Shimura Varieties

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1.1 Modular curves

[DS05] Let $\mathcal{H}$ be the upper half plane. For $N \geq 1$, let $\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$ and $Y(N) := \Gamma(N) \setminus \mathcal{H}$ be the left quotient. Then, $Y(N)$ has a Riemann surface structure, and is called the *modular curve* of level $\Gamma(N)$.

Remark. If $N \geq 2$, then the projection $\mathcal{H} \to Y(N)$ is a local homeomorphism, so that $Y(N)$ is a Riemann surface.

Definition. For an elliptic curve E over $\mathbb{C}$, a *level* $\Gamma(N)$*-structure* is a pair (P, Q) of generators of $E[N]$ satisfying $e_N(P,Q) = e^{\frac{2\pi i}{N}}$, where $e_N : E[N] \times E[N] \to \mu_N$ is the Weil pairing. We denote by $\mathrm{Ell}(\Gamma(N))$ the set of isomorphism classes of elliptic curves over $\mathbb{C}$ with level $\Gamma(N)$-structure.

For $\tau \in \mathcal{H}$, let $E_\tau := \mathbb{C}/\tau\mathbb{Z} \oplus \mathbb{Z}$.

Proposition. $\tau \mapsto (E_\tau, ([N^{-1}], [-\tau N^{-1}])$ *is bijective and induces* $Y(N) \to \mathrm{Ell}(\Gamma(N))$.

Remark. $(E_\tau, ([N^{-1}], [-\tau N^{-1}])) \cong (E_{-\tau^{-1}}, ([-\tau^{-1}N], [N^{-1}]))$.

Remark. If $N \geq 3$, then $Y(N)$ can be regarded as the moduli space of elliptic curves with $\Gamma(N)$-level structure.

Let $\mathcal{H}^\pm := \mathbb{C} \setminus \mathbb{R}$. Then, $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathcal{H}^\pm$. Let $\mathbb{A}^\infty := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the finite adele ring. For a compact open subgroup $K \subset \mathrm{GL}_2(\mathbb{A}^\infty)$, define the double coset space

$$\mathrm{Sh}_K := \mathrm{GL}_2(\mathbb{Q}) \setminus \mathcal{H}^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty)/K.$$

For $N \geq 3$, let $K(N) := \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$.

Proposition. *We have a bijection*

$$\coprod_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N) \setminus \mathcal{H} \to \mathrm{Sh}_{K(N)} : [\tau]_a \mapsto [(\tau, \begin{pmatrix} \widehat{a} & 0 \\ 0 & 1 \end{pmatrix})],$$

where $\widehat{a} \in \widehat{\mathbb{Z}}$ *is the lift of* a.

Remark. To give a moduli interpretation on $\mathrm{Sh}_{K(N)}$, we can remove the condition $e_N(P,Q) = e^{2\pi i/N}$ in the definition of level structures. More generally, Sh_K has a natural scheme structure over $\mathbb{C}$, called the *Shimura variety of level* K *for* $(\mathrm{GL}_2, \mathcal{H}^\pm)$.

Let Ell_K be the set of isogeny classes of $(E, \eta K)$, where E is a complex elliptic curve and

$$\eta : (\mathbb{A}^\infty)^2 \xrightarrow{\sim} V^\infty(E) := (\varprojlim_n E[n]) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Fix $[(E, \eta K)] \in \mathrm{Ell}_K$.

Take $\psi : H_1(E, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^2$. Then, by the Hodge decomposition $H^1(E, \mathbb{C}) \cong H^1(E, \mathcal{O}_E) \oplus H^0(E, \Omega_E)$, we can define a unique $\tau_\psi \in \mathcal{H}^\pm$ such that $\ker \rho_\psi = \mathbb{C}(\tau_\psi, 1)$, where

$$\rho_\psi : \mathbb{C}^2 \to H_1(E, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_1(E, \mathbb{C}) \cong H^1(E, \mathbb{C})^* \cong H^1(E, \mathcal{O}_E)^* \oplus H^0(E, \Omega_E)^* \twoheadrightarrow H^0(E, \Omega_E)^*.$$

Define $g_{\eta,\psi} \in \mathrm{GL}_2(\mathbb{A}^\infty)$ by

$$(\mathbb{A}^\infty)^2 \xrightarrow{\eta} V^\infty(E) \cong (\varprojlim_n E[n]) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\varprojlim_n H_1(E, \mathbb{Z}/n\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{H}_1(E, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\psi \otimes 1} (\mathbb{A}^\infty)^2.$$

Now, it is known that we have a bijection

$$\Phi : \mathrm{Ell}_K \to \mathrm{Sh}_K : [(E, \eta K)] \mapsto [(\tau_\psi, g_{\eta,\psi})].$$

Then, $\mathcal{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}^{\infty})/K$ can be seen as the set of all isogeny classes of $(E, \eta K, \psi)$, and we have the following diagram:

$$\begin{array}{ccccc} \mathrm{Sh}_K & \longleftarrow & \mathcal{H}_{\pm} \times \mathrm{GL}_2(\mathbb{A}^{\infty})/K & \longrightarrow & \mathcal{H}^{\pm} \\ [(E, \eta K)] & \longleftarrow\!\shortmid & [(E, \eta K, \psi)] & \longmapsto & \tau_{\psi} \end{array}$$

1.2 $\mathcal{H}^{\pm}$ for the theory of Shimura varieties

Let $\mathbb{S}$ be the *Deligne torus*, defined as the Weil restriction $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. This is a group scheme over $\mathbb{R}$ characterized such that for $\mathbb{R}$-algebra R, we have $\mathbb{S}(R) = \mathbb{G}_m(\mathbb{C} \otimes_{\mathbb{R}} R) \cong (\mathbb{C} \otimes_{\mathbb{R}} R)^{\times}$. For a real vector space V, the homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)$ corresponds to the Hodge structure on V such that $h(z_1, z_2)v = z_1^{-p} z_2^{-q} v$ if $v \in V^{p,q}$.

Let $h : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}} : a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. If we let X be a $\mathrm{GL}_2(\mathbb{R})$-conjugacy class of h, then $X \to \mathcal{H}^{\pm} : \mathrm{ad}(g)h \mapsto gi$ is bijective. Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2 : a \otimes b \mapsto (ab, a\overline{b})$, we have $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, and we can define

$$\mu_h : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}}^2 \cong \mathbb{S}_{\mathbb{C}} \xrightarrow{h \otimes \mathbb{C}} \mathrm{GL}_{2,\mathbb{C}}.$$

Let M_X be the $GL_2(\mathbb{C})$-conjugacy classes of μ_h, and consider $X \to M_X : \mathrm{ad}(g)h \mapsto \mathrm{ad}(g)\mu_h$. For $\mu \in M_X$, by associate a one-dimensional subspace of $\mathbb{C}^2$ such that $\mu(z)$ acts as the scaling by z, we have $M_X \to \mathbb{P}^1(\mathbb{C})$. Therefore, we can put a complex structure on X by $X \to \mathbb{P}^1(\mathbb{C})$, which is compatible with the one of $\mathcal{H}^{\pm}$.

For $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and $\mu \in M_X$, we determine $\sigma(\mu)$ such that

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \otimes_{\mathbb{C},\sigma} \mathbb{C} \arrow[r, "\mu_{\otimes_{\mathbb{C},\sigma}\mathbb{C}}"] \arrow[d, equal] & \mathrm{GL}_{2,\mathbb{C}} \otimes_{\mathbb{C},\sigma} \mathbb{C} \arrow[d, equal] \\ \mathbb{G}_{m,\mathbb{C}} \arrow[r, "\sigma(\mu)"] & \mathrm{GL}_{2,\mathbb{C}} \end{tikzcd}$$

commutes. Let $\sigma(M_X)$ be the $\mathrm{GL}_2(\mathbb{C})$-conjugacy class of $\sigma(\mu)$. The fixed field determined by $\{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) : \sigma(M_X) = M_x\}$ is called the *relfex field* of M_X. In the case we have seen, the reflex field of M_X is $\mathbb{Q}$, which means that we have a standard model of Sh_K on $\mathbb{Q}$.

Matome:

$$\begin{tikzcd} X \times \mathrm{GL}_2(\mathbb{A}^{\infty}) \arrow[r, two heads] \arrow[d] \arrow[dr] & X \arrow[r, tail] \arrow[d, tail] & M_X \arrow[dl] \\ \mathrm{Sh}_K & \mathbb{P}^1(\mathbb{C}) & \end{tikzcd}$$