Analysis VIII/Linear Differential Equations

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On this course

Purpose: We learn basics of pseudodiffernetial operators.

Grading: The grade will be decided by a final report. The report problems will be distributed later in this course.

References:

- X. Saint Raymond, "Elementary Introduction to the Theory of Pseudodifferential Operators", CRC Press
- H. Kumano-go, "Pseudo-Differential Operators", MIT Press
- A. Martinez, "An Introduction to Semiclassical and Microlocal Analysis", Springer
- M.A. Shubin, "Pseudodifferntial Operators and Spectral Analysis", Springer
- M. Zworski, "Semiclassical Analysis", Amer. Math. Soc.
- N. Lerner, "Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators", Springer

Chapter 1

Oscillatory Integrals

§ 1.1 Introduction

◦ Notation

In this course we use the notation

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}.$$

We usually let $d \in \mathbb{N}$ be the dimension of the **configuration space**. For any **multi-index** $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we define its **length** and **factorial** as

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = (\alpha_1!) \cdot \cdots \cdot (\alpha_d!),$$

respectively. In addition, for any $\alpha, \beta \in \mathbb{N}_0^d$ we let

$$\alpha \le \beta \quad \overset{\text{def}}{\Longleftrightarrow} \quad \alpha_j \le \beta_j \ \text{ for all } j = 1, \ldots, d,$$

and define the **binomial coefficient** as

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \quad \text{if } 0 \le \beta \le \alpha, \qquad \binom{\alpha}{\beta} = 0 \quad \text{otherwise},$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_d - \beta_d)$.

For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \qquad \partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}.$$

Moreover, we introduce the notation

$$D_j = -\mathrm{i}\partial_j, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}.$$

Then, in particular, we have

$$D^\alpha = (-\mathrm{i})^{|\alpha|}\partial^\alpha.$$

Thoughout the course for any $x, \xi \in \mathbb{R}^d$ we write simply

$$x\xi = x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d, \quad x^2 = x \cdot x, \quad |x| = \sqrt{x \cdot x},$$

and we adopt the **Fourier transform**, its inverse defined as extensions from

$$\mathcal{F}u(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} u(x)\,\mathrm{d}x \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d),$$
$$\mathcal{F}^* f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\xi} f(\xi)\,\mathrm{d}\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d),$$

respectively. Note, in particular, for any $u, v \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$

$$(u, v)_{L^2} = (\mathcal{F}u, \mathcal{F}v)_{L^2}, \quad \mathcal{F}^*\xi^\alpha \mathcal{F}u = D^\alpha u,$$

where $(\cdot, \cdot)_{L^2}$ denotes the L^2-**inner product**, being linear and conjugate-linear in the first and second entries, respectively.

Problem. 1. (**Binomial theorem**) Show for any $\alpha \in \mathbb{N}_0^d$ and $x, y \in \mathbb{R}^d$

$$(x+y)^\alpha = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta; \quad \text{In particular,} \quad \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} = 2^{|\alpha|}.$$

2. (**Leibniz rule**) Show for any $\alpha \in \mathbb{N}_0^d$ and $f, g \in C^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha(fg) = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g).$$

◦ **Partial differential operators**

Consider a partial differential operator (PDO) on $\mathbb{R}^d$:

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \qquad a_\alpha \in C^\infty(\mathbb{R}^d).$$

If we let

$$a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

then we can write for any $u \in C_c^\infty(\mathbb{R}^d)$

$$Au(x) = a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(x,\xi) u(y) \, dy d\xi.$$

The last integral makes sense even if we replace the polynomial $a(x, \xi)$ in ξ by a **symbol** growing at most polynomially in $\xi \in \mathbb{R}^d$. That is a **pseudodifferential operator** (ΨDO, or PsDO). We are going to develop a pseudodifferential calculus for an appropriate symbol class, and discuss its applications.

Remark. The last integral has to be interpreted as an iterated integral; The integrand is not integrable in (y, ξ). However, we can also justify it as an **oscillatory integral**, as discussed in the following section.

§ 1.2 Oscillatory Integrals

For any $x \in \mathbb{R}^d$ we let

$$\langle x \rangle = (1 + x^2)^{1/2} \in C^\infty(\mathbb{R}^d).$$

Lemma 1.1. 1. For any $x \in \mathbb{R}^d$

$$\frac{1}{\sqrt{2}}(1 + |x|) \leq \langle x \rangle \leq 1 + |x|.$$

2. For any $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that for any $x \in \mathbb{R}^d$

$$|\partial^\alpha \langle x \rangle| \leq C_\alpha \langle x \rangle^{1-|\alpha|}.$$

3. (**Peetre's inequality**) For any $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$\langle x + y \rangle^s \leq 2^{|s|} \langle x \rangle^{|s|} \langle y \rangle^s.$$

Proof. 1, 2. We omit the proofs.

3. By the assertion 1 we can estimate

$$\begin{aligned}\langle x+y\rangle &\le 1+|x+y| \le 1+|x|+|y| \\ &\le (1+|x|)(1+|y|) \le 2\langle x\rangle\langle y\rangle.\end{aligned}$$

This implies the assertion for $s \ge 0$. The same estimate also implies

$$\langle y\rangle^{-1} \le 2\langle x\rangle\langle x+y\rangle^{-1}.$$

If we replace x by $-x$, and then y by $x+y$, it follows that

$$\langle x+y\rangle^{-1} \le 2\langle x\rangle\langle y\rangle^{-1},$$

which implies the assertion for $s \le 0$. Hence we are done. $\square$

∘ **Oscillatory Integrals**

For any $m, \delta \in \mathbb{R}$ we define the set of **amplitude functions** as

$$A_\delta^m(\mathbb{R}^d) = \left\{ a \in C^\infty(\mathbb{R}^d);\ \forall \alpha \in \mathbb{N}_0^d \ \sup_{x \in \mathbb{R}^d} \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)| < \infty \right\}.$$

For any $k \in \mathbb{N}_0$ define a **seminorm** $|\cdot|_k$ on $A_\delta^m(\mathbb{R}^d)$ as

$$|a|_k = |a|_{k, A_\delta^m} = \sup \left\{ \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)|;\ |\alpha| \le k,\ x \in \mathbb{R}^d \right\}.$$

Remark. Obviously, $A_\delta^m(\mathbb{R}^d)$ is a **Fréchet space** with respect to the family $\{|\cdot|_k\}_{k \in \mathbb{N}_0}$ of seminorms.

Theorem 1.2. Let Q be a non-degenerate real symmetric matrix of order d, and let $m \in \mathbb{R}$ and $\delta < 1$. Then for any $a \in A^m_\delta(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$ there exists the limit

$$I_Q(a) := \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x) \, \mathrm{d}x, \qquad (\spadesuit)$$

and it is independent of choice of $\chi \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exist $k \in \mathbb{N}_0$ and $C > 0$ such that for any $a \in A^m_\delta(\mathbb{R}^d)$

$$|I_Q(a)| \le C|a|_{k, A^m_\delta}.$$

Remark. The last bound implies $I_Q \colon A^m_\delta(\mathbb{R}^d) \to \mathbb{C}$ is continuous.

Proof. Noting that for any $x, y \in \mathbb{R}^d$

$$y\partial\Big(\frac{xQx}{2}\Big) = \frac{1}{2}\sum_{j=1}^{d} y_j(e_jQx + xQe_j) = yQx,$$

we can deduce

$$\mathrm{e}^{\mathrm{i}xQx/2} = {}^tL\mathrm{e}^{\mathrm{i}xQx/2}; \quad {}^tL = \langle x\rangle^{-2}\Big(1 + xQ^{-1}D\Big).$$

Substitute the above identity into the integrand of (♠), and integrate it by parts. Repeat this precedure, and we obtain

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}L^k\Big(\chi(\epsilon x)a(x)\Big)\,\mathrm{d}x$$

for any $k \in \mathbb{N}_0$. Since L is of the form

$$L = c_0 + \sum_{j=1}^{d} c_j\partial_j; \quad c_0 \in A^{-2}_{-1}(\mathbb{R}^d), \quad c_j \in A^{-1}_{-1}(\mathbb{R}^d),$$

there exists $C > 0$ such that for any $\epsilon \in (0, 1)$ and $a \in A^m_\delta(\mathbb{R}^d)$

$$\Big| L^k \Big(\chi(\epsilon x) a(x) \Big) \Big| \le C |a|_{k, A^m_\delta} \langle x \rangle^{m-(1-\delta)k}. \qquad (\heartsuit)$$

We also note there exists a pointwise limit

$$\lim_{\epsilon \to +0} L^k \Big(\chi(\epsilon x) a(x) \Big) = L^k a(x).$$

Then, if we choose $k \in \mathbb{N}_0$ such that $m - (1 - \delta)k < -d$, it follows by the Lebesgue convergence theorem that

$$I_Q(a) = \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} L^k a(x) \, \mathrm{d}x.$$

Certainly the last expression is independent of χ. Combined with $(\heartsuit)$, it also implies the asserted bound. We are done. $\square$

Remarks. 1. The limit (♠) from Theorem 1.2 is called an **oscillatory integral**, and is denoted simply by

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\,\mathrm{d}x.$$

The notation is compatible with the case $a \in L^1(\mathbb{R}^d)$.

2. We can also define the oscillatory integral as

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} L^k a(x)\,\mathrm{d}x,$$

where L^k is from the proof of Theorem 1.2. Practically, in order to compute an oscillatory integral we may implement *any* formal integrations by parts until the integrand gets integrable, see Lemma 1.3.3 and the preceding remark.

Lemma 1.3. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$.

1. For any $c \in \mathbb{R}^d$
$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \mathrm{e}^{\mathrm{i}cQc/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}yQy/2} \Big(\mathrm{e}^{\mathrm{i}cQx} a(y+c)\Big)\,\mathrm{d}y.$$

2. For any real invertible matrix P of order d
$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py) |\det P|\,\mathrm{d}y.$$

3. For any $\alpha \in \mathbb{N}_0^d$
$$\int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) a(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \partial^\alpha a(x)\,\mathrm{d}x.$$

Proof. 1 and 2. We can prove 1 and 2 very similarly, and here we disucss only 2. Let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then by definition of the oscillatory integral

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x &= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\,\mathrm{d}x \\ &= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} \chi(\epsilon Py) a(Py) |\det P|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py) |\det P|\,\mathrm{d}y.\end{aligned}$$

This implies the assertion.

3. Similarly to the above, let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) \chi(\epsilon x) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon \to +0} (-1)^{|\alpha|} \Bigg[\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) \partial^\alpha a(x)\,\mathrm{d}x \\
&\qquad\qquad + \sum_{|\beta| \geq 1} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \Big(\partial^\beta \chi(\epsilon x)\Big) \Big(\partial^{\alpha-\beta} a(x)\Big)\,\mathrm{d}x \Bigg].
\end{aligned}
$$

For the second integral in the above square brackets we can further implement integrations by parts, e.g., by using L from the proof of Theorem 1.2, and then we can verify that it converges to 0 as $\epsilon \to +0$. Thus we obtain the assertion. $\square$

§ 1.3 Expansion Formula

Definition. Let Q be a non-degenerate real symmetric matrix of order d, and let $u \in \mathcal{S}'(\mathbb{R}^d)$. We define

$$\mathrm{e}^{\mathrm{i}DQD/2}u = \mathcal{F}^*\mathrm{e}^{\mathrm{i}\xi Q\xi/2}\mathcal{F}u \in \mathcal{S}'(\mathbb{R}^d).$$

Theorem 1.4. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$. Then

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}yQ^{-1}y/2}a(x+y)\,\mathrm{d}y.$$

Remark. For $a \in A^m_\delta(\mathbb{R}^d)$ we can compute pointwise values of $\mathrm{e}^{\mathrm{i}DQD/2}a$ within the smooth category.

Theorem 1.5. There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^d)$ and $N \in \mathbb{N}$

$$\mathsf{e}^{\mathsf{i}DQD/2}a(x) = \sum_{k=0}^{N-1} \frac{\mathsf{i}^k}{2^k k!}(DQD)^k a(x) + R_N(a)$$

with

$$\left|R_N(a)\right| \leq \frac{C}{2^N N!} \sum_{|\alpha| \leq d+1} \left\|\partial^\alpha (DQD)^N a\right\|_{L^1}.$$

Lemma 1.6. Let Q be a non-degenerate real symmetric matrix of order d. Then

$$\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\Big)(\xi) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2}}\mathrm{e}^{-\mathrm{i}\xi Q^{-1}\xi/2}.$$

Proof. Step 1. We first let $d = 1$. Since $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is continuous, we can proceed as

$$\begin{aligned}
\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}Qx^2/2}\Big)(\xi) &= \lim_{\epsilon\to+0}\Big(\mathcal{F}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)x^2/2}\Big)(\xi)\\
&= \lim_{\epsilon\to+0}\Big(\epsilon-\mathrm{i}Q\Big)^{-1/2}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)^{-1}\xi^2/2}\\
&= \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|Q|^{1/2}}\mathrm{e}^{-\mathrm{i}Q^{-1}\xi^2/2}.
\end{aligned}$$

Thus the assertion for $d = 1$ is verified.

Step 2. There exists an invertible real matrix P such that

$$ {}^{t}PQP = \mathrm{diag}(I_p, -I_q), $$

where I_p, I_q are the identity matrices of order $p, q \in \mathbb{N}_0$ with $p + q = d$, respectively. Changing variables as $x = Py$ and spliting $y = (y', y'') \in \mathbb{R}^p \times \mathbb{R}^q$, we can compute

$$
\begin{aligned}
&\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\Big)(P^{-1}\eta) \\
&= \lim_{\epsilon \to +0} \Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\mathrm{e}^{-\epsilon x({}^{t}P^{-1}P^{-1})x}\Big)(P^{-1}\eta) \\
&= \lim_{\epsilon \to +0} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y\eta}\mathrm{e}^{\mathrm{i}(y'^2 - y''^2)/2}\mathrm{e}^{-\epsilon y^2} |\det P| \, \mathrm{d}y \\
&= |\det P| \mathrm{e}^{\mathrm{i}\pi(\mathrm{sgn}\, Q)/4}\mathrm{e}^{-\mathrm{i}(\eta'^2 - \eta''^2)/2},
\end{aligned}
$$

where in the last equality we use the result from Step 1. Finally let $\eta = P\xi$, and we obtain the assertion. $\square$

Proof of Theorem 1.4. Let $a \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. Then it follows by change of variables, the Plancherel theorem and Lemma 1.6

$$\begin{aligned} \mathrm{e}^{\mathrm{i}DQD/2}a(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\xi Q\xi/2}\Big(\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}y\xi}a(x+y)\,\mathrm{d}y\Big)\,\mathrm{d}\xi \\ &= \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn}Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}}\int_{\mathbb{R}^{2d}}\mathrm{e}^{-\mathrm{i}yQ^{-1}y/2}a(x+y)\,\mathrm{d}y. \end{aligned}$$

Then, since the right-hand side of the asserted identity is continuous on $A_\delta^m(\mathbb{R}^d)$ byTheorem 1.2, we obtain the assertion. $\square$

Proof of Theorem 1.5. Recall by Taylor's theorem for any $N \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\mathrm{e}^{\mathrm{i}t} = \sum_{k=0}^{N-1} \frac{(\mathrm{i}t)^k}{k!} + \frac{\mathrm{i}^N}{(N-1)!} \int_0^t \mathrm{e}^{\mathrm{i}s}(t-s)^{N-1}\,\mathrm{d}s,$$

so that we can write

$$\mathrm{e}^{\mathrm{i}\xi Q\xi/2} = \sum_{k=0}^{N-1} \frac{(\mathrm{i}\xi Q\xi)^k}{2^k k!} + r_N(\xi, h); \quad |r_N(\xi, h)| \le \frac{|\xi Q\xi|^N}{2^N N!}.$$

Substitute the above expansion into the definition of $\mathrm{e}^{\mathrm{i}DQD/2}a$ and implement the Fourier inversion formula, and then

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \sum_{k=0}^{N-1} \frac{\mathrm{i}^k}{2^k k!}(DQD)^k u(x) + R_N(a)$$

with

$$|R_N(a,h)| \leq \frac{1}{(2\pi)^{d/2} 2^N N!} \int_{\mathbb{R}^d} \Big| \Big(\mathcal{F}(DQD)^N a \Big)(\xi) \Big| \, \mathrm{d}\xi.$$

Finally it suffices to show that for any $v \in C_\mathrm{c}^\infty(\mathbb{R}^d)$

$$\|\mathcal{F}v\|_{L^1} \leq C \sum_{|\alpha| \leq d+1} \|\partial^\alpha v\|_{L^1}.$$

However, it is clear since

$$\mathcal{F}v(\xi) = (2\pi)^{-d/2} \langle \xi \rangle^{-2(d+1)} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} (1 + \xi D)^{d+1} v(x) \, \mathrm{d}x.$$

Thus we are done. $\square$

Corollary 1.7 (Stationary phase theorem). There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$, $N \in \mathbb{N}$ and $h > 0$

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/(2h)} a(x)\,\mathrm{d}x$$
$$= \sum_{k=0}^{N-1} \frac{(2\pi)^{d/2} h^{k+d/2} \mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2} (2\mathrm{i})^k k!} \Big((DQ^{-1}D)^k a\Big)(0) + R_N(a,h)$$

with

$$\Big|R_N(a,h)\Big| \le \frac{C h^{N+d/2}}{|\det Q|^{1/2} 2^N N!} \sum_{|\alpha| \le d+1} \Big\|\partial^\alpha (DQ^{-1}D)^N a\Big\|_{L^1}.$$

Proof. The assertion is clear by Theorems 1.4 and 1.5. □

Remarks. 1. As $h \to +0$, the rapid oscillatory factor $\mathrm{e}^{\mathrm{i}xQx/(2h)}$ cancels contributions from the amplitude a. However, the oscillation is slightly milder at the stationary point $x = 0$ of the phase function. This is why the behavior of a at around $x = 0$ dominates the asymptotics.

2. The **semiclassical parameter** $h > 0$, rooted in the **Planck constant**, plays a fundamental role in the **semiclassical analysis**. However, in this course we do not discuss it.

Problem. Show the following extended version of the "pointwise Fourier inversion formula": For any $a \in A_\delta^m(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$ and for any $\alpha \in \mathbb{N}_0^d$ and $x' \in \mathbb{R}^d$

$$(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \xi^\alpha a(x) \,\mathrm{d}x\mathrm{d}\xi = (D^\alpha a)(x').$$

Remark. This is an oscillatory integral on $\mathbb{R}^{2d} = \mathbb{R}_x^d \times \mathbb{R}_\xi^d$, not on $\mathbb{R}^d$, with a phase function

$$-x\xi = 4^{-1}\Big((x-\xi)^2 - (x+\xi)^2\Big)$$

and an amplitude $\mathrm{e}^{\mathrm{i}x'\xi} \xi^\alpha a(x) \in A_\delta^{\max\{m,|\alpha|\}}(\mathbb{R}^{2d})$.

Solution. By Lemma 1.3 it suffices to prove the assertion for $\alpha = 0$. By definition of oscillatory integrals, take any $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then we can compute

$$\begin{aligned}
&(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon \to +0} (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \chi(\epsilon x)\chi(\epsilon\xi) a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon \to +0} (2\pi\epsilon)^{-d/2} \int_{\mathbb{R}^{d}} (\mathcal{F}\chi)((x-x')/\epsilon)\chi(\epsilon x) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon \to +0} (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} (\mathcal{F}\chi)(\eta)\chi(\epsilon(x'+\epsilon\eta)) a(x'+\epsilon\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} a(x')(\mathcal{F}\chi)(\eta)\,\mathrm{d}\eta \\
&= a(x').
\end{aligned}$$

Hence we are done. □