Foundations of Calculus

Ikhan Choi

March 12, 2022

Contents

Part I

Sequences

Chapter 1

Convergence

Exercises

1.1. Every real sequence $(a_n)_{n=1}^{\infty}$ has a monotonic subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.

Chapter 2

Series

2.1 Convergence tests

2.1 (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_kB_k - A_{k-1}B_{k-1}$$

$$\sum_{m<k\le n} A_k b_k = A_nB_n - A_mB_m - \sum_{m<k\le n} a_kB_{k-1}.$$

2.2 (Dirichlet test).

2.3 (Mertens' theorem). If $\sum_{k=0}^\infty a_k$ converges to A absolutely and $\sum_{k=0}^\infty b_k$ converges to B, then their Cauchy product $\sum_{k=0}^\infty c_k$ with $c_k := \sum_{l=0}^k a_l b_{k-l}$ converges to AB.

Proof. Let

$$A_n := \sum_{k=0}^n a_k,\ B_n := \sum_{k=0}^n b_k,\ \text{and } C_n := \sum_{k=0}^n c_k.$$

Consider the regions

$$T_n := \{(k,l) \in \mathbb{Z}^2_{\ge 0} : k + l \le n\}, \qquad R_m : \{(k,l) \in \mathbb{Z}^2_{\ge 0} : k \le m\}.$$

Write

$$\begin{aligned} AB - C_n &= \sum_{k\le m}\sum_{l>n-k} a_k b_l + \sum_{k>m}\sum_{l\ge 0} a_k b_l - \sum_{m<k\le n}\sum_{l\le n-k} a_k b_l \\ &= \sum_{k\le m} a_k(B - B_{n-k}) + \sum_{k>m} a_k B - \sum_{m<k\le n} a_k B_{n-k}. \end{aligned}$$

The first term

$$|\sum_{k\le m} a_k(B-B_{n-k})| \le (\max_k |a_k|)(\sum_{l\ge n-m} |B-B_l|)$$

converges to zero as $n\to\infty$ for fixed m, the second term

$$|\sum_{k>m} a_k B| \le |A-A_m||B|$$

converges to zero as $m\to\infty$ for any n, and finally the third term

$$|\sum_{m<k\le n} a_k B_{n-k}| \le (\sum_{k>m}|a_k|)(\max_l |B_l|)$$

converges to zero as $m\to\infty$ for any n.

Fix m such that the second and third terms are bounded by arbitrary $\frac{\varepsilon}{2}>0$ so that

$$|C_n - AB| \le |\sum_{k\le m} a_k(B-B_{n-k})| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Then, by taking $n\to\infty$, we obtain

$$\limsup_{n\to\infty} |C_n - AB| \le \varepsilon.$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty} C_n = AB. \qquad \square$$

Exercises

2.4. If $a_n \to 0$, then $\frac{1}{n}\sum_{k=1}^n a_k \to 0$.

2.5. If $a_n \ge 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ also diverges.

2.6. If $a_n \downarrow 0$ and $S_n \le 1 + na_n$, then $S_n \le 1$.

Chapter 3

Open sets and closed sets

Exercises

Part II

Real functions

Chapter 4

Continuous functions

Exercises

4.1. The set of local minima of a convex real function is connected.

4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The equation $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.

4.3. A continuous function that takes on no value more than twice takes on some value exactly once.

4.4. Let f be a function that has the intermediate value property. If the preimage of every singleton is closed, then f is continuous.

4.5. * If a sequence of real functions $f_n : [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$, then the sequence has a uniformly convergent subsequence.

Chapter 5

Differentiable functions

Exercises

5.1. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} f'(x) = b$, then $a = 0$.

5.2. Let f be a real C^2 function with $f(0) = 0$ and $f''(0) \neq 0$. Defined a function ξ such that $f(x) = xf'(\xi(x))$ with $|\xi| \leq |x|$, we have $\xi'(0) = 1/2$.

5.3. Let f be a C^2 function such that $f(0) = f(1) = 0$. We have $\|f\| \leq \frac{1}{8}\|f''\|$.

5.4. A smooth function such that for each x there is n having the nth derivative vanish is a polynomial.

5.5. If a real C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.

5.6. Let a real function f be differentiable. For $a < a' < b < b'$ there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b') - f(a') = f'(c')(b' - a')$.

5.7. Let f be a differentiable function on the unit closed interval. If $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

5.8. Let f be a differentiable function on the unit closed interval. If $f(0) = 0$ there is c such that $cf(c) = (1 - c)f'(c)$.

Chapter 6

Analytic functions

Exercises

Part III

Integration

Chapter 7

Riemann integration

Exercises

7.1. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

7.2. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

Chapter 8

Henstock-Kurzweil intergation

Chapter 9

Part IV

Multivariable Calculus

Chapter 10

Frećhet derivatives

10.1 Inverse function theorem

Chapter 11

Differential forms

Chapter 12

Stokes' theorem