Lebesgue Theory

Ikhan Choi

July 24, 2022

Contents

Part I

Measure theory

Chapter 1

Measures and σ-algebras

1.1 Measures

1.1 (Definition of measures). Let $(\Omega, \mathcal{M})$ be a measurable space. A *measure* on $\mathcal{M}$ is a set function $\mu : \mathcal{M} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i)$$

for $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$. Here the squared cup notation reads the disjoint union.

1.2 (Continuity of measures).

1.2 Carathéodory extension

1.3 (Outer measures). Let Ω be a set. An *outer measure* on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty] : \varnothing \mapsto 0$ such that

(i) μ^* is *monotone*: we have
$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \le \mu^*(S_2)$$
for $S_1, S_2 \in \mathcal{P}(\Omega)$,

(ii) μ^* is *countably subadditive*: we have
$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$
for $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

Compairing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty] : \varnothing \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:
$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$
for $S \in \mathcal{P}(\Omega)$ and $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

(b) For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^{\infty} \rho(A_i) : S \subset \bigcup_{i=1}^{\infty} A_i,\ A_i \in \mathcal{A} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

Proof. □

1.4 (Carathéodory measurable sets)**.** Let μ^* be an outer measure on a set Ω. We want to construct a measure by restriction of μ^* on a properly defined σ-algebra. A subset $E \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(\Omega)$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is complete.

Proof. □

1.5 (Carathéodory extension theorem)**.** For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. Consider the following two conditions:

(i) We have the monotone countable subadditivity:

$$A \subset \bigcup_{i=1}^{\infty} A_i \Rightarrow \rho(A) \le \sum_{i=1}^{\infty} \rho(A_i)$$

for $A \in \mathcal{A}$ and $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$.

(ii) For every $B, A \in \mathcal{A}$, and for any $\varepsilon > 0$, there are $\{B_j'\}_{j=1}^{\infty}$ and $\{B_j''\}_{j=1}^{\infty} \subset \mathcal{A}$ such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} B_j' \quad \text{and} \quad B \setminus A \subset \bigcup_{j=1}^{\infty} B_j'',$$

and

$$\rho(B) + \varepsilon > \sum_{j=1}^{\infty} \rho(B_j') + \sum_{j=1}^{\infty} \rho(B_j'').$$

Let $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0,\infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets. The above two conditions give a sufficient condition for μ to be a measure on a σ-algebra containing $\mathcal{A}$.

(a) $\mu^*|_{\mathcal{A}} = \rho$ if (i) is satisfied.

(b) $\mathcal{A} \subset \mathcal{M}$ if (ii) is satisfied.

Proof. (a) Clearly $\mu^*(A) \le \rho(A)$ for $A \in \mathcal{A}$. We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{A_i\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^{\infty} \rho(A_i) \ge \rho(A).$$

Limiting $\varepsilon \to 0$, we get $\mu^*(A) \ge \rho(A)$.

(b) Let $S \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. It is enough to check the inequality $\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \setminus A)$ for S with $\mu^*(S) < \infty$, so we may assume there is a countable family $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. Then, we have $B_i \cap A \subset \bigcup_{j=1}^\infty B'_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty B''_{i,j}$ satisfying

$$\mu^*(S) + \varepsilon > \sum_{i=1}^\infty (\rho(B_i) + \frac{\varepsilon}{2^{i+1}}) > \sum_{i,j=1}^\infty \rho(B'_{i,j}) + \sum_{i,j=1}^\infty \rho(B''_{i,j}) \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Therefore, A is Carathéodory measurable relative to μ^*. □

1.6 (Uniqueness of Carathéodory extensions)**.** The Carathéodory extension theorem provides with a uniqueness theorem for measures.

Proof. □

Exercises

1.7 (Semi-rings and semi-algebras)**.** We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A}$ be a collection of subsets of a set Ω such that $\varnothing \in \mathcal{A}$. We say $\mathcal{A}$ is a semi-ring if it is closed under finite intersection, and the complement is a finite union of elements of $\mathcal{A}$. We say $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over Ω. Suppose a set function $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^\infty A_i\Big) \leq \sum_{i=1}^\infty \rho(A_i)$$

for $(A_i)_{i=1}^\infty \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

1.8 (Monotone class lemma)**.** alternative direct proof method without using Carathéodory extension.

Chapter 2

Measures on the real line

2.1 (Distribution functions).

2.2 (Helly selection theorem).

2.3 (Non-Lebesgue measurable set).

Exercises

2.4 (Steinhaus theorem). Let $\mathbb{E} \subset \mathbb{R}$ be Lebesgue measurable with $\lambda(E) > 0$.

(a) For any $\alpha < 1$, there is an interval $I = [a, b]$ such that $\lambda(E \cap I)/\lambda(I) > \alpha$.

(b) $E - E$ contains an open interval containing zero.

Proof. (a) □

Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

Chapter 3

Measurable functions

3.1 Extended real numbers

3.2 Simple functions

3.1 (Measurability of pointwise limits)**.**

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

$\square$

Every measurable extended real-valued function is a pointwise limit of simple functions.

3.2 (Egorov theorem)**.** Let $f_n : \Omega \to \mathbb{R}$ be a sequence of measurable functions on a finite measure space (Ω, μ) that converges almost everywhere.

(a) For every $\varepsilon > 0$,

$$\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \varepsilon \,\} \uparrow \text{ a full set} \qquad \text{as} \qquad n_0 \to \infty.$$

(b) For $\varepsilon > 0$, there is a measurable $E_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ and f_n is uniformly convergent on E_ε.

Proof. (a) We may assume $f_n \to 0$. The set of convergence is given by

$$\bigcap_{k>0} \bigcup_{n_0>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \varepsilon \,\},$$

which is a full set. We want to get rid of the dependence on the point x of n_0 in the union $\bigcup_{n_0>0}$. Since

$$\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \varepsilon \,\}$$

is increasing as $n_0 \to \infty$ to a full set.

(b) We can find $n_0 = n_0(k, \varepsilon)$ such that

$$\mu(\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\}) > \mu(\Omega) - \frac{\varepsilon}{2^k}.$$

Then,

$$\mu(\bigcap_{k>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\}) > \mu(\Omega) - \varepsilon.$$

If we define

$$E_\varepsilon := \bigcap_{k>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\},$$

then for any $k > 0$ and $x \in E_\varepsilon$, and with the $n_0(k,\varepsilon)$ we have chosen, we have

$$n \geq n_0 \quad \Rightarrow \quad |f_n(x)| < \frac{1}{k}. \qquad \square$$

Exercises

3.3 (Cauchy's functional equation)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x+y) = f(x) + f(y)$, satisfied for all $x, y \in \mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x) = ax$ for all $x \in \mathbb{R}$, where $a := f(1)$.

(a) $f(x) = ax$ for all $x \in \mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If f is conitnuous at a point, then f is linear.

(c) If f is Lebesgue measurable, then f is linear.

Part II

Lebesgue integral

Chapter 4

Convergence theorems

4.1 Definition of Lebesgue integral

4.2 Convergence theorems

4.1 (Monotone convergence theorem).

4.3 Radon-Nikodym theorem

4.4 Modes of convergence

4.2 (Borel-Cantelli lemma).

4.3 (Convergence in measure). Let (X,μ) be a measure space. Let f_n and f be measurable. We say f_n converges to f *in measure* if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{\, x : |f_n(x) - f(x)| > \varepsilon \,\}) = 0.$$

(a) If $f_n \to f$ in L^1, then $f_n \to f$ in measure.

(b) If $f_n \to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ almost everywhere.

Proof. (b) We can extract a subsequence f_{n_k} such that

$$\mu(\{\, x : |f_{n_k} - f| > \tfrac{1}{k} \,\}) > \tfrac{1}{2^k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{\, x : |f_{n_k} - f| > \tfrac{1}{k} \,\}) < \infty,$$

by the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k} \{\, x : |f_{n_k} - f| > \tfrac{1}{k} \,\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. $\square$

Chapter 5

Product measures

5.1 Fubini-Tonelli theorem

5.2 Lebesgue measure on Euclidean spaces

Chapter 6

Measures on metric spaces

6.1 Borel measures

6.2 Riesz-Markov-Kakutani representation theorem

locally compact

6.3 Hausdorff measures

Part III

Linear operators

Chapter 7

Lebesgue spaces

7.1 L^p spaces

7.2 L^1 spaces

7.3 L^2 spaces

7.4 L^∞ spaces

Chapter 8

Bounded linear operators

8.1 Continuity

Schur test

8.2 Density arguments

extension of operators

8.3 Interpolation

weak Lp, marcinkiewicz

Chapter 9

Convergence of linear operators

9.1 Translation and multiplication operators

9.2 Convolution type operators

approximation of identity

9.3 Computation of integral transforms

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega_x \times \Omega_y \to \mathbb{R}$ be such that $\partial_{x_i}f$ is well-defined. Suppose f and $\partial_{x_i}f$ are locally integrable in x and integrable y.

Then,

$$\partial_{x_i} \int f(x,y)\,dy = \int \partial_{x_i} f(x,y)\,dy.$$

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is $L^{\infty}_{\mathrm{loc}}$

(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^{1}_{loc}

(a) f is Lip iff f' is L^{∞}

(b) f is AC iff f' is L^{1}

(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

Lebesgue differentiation theorem