Abstract Harmonic Analysis

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Contents

Part I

Chapter 1

Hopf $*$-algebras

1.1

Multiplier Hopf $*$-algebras
    Algebraic quantum groups
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Chapter 2

Locally compact groups

2.1

2.1 (Non-σ-finite measures). Following technical issues are important

(a) The Fubini theorem

(b) The Radon-Nikodym theorem

(c) The dual space of L^1 space

2.2 (Existence of the Haar measure).

2.3 (Left and right uniformities).

2.4 (Modular functions).

2.5 (Uniformly continuous functions). G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$. $g \in C_c(G)$ is two-sided uniformly continuous.

2.6 (Structures on a locally compact group). For a locally compact group G, consider $A := C_c(G)$. It is a left Hilbert algebra by the existence of the left Haar measure

$$(f * g)(s) := \int f(t)g(t^{-1}s)\,dt, \qquad \langle f, g\rangle := \int \overline{g(s)}f(s)\,ds, \qquad f^\sharp(s) := \delta(s^{-1})\overline{f(s^{-1})}.$$

and is a commutative counital multiplier Hopf $*$-algebra by the group structure.

$$(fg)(s) := f(s)g(s), \qquad \Delta f(s,t) = f(st), \qquad f^*(s) := \overline{f(s)}, \qquad Sf(s) = f(s^{-1}).$$

Since the image of the comultiplication does not belong to $C_c(G) \otimes C_c(G)$, we need to do something unless G is finite. They satisfy a compatibility condition $\langle fg, h\rangle = \langle f, g^*h\rangle$.

With the integral notation $f = \int f(s)\lambda_s\,ds$, we can write

For multipliers, intuitively

We start from this structures.

From now on, we are going to exclude any measure theory and the theory of non-commutative L^p spaces. First, we have the completion $H =: L^2(G)$. Consider two representations

$$\lambda : (C_c(G), *, ^\sharp) \to B(L^2(G)), \qquad m : (C_c(G), \cdot, ^*) \to B(L^2(G)).$$

(a) λ is well-defined.

(b) m is well-defined.

Proof. The multiplication representation m is well-defined because for $f \in C_c(G)$ we have $f^*f \in C_c(G) \subset L^2(G)$ so

$$\|m(f)g\|^2 = \langle fg, fg\rangle = \langle f^*fg, g\rangle, \qquad g \in C_c(G).$$

□

2.2

2.7 (Left convolution algebra $L^1(G)$)**.** Let G be a locally compact group. The representation m defines the von Neumann algebra $m(C_c(G))'' =: L^\infty(G)$ and its predual $L^1(G)$.

(a) There is a natural injection $C_c(G) \to L^1(G)$.

(b) There is a natural Banach $*$-algebra structure on $L^1(G)$ extended from the Hilbert algebra structure of $C_c(G)$.

(c) The Banach algebra $L^1(G)$ has a two-sided approximate unit.

(d) $\alpha : G \to \mathrm{Aut}(L^1(G))$ is point-norm continuous.

(e) $\lambda : G \to U(L^2(G))$ and $\lambda : L^1(G) \to B(L^2(G))$ are strongly continuous.

(f) Convolution inequalities.

(g) Representation theory equivalence.

Proof. Let (U_α) be a directed set of open neighborhoods of the identity e of G. By Urysohn lemma, there is $e_\alpha \in C_c(U)^+$ such that $\|e_\alpha\|_1 = 1$ for each α. We claim that e_α is a two-sided approximate unit for $L^1(G)$. Suppose $g \in C_c(G)$, which is two-sided uniformly continuous. For any $\varepsilon > 0$, take α_0 such that $\|g - \lambda_s g\| < \varepsilon$ and $\|g - \rho_s g\| < \varepsilon$ for all $s \in U_\alpha$ for $\alpha \succ \alpha_0$. Then, we have

$$\begin{aligned}\|e_\alpha * g - g\|_1 &= \int |e_\alpha * g(t) - g(t)|\,dt \le \iint e_\alpha(s)|g(s^{-1}t) - g(t)|\,ds\,dt \\ &= \int_{U_\alpha} e_\alpha(s)\|\lambda_s g - g\|_1\,ds < \varepsilon \int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|g * e_\alpha - g\|_1 &= \int |g * e_\alpha(s) - g(s)|\,ds \le \iint |g(t) - g(s)|e_\alpha(t^{-1}s)\,dt\,ds \\ &= \iint |g(t) - g(ts)|e_\alpha(s)\,dt\,ds = \int \|g - \rho_s g\|_1 e_\alpha(s)\,ds < \varepsilon \int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and they imply $\lim_\alpha \|e_\alpha * g - g\|_1 = \lim_\alpha \|g * e_\alpha - g\|_1 = 0$. We can approximate $f \in L^1(G)$ with compactly supported continuous functions by the $\varepsilon/3$ argument. □

Note that we have

$$\begin{aligned}|\langle \lambda(\xi)\eta, \zeta\rangle|^2 &= |\iint \xi(t)\eta(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\ &\le \iint |\xi(t)||\eta(t^{-1}s)|^2\,ds\,dt \cdot \iint |\xi(t)||\zeta(s)|^2\,ds\,dt \\ &= \|\xi\|_1^2\|\eta\|_2^2\|\zeta\|_2^2\end{aligned}$$

and

$$
\begin{aligned}
|\langle \rho(\xi)\eta, \zeta\rangle|^2 &= |\iint \eta(t)\xi(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\
&\le \iint |\xi(t^{-1}s)||\eta(t)|^2\,ds\,dt \cdot \iint |\xi(t^{-1}s)||\zeta(s)|^2\,ds\,dt \\
&= \|\xi\|_1 \|F\xi\|_1 \|\eta\|_2^2 \|\zeta\|_2^2
\end{aligned}
$$

imply

$$\|\lambda(\xi)\|_{2\to 2} \le \|\xi\|_1, \qquad \|\rho(\xi)\|_{2\to 2} \le \sqrt{\|\xi\|_1 \|F\xi\|_1}.$$

The equalities do not hold, consider $\|\lambda(\xi)\| = \|\widehat{\xi}\|_\infty$ if $G = \mathbb{R}$.

2.8 (Group C^*-algebras). $\overline{\lambda(C_c(G))} =: C_r^*(G)$, $\overline{m(C_c(G))} =: C_0(G)$.

2.9 (Fell absorption principle). Structure operator

2.10 (Fourier algebra). The Fourier algebra is $H * SH =: A(G)$.

2.11 (Fourier-Stieltjes algebra). positive definite functions, Bochner theorem

2.12 (GNS construction for locally compact groups). Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

$$
\begin{tikzcd}
G \arrow[r] & M(G) & & \\
L_1(G) \arrow[ur, hook] \arrow[r, hook] \arrow[d, dashed, "*"] & C^*(G) \arrow[r, two heads] \arrow[d, dashed, "*"] & C_r^*(G) \arrow[r, hook] \arrow[d, dashed, "*"] & L(G) \arrow[d, dashed, "* \text{ with } \sigma w"] \\
L^\infty(G) & B(G) \arrow[l, hook'] & C_r^*(G)^* \arrow[l, hook'] & A(G) \arrow[l, hook'] \\
& C_0(G) \arrow[ul, hook'] & &
\end{tikzcd}
$$

2.3 Pontryagin duality

2.13 (Locally compact abelian groups). Let G be a locally compacy abelian group. Then, we can consider the intersection of L^2 and L^∞ via $A' =: \mathcal{F}^{-1}(L^2(G) \cap L^\infty(G))$.

2.14 (Dual group).

2.15 (Fourier inversion theorem).

2.16 (Plancherel's theorem).

2.4 Structure theorems

2.5 Spectral synthesis

2.17 (Compact groups). Let G be a compact group. Then, $C_c(G) = C(G)$ is a Hopf C^*-algebra.

2.18 (Discrete groups). Let G be a discrete group. Then, $C_c(G)$ is a unital left Hilbert algebra.

Part II

Topological quantum groups

Chapter 3

Kac algebras

Chapter 4

Compact quantum groups

Chapter 5

Locally compact quantum groups

5.1 Multiplicative unitaries

Part III

Representation categories

Chapter 6

Representations of compact groups

6.1 Peter-Weyl theorem

6.2 Tannaka-Krein duality

6.3 Mackey machine

Example of non-compact Lie groups, Wigner classification