Algebraic Number Theory

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Part I

Algebraic numbers

Chapter 1

1.1 Local fields

1.1 (Absolute value)**.** Let K be a field. An *absolute value* or a *multiplicative valuation* on K is a function $|\cdot| : K \to [0, \infty)$ such that

(i) $x = 0$ if $|x| = 0$,

(ii) $|xy| = |x||y|$,

(iii) $|x + y| \leq |x| + |y|$.

Non-archimedean

1.2 (Local fields)**.** A *local field* is a field with an absolute value with the induced topology that is locally compact.

1.3 (Ostrowski theorem)**.**

1.4 (Places)**.**

Chapter 2

Adèles and idèles

Part II

Class field theory

Chapter 3

Local class field theory

3.1 Lubin-Tate theory

3.2 Kronecker-Weber theorem

3.1 (Local Kronecker-Weber theorem). Let $K/\mathbb{Q}_p$ be a finite abelian extension.

Let m be a conductor of a finite abelian extension $K/\mathbb{Q}$. Then, we have a surjection

$$\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$$

by the Kronecker-Weber theorem. For a prime $p \in \mathbb{Z}$ that does not divide m so that p is not ramified, then the decomposition group $G_p \leq \mathrm{Gal}(K/\mathbb{Q})$ is a cyclic group generated by the Frobenius element $x \to x^p$, denoted by Frob_p or $\left(\frac{K/\mathbb{Q}}{p}\right)$. Artin map $I_{\mathbb{Q}}^m \to \mathrm{Gal}(K/\mathbb{Q})$ of $K/\mathbb{Q}$ maps each prime $p \nmid m$ to the Frobenius element Frob_p.

Artin map factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$

Chapter 4

Global class field theory

Chapter 5

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Arithmetic geometry

Part IV

Langlands program

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Modular forms

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L-functions

Riemann $\zeta(s)$

Dedekind $\zeta_K(s)$

Hasse-Weil $\zeta_X(s)$

7.1 Dirichlet L-functions

7.1 (Hecke character)**.** Dirichlet character can be understood as a group homomorphism $\chi : \widehat{\mathbb{Z}}^\times \to \mathbb{C}$ of finite order, which means that there is n such that χ factors through $(\mathbb{Z}/n\mathbb{Z})^\times$.

In order to construct an L-function from a character, we need to extend a character as a function of ideals. We interpret $(\mathbb{Z}/n\mathbb{Z})^\times$ as the ray class group modulo $\mathfrak{m}$.

To extend the order of a character to possibly infinite cases, Hecke character is defined a character of an idele class group $C_K := \mathbb{A}_K^\times/K^\times$.

Dirichlet (Hecke) L-functions for ray-class characters $\chi : C_K \to \mathbb{C}$:

$$L(\chi,s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}\text{ prime}} \frac{1}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s}}$$

Artin L-functions for a Galois representation $\rho : \mathrm{Gal}(L/K) \to GL_n(\mathbb{C})$:

$$L(\rho,s) = \prod_{\mathfrak{p}\text{ prime}} \frac{1}{\det(1-\rho(\mathrm{Frob}_{\mathfrak{p}})N(\mathfrak{p})^{-s})}$$

Elliptic curves $L(E,s)$

Modular forms $L(f,s)$

Chapter 8

Automorphic representations