Stochastic Analysis

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For each $\omega \in \Omega$ the map $t \mapsto B_t(\omega)$ is continuous, but possibly not differentiable.

The meaning of the equation

$$dX(t) = \sigma(X(t))\,dB_t + b(X_t)\,dt$$

is more clarified by the integral equation

$$X(t) = x + \int_0^t \sigma(X(s))\,dB_s + \int_0^t b(x(s))\,ds.$$

Stochastic processes

Definition 1.1 (Filtrated probability space). Let $\mathbb{T} \in \{[0,\infty), [0,T], \mathbb{Z}_{\geq 0} : 0 < T < \infty\}$. A *filtered probability space* is a tuple $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ such that $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{F}_t \subset \mathcal{F}$ is a σ-subfield, and $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$. We say, when $\mathbb{T}$ is continuous, that $\{\mathcal{F}_t\}$ is right continuous if

$$\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} =: \mathcal{F}_{t+}, \qquad t \in \mathbb{T}.$$

Definition 1.2 (Usual condition). A filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ is said to satisfy the *usual condition* if $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{N} = \{A \in \mathcal{F} : P(A) = 0\}$ is a subset of $\mathcal{F}_0$, and $\{\mathcal{F}_t\}$ is right continuous.

Definition 1.3 (Measurability of stochastic processes). Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ be a filtrated probability space. A family of random variables $\{X_t\}_{t\in\mathbb{T}}$ is called a *stochastic process* or a *random process*. From now on, we will consider random *vectors* with $X_t(\omega) \in \mathbb{R}^d$ for each t, ω.

(a) $\{X_t\}$ is called $\{\mathcal{F}_t\}$-*adapted* if X_t is $\mathcal{F}_t$-measurable for each $t \in \mathbb{T}$.

(b) $\{X_t\}$ is called *measurable* if $X : \mathbb{T} \times \Omega \to \mathbb{R}^d$ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}$-measurable.

(c) For $\mathbb{T}$ continuous, $\{X_t\}$ is called *right or left continuous* if for each ω the *sample path* $t \mapsto X_t(\omega)$ is right or left continuous respectively.

(d) For $\mathbb{T}$ continuous, $\{X_t\}$ is called $\{\mathcal{F}_t\}$-*progressively measurable* if for each $t \in \mathbb{T}$ the map $X : [0,t] \times \Omega \to \mathbb{R}^d$ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}_t$-measurable.

(e) For $\mathbb{T}$ continuous, the *predictable σ-field* is the minimal σ-subfield of $(\mathbb{T} \times \Omega, \mathcal{B}(\mathbb{T}) \otimes \mathcal{F})$ such that every real-valued left continuous $\{\mathcal{F}_t\}$-adapted process is measurable.

(f) For $\mathbb{T}$ continuous, a *predictable process* is a stochastic process that is measurable with respect to the predictable σ-field.

Remark. In other words, stochastic process is a random element on $(S^{\mathbb{T}}, \mathcal{B}(S^{\mathbb{T}}))$, which is not standard if $\mathbb{T}$ is uncountable. Also, a continuous stochastic process is a random element on $(C(\mathbb{T}, S), \mathcal{B}(C(\mathbb{T}, S)))$ because the Borel σ-algebra coincides with the induced σ-algebra from $S^{\mathbb{T}}$!

If $\{\mathcal{F}_t\}$-progressive measurable, then measurable and $\{\mathcal{F}_t\}$-adapted.

Proposition 1.5. *If $\{X_t\}$ is left or right continuous and $\{\mathcal{F}_t\}$-adapted, then $\{X_t\}$ is progressively measurable.*

Proof. Suppose $\{X_t\}$ is right $\{\mathcal{F}_t\}$-adapted and fix $t \in \mathbb{T}$. Let $I_k := [\frac{k-1}{n}t, \frac{kt}{n})$, $1 \leq k \leq n-1$, and let $I_n := [\frac{n-1}{n}t, t]$. Let

$$X_s^n(\omega) := \begin{cases} X_{\frac{k}{n}t}(\omega) & ,s \in I_k,\ k \leq n-1 \\ X_t(\omega) & ,s \in I_n \end{cases}.$$

Then, for every Borel set $A \in \mathcal{B}(\mathbb{R}^d)$,

$$(X^n)^{-1}(A) = \bigcup_{k=1}^{n} (I_k \times X_{\frac{k}{n}t}^{-1}(A)) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t.$$

Thus X^n is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable, and we are done because

$$X(s,\omega) = \lim_{n\to\infty} X^n(s,\omega), \qquad (s,\omega) \in [0,t] \times \Omega. \qquad \square$$

Proposition 1.6.

(a) *Let* $\mathbb{T} = [0,\infty)$. *If*

$$I := \{A \times (s,t] : A \in \mathcal{F}_s,\ 0 < s < t\} \cup \{A \times [0,t] : A \in \mathcal{F}_0\},$$

then I *is a* π*-system, which generates the predictable* σ*-field.*

(b) *A predictable process is progressively measurable.*

Definition 1.7 (Stopping times)**.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}})$ be a filtrated measurable space.

(a) A random variable $\tau : \Omega \to \mathbb{T} \cup \{+\infty\}$ is called a $\{\mathcal{F}_t\}$*-stopping time* if for every $t \in \mathbb{T}$ we have $\{\tau \le t\} \in \mathcal{F}_t$.

(b) For $\{\mathcal{F}_t\}$-stopping time τ,

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t,\ \forall t \in \mathbb{T}\}.$$

Remark 1.8.

(a) For $t_0 \in \mathbb{T}$, $\tau \equiv t_0$ is a $\{\mathcal{F}_t\}$-stopping time.

(b) For $\{X_t\}$ an $\mathbb{R}^d$-valued $\{\mathcal{F}_t\}$-adapted process, then for any Borel $E \in \mathbb{R}^d$ the function

$$\sigma_E(\omega) := \inf\{t : X_t(\omega) \in E\},$$

where $\inf\varnothing = \infty$, is a $\{\mathcal{F}_t\}$-stopping time called the *hitting time*.

Proposition 1.9. *Let* τ *be a* $\{\mathcal{F}_t\}$*-stopping time.*

(a) $\mathcal{F}_\tau$ *is a* σ*-field and* τ *is* $\mathcal{F}_\tau$*-measurable.*

(b) *Let* $\mathbb{T} = [0,\infty)$*, and let* $\{\mathcal{F}_t\}$ *be right continuous. Then,* τ *is a* $\{\mathcal{F}_t\}$*-stopping time if and only if* $\{\tau < t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{T}$*. If* τ *is a* $\{\mathcal{F}_t\}$*-stopping time, then* $A \in \mathcal{F}_\tau$ *if and only if* $A \cap \{\tau < t\} \in \mathcal{F}_t$ *for all* $t \in \mathbb{T}$*.*

(c) *Let* $\mathbb{T} = [0,\infty)$*. If* $\{X_t\}$ *is a* $\{\mathcal{F}_t\}$*-progressively measurable and* τ *is* $\{\mathcal{F}_t\}$*-stopping time, then* $X_\tau \mathbf{1}_{\tau<\infty}$ *is* $\mathcal{F}_\tau$*-measurable.*

Proof. (a) If $A \in \mathcal{F}_\tau$, then for every t we have $A \cap \{\tau \le t\} \in \mathcal{F}_t$, so $A^c \cap \{\tau \le t\} = \{\tau \le t\} \setminus (A \cap \{\tau \le t\}) \in \mathcal{F}_t$, which implies $A^c \in \mathcal{F}_\tau$. For countable union, if $(A_n)_{n=1}^\infty \subset \mathcal{F}_\tau$, then $(\bigcup A_n) \cap \{\tau \le t\} \in \mathcal{F}_t$ is clear.

For $a > 0$, we can show $\{\tau \le a\} \in \mathcal{F}_\tau$ since

$$\{\tau \le a\} \cap \{\tau \le t\} = \{\tau \le a \wedge t\} \in \mathcal{F}_{a\wedge t} \subset \mathcal{F}_t.$$

(b) ($\Rightarrow$) $\{\tau < t\} = \bigcup_{n=1}^\infty \{\tau \le t - \frac{1}{n}\} \in \mathcal{F}_t$.
($\Leftarrow$) $\{\tau \le t\} = \bigcap_{n=N}^\infty \{\tau \le t + \frac{1}{n}\} \in \mathcal{F}_{t+\frac{1}{N}}$, so $\{\tau \le t\} \in \mathcal{F}_t$.

(c) For $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in \mathbb{T}$, it suffices to show $\{X_\tau \in A\} \cap \{\tau \le t\} \in \mathcal{F}_t$. Maps

$$\Phi : \{\tau \le t\} \to [0,t] \times \Omega : \omega \mapsto (\tau(\omega), \omega)$$

and

$$X : [0,t] \times \Omega \to \mathbb{R}^d : (t,\omega) \mapsto X_t(\omega)$$

are measurable with respect to $\mathcal{F}_t$, $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$, $\mathcal{B}(\mathbb{R}^d)$, because $\Phi^{-1}([a,b] \times B) = \{\tau \le b\} \cap \{\tau < a\}^c \cap B \in \mathcal{F}_t$, and because of the definition of progressive measurability. □

Proposition 1.10. *Let $\mathbb{T} = [0,\infty)$ and $\{X_t\}$ be a $\{\mathcal{F}_t\}$-progressively measurable process. For Borel $E \subset \mathbb{R}^d$, let σ_E be the hitting time of $\{X_t\}$.*

(a) *If $\{X_t\}$ and $\{\mathcal{F}_t\}$ are right continuous, and if E is open, then σ_E is $\{\mathcal{F}_t\}$-stopping time.*

(b) *If $\{X_t\}$ is continuous and E is closed, then σ_E is $\{\mathcal{F}_t\}$-stopping time.*

Proof. (a) Let $t > 0$. Then,

$$\{\sigma_E < t\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s \in E\} \in \mathcal{F}_t.$$

(b) We show $\{\sigma_E \le t\} \in \mathcal{F}_t$ for each $t > 0$. If we introduce $f_E(x) := d(x,E) = \inf\{|x-y| : y \in E\}$, then f_E is continuous and $f_E(x) = 0$ is equivalent to $x \in E$. Now we can write

$$\{\sigma_E \le t\} = \{\min_{s \in [0,t]} f_E(X_s) = 0\} = \{\inf_{s \in [0,t] \cap \mathbb{Q}} f_E(X_s) = 0\} \in \mathcal{F}_t.$$ □

2 Day 2: October 12

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process $\{B_t\}_{t\geq 0}$ on Ω is called a d-dimensional *standard Brownian motion* if

(i) it is continuous, i.e. each sample path for ω is continuous,

(ii) $B_t - B_s \sim N(0, (t-s)I)$ for $0 \leq s < t$

(iii) $B_{t_{i+1}} - B_{t_i}$ are independent for $0 = t_0 < t_1 < \cdots < t_n < \infty$.

Remark. If we write $B_t = (B_t^1, \cdots, B_t^d)$, then

$$E[(B_t^i - B_s^i)(B_t^j - B_s^j)] = \delta_{ij}(t-s).$$

If $B_0 \equiv x$ for a vector $x \in \mathbb{R}^d$, then we say $\{B_t\}$ is a Brownian motion starts from x, and if $B_0 \equiv \nu$ for a distribution ν on $\mathbb{R}^d$, then we say ν is the initial distribution of $\{B_t\}$.

Proposition 2.2. *Let $\{B_t\}$ be a standatd Brownian motion with initial distribution ν. For $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $A_0, \cdots, A_n \in \mathcal{B}(\mathbb{R}^d)$, we have*

$$P(B_{t_0} \in A_0, \cdots, B_{t_n} \in A_n) = \int \mathbf{1}_{A_0}(x_0) \cdots \mathbf{1}_{A_n}(x_n)$$
$$g_d(t_1 - t_0, x_0, x_1) \cdots g_d(t_n - t_{n-1}, x_{n-1}, x_n)\, d\nu(x_0)\, dx_1 \cdots dx_n,$$

where

$$g_d(t, x, y) := \frac{1}{\sqrt{2\pi t}^d} e^{-\frac{|x-y|^2}{2t}}.$$

Proof.

$$\begin{aligned} P(B_{t_0} \in A_0, \cdots, B_{t_n} \in A_n) &= P(B_{t_0} \in A_0, \cdots, B_{t_0} + \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}}) \in A_n) \\ &= \int \mathbf{1}_{A_0}(y_0)\, d\nu(y_0) \int \mathbf{1}_{A_1}(y_0 + y_1) g_d(t_1 - t_0, 0, y_1)\, dy_1 \\ &\cdots \int \mathbf{1}_{A_n}(y_0 + \sum_{i=1}^n y_i) g_d(t_n - t_{n-1}, 0, y_n)\, dy_n. \end{aligned}$$

Here the matrix of coordinate change $x_0 = y_0$, $x_i = y_0 + \sum_{k=1}^i y_k$ has determinant one. Then we are done. $\square$

Theorem 2.3. *For a d-dimensional stochastic process $\{B_t\}$, TFAE:*

(1) $\{B_t\}$ is a standard Brownian motion starting from zero.

(2) $\{B_t^i\}$ are mutually independent standard Brownian motions starting from zero.

In particular, for its construction the one-dimensional Brownian motion is sufficient.

Remark. For stochastic processes $\{X_t\}$ and $\{Y_t\}$ are said to be independent if and only if for an arbitrarily long sequence $0 = t_0 < \cdots < t_M < \infty$ with $A_0, \cdots A_M$ and $B_0, \cdots, B_M$, we have

$$\begin{aligned} &P(X_{t_0} \in A_0, \cdots, X_{t_M} \in A_M, \quad Y_{t_0} \in B_0, \cdots, Y_{t_M} \in B_M) \\ &\quad = P(X_{t_0} \in A_0, \cdots, X_{t_M} \in A_M) P(Y_{t_0} \in B_0, \cdots, Y_{t_M} \in B_M). \end{aligned}$$

Definition 2.4 (Modification)**.** A stochastic process $\{X_t\}$ is called a *modification* of $\{Y_t\}$ if $P(X_t = Y_t) = 1$ for all $t \geq 0$. We say $\{X_t\}$ and $\{Y_t\}$ are *indistinguishable* if $P(X_t = Y_t : t \geq 0) = 1$.

Proposition 2.5. *If $\{X_t\}$ and $\{Y_t\}$ are right continuous modifications of each other, then they are indistinguishable.*

Proof. By the definition of modifications, the following set is full:

$$\widetilde{\Omega} := \{\omega \in \Omega : X_t(\omega) = Y_t(\omega),\ \forall t \in \mathbb{Q}_{\geq 0}\}.$$

Then, by the right continuity, $\widetilde{\Omega} \subset \{X_t = Y_t : t \geq 0\}$. □

Let $\Omega := (\mathbb{R}^d)^{[0,\infty)}$, and $\mathcal{F} := \sigma(\{\pi_t\})$ be the Borel σ-algebra. It is not a standard Borel space. We will construct a probability measure P on $(\Omega, \mathcal{F})$ such that $\pi_t \sim B_t$ for all t (it means the π_t satisfies the conditions for the Brownian motion only in distribution) and we will find a continuous modification of $\{\pi_t\}$.

Let $\mathcal{T}$ be the set of all strictly increasing finite nonnegative real sequences (t_i) such that $t_0 = 0$. For $\overline{t} = (t_0, \cdots, t_n) \in \mathcal{T}$, consider $\mathcal{F}_{\overline{t}}$ and $\pi_{\overline{t}} : \Omega \to (\mathbb{R}^d)^{n+1}$.

Theorem 2.6 (Kolmogorov extension). *Suppose a probability measure $P_{\overline{t}}$ is given on $(\Omega, \mathcal{F}_{\overline{t}})$ for every $\overline{t} \in \mathcal{T}$. Then, TFAE:*

(1) There is a probability measure P on $(\Omega, \mathcal{F})$ such that $P|_{\mathcal{F}_{\overline{t}}} = P_{\overline{t}}$ for all $\overline{t} \in \mathcal{T}$.

(2) If $\overline{t} \subset \overline{t}'$, then $P_{\overline{t}'}|_{\mathcal{F}_{\overline{t}}} = P_{\overline{t}}$.

Remark. (2) in the above is equivalent to the following: If $\overline{t} = (t_0, t_1, \cdots, t_{i-1}, t_i, t_{i+1}, \cdots, t_n)$ and $\overline{t}_i = (t_0, t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$, for $A \in \mathcal{B}((\mathbb{R}^d)^i)$ and $B \in \mathcal{B}((\mathbb{R}^d)n - i)$, we have

$$P_{\overline{t}}(\pi_{\overline{t}}^{-1}(A \times \mathbb{R}^d \times B)) = P_{\overline{t}_i}(\pi_{\overline{t}_i}^{-1}(A \times B)).$$

Remark. The consistency condition is equivalent to

$$g_d(t_i - t_{i-1}, x_{i-1}, x_i) g_d(t_{i+1} - t_i, x_i, x_{i+1})\, dx_i = g_d(t_{i+1} - t_{i-1}, x_{i-1}, x_{i+1}).$$

It is called the Chapman-Kolmogorov equation.

In fact, we have stronger estimate $E[e^{\varepsilon \|B\|_\gamma^2}] < \infty$.

Theorem 2.7. *Let $\{X_t\}_{t \in [0,T]}$ be a stochastic process on $\mathbb{R}^d$. If there is $\alpha, \beta, C > 0$ such that*

$$E[|X_t - X_s|^\alpha] \leq C|t - s|^{1+\beta}, \qquad 0 \leq s < t \leq T,$$

then there is a modification $\{\widetilde{X}_t\}$ of $\{X_t\}$ such that there is a $\mathcal{F}$-measurable random variable $C(\omega) < \infty$ for each $\omega \in \Omega$ and there is $\gamma \in (0, \frac{\beta}{\alpha})$ satisfying

$$|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)| \leq C(\omega)|t - s|^\gamma, \qquad 0 \leq s < t \leq T.$$

In other words, there is a γ-Hölder continuous modification.

Proof. Suppose $d = T = 1$. Fix $n \in \mathbb{N}$. Then, for $r > 0$ and $k = 1, \cdots, 2^n$,

$$P(|X_{k2^{-n}} - X_{(k-1)2^{-n}}| \geq 2^{-nr}) \leq C 2^{-n(1+\beta-r\alpha)}$$

so that

$$P(\bigcup_{k=1}^{2^n} \{|X_{k2^{-n}} - X_{(k-1)2^{-n}}| \geq 2^{-nr}\}) \leq C 2^{-n(\beta - r\alpha)}.$$

If we let $r = \gamma < \beta/\alpha$, then $A_n := \bigcup_{k=1}^{2^n} \{|X_{k2^{-n}} - X_{(k-1)2^{-n}}| \geq 2^{-nr}\}$ satisfies $\sum_{n=1}^\infty P(A_n) < \infty$, which implies $P(\limsup_{n\to\infty} A_n) = 0$ and $P(\liminf_{n\to\infty} A_n^c) = 1$ by the Borel-Cantelli. Let $\widetilde{\Omega} := \liminf_{n\to\infty} A_n^c$. If we let $N(\omega) := \inf\{n : \omega \in \bigcap_{k=n}^\infty A_k^c\}$, then $\widetilde{\Omega} = \{N < \infty\}$.

We claim that if 2-adic rational number $0 \le s < t \le 1$ satisfies $|t-s| < 2^{-N(\omega)}$, then

$$|X_t(\omega) - X_s(\omega)| \le \frac{2}{1-2^{-\gamma}}|t-s|^\gamma.$$

Assume that the claim is true. Consider a sequence $s = t_0 < \cdots < t_l = t$ such that $t_i - t_{i-1} = 2^{-(N(\omega)+1)}$ for $1 \le i \le l-1$ and $t_l - t_{l-1} \le 2^{-(N(\omega)+1)}$. Then, $l \le 2^{N(\omega)+1} + 1$, and we can estimate as follows: for $\omega \in \widetilde{\Omega}$,

$$\begin{aligned}
|X_t(\omega) - X_s(\omega)| &\le \sum_{i=1}^{l} |X_{t_i}(\omega) - X_{t_{i-1}}(\omega)| \\
&\le \sum_{i=1}^{l} \frac{2}{1-2^{-\gamma}}|t_i - t_{i-1}|^\gamma \\
&\le \frac{2(2^{N(\omega)+1}+1)}{1-2^{-\gamma}}|t_i - t_{i-1}|^\gamma \\
&=: C(\omega)|t_i - t_{i-1}|^\gamma.
\end{aligned}$$

Let $\widetilde{X}(\omega) := 0$ for $\omega \notin \widetilde{\Omega}$ and $\widetilde{X}(\omega) = \lim_{t_n \to t} X_{t_n}(\omega)$ for $\omega \in \widetilde{\Omega}$, where t_n runs through 2-adic rationals. The assumption $E[|X_t - X_s|^\alpha] \le C|t-s|^{1+\beta}$ implies that $X_{t_n} \to X_t$ in probability as $t_n \to t$, we have $P(\widetilde{X}_t = X_t) = 1$ for each t. □

3 Day 3: October 19

Claim. Let $\widetilde{\Omega} \subset \Omega$, $P(\widetilde{\Omega}) = 1$ with $N(\omega) < \infty$ for all $\omega \in \widetilde{\Omega}$. Then, for 2-adic rationals $0 \le s < t \le 1$, we have

$$|X_t(\omega) - X_s(\omega)| < \frac{2}{1 - s^{-\gamma}} |t - s|^{\gamma}.$$

Proof. Suppose first $|t - s| < 2^{N(\omega)}$. Then, there is $m \ge N(\omega)$ such that $2^{-m+1} \le t - s < 2^{-m}$. There are two cases:

$$k2^{-m} < s < (k+1)2^{-m} < t < (k+2)2^{-m}$$

or

$$k2^{-m} < s < t \le (k+1)2^{-m}$$

for some k. See the note. □

σ-subalgebra provides the von Neumann subalgebra together with a conditional expectation.

Proposition 3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. If $X, \varphi(X) \in L^1$, then $E(\varphi(X)|\mathcal{G}) \ge \varphi(E(X|\mathcal{G}))$. In particular, $E(-|\mathcal{G})$ is L^p-bounded.*

Definition 3.2. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space. A stochastic process $\{X_t\}$ is called a $\{\mathcal{F}_t\}$-submartingale if it is $\{\mathcal{F}_t\}$-adapted, $X_t \in L^1$ for each t, and $E(X_t|\mathcal{F}_s) \ge X_s$ for each $s < t$.

Proposition 3.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex.*

(a) *If $\{X_t\}$ is a martingale and $\varphi(X_t) \in L^1$ for all t, then $\{\varphi(X_t)\}$ is a submartingale.*

(b) *If $\{X_t\}$ is a submartingale and $\varphi(X_t) \in L^1$ for all t, and if φ is non-decreasing, then $\{\varphi(X_t)\}$ is a submartingale.*

For example,

- *$\{X_t\}$ is a martingale, then $\{|X_t|\}$ is a submartingale,*
- *$\{X_t\}$ is a non-negative martingale with $X_t \in L^p$, then $\{X_t^p\}$ is a submartingale,*
- *$\{B_t\}$ is a $\{\sigma(\{B_s\} : s \le t)\}$-martingale. Because it is not right continuous, so we need to do something.*

Theorem 3.4 (Doob's inequality)**.** *Let $\{X_t\}$ be a non-negative right continuous $\{\mathcal{F}_t\}$-submartingale.*

(a) *For $a > 0$ and $t > 0$,*

$$P(\sup_{s \le t} X_s \ge a) \le \frac{1}{a} E(X_t | \sup_{s \le t} X_s \ge a).$$

(b) *For $p > 1$ let $X_t \in L^p$. Then,*

$$P(\sup_{s \le t} X_s \ge a) \le \frac{1}{a^p} E(X_t^p)$$

and

$$E((\sup_{s \le t} X_s)^p) \le (\frac{p}{p-1})^p E(X_t^p).$$

(c) *If $\{X_t\}$ is a right continuous $\{\mathcal{F}_t\}$-martingale with $X_T \in L^p$ for some $p > 1$, then*

$$E(\sup_{t \le T} |X_t|^p) \le (\frac{p}{p-1})^p E(|X_T|^p)$$

Proof. (a) Use the discrete version.

$$P(A_n) \le \frac{1}{a} E(X_t | \sup_{s\le t} X_s \ge a)$$

and $\{\sup_{s\le t} X_s > a\} \subset \liminf_n A_n$ implies by Fatou

$$P(\{\sup_{s\le t} X_s > a\}) \le P(\liminf_n A_n) \le \frac{1}{a} E(X_t | \sup_{s\le t} X_s \ge a).$$

Using the right continuity, we can limit

$$P(\{\sup_{s\le t} X_s > a\}) \to P(\{\sup_{s\le t} X_s \ge a\}).$$

(b) Let $X_t^* := \sup_{s\le t} X_s$.

$$\begin{aligned}
E((X_t^*)^p) &= \int_0^\infty p x^{p-1} P(X_t^* > x)\, dx \\
&= \int_0^\infty p x^{p-2} E(X_t : X_t^* > x)\, dx \\
&= pE(X_t \frac{(X_t^*)^{p-1}}{p-1}) \\
&= \frac{p}{p-1} E(X_t (X_t^*)^{p-1}) \\
&\le \frac{p}{p-1} E(X_t^p)^{\frac{1}{p}} E(((X_t^*)^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}}.
\end{aligned}$$

(c) Corollary. □

Lemma 3.5. *Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space. Let σ, τ be $\{\mathcal{F}_t\}$-stopping times such that $\sigma \le \tau$. Then, $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.*

Theorem 3.6 (Doob's optional sampling theorem). *Let $\mathbb{T} = [0, \infty)$. Let $\{X_t\}$ be a right continuous $\{\mathcal{F}_t\}$-submartingale and let $\sigma \le \tau$ be bounded $\{\mathcal{F}_t\}$-stopping times. Then, $E(X_\tau | F_\sigma) \ge X_\sigma$.*

Proof.

$$\sigma_\Delta(\omega) := \inf\{t : \sigma(\omega) \le t,\ t \in \Delta\}.$$

□