Geometry II

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Part I

Smooth surfaces

Chapter 1

Smooth manifolds

1.1 Local coordinates

1.1. Let n be a positive integer. A topological space M is called *locally Euclidean* of dimension n if there is an open cover $\{U_\alpha\}$ of M such that each open set U_α is homeomorphic to an open set of $\mathbb{R}^n$. A *topological manifold* is defined as a paracompact Hausdorff locally Euclidean space.

A *chart* of dimension n on M is a topological embedding $\varphi : U \to \mathbb{R}^n$ of an open subset U of M onto an open subset $\varphi(U)$ of $\mathbb{R}^n$. An *atlas* of dimension n on a topological space is a family $\{\varphi_\alpha\}$ of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ such that $\{U_\alpha\}$ is an open cover of M. By definition, a topological space is locally Euclidean if and only if it admits an atlas.

Two smooth atlases are called *smoothly equivalent* if their addition is a smooth atlas.

Given a smooth atlas, we can define the *smoothness* of a function $f : M \to \mathbb{R}$ with respect to the smooth atlas as follows: we say that f is smooth if its *coordinate representation* $f\varphi_\alpha^{-1} : \varphi_\alpha(U) \to \mathbb{R}$ is smooth for all charts φ_α.

1.2 (Immersions and embeddings)**.** If $\alpha : U \to \mathbb{R}^n$ is a topological embedding, then we can endow with a unique smooth structure on $\operatorname{im}\alpha$ such that α is smooth.(?)

(a) The image of a regular parameterization is an embedded manifold.

(b) Every open subset of a embedded manifold is a embedded manifold.

(c) Monge patch.

(d) The sphere $S^2 = \{\,(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\,\}$ is a regular surface.

(e) The set $\{\,(x,y) \in \mathbb{R}^2 : y^2 = x^3 + x^2\,\}$ is not a regular curve.

(f) The set $\{\,(x,y) \in \mathbb{R}^2 : y = |x|\,\}$ is not a regular curve.

1.2 Space curves

1.3 Space surfaces

Reparametrizations

Theorem 1.3.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 1.3.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 1.3.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 1.3.4. *Metric coefficients*

$$\begin{aligned} \langle \alpha_x, \alpha_x \rangle &=: g_{11} \qquad & \langle \alpha_x, \alpha_y \rangle &=: g_{12} \\ \langle \alpha_y, \alpha_x \rangle &=: g_{21} \qquad & \langle \alpha_y, \alpha_y \rangle &=: g_{22} \end{aligned}$$

Theorem 1.3.5 (Normal coordinates)**.** *...?*

Differentiation of tangent vectors

Definition 1.3.6. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x, y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x, y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 1.3.7 (Gauss formula, Γ_{ij}^k, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$ and $\{L_{ij}\}_{i,j=1}^2$ by the Gauss formula

$$\begin{aligned} \alpha_{xx} &=: \Gamma_{11}^1 \alpha_x + \Gamma_{11}^2 \alpha_y + L_{11}\,\nu, \qquad & \alpha_{xy} &=: \Gamma_{12}^1 \alpha_x + \Gamma_{12}^2 \alpha_y + L_{12}\,\nu, \\ \alpha_{yx} &=: \Gamma_{21}^1 \alpha_x + \Gamma_{21}^2 \alpha_y + L_{21}\,\nu, \qquad & \alpha_{yy} &=: \Gamma_{22}^1 \alpha_x + \Gamma_{22}^2 \alpha_y + L_{22}\,\nu. \end{aligned}$$

The *Christoffel symbols* refer to eight functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij}\,\nu. \end{aligned}$$

Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu \rangle = \frac{1}{2}\partial_X \langle \nu, \nu \rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 1.3.8. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 1.3.9. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y) \rangle = \langle X, -\partial_Y \nu \rangle = \langle \partial_Y X, \nu \rangle = \langle \partial_X Y, \nu \rangle = \langle \mathcal{S}(X), Y \rangle. \qquad \square$$

Theorem 1.3.10. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i \alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j \alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}_i^j X_i$. Then,

$$g_{ik} X^i \mathcal{S}_j^k Y^j = \langle X, \mathcal{S}(Y) \rangle = \langle \partial_X Y, \nu \rangle = X^i Y^j L_{ij}$$

implies $g_{ik} \mathcal{S}_j^k = L_{ij}$. $\square$

Chapter 2

Fundamental forms

2.1 Riemannian metrics

2.2 Gaussian curvatures

Theorema egregium surfaces of constant gaussian curvature

Definition 2.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$\begin{aligned} E &:= \langle \alpha_x, \alpha_x \rangle = g_{11}, & F &:= \langle \alpha_x, \alpha_y \rangle = g_{12}, & G &:= \langle \alpha_y, \alpha_y \rangle = g_{22}, \\ L &:= \langle \alpha_{xx}, \nu \rangle = L_{11}, & M &:= \langle \alpha_{xy}, \nu \rangle = L_{12}, & N &:= \langle \alpha_{yy}, \nu \rangle = L_{22}. \end{aligned}$$

Corollary 2.2.2. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\nu_x = \frac{FM - GL}{EG - F^2}\alpha_x + \frac{FL - EM}{EG - F^2}\alpha_y,$$
$$\nu_y = \frac{FN - GM}{EG - F^2}\alpha_x + \frac{FM - EN}{EG - F^2}\alpha_y.$$

Theorem 2.2.3.

$$\Gamma_{ij}^l = \frac{1}{2} g^{kl}(g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2}(\log g)_x = \Gamma_{11}^1.$$

$$\nu_x \times \nu_y = K\sqrt{\det g}\ \nu.$$

$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$

$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det \begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

2.1 (Gaussian curvature formula). (a) In general,

$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) For orthogonal coordinates such that $F \equiv 0$,

$$K = -\frac{1}{2\sqrt{\det g}}\left(\left(\frac{1}{\sqrt{\det g}} E_y\right)_y + \left(\frac{1}{\sqrt{\det g}} G_x\right)_x \right).$$

(c) For $f(x,y,z)=0$,

$$K = -\frac{1}{|\nabla f|^4}\begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f)\end{vmatrix},$$

where ∇f denotes the gradient $\nabla f = (f_x, f_y, f_z)$.

(d) (Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then

$$K = -\tau^2.$$

(e) (Brioschi) E, F, G describes K.

Proof. (a) Clear.

(b) We have $GM = EM$ and

$$\nu_x = -\frac{L}{E}\alpha_x - \frac{M}{G}\alpha_y, \qquad \nu_y = -\frac{M}{E}\alpha_x - \frac{N}{G}\alpha_y.$$

$$\nu_x \times \nu_y = \frac{LN - M^2}{EG}\alpha_x \times \alpha_y$$

After curvature tensors...

□

2.2 (Computation of Gaussian curvatures). (a) (Monge's patch) For $(x, y, f(x,y))$,

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t) = (r(t), z(t))$ be a plane curve with $r(t) > 0$. If $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

Proof. (b) Let

$$\alpha(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

be a parametrization of a surface of revolution. Then,

$$\begin{aligned}
\alpha_\theta &= (-r(t)\sin\theta, r(t)\cos\theta, 0)\\
\alpha_t &= (r'(t)\cos\theta, r'(t)\sin\theta, z'(t))\\
\nu &= \frac{1}{\sqrt{r'(t)^2 + z'(t)^2}}(z'(t)\cos\theta, z'(t)\sin\theta, -r'(t)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{\theta\theta} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0)\\
\alpha_{\theta t} &= (-r'(t)\sin\theta, -r'(t)\cos\theta, 0)\\
\alpha_{tt} &= (r''(t)\cos\theta, r''(t)\sin\theta, z''(t)).
\end{aligned}$$

Thus we have

$$E = r(t)^2, \quad F = 0, \quad G = r'(t)^2 + z'(t)^2,$$

and

$$L = -\frac{r(t)z'(t)}{\sqrt{r'(t)^2 + z'(t)^2}}, \quad M = 0, \quad N = \frac{r''(t)z'(t) - r'(t)z''(t)}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

□

2.3 (Local isomorphism)**.** Surfaces of the same constant Gaussian curvature are locally isomorphic.

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 3

Geometric surfaces

3.1

3.1 (Geometric manifolds)**.** We are concerned only with metric geometries. A *geometric manifold* is a smooth manifold together with a metric that is

(i) geodesically connected,

(ii) geodesically complete,

(iii) realized by a Riemannian metric,

(iv) locally homogeneous, i.e. every pair of points is connected by an isometry between neigborhoods.

Each condition has been obtained by modifying the first four postulates of Euclid's Elements. On a smooth manifold, we define a *geometric structure* as a locally homogeneous Riemannian metric, which embodies the third and fourth postulates. The completeness is sometimes assumed. A geometric manifold is a connected smooth manifold together with a complete geometric structure.

(a) On a connected smooth surface, a geometric structure is same as a Riemannian metric of constant Gaussian curvature.

(b) The isotropy group of a manifold with a geometric structure is a compact Lie group uniquely determined up to isomorphism.

(c) On a connected smooth manifold, there is a unique complete geometric structure up to...?

3.2. A *geometry* is a smooth G-manifold X such that

1. X is connected and simply connected,
2. G acts on X effectively and transitively,
3. the isotropy group is compact,
4. there exists a cocompactly freely properly discretely acting subgroup of G.

A *maximal geometry* or a *model geometry* is a geometry such that there is no properly large group G' satisfying the above assumptions. We want to classify model geometries up to intertwining maps, or equivalently as subgroups of $\mathrm{Diff}(X)$ up to conjugacy.

(a)

(b) A surface is geometric if and only if it is universally covered by a simply connected geometric surface, which is one of $\mathbb{S}^2$, $\mathbb{E}^2$, and $\mathbb{H}^2$.

(c) A complete geometric structure lifts to a complete geometric structure on the universal covering, and a complete geometric structure on a simply connected manifold is homogeneous.

Proof. Let M be a connected smooth manifold, and let X be the universal covering. A complete locally homogeneous Riemannian metric on M naturally induces a complete homogeneous Riemannian metric on X such that the covering map is a local isometry. Let G be the group of isometries on X. Is the geometry (X,G) maximal? For another complete locally homogeneous Riemannian metric on M, we have (X,G'). In which sense (X,G) and (X,G') are equivalent?

Let (X,G) be a model geometry. If $\dim X = n$, then the isotropy subgroup G_x is compact, we can show that there is a G_x-invariant inner-product on T_xX by using the Haar integral. Thus, G_x is naturally embedded in $\mathrm{GL}(T_xX)$. If (X,G') is a geometry such that $G \subset G'$, then G'_x is also embedded in $\mathrm{GL}(T_xX)$. Does the maximality imply G is isomorphic to $\mathrm{O}(n)$?

□

3.3 (Properly discontinuous actions)**.** A proper action of a discrete group on a locally compact Hausdorff space is said to be *properly discontinuous*.

On locally compact spaces: properly discontinuous action iff orbit counted with multiplicity is locally finite iff orbit is discrete and stabilizer is finite

Fuchsian groups act on H2 properly discontinuously (and faithfully) (and PSL(2,R) has compact stabilizer so that it has finite stabilizer), so the quotient is an orbifold.

Fuchsian group acts on H2 freely iff it is torsion-free because an element of PSL(2,R) has finite order iff it has a fixed point (properly discontinuous action is free iff the stabilizer is trivial at every point iff the quotient is a manifold.)

torsion-free Fuchsian groups conjugate iff quotient space form is isometric?

(convex, locally finite) fundamental domains poincare polygon

3.4. Liebmann theorem Hilbert problem

Part II

Riemann surfaces

Chapter 4

Uniformization

4.1 (Riemann surfaces)**.** A *Riemann surface* is a connected two-dimensional complex manifold.

Recall that a *meromorphic function* on X is a holomorphic function $f : U \to \mathbb{C}$ on an open subset $U \subset X$ such that $X \setminus U$ is discrete closed in X and the Laurent expansion at each point of $X \setminus U$ has the finite principal part. The set of meromorphpic functions $\mathcal{M}(X)$ is a field extension over the complex field $\mathbb{C}$ by the removable singularity theorem.

On a surface, complex structures, conformal structures, geomtric structures of curvature in $\{-1, 0, 1\}$ are all equivalent. The equivalence between the last two is usually called the uniformization.

- $g = 0$: Riemann sphere (spherical) $\to$ Riemann sphere itself
- $g = 1$: complex plane (Euclidean) $\to$ elliptic curves
- $g \geq 2$: open unit disk (hyperbolic) $\to$ hyperbolic surfaces, classified by Fuchsian groups(with which properties?)

4.2. holomorphic actions of $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{SO}(3)$ on the Riemann sphere $\mathbb{CP}^1$.
properly discontinuous actions of $\mathrm{PSL}(2, \mathbb{Z})$ on the complex plane $\mathbb{C}$.

Let $p : Y \to X$ be a non-constant holomorphic map. A point $y \in Y$ is called a *branch point* or a *ramification point* over $x \in X$ if $p(y) = x$ and p is not injective on any neighborhoods V of y.

For a non-constant holomorphic map, it is unbranched if and only if it is locally homeomorphic.

For a proper non-constant holomorphic map, a branch point is isolated in X (closed and discrete), and the multiplicity is finitely well-defined.

4.3. Let $p : Y \to X$ be a local homeomorphism from a topological space Y to a locally complex space X. Then, Y admits a unique complex structure such that p is holomorphic.

Proof. Consider the set of all open embeddings $\psi : V \to \mathbb{C}$ that factor through p and a chart on X. It is a complex atlas since $\psi_\alpha \psi_\beta^{-1} = \varphi_\alpha p(\varphi_\beta p)^{-1} = \varphi_\alpha \varphi_\beta^{-1}$ is biholomorphic The domains V form an open cover since p is a local homeomorphism. Therefore, it defines a complex atlas. The uniqueness follows from the identity theorem. □

For a holomorphic function germ in $\mathcal{O}_x$, there is a maximal analytic continuation. An analytic continuation of a holomorphic function germ in $\mathcal{O}_x$ is a triple of an unbranched non-constant holomorphic map $p : Y \to X$, a holomorphic function $f : Y \to \mathbb{C}$, and a point $y \in Y$ such that $p_* f_y$ is the germ.

dz and $d\bar{z}$ forms a basis of a two-dimensional complex vector space $T_x^* X \otimes_{\mathbb{R}} \mathbb{C} = \Omega_x^1(X, \mathbb{C})$.

Laurent expansion of a holomorphic 1-form on U at an isolated singularity. A *meromorphic 1-form* is defined by the finiteness of the negative part of the Laurent expansion.

Chapter 5

Riemann-Roch theorem

The ultimate goal is to prove a compact Riemann surface is algebraic.

cpt Riemann surfaces, existence of meromorphic, Chow, projective embedding, GAGA, coherence, sheaf cohomology

Not using sheaf cohomology and doing everything in line bundles? How can we define line bundle over an algebraic curve? How can we describe $\mathcal{O}^\times$, $\mathcal{M}^\times$ without sheaves?

How can we translate the following fact in terms of line bundles: sheaf of holomorphic/meromorphic functions and regular/rational functions: the category of coherent sheaves are equivalent!

5.1 Čech cohomology

On a compact Riemann surface X,

(a) $\underline{\mathbb{Z}}$, $\underline{\mathbb{C}}$ are constant sheaves.

(b) $\mathcal{O}$, $\mathcal{O}(D)$, $\mathcal{K}$ are invertible sheaves over $\underline{\mathbb{C}}$.

(c) $\mathcal{M}$ is a locally free sheaf over $\underline{\mathbb{C}}$.

(d) $\underline{\mathbb{Z}}$, $\underline{\mathbb{C}}$, $\mathcal{O}^\times$, $\mathcal{M}^\times$ are sheaves over $\underline{\mathbb{Z}}$.

(e) $\underline{\mathbb{C}}$ is not locally free over $\underline{\mathbb{C}}$.

(f) $\mathcal{O} = \Omega^0$ and $\mathcal{K} = \Omega^1$ is the space of holomorphic functions and holomorphic 1-forms.

5.1 (Leray cover).

5.2 (Dolbeault theorem).

5.2 Divisors and line bundles

Let X be a compact Riemann surface. The Cartier divisor group is just the global section space $\mathrm{Div}(X) := H^0(X, \mathcal{M}^\times/\mathcal{O}^\times)$ of the sheaf $\mathcal{M}^\times/\mathcal{O}^\times$ of abelian groups. The Picard group, defined by the group of line bundles, is just the cohomology group $\mathrm{Pic}(X) := H^1(X, \mathcal{O}^\times)$. We have an exact sequence of abelian groups

$$H^0(X, \mathcal{M}^\times) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to H^1(X, \mathcal{M}^\times).$$

We will prove $H^1(X, \mathcal{M}^\times) = 0$ be showing the surjectivity of $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.

If we use the exponential exact sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^\times \to 0$, we can define the first Chern class $\mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$.

On a scheme, a line bundle is usually defined as the locally free $\mathcal{O}$-module of rank one. On nice schemes, a sheaf would be invertible if and only if it is a line bundle. But more precisely, they are different objects. For a locally free sheaf $\mathcal{F}$ of finite rank, then the total space can be defined as $\operatorname{Spec}(\operatorname{Sym}\mathcal{F}^\vee) \to X$. (Recall that $\mathbb{A}_k^r = \operatorname{Spec} k[x_1, \cdots, x_r] = \operatorname{Spec}\operatorname{Sym}\langle x_1, \cdots, x_r\rangle_k$, where $\langle x_1, \cdots, x_r\rangle_k$ denotes the k-linear span)

Let $\pi : L \to X$ be a line bundle on a compact Riemann surface X. It means that π is a holomorphic map and there is an family $\{f_\alpha\}$ of biholomorphic maps $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$.

5.3 (Divisors)**.** Let X be a compact Riemann surface. A *Weil divisor* D on X is an element of the free abelian group generated by points of X. A *Cartier divisor* is a family $\{f_\alpha\}$ of non-zero meromorphic functions $f_\alpha \in \mathcal{M}^\times(U_\alpha)$ indexed by an open cover $\{U_\alpha\}$ of X such that f_α/f_β is extended to a holomorphic function on $U_\alpha \cap U_\beta$.

By compactness of X, a non-zero meromorphic function $f \in \mathcal{M}^\times(X)$ gives rise to a Weil divisor $(f) := \sum_{P\in X} \operatorname{ord}_P(f)P$, called the *principal divisor* of f.

Let $D = \sum_i n_i P_i$ be a Weil divisor on X. Each point $P \in X$ has a meromorphic function f on an open neighborhood U of P such that $(f) = D$ on U. It implies that there is a collection $\{f_\alpha\}$ of meromorphic functions f_α defined on U_α, where $\{U_\alpha\}$ is an open cover of X, such that f_α/f_β is a well-defined holomorphic functions on $U_\alpha \cap U_\beta$. In other words, a Cartier divisor is assigned to each Weil divisor.

A Cartier divisor defines a line bundle.

Here is a direct way to construct a line bundle from Weil divisors. Note that a field $\mathcal{M}(X) = \mathcal{M}^\times(X) \cup \{0\}$ of meromorphic functions contains the field of constant functions $\mathbb{C}$. For a Weil divisor D on X, define

$$L(D) := \{f \in \mathcal{M}^\times(X) : (f) + D \geq 0\} \cup \{0\}.$$

It is a complex vector subspace of $\mathcal{M}(X)$. We can construct a line bundle $\mathcal{O}(D)$ such that $L(D) = H^0(X, \mathcal{O}(D))$.

5.4. Given $\{P_i\}_{i=1}^n$ points and $\{f_i\}_{i=1}^n$ principal parts, there is a meromorphic function f with pre-described principal parts if and only if for every holomorphic 1-form ω we have $\sum_{i=1}^n \operatorname{Res}(f_i\omega, P_i) = 0$.

5.5.

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

The genus can be defined by $g = h^0(X, \Omega^1)$. For algebraic curves, it can be proved as follows: Assuming the Serre duality, we have $\chi(D) = h^0(D) - h^1(D) = l(D) - l(K - D)$ and $\chi(0) = h^0(0) - h^1(0) = 1 - g$. Then, the Riemann-Roch is boiled down to $\chi(D) = \deg(D) + \chi(0)$, which can be shown inductively.

However, we want to prove a compact Riemann surface is projective as an application of the Riemann-Roch theorem, we need to prove the Riemann-Roch theorem without theory of algebraic curves.

(a) If $\deg D < 0$, then $l(D) = 0$.

Proof. (a) Let $f \in L(D) \setminus \{0\}$. Then, $(f) + D \geq 0$ and $\deg(f) = 0$ imply $\deg D \geq 0$, which is a contradiction.

(b) Let $D = 0$. Then, it follows from $l(K) = g$ and $l(0) = 1$.

Let $D > 0$. We may assume $D = \sum_{i=1}^n n_i P_i$ with $n_i > 0$. (why?) Let

$$V_i := \Big\{ \sum_{k=-n_i}^{-1} c_k (z - P_i)^k : c_k \in \mathbb{C} \Big\}$$

and $V := \bigoplus_{i=1}^n V_i$. (how can we define the principal part of f on Riemann surface?) Then, $\dim V = \deg D$. Define $L(D) \to V$ by principal part at each point p_i.

□

5.6 (Embedding theorem)**.** Let X be a compact Riemann surface.

Let L be a line bundle over X, and let $(s_i)_{i=0}^n$ be a family of sections of L such that every point P of X has a section s_i which does not vanish at P.

The *complete linear system* of a Weil divisor D on X is the set

$$|D| := \{(f) + D : f \in \mathcal{O}(X)\}.$$

Then, by the map $L(D) \setminus \{0\} \to |D| : f \mapsto ...$ the set $|D|$ can be identified with the projective space $(L(D) \setminus \{0\})/\mathbb{C}^\times = \mathbb{CP}^{l(D)-1}$. Let $(f_i)_{i=0}^{l(D)-1}$ be an ordered basis of $L(D)$.

For a linear system Δ of projective dimension $n-1$, we can take (how?) a basis $(e_i)_{i=0}^{n-1}$ such that the following map is well-defined:

$$X \setminus \mathrm{Bl}(\Delta) \to \mathbb{CP}^{n-1} : p \mapsto (e_0 : \cdots : e_{n-1}).$$

5.3 Serre duality

Mittag-Leffler

0O(-D)OO(D)0 for effective D

adjunction formula? hyperplane section?

5.4 Hodge decomposition

5.7 (Finiteness of genus)**.** Let X be a compact Riemann surface.

$$0 \to \underline{\mathbb{C}} \to \mathcal{O} \xrightarrow{d} \mathcal{K} \to 0.$$

$h^-(X,-)$	$\mathbb{C}$	$\mathcal{O}$	$\mathcal{K}$
0	1	1	g
1	$2g$	g	1
2	1	0	

We start from

- $h^0(X,\mathbb{C}) = 1$ by connectedness.
- $h^0(X,\mathcal{O}) = 1$ by the Liouville theorem.

By the Serre duality,

- $h^0(X,\mathcal{K}) = h^1(X,\mathcal{O})$.
- $h^1(X,\mathcal{K}) = h^0(X,\mathcal{O}) = 1$.

Then,

- $h^1(X,\mathbb{C}) = 2g$ by the rank-nullity theorem from $h^1(X,\mathcal{O}) = g$ and $h^0(X,\mathcal{K}) = g$.

Also,

- $h^2(X,\mathbb{C}) = 1$ by the Poincaré theorem.
- $h^2(X,\mathcal{O}) = 0$ by the Dolbeault theorem?

The *Euler characteristic* is defined by

$$\chi := h^0(X,\mathbb{C}) - h^1(X,\mathbb{C}) + h^2(X,\mathbb{C}),$$

and the *genus* can be defined by the formula

$$\chi =: 2-2g.$$

Since $h^0(X,\mathbb{C}) = 1$, $h^2(X,\mathbb{C}) = 1$, by the rank-nullity theorem we have

$$g = h^1(X,\mathcal{O}).$$

Let $D \subset \mathbb{C}$ be an open set. Then, U has the canonical volume form $dx \wedge dy$. Let $L^2(D,\mathcal{O})$ be the completion of sqaure-integrable holomorphic functions on D with respect to the L^2-norm. We have $L^2(D,\mathcal{O}) \subset L^2(D)$.

(a) $L^2(D,\mathcal{O}) \subset \Gamma(D,\mathcal{O}) = \mathcal{O}(D)$.

(b) If $D' \Subset D$, then for every $\varepsilon > 0$ there is a finite-codimensional linear subspace $A \subset L^2(D,\mathcal{O})$ such that

$$\|f\|_{L^2(D')} \le \varepsilon \|f\|_{L^2(D)}, \qquad f \in A.$$

Proof. (a) For $r > 0$, let $U_r := \{z \in U : B(z,r) \subset U\}$. It suffices to show

$$\|f\|_{L^\infty(U_r)} \le \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(U)}, \qquad f \in \mathcal{O}(U).$$

(b)

Take $r > 0$ and a finite subset $\{a_j\}$ of D' such that $D' \subset \bigcup_j B(a_j, \frac{r}{2})$ and $\bigcup_j B(a_j, r) \subset D$. Take n such that $2^{-n-1}|\{a_j\}|^{\frac{1}{2}} < \varepsilon$. Let

$$A := \{f \in L^2(D,\mathcal{O}) : f^{(k)}(a_j) = 0 \text{ for all } k \le n \text{ and } j\}.$$

If $f \in A$, then at a_j we have the power series expansion

$$f(z) = \sum_{k=n}^{\infty} c_k (z-a_j)^k, \qquad z \in B(a_j, r).$$

Thus

$$\begin{aligned}
\|f\|_{L^2(D')}^2 &\le \sum_j \|f\|_{L^2(B(a_j,\frac{r}{2}))}^2 \\
&= \sum_j \sum_{k=n}^{\infty} \frac{\pi(\frac{r}{2})^{2n+2}}{k+1} |c_k|^2 \\
&= 2^{-2n-2} \sum_j \sum_{k=n}^{\infty} \frac{\pi r^{2n+2}}{k+1} |c_k|^2 \\
&= 2^{-2n-2} \sum_j \|f\|_{L^2(B(a_j,r))}^2 \\
&\le 2^{-2n-2} |\{a_j\}| \|f\|_{L^2(D)}^2, \qquad f \in A,
\end{aligned}$$

so the desired inequality follows.

(c) 14.6?

(d)

$$L := \{(\xi,\eta,\zeta) \in Z^1_{L^2}(\mathcal{U},\mathcal{O}) \oplus Z^1_{L^2}(\mathcal{V},\mathcal{O}) \oplus C^0_{L^2}(\mathcal{W},\mathcal{O}) : \eta = \xi + \delta\zeta \text{ on } \mathcal{W}\}.$$

Since the projection$L \to Z^1_{L^2}(\mathcal{V},\mathcal{O})$ is a surjective bounded linear operator, by the Bartle-Graves theorem, there is a non-necessary linear bounded map $Z^1_{L^2}(\mathcal{V},\mathcal{O}) \to L$ which is a right inverse.

□

A method using Hodge decomposition may be more natural?

Chapter 6

Algebraic curves

6.1 Projective varieties

multiplicities, Bezout theorem

6.2 Chow theorem

divisors, line bundles euler characteristic (tangent line bundle degree 2-2g, canonical line bundle 2g-2) $L(D) := \Gamma(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D))$

Jacobian variety (moduli spaces....)

6.1 (Chow theorem)**.** A complex submanifold of a projective space is algebraic.

6.3 Moduli spaces