Von Neumann Algebras

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Contents

Part I

Chapter 1

1.1 Types

partial ordering on projection lattice

1.2 Commutative von Neumann algebras

1.1 (Maximal commutative subalgebras)**.** A commutative von Neumann algebra M is m.a.s.a. if and only if it admits a cyclic vector. In this case, M is spatially isomorphic to some L^∞(if separable?).

separable commutative von Neumann algebra is generated by one self-adjoint element.
hyperstonean sapces

1.3 Direct integral

Type I factors. It possess a minimal projection. It is isomorphic to the whole $B(H)$ for some Hilbert space. Therefore, it is classified by the cardinality of H.

Type II factors. No minimal projection, but there are non-zero finite projections so that every projection can be "halved" by two Murray-von Neumann equivalent projections.

In type II_1 factors, the identity is a finite projection Also, Murray and von Neumann showed there is a unique finite tracial state and the set of traces of projections is $[0,1]$. Examples of II_1 factors include crossed product, tensor product, free product, ultraproduct. Free probability theory attacks the free groups factors, which are type II_1.

In type II_∞ factors. There is a unique semifinite tracial state up to rescaling and the set of traces of projections is $[0,\infty]$.

In type III factors no non-zero finite projections exists. Classified the $\lambda \in [0,1]$ appeared in its Connes spectrum, they are denoted by III_λ. Tomita-Takesaki theory. It is represented as the crossed product of a type II_∞ factor and $\mathbb{R}$.

Amenability, equivalently hyperfiniteness is a very nice condition in von Neumann algebra theory. Group-measure space construction can construct them. There are unique hyperfinite type II_1 and II_∞ factors, and their property is well-known. Fundamental groups of type II factors, discrete group theory, Kazhdan's property (T) are used.

Tensor product factors such as Araki-Woods factors and Powers factors.

Chapter 2

Weights

2.1 Hilbert algebras

2.2 Traces

2.1 (Tracial von Neumann algebras)**.** Let M be a von Neumann algebra. We say M is *tracial* if there is a faithful normal tracial state on M. A tracial factor is also called a II_1 *factor*.

(a) regular representation and anti-linear isometric involution J. $L(G)=\rho(G)'$

(b) A factor M has at most one tracial state, which is normal and faithful.

2.2 (Semi-finite traces)**.** Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x\in M^+$ has a net $x_\alpha\in L^1(M,\tau)^+$ such that $x_\alpha\uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x)=\sup\{\,\tau(y):y\le x,\ y\in L^1(M,\tau)^+\,\}\quad\text{for}\quad x\in M^+.$$

2.3 (Uniformly hyperfinite algebras)**.** Let A be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique tracial state.

(b) Every UHF algebra admits a unique tracial state.

(c) Every hyperfinite

Chapter 3

Modular theory

3.1 Automorphism groups

3.1 (Unitary group). (a) $U(H)$ is strongly* complete.

(b) $U(H)$ is not strongly complete.

(c) $U(H)$ is weakly relatively compact.

Let A be a C*-algebra. Then, $\overline{U(A)\cap B(1,r)}^{s*} = U(A'')\cap B(1,r)$. In particular, $U(A)$ is strongly* dense in $U(A'')$. (Kaplansky?)

Part II

Factors

Chapter 4

Type II factors

ergodic theory, rigidity theory

Chapter 5

Type III factors

Part III

Subfactors

The way how quantum systems are decomposed. Quite combinatorial! And has Galois analogy.

5.1 (Jones index theorem)**.** A *subfactor* of a factor M is a factor N containing 1_M.

Tensor categories and topological invariants of 3-folds. Ergodic flows.
standard invariant, Ocneanu's paragroups, Popa's λ-lattices, Jones' planar algebras