Representation Theory

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Contents

Part I

Finite group representations

Chapter 1

Character theory

1.1 Irreducible representations

1.1 (Definition of group representations).

1.2 (Intertwining maps).

1.3 (Subrepresentations). We say *invariant* or *stable*

1.4 (Irreducible representations). indecomposable and irreducible

1.5 (Maschke's theorem). Let G be a finite group and k be a field. Suppose the characteristic of k does not divide $|G|$. Let V be a finite-dimensional representation of G over k.

(a) Every invariant subspace W of V has a complement W' in V that is also invariant.

(b) V is isomorphic to the direct sum of irreducible representations of G over k.

(c) If $k = \mathbb{R}$ or $\mathbb{C}$, then V admits an inner product such that $W \perp W'$ and $\rho_V(g)$ is unitary for all $g \in G$.

1.6 (Schur's lemma). Let G be a group and k be a field. Let V and W be irreducible representations of G over k. Let $\psi : V \to W$ be an intertwining map.

(a) If $V \ncong W$, then $\psi = 0$.

(b) If $V \cong W$, then ψ is an isomorphism.

(c) If k is algebraically closed and $\dim V < \infty$, then every intertwining map $\psi : V \to V$ is a homothety.

1.2 Group algebra

1.7 (Modules and representations). ring <-> group module <-> representation finitely generated <-> finite dimensional

1.8 (Wedderburn's theorem). central idempotents dimension computation

1.9 (Group algebra). regular representation $k[G]$-module and G-representation correspondence

(a) $\mathbb{C}[G]$ is the direct sum of all irreducible representations.

(b) $|G| = \sum_{[V]\in\widehat{G}}(\dim V)^2$.

1.10. The number of irreducible representations and the number of conjugacy classes double counting on $Z(\mathbb{C}[G])$.

1.3 Characters

1.11 (Space of class functions)**.** Ring and inner product structure on the space of class functions.

(a) $\dim \hom_G(V, W) = \langle \chi_V, \chi_W \rangle$.

(b) Irreducible characters form an orthonormal basis of the space of class functions.

1.12 (Characters classify representations)**.** Let G be a finite group and let $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of G over $\mathbb{C}$.

$$\mathrm{Tr} : \mathbf{Rep}(G) \to \{\text{finite sum of irreducible characters}\}$$

surjectivity: trivial injectivity: Suppose two characters are equal. Maschke -> all characters are sum of irreducible characters Schur -> orthogonality, so the coefficients are all equal irreducible-factor-wisely construct an isomoprhism.

1.13 (Character table)**.** computation of matrix elements by character table abelian group, 1dim rep lifting

S^3	e	(12)	(123)
1	1	1	1
ε	1	-1	1
ρ	2	0	-1

the dual inner product: conjugacy check relation to normal subgroups center of rep
algebraic integer dim of irrep divides group order burnside pq theorem

Chapter 2

Classification of representations

2.1 Symmetric groups

young tableux

2.2 Linear groups over finite fields

GL2 and SL2 over finite fields

2.3 Induced representations

induction and restriction of reps (from and to subgroup) frobenius reciprocity, mackey theory

tensoring, complex, real symmetric, exterior

Chapter 3

Brauer theory

Part II

Lie algebras

Chapter 4

Semisimple Lie algebras

Solvability and nilpotency Engel's theorem Killing forms
Casimir element Weyl's theorem
Cartan subalgebra uniqueness? (conjugacy theorem)

Chapter 5

Root systems

root space decomposition integrality Weyl group
Coxeter graph Dynkin diagram Real forms
Isomorphism theorem
Existence theorem Universal enveloping algebra PBW theorem Verma module

Chapter 6

Representations of Lie algebras

6.1 Representations of $\mathfrak{sl}(2,\mathbb{C})$

6.1 (Pauli matrices)**.** Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis of complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, and $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of real Lie algebra $\mathfrak{so}(3)$.

(b) For a unit vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ has eigenvalues ± 1.

6.2 Highest weight theorem

6.3 Multiplicity formulas

Exercises

6.2 (Triplets and quadraplets)**.** Let (π_2, V_2) be the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of degree two. Consider $V_2 \otimes V_2$. Cartan element S_z. $V_2^{\otimes 3}$.

6.3 (Casimir element)**.** Casimir element decomposes a representation into irreducible representations.

Part III

Lie groups

Chapter 7

Lie correspondence

7.1 Exponential map

7.1 (Exponential map).

7.2 (Surjectivity of exponential map).

7.3 (Lie functor).

7.2 Lie's second theorem

7.4 (Derivative of the exponential map). Let G be a Lie group.

(a)
$$\frac{d}{ds}\exp(sX) = \exp(sX)X$$
for $s \in \mathbb{R}$ and $X \in \mathfrak{g}$.

(b)
$$\frac{\partial}{\partial s}$$

7.5 (Baker-Campbell-Hausdorff formula). Let G be a Lie group. Let $X, Y \in \mathfrak{g}$ such that $\exp(X)\exp(Y)$ Define
$$Z(t) := \log(\exp(X)\exp(tY))$$

7.6. (a) The Lie functor
$$\mathrm{Lie} : \mathrm{LieGrp}_{simple} \to \mathrm{LieAlg}_{\mathbb{R}}$$
is fully faithful.

7.3 Lie's third theorem

7.7 (Ado's theorem).

7.8 (Lie's third theorem). Also called the Cartan-Lie theorem.

(a) The Lie functor
$$\mathrm{Lie} : \mathrm{LieGrp}_{simple} \to \mathrm{LieAlg}_{\mathbb{R}}$$
is essentially surjective.

7.4 Fundamental groups of Lie groups

Chapter 8

Compact Lie groups

8.1 Special orthogonal groups

8.2 Special unitary groups

8.3 Symplectic groups

Exercises

8.1 (Lorentz group). $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$

(a) $O(1,3)$ has four components and $SO^+(1,3)$ is the identity component. Orthochronous $O^+(1,3)$, proper $SO(1,3)$.

Chapter 9

Representations of Lie groups

9.1 Peter-Weyl theorem

9.2 Spin representations

Clifford algebra

Part IV

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