Homological Algebra

Ikhan Choi

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Contents

Part I

Derived categories

Chapter 1

Derived functors

Chapter 2

Differential graded categories

2.1 Chain complexes

2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and suppose $\mathcal{A}$ has enough injectives, that is, every object $A \in \mathcal{A}$ admits a monomorphism $A \to I$ for an injective object I. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

derived category of differential graded category.

2.2 Triangulated categories

2.2 (Triangulated categories)**.** A *triangulated category* is an additive functor $\mathcal{D}$ together with a translation functor $\mathcal{D} \to \mathcal{D} : X \mapsto X[1]$, which is an equivalence of categories, and a collection of distinguished triangles

Chapter 3

Part II

Homotopical algebra

Chapter 4

Model categories

4.1 Model structures

4.1 (Model structures)**.** Let $\mathcal{C}$ be a category. We say a pair $(\mathcal{A}, \mathcal{B})$ of subcategories of $\mathcal{C}$ is a *functorial weak factorization system* if

(i) for $i \in \mathrm{Mor}(\mathcal{A})$ and $p \in \mathrm{Mor}(\mathcal{B})$ there exists $h \in \mathrm{Mor}(\mathcal{C})$ such that the following commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y. \end{tikzcd} \tag{lifting}$$

(ii) there are functors $\alpha : \mathcal{C} \to \mathcal{A}$ and $\beta : \mathcal{C} \to \mathcal{B}$ such that for $f \in \mathrm{Mor}(\mathcal{C})$ the following commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha(f)"'] & & B \\ & M \arrow[ur, "\beta(f)"'] & \end{tikzcd} \tag{factorization}$$

Following the definition of Hovey, a *model structure* on $\mathcal{C}$ is a three subcategories of $\mathcal{C}$ called *weak equivalences*, *cofibrations*, and *fibrations* such that

(i) the weak equivalences satisfy the two-out-of-three law,

(ii) cofibrations and acyclic fibrations form a functorial weak factorization system,

(iii) acyclic cofibrations and fibrations form a functorial weak factorization system.

We denote by $\mathcal{W}$ the subcategory of weak equivalences. A *model category* is a category with small limits and colimits equipped with a model structure.

cofibrant and fibrant replacements.

(a) retract closedness

(b)

4.2 (Homotopy category of a model category)**.** left homotopy and right homotopy, cofibrant-fibrant objects.

4.3 (Derived categories and derived functors)**.** For a functor $F : \mathcal{C} \to \mathcal{D}$ between model categories, a *left derived functor* $LF : h\mathcal{C} \to \mathcal{D}$ is defined as the right Kan extension of F with respect to $\mathcal{C} \to h\mathcal{C}$. It may not exist in general, but there is an equivalent condition which can be easily investigated.

4.2 Quillen functors

4.3 Examples of model structures

4.4 (Model structures on chain complexes). projective and Hurewicz(chain homotopy) model structures on non-negative chain complexes

4.5 (Model structures on topological spaces). Serre and Hurewicz model structures

monoidal, simplicial, pointed, stable model categories

Chapter 5

Simplicial categories

5.1 Simplicial sets

Simplicial methods convert a differential graded category to a simplicial category via the Dold-Kan correspondence, so that a model structure on the differential graded category becomes simplicial. In a simplicial model category we can expand simplicial resolutions explicitly.

5.1 (Simplicial sets).

5.2 (Simplicial complexes). A *simplicial complex* is a set K of non-empty finite subsets of a set V which is closed under subsets. If V is linearly ordered, then we say K is ordered. To every ordered simplicial complex one can associate a simplicial setas follows. Let K_n be the set of all ordered tuples $(v_0, \cdots, v_n)$ such that $v_0 \leq \cdots \leq v_n$ and $\{v_0, \cdots, v_n\} \in K$. Then, for each morphism $\alpha : [n] \to [m]$ in Δ, we can define $\alpha^* : K_m \to K_n$.

5.3 (Dold-Kan correspondence). The Dold-Kan correspondence states that $\mathcal{A}_\Delta \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a categorical equivalence for an abelian category $\mathcal{A}$.

Two descriptions for normalized Moore complexes:

$$0 \to N_\bullet(A) \to C_\bullet(A) \to D_\bullet(A) \to 0.$$

Eilenberg-Maclane functor $K : \mathrm{Ch}(\mathbb{Z}) \to \mathrm{sAb}$ as the right adjoint for the functor $N_\bullet$.

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{R[\cdot]} \mathrm{sMod}_R \xrightarrow{C_\bullet \text{ or } N_\bullet} \mathrm{Ch}(R) \xrightarrow{H_n} \mathrm{Mod}_R$$

5.2 Simplicial model categories

5.4 (Model structures on simplicial sets). Kan and Joyal model structures

Via the Dold-Kan correspondence $\mathcal{A}_\Delta \cong \mathrm{Ch}_{\geq 0}(\mathcal{A})$, the Kan model structure corresponds to the projective model structures

Chapter 6

Infinity categories

6.1 Simplicial sets

Two representative examples: nerves and Kan complexes
infinity categories as simplicially enriched categories

6.1 (Nerves)**.** For an ordinary category as a nerve, two morphisms are homotopic only if they are identical.

6.2 (Kan complexes)**.** A geometric model for infinity groupoids. In a Kan complex, including Sing of a topological space, every morphism is invertible up to homotopy.

Infinity groupoids are usually considered as “spaces”.

6.2 Kan complexes

The *infinity category of spaces*, denoted by Spc, is defined as the homotopy-coherent nerve of the category Kan of Kan complexes.

6.3 Stable infinity categories

examples of stable infinity category: the infinity category of spectra, the dervied category of an abelian category

6.3. A *stable infinity category* is an infinity category such that

(i) there is a zero object,
(ii) every morphism admits a fiber and cofiber,
(iii) a triangle is a fiber sequence if and only if it is a cofiber sequence.

It is known that its homotopy category is tricngulated.

6.4 (Triangulated categories)**.**

6.5 (Differential graded category)**.**