1, \qquad x \in U,$$

we can deduce $x^* \in X^*$ from that $(x^*)^{-1}(\mathbb{D})$ contains a neighborhood U of zero in X_τ, and automatically $x^* \in U^\circ$ by the inequality. Therefore, the compactness of U° in $X^\#_\sigma$ and hence in X^*_σ follows. □

3.2 Compact convex sets

Krein-Milman theorem Choquet theory

Exercises

3.7 (James' space)**.** not reflexive but isometrically isomorphic to bidual

3.8 (Preduals). Let X be a Banach space. A *predual* of X is a Banach space F together with an isometric isomorphism $\varphi : X \to F^*$. Two preduals $\varphi_1 : X \to F_1^*$ and $\varphi_2 : X \to F_2^*$ are said to be equivalent if there is an isometric isomorphism $\theta : F_1 \to F_2$ such that $\theta^* = \varphi_1\varphi_2^{-1}$.

(a) There is a one-to-one correspondence between the equivalence class of preduals of X and the set of closed subspaces X_* of X^* such that B_X is compact and Hausdorff in $(X, \sigma(X, X_*))$. Such a subspace X_* is also called a predual of X.

(b) If X admits a predual $X_* \subset X^*$, then a $\sigma(X, X_*)$-closed subspace V of X also admits a predual $X_*|_V$.

Proof. (a) Goldstine theorem for surjectivity.

(b) It is easy if we apply the part (a). We can show more directly. If we let $V_* := X_*|_V$ the image of X_* under the map $X^* \to V^*$, then we have isometric injections $V \to (V_*)^* \to X$. We can show V is $\sigma(X, X_*)$ dense in $(V_*)^*$, hence the closedness proves the bijectivity of $V \to (V_*)^*$. □

3.9 (Mazur's lemma).

Part II

Banach spaces

Chapter 4

Operators on Banach spaces

4.1 Bounded operators

4.1 (Bounded belowness in Banach spaces). Let $T \in L(X,Y)$ for Banach spaces X and Y. The following statements are equivalent:

(a) T is bounded below.

(b) T is injective and has closed range.

(c) T is a topological isomorphism onto its image.

4.2 (Bounded belowness in Hilbert spaces). Let $T \in B(H,K)$ for Hilbert spaces H and K. The following statements are equivalent:

(a) T is bounded below.

(b) T is left invertible.

(c) T^* is right invertible.

(d) T^*T is invertible.

4.3 (Injectivity and surjectivity of adjoint). Let $T : X \to Y$ be a continuous linear operator between locally convex spaces.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is an embedding.

4.2 Compact operators

$K(X,Y)$ is closed in $B(X,Y)$. $K(X)$ is an ideal of $B(X)$. adjoint is $K(X,Y) \to K(Y^*,X^*)$. integral operators are compact. riesz operator, quasi-nilpotent operator.

4.3 Fredholm operators

4.4. Let E_1 and E_2 be Fréchet spaces. A *Fredholm operator* is a bounded linear operator $F \in B(E_1,E_2)$ which has closed range and the kernels of F and F^* are finite-dimensional. Let $F \in B(E)$.

(a) F is Fredholm if and only if F^* is Fredholm.

(b) F is Fredholm of index zero if $1-F$ is compact.

(c) F is Fredholm if and only if $\pi(F)$ is invertible in $Q(E)$.

Proof. (b) Since $1-F$ and $1-F^*$ are the compact identities on the closed subspaces $\ker F$ and $\ker F^*$ respectively, what remains is to show F has closed range. Consider the continuous dense embedding $V : E/\ker F \to \overline{\operatorname{ran}} F$ induced from F. Suppose that V is not bounded below so that there is a sequence of vectors $x_n \in E$ such that x_n does not converge to zero in $E/\ker F$ but $Fx_n \to 0$ in E. We may assume $(1-F)x_n \to x$ for some $x \in E$ by compactness of K, which implies $x_n \to x$ in the closed subspace $\ker T^\perp$. Since $Tx = x - Kx = 0$, we have $x \in \ker T^\perp \cap \ker T = \{0\}$, so we obtain a contradiction $x_n \to 0$. Thus, V is bounded below, so the range of F is closed. □

4.5 (Atkinson theorem)**.** Let E and F be Banach spaces.

(a) An operator $T \in B(E,F)$ is Fredholm if and only if there is $S \in B(F,E)$ such that $1-ST$ and $1-TS$ is finite-rank.

(b) An operator $T \in B(E)$ is Fredholm if and only if $\pi(T)$ is invertible in $Q(E)$.

Proof. (c) Let F be a Fredholm operator. Note that the induced operator $V : E/\ker F \to \overline{\operatorname{ran}} F$ is a topological isomorphism. Since $\ker F$ and $\operatorname{ran} F$ are complemented, we can define $F' := V^{-1} \oplus 0$. Then, $1 - F'F$ and $1 - FF'$ are of finite-rank.

Conversely if $\pi(F)$ has an inverse $\pi(F')$ in $Q(E)$ for some $F' \in B(E)$, then compactness of $1-F'F$ and $1-FF'$ implies that $F'F$ and FF' are Fredholm. Then, $\ker F \subset \ker(F'F)$ and $\ker F^* \subset \ker((FF')^*)$ are finite-dimensional, F is Fredholm. □

4.6 (Fredholm index)**.** locally constant, in particular, continuous. composition makes the addition of indices.

4.4

Exercises

4.7 (Completely continuous operators)**.** On reflexive spaces, completely continuous operators are same with compact operators.

4.8 (Dunford-Pettis property)**.** A Banach space X is said to have the *Dunford-Pettis property* if all weakly compact operators $T : X \to Y$ to any Banach space Y is completely continuous.

(a) X has the Dunford-Pettis property if and only if for every sequences $x_n \in X$ and $x_n^* \in X^*$ that converge to x and x^* weakly we have $x_n^*(x_n) \to x^*(x)$.

(b) $C(\Omega)$ for a compact Hausdorff space Ω has the Dunford-Pettis property.

(c) $L^1(\Omega)$ for a probability space Ω has the Dunford-Pettis property.

(d) Infinite dimensional reflexive Banach space does not have the Dunfor-Pettis property.

4.9.

(a) (Mazur-Ulam, 1932) A surjective isometry $T : X \to Y$ between normed spaces is affine.

(b) (Mankiewicz, 1972) Let U, V be open sets in X, Y, normed spaces. A surjective isometry $U \to V$ is uniquely extended to a surjective isometry $X \to Y$.

(c) (Mori) A surjective local isometry $T : X \to Y$ between Banach spaces is an isometry, if X is separable. (Use the Baire category)

Solution. (a) T is continuous. It is easy to see for continuous map T that it is affine if and only if T preserves the midpoint. For $x_1 \neq x_2 \in X$ let x_0 be the midpoint. Define inductively

$$C_1 := \{x \in X : \|x - x_1\| = \|x - x_2\| = \frac{1}{2}\|x_1 - x_2\|\}, \qquad C_k := \{x \in C_{k-1} : \sup_{x' \in C_{k-1}} \|x - x'\| \leq \frac{1}{2} \operatorname{diam} C_{k-1}\}.$$

Since $x_0 \in C_{k-1}$ and $x' \in C_{k-1}$ imply $x_0 \in C_k$ by $\|x_0 - x'\| = \frac{1}{2}\|(2x_0 - x') - x'\| \leq \frac{1}{2} \operatorname{diam} C_{k-1}$, and since $\operatorname{diam} C_k \leq \frac{1}{2} \operatorname{diam} C_{k-1}$, we have $\{x_0\} = \bigcup_{k=1}^{\infty} C_k$. It follows that the midpoint can be detected from the metric structure of X, not depending on the linear structure of X. □

Problems

1. If $K \in B(L^2([0,1]))$ is a compact operator, then for any $\varepsilon > 0$ there is a constant $C > 0$ such that

$$\|Kf\|_{L^2} \leq \varepsilon \|f\|_{L^2} + C\|f\|_{L^1}.$$

Proof. Suppose there is $\varepsilon > 0$ such that we have sequence $f_n \in L^2$ satisfying $\|f_n\|_{L^2} = 1$ and $\|Kf_n\|_2 > \varepsilon + n\|f_n\|_1$. Since K is compact, there is a subsequence Kf_{n_j} convergent to $g \neq 0$ in L^2. Then, $\|f_{n_j}\|_{L^1} \to 0$ and $\|f_{n_j}\|_{L^2} \leq 1$ imply $f_{n_j} \to 0$ and hence $Kf_{n_j} \to 0$ weakly in L^2, which implies a contradiction $g = 0$. □

Chapter 5

Tensor products of Banach spaces

5.1 Injective and projective tensor products

5.1 (Realizations). For Banach spaces X and Y, $L(X,Y)$ and $Bi(X,Y) = L(X,Y^*) = L(Y,X^*)$ are naturally Banach spaces. Also we have a natural algebraic inclusions of $X \otimes Y$ into $L(X^*,Y) \leq Bi(X^*,Y^*)$, and $Bi(X,Y)^*$. Also we have a natural algebraic inclusions of $X^* \otimes Y$ into $\mathcal{L}(X,Y) \leq \mathcal{B}(X,Y^*)$.

5.2. Let X and Y be a Banach spaces, and α be a norm on $X \otimes Y$. We say α is a *cross norm* if

$$\alpha(x \otimes y) = \|x\|\|y\|, \qquad x \in X,\ y \in Y,$$

and a cross norm is *reasonable* if the *dual norm* α^* on $X^* \otimes Y^* \subset (X \otimes Y, \alpha)^*$ of α is also a cross norm.

$$\varepsilon(u) := \|u\|_{\mathcal{B}(X^*,Y^*)}, \qquad \pi(u) := \|u\|_{\mathcal{B}(X,Y)^*}.$$

5.3 (Type C and type L spaces).

5.4 (Duals of tensor products).

$$\mathcal{K}(X,Y) \hookrightarrow X^*\widehat{\otimes}_\varepsilon Y \leftarrow X^*\widehat{\otimes}_\pi Y \twoheadrightarrow \mathcal{N}(X,Y).$$

5.2 Vector-valued integrals

harmonic and complex analysis

5.5 (Pettis measurability theorem). Let (Ω,μ) be a measure space and X a Banach space. Let $f : \Omega \to X$ be a function. We say f is *strongly measurable* or *Bochner measurable* if it is a pointwise limit of a sequence of simple functions.

If μ is complete, then all the pointwise convergence discussed here can be relaxed to the almost everywhere convergence.

(a) If f is strongly measurable, then f is Borel measurable.

(b) If f is Borel measurable, then f is weakly measurable.

(c) If f is weakly measurable and separably valued, then f is strongly measurable.

5.6 (Pettis integrals).

$$L^1\widehat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(X^*,L^1) \overset{*}{\hookrightarrow} \mathcal{L}(L^\infty,X^{**}).$$

- Pettis integrable: $L^1\widehat{\otimes}_\varepsilon X$,

- weakly integrable: $\mathcal{L}(X^*, L^1)$,
- Dunford integrable: $\mathcal{L}(L^\infty, X^{**})$,
- Pettis integral: $L^1\widehat{\otimes}_\varepsilon X \cong *^{-1}\mathcal{L}(L^\infty, X) \subset \mathcal{L}(X^*, L^1)$. It defines $L^1\widehat{\otimes}_\varepsilon X \hookrightarrow \mathcal{K}(L^\infty, X_\sigma)$.

(a) The close graph theorem and the existence of an a.e. convergent subsequence of an L^1 convergent sequence proves a weakly integrable function defines an operator in $\mathcal{L}(X^*, L^1)$.

5.7 (Bochner integrals)**.** Let (Ω, μ) be a measure space and X a Banach space. Let $f : \Omega \to X$ be a strongly measurable function. The function f is said to be *Bochner integrable* if there is a net of simple functions $(s_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$\int_\Omega \|f(\omega) - s_\alpha(\omega)\| \, d\mu(\omega) \to 0$$

for $\alpha \in \mathcal{A}$.

For $T \in \mathcal{L}(X, Y)$ and $\mu : L^1(\mu) \to \mathbb{C}$, the commutative diagram for $\alpha \in \{\varepsilon, \pi\}$

$$\begin{tikzcd} L^1(\mu)\widehat{\otimes}_\alpha X \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes T"'] & X \arrow[d, "T"] \\ L^1(\mu)\widehat{\otimes}_\alpha Y \arrow[r, "\mu\otimes\mathrm{id}"] & Y, \end{tikzcd}$$

which is shown with approximation by simple tensors, justifies that T commutes with the integral:

$$T\int f \, d\mu = \int Tf \, d\mu.$$

The space of Bochner integrable functions $L^1\widehat{\otimes}_\pi X$, factoring through $L^1\widehat{\otimes}_\varepsilon X$, is naturally mapped to the space of Pettis integrals $\mathcal{K}(L^\infty, X_\sigma)$.

(a) f is Bochner integrable if and only if $\int \|f(\omega)\| \, d\mu(\omega) < \infty$.

(b) If f is Bochner integrable, then it is Pettis integrable and the integrals coincides.

Bochner integrable => Pettis integrable => weakly(scalarly) integrable

5.8 (Vector measures)**.** If an element of the Dunford integral $\mathcal{L}(L^\infty, X^{**})$, or the Pettis integral $\mathcal{K}(L^\infty, X_\sigma)$, defines a σ-weakly continuous linear operator $L^\infty \to X$, then it is called a vector measure?

5.3 Approximation property

dual is Banach. Basis problem, Mazur' duck.

5.9 (Approximation property)**.** A locally convex space X has the *approximation property* if $F(X)$ is dense in $L_\tau(X)$. (Schafer-Wolff do not assume the convexity of compact sets) Recall that a locally convex space X is called nuclear if $N(X)$ is dense in $L(X)$.

Every compact operator is a limit of finite-rank operators.

(a) An Hilbert space has the AP.

(b) If a locally convex space X has the approximation property, then X has a Schauder basis, and the converse holds if X is separable and Fréchet.

Proof. (a) Let H be a Hilbert space and $K \in K(H)$. Since $\overline{KB_H}$ is a compact metric space, it is separable, which means $\overline{KH}$ is separable. Let $(e_i)_{i=1}^\infty$ be an orthonormal basis of $\overline{KH}$, and let P_n be the orthogonal projection on the space spanned by $(e_i)_{i=1}^n$. If we let $K_n := P_nK$, then $K_n \to K$ strongly and K_n has finite rank. Take any

$\varepsilon > 0$ and find, using the totally boundedness of KB_H, a finite subset $\{x_j\} \subset B_H$ such that for any $x \in B_H$ there is x_j satisfying $\|Kx - Kx_j\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}\|Kx - K_n x\| &\leq \|Kx - Kx_j\| + \|Kx_j - K_n x_j\| + \|P_n(Kx_j - Kx)\| \\ &\leq \frac{\varepsilon}{2} + \|Kx_j - K_n x_j\| + \frac{\varepsilon}{2}.\end{aligned}$$

By taking the supremum on $x \in B_H$, we have

$$\|K - K_n\| \leq \max_j \|Kx_j - K_n x_j\| + \varepsilon,$$

which deduces $K_n \to K$ in norm.

□

Exercises

Tingley problem

Chapter 6

Geometry of Banach spaces

6.1

6.1 (Eberlein-Šmulian theorem).

6.2 (James theorem).

6.3 (Krein-Šmulian theorem for weakly* closed sets). Let X be a Banach space, and let F^* be a convex subset of X^* whose bounded parts are weakly* closed.

(a) If F^* is disjoint to B_{X^*}, then there is $x \in X$ separating F^* and B_{X^*}.

(b) F^* is weakly* closed.

Proof. (a) Note that for any Banach space X, if F is a subset of B_X, then we have a natural contractive linear operator $\ell^1(F) \to X$ and its dual $X^* \to \ell^\infty(F)$. We will construct a subset F of B_X such that F^* induces a subset of $c_0(F)$ and $F^* \cap F^\circ = \varnothing$, where F° denote the complex polar of F. If it exists, the image of F^* in $c_0(F)$ is a convex set disjoint to $B_{c_0(F)}$, so there exists a separating linear functional in $B_{\ell^1(F)}$ by the Hahn-Banach separation, and it induces a linear functional separating F^* and B_{X^*}.

Let $F_0 := \{0\} \subset X$. As an induction hypothesis on n, suppose we have F_k for $0 \le k \le n-1$ are finite subsets of $k^{-1}B_X$ such that

$$F^* \cap nB_{X^*} \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ = \varnothing.$$

If every finite subset F_n of $n^{-1}B_X$ satisfies

$$F^* \cap (n+1)B_{X^*} \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ \cap F_n^\circ \neq \varnothing,$$

then since F^* is weakly* compact on a bounded part by assumption, the finite intersection property leads a contradiction because the intersection of all complex polars F_n° of finite subsets F_n of $n^{-1}B_X$ is nB_{X^*}, the polar of all union of finite subsets F_n of $n^{-1}B_X$. Thus, we have a finite subset F_n of $n^{-1}B_X$ such that

$$F^* \cap (n+1)B_{X^*} \cap \left(\bigcup_{k=0}^{n} F_k\right)^\circ = \varnothing.$$

Then, $F := \bigcup_{k=0}^\infty F_k$ has the property we want.

□

6.4 (Krein-Šmulian theorem for weakly compact sets).

6.5 (Bishop-Phelps theorem).

Let $T : X \to Y$ be a quotient. For each $y^* \in Y^*$, if we take $y \in Y$ such that $\|y\| = 1$ and $\|y^*\| < |\langle y, y^*\rangle| + \varepsilon$ with $x \in X$ such that $Tx = y$, then since $1 = \|Tx\| = \inf_{Tx'=0} \|x - x'\|$, we can find x' such that $Tx' = 0$ and $\|x - x'\| < 1 + \varepsilon$, so $\|y^*\| < |\langle y, y^*\rangle| + \varepsilon = |\langle x - x', T^*y^*\rangle| + \varepsilon < (1 + \varepsilon)\|T^*y^*\| + \varepsilon$ implies that T^* is an isometry.

Let $T : X \to Y$ be an isometry. For each $y^* \in Y^*$, since the Hahn-Banach extension gives $y^{*\prime} \in Y^*$ such that $T^*y^{*\prime} = T^*y^*$ and $\|T^*y^*\| = \|y^{*\prime}\|$, we have $\inf_{T^*y^{*\prime}=0} \|y^* - y^{*\prime}\| = \inf_{T^*y^{*\prime}=T^*y^*} \|y^{*\prime}\| \le \|T^*y^*\|$, so T^* is a quotient.

Part III

Spectral theory

Chapter 7

Operators on Hilbert spaces

7.1

7.1. quadratic form = symmetric bilinear form hermitian form = conjugate-symmetric sesqui-linear form

polarization works for quadratic forms and sesquilinear forms. Cauchy-Schwarz works for positive semi-definite quadratic forms and positive semi-definite hermitian forms

Proof. Let h be a positive semi-definite hermitian form on a complex vector space H. For $\xi, \eta \in H$ and $\varepsilon > 0$, we have

$$\begin{aligned} 0 &\le h\left(\xi - \frac{h(\xi,\eta)}{h(\eta,\eta)+\varepsilon}\eta,\ \xi - \frac{h(\xi,\eta)}{h(\eta,\eta)+\varepsilon}\eta\right)(h(\eta,\eta)+\varepsilon) \\ &= h(\xi,\xi)(h(\eta,\eta)+\varepsilon) - |h(\xi,\eta)|^2 \frac{h(\eta,\eta)+2\varepsilon}{h(\eta,\eta)+\varepsilon} \\ &\le h(\xi,\xi)(h(\eta,\eta)+\varepsilon) - |h(\xi,\eta)|^2, \qquad \xi,\eta \in H. \end{aligned}$$

Limiting $\varepsilon \to 0$, we obtain the Cauchy-Schwarz inequality. □

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

7.2. (a) A Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

7.3 (Riesz representation theorem)**.** Let H be a Hilbert space over a field $\mathbb{K}$, which is either $\mathbb{R}$ of $\mathbb{C}$.

We use the bilinear form $\langle -, - \rangle : X \times X^* \to \mathbb{K}$ of canonical duality. The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each $x^* \in H^*$, there is a unique $x \in H$ such that $\langle y, x^* \rangle = \langle y, x \rangle$ for every $y \in H$.

(b) $H \to H^* : x \mapsto \langle -, x \rangle$ is a natural linear and anti-linear isomorphism if $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$, respectively.

Let H be a separable Hilbert space. Find a positive sequence a_n such that every sequence x_n of unit vectors of H satisfying $|\langle x_i, x_j \rangle| \le a_j$ for all $i < j$ converges weakly to zero.

7.4 (Normal operators)**.** For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. Suppose T is normal.

(a) $\ker T = \ker T^*$.

(b) T is bounded below if and only if T is invertible.

(c) If T is surjective, then T is invertible.

7.5 (Invariant and Reducing subsapces)**.** Let K be a closed subspace of H.

(a) K is reducing for T if and only if K is invariant for T and T^*.

(b) K is reducing for T if and only if $TP = PT$, where P is the orthogonal projection on K.

7.2 Compact operators

spectral theorem of compact normal operators.

linearly spanned by rank-one projections...

$K(H)$ is the unique non-zero proper closed ideal of $B(H)$ if H is separable.

analytic Skolem-Noether theorem: automorphism of $K(H)$ is inner in the identity representation $B(H)$.

states of $K(H)$ are density operators.

7.3

7.6 (Traces)**.** Bounded linear operators t and h on a Hilbert space H are said to be a *trace-class* and a *Hilbert-Schmidt* operator respectively if

$$\sum_i \langle |t|\delta_i, \delta_i\rangle < \infty, \qquad \sum_i \langle |h|^2\delta_i, \delta_i\rangle < \infty,$$

where (δ_i) is an orthonormal basis of H.

(a) The trace does not depend on the choice of the orthonormal basis. The trace is tracial. Finite-rank operators are dense.

(b) $L^2(H)$ is a Hilbert space.

(c) $L^1(H) \to K(H)^* : t \mapsto \mathrm{Tr}(\cdot t)$ is an isometric isomorphism.

(d) $B(H) \to L^1(H)^* : x \mapsto \mathrm{Tr}(x\cdot)$ is an isometric isomorphism.

(e) $t \in B(H)$ is a trace class if and only if $t = \sum_i \lambda_i \theta_{\delta_i', \delta_i}$ for some $(\lambda_i) \in \ell^1(\mathbb{N})$ and orthonormal sequences $(\delta_i), (\delta_i') \subset H$.

Proof. If (δ_i), $(\delta'_{i'})$, and $(\delta''_{i''})$ are any orthonormal bases of H, then by the Parseval theorem,

$$\sum_i \|h\delta_i\|^2 = \sum_{i,i'} |\langle h\delta_i, \delta'_{i'}\rangle|^2 = \sum_{i,i'} |\langle h^*\delta'_{i'}, \delta_i\rangle|^2 = \sum_{i'} \|h^*\delta'_{i'}\|^2,$$

and similarly

$$\sum_{i'} \|h^*\delta'_{i'}\|^2 = \sum_{i''} \|h\delta'_{i''}\|^2.$$

In particular, $\mathrm{Tr}(|h|^2) = \mathrm{Tr}(|h^*|^2)$. For Hilbert-Schmidt operators h_1 and h_2 on H, the polarization deduces

$$\mathrm{Tr}(h_2^*h_1) = \sum_{k=0}^3 i^k \,\mathrm{Tr}(|h_1 + i^k h_2|^2) = \sum_{k=0}^3 i^k \,\mathrm{Tr}(|h_1^* + \overline{i^k} h_2^*|^2) = \sum_{k=0}^3 i^k \,\mathrm{Tr}(|h_2^* + i^k h_1^*|^2) = \mathrm{Tr}(h_1 h_2^*).$$

(c)

(d) For $t \in L^1(H)$, let $t = v|t|$ be the polar decopmosition. Then, the boundedness follows from

$$\begin{aligned}
|\mathrm{Tr}(xt)|^2 &= |\mathrm{Tr}(xv|t|^{\frac12} \cdot |t|^{\frac12})|^2 \\
&\le |\mathrm{Tr}(xv|t|^{\frac12} \cdot |t|^{\frac12} v^* x^*)||\mathrm{Tr}(|t|^{\frac12} \cdot |t|^{\frac12})| \\
&= |\mathrm{Tr}(|t|^{\frac12} v^* x^* \cdot xv|t|^{\frac12})||\mathrm{Tr}(|t|)| \\
&\le \|x^*x\||\mathrm{Tr}(|t|^{\frac12} v^* v|t|^{\frac12})||\mathrm{Tr}(|t|)| \\
&= \|x\|^2|\mathrm{Tr}(|t|)|^2, \qquad x \in B(H).
\end{aligned}$$

We can check the isometry by putting $x := v^*$. For the surjectivity, let $l \in L^1(H)^*$. A sesqui-linear functional σ on H defined by

$$\sigma(\xi, \eta) := l(\theta_{\xi,\eta}), \qquad \xi, \eta \in H$$

is bounded by $\|l\|$, so there is $x \in B(H)$ such that

$$\sigma(\xi,\eta) = \langle x\xi,\eta\rangle, \qquad \xi,\eta \in H.$$

To verify $l = \mathrm{Tr}(x\cdot)$ on $L^1(H)$, we may assume that $t \in L^1(H)$ has the form $t = \theta_{\xi,\eta}$ because the finite-rank operators are dense in $L^1(H)$, and we finally have for an orthonormal basis (δ_i) such that $\eta = \delta_i$ for some i that

$$\mathrm{Tr}(xt) = \mathrm{Tr}(x\theta_{\xi,\eta}) = \sum_i \langle x\theta_{\xi,\eta}\delta_i,\delta_i\rangle = \sum_i \langle\langle\delta_i,\eta\rangle x\xi,\delta_i\rangle = \langle x\xi,\eta\rangle = l(\theta_{\xi,\eta}) = l(t).$$

(e) Applying the polar decomposition and diagonalizing the compact operator $|t|$, we are done. Conversely, if $t = \sum_i \lambda_i \theta_{\delta_i',\delta_i}$, then we can check the diagonalization $t^*t = \sum_i |\lambda_i|^2\theta_{\delta_i}$, so we have $|t| = \sum_i |\lambda_i|\theta_{\delta_i}$, and

$$\mathrm{Tr}(|t|) = \sum_j \langle|t|\delta_j,\delta_j\rangle = \sum_{i,j}\langle|\lambda_i|\theta_{\delta_i}\delta_j,\delta_j\rangle = \sum_{i,j}\langle|\lambda_i|\delta_{ij}\delta_i,\delta_j\rangle = \sum_{i,j}|\lambda_i|\delta_{ij}^2 = \sum_i |\lambda_i| < \infty.$$

□

7.7 (Six locally convex operator topologies)**.** Let H be a Hilbert space.

$$x \mapsto (\|x\xi\|^2 + \|x^*\xi\|^2)^{\frac12}, \qquad x \mapsto \|x\xi\|, \qquad x \mapsto \langle x\xi,\xi\rangle$$

for $\xi \in H$.

$$x \mapsto \Big(\sum_{i=1}^\infty \|x\xi_i\|^2 + \|x^*\xi_i\|^2\Big)^{\frac12}, \qquad x \mapsto \Big(\sum_{i=1}^\infty \|x\xi_i\|^2\Big)^{\frac12}, \qquad x \mapsto \Big|\sum_{i=1}^\infty \langle x\xi_i,\xi_i\rangle\Big|$$

for $(\xi_i) \in \ell^2(\mathbb{N},H)$.

(a) A net T_i converges to T strongly in $B(H)$ if and only if $\|(T_i - T)^{\oplus n}\overline{\xi}\| \to 0$ for all $\overline{\xi} \in H^{\oplus n}$.

(b) A net T_i converges to T σ-strongly in $B(H)$ if and only if $\|(T_i - T)^{\oplus\infty}\overline{\xi}\| \to 0$ for all $\overline{\xi} \in H^{\oplus\infty}$.

7.8 (Continuity of linear functionals)**.** Let l be a linear functional on $B(H)$ for a Hilbert space H.

(a) l is weakly continuous if and only if it is strongly* continuous, and in this case we have

$$l = \sum_i \lambda_i\omega_{e_i,e_i'}, \qquad (\lambda_i) \in c_c, \quad (e_i),(e_i') \subset H \text{ orthonormal.}$$

or equivalently,

$$l = \sum_i \omega_{x_i,y_i}, \qquad (x_i),(y_i) \in c_c(\mathbb{N},H)$$

(b) l is σ-weakly continuous if and only if it is σ-strongly* continuous, and in this case we have

$$l = \sum_i \lambda_i\omega_{e_i,e_i'}, \qquad (\lambda_i) \in \ell^1, \quad (e_i),(e_i') \subset H \text{ orthonormal.}$$

or equivalently,

$$l = \sum_i \omega_{x_i,y_i}, \qquad (x_i),(y_i) \in \ell^2(\mathbb{N},H)$$

(c) For a convex subset of $B(H)$ is (σ-)weakly closed if and only if (σ-)strongly* closed.

Proof. Suppose l is strongly continuous. There exists $\overline{x} \in H^{\oplus n}$ such that

$$|l(T)| \le \|T^{\oplus n}\overline{x}\|.$$

The functional $l : A \to \mathbb{C}$ factors through $H^{\oplus n}$ such that

$$A \xrightarrow{\overline{x}} H^{\oplus n} \to \mathbb{C}.$$

□

7.9.

(σ-) means that we have chosen the standard form. The followings also hold in Hilbert modules.

(a) On a bounded subset of $B(H)$, the weak, strong, strong* topologies coincide with the σ-weak, σ-strong, σ-strong* topologies, respectively.

(b) M is (σ-)strongly* complete, M_1 is (σ-)weakly and (σ-)strongly complete.

(c) On $\mathrm{U}(M)$, the (σ-)weak topology and (σ-)strong* topology coincide. However, it is not (σ-)weakly nor (σ-)stronlgy closed, but is (σ-)strongly* closed.

Suppose $T : X \to Y$ satisfies $\|y\| = \inf_{x\in T^{-1}(y)} \|x\|$. If $R : Y \to Z$ is a function such that $RT : X \to Z$ is a contraction, then R is a contraction, because for $y \in Y$, if we take $x \in X$ such that $Tx = y$ and $\|y\| + \varepsilon > \|x\|$, then

$$\|Ry\| = \|RTx\| \le \|x\| < \|y\| + \varepsilon \to \|y\|, \qquad \varepsilon \to 0.$$

The condition is equivalent to the isometry of T^*?

7.4

spectral radius formula and spectral mapping theorem are reuiqred.

7.10 (Spectral radius formula)**.**

7.11 (Spectral mapping theorem)**.** If $f(A)$ is not invertible, then in the factorization $f(z) = c\prod_k(\lambda_k - z)$, we have $\lambda = \lambda_k \in \sigma(A)$ for some k, so $f(\lambda) = 0$.

If $f(N)$ is not invertible, then there is $x \in H$ such that $\langle f(N)x, x\rangle = 0$....? Then, we can check $f(\langle Nx, x\rangle) = 0$ and $\langle Nx, x\rangle \in \sigma(N)$..?

7.12 (Continuous functional calculus)**.** Let N be a bounded normal linear operator on a Hilbert space H. A *continuous functional calculus* of N is a unital $*$-homomorphism

$$\Phi : C(\sigma(N)) \to B(H) : f \mapsto f(N)$$

such that $\Phi(z) = N$, where $z \in C(\sigma(N))$ denotes the inclusion $\sigma(N) \to \mathbb{C}$.

Proof. We first prove the existence. On the polynomial ring $\mathbb{C}[z,\bar{z}] \subset C(\sigma(N))$ with a conjugate-linear involution $z \mapsto \bar{z}$, we have no issue for defining a unital $*$-homomorphism

$$\Phi_0 : \mathbb{C}[z,\bar{z}] \to B(H) : f \mapsto f(N)$$

such that $\Phi_0(z) = N$ because N is normal. First, $\mathbb{C}[z,\bar{z}]$ is a unital $*$-algebra separating points of $\sigma(N)$ so that it is uniformly dense in $C(\sigma(N))$ by the Stone-Weierstrass theorem. Second, $B(H)$ is complete with respect to the norm topology. Third, Φ_0 is bounded by the spectral radius formula and the spectral mapping theorem for

$$\|f(N)\| = r(f(N)) = \sup_{\lambda\in\sigma(f(N))} |\lambda| = \sup_{\lambda\in\sigma(N)} |f(\lambda)| = \|f\|$$

$$\|f(N)\|^2 = \||f|^2(N)\| = r(|f|^2(N)) = \sup_{\lambda\in\sigma(|f|^2(N))} |\lambda| = \sup_{\lambda\in\sigma(N)} |f(\lambda)|^2 = \||f|^2\| = \|f\|^2$$

Therefore, Φ_0 is extended to a bounded linear map

$$\Phi : C(\sigma(N)) \to B(H) : f \mapsto f(N)$$

such that $\Phi(z) = N$. Now it is enough to check Φ also preserves the multiplication and involution.

□

Chapter 8

Unbounded operators

8.1 Densely defined closed operators

We almost always consider the domain of an unbounded linear operator as the union of all subspaces on which a given operator is continuously well-defined. Between complete spaces, the subspaces may be assumed to be closed.

Densely defined operators can be seen as increasing limits of partially defined continuous linear operators.

For X without condition and Y normable, then the continuity of $T : X_\sigma \to Y_\sigma$ implies the boundedness of $T : X \to Y$ because if we have $x_i \to 0$ and Tx_i is not bounded, then the uniform boundedness principle on Y^* proves Tx_i does not coverges weakly to zero.

We want to realize the graph $\Gamma(T)$ as the strict inductive limit of Fréchet spaces $\Gamma(T_i)$ with barrelled X_i. The topology on $\Gamma(T_i)$ may not come from the topology of $X_\sigma \times Y_\sigma$. If so, by the closed graph theorem, $T_i : X_i \to Y$ are everywhere defined continuous linear operators.

Even if the weak topology on $X \times Y$ is not complete but its weakly closed subspace $\Gamma(T)$ can be seen as a Banach space. Which topology is natural on the graph? For closedness, weak topology is the most natural.

I think the most natural setting for densely defined closed operators is the Fréchet space.

8.1. Let X and Y be topological vector spaces. A *linear operator* from X to Y is a linear map $T : \operatorname{dom} T \to Y$, where $\operatorname{dom} T$ is a linear subspace of X.

8.2. Let X and Y be Fréchet spaces. For a closed operator $T : \operatorname{dom} T \subset X \to Y$, there is an increasing net $T_i : \operatorname{dom} T_i \subset X \to Y$ of closed operators such that $\operatorname{dom} T_i$ is closed and $\Gamma(T) = \bigcup_i \Gamma(T_i)$. (Consider the net of finite-dimensional subspaces) Conversely,

(a) a

8.3 (Adjoint operators)**.** Let (X, X^*) and (Y, Y^*) be dual pairs. Let $T : \operatorname{dom} T \subset X_\sigma \to Y$ be a densely defined linear operator. The *adjoint* of T is defined by a linear operator $T^* : \operatorname{dom} T^* \subset Y^*_\sigma \to X^*$ with domain

$$\operatorname{dom} T^* := \{y^* \in Y^* \mid \operatorname{dom} T \subset X_\sigma \to \mathbb{C} : x \mapsto \langle Tx, y^* \rangle \text{ is continuous}\}$$

such that

$$\langle x, T^* y^* \rangle := \langle Tx, y^* \rangle, \qquad x \in \operatorname{dom} T,\ y^* \in \operatorname{dom} T^*.$$

Consider the dual pair $(X \times Y, Y^* \times X^*)$ with the pairing given by $\langle (x, y), (y^*, x^*) \rangle := \langle x, x^* \rangle + \langle y^*, y \rangle$ for $(x, y) \in X \times Y$ and $(y^*, x^*) \in Y^* \times X^*$.

(a) If $T \subset S$, then $S^* \subset T^*$.

(b) $T^* : \operatorname{dom} T^* \subset Y^*_\sigma \to X^*_\sigma$ is always closed.

(c) T is closable if and only if T^* is densely defined. If it is, then T^{**} is the closure of T.

Proof. Before proofs, we first claim that the defining condition of the adjoint T^* is equivalent to the equality $\operatorname{gra}(-T^*) = (\operatorname{gra} T)^\perp$ in $Y^* \times X^*$ with respect to the pairing. One direction is clear by

$$\langle (x, Tx), (y^*, -T^*y^*)\rangle = \langle x, -T^*y^*\rangle + \langle Tx, y^*\rangle = 0, \qquad x \in \operatorname{dom} T,\ y^* \in \operatorname{dom} T^*.$$

Conversely if $(y^*, x^*) \in (\operatorname{gra} T)^\perp$, then since

$$0 = \langle (x, Tx), (y^*, x^*)\rangle = \langle x, x^*\rangle + \langle Tx, y^*\rangle = \langle x, x^* + T^*y^*\rangle, \qquad x \in \operatorname{dom} T,$$

we have $y^* \in \operatorname{dom} T^*$ from the continuity of $\operatorname{dom} T \subset X_\sigma \to \mathbb{C} : x \mapsto \langle Tx, y^*\rangle = -\langle x, x^*\rangle$, and $x^* = -T^*y^*$ by the definition of adjoint operator T^* and the desity of $\operatorname{dom} T$ in X_σ. Hence the claim $(y^*, x^*) = (y^*, -T^*y^*) \in \operatorname{gra}(-T^*)$ follows.

(a) Clear from the claim.

(b) It is because the complement $(\operatorname{gra} T)^\perp$ is closed in $(Y^* \times X^*)_\sigma = Y^*_\sigma \times X^*_\sigma$.

(c) Suppose T is closable. If $y \in Y$ satisfies $\langle y, y^*\rangle = 0$ for every $y^* \in \operatorname{dom} T^*$, then the equation $\langle (0, y), (y^*, -T^*y)\rangle = 0$ implies $(0, y) \in (\operatorname{gra}(-T^*))^\perp = (\operatorname{gra} T)^{\perp\perp} = \overline{\operatorname{gra} T}$, and the closability of T implies $y = T0 = 0$. It means that $\operatorname{dom} T^*$ separates point of Y, that is, $\operatorname{dom} T^*$ is dense in Y^*_σ.

Conversely, if T^* is densely defined, then we can define the double adjoint $T^{**} : \operatorname{dom} T^{**} \subset X \to Y$, which has the graph $\operatorname{gra} T^{**} = (\operatorname{gra}(-T^*))^\perp = (\operatorname{gra} T)^{\perp\perp} = \overline{\operatorname{gra} T}$, so T has the closure T^{**}.

□

8.4 (Everywhere defined continuous operators)**.** Let (X, X^*) and (Y, Y^*) be dual pairs. We will always consider the weak topologies if not mentioned. Let $T : X \to Y$ be an everywhere defined continuous linear operator. It is clear that T is a homeomorphism if and only if it is either an injective quotient map, a surjective embedding, or a closed embedding of dense range. For a dense injection T, T is a homeomorphism if and only if it is either a quotient map, surjective embedding, or a closed embedding.

(a) T is injective iff T^* has dense range.

(b) T is an embedding iff T^* is surjective.

(c) T is a closed embedding iff T^* is a quotient map.

Proof. (a) If the range of T^* is not dense in X^*, then there is a non-zero $x \in X$ vanishing on the range of T^* by the Hahn-Banach extension, and Tx vanishes by every y^* so that the kernel contains x and is non-zero. Conversely, if T^* has dense range, then T is clearly injective by the continuity of T.

(b) If T is an embedding, then every element of X^* induces a partially defined continuous linear functional on the range of T, and the Hahn-Banach extension gives an everywhere defined continuous linear functional on Y, which is indeed an element of Y^*, so T^* is surjective. Conversely if T^* is surjective, then for a net x_i in X such that $Tx_i \to 0$ in Y, taking $y^* \in Y^*$ for each $x^* \in X^*$ such that $T^*y^* = x^*$, we can check easily that $x_i \to 0$ in X, so T is an embedding.

(c) Suppose T is a closed embedding. Let V^* be a neighborhood of zero in Y^*, and find a finite sequences $x_i \in X$ and $y_j \in Y \setminus TX$ such that $\{Tx_i, y_j\}^\circ \subset V^*$. For $x^* \in X^*$ satisfying $\max_i |\langle x_i, x^*\rangle| \le 1$, since TX is closed in Y, we can find $y^* \in Y^*$ by the Hahn-Banach extension theorem such that $x^* = T^*y^*$ and $\max_j |\langle y_j, y^*\rangle| \le 1$. It implies that $\{x_i\}^\circ \subset T^*\{Tx_i, y_j\}^\circ \subset T^*V^*$, so T^*V^* is a neighborhood of zero in X^*, which means T^* is open, and hence T^* is a quotient map.

Conversely, assume T^* is a quotient map, which is automatically open by the fact that a quotient map onto the coset space of a topological group is open. Let x_i be a net in X such that $Tx_i \to y$ in Y. Since T^*

is open and the absolute polar $\{y\}^\circ$ is a neighborhood of zero in Y^*, so is the image $T^*\{y\}^\circ$ in X^*. Then, $x^* \in X^*$ is contained in this image if and only if there is $y^* \in Y^*$ such that $x^* = T^* y^*$ and $|\langle y, y^*\rangle| \leq 1$, which is equivalent to $\lim_i |\langle x_i, x^*\rangle| \leq 1$ since T is an embedding and T^* is surjective. If we consider a linear functional $x^{**} : x^* \mapsto \lim_i \langle x_i, x^*\rangle$ on X^*, then it is continuous because $\{x^{**}\}^\circ = T^*\{y\}^\circ$ is a neighborhood of zero in X^*, so x^{**} defines $x \in X$ such that $Tx = y$.

(The open mapping theorem on Banach spaces follows from the fact that Y^*_τ if always complete if Y is a Banach space, called the Grothendieck completeness criterion.)

□

8.5 (Densely defined closed operators)**.** Let (X, X^*) and (Y, Y^*) be dual pairs. We will always consider the weak topologies if not mentioned. Let $T : \operatorname{dom} T \subset X \to Y$ be a densely defined closed linear operator. Note that $\ker T$ is closed because if $x_i \in \ker T$ is a net with $x_i \to x$ in X, then $(x_i, Tx_i) = (x_i, 0) \to (x, 0)$ in $X \times Y$, so the closedness of T implies that $Tx = 0$. Consider the induced operator $V : \operatorname{dom} V \subset X/\ker T \to \overline{\operatorname{ran}} T$ into the closure of the range, which is always a densely defined dense injection with $\operatorname{dom} V = \operatorname{dom} T/\ker T$ and $\operatorname{ran} V = \operatorname{ran} T$ as vector spaces. Furthermore, we can decompose V via the graph to construct a diagram of everywhere defined continuous linear operators

$$X \xrightarrow{(1)} X/\ker T \xleftarrow{(2)} \operatorname{gra} V \xrightarrow{(3)} \overline{\operatorname{ran}}\, T \xrightarrow{(4)} Y$$

where (1) is a quotient map, (2) and (3) are dense injections, and (4) is a closed embedding. For the operator T, we can consider the following six conditions.

(a) (1) is a homeomorphism iff T is injective.

(b) (2) is an embedding iff T is continuous.

(c) (2) is a surjection iff T is everywhere defined.

(d) (3) is an embedding iff T is ...

(e) (3) is a surjection iff T has closed range.

(f) (4) is a homeomorphism iff T has dense range.

The proofs are clear.

8.6 (Weak dual operators)**.** Let (X, X^*) and (Y, Y^*) be dual pairs. We will always consider the weak topologies if not mentioned. Let $T : \operatorname{dom} T \subset X \to Y$ be a densely defined closed linear operator. Consider the following daigram.

$$X \xrightarrow{(1)} X/\ker T \xleftarrow{(2)} \operatorname{gra} V \xrightarrow{(3)} \overline{\operatorname{ran}}\, T \xrightarrow{(4)} Y$$

$$Y^* \xrightarrow{(1^*)} Y^*/\ker T^* \xleftarrow{(2^*)} \operatorname{gra} V^* \xrightarrow{(3^*)} \overline{\operatorname{ran}}\, T^* \xrightarrow{(4^*)} X^*$$

We have a topological isomorphism $(X/\ker T)^* \cong \overline{\operatorname{ran}}\, T^*$ since $(X/\ker T)^* \to X^*$ and $\overline{\operatorname{ran}}\, T^* \to X^*$ are closed embeddings with same range, and another topological isomorphism $(\overline{\operatorname{ran}}\, T)^* \cong Y^*/\ker T^*$ since $Y^* \to Y^*/\ker T^*$ and $Y^* \to (\overline{\operatorname{ran}}\, T)^*$ are quotient maps with same kernel. In the above diagram, (1) and (4*) are mutually duals, (4) and (1*) are mutually duals. We can also check the followings easily.

(a) (2) is an embedding iff (2*) is a surjection.

(b) (2) is a surjection iff (2*) is an embedding.

(c) (2) is a homeomorphism iff (2*) is a homeomorphism.

(d) (3) is an embedding iff (3*) is a surjection.

(e) (3) is a surjection iff (3*) is an embedding.

(f) (3) is a homeomorphism iff (3^*) is a homeomorphism.

We may assume $T : \operatorname{dom} T \subset X \to Y$ is a dense injection so that $V = T$ in the proof.

Proof. (a) If $(x, Tx) \mapsto x$ is an embedding, meaning T is continuous, for $y^* \in Y^*$, T^*y^* is a continuous linear functional on X

(e) Suppose $(x, Tx) \mapsto Tx$ is a surjection. If $T^*y_i^* \to 0$, then $\langle Tx, y_i^*\rangle \to 0$ implies $y_i \to 0$, so $(T^*y^*, y^*) \mapsto T^*y^*$ is an embedding.

□

8.7 (Operators on Frećhet spaces)**.** Let (X, X^*) and (Y, Y^*) be dual pairs. Recall that for an everywhere defined linear operator between $X \to Y$ the weak continuity and the Mackey continuity are equivalent.

Let $T : \operatorname{dom} T \subset X \to Y$ be a densely defined closed between Fréchet spaces. Note that (b) and (c) are equivalent, and (d) and (e) are equivalent by the open mapping theorem. We can ask only four conditions: injectivity, boundedness, closed range, and dense range. Note that T is boundedly invertible if and only if it is bijective on its domain.

Proof. T is a Mackey embedding iff T is surjective: If T is a Mackey embedding, then the completeness of the Mackey topology of X_τ implies that the range of T is closed in the Mackey topology, so T is surjective. Conversely, if T is surjective, then the direct application of the open mapping theorem on T implies that T is a Mackey embedding.

□

—

We want to investigate sufficient conditions in order that continuity implies everywhere definedness. If we may assume the closed convex hull of $\{T^*y_i^*\}$ is compact in X^* for any(?) net $y_i^* \to 0$, then

$$|\langle x, T^*y_i^*\rangle| \le |\langle Tx_0, y_i^*\rangle| + |\langle x_0 - x, T^*y_i^*\rangle|$$

can be estimated by taking x_0 independent of i so that $x \in \operatorname{dom} T^{**} = \operatorname{dom} T$. However, in a different way, we can see that it is enough to have the complete Mackey topology. Why is it okay?

—

<Strong bidual>

We have in general $X_\tau^* \ne X_\beta^*$.

bounded below iff 1,2,3 surjective iff 3,4 boundedly invertible iff 1,3,4

For symmetric operators, 4 implies 1 For self-adjoint or normal operators, 1,4 are equivalent

point spectrum if 1 fails residue spectrum if 1 hold but 4 fails continuous spectrum if 1,4 hold but 3 fails

8.8 (Cores)**.**

8.9 (Sum of unbounded operators)**.**

8.10 (Composition of unbounded operators)**.**

8.11 (Inverse of unbounded operators)**.** Let $T : \operatorname{dom} T \subset X \to Y$ be an injective linear operator.

$$\operatorname{dom} T^{-1} := \operatorname{ran} T.$$

8.2 Symmetric and self-adjoint operators

8.12 (Symmetric operators)**.** Let H be a Hilbert space. A densely defined linear operator A on H is called *symmetric* if $A \subset A^*$, that is, $\langle Ax, y\rangle = \langle x, Ay\rangle$ for $x, y \in \operatorname{dom} A$. Let A be a symmetric operator. Then, A is always closable with closure A^{**} since A^* is densely defined because A is densely defined, and since A^* is closed because every adjoint operator is closed. If the closure of A is self-adjoint, then it is called *essentially self-adjoint*. In general, instead of self-adjointness, it is easy to check a given linear operator is symmetric.

(a) Every symmetric extension of A is a restriction of A^*. In particular, A has maximal symmetric extensions.

(b) A self-adjoint operator is maximal, and a maximal symmetric operator is closed.

(c) A symmetric operator is essentially self-adjoint if and only if it has a unique self-adjoint extension.

Proof. (a) . □

Let A be a closed symmetric operator on a Hilbert space H. We want to ask the following questions: Is A self-adjoint? If not, does A admit self-adjoint extensions? Which self-adjoint extension generate the appropriate quantum dynamics?

8.13 (Spectra of closed symmetric operators)**.** Let A be a closed symmetric operator on a Hilbert space H. We have A is injective, bounded, of closed range, of dense range if and only if A^* is of dense range, bounded, of closed range, injective, respectively.

(a) If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\lambda - A$ is injective and has closed range.

(b) $\mathbb{C} \setminus \mathbb{R} \to \mathbb{Z}_{\geq 0} : \lambda \mapsto \dim\ker(\overline{\lambda} - A^*)$ is locally constant.

(c) A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$.

Proof. (a) By the symmetry of A, we have

$$\|(\lambda - A)x\|^2 = \|(\operatorname{Re}\lambda - A)x + i\operatorname{Im}\lambda x\|^2 = \|(\operatorname{Re}\lambda - A)x\|^2 + \|i\operatorname{Im}\lambda x\|^2 \gtrsim \|x\|^2, \qquad x \in \operatorname{dom} A.$$

(b)

(c) If A is self-adjoint, then $A \pm i = A^* \pm i$ is surjective so that $\sigma(A) \subset \mathbb{R}$. Suppose conversely $\sigma(A) \subset \mathbb{R}$ so that $A - \lambda$ is surjective and $A^* - \lambda$ is injective for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $y \in \operatorname{dom} A^*$. By the surjectivity of $A + i$, there is $x \in \operatorname{dom} A$ such that $(A^* + i)y = (A + i)x = (A^* + i)x$, and it implies $y = x$ by the injectivity of $A^* + i$. Therefore, $y \in \operatorname{dom} A$ and $A^* = A$. □

8.14 (Unbounded normal operators)**.** Let N be a normal operator on a Hilbert space H, a densely defined closed operator such that $N^*N = NN^*$. We define the *continuous functional calculus* of N as the $*$-homomorphism $\Phi : C_0(\sigma(N)) \to B(H) : f \mapsto f(N)$ such that

$$Nf(N) = (zf)(N) = \overline{f(N)N}, \qquad N^*f(N) = (\overline{z}f)(N) = \overline{f(N)N^*}$$

for all $f \in C_0(\sigma(N))$ satisfying $zf \in C_0(\sigma(N))$, where $z \in C_0(\sigma(N))$ denotes the inclusion $\sigma(N) \to \mathbb{C}$. Define $R := (1 + N^*N)^{-1}$ and $B := N(1 + N^*N)^{-\frac{1}{2}}$, sometimes called the *resolvent* and the *bounded transform* of N.

(a) R is an everywhere defined bounded self-adjoint operator.

(b) B is an everywhere defined bounded normal operator.

(c) The continuous functional calculus uniquely exists.

Proof. (a) This statement is true for any densely defined closed linear operator T on a Hilbert space, instead of N. We prove $1+T^*T$ is a boundedly invertible self-adjoint operator. Consider the inequality

$$\|(1+T^*T)x\|^2 = \|x\|^2 + 2\|Tx\|^2 + \|T^*Tx\|^2 \geq \|x\|^2, \qquad x \in \operatorname{dom} T^*T.$$

The operator $1+T^*T$ is then clearly injective, and is also surjective because for any $z \in H$, since the graph of T and the swapped graph of $-T^*$ in $H \oplus H$ are mutually orthogonal complements, there is $x \in \operatorname{dom} T$ and $y \in \operatorname{dom} T^*$ such that $(z,0) = (x,Tx) + (-T^*y, y)$, which gives $z = x - T^*y = x - T^*(-Tx) = (1+T^*T)x$.

It is closed because the surjectivity of $1+T^*T$ implies that the inverse $(1+T^*T)^{-1}$ is everywhere defined and bounded by the above inequality, which implies $(1+T^*T)^{-1}$ is closed so that the original operator $1+T^*T$ is also closed. It is densely defined because if $(x, Tx) \in \operatorname{gra} T$ is orthogonal to $\{(y,Ty) : y \in \operatorname{dom} T^*T\}$, then it follows from the surjectivity of $1+T^*T$ that $x = 0$ by

$$0 = \langle x, y\rangle + \langle Tx, Ty\rangle = \langle x, y\rangle + \langle x, T^*Ty\rangle = \langle x, (1+T^*T)y\rangle, \qquad y \in \operatorname{dom} T^*T,$$

so $\operatorname{dom} T^*T$ is dense in $\operatorname{dom} T$, and also in H. The self-adjointness is now clear from $0 \notin \sigma(1+T^*T)$.

(b) The operator R is the everywhere defined bounded linear operator on H such that $\operatorname{ran} R = \operatorname{dom} N^*N$ such that $N^*NR = 1-R$. We also have $0 \leq R \leq 1$. We first prove $\operatorname{ran} R^{\frac{1}{2}} \subset \operatorname{dom} N$ for everywhere definedness of $B = NR^{\frac{1}{2}}$. Fix an element $R^{\frac{1}{2}}x \in \operatorname{ran} R^{\frac{1}{2}}$ and arbitrary $\varepsilon > 0$. Since an injective normal operator $R^{\frac{1}{2}}$ has dense range, we can take $R^{\frac{1}{2}}x_0 \in \operatorname{ran} R^{\frac{1}{2}}$ satisfying $\|x - R^{\frac{1}{2}}x_0\| < \varepsilon$. Then, since

$$\|NRx_0\|^2 = \langle RN^*NRx_0, x_0\rangle = \langle R^{\frac{1}{2}}(1-R)x_0, R^{\frac{1}{2}}x_0\rangle = \langle (1-R)R^{\frac{1}{2}}x_0, R^{\frac{1}{2}}x_0\rangle \leq \|R^{\frac{1}{2}}x_0\|^2,$$

we have $R^{\frac{1}{2}}x \in \operatorname{dom} N$ by limiting $\varepsilon \to 0$ on

$$\begin{aligned}
|\langle R^{\frac{1}{2}}x, N^*y\rangle| &\leq |\langle R^{\frac{1}{2}}(x - R^{\frac{1}{2}}x_0), N^*y\rangle| + |\langle Rx_0, N^*y\rangle| \\
&\leq \|R^{\frac{1}{2}}\|\|x - R^{\frac{1}{2}}x_0\|\|N^*y\| + |\langle NRx_0, y\rangle| \\
&\leq \varepsilon\|N^*y\| + \|NRx_0\|\|y\| \\
&\leq \varepsilon\|N^*y\| + \|R^{\frac{1}{2}}x_0\|\|y\| \\
&\leq \varepsilon\|N^*y\| + (\|x\| + \varepsilon)\|y\| \to \|x\|\|y\|, \qquad y \in \operatorname{dom} N^*.
\end{aligned}$$

Therefore, B is everywhere defined, and the boundedness, by one, automatically follows in the proof.

Now we check the normality. Be cautious that B^* is the closure of a densely defined bounded operator $R^{\frac{1}{2}}N^*$ but not itself. Let $x \in \operatorname{dom} N$. Since $Rx \in \operatorname{ran} R = \operatorname{dom} N^*N \subset \operatorname{dom} N$, we have $N^*NRx = x - Rx \in \operatorname{dom} N$, so NN^*NRx is well-defined in H and

$$RNx = RN(1+N^*N)Rx = R(N + NN^*N)Rx = R(1+NN^*)NRx = R(1+N^*N)NRx = NRx.$$

Thus, $N^*Rx \in \operatorname{dom} N$ implies $RNN^*Rx = NRN^*Rx$, so

$$B^*BRx = R^{\frac{1}{2}}N^*NR^{\frac{3}{2}}x = R^{\frac{1}{2}}(1-R)R^{\frac{1}{2}}x = (R-R^2)x = RNN^*Rx = NRN^*Rx = BB^*Rx.$$

Since the injectivity of R^2 deduces that $R \operatorname{dom} N \supset R \operatorname{ran} R = \operatorname{ran} R^2$ is dense, we have $B^*B = BB^*$.

(c) Observe that

$$\sigma(N) \to \sigma(B) \cap B(0,1) : \lambda \mapsto \lambda(1+|\lambda|^2)^{-\frac{1}{2}}$$

is a homeomorphism, so we can define an faithful non-degenrate representation

$$C_0(\sigma(N)) \to C(\sigma(B)) \to B(H),$$

where $C(\sigma(B)) \to B(H)$ is the continuous functional calculus of the bounded normal operator B. We want to prove it preserves the action of N and N^*, also after extension to Borel functional calculus.

We also want to prove $\pi(C_0(\sigma(N)))''$ is the smallest von Neumann algebra where N is affiliated with.

We also want to prove we can always regard $f(N)$ as normal for finite Borel function f, in particular densely defined and closed by taking closure.

When we do (continuous or Borel) functional calculus of polynomially unbounded functions, it is safe to fix $\xi \in \operatorname{dom}(1+N^*N)^n$ for sufficiently large n, i.e. the common core of operators occuring in computation.

For the continuous functional calculus for finitely generated commutative unital C*-subalgebra of $B(H)$, we can show the existence of common core by collecting the compactly supported functions.

uniqueness..

□

8.15. For an unbounded Borel function f on $\sigma(T)$, there are two methods. One method is the spectral truncation using $f1_{|f|\le n}$. The other method is using $h \in \mathbb{C}(z,\bar{z})$ be such that $h^{-1} : f(\sigma(T)) \to \mathbb{C}$ is an embedding to a bounded set, the inverse bounded transform $z \mapsto z(1-|z|^2)^{-1}$ for example, and take the bounded Borel functional calculus with $h^{-1}\circ f$ and define $f(T) := h(h^{-1}\circ f(T))$. We want to show $f(T)$ is independent of the choice of h.

(a)

8.16. Consider a net $\{F\}$ of all finite subsets of $[0,1]$. Then, $1_F \uparrow 1$ pointwisely in $B_b([0,1])$, but $\omega(1_F)=0$ for Radon measure $\omega \in C([0,1])^*$, which means 1_F does not converge to the unit σ-weakly in $C([0,1])^{**}$.

(a) pointwise bounded sequential convergence
(b) $1_{\{\lambda\}}(N)$ is the projection onto the eigenspace corresponding to λ.
(c) $f(N)$ is approximated in norm by projections.
(d) $f(VNV^*) = Vf(N)V^*$ for V such that V^*V is the identity on $(\ker N)^\perp = \overline{\operatorname{ran}}\, N$.

Proof. (b) We may assume $\lambda = 0$. If $1_{\{\lambda\}}(N)\xi = \xi$, then

$$N\xi = N1_{\{\lambda\}}(N)\xi = (z1_{\{\lambda\}})(N)\xi = (\lambda 1_{\{\lambda\}})(N)\xi = \lambda 1_{\{\lambda\}}\xi = \lambda\xi.$$

Conversely let $N\xi = \lambda\xi$. Define $f_n \in B_b(\mathbb{C})$ such that

$$f_n(z) := \begin{cases} 1-n|z-\lambda| & \text{if } |z-\lambda| \le n^{-1}, \\ 0 & \text{if } |z-\lambda| \ge n^{-1}. \end{cases}$$

It satisfies $0 \le 1-f_n \le n|z-\lambda|$ and $f_n \downarrow 1_{\{\lambda\}}$ poinwisely. Then,

$$\|(1-f_n(N))\xi\|^2 = \langle |1-f_n|^2(N)\xi,\xi\rangle \le n^2\langle |N_\lambda|^2\xi,\xi\rangle = n^{-2}\|(N-\lambda)\xi\|^2 = 0$$

implies the strong convergence $1_{\{\lambda\}}(N)\xi = \lim_n f_n(N)\xi = \xi$.

(d) If N is bounded, then we can check the diagram

$$\begin{array}{ccc} C(\sigma(N)) & \xrightarrow{\Phi_N} & B(H) \\ \| & & \downarrow V\cdot V^* \\ C(\sigma(N)) & \xrightarrow{\Phi_{VNV^*}} & B(VH) \end{array}$$

commutes on the dense $*$-subalgebra $\mathbb{C}[z,\bar{z}]$.

$C_0(\sigma(N)) \to B(VH) : f \mapsto Vf(N)V^*$ satisfies the axiom of functional calculus for VNV^*? We can check it is a unital $*$-homomorphism such that

$$(VNV^*)Vf(N)V^* = VNf(N)V^* = V(zf)(N)V^*$$

□

8.17 (Cayley transform)**.** There is a one-to-one correspondence between the unitary operators from K_+ to K_-, the deficiency subspaces.

If A is a densely defined closed symmetric operator, then

$$Ux := \begin{cases} 0 & \text{if } x \in L^+, \\ (T-i)(T+i)^{-1}x & \text{if } x \in (L^+)^\perp, \end{cases}$$

is a partial isometry with initial and final spaces $(L^+)^\perp$ to $(L^-)^\perp$ such that $\operatorname{dom} A = (1-U)(L^+)^\perp$.

(a) If A is self-adjoint, then $1-U$ is injective and $\operatorname{dom} A = \operatorname{ran}(1-U)$.

(b) The Cayley transform provides a one-to-one correspondence between self-adjoint operators A and unitary operators U satisfying $\ker(1-U) = 0$.

(c)

8.18 (Kato-Rellich theorem)**.**

8.19 (Non-negative symmetric operators)**.** Let A be a non-negative symmetric operator on a Hilbert space H. Define a Hilbert space H_1 by the completion of the inner product space $\operatorname{dom} A$ given such that $\langle x, y\rangle_1 := \langle (1+A)x, y\rangle$. We have a dense inclusion $T : H_1 \to H$ satisfying $Tx = x$ and

$$\langle (1+A)x, y\rangle = \langle T^{-1}x, T^{-1}y\rangle_1 = \langle (T^{-1})^* T^{-1}x, y\rangle = \langle (TT^*)^{-1}x, y\rangle, \qquad x, y \in \operatorname{dom} A,$$

so that $TT^* : H \to H$ is also a dense inclusion, which is a bounded self-adjoint operator. Define a self-adjoint operator $\widetilde{A} := (TT^*)^{-1} - 1 : \operatorname{dom} \widetilde{A} \subset H \to H$ with domain $\operatorname{dom} \widetilde{A} := \operatorname{ran}(TT^*)$. Then, we can check $\widetilde{A}$ extends A as

$$\widetilde{A}x = (TT^*)^{-1}x - x = (1+A)x - x = Ax, \qquad x \in \operatorname{dom} A.$$

The self-adjoint operator $\widetilde{A}$ is called the *Friedrichs extension*.

Krein characterization.

For $x \in H$, let $\langle TT^*x, y\rangle + \langle \widetilde{A}TT^*x, Ay\rangle = 0$ for all $y \in \operatorname{dom} A$.

$$\begin{aligned} 0 &= \langle TT^*x, y\rangle + \langle \widetilde{A}TT^*x, Ay\rangle \\ &= \langle TT^*x, y\rangle + \langle x, Ay\rangle - \langle TT^*x, Ay\rangle \\ &= \langle TT^*x, y\rangle + \langle x, Ay\rangle - \langle \widetilde{A}TT^*x, y\rangle \\ &= \langle TT^*x, y\rangle + \langle x, Ay\rangle - \langle x, y\rangle + \langle TT^*x, y\rangle \\ &= 2\langle TT^*x - x, y\rangle + \langle x, (1+A)y\rangle \end{aligned}$$

8.20 (Multiplication operators)**.** Let (X, μ) be a localizable measure space. For $f \in L^0_{\mathrm{loc}}(X, \mu)$ an almost everywhere finite measurable function, we define a linear operator $m(f)$ on $L^2(X, \mu)$ by multiplication

$$m(f)\xi := f\xi, \qquad \xi \in \operatorname{dom} m(f) := \{\xi \in L^2(X,\mu) : f\xi \in L^2(X,\mu)\}.$$

almost everywhere zero function

(a) $L^0_{\mathrm{loc}}(X, \mu)$ is identified with the set of all normal operators on $L^2(X, \mu)$ affiliated with $m(L^\infty(X, \mu))$.

Proof. (a) We first prove that for a measurable function $f : X \to \mathbb{C}$ the multiplication operator $m(f)$ is a normal operator on $L^2(X, \mu)$ affiliated with $m(L^\infty(X, \mu))$.

The operator $m(f)$ is densely defined since if we let $E_n := f^{-1}(B(0, n)) \subset X$ be a non-decreasing sequence of measurable subsets so that $E_n \uparrow X$ as $n \to \infty$ almost everywhere, then for any $\xi \in L^2(X, \mu)$ we have $E_n\xi \to \xi$ in $L^2(X, \mu)$ as $n \to \infty$ and $E_n\xi \in \operatorname{dom} m(f)$.

The adjoint $m(f)^*$ is densely defined since

The operator $m(f)$ is closed because if ξ_n is a sequence in $\operatorname{dom} m(f)$ such that $\xi_n \to \xi$ and $m(f)\xi_n \to \eta$ in $L^2(X,\mu)$, then we have $\xi \in \operatorname{dom} m(f)$ by limit $n \to \infty$ on

$$\begin{aligned}
|\langle \xi, m(f)^*\zeta\rangle| &\le |\langle \xi-\xi_n, m(f)^*\zeta\rangle| + |\langle \xi_n, m(f)^*\zeta\rangle| \\
&= |\langle \xi-\xi_n, m(f)^*\zeta\rangle| + |\langle m(f)\xi_n, \zeta\rangle| \\
&\le |\langle \xi-\xi_n, m(f)^*\zeta\rangle| + |\langle m(f)\xi_n-\eta, \zeta\rangle| + |\langle \eta, \zeta\rangle| \to |\langle \eta, \zeta\rangle|, \qquad \zeta \in \operatorname{dom} m(f)^*,
\end{aligned}$$

and $m(f)\xi = \eta$ by limit $n \to \infty$ on

$$\begin{aligned}
|\langle m(f)\xi-\eta, \zeta\rangle| &\le |\langle m(f)\xi - m(f)\xi_n, \zeta\rangle| + |\langle m(f)\xi_n-\eta, \zeta\rangle| \\
&= |\langle \xi-\xi_n, m(f)^*\zeta\rangle| + |\langle m(f)\xi_n-\eta, \zeta\rangle| \to 0, \qquad \zeta \in \operatorname{dom} m(f)^*.
\end{aligned}$$

affiliated? easy if we know $m(L^\infty(X,\mu))$ is maximal abelian subalgebra.

Conversely, let T be a normal operator affiliated with $m(L^\infty(X,\mu))$. By the spectral theorem, $1_{[-n,n]}(T) \in m(L^\infty(X,\mu))$. We can construct a non-decreasing sequence of measurable subsets E_n such that $m(1_{E_n}) = 1_{[-n,n]}(T)$.

Let $f_{n,i} := T1_{E_n\cap F_i} \in L^2(F_i,\mu)$. $f_{n,i}$ converges to f locally in measure... in $L^0_{\mathrm{loc}}(X,\mu)$. We need to check $m(f) = T$.

□

8.21 (Polar decomposition)**.** If $T : H \to H$, then for $T = V|T|$, V is a partial isometry which connects from the complement of the kernel to the closure of the range as a unitary. Same for unbounded operator.

$$T = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : (\ker T)^\perp \oplus \ker T \xrightarrow{|T|} (\ker T)^\perp \oplus \ker T \xrightarrow{V} \overline{\operatorname{ran} T} \oplus (\operatorname{ran} T)^\perp$$

T is normal then $(\ker T)^\perp = \overline{\operatorname{ran} T}$.

polar decomposition polar decomposition of symmetric operator? polar decomopsition changes spectrum or domains?

support projection

8.3 Infinitesimal generators

8.22 (Stone theorem)**.** Let $u : \mathbb{R} \to U(H)$ be a weakly continuous unitary flow on a Hilbert space H. Then, there is a unique self-adjoint operator h on H such that $e^{ish} = u_s$ for $s \in \mathbb{R}$.

Proof. Define a one-parameter family of bounded operators

$$h_s := \frac{u_s - 1}{is}, \quad e_s := \frac{1}{s}\int_0^s u_t\,dt, \qquad s \in \mathbb{R}\setminus\{0\},$$

where the integral is justified by the continuous $*$-homomorphism $u : M(\beta\mathbb{R})_\sigma \to B_\sigma(H_\sigma) : s^{-1}1_{[0,s]} \mapsto e_s$ between the weak* topology and the weak operator topology, which extends $u : \mathbb{R} \to U(H)$. Since $s^{-1}1_{[0,s]} \to \delta_0$ in $M(\beta\mathbb{R})$, we have $e_s \to 1$ weakly in $B(H)$. Define a linear operator h on H such that

$$h\xi := \lim_{s\to 0} h_s\xi, \qquad \xi \in \operatorname{dom} h := \{\xi \in H : \lim_{s\to 0} h_s\xi \text{ exists in } H_\sigma\},$$

where the limits are in the weak topology H_σ.

First, $\operatorname{dom} h$ is weakly dense in H because if we choose any $\xi \in H$, then we have $u_f\xi \in \operatorname{dom} h$ for any function $f \in BV(\mathbb{R})$ and we can take an weak* approximate unit of $M(\beta\mathbb{R})$ in $BV(\mathbb{R})$. In more detail, we have $hu_f = iu_\mu$ where $f' = \mu \in M(\mathbb{R})$ by

$$\langle h_s u_f\xi - iu_{f'}\xi, \eta\rangle = \int_{\mathbb{R}} \langle u_t\xi, \eta\rangle \left(\frac{f(t)-f(t-s)}{s}\,dt - d\mu(t)\right) \to 0, \qquad \eta \in H,\ s \to 0.$$

(We did not check yet)

Second, we prove h is weakly closed. As a lemma, we introduce an operator version of the fundamental theorem of calculus formulated as

$$h_s\xi = e_s h\xi, \qquad \xi \in \operatorname{dom} h,\ s \in \mathbb{R}\setminus\{0\},$$

which can be shown by introducing $f(s) := \langle u_s\xi, \eta\rangle$ and $F(s) := \int_0^s f(t)\,dt$ as

$$\begin{aligned}
\langle e_s h\xi, \eta\rangle = \langle h\xi, e_s^*\eta\rangle &= \lim_{t\to 0}\langle h_t\xi, e_s^*\eta\rangle = \lim_{t\to 0}\langle e_s h_t\xi, \eta\rangle \\
&= \lim_{t\to 0}\left\langle \frac{1}{s}\int_0^s u_r \frac{u_t - 1}{it}\,dr\,\xi, \eta\right\rangle \\
&= \frac{1}{is}\lim_{t\to 0}\int_0^s \frac{f(r+t)-f(r)}{t}\,dr \\
&= \frac{1}{is}\lim_{t\to 0}\left(\frac{F(s+t)-F(s)}{t} - \frac{F(t)-F(0)}{t}\right) \\
&= \frac{1}{is}(f(s)-f(0)) = \langle h_s\xi, \eta\rangle, \qquad \eta \in H.
\end{aligned}$$

To prove the claim, if we take a net $\xi_i \in \operatorname{dom} h$ such that $\xi_i \to \xi$ and $h\xi_i \to \xi'$ weakly in H, then for any $\varepsilon > 0$ and for sufficiently small neighborhood U of zero in $\mathbb{R}$ we have

$$\begin{aligned}
|\langle h_s\xi - \xi', \eta\rangle| &\le |\langle h_s(\xi - \xi_i), \eta\rangle| + |\langle h_s\xi_i - e_s\xi', \eta\rangle| + |\langle e_s\xi' - \xi', \eta\rangle| \\
&\le |\langle \xi - \xi_i, h_s^*\eta\rangle| + |\langle h\xi_i - \xi', e_s^*\eta\rangle| + \varepsilon, \qquad s \in U,\ \eta \in H,
\end{aligned}$$

which implies the weak convergence $h_s\xi \to \xi'$ as $s \to 0$ by taking limit on i.

Next, we can prove h is self-adjoint. Let $\xi \in \operatorname{dom} h^*$ and $s \in \mathbb{R}\setminus\{0\}$. Since $\{\eta \in H : e_s^*\eta \in \operatorname{dom} h\}$ is dense because if $\eta \in$ then

$$\cdots,$$

with $he_s^*\eta = h_{-s}\eta$, we have

$$\langle e_s h^*\xi, \eta\rangle = \langle \xi, he_s^*\eta\rangle = \langle \xi, h_{-s}\eta\rangle = \langle h_s\xi, \eta\rangle$$

for dense choices of η in H. It implies

$$|\langle h_s\xi - h^*\xi, \eta\rangle| \le |\langle h_s\xi - e_s h^*\xi, \eta\rangle| + |\langle e_s h^*\xi - h^*\xi, \eta\rangle| = |\langle (e_s - 1)h^*\xi, \eta\rangle| \to 0, \qquad \eta \in H,\ s \to 0,$$

so $\xi \in \operatorname{dom} h$ and $h\xi = h^*\xi$. The claim follows from the same argument but applying conversely.

Finally, we claim that the functional calculus gives $e^{ish} = u_s$ for $s \in \mathbb{R}$. Since the functional calculus is a $*$-homomorphism, e^{ish} is unitary on H for each $s \in \mathbb{R}$. □

Cores and invaraint spaces?

8.23 (Smooth and analytic vectors)**.** Cores

(a) If T is symmetric and D_0 is dense, then $T|_{D_0}$ is essentially self-adjoint.

8.24 (Resolvent convergence)**.**

8.4 Decomposition of spectrum

8.25. Let $T : \operatorname{dom} T \subset E \to F$ be a linear operator between Banach spaces. We define the *point spectrum* and the *continuous spectrum* of T as

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not injective}\}, \qquad \sigma_c(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is a dense inclusion}\},$$

and the *residual spectrum* as $\sigma_r(T) := \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$.

$$\begin{aligned} \sigma &= \sigma_p \cup \sigma_c \cup \sigma_r \\ &= \sigma_{ess} \cup \sigma_d \\ &= \overline{\sigma_{pp}} \cup \sigma_{ac} \cup \sigma_{sc}. \end{aligned}$$

$$\sigma = \sigma_p \sqcup \sigma_c \sqcup \sigma_r = \overline{\sigma_{pp}} \cup \sigma_{ac}\sigma_{sc} = \sigma_d \sqcup \sigma_{ess,5}.$$

Exercises

8.26 (Strict topology)**.** Let H be a Hilbert space. Let $(T_\alpha) \subset B(H)$ and $K \in K(H)$.

(a) The strong* topology and the strict topology agree on bounded sets of $B(H)$.

8.27 (Unitary group)**.** Let H be a Hilbert space.

(a) The weak topology and the strict topology agree on $U(H)$.

8.28 (Bounded increasing nets)**.** Let T_α be a bounded increasing net of bounded self-adjoint operators on H.

(a) T_α converges strictly. In particular, $T_\alpha \to T$ strictly iff $T_\alpha \to T$ weakly.

Proof. Define T such that

$$\langle Tx, y\rangle := \lim_\alpha \sum_{k=0}^{3} i^k \langle T_\alpha(x + i^k y), x + i^k y\rangle.$$

The convergence is due to the monotone convergence in $\mathbb{R}$. We can check it is a well-defined bounded linear operator by considering the bounded sesquilinear form. Then, $T_\alpha \to T$ weakly by definition, and σ-strongly because the net is increasing. □

8.29 (Distributional operators)**.** (a) Every continuous linear operator $T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ naturally defines a closable densely defined operator $T : \operatorname{dom} T \to L^2(\mathbb{R})$ with $\operatorname{dom} T := \mathcal{D}(\mathbb{R})$.

8.30 (Differential operators on intervals)**.** Let $D : \operatorname{dom} D \subset H \to H$ be a linear operator $H := L^2([0,1])$ such that

$$Df(x) := if'(x), \qquad f \in \operatorname{dom} D := C_c((0,1)).$$

It is symmetric.

(a) $\operatorname{dom} \overline{D} = H_0^1((0,1))$.

(b) $\operatorname{dom} D^* = H^1((0,1)) \subset C([0,1])$.

(c) The family of self-adjoint extensions $\{D_\alpha\}$ can be parametrized by $\alpha \in \mathbb{T}$, where

$$\operatorname{dom} D_\alpha = \{f \in H^1((0,1)) : \lambda f(0) = f(1)\}.$$

(d) $\widetilde{D}$ has no self-adjoint extension if

$$\operatorname{dom}\widetilde{D} = C^\infty((0,1)) \cap C_0((0,1]).$$

Proof. (d) has no self-adjoint extension because we have deficiency indices $n^+ = 1$ and $n^- = 0$ (maybe). □

8.31 (Schr̈odinger operators)**.** For the *potential* $V \in L^0_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{D}'(\mathbb{R}^d)$ with the same symbol as the multiplication operator on $L^2(\mathbb{R}^d)$, let H be a linear operator on $L^2(\mathbb{R}^d)$ defined by

$$H\psi := -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \qquad \psi \in \operatorname{dom} H := C_c(\mathbb{R}^d),$$

where $\hbar$ and m are positive real constants. It is called the *Schrödinger operator*, and simply we write $H = -\Delta + V$ by putting $\hbar = 1$ and $m = \frac{1}{2}$.

The eigenvectors associated to the discrete spectrum is called *bound eigenstates*.

8.32 (Hydrogen atom)**.** Consider the Schrödinger operator $H := -\Delta - |x|^{-1}$ defined on $L^2(\mathbb{R}^3)$. We want to investigate the spectral decomposition of H by diagonalization.

(a) H is self-adjoint.

(b) $\sigma_d(H) = \{\}$

Proof. The orbital comes from the diagonalization of the Laplace-Beltrami operator on the unit sphere. □

8.33 (Periodic Schrödinger operators)**.** It is diagonalized to the direct integral of elliptic operators defined on the Brillouin torus.

Proof. □

Chapter 9

Operator theory

9.1 Toeplitz operators

invariant subspace problem Beurling theorem Hardy and Bergman and Bloch spaces JB* triple

Part IV

Operator algebras

Chapter 10

Banach algebras

10.1 Spectra of elements

A *Banach algebra* is a complete normed algebra, and a *unital Banach algebra* is a Banach algebra that is unital and satisfies $\|1\| = 1$. (For a Banach algebra A that is unital, there is a complete algebra renorming such that $\|1\| = 1$.) If an element a of a unital Banach algebra A satisfies $\|1 - a\| < 1$, then we can easily see that a is invertible in A with $\|a^{-1}\| \le (1 - \|1 - a\|)^{-1}$, because the Neumann series $\sum_{k=0}^{\infty}(1-a)^k$ exists in A and defines an inverse of a.

10.2 Ideals

10.1 (Ideals). (a) If I is a left ideal, then A/I is a left A-module.

10.2 (Modular left ideals). A left ideal I is called *modular* if there is $e \in A$ such that $a - ae \in I$ for all $a \in A$. The element e is called a *right modular unit* for I.

(a) I is modular if and only if A/I is unital(?).

(b) A proper modular left ideal is contained in a maximal left ideal.

(c) I is a maximal modular left ideal if and only if I is a modular maximal left ideal.

(d) There is a non-modular maximal ideal in the disk algebra.

10.3 (Closed ideals). (a) closure of proper left ideal is proper left.

(b) maximal modular left ideal is closed.

10.4 (Unitization). Let A be an associative complex algebra. Since A is a module over A itself, there is a algebra homomorphism $A \to L(A)$, so we can define

$$\widetilde{A} := \{ a + \lambda \in L(A) : a \in A,\ \lambda \in \mathbb{C} \}.$$

It is called the *Doroh extension* of A. If A is not unital, then it is usually called the *unitization*.

(a) If A is normed, then $\widetilde{A}$ is a normed algebra such that there is an isometric embedding $A \to \widetilde{A}$.

(b) If A is Banach, then $\widetilde{A}$ is a Banach algebra.

(c) $A \oplus \mathbb{C}$ is topologically isomorphic to $\widetilde{A}$ as normed spaces.

Proof. (a) Since A is normed, the space of bounded operators $L(A)$ has a natural normd algebra structure together with an isometry $A \to L(A)$. Then, $\widetilde{A}$ is a normed $*$-algebra with induced norm

$$\|a+\lambda\|_{L(A)} = \sup_{b\in A} \frac{\|ab+\lambda b\|}{\|b\|}$$

Then, A is a normed $*$-subalgebra of $\widetilde{A}$ because the norm and involution of A agree with $\widetilde{A}$.

(b) Suppose $a_n + \lambda_n$ is Cauchy in $\widetilde{A}$. Since A is complete so that it is closed in $\widetilde{A}$, we can induce a norm on the quotient $\widetilde{A}/A$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|a\| \le \|a+\lambda\| + |\lambda|$ shows that a_n is Cauchy in A.

Since a finite dimensional normed space is always Banach and A is Banach, λ_n and a_n converge. Finally, the inequality $\|a+\lambda\| \le \|a\| + |\lambda|$ implies that $a_n + \lambda_n$ converges.

(c) Check the topology on $A \oplus \mathbb{C}$ in detail... □

unitization, homomorphisms, category(direct sum, product, etc.)

$B(\mathbb{C}^n) = M_n(\mathbb{C})$ is simple, but $B(H)$ is not simple.

10.3 Gelfand theory

10.5 (Spectra of elements in unital Banach algebras)**.** Let a be an element of a unital Banach algebra A. The *spectrum* of a in A is defined to be the set

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A\},$$

and the *resolvent set* of a in A is defined to be its complement $\rho_A(a) := \mathbb{C} \setminus \sigma_A(a)$. If the ambient algebra A is clear in its context, we often omit it to just write $\sigma(a)$ and $\rho(a)$.

(a) $\sigma(a)$ is compact.

(b) $\sigma(a)$ is non-empty.

(c) If A is further a division ring, then $A \cong \mathbb{C}$. This result is called the *Gelfand-Mazur theorem*.

Proof. (a) If $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|a\|$, then $\lambda - a = \lambda(1-\lambda^{-1}a)$ is always invertible because $\|\lambda^{-1}a\| < 1$, so $\lambda \notin \sigma(a)$ and the spectrum is bounded. To show the closedness, it suffices to prove the set of invertibles $A^\times$ is open in A. For $a \in A^\times$, $a+h$ has the inverse $(1+a^{-1}h)^{-1}a^{-1}$ if $h \in A$ is sufficiently small such that $\|a^{-1}h\| < 1$, so we are done.

(b) Suppose the spectrum $\sigma(a)$ is empty so that the resolvent function $\mathbb{C} \to A : \lambda \mapsto (\lambda - a)^{-1}$ is well-defined on the entire domain $\mathbb{C}$. Note that $a \neq 0$. Because $A^\times \to A^\times : a \mapsto a^{-1}$ is norm differentiable in the sense that

$$\frac{\|b^{-1}-a^{-1}-(-a^{-1}(b-a)a^{-1})\|}{\|b-a\|} = \frac{\|(a^{-1}-b^{-1})(b-a)a^{-1}\|}{\|b-a\|} \le \|a^{-1}-b^{-1}\|\|a^{-1}\| \to 0, \qquad b \to a,$$

the resolvent function is weakly continuous and weakly holomorphic on $\mathbb{C}$. Since

$$\|(\lambda-a)^{-1}\| = \|\lambda^{-1}(1-\lambda^{-1}a)^{-1}\| = \left\|\lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}a)^k\right\| \le (2\|a\|)^{-1}\sum_{k=0}^{\infty}2^{-k} = \|a\|^{-1}, \qquad \lambda \in \mathbb{C}\setminus B(0,2\|a\|),$$

the resolvent function is bounded. The Liouville theorem implies that it is weakly constant, which is indeed a constant function because A^* separates points of A. It means that $a \in \mathbb{C}$ and contradicts to $\sigma(a) = \varnothing$.

(c) For any $a \in A$, by the part (b), there must be λ such that $\lambda - a$ is not invertible. In a division ring, zero is the only non-invertible element, so $\lambda = a$. □

10.6 (Spectral radius)**.** Let a be an element of a unital Banach algebra A. The *spectral radius* of a in A is defined to be

$$r(a) := \sup_{\lambda\in\sigma(a)} |\lambda|.$$

(a) $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) $\limsup_n \|a^n\|^{\frac{1}{n}} \le r(a)$, i.e. $r(a) = \lim_n \|a^n\|^{\frac{1}{n}}$.

Proof. (a) Since $(\lambda - a)^{-1} = \lambda^{-1}(1-\lambda^{-1}a)^{-1}$ exists if $|\lambda| > \|a\|$, we have $r(a) \le \|a\|$ for all $a \in \mathcal{A}$. For every $\lambda \in \sigma(a)$ and every integer $n \ge 1$ we have

$$|\lambda|^n = |\lambda^n| \le r(a^n) \le \|a^n\|,$$

and it proves $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) Fix $\omega \in A^*$ and a. Since the resolvent function $\rho(a) \to A : \lambda \mapsto (\lambda - a)^{-1}$ is weakly holomorphic, on the domain $\mathbb{C} \setminus \overline{B(0, r(a))} \subset \rho(a)$ we can consider the Laurent expansion

$$\omega((\lambda - a)^{-1}) = \sum_{k=-\infty}^{\infty} a_k \lambda^k, \qquad |\lambda| > r(a),$$

with a coefficient sequence $a_k \in \mathbb{C}$. Since on a smaller domain $\mathbb{C} \setminus \overline{B(0, \|a\|)}$ we have

$$\omega((\lambda-a)^{-1}) = \omega(\lambda^{-1}(1-\lambda^{-1}a)^{-1}) = \omega(\lambda^{-1}\sum_{k=0}^{\infty} \lambda^{-k}a^k) = \sum_{k=0}^{\infty} \omega(a^k)\lambda^{-k-1}, \qquad |\lambda| > \|a\|,$$

we can determine the coefficients a_k by the identity theorem

$$\omega((\lambda - a)^{-1}) = \sum_{k=0}^{\infty} \omega(a^k)\lambda^{-k-1}, \qquad |\lambda| > r(a).$$

It implies that if we take any positive $\lambda > r(a)$, then the sequence $a^k\lambda^{-k-1}$ in A indexed by k is weakly bounded, hence is bounded in norm by the uniform boundedness principle. Let $\|a^n\| \le C_\lambda \lambda^{n+1}$ for all $n \ge 1$. Then,

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \le \limsup_{n\to\infty} C_\lambda^{\frac{1}{n}} (\lambda^{n+1})^{\frac{1}{n}} = \lambda.$$

If we limit $\lambda \downarrow r(a)$, we are done. □

10.7 (Spectral invariance)**.** For fixed element, smaller the ambient algebra, less "holes" in the spectrum. Let $A \subset B$ be a closed subalgebra containing 1_A. Note that A may be unital even for $1_B \notin A$.

(a) $B^\times$ is clopen in $A^\times \cap B$.

semisimplicity and symmetricity

10.8 (Spectra of commutative unital Banach algebras)**.** Let A be a commutative unital Banach algbera. A *character* or a *one-dimensional representation* of A is a non-zero algebra homomorphism $\omega : A \to \mathbb{C}$. Denote by $\widehat{A}$ or $\sigma(A)$ the set of all characters of A and endow with the weak* topology on $\widehat{A} \subset A^*$. We call this space the *character space* or the *spectrum* of A.

(a) $\widehat{A}$ is a closed subset of the unit sphere of A^*.

(b) If A is generated by $a \in A$ in the sense that $\mathbb{C}[a]$ is dense in A, then $\widehat{A}$ is homeomorphic to $\sigma(a)$.

(c) For $a \in A$ and $\lambda \in \mathbb{C}$, we have $\lambda \in \sigma(a)$ if and only if there is $\omega \in \widehat{A}$ such that $\omega(a) = \lambda$.

Proof. (c) If $\lambda \notin \sigma(a)$, then $\omega(1) = 1$ implies $\lambda - \omega(a) = \omega(\lambda - a) = \omega((\lambda - a)^{-1})^{-1}$ cannot be zero. Conversely, let $\lambda \in \sigma(a)$. Since the closed ideal generated by $\lambda - a$ is proper, there is a maximal ideal I containing $\lambda - a$. By the Gelfand-Mazur theorem, the quotient homomorphism $A \to A/I \cong \mathbb{C}$ defines a one-dimensional representation $\omega \in \widehat{A}$ such that $\omega(\lambda - a) = 0$, so we are done. $\square$

10.9 (Gelfand transform)**.** Let A be a commutative unital Banach algebra. The *Gelfand transform* or the *Gelfand representation* is the algebra homomorphism

$$\Gamma : A \to C(\widehat{A}) : a \mapsto (\omega \mapsto \omega(a)).$$

(a) Γ has the image separating points by definition.

(b) Γ is isometric if and only if $r(a) = \|a\|$ for all $a \in A$.

Proof. (a)

$\square$

10.10 (Non-unital Banach algebras)**.**

10.4 Holomorphic functional calculus

10.11 (Holomorphic functional calculus)**.** Let a be an element of a unital Banach algebra A. Let f be a holomorphic function on a neighborhood U of $\sigma(a)$. Let γ be any positively oriented smooth simple closed curve in U enclosing $\sigma(a)$. Define $f(a) \in A$ by the Bochner integral

$$f(a) := \int_\gamma f(\lambda)(\lambda - a)^{-1}\, d\lambda.$$

Let $\mathrm{Hol}(\sigma(a))$ be the Fréchet algebra of all holomorphic functions on a neighborhood of $\sigma(a)$ endowed with the topology of compact convergence. We define the *holomorphic functional calculus* or the *Dunford-Riesz calculus* by a faithful unital algebra homomorphism $\Phi : \mathrm{Hol}(\sigma(a)) \to A$ such that $\Phi(zf) = a\Phi(f) = \Phi(f)a$ for all $f \in \mathrm{Hol}(\sigma(a))$.

Contour integrals of weakly holomorphic functions. It is a bounded weakly continuous function on the contour, we can define the integral.

(a) $f(a)$ is independent of the choice of γ.

(b) spectral mapping.

(c) power series.

Proof. (a)

$\square$

Exercises

10.12 (Basic properties of spectrum)**.** Let A be a unital Banach algebra.

(a) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. In particular, we cannot have $ab - ba = 1$. The left and right shift operators give an counterexample.

(b) If $\sigma(a)$ is non-empty, then $\sigma(p(a)) = p(\sigma(a))$.

Proof. (a) Intuitively, the inverse of $1 - ab$ is $c = 1 + ab + abab + \cdots$. Then, $1 + bca = 1 + ba + baba + \cdots$ is the inverse of $1 - ba$. $\square$

$C_b(\Omega)$ $\ell^\infty(S)$ $L^\infty(\Omega)$ $B_b(\Omega)$ $A(\mathbb{D})$ $B(X)$

10.13. In $C(\mathbb{R})$, the modular ideals correspond to compact sets.

10.14 (Disk algebra). (a) Every continuous homomorphism is an evaluation.

10.15 (Polynomial convexity). (See Conway)

10.16 (Inclusion relation on spectra). (a) $\sigma(a+b)\subset\sigma(a)+\sigma(b)$ and $\sigma(ab)\subset\sigma(a)\sigma(b)$ for unital cases.

(b) $\sigma(a^{-1})=\sigma(a)^{-1}$ for unital cases.

(c) $r(a)^n=r(a^n)$.

10.17 (Spectral radius function). (a) upper semi-continuous

10.18 (Vector-valued complex function theory). Let Ω be an open subset of $\mathbb{C}$ and X a Banach space. For a vector-valued function $f:\Omega\to X$, we say f is *differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}(f(\lambda)-f(\lambda_0))$$

exists in X for every $\lambda\in\Omega$, and *weakly differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}\langle f(\lambda)-f(\lambda_0),x^*\rangle$$

exists in $\mathbb{C}$ for each $x^*\in X^*$ and every $\lambda\in\Omega$. Then, the followings are all equivalent.

(a) f is differentiable.

(b) f is weakly differentiable.

(c) For each $\lambda_0\in\Omega$, there is a sequence $(x_k)_{k=0}^\infty$ such that we have the power series expansion

$$f(\lambda)=\sum_{k=0}^\infty(\lambda-\lambda_0)^k x_k,$$

where the series on the right hand side converges absolutely and uniformly on any closed ball in Ω centered at λ_0.

10.19 (Exponential of an operator).

Chapter 11

C*-algebras

11.1 Continuous functional calculus

11.1 (C*-algerbas)**.** A *C*-algebra* is a Banach $*$-algebra A such that the norm satisfies the *C*-identity* $\|a^*a\| = \|a\|^2$ for all $a \in A$. We automatically have $\|1\| = 1$ in a unital C*-algebra because $\|1\| = \|1^*1\|^2 = \|1\|^2$. The *standard unitization* or the *Dorroh extension* of a C*-algebra A is a C*-algebra $\widetilde{A}$ defined by a vector space $A \oplus \mathbb{C}$ with the multiplication, involution, and norm are given such that

(a)

Proof. The C*-identity easily follows from the following inequality:

$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\
&= \sup_{\|b\|=1} \|(ab + \lambda b)^*(ab + \lambda b)\| \\
&= \sup_{\|b\|=1} \|b^*((a^*a + \lambda a^* + \overline{\lambda} a)b + |\lambda|^2 y)\| \\
&\le \sup_{\|b\|=1} \|(a^*a + \lambda a^* + \overline{\lambda} a)b + |\lambda|^2 b\| \\
&= \|(a,\lambda)^*(a,\lambda)\|.
\end{aligned}$$

□

11.2 (Normal elements)**.** Let a be an element of a C*-algebra A. We say a is *normal* if $a^*a = aa^*$, *self-adjoint* if $a^* = a$, and *unitary* if A is unital and $a^*a = aa^* = 1$, respectively.

(a) A normal element a is unitary if and only if $\sigma(a) \subset \mathbb{T}$.

(b) A normal element a is self-adjoint if and only if $\sigma(a) \subset \mathbb{R}$.

(c) A normal element a is self-adjoint if and only if $\omega(a) \in \mathbb{R}$ for all ω?

(d) A normal element a is positive if and only if $\omega(a) \ge 0$ for all ω?

Proof. (a) We may assume A is unital. Let $u \in A$ be unitary. We have $\|u\|^2 = \|u^*u\| = \|1\| = 1$ and similarly $\|u^*\| = 1$. If $\lambda \in \sigma(u)$, then $|\lambda| \le \|u\| = 1$ and $\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$ implying $|\lambda^{-1}| \le \|u^*\| \le 1$, we have $|\lambda| = 1$ so that $\sigma(u) \subset \mathbb{T}$. If $\sigma(u) \subset \mathbb{T}$ conversely,

(b) We may assume A is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{k=0}^{\infty} \frac{(ia)^k}{k!} \in A,$$

and the inverse of e^{ia} is e^{-ia}. Since the involution on A is bounded, we can check e^{ia} is unitary by

$$(e^{ia})^* = \sum_{k=0}^{\infty} \frac{(-ia)^k}{k!} = e^{-ia}.$$

For every $\omega \in \sigma(A)$, then by the part (a) the equality

$$e^{-\operatorname{Im}\omega(a)} = |e^{i\omega(a)}| = |\omega(e^{ia})| = 1$$

proves $\omega(a) \in \mathbb{R}$, hence $\sigma(a) \subset \mathbb{R}$. □

11.3 (Gelfand-Naimark representation theorem)**.** Let A be a commutative C*-algebra. Consider the Gelfand transform $\Gamma : A \to C_0(\widehat{A})$.

(a) Γ is a $*$-isomorphism.

Proof. (a) The Gelfand transform Γ is a $*$-homomorphism since each $\omega \in \sigma(A)$ is a homomorphism, and it preserves involution as

$$\omega(a^*) = \omega(\operatorname{Re} a - i \operatorname{Im} a) = \omega(\operatorname{Re} a) - i\omega(\operatorname{Im} a) = \overline{\omega(\operatorname{Re} a) + i\omega(\operatorname{Im} a)} = \overline{\omega(a)}, \qquad a \in A,$$

because ω sends a self-adjoint element to a number contained in its spectrum, which is real. It is also an isometry since

$$\|\Gamma(a)\| = \sup_{\omega\in\sigma(A)} |(\Gamma(a))(\omega)| = \sup_{\omega\in\sigma(A)} |\omega(a)| = r(a), \qquad a \in A,$$

implies that we have

$$\|\Gamma(a)\|^2 = \|\Gamma(a^*a)\| = r(a^*a) = \lim_{n\to\infty} \|(a^*a)^{2^n}\|^{\frac{1}{2^n}} = \|a^*a\| = \|a\|^2, \qquad a \in A$$

by the C*-identity and the spectral radius formula. Thus, the image $\Gamma(A)$ of an isometric $*$-homomorphism Γ is a closed unital $*$-subalgebra of $C(\sigma(A))$, and it separates points by definition. Then, $\Gamma(A)$ is dense in $C(\sigma(A))$ by the Stone-Weierstrass theorem, which implies $\Gamma(A) = C(\sigma(A))$. □

11.4 (Continuous functional calculus)**.** Let A be a unital C*-algebra, and $a \in A$ a normal element. Then, we have a $*$-isomorphism

$$C(\sigma(a)) \to C^*(1, a) : \mathrm{id}_{\sigma(a)} \mapsto a$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

joint spectrum

(a) spectral mapping: $\lambda \in \sigma_p(a)$ implies $f(\lambda) \in \sigma_p(f(a))$, $\lambda \in \sigma(a)$ iff $f(\lambda) \in \sigma(f(a))$, composition, ...

11.5 ($*$-homomorphisms)**.** Let $\varphi : A \to B$ be a $*$-homomorphism between C*-algerbas.

(a) φ is determined by self-adjoint elements.

(b) $\|\varphi\| = 1$ if φ is non-trivial.

(c) The quotient of A by a closed ideal I is a C*-algebra.

(d) If φ is injective, then it is an isometry.

(e) If φ has dense range, then it is surjective.

Proof. Let $\varphi : A \to B$ be an injective $*$-homomorphism. We may assume A and B are commutative so that $A = C_0(X)$ and $B = C_0(Y)$. Then, φ induces a continuous surjective pointed map $Y_+ \to X_+$. The pullback map is an isometry.

For the surjectivity, quotient out by kernel. □

11.2 States

11.6 (Positive elements)**.** Let a, b be elements of a C*-algebra A. We say a is *positive* and write $a \geq 0$ if it is normal and $\sigma(a) \subset \mathbb{R}_{\geq 0}$, and the set of all positive elements of A is denoted by A^+. If we define a relation $a \leq b$ as $b - a \geq 0$, then we can see that it is a partial order on A.

(a) $a \geq 0$ if and only if $\|\lambda - a\| \leq \lambda$ for some $\lambda \geq \|a\|$.

(b) If $a \geq 0$ and $\sigma(b) \subset \mathbb{R}_{\geq 0}$, then $\sigma(a + b) \subset \mathbb{R}_{\geq 0}$.

(c) $a \geq 0$ if and only if $a = b^*b$ for some $b \in A$.

Proof. (c) If $a \geq 0$, then $b := a^{\frac{1}{2}}$ gives $a = b^*b$. Conversely, if we let $c := b(b^*b)_-$, then $c^*c = -(b^*b)_-^3 \leq 0$, so we have $cc^* \leq 0$ and $c^*c + cc^* \leq 0$ because $\sigma(c^*c) = \sigma(cc^*)$. However, $c^*c + cc^* = 2(\mathrm{Re}\, c)^2 + 2(\mathrm{Im}\, c)^2 \geq 0$, thus we have $c = \mathrm{Re}\, c + i \,\mathrm{Im}\, c = 0$, which implies $(b^*b)_- = -(c^*c)^{\frac{1}{3}} = 0$. □

11.7 (Operator monotone operations)**.** (a) If $0 \leq a \leq b$, then $a^{-1} \geq b^{-1}$.

(b) If $a \leq b$, then $cac^* \leq cbc^*$.

(c) If $0 \leq a \leq B$, then $a^p \leq b^p$ for $0 \leq p \leq 1$.

11.8 (Standard approximate units)**.** For a von Neumann algebra or a multiplier algebra, we can ask an approximate unit is bounded, directed, or countable. If we consider only positive elements for an approximate unit, then countable => directed => bounded.

Let M be a σ-finite von Neumann algebra, and $\mathcal{A}$ be a σ-weakly dense $*$-subalgebra of M. Using the Kaplansky density theorem and the σ-strong metrizability of the bounded part of M, take a bounded approximate unit $b_n \in \mathcal{A}^+$ in M such that $\|b_n\| \leq 1$ for all n and $b_n \to 1$ σ-strongly. If we let p_n be the support projection of $\sum_{k \leq n} b_k$, then $p_n \uparrow 1$. Inductively define a sequence $e_n \in \mathcal{A}^+$ by

$$e_0 := 1 - (1 - b_0)^{k_0}, \qquad e_n := 1 - ((1 - e_{n-1})(1 - b_n)(1 - e_{n-1}))^{k_n},$$

where $k_n \in \mathbb{Z}_{>0}$ is taken such that $\omega(p_n - e_n) < n^{-1}$, which can be done since $((1-e_{n-1})(1-b_n)(1-e_{n-1}))^k \to 1-p_n$ σ-strongly as $k \to \infty$. Then, $e_n \uparrow 1$ σ-weakly, hence σ-strongly by the monotone convergence theorem.

Let $I := \{a \in A^+ : \|a\| < 1\}$. It is directed since if $a, b \in I$, then

$$c := (a(1-a)^{-1} + b(1-b)^{-1})(1 + a(1-a)^{-1} + b(1-b)^{-1})^{-1}$$

belongs to I with $a \leq c$ and $b \leq c$. Define the *standard approximate unit* of the C*-algebra A as a net $e_i \in A$ indexed on I by $e_i := i$. If we fix any $a \in A^+$ and $\varepsilon > 0$, then for any sufficiently advancing i such that $e_i \geq a(a + \varepsilon)^{-1}$, by letting $\varepsilon \to 0$ on

$$\|a - e_i a\|^2 = \|a(1 - e_i)^2 a\| \leq \|a(1 - e_i)a\| \leq \|a(\varepsilon(a + \varepsilon)^{-1})a\| \leq \varepsilon\|a\|,$$

we can check $e_i a \to a$ in norm of A.

(a)

11.9 (States)**.** Let A be a C*-algebra. We say a linear functional $\omega \in A^*$ is *self-adjoint* if $\omega(a^*) = \overline{\omega(a)}$, and *positive* if $\omega(a^*a) \geq 0$, for all $a \in A$. A *state* of A is defined as a normalized positive linear functional on A, that is, $\omega \in (A^*)^+$ with $\|\omega\| = 1$.

(a) For $\omega \in A^*$, ω is positive if and only if $\omega(e_i) \to \|\omega\|$.

(b) Let V be a closed linear subspace of A containing the unit of A. If $\omega_0 : V \to \mathbb{C}$ satisfies $\omega_0(1) = 1$ and $\|\omega_0\| = 1$, then ω_0 is extended to a state of A.

Proof. □

11.10 (Pure states)**.** Let A be a C^*-algebra. A state ω of A is called *pure* if every positive linear functional on A dominated by ω is a scalar multiple of ω.

Let ω be a state of A.

(a) If ω is multiplicative, then it is pure.

(b) If ω is pure, then its restriction on the center is multiplicative.

Proof. (a)

(b) Fix $z \in Z(A)$ with $0 \le z \le 1$ and define $\omega_z \in A^*$ such that $\omega_z(a) := \omega(za)$ for all $a \in A$. Then, ω_z is positive by $\omega_z(a) = \omega(z^{\frac{1}{2}} a z^{\frac{1}{2}})$ for $a \in A$, and the inequality

$$\omega_z(a) = \omega(a^{\frac{1}{2}} z a^{\frac{1}{2}}) \le \omega(a^{\frac{1}{2}} a^{\frac{1}{2}}) = \omega(a), \qquad a \in A^+$$

implies that there is $\lambda \in \mathbb{R}_{\ge 0}$ such that $\omega_z = \lambda\omega$ by the assumption that ω is pure. For the standard approximate unit e_i of A, we have $\omega(z) = \lim_i \omega_z(e_i) = \lambda \lim_i \omega(e_i) = \lambda$, so

$$\omega(za) = \omega_z(a) = \lambda\omega(a) = \omega(z)\omega(a), \qquad a \in A.$$

Since a C^*-algebra is linearly generated by positive elements in the closed unit ball, it implies that ω is multiplicative on the center $Z(A)$. □

11.11 (Probability regular Borel measures)**.** We investigate states of the commutative C^*-algebra $C_0(X)$, where X is a locally compact Hausdorff space.

11.12 (Vector states)**.** We investigate states of the C^*-algebra $K(H)$ of compact operators on a Hilbert space H.

11.3 Representations

11.13 (Non-degenerate representations)**.** Let A be a C^*-algebra. A *representation* of A on a Hilbert space H is a $*$-homomorphism $\pi : A \to B(H)$. We say a representation $\pi : A \to B(H)$ is *non-degenerate* if $\pi(A)H$ is dense in H.

(a) Every representation has a unique non-degenerate subrepresentation.

(b) The following statements are equivalent:

  (i) π is non-degenerate.

  (ii) For each $\xi \in H$ there is $a \in A$ such that $\pi(a)\xi \ne 0$.

  (iii) $\pi(e_i) \to 1$ strongly for an approximate unit e_i of A.

11.14 (Cyclic representations)**.** *cyclic* if there is a vector $\Omega \in H$ such that $A\Omega$ is dense in H. Cyclic decomposition

11.15 (Irreducible representations)**.** *irreducible* if there is no proper closed subspace $K \subset H$ such that $\pi(A)K \subset K$. The following statements are equivalent:

(i) π is irreducible if and only if $\pi(A)' = \mathbb{C}$.

(ii) π is irreducible if and only if every non-zero vector in H is cyclic.

11.16 (Gelfand-Naimark-Segal representation)**.** Let A be a C^*-algebra, and ω be a state on A. The *left kernel* of ω is defined to be

$$\mathfrak{n}_\omega := \{a \in A : \omega(a^*a) = 0\}.$$

(a) $\mathfrak{n}_\omega$ is a left ideal of A.

(b) $\langle a + \mathfrak{n}_\omega, b + \mathfrak{n}_\omega \rangle := \omega(b^*a)$ is an inner product on $A/\mathfrak{n}_\omega$.

(c) There is a unique representation $\pi_\omega : A \to B(H_\omega)$ such that $\pi_\omega(a)(b + \mathfrak{n}_\omega) := ab + \mathfrak{n}_\omega$ for $a, b \in A$.

(d) $\pi_\omega : A \to B(H_\omega)$ is a cyclic representation.

11.4 Ideals

For a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0\ ,$$

we have

$$\begin{tikzcd}
PS(I) \arrow[r, hook] \arrow[d, two heads] & PS(A) \arrow[d, two heads] & PS(B) \arrow[l, hook'] \arrow[d, two heads] \\
\widehat{I} \arrow[r, hook] \arrow[d, two heads] & \widehat{A} \arrow[d, two heads] & \widehat{B} \arrow[l, hook'] \arrow[d, two heads] \\
\mathrm{Prim}(I) \arrow[r, hook, "\text{open}"'] & \mathrm{Prim}(A) & \mathrm{Prim}(B) \arrow[l, hook', "\text{closed}"]
\end{tikzcd}$$

11.17 (Modular maximal left ideals)**.**

11.18 (Primitive ideals)**.** hull kernel topology

$$PS(A) \cong \{(\pi, \psi)\}/\sim_u, \qquad \widehat{A} \cong \{\pi\}/\sim_u .$$

A	$PS(A)$	$\widehat{A}$	$\mathrm{Prim}(A)$
$C(X)$	X	X	X
$K(H)$	PH	$*$	$*$
$\widetilde{K}(H)$	?	?	$\{0, K(H)\}$
$B(H)$			

(a) $\mathrm{Prim}(A)$ is locally compact T_0 space.

(b) Two maps $PS(A) \to \widehat{A} \to \mathrm{Prim}(A)$ are continuous surjective open maps

(c) If A is type I, then $\widehat{A} \to \mathrm{Prim}(A)$ is an homeomorphism.

Every morphism $A \to M(B)$ induces the following?:

$$\begin{tikzcd}
PS(B) \arrow[r, two heads] \arrow[d] & \widehat{B} \arrow[r, two heads] \arrow[d] & \mathrm{Prim}(B) \arrow[d] \\
PS(A) \arrow[r, two heads] & \widehat{A} \arrow[r, two heads] & \mathrm{Prim}(A).
\end{tikzcd}$$

Exercises

11.19 (Projections in $M_2(\mathbb{C})$)**.** The space of self-adjoint elements in $M_2(\mathbb{C})$ is a real vector space spanned by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) $(p-q)^2 = \frac{1}{2}$.

(b) If we let $\lambda_\pm$ be the eigenvalues of $ap+bq$, then $\lambda_+ + \lambda_- = a+b$ and $\lambda_+ - \lambda_- = \sqrt{a^2+b^2}$.

(c) Every functional calculus $f(x)$ of self-adjoint x is a linear combination of x and 1.

(d) $ap+bq+c \geq 0$ if and only if $a+b+2c \geq \sqrt{a^2+b^2}$.

(e) Every projection of rank one is given by $ap+bq+(1-a-b)/2$ for $a^2+b^2=1$.

11.20 (Operator monotone square). Let A be a C*-algebra in which the square function is operator monotone, that is, $0 \leq a \leq b$ implies $a^2 \leq b^2$ for any positive elements a and b in A. We are going to show that A is necessarily commutative. Let a and b denote arbitrary positive elements of A.

(a) Show that $ab+ba \geq 0$.

(b) Let $ab = c+id$ where c and d are self adjoints. Show that $d^2 \leq c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \leq c^2$. Show that $c^2d^2 + d^2c^2 - 2\lambda d^4 \geq 0$.

(d) Show that $\lambda(cd+dc)^2 \leq (c^2-d^2)^2$.

(e) Show that $\sqrt{\lambda^2+2\lambda-1}\cdot d^2 \leq c^2$ and deduce $d=0$.

(f) Extend the result for general exponent: A is commitative if $f(x)=x^\beta$ is operator monotone for $\beta > 1$.

11.21 (States on unitization). Let A be a non-unital C*-algebra and $\widetilde{A}$ be its unitization. Let $\widetilde{\omega} = \omega \oplus \lambda$ be a bounded linear functional on $\widetilde{A}$, where $\omega \in A^*$ and $\lambda \in \mathbb{C}^* = \mathbb{C}$.

Since A is hereditary in $\widetilde{A}$, the extension defines a well-defined injective map $S(A) \to S(\widetilde{A})$. We can identify $PS(A)$ as a subset of $PS(\widetilde{A})$ whose complement is a singleton.

(a) $\widetilde{\rho}$ is positive if and only if $\lambda \geq 0$ and $0 \leq \rho \leq \lambda$.

(b) $\widetilde{\omega}$ is a state if and only if $\lambda = 1$ and $0 \leq \omega \leq 1$.

(c) $\widetilde{\omega}$ is a pure state if and only if $\lambda = 1$ and ω is either a pure state or zero.

11.22 (Representations of $C_0(X)$). Let $A = C_0(X)$ and μ be a state on A, a regular Borel probability measure on a locally compact Hausdorff space X.

(a) The left kernel of μ is $\mathfrak{n}_\mu = \{f \in A : f|_{\operatorname{supp}\mu} = 0\}$.

(b) $H_\mu = L^2(X,\mu)$.

(c) The canonical cyclic vector is the unity function on X.

11.23 (Representations of $K(H)$).

11.24 (Automorphism group of $K(H)$ and $B(H)$).

11.25 (Approximate eigenvectors).

11.26 (Kadison transitivity theorem).

11.27 (Hereditary C*-algebras).

11.28 (Extreme points of the ball). Let A be a C*-algebra and let B_A be the closed unit ball of A.

(a) Extreme points of $A_+ \cap B_A$ is the projections in A.

(b) Extreme points of $A_{sa} \cap B_A$ is the self-adjoint unitaries in A.

(c) Every extreme point of B_A is a partial isometry.

11.29 (Category of commutative C*-algebras).

$$\begin{tikzcd}
 & \mathrm{CH} \arrow[r] \arrow[d, shift right=1.5, "\text{disjoint base}"'] \arrow[d, phantom, "\dashv"] & \mathrm{LCH} \arrow[r] & \mathrm{cpltH} \\
\mathrm{LCH}_{\mathrm{prop}} \arrow[r] & \mathrm{CH}_* \arrow[u, shift right=1.5, "\text{forgetful}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{uCC^*Alg}_{\mathrm{unital}} \arrow[r] \arrow[d, shift right=1.5, "\text{inclusion}"'] \arrow[d, phantom, "\vdash"] & \mathrm{CC^*Alg}_{\mathrm{mor}} \arrow[r] & \mathrm{locCC^*Alg} \\
\mathrm{CC^*Alg}_{\mathrm{nondeg}} \arrow[r] & \mathrm{CC^*Alg} \arrow[u, shift right=1.5, "\text{unitization"'] & &
\end{tikzcd}$$

The unitization is left adjoint to the inclusion functor.

Proof. Let X and Y be locally compact Hausdorff spaces. We show the continuous maps $\varphi^* : X \to Y$ corresponds to non-degenerate $*$-homomorphisms $\varphi : C_0(Y) \to M(C_0(X)) \cong C_b(X) \cong C(\beta X)$. If $f^* : X \to Y$ is continuous, then $\varphi : C_0(Y) \to C_b(X)$ is well-defined, which is non-degenerate since $e_i \in C_0(Y)$ with $e_i \uparrow 1$ so that for $f \in C_0(X)$ and arbitrary $\varepsilon > 0$ we have a compact $K \subset X$ such that

$$\begin{aligned}
\|f - \varphi(e_i)f\| &= \sup_{x \in X} |(1 - e_i(\varphi^*(x)))f(x)| \\
&\le \sup_{x \in K} |1 - e_i(\varphi^*(x))| \|f\| + \varepsilon,
\end{aligned}$$

hence the Dini theorem proves $\|f - \varphi(e_i)f\| \to 0$. Conversely, if $\varphi : C_0(Y) \to M(C_0(X))$ is non-degenerate, then the dual gives $\varphi^* : X \subset \mathrm{Prob}(\beta X) \to \mathrm{Prob}(Y)$, and the image of x is pure on $M(Y)$ since it defines a character on Y by

$$\varphi^*(x)(fg) = \varphi(fg)(x) = \varphi(f)(x)\varphi(g)(x) = \varphi^*(x)(f)\varphi^*(x)(g), \qquad f, g \in C_0(Y).$$

□

Problems

1. A C-algebra is commutative if and only if a function $f(x) = x(1+x)^{-1}$ is operator subadditive.

2. On a normed algebra, there is a unique C*-algebra structure.

Chapter 12

Von Neumann algebras

12.1 Normal states

12.1 (Von Neumann algebras). A *von Neumann algebra* on a Hilbert space H is a σ-weakly closed unital $*$-subalgebra of $B(H)$. We will see later that a $*$-subalgebra of $B(H)$ is weakly closed if and only if it is σ-strongly* closed. A linear map between von Neumann algebras on H is called *normal* if it is continuous between σ-weak topologies. We denote by M_* the space of normal linear functionals on M.

(a) M_* is a predual of M.

(b) A state on M is normal if and only if it is completely additive. As a corollary, a positive linear map between von Neumann algebras is normal if it is σ-weakly continuous on bounded parts or commutative von Neumann subalgebras.

Proof. (a) To prove that M_* is a predual of M, we need to show that M_* is Banach and the map $M \to (M_*)^*$ is an isometric isomorphism. The cokernel of the kernel of $B(H)_* \to M_*$ gives a Banach quotient map $B(H)_* \to F$, whose dual $F^* \to B(H)$ is an isometry. The Hahn-Banach separation proves that the image of the isometry $F^* \to B(H)$ is the σ-weak closure of M. Here the σ-weak closedness of M is not necessary. Thus, by taking dual for weakly* dense isometry $M \to F^*$ we get an isometry $F \to M^*$. We know that $B(H)_* \to F$ is surjective by construction of F. This implies F can be identified with M_* by definition of M_*, hence M_* is Banach.

Since the norm topology is stronger than the topology generated by M on M_*, the norm-$\sigma(M_*, M)$ continuous bijection $M_* \to M_*$ implies a weakly* dense isometry $M \to (M_*)^*$. Because the weak* topology on $(M_*)^*$ is induced from the σ-weak topology of $B(H)$ via the dual map $(M_*)^* \to (B(H)_*)^* = B(H)$, the σ-weak closedness of M implies the bijectivity of $M \to (M_*)^*$. The norms are induced from $B(H)$, they are isometrically isomorphic.

(b) A normal state is clearly completely additive. Let ω be a completely additive state on M. Let $\{p_i\}$ be a maximal family of orthogonal projections of M such that there exist vectors $\{\xi_i\}$ in H satisfying $\omega(p_i x p_i) \le \langle x\xi_i, \xi_i\rangle$ for all $x \in M^+$, and suppose $p := \sum_i p_i < 1$. For any fixed $\xi \in (1-p)H$ with $\|\xi\| = 1$, let $\{q_j\}$ be a maximal family of orthogonal subprojections of $1-p$ such that $\omega(q_j) > \langle q_j\xi, \xi\rangle$, and suppose further $q := \sum_j q_j = 1-p$. Then, we have

$$\omega(1-p) = \omega(q) \ge \sum_j \omega(q_j) > \sum_j \langle q_j\xi, \xi\rangle = \langle q\xi, \xi\rangle = \langle (1-p)\xi, \xi\rangle = \|\xi\|^2 = 1,$$

which contradicts to $\omega(1-p) \le 1$, so we have $q < 1-p$. Here we did not use the complete additivity of ω yet. Since every subprojection r' of $r := 1-p-q$ satisfies $\omega(r') \le \langle r'\xi, \xi\rangle$, and since every positive element is approximated by finite linear combinations of projections in norm, we have $\omega(rxr) \le \langle rxr\xi, \xi\rangle$ for all $x \in M^+$. If we consider $\{p_i\} \cup \{r\}$ with the corresponding vector $r\xi$, then it contradicts to the maximality of

$\{p_i\}$, so we have $p = 1$. If we let $p_J := \sum_{i\in J} p_i$ for any finite subset J of I, then the complete additivity of ω implies the norm convergence $p_J\omega \to \omega$ in M^* by

$$|(\omega - p_J\omega)(x)|^2 = |\omega(x(1-p_J))|^2 \le \omega(1)\omega((1-p_J)x^*x(1-p_J)) \le \|x\|^2\omega(1-p_J) \to 0$$

as $J \to I$. For each i, the linear functional $p_i\omega$ is normal and so is $p_J\omega$ because it is σ-strongly continuous by

$$|\omega(xp_i)|^2 \le \omega(1)\omega(p_ix^*xp_i) \le \langle x^*x\xi_i, \xi_i\rangle = \|x\xi_i\|, \qquad x \in M,$$

so we have $\omega \in M_*$ because M_* is Banach. □

12.2 (Normal cyclic representations)**.** Let M be a von Neumann algebra on a Hilbert space H. A vector $\Omega \in H$ is called *separating* if $x\Omega = 0$ and $x \ge 0$ imply $x = 0$.

Properties for a faithful unital normal representation:

- admits a cyclic vector
- admits a separating vector
- admits a cyclic separating vector
- every normal state is a vector state

classification of cyclic normal representation
separability and σ-finiteness and the existence of separating vectors

(a) The associated cyclic representation of a normal state is normal.

(b) ξ is separating if and only if it is cyclic for M', and it is equivalent that the vector functional ω_ξ is a faithful normla state of M.

(c) sufficiently large representation, dependence of weak and strong topologies.

(d) Radon-Nikodym

A $*$-isomorphism between von Neumann algebras is normal.

12.3. Jordan decomposition preserves normality?

12.2 Density theorems

12.4 (Double commutant theorem)**.** Let H be a Hilbert space. The *commutant* of a subset $A \subset B(H)$ is the von Neumann algebra A' on H consisting of all elements of $B(H)$ that commute every $a \in A$. Let A be a non-degenerate $*$-subalgebra of $B(H)$. By the double commutant theorem, one can describe the von Neumann algebra generated by A in $B(H)$ purely algebraically in terms of commutants.

(a) A'' is the strong closure of A.

(b) A'' is the σ-strong* closure of A.

(c) A σ-strongly* closed $*$-subalgebra of $B(H)$ is weakly closed.

Proof. (a) The strong closedness of A'' is clear, so take $x \in A''$. We claim $x\xi \in \overline{A\xi}$ for $\xi \in H$. Let $p \in B(H)$ be the projection onto $\overline{A\xi}$. First, we obtain $p \in A'$ because for any $a \in A$ the left action of p fixes ap and a^*p since their ranges are in $A\xi \subset pH$, and $pap = ap$ and $pa^*p = a^*p$ imply $ap = pa$. Next, since $(1-p)\xi$ is orthogonal to the dense subspace AH of H, we have $p\xi = \xi$. Therefore, $px\xi = xp\xi = x\xi$ implies $x\xi \in pH = \overline{A\xi}$.

(b) Since A'' is weakly closed and A is self-adjoint, it suffices to show A is σ-strongly dense in A''. Consider the diagonal inclusion $B(H) \to B(\ell^2 \otimes H)$, which is an injective unital normal $*$-homomorphism. Then, A is non-degenerately represented also in $B(\ell^2 \otimes H)$, and we can check that the double commutant of A does not change in the new representation $A \to B(\ell^2 \otimes H)$. One way to check this is using (a). Now by the part (a) for arbitrary vector in $\ell^2 \otimes H$, we deduce the desired result.

(c) Let M be a σ-strongly* closed $*$-subalgebra of $B(H)$. We may assume that M is non-degenerate in $B(H)$. Then, by the part (b) we have $M'' = M$, which is strongly closed by the part (a). Since a strongly closed convex set is weakly closed, M is weakly closed. □

12.5 (Kaplansky density theorem)**.** Let $f : F \to \mathbb{C}$ be a continuous function on a closed subset F of $\mathbb{C}$. We say f is *strongly continuous* if for every net $x_i \in B(H)$ of normal operators with the spectra $\sigma(x_i) \subset F$ for all i, the strong convergence $x_i \to x$ implies the strong convergence $f(x_i) \to f(x)$.

(a) Since we only consider normal operators, strong and strong* have no difference.

(b) The image of a von Neumann algebra under a normal $*$-homomorphism is a von Neumann algebra.

Proof. Let $A \subset C(F)$ be the set of all strongly continuous functions. We can check A is a $*$-algebra containing the polynomial algebra $\mathbb{C}[z,\bar{z}]$. We will prove that $C_0(F) \subset A$ using the Stone-Weierstrass theorem.

Now we have

$$C_0(F) \cup \mathbb{C}[z,\bar{z}] \subset A \subset C(F).$$

If g is bounded and continuous on F, then

$$g(z) = \frac{g(z)}{1+|z|^2} + \bar{z}\frac{zg(z)}{1+|z|^2}, \qquad z \in F$$

implies $g \in C_0(F) + \bar{z}C_0(F) \subset A$.

□

12.6 (Approximate units for von Neumann algebras)**.** Let M be a von Neumann algebra on a Hilbert space H. Let A be a σ-weakly dense $*$-subalgebra.

(a) There is a net $e_i \in A_1^+$ such that $e_i \to 1$ σ-strongly*.

(b) If either A is hereditary in the sense that $AMA \subset A$ or M is countably decomposable, then we may assume $e_i \uparrow 1$.

(a) If φ is a normal $*$-homomorphism, then its image is a von Neumann algebra on H. (Kaplansky density is needed)

12.3 Projections

12.7 (Projection lattices)**.** Let M be a von Neumann algebra. Let $P(M)$ be the partially ordered set of all projections in M, called the *projection lattice* of N

(a) The linear span of $P(M)$ is σ-weakly dense in M.

(b) $P(M)$ is a complete orthomodular lattice.

(c)

$1 \le s_l(x) + s_r(1-x)$

Since $\ker x \cap \ker y \subset \ker(x+y)$, we have $s_r(x+y) \le s_r(x) \vee s_r(y)$.

$p \wedge s_l(x) = s_l(px)$

12.8 (Support projections for operators)**.** Let M be a von Neumenna algebra on a Hilbert space H. The *left support projection* or the *range projection* of $x \in M$ is the minimal projection $s_l(x) \in M$ such that $s_l(x)x = x$.

We have $s_r(x) = s_l(x^*)$. The projections $s_l(x)$ and $1-s_r(x)$ are also called the *range* and *kernel* projections of x, respectively.

Riesz refinement?

(a) The left support projection $s_l(x) \in M$ of $x \in M$ uniquely exists.

(b) We have $s_l(x)H = \overline{xH}$ and $Ms_l(x) = \overline{Mx}$.

(c) $x^*yx = 0$ if and only if $s_l(x)ys_l(x) = 0$ for every $y \in M$, and we have $s_r(x) = s(x^*x) = s(|x|)$. In particular, $s_l(x) = s_r(x)$ if x is normal.

(d) If $x, y \in M$ satisfies $x^*x \le y^*y$, then there is a unique $v \in M$ such that $x = vy$ and $s_r(v) \le s_l(y)$.

(e) For $x \in M$, there is unique partial isometry $v \in M$ such that the polar decomposition $x = v|x| = |x^*|v$ holds with $v^*v = s_r(x)$ and $vv^* = s_l(x)$. Moreover, $x^* = v^*|x^*| = |x|v^*$.

Proof. (a) Let $x \in M$. We may assume $0 \le x \le 1$. Then, $(xx^*)^{2^{-n}}$ is a bounded increasing sequence in M, so it converges strongly to some $s \in M_+$. We can check that s is a projection by

$$s^2 = \cdots = s.$$

Now we show that the projection s is the left support projection of x.

(d) The operator $v_0 : \overline{yH} \to \overline{xH} : y\xi \mapsto x\xi$ is well defined because

$$\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle \le \langle y^*y\xi, \xi\rangle = \|y\xi\|^2.$$

If we let $v := v_0 s_l(y)$, then we can easily check $x = vy$, and since $v(1-s_l(y))v^* = 0$ implies $s_r(v)(1-s_l(y))s_r(v) = 0$, we have $s_r(v) \le s_l(y)$.

For the uniqueness, if $v' \in B(H)$ satisfies $y = v'x$ and $v' = v's_l(y)$, then $y^*(v-v')^*(v-v')y = (x-x)^*(x-x) = 0$ implies $0 = s_l(y)(v-v')^*(v-v')s_l(y) = (v-v')^*(v-v')$, so the uniqueness of v in $B(H)$ follows. If $u \in M'$ is any unitary, then $uvu^* \in B(H)$ is a partial isometry satisfying the same properties $(uvu^*)x = uvxu^* = uyu^* = y$ and $(uvu^*)s_l(y) = uvs_l(y)u^* = uvu^*$, so the uniqueness implies $uvu^* = v$. Since unitaries of M' span M', we have $v \in M'' = M$.

(e) Since $x^*x \le |x|^*|x|$, there is $v \in M$ such that $x = v|x|$ and $v = vs(|x|) = vs_r(x)$. Then, $s_r(x) - v^*v = s_r(x)(1-v^*v)s_r(x) = 0$ because $|x|(1-v^*v)|x| = |x|^2 - |x|^2 = 0$, and $s_l(x) - vv^* = s_l(x)(1-vv^*)s_l(x) = 0$ because $x^*(1-vv^*)x = |x|^2 - |x|^2 = 0$ and $s_l(v) = s_l(x)$. The uniqueness of v follows from the part (d) since $s_r(x) = v^*v$ implies $s_r(v) = s_r(v^*v) = s_r(s_r(x)) = s_r(x) = s(|x|)$.

The equality $xv^* = |x^*|$ follows from $xv^* = v|x|v^* \ge 0$ and $|xv^*|^2 = vx^*xv^* = v|x|^2v^* = xx^* = |x^*|^2$. □

12.9 (Support projections for linear functionals)**.** Let M be a von Neumann algebra on a Hilbert space H. Note that $x \in M$ canonically acts on $\omega \in M_*$ from left as $(x\omega)(\cdot) := \omega(\cdot x)$. The *left support projection* of $\omega \in M_*$ is the minimal projection $s_l(\omega) \in M$ such that $s_l(\omega)\omega = \omega$.

(a) The left support projection exists $s_l(\omega) \in M$ of $\omega \in M_*$ uniquely exists.

(b) If $\omega \ge 0$, then $s(\omega)$ can be characterized by $\mathfrak{n}_\omega = M(1-s(\omega))$. $\mathfrak{n}_{|\omega|} = M(1-s(\omega))$ for any $\omega \in M_*$?

(c) $s(\omega_\xi)H = \overline{M'\xi}$.

Proof. (a)

(b) Let $p \in M$ be the projection such that $\mathfrak{n}_\omega = Mp$. Since $\omega(p) = 0$ implies $\omega(xp)^2 \le \omega(xx^*)\omega(p) = 0$ for any $x \in M$, the left action of $1-p$ fixes ω, so $1-p \ge s(\omega)$ by the minimality of $s(\omega)$. Conversely, we have $\omega(1-s(\omega)) = \omega(1) - (s(\omega)\omega)(1) = 0$ so that $1-s(\omega) \in Mp$, hence $(1-s(\omega))p = p$ and $1-s(\omega) \le p$. Therefore, $1-s(\omega) = p$.

□

12.10 (σ-weakly closed subalgebras)**.** Let M be a von Neumann algebra on a Hilbert space H.

(a) If $\mathfrak{n}$ is a σ-strongly* closed left ideal of M, then there is a unique projection $p \in M$ such that $\mathfrak{n} = Mp$.

(b) If V is a left invariant closed subspace of M_*, then there is a unique projection $p \in M$ such that $V = M_*p$.

Proof. (a) If we define $\mathfrak{a} := \mathfrak{n}^* \cap \mathfrak{n}$, then it is a σ-strongly* closed $*$-subalgebra, that is, a von Neumann subalgebra of M, and it admits a unit $p \in M$. Since $p \in \mathfrak{n}$ and $\mathfrak{n}$ is a left ideal, we have $Mp \subset \mathfrak{n}$. Conversely, if $x \in \mathfrak{n}$, then $x^*x \in \mathfrak{a}$ implies $|x| \in \mathfrak{a}$, and by the polar decomposition $x = v|x|$ we have $x = v|x| = v|x|p \in Mp$. Therefore, $\mathfrak{n} = Mp$. If two projections p and q in M satisfy $Mp = Mq$, then since there is a unique unit in a σ-strongly* closed $*$-algebra $pMp = qMq$, hence $p = q$ and the uniqueness follows.

□

12.11. The *central support projection* of $x \in M$ is the smallest central projection $z(x)$ that fixes x from left or right.

existence We have $z(x)H = MxH$.

cyclic projection

12.4 Predual

12.12 (Jordan decomposition for C*-algebras)**.** Let A be a C*-algebra

(a) For a normal element $a \in A$ there is a state ω of A such that $|\omega(a)| = \|a\|$.

(b) A self-adjoint bounded linear functional is uniquely represented as the difference of two positive linear functional.

Proof. Note that we will not prove the existence of such a state by the Hahn-Banach extension.

It is trivial if $a = 0$, so we let $a \neq 0$, and assume $\|a\| = 1$ without loss of generality. We may assume A is unital because a state $\widetilde{\omega}$ of the unitization $\widetilde{A}$ satisfying $\widetilde{\omega}(a) = 1$ restricts to a state $\omega := \widetilde{\omega}|_A$ of A, as the norm is exactly one by

$$1 = |\omega(a)| \le \|\omega\| \le \|\widetilde{\omega}\| = 1.$$

Since $\sigma(a) \cup \{0\}$ is compact, there is $\lambda \in \sigma(a)$ such that $|\lambda| = 1$. The Dirac measure δ_λ induces a state ω_0 of $C^*(1, a)$ via the continuous functional calculus satisfying $\omega_0(a) = \lambda$. By the Hahn-Banach extension, there is a linear functional $\omega \in A^*$ of $\omega_0 \in C^*(1, a)^*$ that is normalized as $\|\omega\| = \|\omega_0\| = 1$ and positive with $\omega(1) = \omega_0(1) = 1$, which means ω is a state of A. Fianlly, since $|\omega(a)| = |\omega_0(a)| = |\lambda| = 1$, we are done.

(b) We may assume A is unital since the positivity of linear functionals does not change under the restriction from the standard unitization $\widetilde{A}$ onto A, so that $S(A)$ and $-S(A)$ are weakly* compact in A^*. Then, we are enough to show

$$(A^*)_1^{sa} = \operatorname{conv}(S(A) \cup -S(A)).$$

Note that the right-hand is weakly*-compact because if $(1 - t_i)\omega_i^+ - t_i\omega_i^-$ is a net in the convex hull of $S(A) \cup -S(A)$, then we can find a subnet such that ω_i^+ and ω_i^- converge weakly* in $S(A)$ and t_i converges in $[0, 1]$, which implies the extracted subnet converges weakly* in the convex hull.

Since one inclusion is clear, suppose there exists $\omega \in (A^*)_1^{sa}$ which is not contained in the weakly* closed compact set $\operatorname{conv}(S(A) \cup -S(A))$. By the Hahn-Banach separation, and by the fact that the real dual $(A^{sa})^*$ can be identified with the self-adjoint part $(A^*)^{sa}$ of the complex dual of A as real locally convex spaces, there is $a \in A^{sa}$ such that

$$\sup_{\omega' \in S(A)} |\omega'(a)| = \sup_{\omega' \in S(A) \cup -S(A)} \omega'(a) < \omega(a).$$

If we take $\omega' \in S(A)$ such that $|\omega'(a)| = \|a\|$ using the part (b), then it leads to a contradiction, hence the claim follows. □

12.13 (Sherman-Takeda theorem)**.** Let A be a C*-algebra, and let $\pi_u : A \to B(H_u)$ be the universal representation of A constructed as the direct sum of the cyclic representations associated to all the states of A. Let $M := \pi_u(A)''$ be the von Neumann algebra on H_u generated by $\pi_u(A)$. The bidual A^{**} is called the *enveloping von Neumann algebra* of a C*-algebra A.

(a) A^{**} is canonically a von Neumann algebra on a H_u such that the canonical embedding $A \to A^{**}$ induces an isometric isomorphism $(A^{**})_* \to A^*$.

(b) A^{**} enjoys a universal property in the sense that every $*$-homomorphism $\varphi : A \to N$ to a von Neumann algebra N has a unique normal extension $\widetilde{\varphi} : A^{**} \to N$ of φ.

Proof. (a) Consider the following adjoint maps

$$\pi_u : A \to M, \qquad \pi_u^* : M_* \to A^*, \qquad \pi_u^{**} : A^{**} \to M.$$

The adjoint $\pi_u^* : M_* \to A^*$ is an isometry since

$$\|\pi_u^*(\omega)\| = \sup_{\substack{\|a\|\le 1\\ a\in A}} |\omega(\pi_u(a))| = \sup_{\substack{\|x\|\le 1\\ x\in M}} |\omega(x)| = \|\omega\|, \qquad \omega \in M_*$$

by the non-degeneracy of the representation π_u and the Kaplansky density. It is also surjective because if we take $\omega \in A^*$, which can be assumed to be a state by the Jordan decomposition we proved, then the universal representation π_u has the cyclic representation associated to ω as a subrepresentation, so ω is given by a vector state in π_u, which means that it gives rise to a normal state of M which extends ω via π_u. Now we have the isometric isomorphism $\pi_u^{**} : A^{**} \to M$, and the $*$-algebra structure on A^{**} can be determined from the von Neumann algebra M on H_u, which is unique by the Kaplansky density.

(b) We can define $\widetilde{\varphi}$ as the bitranspose of $\varphi : A \to N_{\sigma w}$, and it is a unique extension because A is σ-weakly dense in A^{**}. □

12.14 (Conditional expectations)**.** Let B be a C*-subalgebra of a C*-algebra A. A *conditional expectation* is defined as a normalized positive B-bimodule map $\varepsilon : A \to B$.

(a) A contractive retraction $\varepsilon : A \to B$ is a conditional expectation.

(b) A conditional expectation is completely positive.

Proof. (a) Taking bidual, we may assume A and B are von Neumann algebras M and N on some Hilbert spaces. Let $x \in M$ and $p \in N$ be a projection. Since the linear span of projections is norm dense in a von Neumann algebra, it is enough to show $p\varepsilon(x) = \varepsilon(px)$ and $\varepsilon(xp) = \varepsilon(x)p$. Since $p\varepsilon$ is idempotent, taking the limit $t \to \infty$ on

$$\begin{aligned}(1+t)^2\|p\varphi((1-p)x)\|^2 &= \|p\varepsilon((1-p)x) + tp\varepsilon((1-p)x))\|^2\\ &= \|p\varepsilon((1-p)x) + tp\varepsilon(p\varepsilon((1-p)x))\|^2\\ &\le \|(1-p)x + tp\varepsilon((1-p)x)\|^2\\ &= \|(1-p)x\|^2 + t^2\|p\varepsilon((1-p)x)\|^2,\end{aligned}$$

we have $p\varepsilon((1-p)x) = 0$. Let q be the unit of N. Substituting $q-p$ and q in the place of p respectively, we obtain

$$(1-p)\varepsilon((1-q+p)x) = 0, \qquad \varepsilon((1-q)x) = 0,$$

which imply $(1-p)\varepsilon(px) = 0$, hence for any $x \in M$ we have

$$p\varepsilon(x) = p\varepsilon(px) = \varepsilon(px).$$

Similarly we can show $\varepsilon(x(1-p))p = 0$ and $\varepsilon(xp)(1-p) = 0$, we are done.

(b) It follows easily from

$$\sum_{i,j} b_i^* \varepsilon(a_i^* a_j) b_j = \sum_{i,j} \varepsilon(b_i^* a_i^* a_j b_j) = \varepsilon(\sum_{i,j} b_i^* a_i^* a_j b_j) = \varepsilon(|\sum_j a_j b_j|^2) \geq 0,$$

where $[a_i] \in A^n$ and $[b_i] \in B^n$ for $n \geq 1$. $\square$

12.15 (Sakai theorem)**.** Let M be a W*-*algebra* or just a *von Neumann algebra* in the intrinsic sense that it does not depend on the choice of Hilbert spaces where it acts, that is, a C*-algebra together with a predual $M_* \subset M^*$. Consider the canonical weakly* dense embedding $M \subset M^{**}$ and a faithful unital normal representation of M^{**}, constructed by the Sherman-Takeda theorem for example. We will show that every W*-algebra M is canonically embedded in M^{**} as a weakly* closed $*$-subalgebra, but in a different way as the canonical embedding, so that M admits a faithful unital normal representation. In this context, a von Neumann algebra on a Hilbert space is equivalent to just a W*-algebra together with a faithful unital normal representation.

(a) There is an injective $*$-homomorphism $\pi : M \to M^{**}$ with weakly* closed image.

(b) π is a topological embedding with respect to $\sigma(M, M_*)$ and $\sigma(M^{**}, M^*)$.

(c) A predual of a C*-algebra is unique in the dual space if it exists.

Proof. (a) Note that the canonical embedding $M \to M^{**}$ between C*-algberas is not continuous with respect to the weak* topologies. For the inclusion $M_* \to M^*$, we have the dual map $\varepsilon : M^{**} \to M$ that is contractive and idempotent onto M, and hence is a M-bimodule map. Since $\varepsilon : M^{**} \to M$ is continuous between the weak*-topologies, and since M is weakly* dense in M^{**}, we can check that the kernel of ε is a weakly* closed ideal of M^{**}, so we have a central projection $z \in M^{**}$ such that $\ker \varepsilon = (1-z)M^{**}$. Define $\pi : M \to M^{**} : x \mapsto zx$. It is a $*$-homomorphism because z is central, and is injective because π is a right inverse of ε. Furethermore, the idempotence of ε implies that $zM^{**} = zM$, so the image $\pi(M) = zM = zM^{**}$ is weakly* closed in M^{**}.

(b) Note that $\pi : M \to M^{**}$ is continuous with respect to the norm topology of M and the weak* topology of M^{**} so that its adjoint can have the form $\pi^* : M^* \to M^*$. For π to be an embedding between weak* topologies, it suffices to prove $\pi^*(M^*) = M_*$. Suppose not and take $\varphi \in M^*$ satisfying $\pi^*(\varphi) \notin M_*$. Because M_* is norm closed in M^*, there is $x \in M^{**}$ by the Hahn-Banach extension theorem such that $\pi^*(\varphi)(x) \neq 0$ and $\omega(x) = 0$ for all $\omega \in M_*$. Since $\omega(\varepsilon(x)) = \omega(x) = 0$ for every $\omega \in M_*$ from the definition of ε, and since M_* separates points of M, we have $\varepsilon(x) = 0$ so that $zx = 0$. If we take a net $e_i \in M$ such that $e_i \to z$ weakly* in M^{**}, then the σ-weak continuity of the multiplication by z implies a contradiction

$$\pi^*(\varphi)(x) = \lim_i \pi^*(\varphi)(e_i) = \lim_i \varphi(ze_i) = \varphi(zx) = 0,$$

so we can conclude $\pi^*(M^*) \subset M_*$. Conversely, if $\omega \in M_*$, then we have $\pi^*(\omega)(x) = \omega(zx) = \omega(x)$ for every $x \in M$ because $(1-z)x \in \ker \varepsilon$ acts on M_* trivially by definition of ε, so $\omega = \pi^*(\omega) \in \pi^*(M^*)$.

(c) Since $*$-isomorphism between von Neumann algebras is automatically normal, we can recover the predual by taking adjoint for the identity map on M. $\square$

12.5 Types

For von Neumann algebras, we want to compare order topology, measure topology, and operator topologies. - commutative and non-commutative. - M, M^{sa}, M_1^{sa}. - closedness for convex sets or bounded sets. - continuity of linear operators and functionals.

abelian, finite, purely infinite(every non-zero subprojection is infinite), properly infinite(every non-zero central subprojection is infinite)

central projection = union of components central support = a kind of minimal union of components centrally orthogonal

12.16 (Comparison of projections). l

(a) For projections p and q in M, there is a central projection z in M such that $pz \precsim qz$ and $p(1-z) \succsim q(1-z)$.

12.17 (Finite projections). A projection in a von Neumann algebra is called *finite* if there is no proper Murray-von Neumann equivalent subprojection. Let p and q be projections in a von Neumann algebra M.

(a) If M is finite, then $p \sim q$ implies $1-p \sim 1-q$.

(b) If p and q are finite, then $p \vee q$ is finite.

(c) An abelian projection is finite.

12.18 (Types of von Neumann algebras). A von Neumann algebra is called

(i) *type I* if every non-zero central projection has a non-zero abelian subprojection,

(ii) *type II* if every non-zero central projection has a non-zero finite subprojection and there is no non-zero abelian projection,

(iii) *type III* if there is no non-zero finite projection.

(a) Every von Neumann algebra M is uniquely decomposed into the direct sum of three von Neumann algebras of each type.

V.1.35. For a purely non-abelian von Neumann algebra, every projection is the sum of two equivalent orthogonal projections.

12.19 (Type I). Let M be a von Neumann algebra of type I. Then, there are families $\{M_\kappa\}_\kappa$ and $\{H_\kappa\}_\kappa$ of commutative von Neumann algebras M_κ and Hilbert spaces H_κ satisfying $\dim H_\kappa = \kappa$, indexed by cardinals κ, such that

$$M \cong \bigoplus_{\kappa \in \mathrm{Card}} M_\kappa \overline{\otimes} B(H_\kappa).$$

If M is a factor, then $M \cong B(H)$ for a Hilbert space H.

(a) A minimal projection is abelian and a non-zero abelian projection in a factor is minimal.

12.20 (Semi-finite and tracial von Neumann algebras). Let M be a von Neumann algebra. We say M is *semi-finite* if it admits a faithful semi-finite normal trace, and *tracial* if it admits a faithful normal tracial state.

(a) regular representation and antilinear isometric involution J. $L(G) = \rho(G)'$

(b) M is semi-finite if and only if type III does not occur in the direct sum.

(c) A factor M has at most one tracial state, which is normal and faithful.

(d) A factor is tracial if and only if it is type II_1.

12.21. Let M be a von Neumann algebra. A *center-valued trace* is $M^+ \to \widehat{Z(M)}^+$.

(a) There is a faithful semi-finite normal center-valued trace on M if and only if M is semi-finite.

(b) If M is a semi-finite factor, then a projection $p \in M$ is finite if and only if $\tau(p) < \infty$.

12.22 (Semi-finite traces). Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x \in M^+$ has a net $x_\alpha \in L^1(M,\tau)^+$ such that $x_\alpha \uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x) = \sup\{\, \tau(y) : y \le x,\ y \in L^1(M,\tau)^+ \,\} \quad \text{for} \quad x \in M^+.$$

12.23 (Uniformly hyperfinite algebras). Let A be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique tracial state.

(b) Every UHF algebra admits a unique tracial state.

(c) Every hyperfinite

Exercises

12.24 (Extremally disconnected space). $\sigma(B^\infty(\Omega))$ is extremally disconnected.

resolution of identity normal operator theories: multiplicity, invariant subspaces L^∞ representation

σ-weakly closed left ideal has the form Mp. II.3.12

Let $\mathfrak{m}$ be an algebraic ideal of a von Neumann algebra M, and $\overline{\mathfrak{m}}$ be its σ-weak closure. If $x \in (\overline{\mathfrak{m}})_+$, then there is an increasing net $(x_i) \subset \mathfrak{m}$ converges to x strongly. II.3.13

binary expansion and hereditary subalgebras