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1 Topological group action

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1.1. Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map.

- (a) $p^{-1}(p(A)) = \bigcup_{g \in G} gA$ for any $A \subset X$.
- (b) p is open.
- (c) If $x \neq gx$, then there is an open neighborhood U of x such that gU is disjoint to U .

Proof. (c) Since X is Hausdorff, there is disjoint open neighborhoods U_0 and U_1 respectively of x and gx . Then, $U := g^{-1}(gU_0 \cap U_1) \subset U_0$ and $gU = gU_0 \cap U_1 \subset U_1$ are disjoint. \square

1.2. Let $f : X \rightarrow Y$ be continuous. We say f is *proper* if $f^{-1}(K)$ is compact for compact K . We say f is *Bourbaki-proper* if it is closed and proper. If X is Hausdorff and Y is locally compact, then two notions are equivalent.

1.3 (Properly discontinuous actions). Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map. This action is called *properly discontinuous* if for every compact $K \subset X$ only finite gK intersect K .

- (a) If Γ is discrete, then orbits are locally finite.
- (b) If orbits are locally finite, then Γ acts properly discontinuously.
- (c) Suppose the stabilizer is always finite. If Γ act properly discontinuously then Γ is discrete.

compact stabilizer condition: If Γ is discrete, then for $z \in \mathbb{H}^2$ and a compact set $K \subset \mathbb{H}^2$, $\{g \in \Gamma : gz \in K\}$ is finite, so properly discontinuous.

1.4 (Covering space actions). Let G be a topological group acting on a topological space X . Let $p : X \rightarrow X/G$ be the quotient map. This action is called a *covering space action* if every $x \in X$ has a neighborhood U such that gU are all disjoint for $g \in G$.

- (a) A properly discontinuous and free action is a covering space action, if X is locally compact and Hausdorff.
- (b) A covering space action is properly discontinuous.
- (c) A covering space action is free.

Proof. (a) Fix $x \in X$ and let K be a compact neighborhood of x . By the proper discontinuity, there is a finite subset $F \subset G$ such that gK intersects K only for $g \in F$. Because the action is free, for every $g \in F \setminus \{1\}$ there is an open neighborhood U_g of x such that $gU_g \cap U_g = \emptyset$. Then, $U := K^\circ \cap \bigcap_{g \in F \setminus \{1\}} U_g$ satisfies $gU \cap U = \emptyset$.

(b)

\square

2 Hyperbolic plane geometry

2.1 Fuchsian groups

Classification of elements. An abelian Fuchsian group is cyclic. Elliptic point is discrete

2.2 Fundamental domain

2.1 (Fundamental domain). Let Γ be a Fuchsian group. An open set $D \subset \mathbb{H}^2$ is called a *fundamental domain* of Γ if

- (i) $\{g(D) : g \in \Gamma\}$ are pairwise disjoint,
- (ii) $\{g(\overline{D}) : g \in \Gamma\}$ covers \mathbb{H}^2 .

2.2 (Dirichlet domain). Let Γ be a Fuchsian group. Let $z_0 \in \mathbb{H}^2$ be a point that is not fixed by any isometry in $\Gamma \setminus \{e\}$, i.e. a non-elliptic point. The *Dirichlet domain* of Γ with center z_0 is defined as the set

$$D := \bigcap_{g \in \Gamma \setminus \{e\}} \{z \in \mathbb{H}^2 : d(z, z_0) < d(z, gz_0)\}.$$

We denote by \overline{D} and ∂D the closure and the boundary of D in $\overline{\mathbb{H}^2}$.

- (a) There exists a non-elliptic point in \mathbb{H}^2 .
- (b) $\{g(\overline{D}) : g \in \Gamma\}$ is a locally finite. It is called the *Dirichlet tessellation*.
- (c) D is a convex fundamental domain of Γ .

Proof. (a) Elliptic points are countably many.

(b) There are finitely many $g \in \Gamma$ satisfying $B(z_0, r) \cap g(\overline{D}) \neq \emptyset$, since this condition implies $gz_0 \in B(z_0, 2r)$. \square

2.3 (Boundary and edges of Dirichlet domain). Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . A subset $l \subset \overline{\mathbb{H}^2}$ is called an *edge* of D if $l = g(\overline{D}) \cap \overline{D}$ for some $g \in \Gamma \setminus \{e\}$ and $|l| > 1$.

- (a) For $g \in \Gamma \setminus \{e\}$, the set $g(\overline{D}) \cap \overline{D}$ has the three cases: the null set, one point, or a geodesic segment.
- (b) If l is an edge, then there is unique $g \in \Gamma \setminus \{e\}$ such that $l = g(\overline{D}) \cap \overline{D}$.
- (c) The intersection of two distinct edges is one point or the null set.
- (d) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{g \in \Gamma \setminus \{e\}} g(\overline{D}) \cap \overline{D}.$$

- (e) We have

$$\partial D \cap \mathbb{H}^2 \subset \bigcup_{l: \text{edge}} l.$$

Proof. (d) Let $z \in \partial D \cap \mathbb{H}^2$. Since $d(z, z_0) \leq d(z, gz_0)$ for all $g \in \Gamma \setminus \{e\}$ but $d(z, z_0) \geq d(z, gz_0)$ for some $g \in \Gamma \setminus \{e\}$, there is $g \in \Gamma \setminus \{e\}$ such that $d(z, z_0) = d(z, gz_0)$. By sending z_0 and gz_0 to $\pm 1 + i$ with an isometry so that z is sent to a point on a imaginary axis, we can check for each n that we have $B(z, 1/n) \cap (\mathbb{H}^2 \setminus \overline{D}) \neq \emptyset$. Since $B(z, 1/n) \setminus \overline{D}$ is a non-empty open set in $\mathbb{H}^2 \setminus \overline{D}$, and since

$$\mathbb{H}^2 \setminus \overline{D} \subset \mathbb{H}^2 \setminus D = \overline{\bigcup_{g \in \Gamma \setminus \{e\}} g(D)},$$

we can deduce that $B(z, 1/n)$ intersects with $g(D)$ for some $g \in \Gamma \setminus \{e\}$.

Combining this result with the local finiteness of $\{g(D) : g \in \Gamma\}$, the sequence of sets

$$\{g \in \Gamma \setminus \{e\} : B(z, 1/n) \cap g(D) \neq \emptyset\}$$

indexed by n consists of non-empty finite subsets of $\Gamma \setminus \{e\}$ that are non-increasing. By the pigeonhole principle, there exists $g \in \Gamma \setminus \{e\}$ such that $B(z, 1/n) \cap g(D) \neq \emptyset$ for all n , which allows to extract a sequence $z_n \in g(D)$ that converges to z , which implies $z \in g(\overline{D})$.

(e) Suppose $z \in \partial D \cap \mathbb{H}^2$ is not contained in any edges. Let Z be the set of all $g \in \Gamma \setminus \{e\}$ such that $\{z\} = g(\overline{D}) \cap \overline{D}$. For $g \in \Gamma \setminus (Z \cup \{e\})$, $g(\overline{D}) \cap \overline{D}$ is the null set, one point, or an edge, and any of possibility does not contain z . Therefore,

$$(\partial D \setminus \{z\}) \cap \mathbb{H}^2 = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \mathbb{H}^2$$

by the part (d). Change the restriction \mathbb{H}^2 to a compact ball as

$$(\partial D \setminus \{z\}) \cap \overline{B(z, 1)} = \bigcup_{g \in \Gamma \setminus (Z \cup \{e\})} (g(\overline{D}) \cap \overline{D}) \cap \overline{B(z, 1)}.$$

Then, the left-handed side is homeomorphic to $[-1, 0) \cup (0, 1]$ or $(-1, 1)$ since ∂D is homeomorphic to S^1 , but the right-handed side is compact because the union becomes finite due to the local finiteness. This is a contradiction, so z is contained in an edge. \square

2.4 (Finitely generated Fuchsian group). Let Γ be a Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . Let W be the set of all $g \in \Gamma \setminus \{e\}$ such that $g(\overline{D}) \cap \overline{D}$ is an edge.

- (a) W generates Γ .
- (b) If Γ is finitely generated, then W is finite.
- (c) If W is finite, then Γ is finitely generated.

2.5 (Siegel's theorem). Finite area iff finitely generated.

- (a) If Γ is finitely generated, then

$$\partial D = \bigcup_{l: \text{edge}} l.$$

2.3 Side pairing and cycle conditions

2.6 (Side pairing condition). Let Γ be a finitely generated Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . We have seen that ∂D consists of finitely many edges. A point $v \in \partial D$ is called a *vertex* if it either

- (i) the intersection of two edges, or
- (ii) the fixed point of elliptic isometry $g \in \Gamma$ of order two.

Let $v_0, v_1, \dots, v_n = v_0$ be vertices, indexed along the boundary counterclockwise. A *side* is geodesic segments s_i connecting v_i and v_{i+1} .

- (a) For each side s of D , there is unique $g_s \in \Gamma$ such that $g_s^{-1}(s)$ is another side of D . The isometry g_s is called the *side pairing isometry* of the side s .
- (b) The side pairing isometry of $g_s^{-1}(s)$ is g_s^{-1} .
- (c) The number of sides n is always even.

Proof. \square

2.7 (Cycle condition). Let Γ be a finitely generated Fuchsian group, and let D be a Dirichlet domain of Γ with center z_0 . Let V and S be the set of all vertices and sides of D , respectively. Define $\sigma : V \rightarrow V$ and $\sigma : S \rightarrow S$ which use same notation such that $\sigma(v_i) = v_{j+1}$ and $\sigma(s_i) := s_{j+1}$ where $s_j = g_s^{-1}(s_i)$. The map σ can be seen as an element of the symmetric group S_n .

- (a) Suppose $v_0 \in \mathbb{H}^2$ and $s = s_0$. Let m be the minimal positive integer such that $\sigma^m(s) = s$. Then, $g_{\sigma^{m-1}(s)} \cdots g_{\sigma(s)} g_s$ is either the identity or elliptic.

(b) Suppose $v_0 \in \partial \mathbb{H}^2$.

2.8 (Genus two surface).

2.9 (Modular group). Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ be the modular group and choose the origin $2i$. $v_0 = i$, $v_1 = e^{\pi i/3}$, $v_2 = \infty$, $v_3 = e^{2\pi i/3}$. $g_{s_0} = S$, $g_{s_1} = T$, $g_{s_2} = T^{-1}$, $g_{s_3} = S^{-1}$. $\sigma = (13)$. The elliptic cycle condition: (0) defines $SS = 1$, (13) defines $(S^{-1}T)^3 = 1$

2.4 The Poincaré polygon theorem

2.10 (Definition of polygon). (a)

2.11 (Side pairing identification). Let P be an open polygon. Define cycles of each vertex. Let $Y := \bar{P}/\sim$ and $\tilde{Y} := (\Gamma \times \bar{P})/\sim$. Define $\Pi : \tilde{Y} \rightarrow Y$.

(a)

2.12 (Elliptic cycle condition). Let P be an open polygon with a side pairing identification. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P .

(a) P satisfies the elliptic cycle condition.

(b) D is a local homeomorphism.

(c) D is a covering map onto its image.

Proof. (a) \Rightarrow (b)

(b) \Rightarrow (c) We claim p has the path lifting property, which is unique because it is a local homeomorphism. Let $w : [0, 1] \rightarrow \text{im } D$, and $\tilde{w} : [0, \tau] \rightarrow \tilde{Y}$ its maximal extension. Write $\tilde{w}(t) = [g(t), z(t)]$ and $w(\tau) = gz$. Define $\tilde{w}(\tau) := [g, z]$. Then,

$$D\tilde{w}(\tau) = D([g, z]) = gz = w(\tau).$$

Let U be an open neighborhood of $[g, z]$ in \tilde{Y} such that $D|_U$ is a homeomorphism and $D(U)$ is open in \mathbb{H}^2 . Then, as $t \rightarrow \tau$,

$$p\tilde{w}(t) = w(t) \rightarrow w(\tau) = p\tilde{w}(\tau)$$

implies

$$\tilde{w}(t) \rightarrow \tilde{w}(\tau),$$

so $\tilde{w} : [0, \tau] \rightarrow \tilde{Y}$ is a continuous extension of $w : [0, \tau] \rightarrow \mathbb{H}^2$. Therefore, D is a local homeomorphism that has the unique path lifting property, so it is a covering map onto its image. \square

2.13 (Finite cycle condition). Let P be an open polygon with a side pairing identification. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P . Consider D and Π such that

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{D} & \mathbb{H}^2 \\ \downarrow \Pi & & \\ Y & & \end{array}.$$

(a) If every cycle of finite points is finite, then $\text{im } D$ is open.

(b) If every cycle is finite, then there is a metric ρ on Y such that $[z_n] \rightarrow [z]$ in ρ if and only if $h_n z_n \rightarrow z$ in \mathbb{H}^2 for a sequence $h_n \in \Gamma$.

Proof.

$$\rho(x, y) := \inf \sum, \\ \inf_{h \in \Gamma} d(h^{-1}z, z') = \rho([z], [z'])$$

\square

2.14 (Parabolic cycle condition). Let P be an open polygon with a side pairing identification such that every cycle is finite. Let Γ be a subgroup of $\text{Isom}^+(\mathbb{H}^2)$ generated by side pairing isometries of P . Suppose

- (a) P satisfies the parabolic cycle condition,
- (b) M is a complete metric space.
- (c) D is surjective.

Proof. (b) \Rightarrow (c) Let $w \in \partial(\text{im } D)$ so that we have $[g_n, z_n] \in \tilde{Y}$ such that $g_n z_n \rightarrow w$ in \mathbb{H}^2 . Since $g_n z_n$ is Cauchy, $[z_n]$ is also Cauchy, so we have a limit $[z_n] \rightarrow [z]$ in Y . Then, there exists a sequence $h_n \in \Gamma$ such that $h_n z_n \rightarrow z$ in \mathbb{H}^2 , which implies $g_n h_n^{-1} z \rightarrow w$ in \mathbb{H}^2 and $w \in \overline{\Gamma z}$. Since $\text{im } D$ is open and $\overline{P} \subset \text{im } D$, there is $\varepsilon > 0$ such that $B(z, \varepsilon) \subset \text{im } D$. There is $g \in \mathbb{H}^2$ such that $d(gz, w) < \varepsilon$, which implies $g^{-1}w \in B(z, \varepsilon)$. Because Γ acts on $\text{im } D$, we can conclude $w \in \text{im } D$. \square

If P satisfies the cycle conditions,

- (a) Γ is discrete.
- (b) Γ is given by the presentation $\langle S | R \rangle$, where S is the set of side-pairing isometries and R is the set of cycle relations.
- (c) P is a fundamental domain of Γ
- (d) $Y \cong \mathbb{H}^2 / \Gamma$.

2.5 Geometric structures

2.15 (Pseudogroup). Let X be a topological space. A *pseudogroup* on X is a wide subgroupoid Γ of $\text{Homeo}(X)$ such that $U \mapsto \{g \in \Gamma : \text{dom } g = U\}$ is a separated presheaf.

- (a) For $G \leq \text{Homeo}(X)$, $\{g|_U : g \in G, U \in \mathcal{T}\}$ is a pseudogroup on X , and we write this pseudogroup as (G, X) .

G does not act on (G, X) -manifold!

Model geometry G -manifold X that are connected, transitive, simply connected, and has compact stabilizers. The group G is assumed to be maximal for a space X .

A geodesically connected and geodesically complete space is rigid.

2.16 (Continuation of path). Let M be a (G, X) -manifold, $\varphi : U \rightarrow X$ a chart, and $\gamma : [0, 1] \rightarrow M$ a path. There is a partition $(t_i)_{i=0}^m$ of the interval $[0, 1]$ with $t_0 = 0$, $t_m = 1$ and a sequence of chart $(\varphi_i : U_i \rightarrow X)_{i=0}^{m-1}$ with $\varphi_0 = \varphi$ such that $\gamma([t_{i-1}, t_i]) \subset U_i$. Since $\tau_{i+1,i} \circ \varphi_{i+1} \circ \gamma(t_{i+1}) = \varphi_i \circ \gamma(t_{i+1})$, we can define a path $\hat{\gamma} : [0, 1] \rightarrow X$ by

$$\hat{\gamma}(t) := \tau_{1,0} \circ \cdots \circ \tau_{i,i-1} \circ \varphi_i \circ \gamma|_{[t_i, t_{i+1}]}(t)$$

for $t \in [t_i, t_{i+1}]$, where $\tau_{i+1,i} = \varphi_i \circ \varphi_{i+1}^{-1}$ are transition maps. The path $\hat{\gamma}$ is called the *continuation* of $\varphi \circ \gamma$.

- (a) $\hat{\gamma}$ does not depend on the choice of the sequence of charts when the partition is given.
- (b) $\hat{\gamma}$ does not depend on the choice of the partition.
- (c) If γ_0 and γ_1 are homotopic fixing endpoints, then their continuations are also homotopic fixing endpoints.

2.17 (Developing map). Let M be a connected (G, X) -manifold, \tilde{M} the universal covering of M , and $\varphi : U \rightarrow X$ a chart on \tilde{M} .

2.18 (Holonomy). Let M be a connected (G, X) -manifold, \tilde{M} the universal covering of M , and $\varphi : U \rightarrow X$ a chart on \tilde{M} .

$$h : \pi_1(M) \rightarrow G.$$

- (a) If $f_0, f_1 : \tilde{M} \rightarrow X$ are (G, X) -maps, then there is a unique $g \in G$ such that $f_1 = gf_0$.
- (b) For $H \leq G$, M admits a (H, X) -structure if and only if $\text{im } h \subset H$.

surjectivity of a map from torsion-free discrete subgroups of G to complete (G, X) -manifolds? (up to homeomorphism, up to geometric structure)

Definition 2.1 (Several definitions of hyperbolic manifolds). Let $G = \text{Isom}^+(\mathbb{H}^n)$ and X a n -manifold. Then, X is a hyperbolic manifold if one of the following satisfied...?:

1. It admits a hyperbolic atlas, and it is “complete”
2. It is homeomorphic to \mathbb{H}^n/Γ for a torsion-free discrete subgroup Γ of G .
3. It is a geodesically complete Riemannian manifold with constant sectional curvature -1.

Thurston geometry is a three-dimensional model geometry on which a closed 3-manifold has a geometric structure modelled.

oriented prime closed 3-manifolds

3 Universal coefficient theorem

Lemma 3.1. *Suppose we have a flat resolution*

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

Then, we have a exact sequence

$$\cdots \rightarrow 0 \rightarrow \text{Tor}_1^R(A, B) \rightarrow P_1 \otimes B \rightarrow P_0 \otimes B \rightarrow A \otimes B \rightarrow 0.$$

Theorem 3.2. *Let R be a PID. Let C_\bullet be a chain complex of flat R -modules and G be a R -module. Then, we have a short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0,$$

which splits, but not naturally.

1. We have a short exact sequence of chain complexes

$$0 \rightarrow Z_\bullet \rightarrow C_\bullet \rightarrow B_{\bullet-1} \rightarrow 0$$

where every morphism in Z_\bullet and B_\bullet are zero. Since modules in $B_{\bullet-1}$ are flat, we have a short exact sequence

$$0 \rightarrow Z_\bullet \otimes G \rightarrow C_\bullet \otimes G \rightarrow B_{\bullet-1} \otimes G \rightarrow 0$$

and the associated long exact sequence

$$\rightarrow H_n(B; G) \rightarrow H_n(Z; G) \rightarrow H_n(C; G) \rightarrow H_{n-1}(B; G) \rightarrow H_{n-1}(Z; G) \rightarrow$$

where the connecting homomorphisms are of the form $(i_n: B_n \rightarrow Z_n) \otimes 1_G$ (It is better to think diagram chasing than a natural construction). Since morphisms in B and Z are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n(C; G) \rightarrow B_{n-1} \otimes G \rightarrow Z_{n-1} \otimes G \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(H_n, G) \rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(H_{n-1}, G) \rightarrow 0.$$

For splitting, □

2. Since R is PID, we can construct a flat resolution of G

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow G \rightarrow 0.$$

Since modules in C_\bullet are flat so that the tensor product functors are exact and $P_1 \rightarrow P_0$ and $P_0 \rightarrow G$ induce the chain maps, we have a short exact sequence of chain complexes

$$0 \rightarrow C_\bullet \otimes P_1 \rightarrow C_\bullet \otimes P_0 \rightarrow C_\bullet \otimes G \rightarrow 0.$$

Then, we have the associated long exact sequence

$$\rightarrow H_n(C; P_1) \rightarrow H_n(C; P_0) \rightarrow H_n(C; G) \rightarrow H_{n-1}(C; P_1) \rightarrow H_{n-1}(C; P_0) \rightarrow .$$

Since flat tensor product functor commutes with homology functor from chain complexes, we have

$$\rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n(C; G) \rightarrow H_{n-1} \otimes P_1 \rightarrow H_{n-1} \otimes P_0 \rightarrow .$$

Since

$$0 \rightarrow \text{Tor}_1^R(G, H_n) \rightarrow H_n \otimes P_1 \rightarrow H_n \otimes P_0 \rightarrow H_n \otimes G \rightarrow 0$$

for all n , the exact sequence splits into short exact sequence by images

$$0 \rightarrow H_n \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}_1^R(G, H_{n-1}) \rightarrow 0.$$

□

Proof 3. By tensoring G , we get the following diagram.

$$\begin{array}{ccccc}
 H_n \otimes G & & & & H_{n-1} \otimes G \\
 \searrow & & & & \nearrow \\
 & \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G & \\
 \nearrow & \searrow & & \nearrow & \searrow \\
 C_n \otimes G & & \text{im } \partial_n \otimes G & & C_{n-1} \otimes G \\
 & \nearrow & & \nwarrow & \\
 & \text{Tor}_1(H_{n-1}, G) & & &
 \end{array}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernels are preserved, but monomorphisms and kernels are not. Especially, $\text{coker } \partial_{n+1} \otimes G = \text{coker}(\partial_{n+1} \otimes 1_G)$ is important.

Consider the following diagram.

$$\begin{array}{ccccc}
 H_n(C; G) & & H_n \otimes G & & \\
 \searrow & & \downarrow & & \\
 & \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G & \\
 & \downarrow & \nearrow & \downarrow & \nearrow \\
 & \text{im } \partial_n \otimes G & & \text{im}(\partial_n \otimes 1_G) & \\
 & \nearrow & & \nwarrow & \\
 & \text{Tor}_1(H_{n-1}, G) & & &
 \end{array}$$

$\text{ker } \partial_{n-1} \otimes G \xrightarrow{\text{monic!}} C_{n-1} \otimes G$

Since $\text{ker } \partial_{n-1}$ is free,

If we show $\text{im}(\partial_n \otimes 1_G) \rightarrow \text{ker } \partial_{n-1} \otimes G$ is monic, then we can get

$$\begin{aligned}
 H_n(C; G) &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{im}(\partial_n \otimes 1_G)) \\
 &= \text{ker}(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{ker } \partial_{n-1} \otimes G).
 \end{aligned}$$

4 Fundamental differential geometry

4.1 Manifold and Atlas

Definition 4.1. A locally Euclidean space M of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to an open subset of \mathbb{R}^d .

Definition 4.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 4.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of \mathbb{R}^d . A chart is often written by a pair (U, φ) .

Definition 4.4. An *atlas* \mathcal{F} is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 4.5. A *differentiable manifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

4.2 Definition of Differentiable Structure

Definition 4.6. An atlas \mathcal{F} is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each transition function $\tau_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 4.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 4.8. An differentiable atlas \mathcal{F} is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in \mathcal{F} , then $(U, \varphi) \in \mathcal{F}$.

Definition 4.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manifold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M . When the charts is already equipped on M , it is natural to define a function $f : M \rightarrow \mathbb{R}$ differentiable if the functions $f \circ \varphi^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas \mathcal{F} , then f is also differentiable on any atlases which is equivalent to \mathcal{F} by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 4.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 4.10. A continuous function $f : M \rightarrow N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

4.3 Curves

Definition 4.11. For $f : M \rightarrow \mathbb{R}$ and (U, ϕ) a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 4.12. Let $\gamma: I \rightarrow M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi: M \rightarrow N$ be a smooth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f: M \rightarrow \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt}f \circ \gamma.$$

Proposition 4.1. Let $\gamma: I \rightarrow M$ be a smooth curve on a manifold M . The notation $\dot{\gamma}^\mu$ is not confusing thanks to

$$(\dot{\gamma})^\mu = (\dot{\gamma}^\mu).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt}x^\mu \circ \gamma.$$

4.4 Connection computation

$$\begin{aligned}\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\ &= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\ &= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma_{\mu\nu}^\lambda \partial_\lambda) \\ &= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda \right) \partial_\nu.\end{aligned}$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ .

$$Y_{;\mu}^\nu = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \quad Y_{;\mu}^\nu = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda.$$

Theorem 4.2. For Levi-civita connection for g ,

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Proof.

$$\begin{aligned}(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} \\ (\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma_{jk}^l g_{li} - \Gamma_{jl}^l g_{kl} \\ (\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}\end{aligned}$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$. Thus,

$$\begin{aligned}\Gamma_{ij}^l g_{kl} &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}). \\ \Gamma_{ij}^l &= \frac{1}{2}g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).\end{aligned}$$

□

4.5 Geodesic equation

Theorem 4.3. *If c is a geodesic curve, then components of c satisfies a second-order differential equation*

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0.$$

Proof. Note

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\lambda \partial_\lambda) = (\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu + \dot{\gamma}^\nu \dot{\gamma}^\lambda \Gamma_{\nu\lambda}^\mu) \partial_\mu.$$

Since

$$\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu = \dot{\gamma}(\dot{\gamma}^\mu) = d\dot{\gamma}^\mu(\dot{\gamma}) = d\dot{\gamma}^\mu \circ d\gamma\left(\frac{\partial}{\partial t}\right) = d\dot{\gamma}^\mu\left(\frac{\partial}{\partial t}\right) = \ddot{\gamma}^\mu,$$

we get a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0$$

for each μ . □

5 Vector calculus on spherical coordinates

$$\begin{aligned} V &= (V_r, V_\theta, V_\phi) \\ &= V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi} \quad (\text{normalized coords}) \\ &= V_r \frac{\partial}{\partial r} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} V_\phi \frac{\partial}{\partial \phi} \quad (\Gamma(TM)) \\ &= V_r dr + r V_\theta d\theta + r \sin \theta V_\phi d\phi \quad (\Omega^1(M)) \\ &= r^2 \sin \theta V_r d\theta \wedge d\phi + r \sin \theta V_\theta d\phi \wedge dr + r V_\phi dr \wedge d\theta \quad (\Omega^2(M)). \end{aligned}$$

$$\nabla \cdot V = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (r \sin \theta V_\theta) + \frac{\partial}{\partial \phi} (r V_\phi) \right]$$

$$\Delta u = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} u \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} u \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} u \right) \right]$$

Let (ξ, η, ζ) be an orthogonal coordinate that is *not* normalized. Then,

$$\sharp = g = \text{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2)$$

$$\hat{x} = \|\partial_x\|^{-1} \partial_x = \|\partial_x\| dx = \|\partial_y\| \|\partial_z\| dy \wedge dz$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r d\theta, \quad r \sin \theta d\phi, \quad (r d\theta) \wedge (r \sin \theta d\phi), \quad (r \sin \theta d\phi) \wedge (dr), \quad (dr) \wedge (r d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[\frac{1}{\|\partial_x\|} \frac{\partial}{\partial x} \cdot -, \frac{1}{\|\partial_y\|} \frac{\partial}{\partial y} \cdot -, \frac{1}{\|\partial_z\|} \frac{\partial}{\partial z} \cdot - \right] \\
\text{curl} : \nabla &= \left[\frac{1}{\|\partial_y\| \|\partial_z\|} \left(\frac{\partial}{\partial y} (\|\partial_z\| \cdot -) - \frac{\partial}{\partial z} (\|\partial_y\| \cdot -) \right), \right. \\
&\quad \frac{1}{\|\partial_z\| \|\partial_x\|} \left(\frac{\partial}{\partial z} (\|\partial_x\| \cdot -) - \frac{\partial}{\partial x} (\|\partial_z\| \cdot -) \right), \\
&\quad \left. \frac{1}{\|\partial_x\| \|\partial_y\|} \left(\frac{\partial}{\partial x} (\|\partial_y\| \cdot -) - \frac{\partial}{\partial y} (\|\partial_x\| \cdot -) \right) \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} (\|\partial_y\| \|\partial_z\| \cdot -), \frac{\partial}{\partial y} (\|\partial_z\| \|\partial_x\| \cdot -), \frac{\partial}{\partial z} (\|\partial_x\| \|\partial_y\| \cdot -) \right] \\
\Delta &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[\frac{\partial}{\partial x} \left(\frac{\|\partial_y\| \|\partial_z\|}{\|\partial_x\|} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\|\partial_z\| \|\partial_x\|}{\|\partial_y\|} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\|\partial_x\| \|\partial_y\|}{\|\partial_z\|} \frac{\partial}{\partial z} \right) \right]
\end{aligned}$$

$$\text{grad} = \frac{1}{\|\cdot\|^1} (\nabla) \cdot \|\cdot\|^0$$

$$\text{curl} = \frac{1}{\|\cdot\|^2} (\nabla \times) \cdot \|\cdot\|^1$$

$$\text{div} = \frac{1}{\|\cdot\|^3} (\nabla \cdot) \cdot \|\cdot\|^2$$

6 Bundles

Show that S^n has a nonvanishing vector field if and only if n is odd.

Solution. Since S^n is embedded in \mathbb{R}^{n+1} , the tangent bundle TS^n can be considered as an embedded manifold in $S^n \times \mathbb{R}^{n+1}$ which consists of (x, v) such that $\langle x, x \rangle = 1$ and $\langle x, v \rangle = 0$, where the inner product is the standard one of \mathbb{R}^{n+1} .

Suppose n is odd. We have a vector field $(x_1, x_2, \dots, x_{n+1}; x_2, -x_1, \dots, -x_n)$ which is nonvanishing. Conversely, suppose we have a nonvanishing vector field X . Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \rightarrow S^n \times \mathbb{R}^{n+1} \rightarrow \phi \mathbb{R}^{n+1} \rightarrow S^n.$$

The last map can be defined since X is nowhere zero. Since this map satisfies $\langle x, \phi(x) \rangle = 0$ for all $x \in S^n$, we can define homotopies from ϕ to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree, $+1$, so n is odd. \square

Proposition 6.1. *Independent commuting vector fields are realized as partial derivatives in a chart.*

Proposition 6.2. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent involutive vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent. (Maybe)*

Proposition 6.3. *Let $\{\partial_1, \dots, \partial_k\}$ be an independent commuting vector fields. We can find independent commuting $\{\partial_{k+1}, \dots, \partial_n\}$ such that union is independent and commuting. (Maybe)*

The following theorem says that image of immersion is equivalent to kernel of submersion.

Proposition 6.4. *An immersed manifold is locally an inverse image of a regular value.*

Proposition 6.5. *A closed submanifold with trivial normal bundle is globally an inverse image of a regular value.*

Proof. It uses tubular neighborhood. Pontryagin construction? □

Proposition 6.6. *An immersed manifold is locally a linear subspace in a chart.*

Proposition 6.7. *Distinct two points on a connected manifold are connected by embedded curve.*

Proof. Let $\gamma : I \rightarrow M$ be a curve connecting the given two points, say p, q .

Step [.1]Constructing a piecewise linear curve For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\text{im } \gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$ for all $1 \leq i \leq n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i, j) : U_{t_i} \cap U_{t_j} \neq \emptyset\}$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \leq i \leq n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q .

Step [.2]Smoothing the curve □

Proposition 6.8. *Let M is an embedded manifold with boundary in N . Any kind of sections on M can be extended on N .*

Proposition 6.9. *Every ring homomorphism $C^\infty(M) \rightarrow \mathbb{R}$ is obtained by an evaluation at a point of M .*

Proof. Suppose $\phi : C^\infty(M) \rightarrow \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Proposition 6.10. *The set of points that is geodesically connected to a point is open.*