Fiber Bundles

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Contents

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References: Steenrod, *The topology of fiber bundles*, and Tamaki, *Fiber bundles and homotopy* (Japanese)

1. Introduction

An ultimate goal of topology is to classify topological spaces, up to homeomorphism. If you want to show two spaces are homeomorphic, we should construct a homeomorphism: *Shokuninwaza* (wild knot, Casson handle). If you want to show two spaces are not homeomorphic, then we can investigate topological *properties*, and as their quantitative comparison, we can investigate topological *invariants* Some examples include

- the number of connected componenets,
- the Euler characteristic,
- homology groups,
- homotopy groups,
- the minimal number of open contractible sets to cover the spaces (Lusternik-Schnirelmann category, topological complexity),
- Gelfand-Naimark theorem: $C(X) \cong C(Y)$ implies $X \cong Y$ if they are compact Hausdorff.

We will restrict objects to study. For example, metric spaces, manifolds, CW-complexes. As the assumptions change, invariants may have different appearances. For a manifold X,

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rk}_{\mathbb{Z}} H_q(X) = \sum_{q=0}^{\infty} (-1)^q b_q(X).$$

For a CW-complex X,

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q (\text{the number of } q\text{-cells}).$$

Let M be an connected closed n-dimensional manifold. Some classification results are as follows(up to both homeomorphisms and diffeomorphisms, because $d \leq 2$):

- ($n = 0$) $M \cong *$, and $\chi(*) = 1$.
- ($n = 1$) $M \cong S^1$, and $\chi(S^1) = 0$.
- ($n = 2$)
  - If M is orientable, then $M \cong \Sigma_g$ for $g \geq 0$, and $\chi(\Sigma_g) = 2 - 2g$.
    $\Sigma_0 \cong S^2$, $\Sigma_1 \cong T^2$.
  - If M is not orientable, then $M \cong (\mathbb{RP}^2)^{\#h}$ for $h \geq 1$, and $\chi((\mathbb{RP}^2)^{\#h}) = 2 - h$.
    $\mathbb{RP}^2 (\cong \text{Möbius strip} \cup D^2)$, $K = \mathbb{RP}^2 \# \mathbb{RP}^2$

Problem 1. Show $\mathbb{RP}^2 \# T^2 \cong \mathbb{RP}^2 \# K$.

Here are some facts about triangulability:

- Cairns(1935), Whitehead (1940): every C^1-manifold is triangulable (unique as a PL-manifold).
- Rado(1925, $n = 2$), Moise(1952, $n = 3$): for $n \leq 3$, every C^0-manifold is triangulable (unique as a PL-manifold).
- Kirby-Siebermann(1966, $n \geq 5$): for $n \geq 4$, there is a non-triangulable PL-manifold.

- Donaldson, Freedman, Casson: for $n = 4$, there is a non-triangulable manifold as a topological space.
- Manolescu(2013): for $n \geq 5$, there is a non-triangulable manifold as a topological space.

Orientability? For a connected closed surface S, it is orientable iff $H_2(S) \cong \mathbb{Z}$, not orientable iff $H_2(S) \cong 0$. The generator of $H_2(S)$ is called the fundamental class. Orientability asks if the tubular neighborhood of every simple closed curve is homeomorphic to an anulus. It is described by the first Stiefel-Whitney class:

$$w_1(S) \in H^1(S; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H^1(S), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(S), \mathbb{Z}/2\mathbb{Z}).$$

Euler characteristic of manifolds

(0) Odd-dimensional manifolds

Theorem. *For an odd-dimensional closed connected manifold,* $\chi(M^{2n+1}) = 0$.

Proof. If orientable, then $b_0(M) = 1$, $b_3(M) = 1$, $b_1(M) = b_2(M)$ by the Poincaré duality. If not, a double cover is orientable, and $\chi(\widetilde{M}) = 2\chi(M)$. □

(1) Gauss-Bonnet theorem

Theorem (Gauss-Bonnet)**.** *If a smooth manifold M^n embeds into $\mathbb{R}^{n+1}$ (hypersurface), then it is orientable and the Euler characteristic is given by*

$$\chi(M) = \frac{2}{\mathrm{vol}(S^n)} \int_M K \, d\,\mathrm{vol}_M \, .$$

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We have a cohomological interpretation. In the Chern-Weil theory, we have a generalized version of the Gauss-Bonnet theorem for a general compact manifold using the theory of connections. We can interpret $2\,\mathrm{vol}(S^n)^{-1} K \cdot d\,\mathrm{vol}_M$ as a differential form which provides with the Euler characteristic. In the context of the de Rham theorem, we will eventually call the equivalence class of this differential form as the *Euler class*.

(2) Poincaré-Hopf theorem

Let M^n be a orientable connected smooth closed manifold. Let X be a smooth vector field on M such that there are only finitely many zeros $\{p_1, \cdots, p_m\}$. For each p_j, define the index $\mathrm{Ind}(X, p_j)$ as follows: seeing X as a vector field on $\varphi_j(U_j)$ for a chart (U_j, φ_j) not containing zeros of X but p_j and mapping p_j to zero in $\mathbb{R}^n$, we define $\mathrm{Ind}(X, p_j) = \deg f_j$, where $f_j : S_\varepsilon (\approx S^{n-1}) \to S^{n-1} : x \mapsto X_x / \|X_x\|$.

Example. Let $n = 2$. We have indices $1, 1, 1, -1, 0, 2$ for

$$X_1(x,y) = (x,y), \quad X_2(x,y) = (-x,-y), \quad X_3(x,y) = (-y,x),$$

$$X_4(x,y) = (-x,y), \quad X_5(x,y) = \sqrt{x^2+y^2}(1,1), \quad X_6(x,y) = (x^2-y^2, 2xy).$$

Theorem (Poincaré-Hopf)**.**

$$\sum_{j=1}^{m} \mathrm{Ind}(X, p_j) = \chi(M).$$

We have a cohomological interpretation. Let $c = \sum_{j=1}^{m} \mathrm{Ind}(X, p_j) p_j$ be a singular 0-cycle on M. Then, the Poincaré-Hopf theorem states that we have

$$\begin{array}{ccc} H_0(M) & \xrightarrow{\sim} & \mathbb{Z} \\ p_j & \mapsto & 1 \\ c & \mapsto & \chi(M). \end{array}$$

By the Poincaré duality, we can identify the homology class $[c]$ with a de Rham cohomology class, and the above map is just an integration map.

The cycle c tells us the information of intersections of X and zero section(of the tangent bundle). If TM is trivial, then the zero section does not self-intersection(?) so that $c = 0$. The Euler characteristic measures the twist of a bundle, and the characteristic class generalizes this wakugumi.

2. Fiber bundles

From now we will only consider paracompact Hausdorff spaces. Recall that a space is paracompact iff for every open cover there is a locally finite refinement.

Example. Open sets of $\mathbb{R}^n$, metric spaces, CW-complexes, countable inductive limit of compact spaces are paracompact.

Theorem 2.1. *For any open cover of a paracompact Hausdorff space X, there is a partition of unity subordinate to it.*

Problem 2. Prove the above theorem.

Definition 2.2. Let B be connected(for simplicity). A map $E \to B$ is called a fiber bundle with fiber F, or just a F-bundle, if it is locally trivial: every point $x \in B$ has an open neighborhood U_x such that there is a homeomorphism $\varphi : p^{-1}(U_x) \to U_x \times F$ with $p = \mathrm{pr}_{U_x} \circ \varphi$.

For each $y \in B$ $E_y := p^{-1}(y)$ is homeomorphic to F, and is called the fiber at y. Also, E and B are called the total space and the base space. We somtimes write as $\xi = (F \to E \xrightarrow{p} B)$.

Example.

(a) We say $\mathrm{pr}_1 : B \times F \to B$ is the product or bundle.

(b) $p : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z} : t \mapsto [t]$ is a $\mathbb{Z}$-bundle. In general, a fiber bundle with a discrete fiber is called a covering space.

(c) $p_1 : S^n \to \mathbb{RP}^n = S^n/(x \sim -x)$ is a $\mathbb{Z}/2\mathbb{Z}$-bundle.

(d) $p : S^{2n+1} \to \mathbb{CP}^n = S^{2n+1}/(z \sim uz)$ for $u \in S^1$ is a S^1-bundle. (a generalization of Hopf bundles)

(e) Let M^n be a smooth manifold. Then, the tangent and the contangent bundles are $\mathbb{R}^n$-bundles.

Problem 3. Show that $p : S^{2n+1} \to \mathbb{CP}^n$ is a S^1-bundle by checking concretely its local triviality.

Definition 2.3. If F, E, B are C^r, $p : E \to B$ is C^r, and the local trivialzation is C^r, then we say the fiber bundle is C^r.

Definition 2.4. For $\xi_1 = (F \to E_1 \xrightarrow{p_1} B_1)$, $\xi_2 = (F \to E_2 \xrightarrow{p_2} B_2)$, a bundle map $\Phi = (\widetilde{f}, f) : \xi_1 \to \xi_2$ is a pair of maps $\widetilde{f} : E_1 \to E_2$ and $f : B_1 \to B_2$ such that $f \circ p_1 = p_2 \widetilde{f}$ and the restriction $\widetilde{f} : p_1^{-1}(b) \to p_2^{-1}(f(b))$ is a homeomorphism for every $b \in B$.

If both f and $\widetilde{f}$ are homeomorphisms, then Φ is called a bundle isomorphism. If a bundle is isomorphic to a product bundle, then it is called to be trivial.

Problem 4 For a bundle map Φ, is $\widetilde{f}$ homeomorphic if f is homeomorphic? (If we are doing in the category of smooth manifolds, then the inverse function theorem may be helpful.)

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Transition maps and structure groups

Let $\xi = (F \to E \xrightarrow{p} B)$ be an F-bundle. We have an open cover $\{U_\alpha\}$ such that for each α we have a local trivialization $p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$. For $U_\alpha \cap U_\beta \neq \varnothing$, we have a map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F,$$

by which we can define $\widetilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}(b, f) =: (b, \widetilde{g}_{\alpha\beta}(b, f))$. The map $\widetilde{g}_{\alpha\beta}$ is continuous, and we have for each b a homeomorphism

$$g_{\alpha\beta}(b) : F \to F : f \mapsto \widetilde{g}(b, f),$$

that is, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$. If we endow the compact-open topology on $\mathrm{Homeo}(F)$, then $g_{\alpha\beta}$ is continuous.

From definition, $g_{\alpha\beta}(b) \circ g_{\beta\alpha}(b) = \mathrm{id}_F$ for $b \in U_\alpha \cap U_\beta \neq \varnothing$, and $g_{\alpha\beta}(b) \circ g_{\beta\gamma}(b) = g_{\alpha\gamma}(b)$ for $b \in U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ (Note that the second relation implies the first.). The second condition is called the cocycle condition. The maps $\{g_{\alpha\beta}\}$ are called transition maps.

Theorem 2.5. *Let $\{U_\alpha\}$ be an open cover of a connected space B. Suppose we have a collection of continuous maps*

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)\}_{(\alpha,\beta):U_\alpha \cap U_\beta \neq \varnothing}$$

satisfying the cocycle condition.

(♠) Suppose also that F is locally compact, or there exists a topological transformation group G(i.e. G is a topological group such that the group action $G \times F \to F$ is continuous) with

$$\bigcup_{\alpha,\beta} g_{\alpha\beta}(U_\alpha \cap U_\beta) \subset G \subset \mathrm{Homeo}(F).$$

Then, there exists a unique F- bundle $(F \to E \xrightarrow{p} B$ such that it is locally trivializable over $\{U_\alpha\}$ and $\{g_{\alpha\beta}\}$ is the transition maps of the bundle.

The viewpoint of the above theorem is more likely to be the physicist's way of defining manifolds in the sense that they sometimes deifne a manifold as a collection of open subsets of a Euclidean space and transition maps between them.

The condition (♠) gaurantees for the second map in

$$\widetilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times \mathrm{Homeo}(F) \times F \to (U_\alpha \cap U_\beta) \times F$$

$$(b, f) \mapsto (b, g_{\alpha\beta}(b), f) \mapsto (b, g_{\alpha\beta}(f))$$

to be continuous.

Proof. (Sketch) Define

$$\widetilde{E} := \coprod U_\alpha \times F$$

and $E := \widetilde{E}/\sim$, where the equivalence relation $\sim$ is generated by: for each $(b_1, f_1) \in U_\alpha \times F$ and $(b_2, f_2) \in U_\beta \times F$ we have $(b_1, f_1) \sim (b_2, f_2)$ iff $b_1 = b_2$ and $f_1 = g_{\alpha\beta}(b_2)(f_2)$. Let $\pi : \widetilde{E} \to E$ be the canonical projection. Define also

$$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F : [(b, f) \in U_\alpha, F] \mapsto (b, f),$$

which are homeomorphisms by the assumption (♠), satisfying $\mathrm{pr}_1 \circ \varphi_\alpha = p$. □

For the second condition in (♠), G is called a structure group of the F-bundle. From now on, whenever we consider a fiber bundle along with a structure group G, we assume it includes the data of local trivialization.

Remark. We will always think of G for bundle maps between fiber bundles with structure group G. We will frequently consider the maximal transition data and compatible(i.e. satisfying the cocycle condition) local trivializations.

Example.

1. Let $F = V \cong \mathbb{R}^n$ be a real vector space, and $G \in \{\mathrm{GL}(V), \mathrm{SL}(V)\}$ or $G \in \{\mathrm{O}(V), \mathrm{SO}(V)\}$ with a fixed inner product on V. These fiber bundles are called real vector bundles.
2. Let $F = V \cong \mathbb{C}^n$ be a complex vector space, and $G \in \{\mathrm{GL}_{\mathbb{C}}(V)\}$ or $G \in \{\mathrm{U}(V)\}$ with a fixed inner product on V. These fiber bundles are called complex vector bundles.
3. $F = G$ be a Lie group. Then, G-bundle with structure group G is called a principal bundle.
4. Let F be a nice smooth manifold and $G = \mathrm{Diff}^{C^\infty}(F)$ be the group of smooth diffeomorphisms together with the Fréchet topology. Then, we have smooth F-bundles.

Definition 2.6. Let G be a structure group and B be a topological space. If an F-bundle $\xi = (F \to E \to B, G)$ and an F'-bundle $\xi = (F' \to E' \to B, G)$ has the same transition data, then they are called associated bundles.

Example. Let $F = \mathbb{R}^n$ be a real vector space with the standard inner product. Let $G = O(n)$. With $S^{n-1} \subset F$, the sphere bundle inside a real vector bundle is an associated bundle of the original real vector bundle. In particular for $n = 2$ and $G = \mathrm{SO}(2)$, then the circle bundle can be recognized as a principal $SO(2)$-bundle associated to a real plane bundle, and if we see the plane bundle as a complex line bundle, then it corresponds to a pricipal $U(1)$-bundle.

Proposition 2.7. *Let G be a topological group and $\xi = (G \to E \to B, G)$ be a principal G-bundle. Then, there is a natural right action of G on E which is free and the orbit space E/G is homeomorphic to B(transitively act on each fiber).*

Proof. Let $u \in E$ and φ_α a local trivialization containing u such that

$$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G : u \mapsto (p(u), h).$$

We can check the well-definedness of $ug = \varphi_\alpha^{-1}(p(u), hg)$ by

$$\varphi_\beta(ug) = \varphi_\beta \circ \varphi_\alpha^{-1}(p(u), hg) = (p(u), g_{\beta\alpha}(p(u))(hg)) = (p(u), h'g).$$

The right action of G on G is continuous, free, and transitive. The right action of G on E is continuous and free, and $\overline{p} : E/G \to B$ is continuous and bijective. □

Problem 5. Show that $\overline{p}^{-1}$ is also continuous.

Remark. A principal G-bundle may also be defined as follows: a G-bundle such that (1) there is a continuous free right action of G on E which is (2) fiber-preserving and fiberwise transitive, and (3) we can choose *G-equivariant* local trivialization such that $\varphi_\alpha(u) = (p(u), h)$ implies $\varphi_\alpha(ug) = (p(u), hg)$.

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Let G be a topological group. A pricipal G-bundle $(G \to E \to B, G)$ has a continuou free action of G on E.

Remark. For two principal G-bundles, $(\widetilde{f}, f)$ is a bundle map if and only if $\widetilde{f}$ is G-equivariant.

Definition 2.8. Let $\xi = (F \to E \xrightarrow{p} B)$ be a fiber bundle. A continuous map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$ is called a section or a cross section.

An important question asks if there is a section globally defined on the whole B.

Proposition 2.9. *Let $\xi = (G \to E \to B, G)$ be a principal G-bundle. Then, ξ is trivial if and only if it admits a global section.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let $s : B \to E$ be a global section. Define

$$\Phi : B \times G \to E : (b, g) \mapsto s(b)g.$$

Then, it is an G-equivariant isomorphism. $\square$

Let X be a right G-space which is free. Then, is X/G a principal G bundle? We have two problems:

(a) Is the inverse image(=orbit) of each point of X/G homeomorphic to G? No, the dynamics $\mathbb{T}^2 \curvearrowleft \mathbb{R}$ with irrational slope.

(b) Does it satisfy the local triviality? No, the translation $\mathbb{R} \leftarrow \mathbb{Q}$.

Proposition 2.10. *Let X be a right G-space which is free. The quotient map $\pi : X \to X/G$ defines a principal G-bundle if and only if $X \curvearrowleft G$ strongly freely(i.e. $X \times X \to G : (x, xg) \mapsto g$ is continuous) and there is a local trivialization for some $y \in X/G$.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$)

$$\pi^{-1}(U) \to U \times G : s(x)g \mapsto (x, g)$$

is continuous by the strongly free action. It defines local trivializations. $\square$

Theorem 2.11 (Gleason, 1950). *Let M be a smooth manifold and G a compact Lie group which gives a free right smooth action on M. Then, M/G is a smooth manifold such that $M \to M/G$ is a principal G-bundle.*

(Compactness of G implies the properness of the action, and smoothness implies the local triviality)

Corollary 2.12 (Samelson, 1941). *Let H be a compact Lie subgroup of a Lie group G. Then, $G \to G/H$ is a principal H-bundle. In fact, it is sufficient for H to be a closed subgroup of G, even if it is not compact.*

Example.

(a) With an action $S^{2n+1} \curvearrowleft S^1$ such that $(z_0, \cdots, z_n)w = (z_1 w, \cdots, z_n w)$, we have an S^1-bundle

$$S^{2n+1} \to \mathbb{CP}^n : (z_0, \cdots, z_n) \mapsto [z_0 : \cdots : z_n].$$

It is a general Hopf bundle.

(b) For $k \leq n$, the Stiefel variety is

$$V_k(\mathbb{R}^n) := \{M \in M_{n,k}(\mathbb{R}) : \mathrm{rk}\, M = k\}.$$

Also define

$$V_k^O(\mathbb{R}^n) := \{M \in V_k(\mathbb{R}^n) : \text{column vectors of } M \text{ are orthonormal}\}$$

and the Grassmannian manifold

$$G_k(\mathbb{R}^n) := \{k\text{-dimensional subspaces of } \mathbb{R}^n\}.$$

Stiefel varieties can be realized as principal bundles on Grassmannian manifolds.

With an action $V_k(\mathbb{R}^n) \curvearrowleft \mathrm{GL}(k, \mathbb{R})$ such that $(v_1, \cdots, v_k)X = (v_1X, \cdots, v_kX)$, we have $G_k(\mathbb{R}^n) \cong V_k(\mathbb{R}^n)/\mathrm{GL}(k, \mathbb{R})$ and $G_k(\mathbb{R}^n) \cong V_k^O(\mathbb{R}^n)/\mathrm{O}(k)$. Then, $(\mathrm{O}(k) \to V_k^O(\mathbb{R}^n) \to G_k(\mathbb{R}^n))$ and $(\mathrm{GL}(k, \mathbb{R}) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n))$ are principal bundles.

(c) As a complex version of (b), we have principal bundles $(\mathrm{U}(k) \to V_k^U(\mathbb{C}^n) \to G_k(\mathbb{C}^n))$ and $(\mathrm{GL}(k, \mathbb{C}) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n))$.

Theorem 2.13. *Let M be smooth manifold and suppose we have a transitive smooth left action of a Lie group G on M. Let H be* the *isotropy group. Then, $G/H \to M$ defines a diffeomorphism and $(H \to G \to M)$ is a principal bundle. Such M is called a homogeneous space.*

Example. With an action $\mathrm{SO}(n) \curvearrowright S^{n-1}$, since the isotropy group is isomorphic to $\mathrm{SO}(n-1)$, we have a principal bundle $SO(n-1) \to SO(n) \to S^n$.

We can also see the examples above(Grassmann and Steifel manifolds) as principal bundles on homogeneous spaces with a diffeomorphsim $\mathrm{O}(n-k) \setminus \mathrm{O}(n) \to V_k^O(\mathbb{R}^n) : [A] \mapsto (Ae_1, \cdots, Ae_k)$ and $\mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k) \cong G_k(\mathbb{R}^n)$: principal $\mathrm{O}(k)$-bundle

$$\begin{tikzcd}
\mathrm{O}(k) \arrow[r] & V_k^O(\mathbb{R}^n) \arrow[r, "\sim"] \arrow[d] & \mathrm{O}(n)/\mathrm{O}(n-k) \\
& G_k(\mathbb{R}^n) \arrow[r, "\sim"] & (\mathrm{O}(n)/\mathrm{O}(n-k))/\mathrm{O}(k).
\end{tikzcd}$$

We also have a complex version.

5 Day 5: May 8

Principal bundles and associated bundles

Let G be a topological group and $\xi = (G \to E \xrightarrow{p} B, G)$ be a principal G-bundle. Let $\{U_\alpha\}$ be an open cover of B. Let G effectively act on F from left as a transformation group, i.e. there is an injective group homomorphism $\sigma : G \to \mathrm{Homeo}(F)$ such that the action $G \times F \to F$ is continuous. Define

$$E \times_G F := E \times F/(eh, f) \sim (e, \sigma(h)f)$$

and

$$\pi : E \times_G F \to B : [e, f] \mapsto p(e).$$

This map is well-defined and continuous so that $\eta = (F \to E \times_G F \xrightarrow{\pi} B, G)$ is a fiber bundle with structure group G and fiber F.

In fact, if $\{g_{\alpha\beta}\}$ is the transition maps of ξ, then the transition maps of η are given by $\{\sigma \circ g_{\alpha\beta}\}$.

Conversely, let $\widetilde{\eta} = (F \to \widetilde{E} \to B, G)$ be a fiber bundle with structure group G and fiber F. If we construct principal G-bundle ξ with the transition data $\{g_{\alpha\beta}\}$ of $\widetilde{\eta}$, then η and $\widetilde{\eta}$ are isomorphic.

Remark. If $\sigma : G \to \mathrm{Homeo}(F)$ is not injective, then $\eta = (F \to E \times_G F \to B$ is a $G/\ker\sigma$-bundle with fiber F. It can be seen as a generalized version of assoicated bundles.

Example. Let M^n be a smooth manifold and $p : TM \to M$ be the tangent bundle with structure group $\mathrm{GL}(n, \mathbb{R})$. For each $x \in M$, consider

$$F_x := \{[v_1, \cdots, v_n] : \text{ordered bases of } T_xM\}$$

and $FM := \bigcup_{x\in M} F_x \curvearrowleft \mathrm{GL}(n, \mathbb{R})$. We call $FM \to M$ the tangent frame bundle.

Theorem (2.14)**.**

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{real vector bundles of rank } n \textit{ on } B \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{principal } \mathrm{GL}(n,\mathbb{R})\textit{-bundles on } B \end{array} \right\}.$$

Proof. Transition maps. □

Example. The tautological vector bundle γ_k is defined as $\mathbb{R}^k \to E_k \xrightarrow{\mathrm{pr}_1} G_k(\mathbb{R}^n)$, where

$$E_k := \{(W, p) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n : p \in W\}.$$

This is the vector bundle associated to the canonical principal $\mathrm{GL}(n, \mathbb{R})$-bundle on $G_k(\mathbb{R}^n)$.

Reduction of structure groups

Definition (2.15)**.** Let H be a closed subgroup G. We say the structure group of a G-bundle ξ with fiber F can be reduced to H if ξ is isomorphic to a H-bundle with fiber F. In other words, we have a collection of H-valued transition maps on an appropriately taken open cover on the base space.

Example.

(a) Let $H := \mathrm{Homeo}^+(F) \subset G := \mathrm{Homeo}(F)$. A bundle with fiber F is orientable if and only if the structure group can be reduced to H.

(b) Let $H := \mathrm{O}(n) \subset G := \mathrm{GL}(n, \mathbb{R})$. A vector bundle of rank n has a Euclidean metric(it is a Riemannian metric if smooth) if and only if the structure group of the associated principal G-bundle can be reduced to H.

($\Rightarrow$) Suppose a vector bundle ξ has a collection of $\mathrm{O}(n)$-valued transition maps on a sufficiently refiend open cover, and the local trivialization is written by $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$. Then, for $x, y \in E_b$ and $b \in U_\alpha \subset B$, the symmetric bilinear form

$$(x, y)_b := (\mathrm{pr}_2 \circ \varphi_\alpha(x), \mathrm{pr}_2 \circ \varphi_\alpha(y))_{\mathbb{R}^n}$$

is a well-defined inner product.

($\Leftarrow$) Suppose a Euclidean metric on a vector bundle ξ of rank n is given. Since $p^{-1}(U_\alpha) \to U_\alpha$ is trivial, we can take sections $(s_i)_{i=1}^n$ on U_α which are linearly independent at each point of U_α. Using the given Euclidean metric, we can apply the Gram-Schmidt algorithm to get another set of sections $(e_i)_{i=1}^n$ which form an orthonormal basis at each point of U_α. With these sections we can construct new local trivializations, having $\mathrm{O}(n)$-valued transition maps.

(Another remark) Since every vector bundle over a paracompact space B admits a Euclidean metric, the structure group of every principal $\mathrm{GL}(n, \mathbb{R})$-bundle can be reduced to $\mathrm{O}(n)$.

(c) For a complex version, a complex vector bundle of rank n admits a Hermitian metric if and only if the structure group $\mathrm{GL}(n, \mathbb{C})$ can be reduced to $\mathrm{U}(n)$. Similarly, the reduction is always possible if B is paracompact.

6 Day 6: May 15

3. Classification of principal bundles

Pullback bundles

Let $\xi = (F \to E \to B)$ and $f : A \to B$ a map. Then, the pullback bundle of ξ by f is $(F \to f^*E \to A)$, where $f^*E := \{(a,e) \in A \times F : f(a) = p(e)\}$. For example, if $\iota : A \hookrightarrow B$ is an inclusion, then the restriction is defined by the pullback $\xi|_A := \iota^*\xi = (F \to p^{-1}(A) \to A)$. For $i : \mathrm{Gt}_k(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_k(\mathbb{R}^n)$ with $k < n' < n$, if we define $E_{n,k} := \{(W,p) \in \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \mid p \in W\}$, then $i^*E_{n,k} = E_{n',k}$. If η is a pricnipal G-bundle and F is a left G-space, then $f^*(\eta \times_G F) = f^*\eta \times_G F$.

Lemma (3.1). *Let $\xi = (F \to E_1 \xrightarrow{p_1} B_1, G)$ and $\eta = (F \to E_2 \xrightarrow{p_2} B_2, G)$. For a map $f : B_1 \to B_2$, then a bundle map $(\widetilde{f}, f) : \xi \to \eta$ exists if and only if $\xi \cong f^*\eta$ over B_1.*

Proof. $(\Rightarrow)$

$$E_1 \to f^*E : x \mapsto (p_1(x), \widetilde{f}(x)).$$

$(\Leftarrow)$

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & f^*E_2 \arrow[rr] \arrow[dl] & & E_2 \arrow[d] \\ & B_1 \arrow[rrr, "f"] & & & B_2 \end{tikzcd}$$

□

Exercise. 6.

(1) Check that $f^*\eta$ over B_1 is an F-bundle which have G as the structure group.

(2) Fill the gap of the above proof of Lemma 3.1.

Theorem (3.2). *Let $\xi = (F \to E \to B, G)$ be a bundle and A be a paracompact space. If there is a homotopy $F : A \times I \to B : (x,t) \mapsto f_t(x)$, then $f_0^*\xi \cong f_1^*\xi$ over B_1.*

Corollary. *Let A, B be paracompact homotopy equivalent spaces. Then, there is a one-to-one correspondence between the sets of equivalence classes of bundles with fiber F and structure group G over A and B respectively.*

In the following lemmas, let $\xi = (F \to E \to A \times I, G)$ and A a paracompact space.

Lemma (Step 1 of 3.2). *If $\xi|_{A \times [a,b]}$ and $\xi|_{A \times [b,c]}$ are trivial, then $\xi|_{A \times [a,c]}$ is also trivial, where $0 \le a < b < c \le 1$.*

Proof. □

Lemma (Step 2 of 3.2). *Each point $a \in A$ has an open neighborhood U of a such that $\xi|_{U \times I}$ is trivial.*

Proof. □

Lemma (Step 3 of 3.2). *If we define $r : A \times I \to A \times I : (a,t) \mapsto (a,1)$, then $\xi \cong r^*\xi$.*

Proof. □

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The total space EG is always contractable.

Note that the base space BG need not be paracompact, but in most cases BG is paracompact, e.g. compact Lie groups and discrete groups.

Proposition (3.7)**.** *Let B_1, B_2 be paracompact spaces and ξ_1, ξ_2 be the universal principal G-bundles of B_1, B_2, respectively. Then, there is a homotopy equivalence $f : B_1 \to B_2$ such that $f^*\xi_2 = \xi_1$. In particular, BG is uniquely defined for a homotopy equivalence of paracompact spaces.*

Proof. By the universality, we have maps $f_1 : B_1 \to B_2$ and $f_2 : B_2 \to B_1$ such that $f_1^*\xi_2 = \xi_1$ and $f_2^*\xi_1 = \xi_2$. Then, $\mathrm{id}^*\xi_1 = \xi_1 = (f_2 \circ f_1)^*\xi$ implies $f_2 \circ f_1 \simeq \mathrm{id}$ by Theorem 3.4. □

Theorem (3.8)**.** *Let $\xi = (G \to E \to B, G)$ be a principal G-bundle on a paracompact space B. If E is contractible, then ξ is universal principal G-bundle.*

Proof. For a paracompact X, we want to show the correspondence $[X, B] \to \mathrm{Prin}_G(X) : [f] \mapsto f^*\xi$ is bijective.

(Surjectivity) Let $\eta = (G \to E_1 \to X, G)$ be a principal G-bundle on X. Define a left action $G \curvearrowright E$ such that $ge := eg^{-1}$ using the principal right action on E. Let $(\eta, \xi) = (E \to E_1 \times_G E \to X)$ be the associated bundle of η. Since E is contractible, it admits a global section by Theorem 3.9, which implies the existence of a bundle map $(\widetilde{f}, f) : \eta \to \xi$. Thus, $f^*\xi \cong \eta$.

Problem 8. For two principal G-bundles $\xi_i = (G \to E_i \to B_i, G)$ with $i = 1, 2$, show that there is a bundle map $(\widetilde{f}, f) : \xi_1 \to \xi_2$ if and only if the associated bundle $(\xi_1, \xi_2) := (E_2 \to E_1 \times_G E_2 \to B_1)$ has a global section.

(Injectivity) Suppose $f_0, f_1 : X \to B$ satisfies $f_0^*\xi \cong f_1^*\xi$. Then, there eixst bundle maps $(\widetilde{f_i}, f_i) : f_i^*\xi \to \xi$ and a bundle isomorphism $(\widetilde{h}, \mathrm{id}_X) : f_0^*\xi \to f_1^*\xi$. Define $\xi := (G \to f_0^*E \times [0,1] \xrightarrow{\pi\times\mathrm{id}} X \times [0,1], G)$, where $\pi : f_0^*\xi \to X$. Also define a partial bundle map $s : \zeta|_{X\times([0,\frac{1}{2})\cup(\frac{1}{2},1])} \to \xi$ such that

$$s(z,t) := \begin{cases} \widetilde{f_0}(z), & t < \frac{1}{2}, \\ \widetilde{f_1}(\widetilde{h}(z)), & t > \frac{1}{2} \end{cases}.$$

We can see s as a section of the associated bundle (ζ, ξ) on $X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1])$. By Theorem 3.9, there exists a section s' on $X \times [0,1]$ such that $s'|_{X\times\{0,1\}} = s|_{X\times\{0,1\}}$. The bundle map $S : \zeta \to \xi$ corresponding to s' induces a map $X \times [0,1] \to B$, and it is a homotopy between f_0 and f_1. □

Theorem (3.9)**.** *Let A be a closed subset of a paracompact Hausdorff space B, and C be a contractible space. Let $\zeta = (C \to E \to B)$ be a C-bundle. For a section $s : N \to E$ on an open neighborhood of N of A, there exists a section $S : B \to E$ such that $S|_A = s|_A$. In particular, if we take $A = N = \varnothing$, then $S : B \to E$ always exists.*

Proof. Uploaded on ITC-LMS. □

Example. Recall that $\xi = (\mathrm{O}(k) \to V_k^O(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n), \mathrm{O}(k))$ is a principal $\mathrm{O}(k)$-bundle, where

$$V_k^O(\mathbb{R}^n) = \{M \in M_{n\times k}(\mathbb{R}) \mid \mathrm{rk}(M) = k, \text{orthonormal columns}\} \cong \mathrm{O}(n)/\mathrm{O}(n-k).$$

Here, we refer to a well-known fact: $\mathrm{Gr}_k(\mathbb{R}^n)$ has a natural cell decomposition(Schubert) so that $\mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^{n+1})$ induced by the embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$ is a cellular map.

Let $\xi_{\mathrm{O}(k)} = (\mathrm{O}(k) \to V_k^O(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty), \mathrm{O}(k))$. (space of k-dimensional subspaces of a separable infinite dimensional Hilbert space..?)

Theorem (3.10)**.** *$\xi_{\mathrm{O}(k)}$ is a universal principal $\mathrm{O}(k)$-bundle.*

Proof. By Theorem 3.8, enough to show $V_k^O(\mathbb{R}^\infty)$ is contractible.

Define unilateral k-shift $f : \mathbb{R}^\infty \to \mathbb{R}^\infty$, where $\mathbb{R}^\infty = \bigcup_{n\geq 1} \mathbb{R}^n$. With the inner product on $\mathbb{R}^\infty$, we can check

$$h_t(v) := \mathrm{Schmidt}[(1-t)v_1 + tf(v_1), \cdots, (1-t)v_k + tf(v_k)]$$

is well-defined homotopy such that $h_0 = \mathrm{id}$ and $h_1(V_k^O(\mathbb{R}^\infty)) = V_k^O(\{0\}^k \times \mathbb{R}^\infty)$. We also have

$$h'_t : V_k^O(\{0\}^k \times \mathbb{R}^\infty) \to V_k^O(\mathbb{R}^\infty)$$

such that

$$h'_t(w_1, \cdots, w_k) \mapsto \mathrm{Schmidt}[te_1 + (1-t)w_1, \cdots, te_k + (1-t)w_k],$$

and by connecting them, we are done. □

With the same reasoning,

$$\xi_{\mathrm{GL}(k,\mathbb{R})} = (\mathrm{GL}(k,\mathbb{R}) \to V_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty), \mathrm{GL}(k,\mathbb{R}),$$
$$\xi_{\mathrm{U}(k)} = (\mathrm{U}(k) \to V_k^U(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty), \mathrm{U}(k),$$
$$\xi_{\mathrm{GL}(k,\mathbb{C})} = (\mathrm{GL}(k,\mathbb{C}) \to V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty), \mathrm{GL}(k,\mathbb{C})$$

are all universal.

Example. Let G be a compact Lie group with closed embedding $G \hookrightarrow \mathrm{U}(k)$. Then, the bundle

$$G \to V_k^U(\mathbb{C}^\infty) \to V_k^U(\mathbb{C}^\infty)/G$$

has a paracompact base space, hence it is universal since $V_k^U(\mathbb{C}^\infty)$ is contractible. If we write $EG = V_k^U(\mathbb{C}^\infty)$ and $BG = V_k^U(\mathbb{C}^\infty)/G$, then $U(k)/G \to BG \to BU(k)$ is a bundle.

In differential geoemtry, we sometimes stop to limit and take a sufficiently large but finite n when, for example, considering $V_k^U(\mathbb{C}^\infty)/G = \lim_{\to} V_k^U(\mathbb{C}^n)/G$.

Let G act F faithfully.

Definition (3.11)**.** For $\xi = (F \to E \to B, G)$ with B paracompact, if $c(\xi_1) = f^*(c(\xi_2)) \in H^*(B;R)$ for every bundle map $(\widetilde{f}, f) : \xi_1 \to \xi_2$, then c is called a *characteristic class* of bundles with structure group G and fiber F with coefficient R.

From definition, $\xi_1 \cong \xi_2$ implies $c(\xi_1) = c(\xi_2)$, and for trivial ξ, $c(\xi) = 0$.

Proposition (3.12)**.** *Let G be a topological group such that we can take paracompact BG. Then,*

$$\left\{ \begin{array}{c} \textit{characteristic classes} \\ \textit{of } G, F, R \end{array} \right\} = H^*(BG, R).$$

Let $a \in H^*(BG, R)$. For $\xi = (F \to E \to B, G)$, if we define $a(\xi) := f^*(a)$ for $f : B \to BG$, then $a(\xi)$ is a characteristic class.

9 Day 9: June 12

4. Characteristic classes of vector bundles

Let X be a paracompact space. Then,

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{rank } n\ \mathbb{K}\text{-vector bundles on } X \end{array} \right\} \longleftrightarrow [X, B\mathrm{GL}(n,\mathbb{K})].$$

Note that $BGL(n,\mathbb{K}) = \mathrm{Gr}_n(\mathbb{K}^\infty)$.

Example. Let $n = 1$, $\mathbb{K} = \mathbb{C}$. Since $G = \mathrm{GL}(1,\mathbb{C})$ and we can reduct it to $\mathrm{U}(1)$, which, in fact, has the following homeomorphism:

$$BG = B\mathrm{GL}(1,\mathbb{C}) \cong B\mathrm{U}(1) \cong \mathbb{CP}^\infty.$$

We will see later about these isomorphisms.

$$H^*(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z}, & * \text{ even} \\ 0, & * \text{ odd} \end{cases},$$

and we have a rign isomorphism $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[c_1]$, where $c_1 \in H^2(\mathbb{CP}^\infty)$ is the element given by the cocycle $e^2 \mapsto -1$. Here,

$$\mathbb{CP}^\infty = \bigcup_{n\geq 1} \mathbb{CP}^n = e_0 \cup e^2 \cup e^3 \cup \cdots$$

represents the CW structure.

Theorem (4.1)**.** *Let X be a CW complex. If we see $\mathrm{Vect}^1_{\mathbb{C}}(X)$ as an abelian group with tensor product, and if we denote by γ_1 the tautological bundle, then we have group isomorphisms*

$$\begin{array}{ccccc} \mathrm{Vect}^1_{\mathbb{C}}(X) & \xrightarrow{=} & [X, B\mathrm{U}(1)(=\mathbb{CP}^\infty)] & \xrightarrow{\cong} & H^2(X) \\ & & & & \\ f^*\gamma_1 & \longmapsto & f & \longmapsto & f^*(c_1) =: c_1(\xi). \end{array}$$

The element $c_1 \in H^2(\mathbb{CP}^\infty)$ is called the *first Chern class*, and also called the *Euler class* for oriented real plane bundles. We have

$$\mathrm{Vect}^1_{\mathbb{C}}(\mathbb{CP}^\infty) = [\mathbb{CP}^\infty, \mathbb{CP}^\infty] \cong H^2(\mathbb{CP}^\infty) \cong \mathbb{Z},$$

$$\gamma_1 \mapsto \mathrm{id} \mapsto c_1 \mapsto -1.$$

Basics of homotopy theory

Let (X, x_0) be a based space.

$$\pi_n(X, x_0) := [(S^n, b), (X, x_0)] = [(I^n, \partial I^n), (X, x_0)].$$

The homotopy group $\pi_n(X, x_0)$ is indeed given a group structure from the componentwise composition along $(I_n, \partial I_n)$. It is abelian for $n \geq 2$. By the obstruction theory, we have for $f : (S^n, b) \to (X, x_0)$ that $[f] = 0 \in \pi_n(X, x_0)$ if and only if it has an extension $D^{n+1} \to X$.

Eilenberg-MacLane complex

Definition (4.2)**.** Let G be a group and $n \in \mathbb{Z}$ with $n \geq 1$, such that G is abelian if $n \geq 2$. Then, a (homotopy class of a) topological space X which satisfies $\pi_n(X) = G$ and $\pi_i(X) = 0$ for $i \neq n$ is called the *Eilenberg-MacLane space* and denoted by $K(G, n)$.

Proposition (4.3)**.**

(a) *For every group G, there is $K(G, 1)$ space which is a CW complex.*

(b) *For every abelian group G and $n \geq 2$, there is $K(G, n)$ space which is a CW complex.*

(c) *The space $K(G, n)$ is unique up to homotopy.*

Example.

$$K(\{1\}, 1) = \{*\} = \text{any contractible spaces}, \quad K(\mathbb{Z}, 1) = S^1, \quad K(\mathbb{Z}^n, 1) = T^n, \quad K(F_n, 1) = \bigvee_{n \in \mathbb{N}} S^1.$$

There is a long exact sequence

$$\begin{tikzcd}
 & \mathbb{Z} & \mathbb{R} & S^1 \\
\pi_3 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_2 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_1 & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[dll] \\
\pi_0 & \mathbb{Z} & &
\end{tikzcd}$$

Since the universal cover of $\bigvee_{\mathbb{N}} S^1$, given by the Cayley graph, is contractible so that we have $\pi_i(\bigvee_{\mathbb{N}} S^1) = 0$ for $i \geq 2$.

$$K(\mathbb{Z}, 2) = \mathbb{CP}^\infty, \quad K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{RP}^\infty.$$

$$\begin{tikzcd}
 & S^1 & S^\infty & \mathbb{CP}^\infty \\
\pi_3 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_2 & 0 \arrow[r, hook] & 0 \arrow[r] & \mathbb{Z} \arrow[dll] \\
\pi_1 & \mathbb{Z} \arrow[r, hook] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_0 & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S^0 & S^\infty & \mathbb{RP}^\infty \\
\pi_3 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_2 & 0 \arrow[r, hook] & 0 \arrow[r] & 0 \arrow[dll] \\
\pi_1 & 0 \arrow[r, hook] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[dll] \\
\pi_0 & \mathbb{Z}/2\mathbb{Z} & &
\end{tikzcd}$$

Partial(?) proof of Proposition 4.3. For every B which is $K(G, 1)$, we have $\xi = (G \to \widetilde{B} \to B, G)$ for a discrete G and the universal cover B. The homotopy long exact sequence implies $\pi_i(\widetilde{B}) = \{1\}$ for all $i \geq 1$, then Whitehead theorem says that $\widetilde{B}$ is contractible. Therefore, ξ is a universal principal G-bundle, in particular,

$$K(G, 1) = BG.$$

Since BG is unique up to homotopy, so is $K(G, 1)$.

For the existence, we have two ways. For the first method, we can slightly modify the Milnor construction of BG. For the second method, if we wright $G = \langle g_\alpha \mid r_\beta \rangle_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$, then we can attach 2-cells on $X_1 := K(F_{\mathcal{A}}, 1)$ to construct $X_2 := \bigvee_{\alpha \in \mathcal{A}} S^1 \cup (\text{2-cells})$ such that each attaching corresponds to r_β. Then, X is a CW complex that satisfies $\pi_1(X) = G$. Put 3-cells and 4-cells and so on. $\square$

Example. How to construct $K(\mathbb{Z}/2\mathbb{Z}, 1)$ in the second method: Note that $\mathbb{Z}/2\mathbb{Z} = \langle x \mid x^2 \rangle$ and $X_1 = S^1$. Then, we can show $X_n = \mathbb{RP}^n = e^0 \cup \cdots \cup e^n$ using $\mathbb{Z}/2\mathbb{Z} \to S^n \to \mathbb{RP}^n$.

Problem. Investigate the proof of Proposition 4.3.

Theorem (4.4)**.** *Let G be an abelian group. For every CW complex X, we have a bijection*

$$[X, K(G,n)] \xrightarrow{\sim} H^n(X,G) : f \mapsto f^*(\iota).$$

Here,

$$H^n(K(G,n),G) \cong \mathrm{Hom}(H_n(K(G,n)),G) = \mathrm{Hom}(\pi_n(K(G,n)),G) = \mathrm{Hom}(G,G),$$

where the first isomorphism follows from the Hurewicz theorem. Fundamental class of $K(G,n)$.

Now we will show

$$c_1(\xi_1 \otimes \xi_2) = c_1(\xi_1) + c_1(\xi_2).$$

Suppose $g_i : X \to \mathbb{CP}^\infty$ are classfying maps of ξ_i, i.e. $g_i^*\gamma_1 = \xi_i$. Let $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be canonical projections. Then, $p_1^*\gamma_1 \otimes p_2^*\gamma_1$ is a line bundle on $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, let $h : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ be the classifying map, i.e. $h^*\gamma \cong p_1^*\gamma_1 \otimes p_2^*\gamma_1$. Consider

$$g : X \xrightarrow{\Delta} X \times X \xrightarrow{g_1 \times g_2} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{h} \mathbb{CP}^\infty.$$

Then,

$$\begin{aligned}
g^*\gamma_1 &= \Delta^*(g_1 \times g_2)^* h^*\gamma_1 \\
&\cong \Delta^*(g_1 \times g_2)^*(p_1^*\gamma_1 \otimes p_2^*\gamma_1) \\
&\cong \Delta^*(g_1 \times g_2)^* p_1^*\gamma_1 \otimes \Delta^*(g_1 \times g_2)^* p_2^*\gamma_1 \\
&= g_1^*\gamma_1 \otimes g_2^*\gamma_1 \\
&\cong \xi_1 \otimes \xi_2.
\end{aligned}$$

Therefore,

$$c_1(\xi_1 \otimes \xi_2) = c_1(g^*\gamma_1) = g^*(c_1(\gamma_1)) = \Delta^*(g_1 \times g_2)^* h^* c_1(\gamma_1).$$

Consider an isomorphism

$$H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \xrightarrow{i_1^* \oplus i_2^*} H^2(\mathbb{CP}^\infty \times \{p\}) \oplus H^2(\{p\} \times \mathbb{CP}^\infty).$$

such that $(i_1^* \oplus i_2^*)(p_1^* a + p_2^* b) = (a,b)$. Then, we have

$$\begin{aligned}
i_1^* h^* c_1(\gamma_1) &= c_1(i_1^* h^*\gamma_1) = c_1(i_1^*(p_1^*\gamma \otimes p_2^*\gamma 1)) \\
&= c_1(i_1^* p_1^*\gamma_1 \otimes i_1^* p_2^*\gamma_1) \\
&= c_1(\gamma_1 \otimes \mathbb{C}) = c_1(\gamma_1).
\end{aligned}$$

Similarly, $i_2^* h^* c_1(\gamma_1) \cong c_1(\gamma_1)$. Therefore, $h^* c_1(\gamma_1) = p_1^* c_1(\gamma_1) + p_2^* c_1(\gamma_1)$, and

$$\begin{aligned}
c(\xi_1 \otimes \xi_2) &= \Delta^*(g_1 \times g_2)^* h^* c_1(\gamma_1) \\
&= \Delta^*(g_1 \times g_2)^*(p_1^* c_1(\gamma_1) + p_2^* c_1(\gamma_1)) \\
&= g_1^* c_1(\gamma_1) + g_2^* c_1(\gamma_1) \\
&= c_1(g_1^*\gamma_1) + c_1(g_2^*\gamma_1) \\
&= c_1(\xi_1) + c_1(\xi_2).
\end{aligned}$$

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Theorem (4.5). *Let X be a CW complex. If we see $\mathrm{Vect}^1_{\mathbb{R}}(X)$ as an abelian group with tensor product, and if we denote by w_1 the tautological bundle, then we have group isomorphisms*

$$\begin{array}{ccccc} \mathrm{Vect}^1_{\mathbb{R}}(X) & \xrightarrow{=} & [X, BO(1)] & \xrightarrow{\cong} & H^1(X, \mathbb{Z}/2\mathbb{Z}) \\ \\ \xi & \longmapsto & f & \longmapsto & f^*(w_1) =: w_1(\xi). \end{array}$$

Here we note that $BO(1) = \mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$. If $w_1(\xi) \neq 0$, then from

$$w_1(\xi) \in H_1(X, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_1(X), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2\mathbb{Z}),$$

there is a loop γ on X such that $w_1(\gamma^*\xi) \neq 0$. It means that on the loop γ the line bundle is given by the Möbius strip.

Thom isomorphism

The Lie groups $\mathrm{SO}(n) \subset \mathrm{SL}(n, \mathbb{R}) \subset \mathrm{GL}^+(n, \mathbb{R})$ are all homotopy equivalent for $n \geq 1$. Let G be any of them. Let $\xi = (\mathbb{R}^n \to E \to B, G)$ be a vector bundle with paracompact B. Suppose ξ is oriented, i.e. there is a (maximal) collection of local trivializations $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ with $A_x \in G$ that is globally determined such that whenever $U_\alpha \cap U_\beta \neq \varnothing$ we have

$$\varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n : (x, v) \mapsto (x, A_x v).$$

An *orientation* on a n-dimensional real vector space V is defined as an element of

$$\{\text{ordered bases of } V\}/\mathrm{GL}^+(n, \mathbb{R}) = ((\bigwedge^n V) \setminus \{0\})/\mathbb{R}_{>0}.$$

Write $V_0 := V \setminus \{0\}$.

Let $\sigma : \Delta^n \hookrightarrow V$ be an orientation preserving embedding such that $\frac{1}{n+1}\sum_{i=0}^n e_i \mapsto 0 \in V$, where the orientation on $\Delta^n \subset \mathbb{R}^{n+1}$ is given by $\langle e_1 - e_0, \cdots, e_n - e_{n-1}\rangle$. Note that

$$H_n(V, V_0) \cong H_{n-1}(V_0) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Then $[\sigma] \in H_n(V, V_0)$ is a generator. Take a dual $U_V \in H^n(V, V_0) \cong \mathbb{Z}$ such that $\langle U_V, [\sigma]\rangle = 1$. In other words, $H^n(V, V_0)$ determines an orientation of V.

Let $\xi = (\mathbb{R}^n \to E \to B, \mathrm{GL}^+(n, \mathbb{R})$. Let $\mathbb{R}^n$ be a vector space oriented by $\langle e_1, \cdots, e_n\rangle$. Then, orientations on $E_x = p^{-1}(x)$ are induced for each $x \in B$ via arbitrarily taken local trivializations. The orientation does not depend on the choice of local trivializations in the structure group preserves the orientation.

Theorem (4.6, Thom's isomorphism theorem). *Let B be a paracompact, arc-connected space. There is a unique $t \in H^n(E, E_0)$, called the* Thom class, *such that*

$$H^n(E, E_0) \to H^n(E_x, E_{x,0}) : t \mapsto U_x.$$

Also, we have an isomorphism, called the Thom isomorphism,

$$H^j(E) \to H^{j+n}(E, E_0) : [\sigma] \mapsto [\sigma] \smile t$$

for each j. The above theorem holds for PID coefficients.

See also the Leray-Hirsch theorem. We prove the Thom isomorphism theorem in the following substeps:

(a) If

Proof. Suppose first that $E = B \times \mathbb{R}^n$. By the (weak) Künneth theorem,

$$H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n_0) \xrightarrow{\times} H^*(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$$

is a ring isomorphism Since $H^i(\mathbb{R}^n, \mathbb{R}^n_0) = 0$ for $i \neq n$ and $H^n(\mathbb{R}^n, \mathbb{R}^n_0) = \mathbb{Z}$, the above cross product means that

$$H^j(B) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n_0) \xrightarrow{\times} H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$$

is an isormorphism between abelian groups. We can take a generator $e^n \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ inductively by the following isomorphism:

$$H^{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}_0) \otimes H^1(\mathbb{R}, \mathbb{R}_0) \xrightarrow{\times} H^n(\mathbb{R}^n, \mathbb{R}^n_0) : e^{n-1} \otimes e_1 \mapsto e^n.$$

(Consider $H^*(X, A) \otimes H^*(Y, B) \xrightarrow{\times} H^*(X \times Y, (X \times B) \cup (A \times Y))$) Then, every morphism of the following diagram is an isomorphism:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & H^0(B) \arrow[r, "\times e^n"] & H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \\
\mathbb{Z} \arrow[r] & H^0(\{x_0\}) \arrow[r, "\times e^n"] & H^n(\{x_0\} \times \mathbb{R}^n, \{x_0\} \times \mathbb{R}^n_0).
\end{tikzcd}$$

Therefore $t = 1 \times e^n$ is unique element satisfying the theorem. The condition is indeed satisfied by checking the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
H^j(B) \otimes H^0(\mathbb{R}^n) \arrow[r] & H^j(B \times \mathbb{R}^n) \arrow[r, "\smile t"] & H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \\
H^j(B) \arrow[u] \arrow[urr, "\times e^n"] & & .
\end{tikzcd}$$

Now consider B is decomposed into $B = B_1 \cup B_2$, where the theorem holds on B_1 and B_2.

□

Definition (4.7)**.** Let $\xi = (\mathbb{R}^n \to E \to B, \mathrm{GL}^+(n, \mathbb{R}))$ be an oriented real vector bundle. We define the *Euler class* $e(\xi) \in H^n(B)$ such that

$$H^n(E, E_0) \to H^n(E) \xrightarrow{s^*, \cong} H^n(B) : t \mapsto e(\xi).$$

For the case $R = \mathbb{Z}/2\mathbb{Z}$, then the orientation is not necessary to be given on ξ, and then we can define the $\mathbb{Z}/2\mathbb{Z}$-coefficient Euler class.

Theorem (4.8)**.** *The Euler class is a characteristic class for oriented real vector bundles.*

Proof. The naturality of $H^n(E, E_0) \to H^n(B)$. The pullback of the Thom class, preseves fibers. □