Algebraic Structures

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Contents

Part I

Groups

Chapter 1

Subgroups

subgroups homomorphisms, image, kernel, inverse images normality, quotient, coset counting direct sum, direct product

Chapter 2

Group actions

2.1 Orbits and stabilizers

Invariants on orbit space. The size and number of orbits.

2.1 (Transitive actions). stabilizer of an action is well defined

2.2 (Free actions). no fixed point, trivial stabilizer for any point, every orbit has 1-1 correspondence to group

2.2 Action by conjugation

2.3 Action by left multiplication

2.4 Automorphism groups

2.3 (Outer automorphism group). duality for center

Chapter 3

Symmetry groups

elements by order elements by conjugacy class subgroups by conjugacy class

3.1 Cyclic groups

3.2 Symmetric groups

3.3 Matrix groups

dihedral groups

Exercises

3.1. Let G be a finite group. If $G/Z(G)$ is cylic, then G is abelian.

3.2. Let G be a finite group. If the cube map $x \mapsto x^3$ is a surjective endomorhpism, then G is abelian.

3.3. Show that a finite symmetric group has two generators.

3.4. Show that a group of order $2p$ for a prime p has exactly two isomorphic types.

3.5. Show that a group G is abelian if $|G| = p^2$ for a prime p.

3.6. Let G be a finite group of order n and p the smallest prime divisor of n. Show that a subgroup of G of index p is normal in G.

3.7 (Primitive roots)**.** We find all n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic.

3.8 (p-groups)**.** (a) A nontrivial normalizer of a p-group meets its center out of identity.

(b) A proper subgroup of a finite p-group is a proper subgroup of its normalizer. In particular, every finite p-group is nilpotent.

3.9. Show that a finite group G satisfying $\sum_{g\in G} \operatorname{ord}(g) \le 2n$ is abelian.

3.10. Show that the order of a group with trivial automorphism group is at most two.

3.11. Find all homomorphic images of A_4 up to isomorphism.

Problems

1.

Part II

Rings

Chapter 4

Ideals

Chapter 5

Integral domains

Exercises

5.1. Show that a finite integral domain is a field.

5.2. Show that every ring of order p^2 for a prime p is commutative.

5.3. Show that a semiring with multiplicative identity and cancellative addtion has commutative addition.

5.4. Show that the complement of a saturated monoid in a commutative ring is a union of prime ideals.

Chapter 6

Polynomial rings

6.1 Irreducible polynomials

relation to maximal ideals Irreducibles over several fields

Part III

Modules

Chapter 7

Exact sequences

free modules inj, proj

Chapter 8

Hom set and tensor products

hom and duality tensor product algebras?

Chapter 9

Modules over a principal ideal domain

invariant factors and elementary divisors

Part IV

Vector spaces

Chapter 10

10.1 Dual space

10.1 (Double dual space).

10.2 Bilinear and sesquilinear forms

10.2 (Polarization identity). (a) Let F be a field of characteristic not 2. If $\langle -,- \rangle$ is a symmetric bilinear form, then

$$\langle x, y \rangle = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2).$$

(b) Let $F = \mathbb{C}$. If $\langle -,- \rangle$ is a sesquilinear form, then

$$\langle x, y \rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

(c) isometry check

10.3 (Cauchy-Schwarz inequality). (a) Let $F = \mathbb{R}$. If $\langle -,- \rangle$ is a positive semi-definite symmetric bilinear form, then

(b) Let $F = \mathbb{C}$. If $\langle -,- \rangle$ is a positive semi-definite Hermitian form, then

10.4 (Dual space identification). Let $\langle -,- \rangle$ be a non-degenerate bilinear form

10.3 Adjoint

10.5 (Adjoint linear transforms).

Chapter 11

Normal forms

11.1 Rational canonical form

11.1 (Finitely generated $\mathbb{F}[x]$-modules).

11.2 (Cyclic subspaces).

11.2 Jordan normal form

11.3 Conjugacy classes in matrix groups

11.3 (Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p)$). The conjugacy classes are classified by the Jordan normal forms. There are four cases: for some a and b in $\mathbb{F}_p$,

(a) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$: $\binom{p-1}{2} = \frac{(q-1)(q-2)}{2}$ classes of size $\frac{|G|}{(q-1)^2} = q(q+1)$.

(b) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$: $q-1$ classes of size 1.

(c) $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$: $q-1$ classes of size $\frac{|G|}{q(q-1)} = q^2-1$.

(d) otherwise, the eigenvalues are in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. In this case, the number of conjugacy classes is same as the number of monic irreducible qudratic polynomials over $\mathbb{F}_p$; $\frac{|\mathbb{F}_{p^2}|-|\mathbb{F}_p|}{2} = \frac{p(p-1)}{2}$ classes. Their size is $\frac{p(p-1)}{2}$.

11.4 Spectral theorems

Exercises

Chapter 12

Tensor algebras

Exterior algebras Symmetric algebras