Part I
Algebra

1 Algebraic structures

Groups

1.1. Show that a finite symmetric group has two generators.

1.2. Show that a group of order $2p$ for a prime p has exactly two isomorphic types.

1.3. Show that a group G is abelian if $|G| = p^2$ for a prime p.

1.4. Show that a group G is abelian if it has a surjective cube map.

1.5. Let G be a finite group of order n and p the smallest prime divisor of n. Show that a subgroup of G of index p is normal in G.

1.6. Find all n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic.

1.7. Show that a nontrivial normalizer of a p-group meets its center out of identity.

1.8. Show that a proper subgroup of a finite p-group is a proper subgroup of its normalizer. In particular, every finite p-group is nilpotent.

1.9. Show that a finite group G satisfying $\sum_{g\in G} \operatorname{ord}(g) \le 2n$ is abelian.

1.10. Show that the order of a group with trivial automorphism group is either 1 or 2.

1.11. Find all homomorphic images of A_4 up to isomorphism.

1.12. Show that in a group of order 105 is a single Sylow p-subgroup for $p = 5, 7$.

1.13. Show that the number of Sylow p-subgroups of $\mathrm{SL}_3(\mathbb{F}_p)$ is $(p^2+p+1)(p+1)$.

Rings

1.14. Show that a finite integral domain is a field.

1.15. Show that every ring of order p^2 for a prime p is commutative.

1.16. Show that a semiring with multiplicative identity and cancellative addtion has commutative addition.

1.17. Show that the complement of a saturated monoid in a commutative ring is a union of prime ideals.

Vector spaces

1.18. Show that two matrices AB and BA have same nonzero eigenvalues whose both multiplicities are coincide...

1.19. Show that if A is a square matrix whose characteristic polynomial is minimal then a matrix commuting A is a polynomial in A.

1.20. Show that the order of 2×2 integer matrices divide 12 if it is finite.

1.21. Let X be a square matrix. Show that there is another matrix Y such that $X+Y$ is invertible.

1.22. Show that a determinant-preserving linear map is rank-preserving.

1.23. For a square matrix A such that $(A-\lambda)^2 = 0$, $A^{2^n} = 2^n \lambda^{2^n-1} A - (2^n-1)\lambda^{2^n}$.

2 Number theory

2.1. Show that there is no integral solution of the equation $x^7 + 7 = y^2$.

2.2. Show that if $(x^2 + y^2 + z^2)/(xy + yz + zx)$ is an integer, then it is not divided by 3.

2.3. Show that there is no non-trivial integral solution of $x^4 - y^4 = z^2$.

Part II

Geometry and Topology

1 Classical geometry

2 Smooth surfaces

3 Differential topology

3.1. Show that the tangent space of the unitary group at the identity is identified with the space of skew Hermitian matrices.

3.2. Prove the Jacobi formula for matrix.

3.3. Show that S^3 and T^2 are parallelizable.

3.4. Show that $\mathbb{R}P^n = S^n/Z_2$ is orientable if and only if n is odd.