Quantum field theory II

- conformal field theory in two dimensions

  " String Theory " by Joseph Polchinski
  (Cambridge University Press , 1998)
  Chapter 2

- application to string field theory

  " Analytic Methods in Open String Field Theory " ,
  by Yuji Okawa
  Prog. Theor. Phys. 128, 1001 - 1060 (2012)

§1 World-sheet actions
(Polchinski §1.2)

notation

the signature of the metric $(- + + \cdots +)$

$c = 1, \quad \hbar = 1$

relativistic point particle

in D flat spacetime dimensions

$$X^i(X^0) \qquad i = 1, \cdots, D-1$$

$$\rightarrow X^\mu(\tau) \qquad \mu = 0, 1, \cdots, D-1$$

τ ; parameter

reparameterization invariance

$$X'^\mu(\tau'(\tau)) = X^\mu(\tau)$$

action

$$S_{pp} = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

$$\dot{X}^\mu = \frac{dX^\mu}{d\tau}$$

$$\delta S_{pp} = -m \int d\tau \, \dot{u}_\mu \delta X^\mu$$

$$u^\mu = \frac{\dot{X}^\mu}{\sqrt{-\dot{X}^\nu \dot{X}_\nu}}$$

world-line metric $\gamma_{\tau\tau}(\tau)$

$$S'_{pp} = -\frac{1}{2}\int d\tau \sqrt{-\gamma}\,(\gamma^{\tau\tau}\dot{X}^{\mu}\dot{X}_{\mu} + m^2)$$

$$\gamma^{\tau\tau} = (\gamma_{\tau\tau})^{-1}$$

$$\gamma = \det \gamma_{\tau\tau} = \gamma_{\tau\tau}$$

$$\sqrt{-\gamma} = \sqrt{-\gamma_{\tau\tau}}$$

$$\sqrt{-\gamma}\,\gamma^{\tau\tau} = \sqrt{-\gamma_{\tau\tau}}\,(\gamma_{\tau\tau})^{-1} = -\frac{1}{\sqrt{-\gamma_{\tau\tau}}}$$

$$\eta(\tau) = \sqrt{-\gamma_{\tau\tau}(\tau)}$$

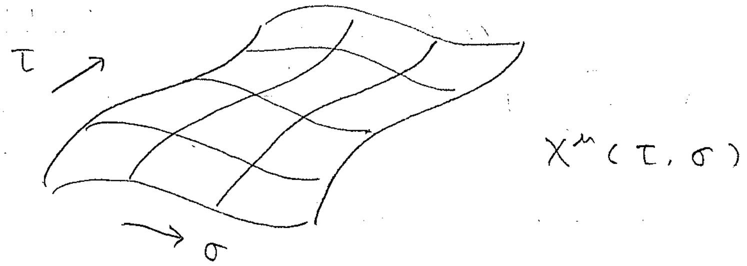
$$S'_{pp} = \frac{1}{2}\int d\tau\,(\eta^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - \eta m^2)$$

reparameterization invariance

$$\eta'(\tau')\,d\tau' = \eta(\tau)\,d\tau$$

$$\delta\eta \Rightarrow \quad \eta^2 = -\frac{\dot{X}^{\mu}\dot{X}_{\mu}}{m^2}$$

relativistic string



the Nambu - Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det h_{ab}}$$

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu \qquad a, b = (\tau, \sigma)$$

$$T = \frac{1}{2\pi\alpha'} \quad ; \text{ tension}$$

the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X_\mu$$

$$\gamma^{ab}\gamma_{bc} = \delta^a_c$$

$$\gamma = \det \gamma_{ab}$$

diffeomorphism invariance

(reparameterization invariance)

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma)$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma)$$

Weyl invariance

$$X'^\mu(\tau, \sigma) = X^\mu(\tau, \sigma)$$

$$\gamma'_{ab}(\tau, \sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau, \sigma)$$

gauge fixing $\Rightarrow$ γ_{ab} ; constant (locally)

(2021 5分休憩)

§ 2 Conformal field theory

§ 2.1 Massless scalars in two dimensions

$X^\mu(\sigma^1, \sigma^2)$: D free scalar fields

action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, (\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu)$$

σ^1, σ^2 : Euclidean metric

$X^0, X^1, \cdots, X^{D-1}$: Minkowski metric

complex coordinates

$$z = \sigma^1 + i\sigma^2, \qquad \bar{z} = \sigma^1 - i\sigma^2$$

$$\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$$

$$\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

v^a : vector

$$v^z = v^1 + iv^2, \qquad v^{\bar{z}} = v^1 - iv^2$$

$$v_z = \frac{1}{2}(v^1 - iv^2), \qquad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2)$$

metric

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0,$$

$$g^{z\bar{z}} = g^{\bar{z}z} = 2, \qquad g^{zz} = g^{\bar{z}\bar{z}} = 0$$

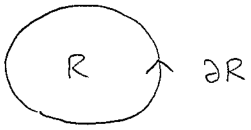
$$d^2z = 2d\sigma^1 d\sigma^2$$

$$\int d^2z \ \delta^2(z,\bar{z}) = 1$$

$$\delta^2(z,\bar{z}) = \frac{1}{2}\delta(\sigma^1)\delta(\sigma^2)$$

the divergence theorem

$$\int_R d^2z \ (\partial_z v^z + \partial_{\bar{z}} v^{\bar{z}}) = i\oint_{\partial R} (v^z d\bar{z} - v^{\bar{z}} dz)$$



$$S = \frac{1}{2\pi\alpha'}\int d^2z \ \partial X^\mu \bar{\partial} X_\mu$$

the equation of motion

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0$$

$\partial X^\mu(z)$: holomorphic

$\bar{\partial} X^\mu(\bar{z})$: antiholomorphic

(2015, 2016 5分休憩)

expectation values

$$\langle \mathcal{F}[X] \rangle = \int [dX] \ e^{-S} \mathcal{F}[X]$$

↑ path integral

$$0 = \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} e^{-S}$$

$$= - \int [dX] \, e^{-S} \frac{\delta S}{\delta X_\mu(z,\bar{z})}$$

$$= - \left\langle \frac{\delta S}{\delta X_\mu(z,\bar{z})} \right\rangle$$

$$= \frac{1}{\pi\alpha'} \left\langle \partial\bar{\partial} X^\mu(z,\bar{z}) \right\rangle$$

$$\langle \partial\bar{\partial} X^\mu(z,\bar{z}) \underset{\nwarrow}{\cdots} \rangle = 0$$

insertions

(no insertions at z)

$\partial\bar{\partial} X^\mu(z,\bar{z}) = 0$: operator equation.

$$0 = \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} \left[e^{-S} X^\nu(z',\bar{z}') \right]$$

$$= \int [dX] \, e^{-S} \Big[\eta^{\mu\nu} \delta^2(z-z', \bar{z}-\bar{z}') + \frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') \Big]$$

$$\frac{1}{\pi\alpha'} \partial_z \partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}') = -\eta^{\mu\nu} \delta^2(z-z', \bar{z}-\bar{z}')$$

holds as an operator equation. ┘

① 2021 10/8 (金)

normal ordering

$$:X^\mu(z,\bar{z}): \; = \; X^\mu(z,\bar{z})$$

$$:X^\mu(z_1,\bar{z}_1)\, X^\nu(z_2,\bar{z}_2):$$

$$= X^\mu(z_1,\bar{z}_1)\, X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu} \ln|z_1-z_2|^2$$

$$:X^{\mu_1}(z_1,\bar{z}_1)\, X^{\mu_2}(z_2,\bar{z}_2)\, X^{\mu_3}(z_3,\bar{z}_3):$$

$$= X^{\mu_1}(z_1,\bar{z}_1)\, X^{\mu_2}(z_2,\bar{z}_2)\, X^{\mu_3}(z_3,\bar{z}_3)$$

$$+\left(\frac{\alpha'}{2}\eta^{\mu_1\mu_2} \ln|z_1-z_2|^2\, X^{\mu_3}(z_3,\bar{z}_3)\right.$$

$$\left. + \text{2 permutations}\right)$$

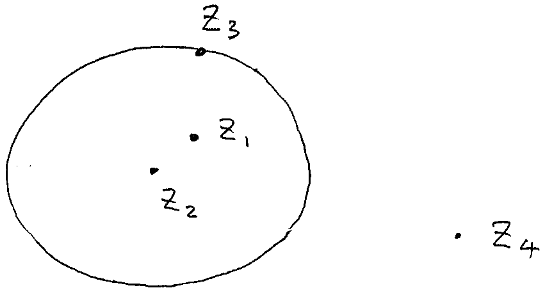
$$:\mathcal{F}: \; = \exp\left(\frac{\alpha'}{4}\int d^2z_1\, d^2z_2 \ln|z_1-z_2|^2 \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)} \frac{\delta}{\delta X_\mu(z_2,\bar{z}_2)}\right)\mathcal{F}$$

$$\partial\bar{\partial} \ln|z|^2 = 2\pi\, \delta^2(z,\bar{z})$$

$$\Rightarrow \quad \partial_{z_1}\partial_{\bar{z}_1} :X^\mu(z_1,\bar{z}_1)\, X^\nu(z_2,\bar{z}_2): \; = 0$$

§2.2 The operator product expansion (OPE)

$$\langle A_{i_1}(z_1, \bar{z}_1) \ \cdots \ A_{i_n}(z_n, \bar{z}_n) \rangle$$



$$A_i(\sigma_1) A_j(\sigma_2) = \sum_k C^k{}_{ij}(\sigma_1 - \sigma_2) A_k(\sigma_2)$$

$$\langle A_i(\sigma_1) A_j(\sigma_2) \cdots \rangle$$
$$= \sum_k C^k{}_{ij}(\sigma_1 - \sigma_2) \langle A_k(\sigma_2) \cdots \rangle$$

$$X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2)$$
$$= -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z_1 - z_2|^2 + :X^\nu X^\mu(z_2, \bar{z}_2):$$
$$+ \sum_{k=1}^{\infty} \frac{1}{k!} \Big[(z_1 - z_2)^k :X^\nu \partial^k X^\mu(z_2, \bar{z}_2):$$
$$+ (\bar{z}_1 - \bar{z}_2)^k :X^\nu \bar{\partial}^k X^\mu(z_2, \bar{z}_2): \Big]$$

① 2015 9/15 (火)
① 2016 9/27 (火)

$$:\partial X^{\mu}(z)\,\partial X_{\mu}(z):\;:\partial' X^{\nu}(z')\,\partial' X_{\nu}(z'):$$

$$= 2\,\eta^{\mu\nu}\eta_{\mu\nu}\left(-\frac{\alpha'}{2}\,\partial\partial'\ln|z-z'|^2\right)^2$$

$$+\,4\,\eta_{\mu\nu}\left(-\frac{\alpha'}{2}\,\partial\partial'\ln|z-z'|^2\right):\partial X^{\mu}(z)\,\partial' X^{\nu}(z'):$$

$$+\,:\partial X^{\mu}(z)\,\partial X_{\mu}(z)\,\partial' X^{\nu}(z')\,\partial' X_{\nu}(z'):$$

$$\sim\;\frac{D\alpha'^2}{2}\,\frac{1}{(z-z')^4}-\frac{2\alpha'}{(z-z')^2}:\partial' X^{\mu}(z')\,\partial' X_{\mu}(z'):$$

$$-\,\frac{2\alpha'}{z-z'}:\partial'^2 X^{\mu}(z')\,\partial' X_{\mu}(z'):$$

equal up to nonsingular terms

$$:e^{ik_1\cdot X(z,\bar z)}:\;:e^{ik_2\cdot X(0,0)}:$$

$$= \exp\left(\frac{\alpha'}{2}\,k_1\cdot k_2\,\ln|z|^2\right):e^{ik_1\cdot X(z,\bar z)}\,e^{ik_2\cdot X(0,0)}:$$

$$= |z|^{\alpha' k_1\cdot k_2}:e^{ik_1\cdot X(z,\bar z)}\,e^{ik_2\cdot X(0,0)}:$$

$$= |z|^{\alpha' k_1\cdot k_2}:e^{i(k_1+k_2)\cdot X(0,0)}\left[1+O(z,\bar z)\right]:$$

§ 2.3 Ward identities and Noether's theorem

$\phi_\alpha(\sigma)$ general fields in d dimensions

symmetry

$$[d\phi'] \, e^{-S[\phi']} = [d\phi] \, e^{-S[\phi]}$$

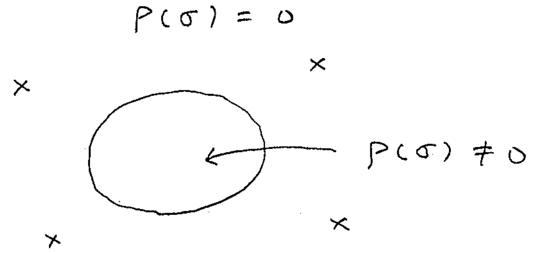
under

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \underset{\nwarrow \; O(\epsilon)}{\delta \phi_\alpha(\sigma)}$$

change of variables

$$\phi'_\alpha(\sigma) = \phi_\alpha(\sigma) + \rho(\sigma) \, \delta\phi_\alpha(\sigma)$$

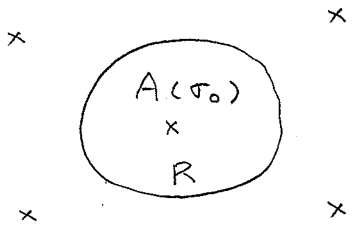
$$[d\phi'] \, e^{-S[\phi']}$$
$$= [d\phi] \, e^{-S[\phi]} \left[1 + \frac{i\epsilon}{2\pi} \int d^d\sigma \sqrt{g} \, j^a(\sigma) \, \partial_a \rho(\sigma) + O(\epsilon^2) \right]$$



$$0 = \int [d\phi'] \, e^{-S[\phi']} \cdots - \int [d\phi] \, e^{-S[\phi]} \cdots$$
$$= \frac{\epsilon}{2\pi i} \int d^d\sigma \sqrt{g} \, \rho(\sigma) \, \langle \nabla_a j^a(\sigma) \cdots \rangle$$

$\nabla_a j^a(\sigma) = 0$ as an operator equation

Noether's theorem



$$\rho(\sigma) = \begin{cases} 1 & \sigma \in R \\ 0 & \sigma \notin R \end{cases}$$

$$\delta A(\sigma_0) + \frac{\epsilon}{2\pi i} \int_R d^d\sigma \sqrt{g}\, \nabla_a j^a(\sigma) A(\sigma_0) = 0$$

$$\int_{\partial R} dA\, n_a j^a A(\sigma_0) = \frac{2\pi}{i\epsilon} \delta A(\sigma_0)$$

two flat dimensions

$$\oint_{\partial R} (j dz - \tilde{j} d\bar{z}) A(z_0, \bar{z}_0) = \frac{2\pi}{\epsilon} \delta A(z_0, \bar{z}_0)$$

$$j \equiv j_z, \quad \tilde{j} \equiv j_{\bar{z}}$$

When $\bar{\partial} j = 0, \quad \partial \tilde{j} = 0,$

$$\underset{z \to z_0}{\mathrm{Res}}\, j(z) A(z_0, \bar{z}_0) + \underset{\bar{z} \to \bar{z}_0}{\overline{\mathrm{Res}}}\, \tilde{j}(\bar{z}) A(z_0, \bar{z}_0)$$

coefficient of $\frac{1}{z - z_0}$ ↖   coefficient of $\frac{1}{\bar{z} - \bar{z}_0}$ ↗

$$= \frac{1}{i\epsilon} \delta A(z_0, \bar{z}_0)$$

examples

spacetime translation $\delta X^\mu = \epsilon a^\mu$

$$\delta S = \frac{\epsilon a_\mu}{2\pi\alpha'} \int d^2\sigma \ \partial^a X^\mu \partial_a \rho$$

$$j^\mu_a = \frac{i}{\alpha'} \partial_a X^\mu$$

$$j^\mu(z) :e^{ik\cdot X(0,0)}: \sim \frac{k^\mu}{2z} :e^{ik\cdot X(0,0)}:$$

$$\tilde{j}^\mu(\bar{z}) :e^{ik\cdot X(0,0)}: \sim \frac{k^\mu}{2\bar{z}} :e^{ik\cdot X(0,0)}:$$

world-sheet translation

$$\delta\sigma^a = \epsilon v^a$$

$$\delta X^\mu = -\epsilon v^a \partial_a X^\mu$$

$$j_a = i v^b T_{ab}$$

$$T_{ab} = -\frac{1}{\alpha'} :\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\delta_{ab} \partial_c X^\mu \partial^c X_\mu\right):$$

the world-sheet energy-momentum tensor

(2015, 2021 5分休憩)

§ 2.4 Conformal invariance

T_{ab} : traceless $\quad T_a{}^a = 0 \quad T_{z\bar{z}} = 0$

$\partial^a T_{ab} = 0 \quad \Rightarrow \quad \bar{\partial} T_{zz} = 0 , \quad \partial T_{\bar{z}\bar{z}} = 0$

$$T(z) \equiv T_{zz}(z), \qquad \tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$$

For the free massless scalar,

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu:$$

tracelessness $\Rightarrow$ larger symmetry

$$j(z) = i \underset{\substack{\uparrow \\ \text{holomorphic}}}{v(z)} T(z), \qquad \tilde{j}(\bar{z}) = i v(z)^* \tilde{T}(\bar{z})$$

$$T(z) X^\mu(0) \sim \frac{1}{z} \partial X^\mu(0), \quad \tilde{T}(\bar{z}) X^\mu(0) \sim \frac{1}{\bar{z}} \bar{\partial} X^\mu(0)$$

$$\Rightarrow \quad \delta X^\mu = -\epsilon v(z) \partial X^\mu - \epsilon v(z)^* \bar{\partial} X^\mu$$

infinitesimal coordinate transformation

$$z' = z + \epsilon v(z)$$

the finite transformation

$$X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}), \quad z' = \underset{\substack{\uparrow \\ \text{holomorphic}}}{f(z)}$$

conformal transformation

conformal invariance

conformal field theory (CFT)

Consider $z' = \zeta z$

ζ : complex rotation + rescaling

$$A'(z', \bar{z}') = \zeta^{-h} \bar{\zeta}^{-\tilde{h}} A(z, \bar{z})$$

$(h, \tilde{h})$: weights

$h + \tilde{h}$ dimension

$h - \tilde{h}$ spin

the conformal transformation of A

$$T(z) A(0,0) \sim \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^{(n)}(0,0)$$

$$T(z) A(0,0) = \cdots + \frac{h}{z^2} A(0,0) + \frac{1}{z} \partial A(0,0) + \cdots$$

primary field (tensor operator)

$$\mathcal{O}'(z', \bar{z}') = (\partial_z z')^{-h} (\partial_{\bar{z}} \bar{z}')^{-\tilde{h}} \mathcal{O}(z, \bar{z})$$

$$T(z) \mathcal{O}(0,0) = \frac{h}{z^2} \mathcal{O}(0,0) + \frac{1}{z} \partial \mathcal{O}(0,0) + \cdots$$

weights	$(h, \tilde{h})$
X^μ	$(0, 0)$
∂X^μ	$(1, 0)$
$\bar{\partial} X^\mu$	$(0, 1)$
$\partial^2 X^\mu$	$(2, 0)$
$:e^{ik\cdot X}:$	$\left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right)$

$\partial^2 X^\mu$ is not a primary field.

The OPE of the energy-momentum tensor with itself

$$T(z)\ T(0) \sim \frac{D}{2z^4} + \frac{2}{z^2}\,T(0) + \frac{1}{z}\,\partial T(0)$$

In general,

$$T(z)\ T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}\,T(0) + \frac{1}{z}\,\partial T(0)$$

c : central charge

$$(\partial_z z')^2\, T'(z') = T(z) - \frac{c}{12}\{z', z\}$$

$$\{f, z\} = \frac{2\partial^3_z f\, \partial_z f - 3\,\partial^2_z f\, \partial^2_z f}{2\,\partial_z f\, \partial_z f}$$

the Schwarzian derivative

② 2015 9/22 (火)
② 2016 10/4 (火)

§2.5 Free conformal field theories

linear dilaton CFT

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: + V_\mu \partial^2 X^\mu$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: + V_\mu \bar{\partial}^2 X^\mu$$

V_μ : constant D-vector

$$c = \tilde{c} = D + 6\alpha' V_\mu V^\mu$$

bc CFT

b, c : anticommuting fields

$$S = \frac{1}{2\pi} \int d^2 z \, b \bar{\partial} c$$

weights

$b : (\lambda, 0)$ $c : (1-\lambda, 0)$

$$\bar{\partial} c(z) = 0$$
$$\bar{\partial} b(z) = 0$$
$$\bar{\partial} b(z) c(0) = 2\pi \delta^2(z, \bar{z})$$

② 2021 10/15 (金)

$$\bar{\partial} \frac{1}{z} = \partial \frac{1}{\bar{z}} = 2\pi \delta^2(z, \bar{z})$$

$$:b(z_1) c(z_2): = b(z_1) c(z_2) - \frac{1}{z_1 - z_2}$$

The operator products

$$b(z_1)\, c(z_2) \sim \frac{1}{z_1 - z_2}$$

$$c(z_1)\, b(z_2) \sim \frac{1}{z_1 - z_2}$$

$$b(z_1)\, b(z_2) = O(z_1 - z_2)$$
$$c(z_1)\, c(z_2) = O(z_1 - z_2)$$

$$T(z) = :(\partial b)c: - \lambda \partial(:bc:)$$
$$\tilde{T}(\bar{z}) = 0$$
$$c = -3(2\lambda - 1)^2 + 1 , \quad \tilde{c} = 0$$

$$\left(\begin{array}{l} \tilde{b}\tilde{c} \ \text{CFT} \\ \qquad S = \frac{1}{2\pi} \int d^2z\, \tilde{b}\partial\tilde{c} \end{array} \right)$$

ghost number symmetry

$$\delta b = -i\epsilon b , \quad \delta c = i\epsilon c$$
$$j = - :bc:$$

$$T(z)\, j(0) \sim \frac{1-2\lambda}{z^3} + \frac{1}{z^2} j(0) + \frac{1}{z} \partial j(0)$$

$$(\partial_z z')\, j_{z'}(z') = j_z(z) + \frac{2\lambda - 1}{2} \frac{\partial_z^2 z'}{\partial_z z'}$$

§ 2.6 The Virasoro algebra

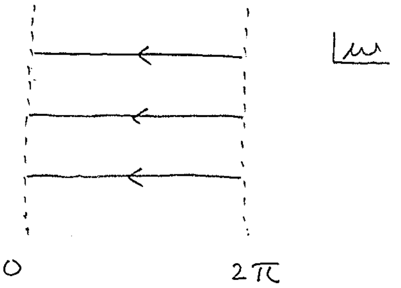
the closed string

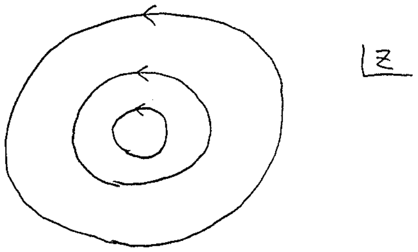
$$\sigma^1 \sim \sigma^1 + 2\pi \ , \quad -\infty < \sigma^2 < \infty$$

$\uparrow$ identification

$$w = \sigma^1 + i\sigma^2 \qquad w \sim w + 2\pi$$

$$z = e^{-iw} = e^{-i\sigma^1 + \sigma^2}$$





Laurent expansion

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

$$\tilde{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

Virasoro generators

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T_{zz}(z)$$

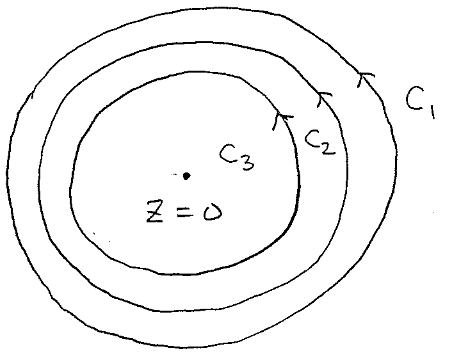
$$\left(T_{ww}(w) = -\sum_{m=-\infty}^{\infty} e^{im\sigma^1 - m\sigma^2} T_m \right)$$

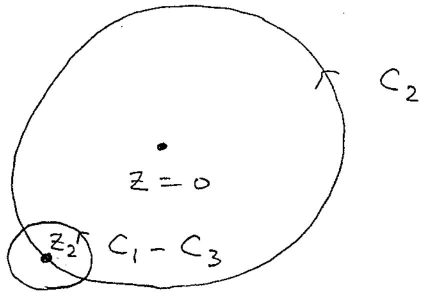
The algebra of charges

$$Q_i\{C\} = \oint_C \frac{dz}{2\pi i}\, j_i(z) \qquad i = 1, 2$$

bosonic

$$Q_1\{C_1\}\, Q_2\{C_2\} - Q_1\{C_3\}\, Q_2\{C_2\}$$





$$[Q_1, Q_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname*{Res}_{z_1 \to z_2} j_1(z_1)\, j_2(z_2)$$

$$[Q, A(z_2, \bar{z}_2)]$$

$$= \operatorname*{Res}_{z_1 \to z_2} j(z_1)\, A(z_2, \bar{z}_2) = \frac{1}{i\epsilon}\, S A(z_2, \bar{z}_2)$$

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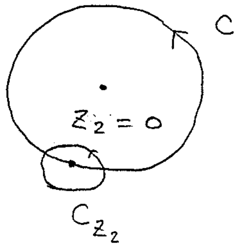
$$[L_m, L_n]$$

$$= \oint_C \frac{dz_2}{2\pi i} z_2^{n+1} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} z_1^{m+1} T(z_1) T(z_2)$$



$$T(z_1) T(z_2)$$

$$\sim \frac{c}{2} \frac{1}{(z_1 - z_2)^4} + \frac{2}{(z_1 - z_2)^2} T(z_2) + \frac{1}{z_1 - z_2} \partial T(z_2)$$

$$z_1^{m+1} = z_2^{m+1} + (m+1) z_2^m (z_1 - z_2)$$

$$+ \frac{m(m+1)}{2} z_2^{m-1} (z_1 - z_2)^2$$

$$+ \frac{m^3 - m}{6} z_2^{m-2} (z_1 - z_2)^3 + O((z_1 - z_2)^4)$$

$$[L_m, L_n]$$

$$= \oint_C \frac{dz_2}{2\pi i} z_2^{n+1} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} \Big[\frac{c}{12} (m^3 - m) z_2^{m-2} \frac{1}{z_1 - z_2}$$

$$+ 2(m+1) z_2^m T(z_2) \frac{1}{z_1 - z_2}$$

$$+ z_2^{m+1} \partial T(z_2) \frac{1}{z_1 - z_2} \Big]$$

$$= \oint_C \frac{dz_2}{2\pi i} \Big[\frac{c}{12} (m^3 - m) z_2^{m+n-1} + 2(m+1) z_2^{m+n+1} T(z_2)$$

$$+ \underbrace{z_2^{m+n+2} \partial T(z_2)}_{\shortparallel} \Big]$$

$$- (m+n+2) z_2^{m+n+1} T(z_2)$$

$$= \oint_C \frac{dz_2}{2\pi i} \Big[\frac{c}{12} (m^3 - m) z_2^{m+n-1} + (m-n) z_2^{m+n+1} T(z_2) \Big]$$

the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

(2015, 2016　5分休憩)

properties

$$[L_0, L_n] = -n L_n$$

$$L_0 |\varphi\rangle = h |\varphi\rangle$$

$$\Rightarrow L_0 L_n |\varphi\rangle = [L_0, L_n]|\varphi\rangle + L_n L_0 |\varphi\rangle$$

$$= -n L_n |\varphi\rangle + h L_n |\varphi\rangle = (h-n) L_n |\varphi\rangle$$

$SL(2, \mathbb{R})$

$$[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1},$$

$$[L_1, L_{-1}] = 2 L_0$$

$\mathcal{O}(z)$: a primary field of weight $(h, 0)$

↖ holomorphic

$$\mathcal{O}(z) = \sum_{m=-\infty}^{\infty} \frac{\mathcal{O}_m}{z^{m+h}}$$

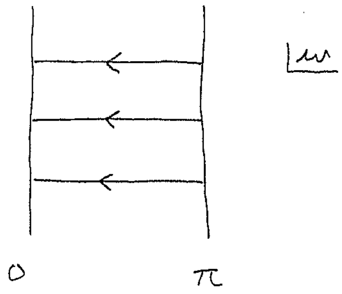
$$[L_m, \mathcal{O}_n] = [(h-1)m - n]\mathcal{O}_{m+n}$$

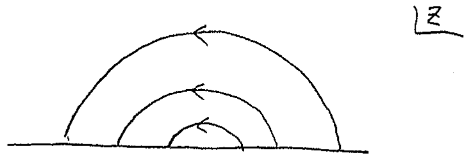
$$h = 1, \ n = 0 \Rightarrow [L_m, \mathcal{O}_0] = 0 \quad \lrcorner$$

③ 2016 10/11 (火)

the open string （2016年度 省略）（2021年度 省略）

$0 \leq \mathrm{Re}\, w \leq \pi$, $\mathrm{Im}\, z \geq 0$





$$T_{ab}\, \underset{\substack{\uparrow \\ \text{normal vector}}}{n^a}\, \underset{\substack{\nwarrow \\ \text{tangent vector}}}{t^b} = 0 \quad \text{at a boundary}$$

$$T_{ww} = T_{\bar{w}\bar{w}} \qquad \mathrm{Re}\, w = 0, \pi$$

$$T_{zz} = T_{\bar{z}\bar{z}} \qquad \mathrm{Im}\, z = 0$$

the doubling trick

$$T_{zz}(z) \equiv T_{\bar{z}\bar{z}}(\bar{z}') \qquad \mathrm{Im}\, z < 0 , \quad z' = \bar{z}$$

$$L_m = \frac{1}{2\pi i} \int_{\substack{c \\ \nwarrow \\ \text{semi-circle}}} (dz\, z^{m+1} T_{zz} - d\bar{z}\, \bar{z}^{m+1} T_{\bar{z}\bar{z}})$$

$$= \frac{1}{2\pi i} \oint dz\, z^{m+1} T_{zz}(z)$$

§ 2.7 Mode expansion

Free scalars

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}$$

$$\bar\partial X^\mu(\bar z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde\alpha_m^\mu}{\bar z^{m+1}}$$

$$\alpha_m^\mu = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} z^m \partial X^\mu(z)$$

$$\tilde\alpha_m^\mu = -i\sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar z}{2\pi i} \bar z^m \bar\partial X^\mu(\bar z)$$

single-valuedness of X^μ $\Rightarrow$ $\alpha_0^\mu = \tilde\alpha_0^\mu$

the spacetime momentum p^μ

$$p^\mu = \frac{1}{2\pi i}\oint \left(dz \frac{i}{\alpha'} \partial X^\mu(z) - d\bar z \frac{i}{\alpha'} \bar\partial X^\mu(\bar z) \right)$$

$$= \frac{1}{\sqrt{2\alpha'}}\alpha_0^\mu + \frac{1}{\sqrt{2\alpha'}}\tilde\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\tilde\alpha_0^\mu$$

$$X^\mu(z,\bar z) = x^\mu - \frac{i\alpha'}{2} p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde\alpha_m^\mu}{\bar z^m}\right)$$

③ 2015 9/29 (水)

③ 2021 10/22 (金)

commutation relations

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\,\delta_{m,-n}\,\eta^{\mu\nu}$$
$$[\tilde{\alpha}^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}] = m\,\delta_{m,-n}\,\eta^{\mu\nu}$$
$$[x^{\mu}, p^{\nu}] = i\,\eta^{\mu\nu}$$

$$\alpha^{\mu}_{n}\,|0;k\rangle = 0 \qquad n>0$$
$$\tilde{\alpha}^{\mu}_{n}\,|0;k\rangle = 0 \qquad n>0$$
$$p^{\mu}\,|0;k\rangle = k^{\mu}\,|0;k\rangle$$

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha^{\mu}_{m-n}\,\alpha_{\mu n} \qquad (m \neq 0)$$

$$L_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty}(\alpha^{\mu}_{-n}\,\alpha_{\mu n}) + \underset{\substack{\uparrow \\ \text{normal ordering constant}}}{a^{X}}$$

$$2L_0\,|0;0\rangle = (L_1 L_{-1} - L_{-1} L_1)\,|0;0\rangle = 0$$
$$\Rightarrow\quad a^{X} = 0$$

creation-annihilation normal ordering

lowering operators ; α^{μ}_{n}, $\tilde{\alpha}^{\mu}_{n}$, p^{μ}

raising operators ; α^{μ}_{-n}, $\tilde{\alpha}^{\mu}_{-n}$, x^{μ}

$(n>0)$

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha^{\mu}_{m-n}\,\alpha_{\mu n}:$$

$|z| > |z'|$

$$X^\mu(z, \bar{z})\, X^\nu(z', \bar{z}')$$

$$= {}^{\circ}_{\circ}\, X^\mu(z, \bar{z})\, X^\nu(z', \bar{z}')\, {}^{\circ}_{\circ} + \frac{\alpha'}{2}\eta^{\mu\nu}\left[-\ln|z|^2 + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{z'^m}{z^m} + \frac{\bar{z}'^m}{\bar{z}^m}\right)\right]$$

$$= {}^{\circ}_{\circ}\, X^\mu(z, \bar{z})\, X^\nu(z', \bar{z}')\, {}^{\circ}_{\circ} - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z - z'|^2$$

$$\underset{\uparrow}{{}^{\circ}_{\circ}}\, X^\mu(z, \bar{z})\, X^\nu(z', \bar{z}')\, {}^{\circ}_{\circ} = :X^\mu(z, \bar{z})\, X^\nu(z', \bar{z}')\underset{\uparrow}{:}$$

creation - annihilation normal ordering

conformal normal ordering

bc CFT

$$b(z) = \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}} , \qquad c(z) = \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}}$$

$$\{ b_m , c_n \} = \delta_{m,-n}$$

$|\downarrow\rangle$: $\quad b_n |\downarrow\rangle = 0 , \quad c_n |\downarrow\rangle = 0 \qquad n > 0$

$\qquad b_0 |\downarrow\rangle = 0$

$|\uparrow\rangle$: $\quad |\uparrow\rangle = c_0 |\downarrow\rangle$

creation - annihilation normal ordering

lowering operators : b_n , c_n , b_0

raising operators : b_{-n} , c_{-n} , c_0

$(n > 0)$

$$L_m = \sum_{n=-\infty}^{\infty} (m\lambda - n) :b_n c_{m-n}: + \delta_{m,0}\, a^g$$

$$2 L_0 |\downarrow\rangle = (L_1 L_{-1} - L_{-1} L_1) |\downarrow\rangle$$

$\qquad\qquad L_1 |\downarrow\rangle = 0$

$$= \lambda b_0 c_1 [(1-\lambda) b_{-1} c_0] |\downarrow\rangle$$

$$= \lambda(1-\lambda) |\downarrow\rangle$$

$$\Rightarrow a^g = \frac{\lambda(1-\lambda)}{2}$$

the ghost number

$$N^g = -\frac{1}{2\pi i}\int_0^{2\pi} dw\, j_w$$

the w - coordinate

$$j_w = - :bc:$$

$$= \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n) + c_0 b_0 - \frac{1}{2}$$

$$[N^g, b_m] = -b_m, \quad [N^g, c_m] = c_m$$

$$N^g|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle, \quad N^g|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$$

（2016、2021 5分 休憩）

open strings （2016年度省略）（2021年度省略）

the Neumann boundary condition

$$\partial_z X^\mu = \partial_{\bar{z}} X^\mu \qquad \mathrm{Im}\, z = 0$$

$$\Rightarrow \quad \alpha_m^\mu = \tilde{\alpha}_m^\mu$$

$$\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

↖ the spacetime momentum

$$X^\mu(z, \bar{z}) = x^\mu - i\alpha' p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{\alpha_m^\mu}{m} \left(\frac{1}{z^m} + \frac{1}{\bar{z}^m} \right)$$

the bc theory

$$c(z) = \tilde{c}(\bar{z}), \quad b(z) = \tilde{b}(\bar{z}) \qquad \mathrm{Im}\, z = 0$$

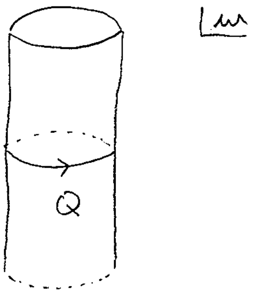
the doubling trick

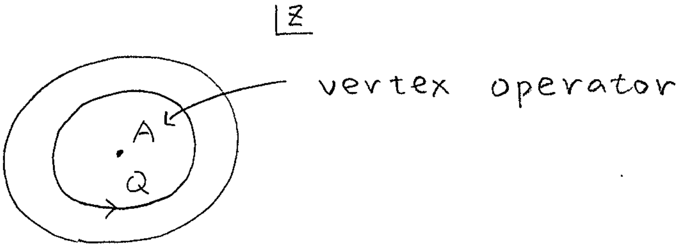
$$c(z) \equiv \tilde{c}(\bar{z}'), \quad b(z) \equiv \tilde{b}(\bar{z}')$$

$$\mathrm{Im}\, z < 0, \quad z' = \bar{z}$$

（2015 5分休憩）

§ 2.8 Vertex operators





$|A\rangle \leftarrow$ the initial state

the state-operator correspondence

the space of states $\xleftrightarrow[\text{isomorphism}]{}$ the set of local operators

What is $|1\rangle$?

1 : the unit operator

$$\alpha^\mu_m |1\rangle \longleftrightarrow i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} z^m \partial X^\mu(z)$$

$$\alpha^\mu_m |1\rangle = 0 \ , \quad \tilde{\alpha}^\mu_m |1\rangle = 0 \quad \text{for } m \geqq 0$$

$$\Rightarrow \quad |1\rangle = |0;0\rangle$$

We use this to fix the normalization of $|0;0\rangle$.

$$\alpha^\mu_{-m} |1\rangle \cong i\sqrt{\frac{2}{\alpha'}} \frac{1}{(m-1)!} \partial^m X^\mu(0) \qquad m \geqq 1$$

$$|0;k\rangle \cong \, :e^{ik\cdot X(0,0)}:$$

the bc theory with $\lambda = 2$

$$b_m |1\rangle = 0 \qquad m \geqq -1$$

$$c_m |1\rangle = 0 \qquad m \geqq 2$$

$$\Rightarrow \quad |1\rangle = b_{-1} |\downarrow\rangle$$

$$|\downarrow\rangle \cong c(0)$$

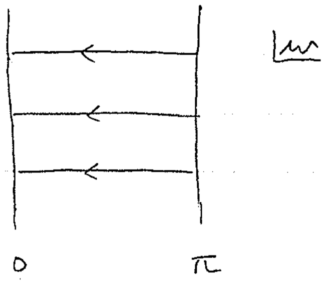
the ghost number in the z-coordinate

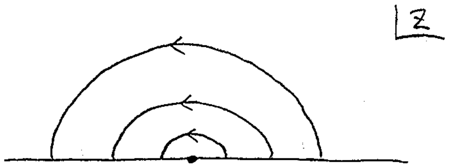
$$Q^g \equiv \oint \frac{dz}{2\pi i} j_z = N^g + \frac{3}{2}$$

$$\frac{2\lambda - 1}{2} \frac{\partial_z^2 w}{\partial_z w} = -\frac{3}{2}\frac{1}{z} \quad \nearrow$$

We usually use Q^g rather than N^g
not only for vertex operators
but also for states.

the open string （2016年度 省略）（2021年度 省略）

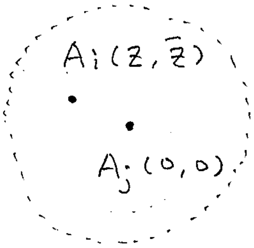


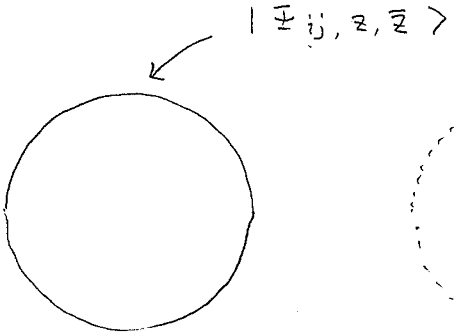


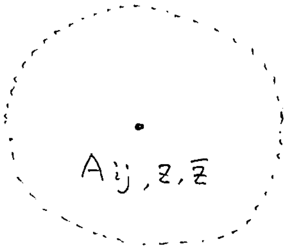
States on the interval $\longleftrightarrow$ isomorphism local operators on the boundary

§ 2.9 More on states and operators

the OPE

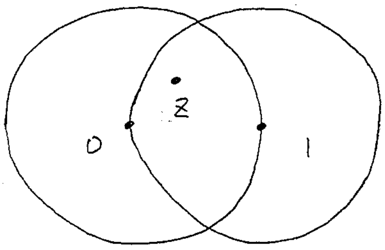






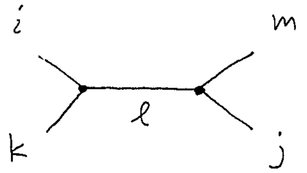
$$A_{ij,z,\bar z} = \sum_k z^{h_k - h_i - h_j}\, \bar z^{\tilde h_k - \tilde h_i - \tilde h_j}\, C^k{}_{ij}\, A_k$$

associativity

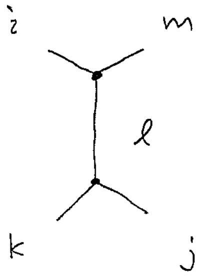


$A_i(0,0)\ A_j(1,1)\ A_k(z,\bar z)$

$\sum_\ell$



$=$ $\sum_\ell$



$c^\ell{}_{ik}\, c^m{}_{\ell j}\, A_m$ $\qquad$ $c^\ell{}_{jk}\, c^m{}_{\ell i}\, A_m$

the Virasoro algebra and highest weight states

$$|A\rangle \cong A(0,0)$$

$$L_m|A\rangle \cong L_m \cdot A(0,0)$$

$$\equiv \oint \frac{dz}{2\pi i} z^{m+1} T(z) A(0,0)$$

$$L_{-1}\cdot A = \partial A, \quad \tilde{L}_{-1}\cdot A = \bar{\partial} A$$

$$L_0 \cdot A = hA, \quad \tilde{L}_0 \cdot A = \tilde{h} A$$

$$|\phi\rangle \cong \phi(0,0)$$

↖ a primary field of weight $(h, \tilde{h})$

$$L_0|\phi\rangle = h|\phi\rangle, \quad \tilde{L}_0|\phi\rangle = \tilde{h}|\phi\rangle$$

$$L_m|\phi\rangle = 0, \quad \tilde{L}_m|\phi\rangle = 0 \qquad m > 0$$

highest weight state

$$L_m|1\rangle = 0, \quad \tilde{L}_m|1\rangle = 0 \qquad m \geqq -1$$

$|1\rangle$: the $SL(2,\mathbb{C})$ - invariant state

↖ L_1, L_0, L_{-1}

$\tilde{L}_1, \tilde{L}_0, \tilde{L}_{-1}$

(the $SL(2,\mathbb{C})$ - invariant vacuum)

④ 2015 10/6 (金)

④ 2016 10/18 (火)

④ 2021 10/29 (金)

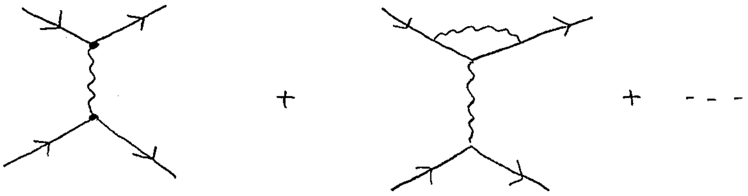
§3 Analytic methods in open string field theory

§3.1 The basics of open string field theory

Quantum field theory

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \not{D} \Psi + \cdots \right]$$

$$\big\downarrow \text{ perturbation theory}$$

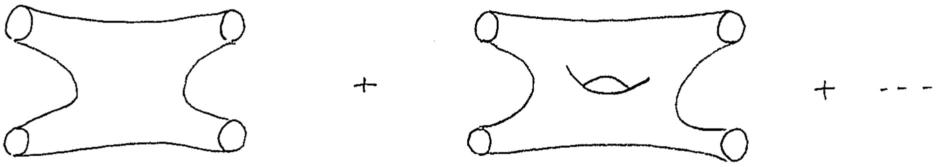


Feynman diagrams

String theory

?

$$\big\downarrow$$



perturbation theory
for on-shell scattering amplitudes
independently defined for each
consistent background

consistent perturbation theory
including gravity

Various approaches to nonperturbative formulations

matrix models
the AdS/CFT correspondence
string field theory

the 1/N expansion

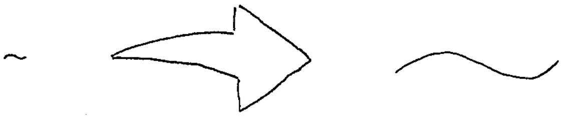
$\boxed{\text{open}}$ string

closed string

$\boxed{\text{bosonic}}$ string

superstring

Open bosonic string theory
flat spacetime in 26 dimensions
Neumann boundary conditions



ground state
tachyonic scalar field $T(k)$

$$\text{mass}^2 = -\frac{1}{\alpha'} \quad \left(\text{tension} = \frac{1}{2\pi\alpha'} \right)$$

first-excited states
massless vector field $A_\mu(k)$

higher-excited states
massive

$$\text{mass}^2 = \frac{1}{\alpha'}, \; \frac{2}{\alpha'}, \; \frac{3}{\alpha'}, \; \cdots$$

Degrees of freedom of string field theory

$$\{ T(k),\ A_\mu(k),\ \cdots \}$$

$$\downarrow$$

$$S[\ T(k),\ A_\mu(k),\ \cdots\ ]$$

The free theory

$$S = - \int \frac{d^{26}k}{(2\pi)^{26}} \Big[\frac{1}{2} T(-k) (k^2 - \frac{1}{\alpha'}) T(k) + \frac{1}{2} A_\mu(-k) (k^2 \eta^{\mu\nu} - k^\mu k^\nu) A_\nu(k) \Big]$$

or

$$S = - \int \frac{d^{26}k}{(2\pi)^{26}} \Big[\frac{1}{2} T(-k) (k^2 - \frac{1}{\alpha'}) T(k) + \frac{1}{2} A_\mu(-k)\, k^2 A^\mu(k) + i B(-k)\, k^\mu A_\mu(k) + \frac{1}{2} B(-k) B(k) \Big]$$

the equations of motion

$$\begin{cases} (k^2 - \frac{1}{\alpha'}) T(k) = 0 \\ k^2 A_\mu(k) - i k_\mu B(k) = 0 \\ B(k) + i k^\mu A_\mu(k) = 0 \end{cases}$$

the gauge transformations

$$\begin{cases} \delta_\Lambda A_\mu(k) = i k_\mu \Lambda(k) \\ \delta_\Lambda B(k) = k^2 \Lambda(k) \end{cases}$$

SU(2) gauge fields $\qquad$ a single 2 × 2 matrix field

$$A_\mu^a(x),\ a = 1, 2, 3 \longrightarrow A_\mu(x) = \frac{1}{2}\sum_{a=1}^{3} A_\mu^a(x)\,\sigma^a$$

$\{T(k),\ A_\mu(k),\ B(k),\ \cdots\} \rightarrow$ string field Ψ

= a state in the matter + bc ghost CFT
of ghost number 1

$$\Psi = \int \frac{d^{26}k}{(2\pi)^{26}} \Big[\frac{1}{\sqrt{\alpha'}} T(k)\, c_1 |0;k\rangle + \frac{1}{\sqrt{\alpha'}} A_\mu(k)\, \alpha_{-1}^\mu c_1 |0;k\rangle + \frac{i}{\sqrt{2}} B(k)\, c_0 |0;k\rangle + \cdots \Big]$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\,\eta^{\mu\nu}\,\delta_{n+m,0}$$

$$\{b_n, c_m\} = \delta_{n+m,0}$$

$$\alpha_n^\mu |0;k\rangle = 0 \quad \text{for } n > 0$$

$$b_n |0;k\rangle = 0 \quad \text{for } n > -2$$

$$c_n |0;k\rangle = 0 \quad \text{for } n > 1$$

$$p_\mu |0;k\rangle = k_\mu |0;k\rangle$$

the equations of motion : $Q_B \Psi = 0$

$\qquad\qquad\qquad\qquad$ ↖ the BRST operator

$$Q_B = \sum_{n=-\infty}^{\infty} c_n \overset{\text{matter}}{L_{-n}^{(m)}} + \frac{1}{2}\sum_{n,m=-\infty}^{\infty} (m-n) :c_m c_n b_{-m-n}: - c_0$$

$$:c_n b_{-n}: = \begin{cases} c_n b_{-n} & \text{for } n \leq 0 \\ -b_{-n} c_n & \text{for } n > 0 \end{cases}$$

$$L_n^{(m)} = \frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_m^\mu \alpha_{n-m}^\nu:\, \eta_{\mu\nu}, \qquad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

the gauge transformations : $\delta_\Lambda \Psi = Q_B \Lambda$

$$\Lambda = \frac{i}{\sqrt{2}\,\alpha'} \int \frac{d^{26}k}{(2\pi)^{26}} \Lambda(k) |0;k\rangle$$

(2015, 2016 5分休憩)

the action

$$S = -\frac{1}{2} \langle \Psi, Q_B \Psi \rangle$$

$\langle A, B \rangle$: the BPZ inner product

$$|0;k\rangle \rightarrow \langle 0;k| \quad \left[\begin{array}{l} \text{not } \langle 0;-k| \\ \text{no complex conjugation} \end{array} \right]$$

$$\mathcal{O}|A\rangle \rightarrow \begin{cases} \langle A|\mathcal{O}^{\star} & \mathcal{O}: \text{Grassmann even} \\ (-1)^{A} \langle A|\mathcal{O}^{\star} & \mathcal{O}: \text{Grassmann odd} \end{cases}$$

$|0;k\rangle$: Grassmann even

$$(\alpha_n^\mu)^{\star} = (-1)^{n+1} \alpha_{-n}^{\mu}$$

$$(c_n)^{\star} = (-1)^{n+1} c_{-n}$$

$$(b_n)^{\star} = (-1)^{n} b_{-n}$$

$$\langle 0;k| c_{-1} c_0 c_1 |0;k'\rangle = (2\pi)^{26} \delta^{(26)}(k+k')$$

$$\langle A, B \rangle = \langle A|B \rangle$$

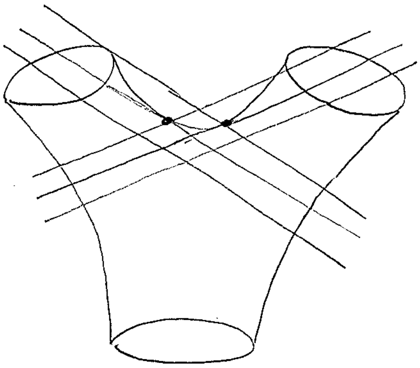
$$\langle \mathcal{O}A, B \rangle = \begin{cases} \langle A, \mathcal{O}^{\star} B \rangle & \mathcal{O}: \text{Grassmann even} \\ (-1)^{A} \langle A, \mathcal{O}^{\star} B \rangle & \mathcal{O}: \text{Grassmann odd} \end{cases}$$

important properties

$$\begin{cases} \langle A, B \rangle = (-1)^{AB} \langle B, A \rangle \\ \langle Q_B A, B \rangle = -(-1)^{A} \langle A, Q_B B \rangle \quad (Q_B^{\star} = -Q_B) \\ Q_B^2 = 0 \end{cases}$$

(2021　5分休憩)

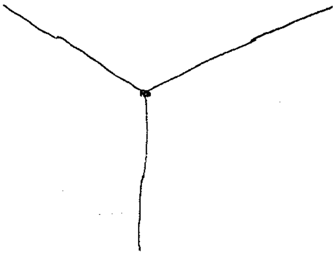
The interacting theory



No Lorentz-invariant interaction point

⇓

The form of the interactions is uniquely determined.



Many interacting theories for a given free theory

U(N) gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda$$

$$\rightarrow \delta A_\mu = \partial_\mu \Lambda + i (\Lambda A_\mu - A_\mu \Lambda)$$

⑤ 2016 10/25 (火)

Open bosonic string field theory

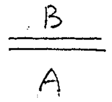
Witten, Nucl. Phys. B268 (1986) 253

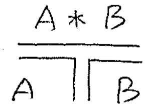
$$S = -\frac{1}{\alpha'^3 g_T^2}\left[\frac{1}{2}\langle \Psi, Q_B \Psi\rangle + \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle\right]$$

↗ the coupling constant ↑ Witten's star product

$$\left[\; g_T = g_o'\sqrt{\frac{2}{\alpha'}} = \frac{g_o}{\alpha'} \;\right]$$

$\langle A, B\rangle \sim$





noncommutative

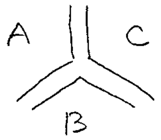
$A * B \neq B * A$ ┘

⑤ 2015 10/13 (火)

analogy

String field	↔	matrix
A		A_{ij}
$A * B$		$(AB)_{ij} = A_{ik}B_{kj}$
$\langle A, B\rangle$		$\mathrm{tr}\, AB = A_{ij}B_{ji}$ ┘

⑤ 2021 11/5 (金)



$$\langle A, B * C\rangle = \langle A * B, C\rangle$$

$$Q_B(A * B) = Q_B A * B + (-1)^A A * Q_B B$$

$$(A * B) * C = A * (B * C)$$

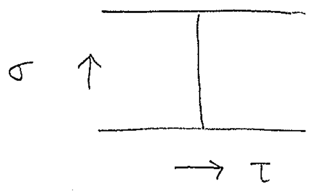
$$Q_B\Psi + \Psi * \Psi = 0$$

$$\delta_\Lambda \Psi = Q_B\Lambda + \Psi * \Lambda - \Lambda * \Psi$$

$\delta_\Lambda S = 0$ Chern-Simons-like

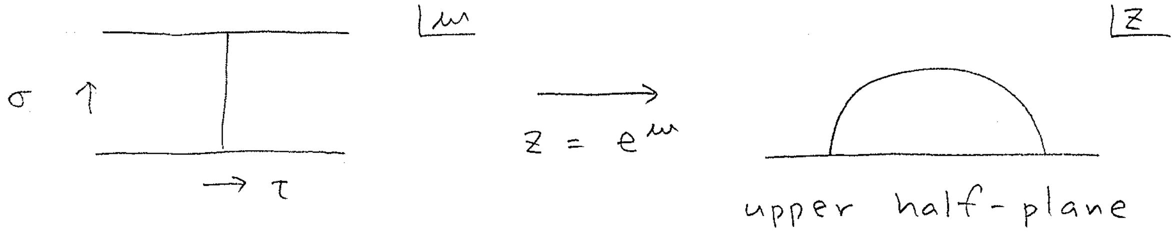
CFT description

string field = state in a boundary CFT



conformal transformations

$$z = x + iy \longrightarrow z' = x' + iy' = f(z)$$



$w = \tau + i\sigma$

upper half-plane
(UHP)

$$\partial\bar{\partial} X^\mu(z,\bar{z}) = 0 \qquad (\partial \equiv \partial_z, \quad \bar{\partial} \equiv \partial_{\bar{z}})$$

$$\bar{\partial} b(z) = 0, \qquad \partial \tilde{b}(\bar{z}) = 0$$

$$\bar{\partial} c(z) = 0, \qquad \partial \tilde{c}(\bar{z}) = 0$$

boundary conditions at $\operatorname{Im} z = 0$

$$\partial X^\mu(z) = \bar{\partial} X^\mu(\bar{z}), \quad b(z) = \tilde{b}(\bar{z}), \quad c(z) = \tilde{c}(\bar{z})$$

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\alpha_n^\mu}{n}\left(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\right)$$

$$b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+2}}, \qquad \tilde{b}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{b_n}{\bar{z}^{n+2}}$$

$$c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}, \qquad \tilde{c}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{c_n}{\bar{z}^{n-1}}$$

the doubling trick

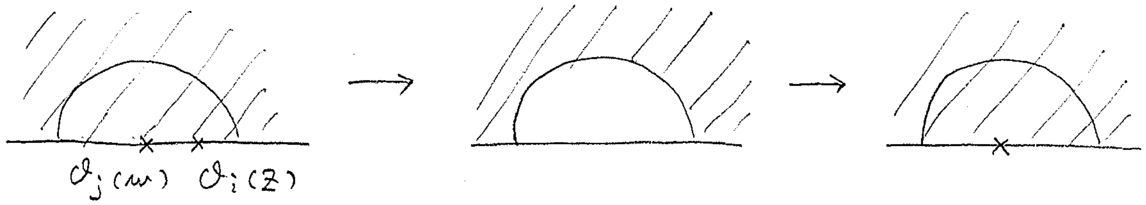
$$\left\{\begin{array}{l} \partial X^\mu(z) \equiv \bar\partial X^\mu(\bar z') \\ b(z) \equiv \tilde b(\bar z') \\ c(z) \equiv \tilde c(\bar z') \end{array}\right. \qquad \text{for } \operatorname{Im} z < 0,\ z' = \bar z$$

$$\alpha_n^\mu = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i} z^n \partial X^\mu(z) \qquad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

$$b_n = \oint \frac{dz}{2\pi i} z^{n+1} b(z)$$

$$c_n = \oint \frac{dz}{2\pi i} z^{n-2} c(z)$$

the operator product expansion (OPE)



$$\mathcal{O}_i(z)\, \mathcal{O}_j(w) = \sum_k c^k{}_{ij}(z-w)\, \mathcal{O}_k(w)$$

$$\partial X^\mu(z)\, \partial X^\nu(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z-w)^2} + :\partial X^\mu(z)\, \partial X^\nu(w):$$

$$= -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z-w)^2} + \sum_{k=0}^{\infty}\frac{1}{k!}(z-w)^k :\partial^{k+1} X^\mu \partial X^\nu:(w)$$

$$b(z)\, c(w) = \frac{1}{z-w} + :b(z)\, c(w):$$

$$= \frac{1}{z-w} + \sum_{k=0}^{\infty}\frac{1}{k!}(z-w)^k :(\partial^k b) c:(w)$$

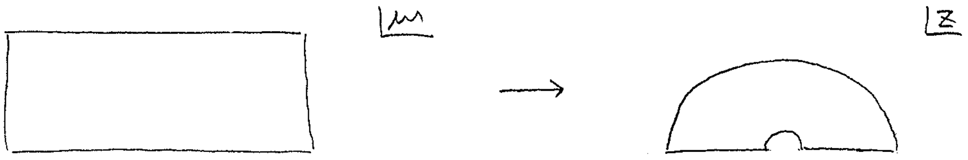
the singular part of the OPE

$$\partial X^{\mu}(z)\,\partial X^{\nu}(w) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z-w)^2}$$

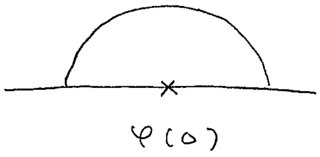
$$b(z)\,c(w) \sim \frac{1}{z-w}$$

(2016 5分休憩)

the state - operator correspondence



state $|\varphi\rangle$
in the Fock space $\longleftrightarrow$



operator

Consider $|\mathbb{1}\rangle$ $\qquad\mathbb{1}$: the unit operator

$\alpha^{\mu}_{n}|\mathbb{1}\rangle = 0$ for $n \geqq 0$

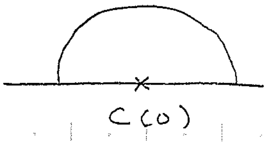
$b_n|\mathbb{1}\rangle = 0$ for $n \geqq -1$

$c_n|\mathbb{1}\rangle = 0$ for $n \geqq 2$

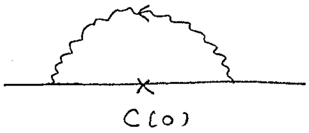
$$\Rightarrow \quad |\mathbb{1}\rangle = |0;0\rangle \equiv |0\rangle$$

$c_1|0\rangle \qquad \displaystyle\oint \frac{dz}{2\pi i}\frac{1}{z}\,c(z) = c(0)$

$c_1|0\rangle \longleftrightarrow$



$b_{-1} c_1 |0\rangle$



$$\oint \frac{dz}{2\pi i} b(z) c(0) = \oint \frac{dz}{2\pi i} \frac{1}{z} = 1 \qquad b_{-1} c_1 |0\rangle = |0\rangle$$

$$(\ b_{-1} c_1 |0\rangle = \{ b_{-1}, c_1 \} |0\rangle = |0\rangle \ )$$

（2021 5分 休憩）

Conformal transformations

$$z \longrightarrow z' = f(z)$$

operator $\varphi(z) \longrightarrow f \circ \varphi(z)$

the conformal invariance

$$\langle \varphi_1(z_1) \varphi_2(z_2) \cdots \varphi_n(z_n) \rangle_\Sigma$$
$$= \langle f\circ\varphi_1(z_1) \ f\circ\varphi_2(z_2) \cdots f\circ\varphi_n(z_n) \rangle_{f\circ\Sigma}$$

a primary field of weight h

$$\mathcal{O}(z) \longrightarrow f\circ\mathcal{O}(z) = \left(\frac{df(z)}{dz}\right)^h \mathcal{O}(f(z))$$

$$\partial X^\mu(z) \longrightarrow f\circ\partial X^\mu(z) = \frac{df(z)}{dz} \partial X^\mu(f(z))$$

$$b(z) \longrightarrow f\circ b(z) = \left(\frac{df(z)}{dz}\right)^2 b(f(z))$$

$$c(z) \longrightarrow f\circ c(z) = \left(\frac{df(z)}{dz}\right)^{-1} c(f(z))$$

$$\partial c(z) \longrightarrow f\circ\partial c(z) = \partial c(f(z)) - \frac{f''(z)}{f'(z)^2} c(f(z))$$

$$f'(z) = \frac{df(z)}{dz}, \qquad f''(z) = \frac{d^2 f(z)}{dz^2}$$

⑥ 2016 11/1 (火) （2015 5分 休憩）

the energy-momentum tensor

$$T(z) \equiv T_{zz}(z), \quad \tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}), \quad T_{z\bar{z}} = 0$$

the conservation

$$\bar{\partial} T(z) = 0, \quad \partial \tilde{T}(\bar{z}) = 0$$

the boundary condition

$$T(z) = \tilde{T}(\bar{z}) \quad \text{at} \quad \mathrm{Im}\, z = 0$$

the expansions

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}, \quad \tilde{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{L_n}{\bar{z}^{n+2}}$$

L_n : the Virasoro generators

the doubling trick

$$T(z) \equiv \tilde{T}(\bar{z}') \quad \text{for} \quad \mathrm{Im}\, z < 0, \quad z' = \bar{z}$$

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

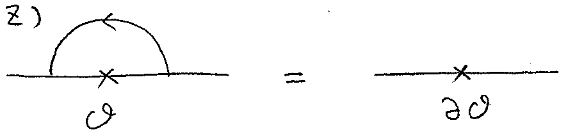
$$T^{(m)}(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : (z)$$

$$T^{(bc)}(z) = :(\partial b) c : (z) - 2 \partial (: bc :) (z)$$

$$T(z) \mathcal{O}(w) \sim \frac{h}{(z-w)^2} \mathcal{O}(w) + \frac{1}{z-w} \partial \mathcal{O}(w)$$

$\mathcal{O}$: a primary field of weight h

$$\oint \frac{dz}{2\pi i} T(z)$$



$$T(z) T(w) \sim \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w)$$

c : the central charge

X^μ ; $c = 1$

bc ghost ; $c = -26$

the BRST operator

$$Q_B = \oint \frac{dz}{2\pi i} j_B(z)$$

the BRST current

$$j_B(z) = c\, T^{(m)}(z) + :bc\partial c:(z) + \frac{3}{2}\partial^2 c(z)$$

$$\nwarrow \; c = 26$$

a primary field of weight 1

⑥ 2021 11/12(金)

properties

$$Q_B^2 = 0$$

$$\oint \frac{dz}{2\pi i} j_B(z) \longrightarrow \oint \frac{dz'}{2\pi i} j_B(z') \qquad z' = f(z)$$

$$Q_B \cdot b(w) \equiv \oint \frac{dz}{2\pi i} j_B(z)\, b(w) = T(w)$$

$$\nwarrow \; \odot_w$$

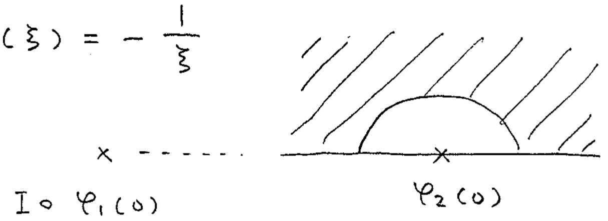
$$Q_B \cdot T(w) = 0$$

$$\{Q_B, b_n\} = L_n, \quad [Q_B, L_n] = 0$$

the BPZ inner product

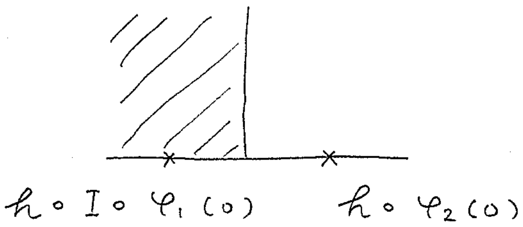
$\langle \varphi_1 , \varphi_2 \rangle$

$$I(\xi) = -\frac{1}{\xi}$$



$$\langle \varphi_1 , \varphi_2 \rangle = \langle I \circ \varphi_1(0) \ \varphi_2(0) \rangle_{UHP}$$

$$\big\downarrow \qquad h(z) = -\frac{z+1}{z-1}$$



$$h \circ I(\xi) = h(I(\xi)) = \frac{\xi - 1}{\xi + 1}$$

$$\langle \varphi_1 , \varphi_2 \rangle = \langle h \circ I \circ \varphi_1(0) \quad h \circ \varphi_2(0) \rangle_{UHP}$$

$$h \circ I \circ h^{-1}(z) = -\frac{1}{z}$$

$$\begin{aligned} &\langle \varphi_2 , \varphi_1 \rangle \\ &= \langle h \circ I \circ h^{-1} \circ h \circ I \circ \varphi_2(0) \quad h \circ I \circ h^{-1} \circ h \circ \varphi_1(0) \rangle_{UHP} \\ &= (-1)^{\varphi_1 \varphi_2} \langle h \circ I \circ \varphi_1(0) \quad h \circ \varphi_2(0) \rangle_{UHP} \\ &= (-1)^{\varphi_1 \varphi_2} \langle \varphi_1 , \varphi_2 \rangle \end{aligned}$$

$$A_\mu(x) = \frac{1}{2}\sum_{a=1}^{3} A_\mu^a(x)\,\sigma^a \qquad A_\mu^a(x) = \mathrm{tr}\,\sigma^a A_\mu(x)$$

We will often define Φ by giving $\langle \varphi, \Phi \rangle$ for all φ in the Fock space. ⌟

⑥ 2015 10/20 (火)

<u>example</u>

$$\langle T, Q_B T \rangle \quad \text{with} \quad T = c_1 |0\rangle$$

$$Q_B \cdot c(w) = \oint \frac{dz}{2\pi i}\, j_B(z)\, c(w)$$

$$= \oint \frac{dz}{2\pi i}\, :bc\partial c:(z)\, c(w)$$

$$= \oint \frac{dz}{2\pi i}\, \frac{1}{z-w}\, c\partial c(z) = c\partial c(w)$$

$$f\circ(Q_B\cdot\varphi(\xi)) = f\circ\left[\oint \frac{dz}{2\pi i}\, j_B(z)\,\varphi(\xi)\right]$$

$$= \oint \frac{dz'}{2\pi i}\, j_B(z')\, f\circ\varphi(\xi)$$

$$z' = f(z)$$

$$f\circ(Q_B\cdot\varphi(\xi)) = Q_B\cdot(f\circ\varphi(\xi))$$

$$\langle T, Q_B T\rangle$$

$$= \langle h\circ I\circ c(0) \quad h\circ(Q_B\cdot c(0))\rangle_{UHP}$$

$$= \langle \underbrace{h\circ I\circ c(0)}_{\frac{1}{2}c(-1)} \quad \underbrace{Q_B\cdot(\underbrace{h\circ c(0)}_{\frac{1}{2}c(1)})}_{\frac{1}{2}c\partial c(1)}\rangle_{UHP}$$

$$= \frac{1}{4}\langle c(-1) \quad c\partial c(1)\rangle_{UHP}$$

$\langle c(z_1)\ c(z_2)\ c(z_3) \rangle_{UHP}$
$= (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$
up to the spacetime volume factor

$\langle T, Q_B T \rangle_{\text{density}} = -1$

(2016 , 2021　5分休憩)

BPZ conjugate

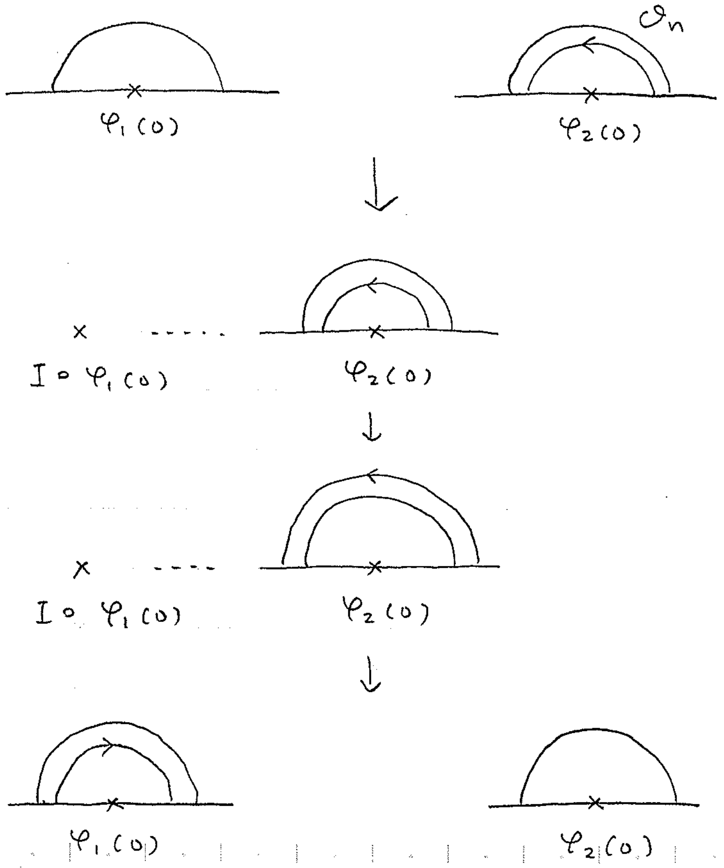
$\langle \varphi_1, \mathcal{O}_n \varphi_2 \rangle$

$\mathcal{O}(z)$: a primary field of weight h

$$\mathcal{O}(z) = \sum_{n=-\infty}^{\infty} \frac{\mathcal{O}_n}{z^{n+h}}$$

$$\mathcal{O}_n = \oint \frac{dz}{2\pi i}\, z^{n+h-1}\, \mathcal{O}(z)$$

(Grassmann even , the doubling trick)



$$z' = -\frac{1}{z} \qquad z = -\frac{1}{z'} \qquad \frac{dz}{dz'} = \frac{1}{z'^2}$$

$$\oint \frac{dz}{2\pi i} \, z^{n+h-1} \, \mathcal{O}(z)$$

$$\longrightarrow \quad \underset{\substack{\uparrow \\ \curvearrowright \, \rightarrow \, \curvearrowleft}}{-} \oint \frac{dz'}{2\pi i} \, \frac{dz}{dz'} \left(-\frac{1}{z'}\right)^{n+h-1} \left(\frac{dz}{dz'}\right)^{-h} \mathcal{O}(z')$$

$$= (-1)^{n+h} \oint \frac{dz'}{2\pi i} \, z'^{-2(1-h)} \, z'^{-n-h+1} \, \mathcal{O}(z')$$

$$= (-1)^{n+h} \oint \frac{dz'}{2\pi i} \, z'^{-n+h-1} \, \mathcal{O}(z')$$

$$= (-1)^{n+h} \, \mathcal{O}_{-n}$$

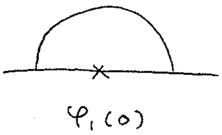
$$\langle \varphi_1 , \mathcal{O}_n \varphi_2 \rangle = (-1)^{n+h} \langle \mathcal{O}_{-n} \varphi_1 , \varphi_2 \rangle$$

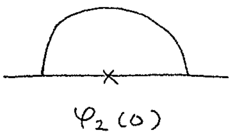
$$\mathcal{O}_n^{\star} = (-1)^{n+h} \mathcal{O}_{-n}$$

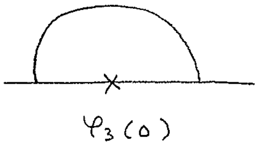
In particular, $Q_B^{\star} = -Q_B$.

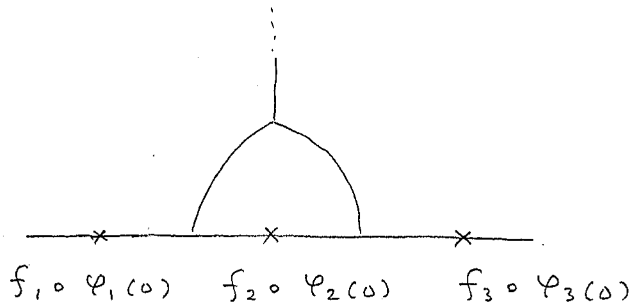
The star product

$\langle \varphi_1 , \varphi_2 * \varphi_3 \rangle$









$$\langle \varphi_1 , \varphi_2 * \varphi_3 \rangle = \langle f_1 \circ \varphi_1(0) \quad f_2 \circ \varphi_2(0) \quad f_3 \circ \varphi_3(0) \rangle_{UHP}$$

$$f_1(\xi) = \tan\left[\frac{2}{3} \left(\arctan \xi - \frac{\pi}{2} \right) \right]$$

$$f_2(\xi) = \tan\left(\frac{2}{3} \arctan \xi \right)$$

$$f_3(\xi) = \tan\left[\frac{2}{3} \left(\arctan \xi + \frac{\pi}{2} \right) \right]$$

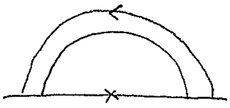
$$\langle \varphi_1 * \varphi_2 , \varphi_3 \rangle \underset{\Uparrow}{=} \langle \varphi_1 , \varphi_2 * \varphi_3 \rangle$$

$$\tilde{f}(z) = \tan\left(\arctan z + \frac{\pi}{3} \right) = \frac{z + \sqrt{3}}{1 - \sqrt{3} z}$$

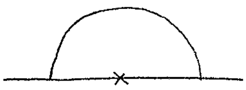
$$\tilde{f} \circ f_1 = f_2 , \quad \tilde{f} \circ f_2 = f_3 , \quad \tilde{f} \circ f_3 = f_1$$

$$\langle Q_B \varphi_1 , \ \varphi_2 * \varphi_3 \rangle = -(-1)^{\varphi_1} \langle \varphi_1 , \ Q_B (\varphi_2 * \varphi_3) \rangle$$

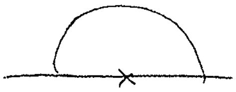
$$\langle Q_B \varphi_1 , \ \varphi_2 * \varphi_3 \rangle$$
$$= \langle f_1 \circ (Q_B \cdot \varphi_1 (0)) \ f_2 \circ \varphi_2 (0) \ \ f_3 \circ \varphi_3 (0) \rangle_{UHP}$$
$$= \langle Q_B \cdot (f_1 \circ \varphi_1 (0)) \ f_2 \circ \varphi_2 (0) \ \ f_3 \circ \varphi_3 (0) \rangle_{UHP}$$



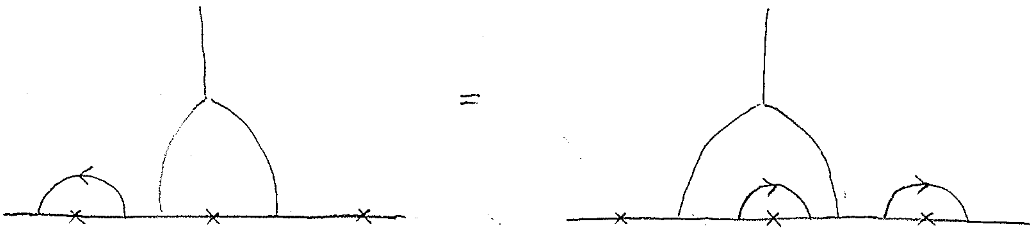
$\varphi_1(0)$



$\varphi_2(0)$



$\varphi_3(0)$

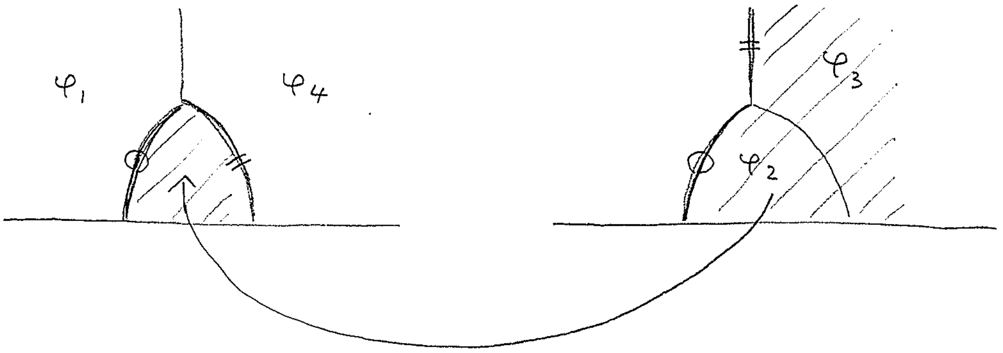


$$\Rightarrow \quad Q_B (\varphi_2 * \varphi_3) = (Q_B \varphi_2) * \varphi_3 + (-1)^{\varphi_2} \varphi_2 * (Q_B \varphi_3) \quad ⌋$$

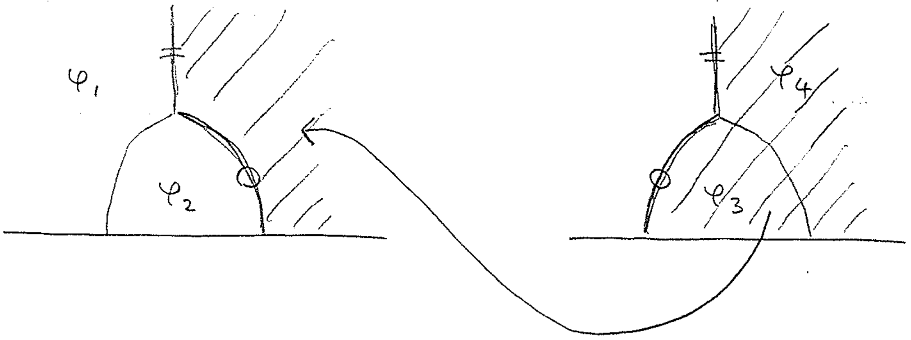
⑦ 2016 11/15 (火)

(2015 5分 休憩)

$\langle \varphi_1, (\varphi_2 * \varphi_3) * \varphi_4 \rangle$



$\langle \varphi_1, \varphi_2 * (\varphi_3 * \varphi_4) \rangle$

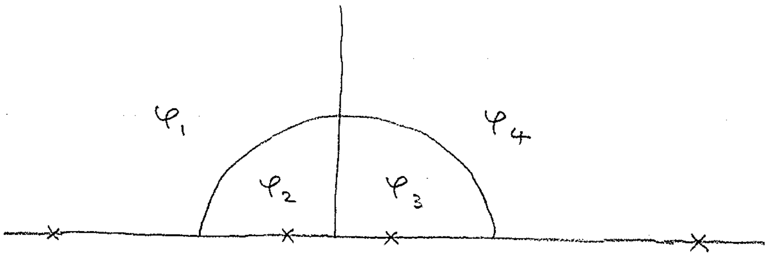


$$\langle \varphi_1, (\varphi_2 * \varphi_3) * \varphi_4 \rangle = \langle \varphi_1, \varphi_2 * (\varphi_3 * \varphi_4) \rangle$$

$$= \langle g_1 \circ \varphi_1(0) \;\; g_2 \circ \varphi_2(0) \;\; g_3 \circ \varphi_3(0) \;\; g_4 \circ \varphi_4(0) \rangle_{\mathrm{UHP}}$$

$$g_1(\xi) = \tan\left[\frac{1}{2}\left(\arctan\xi - \frac{3\pi}{4}\right)\right], \quad g_2(\xi) = \tan\left[\frac{1}{2}\left(\arctan\xi - \frac{\pi}{4}\right)\right]$$

$$g_3(\xi) = \tan\left[\frac{1}{2}\left(\arctan\xi + \frac{\pi}{4}\right)\right], \quad g_4(\xi) = \tan\left[\frac{1}{2}\left(\arctan\xi + \frac{3\pi}{4}\right)\right]$$



example

$$\langle T, T * T \rangle \quad \text{with} \quad T = c_1|0\rangle$$

$$f_1 \circ c(0) = \frac{3}{8} c(-\sqrt{3}), \quad f_2 \circ c(0) = \frac{3}{2} c(0)$$

$$f_3 \circ c(0) = \frac{3}{8} c(\sqrt{3})$$

$$\langle T, T * T \rangle = \frac{27}{128} \langle c(-\sqrt{3}) \, c(0) \, c(\sqrt{3}) \rangle_{UHP}$$

$$\langle T, T * T \rangle_{density} = -\frac{81\sqrt{3}}{64}$$

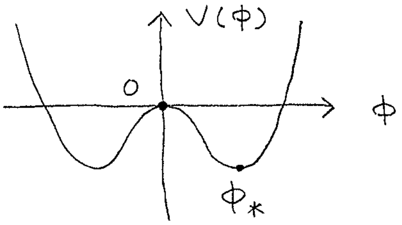
another convention for the star product

$$\langle \varphi_1, \varphi_2 * \varphi_3 \rangle$$
$$= \langle f_3 \circ \varphi_3(0) \; f_2 \circ \varphi_2(0) \; f_1 \circ \varphi_1(0) \rangle_{UHP}$$

⑦ 2021 11/19 (金)
(間違えて 10分間 延長)

§ 3.2 Tachyon condensation

ϕ : scalar field

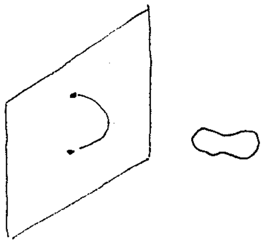


$$\left.\frac{dV(\phi)}{d\phi}\right|_{\phi=\phi_*} = 0$$

$$\left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\phi_*} > 0$$

$$\phi = \phi_* + \delta\phi$$

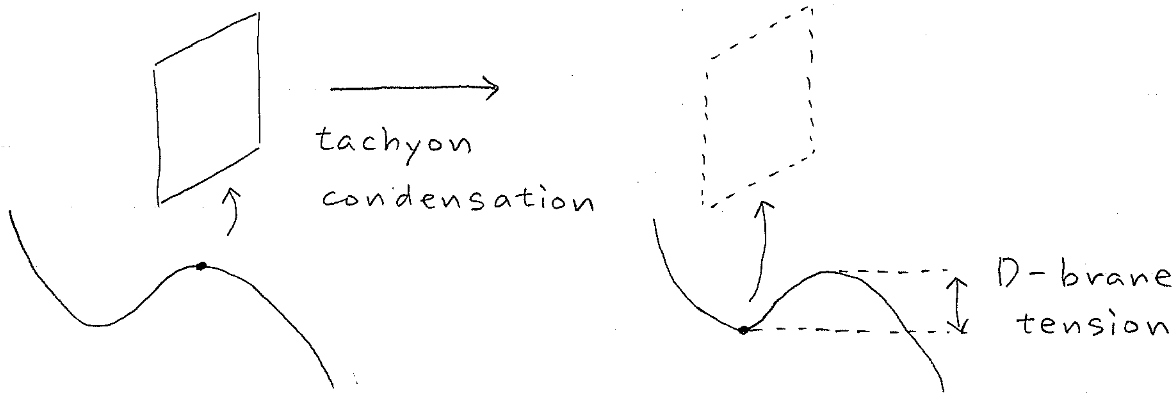
open strings = excitations on solitonic extended objects in string theory called D-branes

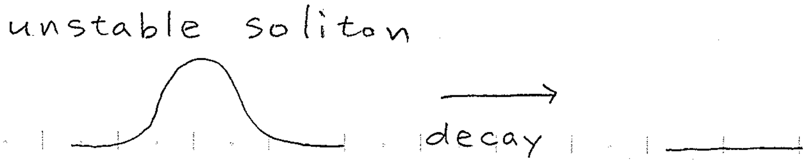


our example ; D25-brane

Sen's conjecture

The instability associated with the open string tachyon corresponds to the decay of the D-brane.





⑦ 2015 10/27 (火)

Tachyon condensation in open string field theory
Sen and Zwiebach, hep-th/9912249

Level truncation

Level 0 only the zero mode of the tachyon t

$$\Psi = t\, c_1 |0\rangle$$

$$V(t) = \frac{1}{\alpha'^3 g_T^2}\left[\frac{1}{2} t^2 \langle T, Q_B T\rangle + \frac{1}{3} t^3 \langle T, T * T\rangle\right]_{\text{density}}$$

$$= \frac{1}{\alpha'^3 g_T^2}\left(-\frac{1}{2} t^2 - \frac{27\sqrt{3}}{64} t^3\right)$$

D25-brane tension $T_{25} = \frac{1}{2\pi^2 \alpha'^3 g_T^2}$

$$\frac{V(t)}{T_{25}} = 2\pi^2\left(-\frac{1}{2}t^2 - \frac{27\sqrt{3}}{64} t^3\right)$$

$$\frac{dV(t)}{dt} = 0 \quad \text{at} \quad t = t_* = -\frac{64}{81\sqrt{3}}$$

$$\frac{V(t_*)}{T_{25}} = -\frac{4096\pi^2}{59049} \simeq -0,68$$

68 %

Level 2 $\Psi = t\, c_1|0\rangle + u\, c_{-1}|0\rangle + v\, L_{-2}^{(m)} c_1 |0\rangle$

bc ghosts and the matter
energy-momentum tensor

$b_0 \Psi = 0$ Siegel gauge

$$\frac{\partial V(t,u,v)}{\partial t} = 0, \quad \frac{\partial V(t,u,v)}{\partial u} = 0, \quad \frac{\partial V(t,u,v)}{\partial v} = 0$$

at $(t,u,v) = (t_*, u_*, v_*)$

$$\frac{V(t_*, u_*, v_*)}{T_{25}} \simeq -0,96$$

96 %

level	V_* / T_{25}
4	−0.9878218
6	−0.9951771
8	−0.9979302
10	−0.9991825
12	−0.9998223

Evidence for Sen's conjecture

Evidence for open string field theory

analytic solution   Schnabl, hep-th/0511286

(2016, 2021 5分休憩)

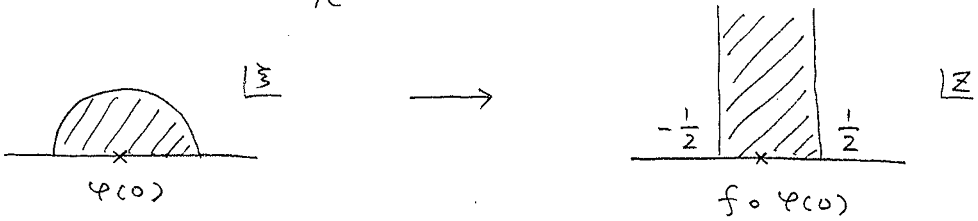
Sliver frame

Rastelli and Zwiebach, hep-th/0006240

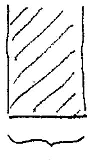
a convenient coordinate

to describe the star product

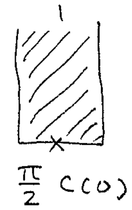
$$z = f(\xi) = \frac{2}{\pi} \arctan \xi$$



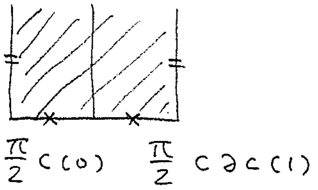
$|0\rangle \longleftrightarrow$



$T = c_1 |0\rangle \longleftrightarrow$



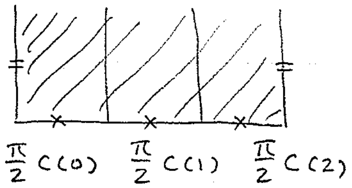
$\langle T, Q_B T \rangle =$



$= -1$

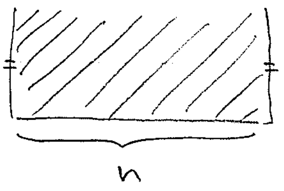
up to the spacetime volume factor

$\langle T, T * T \rangle =$

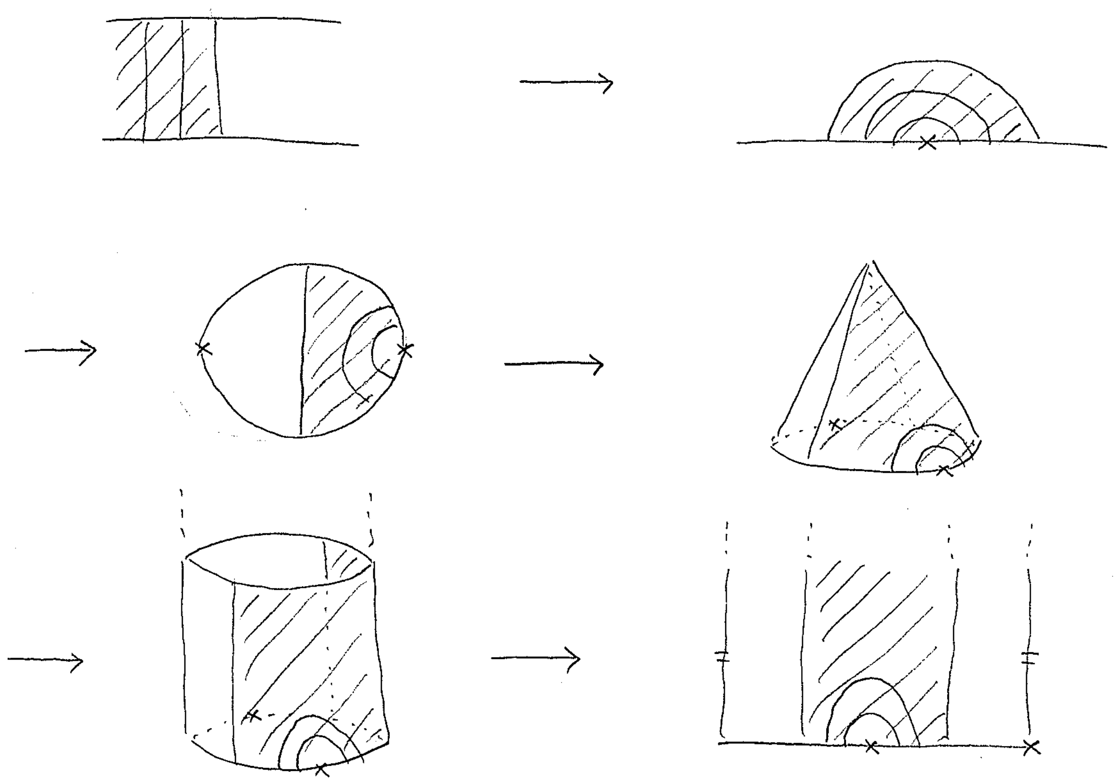


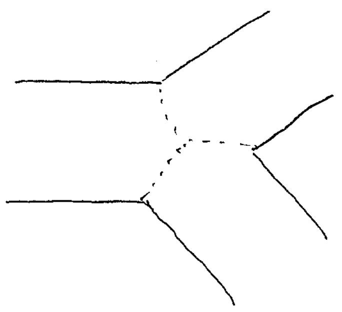
$= -\frac{81\sqrt{3}}{64}$

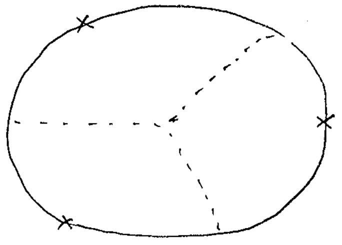
$\langle c(z_1)\, c(z_2)\, c(z_3) \rangle_{\text{density}}$ on

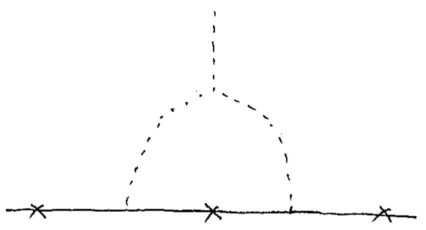


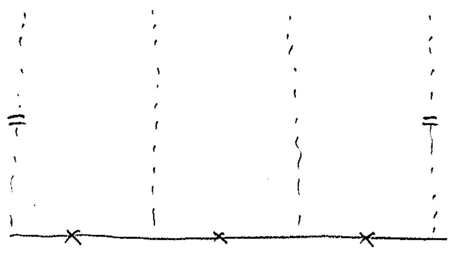
$$= \left(\frac{n}{\pi}\right)^3 \sin\frac{\pi(z_1 - z_2)}{n} \sin\frac{\pi(z_1 - z_3)}{n} \sin\frac{\pi(z_2 - z_3)}{n}$$





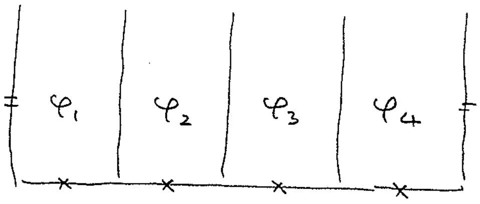






associativity

$$\langle \varphi_1, (\varphi_2 * \varphi_3) * \varphi_4 \rangle = \langle \varphi_1, \varphi_2 * (\varphi_3 * \varphi_4) \rangle$$



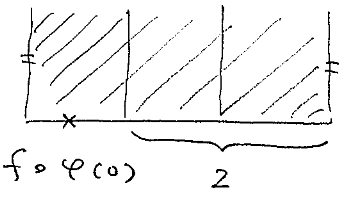
⌋ ⑧ 2016 11/22 (火)

(2015 5分休憩)

the wedge state W_α

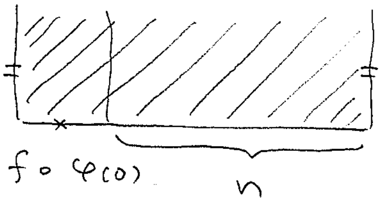
Consider $|0\rangle * |0\rangle$.

$$\langle \varphi, \underset{\substack{\uparrow \\ |0\rangle * |0\rangle}}{0 * 0} \rangle =$$

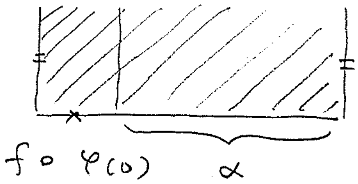


$$\underbrace{|0\rangle * |0\rangle * \cdots * |0\rangle}_{n}$$

$$\langle \varphi, 0 * 0 * \cdots * 0 \rangle =$$



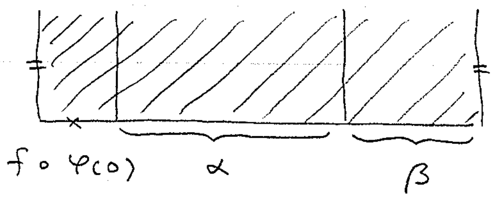
$$\langle \varphi, W_\alpha \rangle =$$



$(\alpha ;\ \text{real},\ \alpha \geqq 0)$

$W_1 = |0\rangle$, $W_2 = |0\rangle * |0\rangle$

$$\langle \varphi, W_\alpha * W_\beta \rangle =$$



$$= \langle \varphi, W_{\alpha+\beta} \rangle$$

$$\boxed{W_\alpha * W_\beta = W_{\alpha+\beta}}$$

wedge-based states :
wedge states with operator insertions

the sliver state W_∞

$$W_\infty = \lim_{\alpha \to \infty} W_\alpha \quad (\text{subtle})$$

$$\left(\lim_{\alpha \to \infty} \langle \varphi, W_\alpha \rangle = \text{finite} \right)$$

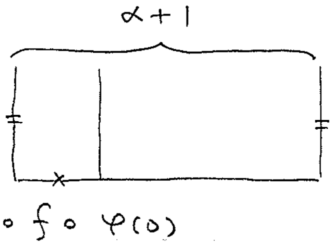
Formally, $W_\infty * W_\infty = W_\infty$

star-algebra projector

derivatives of the wedge state

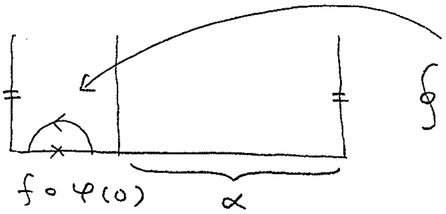
$$W_\alpha' = \frac{dW_\alpha}{d\alpha} = \lim_{\beta\to 0} \frac{W_{\alpha+\beta} - W_\alpha}{\beta}$$

$\langle \varphi, W_{\alpha+\beta} \rangle =$



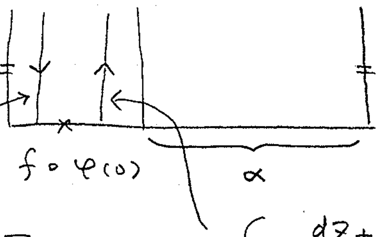
$$g(z) = \frac{\alpha+1}{\alpha+\beta+1} z = z - \frac{\beta}{\alpha+1} z + O(\beta^2)$$

$\langle \varphi, W_\alpha' \rangle = -\dfrac{1}{\alpha+1}$



$\oint \dfrac{dz}{2\pi i} z\, T(z)$

$= -\dfrac{1}{\alpha+1}$



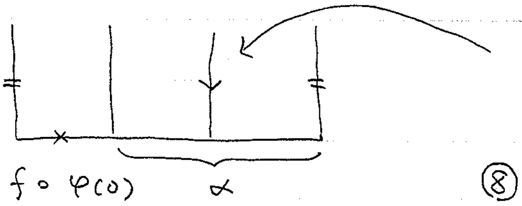
$\int_\downarrow \dfrac{dz_-}{2\pi i} z_-\, T(z_-)$ $\quad$ $\int_\uparrow \dfrac{dz_+}{2\pi i} z_+\, T(z_+)$

$$= \int_\downarrow \frac{dz_+}{2\pi i} (z_+ - \alpha - 1)\, T(z_+) \quad (z_+ = z_- + \alpha + 1)$$

$$-\frac{1}{\alpha+1} \int_\uparrow \frac{dz_+}{2\pi i} z_+\, T(z_+) - \frac{1}{\alpha+1} \int_\downarrow \frac{dz_+}{2\pi i} (z_+ - \alpha - 1)\, T(z_+)$$

$$= \int_\downarrow \frac{dz_+}{2\pi i} T(z_+)$$

$\langle \varphi, W_\alpha' \rangle =$

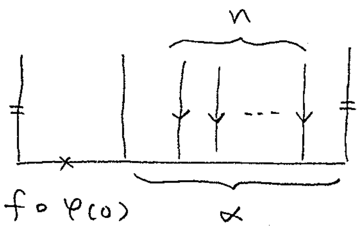


$\int \dfrac{dz}{2\pi i} T(z)$

⑧ 2021 12/3 (金)

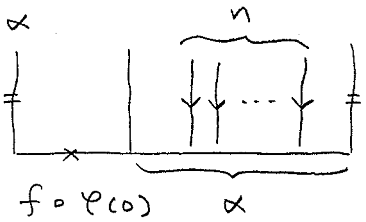
$(\text{cf.}\ \partial_t e^{-tL_0} = -L_0 e^{-tL_0})$

$\langle \varphi, W_\alpha^{(n)} \rangle =$



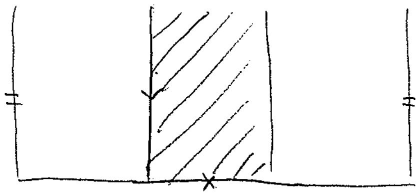
$(\text{cf. } [L_{-1}, L_{-1}] = 0)$

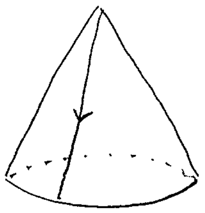
$\langle \varphi, W_{\alpha+\beta} \rangle = \sum_{n=0}^{\infty} \frac{\beta^n}{n!}$

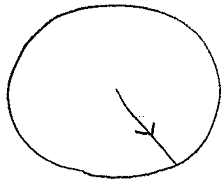


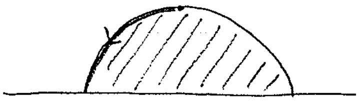
⊥

⑧ 2015 11/3 (火)









midpoint → boundary

$$z = f(\xi) = \frac{2}{\pi} \arctan \xi \qquad f'(\xi) = \frac{2}{\pi} \frac{1}{\xi^2 + 1}$$

$$\int \frac{dz}{2\pi i} T(z) \rightarrow \int \frac{d\xi}{2\pi i} \left(\frac{dz}{d\xi} \right)^{-1} T(\xi)$$

$$= \frac{\pi}{2} \int \frac{d\xi}{2\pi i} (\xi^2 + 1) T(\xi)$$

half integral of

$$\frac{\pi}{2} (L_1 + L_{-1}) = \frac{\pi}{2} K_1$$

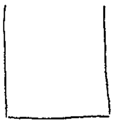
(2016 5分 休憩)

eigenstates of the operator L

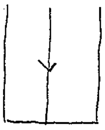
$$L = \oint \frac{dz}{2\pi i}\, z\, T(z) = \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} T(\xi)$$

$$= L_0 + \frac{2}{3} L_2 - \frac{2}{15} L_4 + \cdots$$

$L|0\rangle = 0$

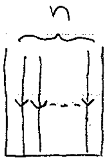


$L = 0$

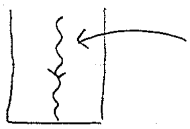


$$\int_{\downarrow} \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\, z\, T(z) T(w) = \int_{\downarrow} \frac{dw}{2\pi i} T(w)$$

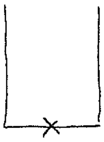
$(\text{cf. } [L_0, L_{-1}] = L_{-1}) \Rightarrow L = 1$



$L = n$



$$\int \frac{dz}{2\pi i} b(z) \qquad L = 1$$

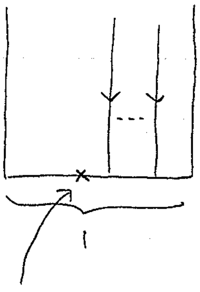


$c\partial^2 c\, \partial^5 c$

$$L = \underbrace{2 + 5}_{\#\text{ of }\partial\text{'s}} - \underset{\uparrow}{3} = 4$$

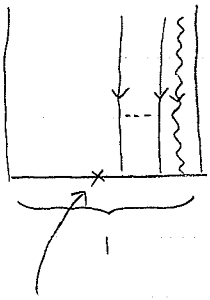
\# of c's

Consider a set of L eigenstates in the form

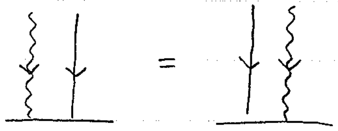


$\prod_i \partial^{n_i} c$

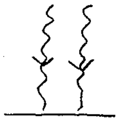
or



$\prod_i \partial^{n_i} c$



cf. $[L_{-1}, b_{-1}] = 0$

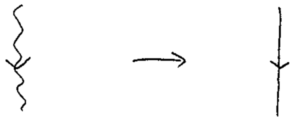


$= 0$

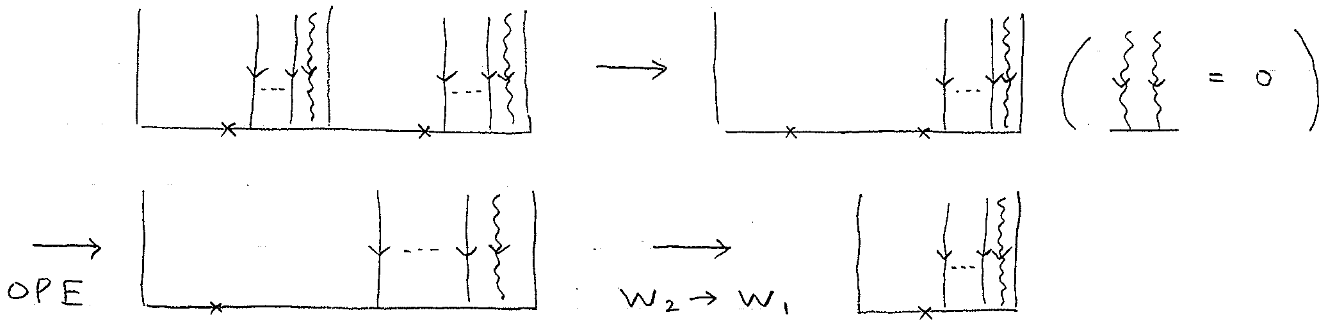
cf. $b_{-1}^2 = 0$

action of Q_B

$$\prod_i \partial^{n_i} c \longrightarrow \prod_i \partial^{n_i'} c$$



Star product

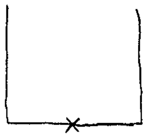


$$\Phi_1 * \Phi_2 = \sum_i c_i \Phi_i \qquad h_i \geq h_1 + h_2$$

$L = h_1$ $L = h_2$ $c_i \neq 0$ $L = h_i$

ghost number = 1

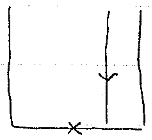
$L = -1$



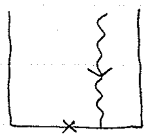
c

$L = 0$

∂c



c

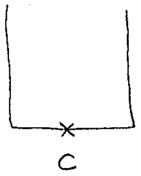


$c\partial c$

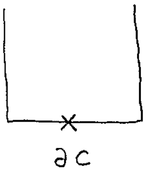
Schnabl gauge

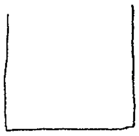
$B\Psi = 0$

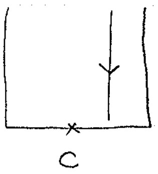
$$B = \oint \frac{dz}{2\pi i}\, z\, b(z) = \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)}\, b(\xi)$$

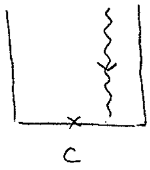
B 

$= 0 \qquad \oint \frac{dz}{2\pi i}\, z\, b(z)\, c(0) = 0$

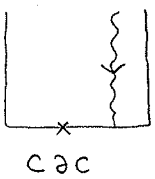
B 

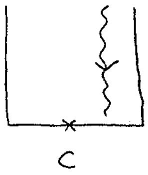
$=$ 

B 

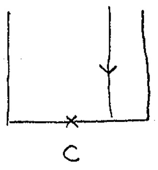
$= -$ 

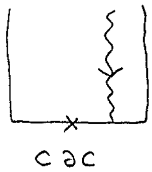
$(\text{cf. } [b_0, L_{-1}] = b_{-1})$

B 

$= -$ 

$\oint \frac{dz}{2\pi i}\, z\, b(z)\, c\partial c(0) = -c(0)$

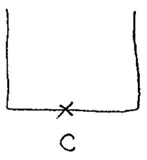
$B\Big($ 

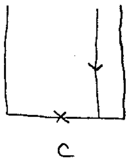
$-$ 

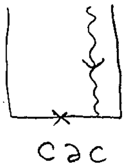
$\Big) = 0$

(2015 5分休憩)

In fact,

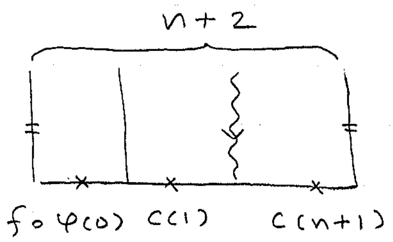


$= \Psi_0$, 

$-$ 

$= \Psi_0'$

where

$\langle \varphi, \Psi_n \rangle =$ 

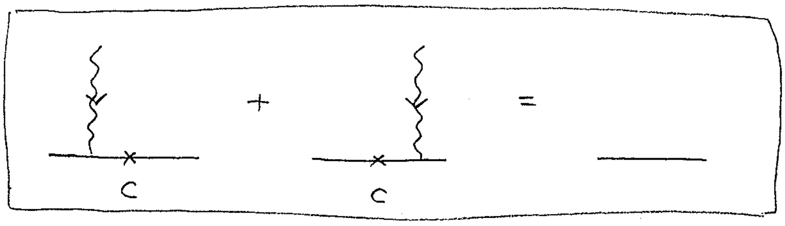
$$\left\langle f\circ\varphi(0)\, c(1) \int \frac{dz}{2\pi i}\, b(z)\, c(n+1) \right\rangle$$

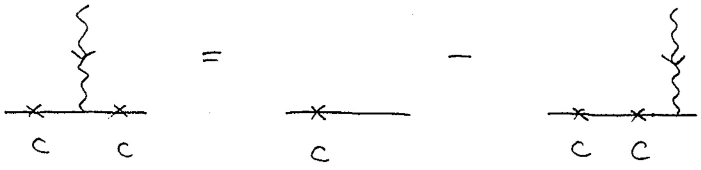
(2021 5分休憩)

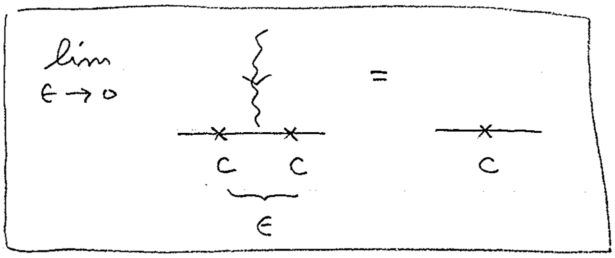
⑨ 2016. 11/29 (火)

$$\int_{\downarrow} \frac{dz}{2\pi i}\, b(z)\, c(0) + c(0) \int_{\downarrow} \frac{dz}{2\pi i}\, b(z)$$

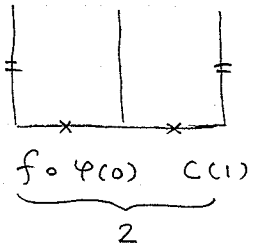
$$= \oint \frac{dz}{2\pi i}\, b(z)\, c(0) = 1$$





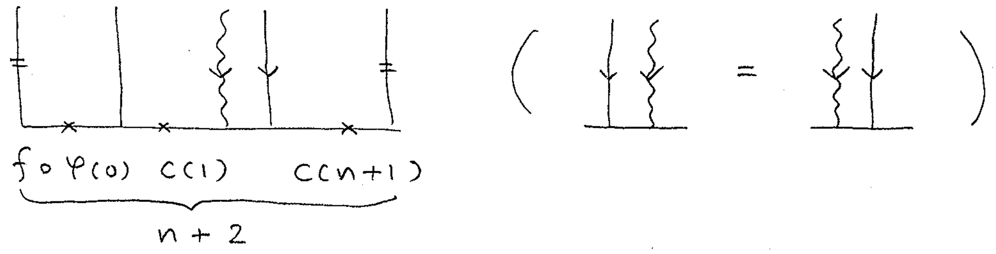


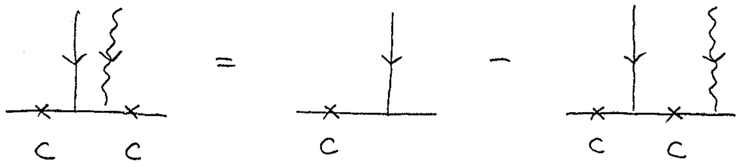
$$\langle \varphi, \Psi_0 \rangle \equiv \lim_{n\to 0} \langle \varphi, \Psi_n \rangle =$$

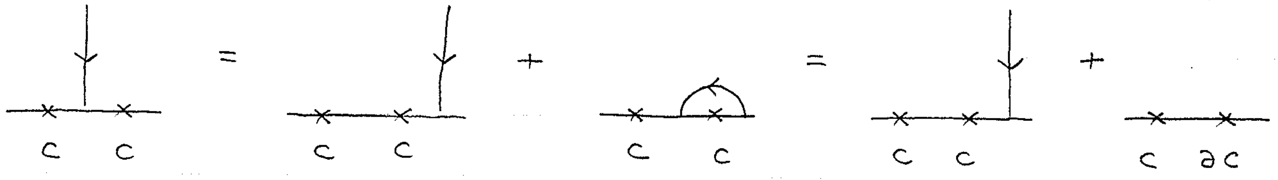


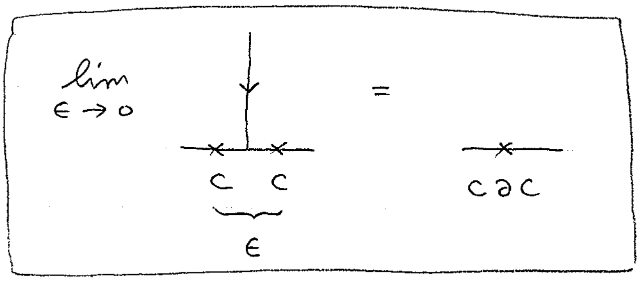
$$\langle \varphi, \Psi_n' \rangle \equiv \langle \varphi, \frac{d\Psi_n}{dn} \rangle$$

=



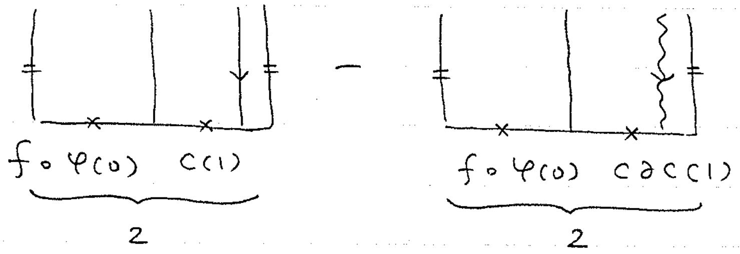






$$\langle \varphi, \Psi_0' \rangle \equiv \lim_{n \to 0} \langle \varphi, \Psi_n' \rangle$$

=



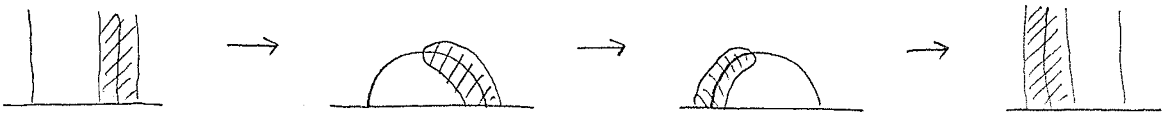
$B\Psi_n = 0$ for any $n \Rightarrow B\Psi_0 = 0,\ B\Psi_0' = 0, \cdots, B\Psi_0^{(n)} = 0, \cdots$

Proof

$$\langle \varphi, B\Psi_n \rangle = - \langle B^\star \varphi, \Psi_n \rangle$$

↖ ghost number 3

$$z \xrightarrow{f^{-1}} \xi \xrightarrow{I} \xi' \xrightarrow{f} z' \qquad I(\xi) = -\frac{1}{\xi}$$

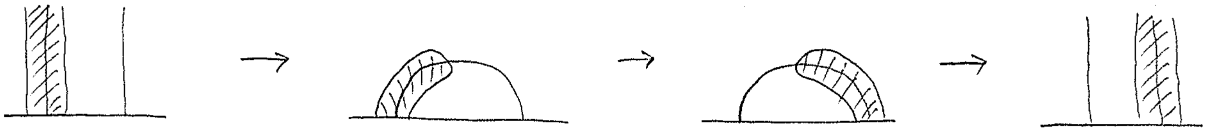


$z' = z - 1$ ┘

$$\int_{\uparrow} \frac{dz}{2\pi i}\, z\, b(z) \to \int_{\uparrow} \frac{dz'}{2\pi i} (z' + 1)\, b(z')$$

⑨ 2021 12/10 (金)

$$= \int_{\uparrow} \frac{dz''}{2\pi i} (z'' - n - 1)\, b(z'') \qquad z'' = z' + n + 2$$

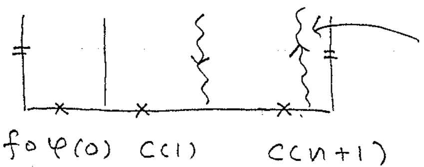


$z' = z + 1$

$$\int_{\downarrow} \frac{dz}{2\pi i}\, z\, b(z) \to \int_{\downarrow} \frac{dz'}{2\pi i} (z' - 1)\, b(z')$$

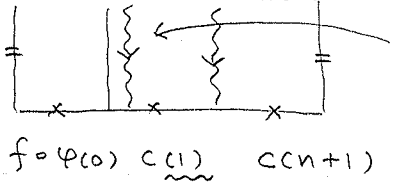
$\langle B^\star \varphi, \Psi_n \rangle$

=



$$\int \frac{dz}{2\pi i} \underline{(z - n - 1)}\, b(z)$$

−



$$\int \frac{dz}{2\pi i} \underline{(z - 1)}\, b(z)$$

= 0

┘ ⑨ 2015 11/17 (火)

Schnabl's analytic solution

$$\Psi = -\psi_0 + \frac{1}{2}\psi_0' - \frac{1}{6}\frac{\psi_0''}{2!} + \frac{1}{30}\frac{\psi_0^{(4)}}{4!} - \frac{1}{42}\frac{\psi_0^{(6)}}{6!}$$

$$+ \frac{1}{30}\frac{\psi_0^{(8)}}{8!} - \frac{5}{66}\frac{\psi_0^{(10)}}{10!} + \frac{\textcircled{691}}{2730}\frac{\psi_0^{(12)}}{12!} - \frac{7}{6}\frac{\psi_0^{(14)}}{14!} + \cdots$$

$$\boxed{\Psi = -\sum_{n=0}^{\infty}\frac{B_n}{n!}\psi_0^{(n)}} \quad \text{(conjecture)}$$

B_n : the Bernoulli number $\quad \dfrac{x}{e^x - 1} = \displaystyle\sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$

the Euler - Maclaurin sum formula

$$\sum_{k=0}^{\infty}\frac{B_k}{k!}\left[f^{(k)}(b) - f^{(k)}(a)\right] = \sum_{n=a}^{b-1} f'(n)$$

$$\sum_{k=0}^{\infty}\frac{B_k}{k!}\left(\psi_{N+1}^{(k)} - \psi_0^{(k)}\right) = \sum_{n=0}^{N}\psi_n'$$

$$\Psi = -\sum_{k=0}^{\infty}\frac{B_k}{k!}\psi_0^{(k)} = \sum_{n=0}^{N}\psi_n' - \sum_{k=0}^{\infty}\frac{B_k}{k!}\psi_{N+1}^{(k)}$$

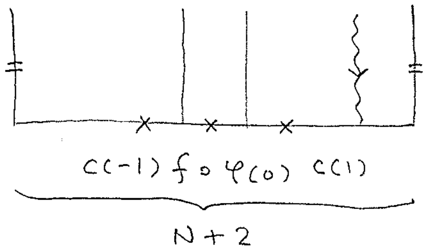
$$\boxed{\Psi = \lim_{N\to\infty}\left[\sum_{n=0}^{N}\psi_n' - \psi_N\right]}$$

(2016 5分 休憩)

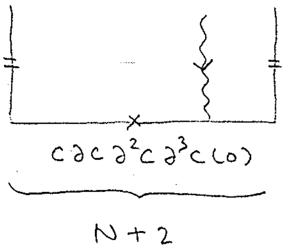
$\lim_{N\to\infty} \langle \varphi, \psi_N \rangle = 0$ for any φ in the Fock space

the "phantom" piece

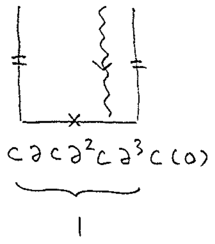
$\langle \varphi, \psi_N \rangle =$



Generically, $\underbrace{c(-1)\ f\circ\varphi(0)\ c(1)}_{\text{ghost number } = 4} \propto c\partial c\partial^2 c\partial^3 c + \cdots$



$= \frac{1}{(N+2)^3}$

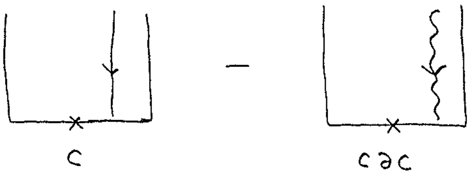


$\langle \varphi, \psi_N \rangle \sim O\left(\frac{1}{N^3}\right)$

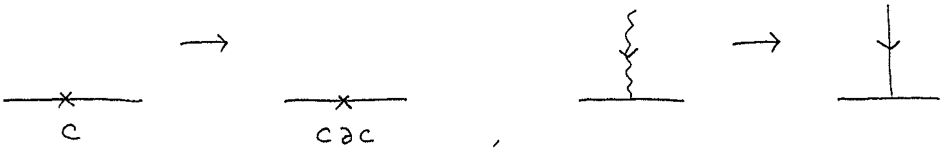
Formally, $\Psi = \sum_{n=0}^{\infty} \psi'_n$

$$\begin{cases} Q_B \psi'_0 = 0 \\ Q_B \psi'_{n+1} = -\sum_{m=0}^{n} \psi'_m * \psi'_{n-m} \end{cases} \Rightarrow Q_B \Psi + \Psi * \Psi = 0$$

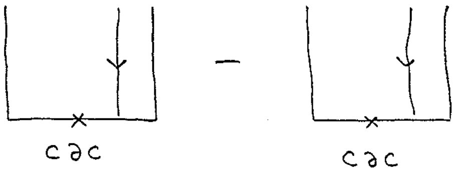
$\Psi_0' =$



$Q_B :$

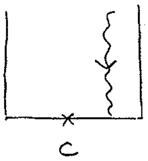


$Q_B \Psi_0' =$

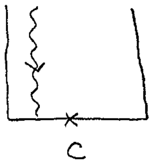


$= 0$

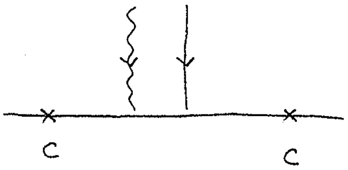
In fact, $\Psi_0' = - Q_B$



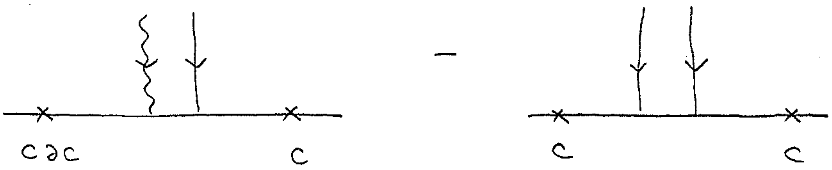
$= Q_B$



$\Psi_{n+1}' =$



$Q_B \Psi_{n+1}' =$



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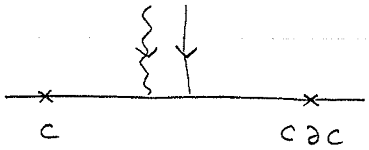
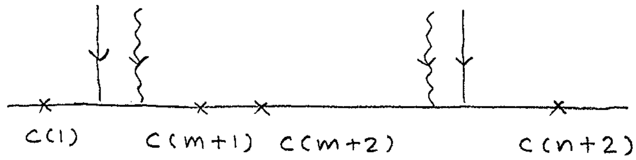
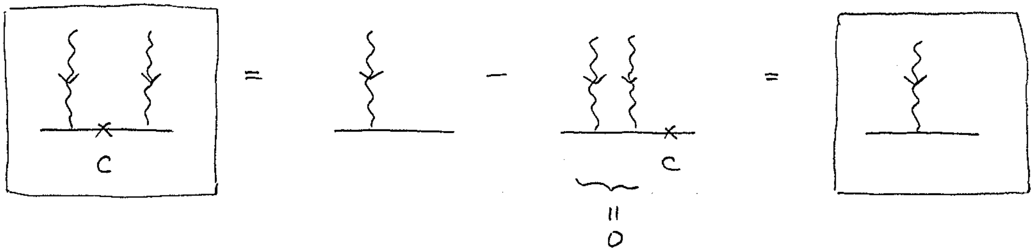
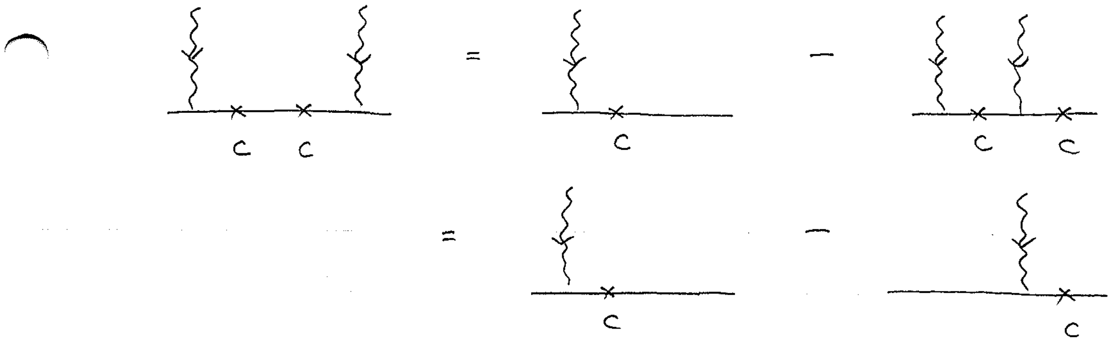


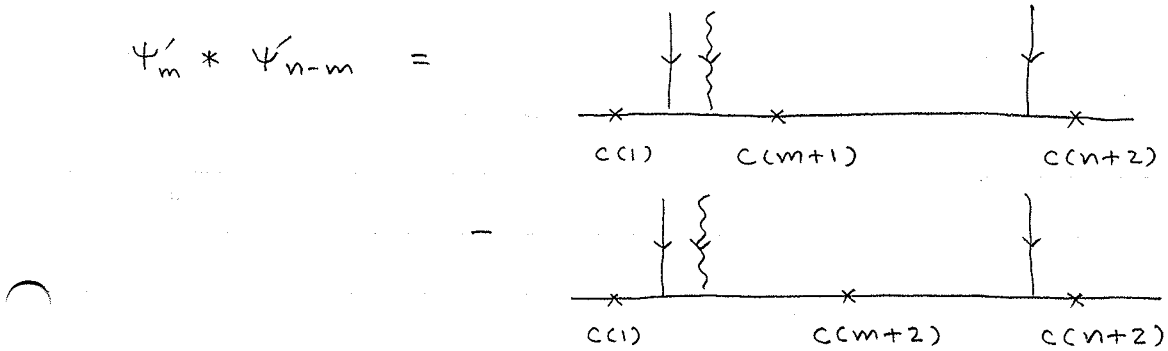
$\Psi'_m * \Psi'_{n-m} =$



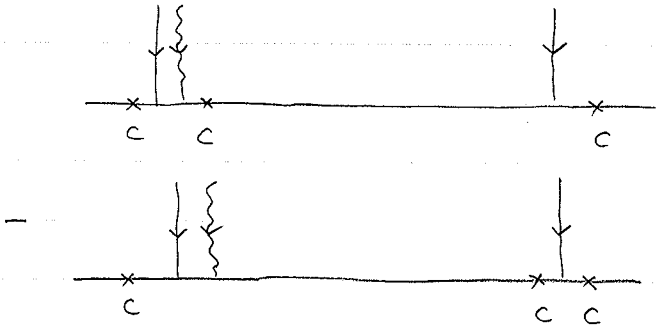


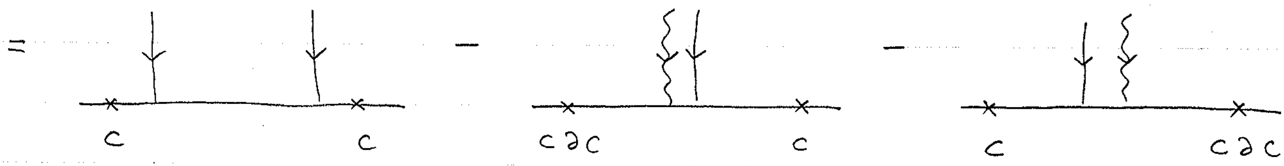


$\Psi'_m * \Psi'_{n-m} =$



$\sum_{m=0}^{n} \Psi'_m * \Psi'_{n-m} =$





$$Q_B \Psi'_{n+1} = - \sum_{m=0}^{n} \Psi'_m * \Psi'_{n-m}$$

We have actually shown that $Q_B \Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0$ for

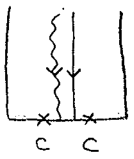
$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \psi'_n$$

with any λ.

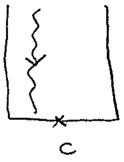
(2015, 2021 5分休憩)

$|\lambda| < 1$: pure gauge

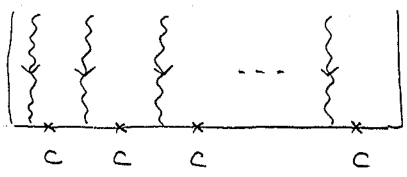
$\psi'_0 = Q_B \Psi$ $\psi'_0 =$



$\Psi =$



$\Psi^n =$



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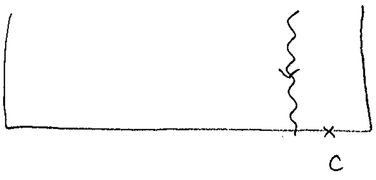
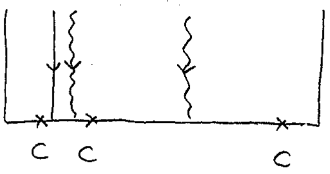
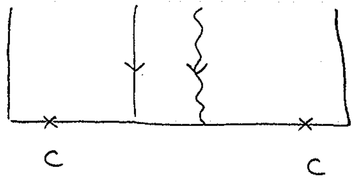


(We will often omit the star symbol.)

$(Q_B \Psi)\, \Psi^n =$



$=$



$$\psi'_n = (Q_B \Psi)\, \Psi^n$$

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \psi'_n = \sum_{n=0}^{\infty} \lambda^{n+1} (Q_B \Psi)\, \Psi^n$$

$$= \lambda\, (Q_B \Psi) \frac{1}{1 - \lambda \Psi}$$

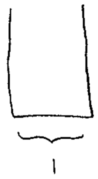
$$e^{-\Lambda} (Q_B e^{\Lambda}) = -(Q_B e^{-\Lambda})\, e^{\Lambda} \qquad e^{\Lambda} = \frac{1}{1 - \lambda \Psi}$$

⑩ 2016 12/6 (火)

KBc subalgebra

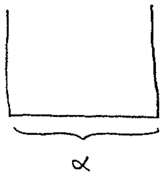
Okawa, hep-th/0603159

$= K$ (even) $|0\rangle =$

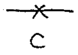


$= e^{K}$

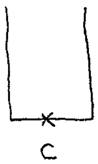
$W_\alpha =$



$= e^{\alpha K}$

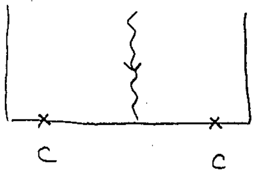


$= c$ (odd) $\psi_0 =$



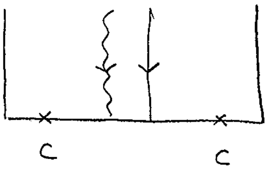
$= e^{\frac{K}{2}} c\, e^{\frac{K}{2}}$

$= B$ (odd) $\psi_n =$



$= e^{\frac{K}{2}} c B\, e^{nK} c\, e^{\frac{K}{2}}$

$\psi_n' =$



$= e^{\frac{K}{2}} c B K e^{nK} c\, e^{\frac{K}{2}}$

$$\boxed{\begin{array}{c} [K, B] = 0, \quad \{B, c\} = 1, \quad c^2 = 0, \quad B^2 = 0 \\ Q_B B = K, \quad Q_B K = 0, \quad Q_B c = cKc \end{array}}$$

$\uparrow$

$Q_B \cdot c = c\partial c$

$$\Psi_\lambda = \lambda e^{\frac{K}{2}} c \frac{BK}{1-\lambda e^{K}} c\, e^{\frac{K}{2}} = f(K)\, c \frac{BK}{1-f(K)^2} c\, f(K)$$

with $f(K) = \sqrt{\lambda}\, e^{\frac{K}{2}}$

$$Q_B \Psi_\lambda = Q_B \left[f(K)\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K) \right]$$

$$= f(K)\, cKc\, \frac{BK}{1-f(K)^2}\, c\, f(K) - f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K)$$

$$+ f(K)\, c\, \frac{BK}{1-f(K)^2}\, cKc\, f(K)$$

$$\Psi_\lambda^2 = f(K)\, c\, \frac{KB}{1-f(K)^2}\, c\, f(K)^2\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K)$$

$$Bc\, f(K)^2\, cB = Bc\, f(K)^2 - Bc\, f(K)^2\, Bc$$

$$= Bc\, f(K)^2 - f(K)^2\, Bc = [Bc, f(K)^2]$$

$$= -[Bc, 1-f(K)^2]$$

$$\Psi_\lambda^2 = - f(K)\, c\, \frac{K}{1-f(K)^2}\, [Bc, 1-f(K)^2]\, \frac{K}{1-f(K)^2}\, c\, f(K)$$

$$= - f(K)\, c\, \frac{K}{1-f(K)^2}\, BcKc\, f(K)$$

$$+ f(K)\, cK\underset{\sim}{Bc}\, \frac{K}{1-f(K)^2}\, c\, f(K)$$

$$= - f(K)\, c\, \frac{BK}{1-f(K)^2}\, cKc\, f(K)$$

$$+ f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K)$$

$$- f(K)\, cKc\, \frac{BK}{1-f(K)^2}\, c\, f(K)$$

$$\Rightarrow \quad Q_B \Psi_\lambda + \Psi_\lambda^2 = 0$$

<u>Note</u> $Q_B \Psi_\lambda + \Psi_\lambda^2 = 0$ for any $f(K)$

$$\frac{K}{1-f(K)^2} = \frac{K}{1-\lambda e^{K}} = O(K) \quad \text{for } |\lambda| < 1$$

$$\lambda = 1 \ : \quad \frac{K}{1-e^{K}} = O(K^0)$$

$$\frac{K}{1-e^{K}} = -\sum_{n=0}^{\infty} \frac{B_n K^n}{n!}$$

$$\Psi_{\lambda=1} = -\sum_{n=0}^{\infty} \frac{B_n}{n!} e^{\frac{K}{2}} c B K^n c e^{\frac{K}{2}} = -\sum_{n=0}^{\infty} \frac{B_n}{n!} \psi_0^{(n)}$$

$$\frac{K}{1-e^{K}} = \sum_{n=0}^{\infty} K e^{nK}$$

$$\sum_{n=0}^{N} K e^{nK} = \frac{K(1-e^{(N+1)K})}{1-e^{K}}$$

$$\frac{K}{1-e^{K}} = \sum_{n=0}^{N} K e^{nK} + \frac{K}{1-e^{K}} e^{(N+1)K}$$

$$\lim_{N\to\infty} e^{NK} = W_\infty \quad : \text{finite}$$

$$K^n e^{NK} \xrightarrow[N\to\infty]{} 0 \qquad n : \text{positive integer}$$

$$\frac{K}{1-e^{K}} = \lim_{N\to\infty} \left[\sum_{n=0}^{N} K e^{nK} - e^{NK} \right]$$

⑩ 2021 12/17 (金)

the trace notation

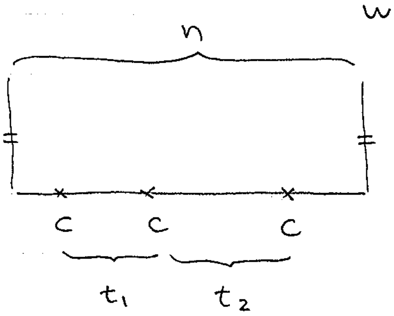
$$\mathrm{tr}\, AB = \langle A, B \rangle_{\text{density}}$$

BPZ inner products which consist of K, B, and c can be reduced to $\mathrm{tr}\, c e^{t_1 K} c e^{t_2 K} c e^{t_3 K}$ and $\mathrm{tr}\, c e^{t_1 K} c e^{t_2 K} c e^{t_3 K} c e^{t_4 K} B$.

$$\mathrm{tr}\, c e^{t_1 K} c e^{t_2 K} c e^{t_3 K}$$

$$= -\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n} \sin\frac{\pi t_2}{n} \sin\frac{\pi t_3}{n}$$

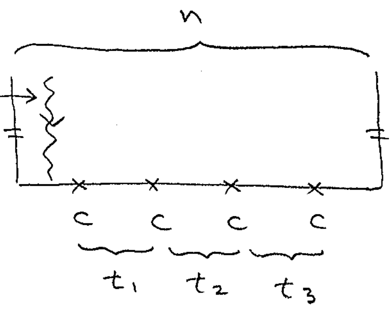
with $n = t_1 + t_2 + t_3$



┘ ⑩ 2015 12/1 (火)

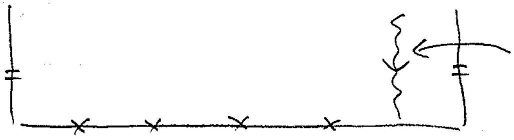
Consider

$$\int \frac{dz}{2\pi i}\, z\, b(z)$$



$$n = t_1 + t_2 + t_3 + t_4$$

=



$$\int \frac{dz}{2\pi i} (z-n)\, b(z)$$

$$\mathrm{tr}\; c\, e^{t_1 K}\, c\, e^{t_2 K}\, c\, e^{t_3 K}\, c\, e^{t_4 K}\, B$$

$$= -\frac{t_1}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi(t_1+t_2)}{n}\, \sin\frac{\pi t_3}{n}\, \sin\frac{\pi t_4}{n}$$

$$+ \frac{t_1+t_2}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n}\, \sin\frac{\pi(t_2+t_3)}{n}\, \sin\frac{\pi t_4}{n}$$

$$- \frac{t_1+t_2+t_3}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n}\, \sin\frac{\pi t_2}{n}\, \sin\frac{\pi(t_3+t_4)}{n}$$

energy

$$K_2 = \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{N} \langle \psi_n', Q_B \psi_m' \rangle_{\text{density}} = \frac{1}{2} - \frac{1}{\pi^2}$$

$$K_1 = \lim_{N\to\infty} \sum_{m=0}^{N} \langle \psi_N, Q_B \psi_m' \rangle_{\text{density}} = \frac{1}{2} + \frac{2}{\pi^2}$$

$$K_0 = \lim_{N\to\infty} \langle \psi_N, Q_B \psi_N \rangle_{\text{density}} = \frac{1}{2} + \frac{2}{\pi^2}$$

$$\langle \Psi, Q_B \Psi \rangle_{\text{density}} = K_2 - 2K_1 + K_0 = -\frac{3}{\pi^2}$$

The phantom piece ψ_N is necessary for

$$\langle \Psi, Q_B \Psi \rangle + \langle \Psi, \Psi^2 \rangle = 0.$$

(2016 5分休憩)

Phantomless solution

Erler and Schnabl, arXiv: 0906.0979

Choose $f(K) = \frac{1}{\sqrt{1-K}}$

$$\frac{K}{1-f(K)^2} = \frac{K}{1-\frac{1}{1-K}} = K-1$$

$$\Psi = \frac{1}{\sqrt{1-K}} \, cB(K-1)c \, \frac{1}{\sqrt{1-K}}$$

$$\tilde{\Psi} = \sqrt{1-K} \, \Psi \, \frac{1}{\sqrt{1-K}} = cB(K-1)c \, \frac{1}{1-K}$$

$$\frac{1}{\sqrt{1-K}} = \frac{1}{\sqrt{\pi}} \int_0^\infty dt \, \frac{e^{-t}}{\sqrt{t}} \, e^{tK}$$

$$\frac{1}{1-K} = \int_0^\infty dt \, e^{-(1-K)t} = \int_0^\infty dt \, e^{-t} e^{tK}$$

the normalized potential

$$\hat{V}(\Psi) = \frac{V(\Psi)}{T_{25}} = 2\pi^2 \mathrm{tr}\left[\frac{1}{2}\Psi Q_B \Psi + \frac{1}{3}\Psi^3\right]$$

($\hat{V}(\Psi) = -1$ at the tachyon vacuum)

$$\hat{V}(\Psi) = \hat{V}(\tilde{\Psi})$$

$$\tilde{\Psi} = cBKc\frac{1}{1-K} - c\frac{1}{1-K}$$

$$= [Q_B(Bc)]\frac{1}{1-K} - c\frac{1}{1-K}$$

$$Q_B\tilde{\Psi} = -(Q_B c)\frac{1}{1-K} = -cKc\frac{1}{1-K}$$

$$\tilde{\Psi} Q_B \tilde{\Psi} = -[Q_B(Bc)]\frac{1}{1-K}(Q_B c)\frac{1}{1-K} + c\frac{1}{1-K}cKc\frac{1}{1-K}$$

$$= -Q_B\left[Bc\frac{1}{1-K}(Q_B c)\frac{1}{1-K}\right] + c\frac{1}{1-K}cKc\frac{1}{1-K}$$

$$\hat{V}(\Psi) = \frac{\pi^2}{3}\mathrm{tr}\,\tilde{\Psi} Q_B \tilde{\Psi} = \frac{\pi^2}{3}\mathrm{tr}\, c\frac{1}{1-K}cKc\frac{1}{1-K}$$

$$\mathrm{tr}\; c\, e^{t_1 K}\, c K c\, e^{t_2 K}$$

$$= -\left(\frac{t_1+t_2}{\pi}\right)^2 \sin\frac{\pi t_1}{t_1+t_2}\;\sin\frac{\pi t_2}{t_1+t_2}$$

$$= -\left(\frac{t_1+t_2}{\pi}\right)^2 \sin^2\frac{\pi t_1}{t_1+t_2}$$

$$\hat{V}(\Psi) = -\frac{\pi^2}{3}\int_0^\infty dt_1 \int_0^\infty dt_2\; e^{-t_1} e^{-t_2} \left(\frac{t_1+t_2}{\pi}\right)^2 \sin^2\frac{\pi t_1}{t_1+t_2}$$

$$\begin{cases} u = t_1 + t_2 \\ v = \dfrac{t_1}{t_1+t_2} \end{cases}$$

$$\left[\;\begin{cases} t_1 = uv \\ t_2 = u(1-v) \end{cases} \\ dt_1\, dt_2 = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} du\, dv = u\, du\, dv \;\right]$$

$$dt_1\, dt_2 = u\, du\, dv$$

$$\hat{V}(\Psi) = -\frac{1}{3}\underbrace{\int_0^\infty du\; u^3 e^{-u}}_{\Gamma(4)\,=\,6}\;\underbrace{\int_0^1 dv\; \sin^2 \pi v}_{\frac{1}{2}} = -1$$

⑪ 2016 12/13（火）

（2015, 2021 5分休憩）

§ 3. 3 Analytic solutions for marginal deformations

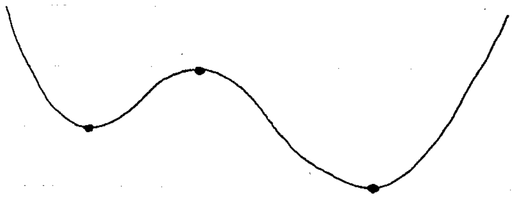
a boundary CFT with $c = 26$

→ a consistent open string background

The perturbation theory is independently defined for each consistent background.

a universal set of degrees of freedom?

Using string field theory, we can in principle discuss various backgrounds.



Open string field theory can be formulated for any boundary CFT with $c = 26$.

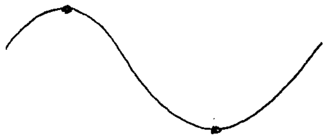
However, we need to choose one boundary CFT.

different backgrounds → classical solutions

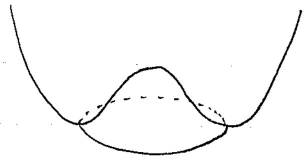
$$\begin{tikzcd} \text{the equation of motion of the spacetime theory} \arrow[d, leftrightarrow] \\ \text{conformal invariance of the world-sheet theory} \\ \text{gauge invariance} \arrow[uu, bend left] \end{tikzcd}$$

Marginal deformations with regular operator products

tachyon condensation



marginal deformations of the boundary CFT



We expect a one-parameter family of solutions to string field theory.

The deformation of the boundary CFT

$$S_{BCFT} \rightarrow S_{BCFT} + \lambda \int_{\text{boundary}} dt \; V(t)$$

is marginal if V is a primary field of weight 1.

examples

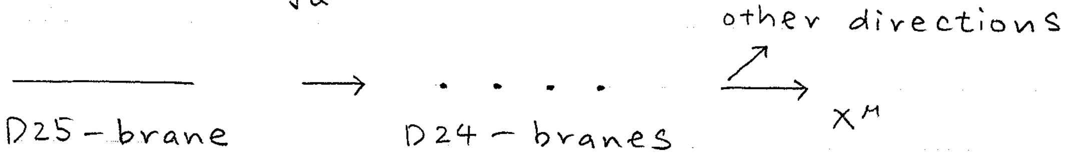
$$V(t) = \frac{i}{\sqrt{2\alpha'}} \partial_t X^\mu(t)$$

constant mode of the gauge field

$$V(t) = \frac{1}{\sqrt{2\alpha'}} \partial_\perp X^\alpha(t)$$

transverse coordinate of the D-brane

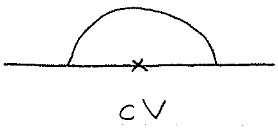
$$V(t) = \sqrt{2} : \cos \frac{X^\mu(t)}{\sqrt{\alpha'}} :$$



For any primary field V of weight 1 in the matter sector, cV is BRST closed:

$$Q_B \cdot cV(t) = \oint \frac{dz}{2\pi i} j_B(z)\, cV(t) = 0 .$$

$\Psi^{(1)}$ =



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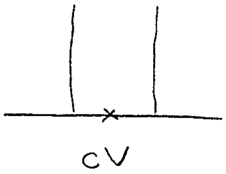


$(f \circ cV(t) = cV(f(t)))$

$$Q_B \Psi^{(1)} = 0$$

When the deformation is exactly marginal, we expect a solution of the form

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \Psi^{(n)}$$

to $Q_B \Psi + \Psi * \Psi = 0$.

$$Q_B \Psi^{(1)} = 0$$
$$Q_B \Psi^{(2)} = - \Psi^{(1)} * \Psi^{(1)}$$
$$\vdots$$
$$Q_B \Psi^{(n)} = - \sum_{m=1}^{n-1} \Psi^{(m)} * \Psi^{(n-m)}$$

Formally, $\Psi^{(2)} = - \frac{b_0}{L_0} [\Psi^{(1)} * \Psi^{(1)}]$.

⑪ 2015 12/8 (火)

(2016 5分休憩)

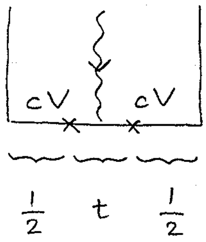
Analytic solutions for regular operator products
($V(t_1)\, V(t_2) \cdots V(t_n)$: regular)
hep-th/0701248 by Schnabl
hep-th/0701249
by Kiermaier, Okawa, Rastelli and Zwiebach

$$\Psi^{(2)} = -\frac{B}{L}\left[\Psi^{(1)} * \Psi^{(1)}\right]$$

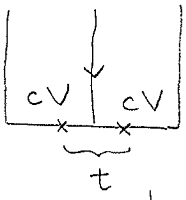
$$\underset{\substack{\uparrow \\ \text{Subtle} \\ \text{(later)}}}{=} -\int_0^\infty dT\ B\, e^{-TL}\left[\Psi^{(1)} * \Psi^{(1)}\right]$$

(a few nontrivial steps)

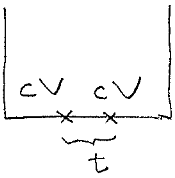
$$= \int_0^1 dt$$



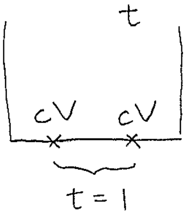
$Q_B \Psi^{(2)} = -\int_0^1 dt$



$= -\int_0^1 dt \frac{\partial}{\partial t}$



$= -$



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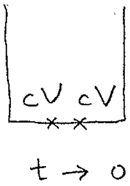


$\parallel$

$- \Psi^{(1)} * \Psi^{(1)}$

$\lim_{\epsilon \to 0} cV(0)\, cV(\epsilon) = 0 \Rightarrow Q_B \Psi^{(2)} = -\Psi^{(1)} * \Psi^{(1)}$

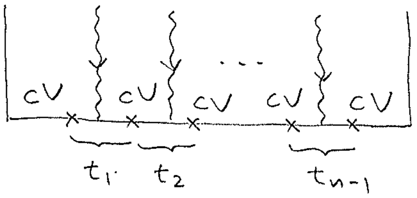
If $\lim_{\epsilon \to 0} V(0)\, V(\epsilon)$ is finite or vanishing,

$\Psi^{(2)}$ is finite and $Q_B \Psi^{(2)} = -\Psi^{(1)} * \Psi^{(1)}$.

Note that $B \Psi^{(2)} = 0$. ⊣

⑪ 2021 12/24 (金)

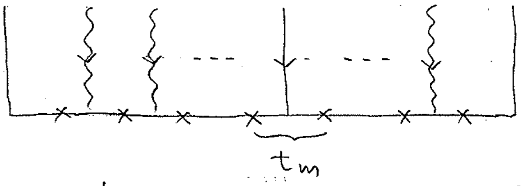
$$\Psi^{(n)} = \int d^{n-1}t$$



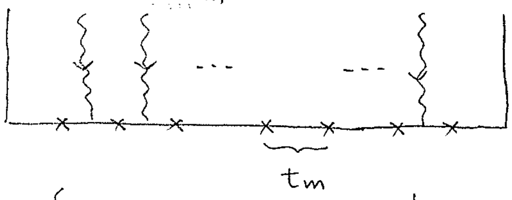
$$\uparrow$$

$$\int d^{n-1}t = \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_{n-1}$$

$$Q_B \Psi^{(n)} = -\sum_m \int d^{n-1}t$$



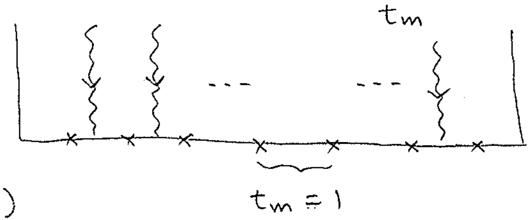
$$= -\sum_m \int d^{n-1}t \, \frac{\partial}{\partial t_m}$$



$$= -\sum_m \int d^{n-2}t$$

$\uparrow$

(no dt_m)



$$= -\sum_m \Psi^{(m)} * \Psi^{(n-m)}$$

example

$$V(t) = \, :\exp\left[\frac{1}{\sqrt{\alpha'}} X^0(t)\right]:$$

rolling tachyon

an exact time-dependent solution incorporating all α' corrections.

Note that $B\Psi = 0$.

⑫ 2016 12/20 (火)

singular operator products

a typical example

$$V(t_1)\ V(t_2) \sim \frac{1}{(t_1-t_2)^2}$$

① Regularize the integral of t_i as

$$\int_0^1 dt_i \rightarrow \int_\epsilon^1 dt_i .$$

② Add counterterms.

③ $\epsilon \rightarrow 0$

$\Psi^{(2)}$ and $\Psi^{(3)}$ were constructed.

obstruction when the deformation is not exactly marginal

$B\Psi \neq 0$ no solution in Schnabl gauge

(2015 5分休憩)

Solutions for general marginal deformations
arXiv: 0707.4472 by Kiermaier and Okawa
(arXiv: 0704.2222 by Fuchs, Kroyter and Potting)

unintegrated vertex operator $cV(t)$

$$Q_B \cdot cV(t) = 0$$

integrated vertex operator

$$V(a,b) = \int_a^b dt\, V(t)$$

$$Q_B \cdot \int_a^b dt\, V(t) = \int_a^b dt\, \partial_t [cV(t)]$$

$$= cV(b) - cV(a)$$

previous $\Psi^{(n)}$

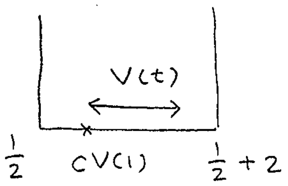
n unintegrated vertex operators
$n-1$ integrals of moduli
$n-1$ b-ghost insertions

↓

1 unintegrated vertex operator
$n-1$ integrated vertex operators

new solution (with regular operator products)

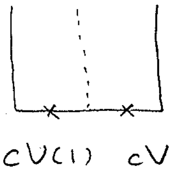
$\Psi_L^{(2)}$: $\int_1^2 dt$



$$Q_B \cdot \int_1^2 dt \;\; cV(1) \;\; V(t) \;=\; - \int_1^2 dt \;\; cV(1) \;\; \partial_t \left[cV(t) \right]$$

$$= - \; cV(1) \;\; cV(2)$$

$-$

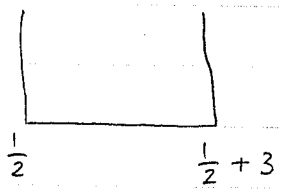


$$= - \; \Psi^{(1)} * \Psi^{(1)}$$

$$\Psi_L^{(3)} : \; \frac{1}{2} \; cV(1) \left[\int_1^3 dt \; V(t) \right]^2 - \frac{1}{2} \; cV(1) \left[\int_2^3 dt \; V(t) \right]^2$$

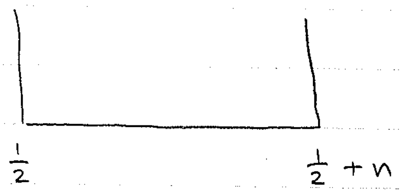
$$= \int_1^2 dt_1 \int_{t_1}^3 dt_2 \;\; cV(1) \;\; V(t_1) \;\; V(t_2)$$

on



$$\Psi_L^{(n)} : \; \int_1^2 dt_1 \int_{t_1}^3 dt_2 \int_{t_2}^4 dt_3 \; \cdots \int_{t_{n-2}}^n dt_{n-1} \;\; cV(1) \;\; V(t_1) \; V(t_2) \; \cdots \; V(t_{n-1}) \;)$$

on



$\Psi_L^{(n)}$ ingredients : $e^{\lambda V(a,b)}$, $\lambda cV(a) e^{\lambda V(a,b)}$

fixed wedge state W_n

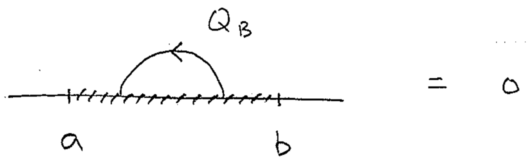
We only need the relations

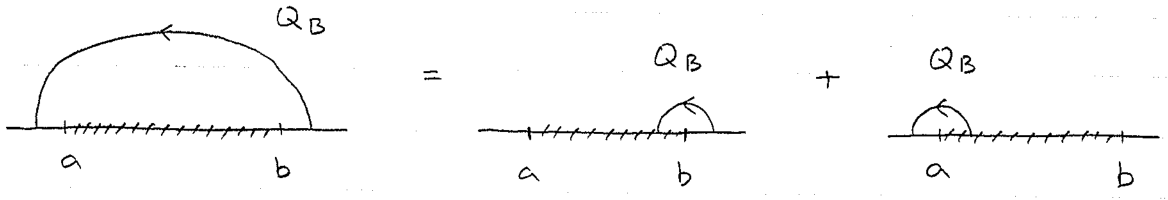
$$Q_B \cdot e^{\lambda V(a,b)}$$
$$= e^{\lambda V(a,b)} \lambda cV(b) - \lambda cV(a) e^{\lambda V(a,b)}$$

and

$$Q_B \cdot [\lambda cV(a) e^{\lambda V(a,b)}]$$
$$= -\lambda cV(a) e^{\lambda V(a,b)} \lambda cV(b) .$$

These relations generalize to the singular case.





change of boundary conditions

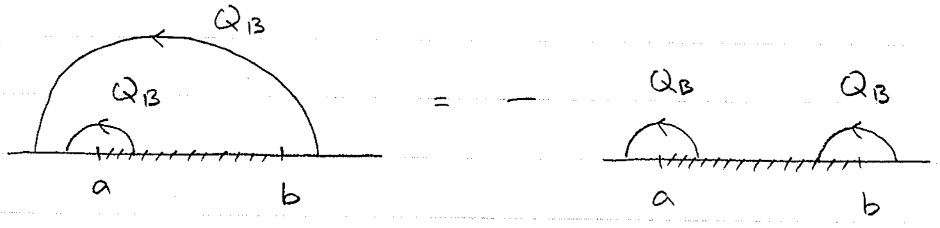
$$e^{\lambda V(a,b)} = 1 + \lambda \int_a^b dt\, V(t) + \frac{\lambda^2}{2} \int_a^b dt_1 \int_a^b dt_2\, V(t_1) V(t_2) + \cdots$$

↓

$[e^{\lambda V(a,b)}]_r$ the singular case

↖ renormalization

$$Q_B \cdot [e^{\lambda V(a,b)}]_r$$
$$= [e^{\lambda V(a,b)} O_R(b)]_r - [O_L(a) e^{\lambda V(a,b)}]_r$$



$$Q_B \cdot [O_L(a) e^{\lambda V(a,b)}]_r = -[O_L(a) e^{\lambda V(a,b)} O_R(b)]_r$$

$O_L = \lambda cV + O(\lambda^2)$, $O_R = \lambda cV + O(\lambda^2)$ ⌟

⑫ 2015 12/15 (火)

⑫ 2022 1/7 (金) ⑬ 2017 11/10 (火)