Homological Algebra

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1 Day 1: April 6

1. Modules

References: Atsushi Shiho, Yukiyoshi Kawada

1.1. R-modules

Definition 1.1. Let R be a ring with 1. A (left) R-module is an abelian group M with a map $R \times M \to M : (a, x) \mapsto ax$ satisfying $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ab)x = a(bx)$, $1x = x$.

Example 1.1. (a) Every abelian group is a $\mathbb{Z}$-module. The R-module structures on an abelian group M has 1-1 correspondence with the ring homomorphisms $R \to \mathrm{End}_{\mathbb{Z}}(M)$.

(b) $M = C^{\infty}(\mathbb{R})$, $R = \mathbb{R}[T]$ a polynomial ring, $R \times M \to M : (P(T), f(x)) \mapsto P(\frac{d}{dx})f(x)$.

Definition 1.2. A (left) R-submodule of M is a subgroup $N \subset M$ such that $ax \in N$ for $a \in R$, $x \in N$. A (left) R-homomorphism is a group homomorphism $M \to N$ which preserves the action of R.

Example 1.2. (a) $M = C^{\infty}(\mathbb{R})$, $R = \mathbb{R}[T]$, then $\varphi : M \to M : f(x) \mapsto f(x+1)$ is an R-homomorphism.

Definition 1.3. Let $f : M \to N$ be an R-homomorphism. The kernel of f is $\ker f := \{x \in M : f(x) = 0\} \xrightarrow{i} M$, and the cokernel of f is $N \xrightarrow{p} \operatorname{coker} f := N/\operatorname{im} f$, where the image is $\operatorname{im} f := \{f(x) \in N : x \in M\} \xrightarrow{j} N$.

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & M \arrow[rr, "f"] \arrow[dr, "f"'] & & N \arrow[r, "p"] & \operatorname{coker} f \\
& & \operatorname{im} f \arrow[ur, "j"'] & &
\end{tikzcd}$$

On each of them, there is a unique R-module structure such that the each map i, j, p becomes an R-homomorphism respectively.

Theorem 1.1 (Universal property)**.** *For the above setting, note that $fi = 0$ and $pf = 0$. If an R-homomorphism $g : M' \to M$ satisfies $fg = 0$, then there is a unique R-homomorphism $h : M' \to \ker f$ such that $g = ih$. If an R-homomorphism $g : N \to N'$ satisfies $gf = 0$, then there is a unique R-homomorphism $h : \operatorname{coker} f \to N'$ such that $g = hp$.*

1.1 Commutative diagrams and exact sequences

Definition 1.4 (Diagram)**.** Among some R-modules suppose we have R-homomorphisms as the following diagram:

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "f_3"'] & M_2 \arrow[d, "g_1"] \arrow[dl, "f_2"'] \\
M_3 \arrow[r, "g_2"] & M_4
\end{tikzcd} .$$

Then, if the compositions sharing each source and target coincide, then we say the diagram is commutative. For example, we say the triangle formed by M_2, M_3, M_4 is commutative iff $g_1 = g_2 f_2$.

Definition 1.5 (Sequence)**.** A sequence is a diagram of R-modules placed linearly as

$$\cdots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \longrightarrow \cdots .$$

If $\operatorname{im} f_n = \ker f_{n+1}$ for all n, then we say the sequence is exact.

Example 1.3. (a) $f : M \to N$ is injective iff $0 \to M \xrightarrow{f} N$ is exact. $f : M \to N$ is surjective iff $M \xrightarrow{f} N \to 0$ is exact.

(b)

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \longrightarrow 0$$

is exact.

(c)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is exact.

(d)

$$0 \longrightarrow \mathbb{R}\cos x \oplus \mathbb{R}\sin x \xrightarrow{n} C^\infty(\mathbb{R}) \xrightarrow{\frac{d^2}{dx^2}+1} C^\infty(\mathbb{R}) \longrightarrow 0$$

is exact.

Proposition 1.2 (Five lemma)**.** *Suppose each row is exact in the folloing commutative diagram:*

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & M_5 \arrow[d, "h_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5.
\end{tikzcd}$$

Then,

(a)

(b)

(c)

Proof. (a) We will show $x \in \ker h_3$ is in the image of $f_2 f_1$: $h_3(x) = 0 \implies f_3(x) = 0 \implies x = f_2(y) \implies g_2 h_2(y) = 0 \implies h_2(y) = g_1(z) \implies z = h_1(u) \implies f_1(u) = y$. Then, $x = f_2(y) = f_2 f_1 = 0$.

(b) Similar.

(c) Clear. $\square$

Proposition 1.3 (Snake lemma)**.** *Suppose the second and the third rows are exact in the following commutative diagram:*

$$\begin{array}{ccccc}
 & \ker h_1 & \ker h_2 & \ker h_3 & \\
 & M_1 & M_2 & M_3 & 0 \\
0 & N_1 & N_2 & N_3 & \\
 & \operatorname{coker} h_1 & \operatorname{coker} h_2 & \operatorname{coker} 3 &
\end{array}$$

(a) *There is $\delta : \ker h_3 \to \operatorname{coker} h_1$ such that*

$$\ker h_1 \xrightarrow{k_1} \ker h_2 \xrightarrow{k_2} \ker h_3 \xrightarrow{\delta} \operatorname{coker} h_1 \xrightarrow{l_1} \operatorname{coker} h_2 \xrightarrow{l_2} \operatorname{coker} 3$$

is exact. Here k_1, k_2, l_1, l_2 are induced from f_1, f_2, g_1, g_2, respectively. The element $\delta(x)$ is determined by u such that $x = f_2(y)$, $z = h_2(y)$, $z = g_1(u)$, and we can check that u does not depend on the choice of y.

(b)

Proof. (a) We have to show the well-definedness of δ, ker ⊂ im, and im ⊂ ker. Skip. □

In the general abelian cateogies, the five lemma and the snake lemma hold but the proofs become more complicated.

1.2 Direct sum, direct product, inductive limit, direct limit

Definition 1.6. Let M_λ be a family of R-modules. The direct product is

$$\prod_\lambda M_\lambda := \{(x_\lambda) : x_\lambda \in M_\lambda\} \twoheadrightarrow M_\lambda,$$

and the direct sum is the submodule of the direct product such that

$$\bigoplus_\lambda M_\lambda := \{(x_\lambda) : x_\lambda = 0 \text{ but finitely many}\} \hookleftarrow M_\lambda$$

Proposition 1.4 (Universal property). (a) *For $f_\mu : M_\mu \to N$ there is unique $f : \bigoplus_\lambda M_\lambda \to N$ such that $f\iota_\mu = f_\mu$.*

(b) *For $g_\mu : N \to M_\mu$ there is unique $g : N \to \prod_\lambda M_\lambda$ such that $p_\mu g = g_\mu$.*

Remark. (a) The direct sum and direct product is unique up to isomorphism by the universal property.

(b) For R-homomorphisms $f_\lambda : M_\lambda \to N_\lambda$ we can induce $\prod_\lambda f_\lambda : \prod_\lambda M_\lambda \to \prod_\lambda N_\lambda$ and $\bigoplus_\lambda f_\lambda : \bigoplus_\lambda M_\lambda \to \bigoplus_\lambda N_\lambda$.

(c) In the category of modules, even for infinite indices, direct product and sum commute with the kernel, cokernel, and image. In an abelian category, we may not have infinie direct product/sum.

(d) exactness also preserved under products and sums

2 Day 2: April 13

Let $(\Lambda, \prec)$ be a totally ordered set. By a direct system, we refer the family of R-modules M_λ for each $\lambda \in \Lambda$ and the family of R-homomorphisms $\tau_{\mu\lambda} : M_\lambda \to M_\mu$ for $\lambda \prec \mu$ such that $\tau_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ and $\tau_{\kappa\lambda} = \tau_{\kappa\mu}\tau_{\mu\lambda}$ for $\lambda \prec \mu \prec \kappa$.

Example 2.1 (1.3.3.). (a) Let $\Lambda = \mathbb{N}$ and $n \prec m \iff n \mid m$, $M_n = \mathbb{Z}$ and $\tau_{mn}(z) : M_n \to M_m : z \mapsto (m/n)z$.

(b) Let M be a R-module, $\{M_\lambda\}$ are finitely generated R-submodules of M, and $\lambda \prec \mu \iff M_\lambda \subset M_\mu$, with $\tau_{\mu\lambda}$ inclusions.

Definition 2.1.

$$\varinjlim M_\lambda = \varinjlim (M_\lambda, \tau_{\mu\lambda}) := \operatorname{coker}\Big(\bigoplus_{\substack{(\lambda,\mu)\in\Lambda \\ \lambda\prec\mu}} M_\lambda \xrightarrow{\Phi} \bigoplus_{\lambda\in\Lambda} M_\lambda\Big),$$

where $\Phi((x_{\lambda\mu})) = \sum_{\lambda\prec\mu} \iota_\mu \tau_{\mu\lambda}(x_{\lambda\mu}) - \iota_\lambda(x_{\lambda\mu})$, and $\iota_\lambda : M_\lambda \to \bigoplus_\lambda M_\lambda$ is a componentwise embedding. That is, we want to identify $x \in M_\lambda$ and $\tau_{\mu\lambda}(x) \in M_\mu$ with the map Φ.

Proposition 2.1 (1.3.4). *Let* $\tau_\mu : M_\mu \xrightarrow{\iota_\mu} \bigoplus_\lambda M_\lambda \twoheadrightarrow \lim_{\longrightarrow} M_\lambda$.

(a) $\tau_\mu = \tau_\kappa \tau_{\kappa\mu}$.

(b) $M_\mu \xrightarrow{f_\mu} N$ *for* $\mu \in \Lambda$ *are R-homomorphisms, and they satisfy* $f_\mu = f_\kappa \tau_{\kappa\mu}$. *Then, there is a unique* $f : \lim_{\longrightarrow} M_\lambda \to N$ *such that* $f_\mu = f\tau_\mu$

For each example in 1.3.3, $\mathbb{Q}$ and M are the direct limits because it satisfies the universal property (1.3.4(b))

Remark. (1) The direct limit is unique by the universal property up to isomorphism.

(2) If $f_\lambda : M_\lambda \to M'_\lambda$ are R-homomorphism such that

$$\begin{tikzcd} M_\lambda \arrow[r, "f_\lambda"] \arrow[d] & M\lambda' \arrow[d] \\ M_\mu \arrow[r, "f_\mu"] & M'_\mu \end{tikzcd}$$

commutes for all $\lambda \prec \mu$, then there is a unique f such that

$$\begin{tikzcd} \bigoplus_{\lambda\prec\mu} M_\lambda & \bigoplus_\lambda M_\lambda & \lim_{\longrightarrow} M_\lambda \arrow[d, "\exists! f"] & 0 \\ \bigoplus_{\lambda\prec\mu} M'_\lambda & \bigoplus_\lambda M'_\lambda & \lim_{\longrightarrow} M'_\lambda & 0 \end{tikzcd}$$

commutes, and f is denoted by $\lim_{\longrightarrow} f_\lambda$. It is by the universal property of cokernel.

Definition 2.2 (1.3.6). A preordered set Λ is a directed set if $\forall \lambda, \lambda' \in \Lambda$, there is $\mu \in \Lambda$ such that $\lambda, \lambda' \prec \mu$.

Proposition 2.2. *If* Λ *is a directed set, then there is a 1-1 correspondence*

$$(\coprod_\lambda M_\lambda)/\sim \longrightarrow \varinjlim M_\lambda : [x_\lambda] \mapsto \tau_\lambda(x_\lambda),$$

where $x_\lambda \sim y_{\lambda'}$ *iff there is* $\mu \succ \lambda, \lambda'$ *such that* $\tau_{\mu\lambda}(x_\lambda) = \tau_{\mu\lambda'}(y_{\lambda'})$.

Proposition 2.3. *If*

$$L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \longrightarrow 0$$

is exact, then

$$\operatorname{colim} L_\lambda \longrightarrow \operatorname{colim} M_\lambda \longrightarrow \operatorname{colim} N_\lambda \longrightarrow 0$$

is exact.

Proof. We cannot use the snake lemma directly on

$$\begin{tikzcd} \bigoplus_{\lambda\prec\mu} L_\lambda \arrow[r] & \bigoplus_{\lambda\prec\mu} M_\lambda \arrow[r] & \bigoplus_{\lambda\prec\mu} N_\lambda \arrow[r] & 0 \\ \bigoplus_\lambda L_\lambda \arrow[r] & \bigoplus_\lambda M_\lambda \arrow[r] & \bigoplus_\lambda N_\lambda \arrow[r] & 0 \\ \operatorname{colim} L_\lambda \arrow[r] & \operatorname{colim} M_\lambda \arrow[r] & \operatorname{colim} N_\lambda \arrow[r] & 0. \end{tikzcd}$$

So,

$$0 \longrightarrow \bigoplus_{\lambda \prec \mu} \ker g_\lambda \longrightarrow \bigoplus_{\lambda \prec \mu} M_\lambda \longrightarrow \bigoplus_{\lambda \prec \mu} N_\lambda \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_\lambda \ker g \longrightarrow \bigoplus_\lambda M_\lambda \longrightarrow \bigoplus_\lambda N_\lambda \longrightarrow 0$$

$$\operatorname{colim} \ker g_\lambda \longrightarrow \operatorname{colim} M_\lambda \longrightarrow \operatorname{colim} N_\lambda \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{\lambda \prec \mu} \ker f_\lambda \longrightarrow \bigoplus_{\lambda \prec \mu} L_\lambda \longrightarrow \bigoplus_{\lambda \prec \mu} \operatorname{im} f_\lambda \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_\lambda \ker f \longrightarrow \bigoplus_\lambda L_\lambda \longrightarrow \bigoplus_\lambda \operatorname{im} f_\lambda$$

$$\operatorname{colim} \ker f_\lambda \longrightarrow \operatorname{colim} L_\lambda \longrightarrow \operatorname{colim} \operatorname{im} f_\lambda$$

???????????? □

Example 2.2. Examples of inverse limit

(a) projection $\mathbb{Z}/p^m\mathbb{Z}\mathbb{Z}/p^n\mathbb{Z}$ for $m > n$.

(b) restriction $C^\infty((-r, r)) \to C^\infty((-r', r')$ for $r' > r$.

Ab-enriched: preadditive binary biproduct: semiadditive existence of ker/coker normality of mono/epi constructions: universals(products, limits, representables)