Homological Algebra

Ikhan Choi

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Contents

Part I

Abelian categories

Chapter 1

Category of modules

A left R-module P is projective if and only if the left exact functor $\mathrm{Hom}_R(P,-)$ is exact.

A left R-module I is injective if and only if the left exact contravariant functor $\mathrm{Hom}_R(-,I)$ is exact.

projective

- direct sum of projectives is projective
  (lem) if free, then projective
- PID: projective iff free (note sub of free is free in PID)
- projective iff direct summand of a free
- every module is a quotient of a free module

injective

- direct product of injectives is injective
  (lem) M injective iff $\mathrm{Hom}_R(R,M) \to \mathrm{Hom}_R(I,M)$ surj
- PID: injective iff divisible $(\cdots a : M \to M$ surj)
  (lem) $\mathrm{Hom}_Z(R,M)$ is injective if M is injective $\mathbb{Z}$-module
- every module is embedded in injective

flat

- PID: flat iff $(\cdot a : M \to M$ inj)
- M flat iff $\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$ is injective
- M flat iff $I \otimes M \to R \otimes M$ inj
- if projective, then flat

continuity of functors

1.1 (Tor functor)**.** Let R be a ring and M be a left R-module. We define the *Tor functor* as the left derived functor of the right exact functor $-\otimes_R M : \mathrm{Mod}\text{-}R \to \mathrm{Ab}$

$$\mathrm{Tor}_n^R(N,M) := H_n(P_\bullet \otimes_R M),$$

where $P_\bullet$ is a projective resolution of a right R-module N.

(a) In fact, the Tor functor may be defined by the left derived functor of the right exact functor $M \otimes_R - : R\text{-}\mathrm{Mod} \to \mathrm{Ab}$ for a right R-module M.

(b) In fact, only for Tor functors, we may only assume $P_\bullet$ is a flat resolution. (Flat resolution lemma)

1.2 (Ext functor)**.** Let R be a ring and M be a left R-module. We define the *Ext functor* as the right derived functor of left exact functor $\mathrm{Hom}_R(M,-)$

$$\mathrm{Ext}^n_R(M,N) := H^n(M,I^\bullet),$$

where $I^\bullet$ is an injective resolution of N.

(a) In fact, the Ext functor may be defined by the right derived functor of the left exact contravariant functor $\mathrm{Hom}(-,M)$.

long exact seuqence

1.3 (Universal coefficient theorem)**.** Let R be a ring. Let $C_\bullet$ be a chain complex of flat right R-modules and M be a left R-module.

Proof. We first prove the Künneth formula. Note that modules in $Z_\bullet$ and $B_\bullet$ are also flat. We start from that we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0.$$

We have a short exact sequence of chain complexes

$$\mathrm{Tor}_1^R(B_{\bullet-1},M) \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since modules in $B_{\bullet-1}$ are flat so that $\mathrm{Tor}_1^R(B_{\bullet-1},M) = 0$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since $H_n(C_{\bullet-1}) = H_{n-1}(C_\bullet)$ for any chain complex C, we have a long exact sequence

$$H_n(B_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(C_\bullet \otimes_R M) \to H_{n-1}(B_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet \otimes_R M).$$

Since every morphism in $B_\bullet$ and $Z_\bullet$ is zero, we have an exact sequence

$$B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet \otimes_R M) \to B_{n-1} \otimes_R M \xrightarrow{f_{n-1}} Z_{n-1} \otimes_R M.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{coker} f_n \to H_n(C_\bullet \otimes_R M) \to \ker f_{n-1} \to 0.$$

Since

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

is a flat resolution of $H_n(C_\bullet)$, by the flat resolution lemma, we have a long exact sequence

$$\mathrm{Tor}_1^R(Z_n,M) \to \mathrm{Tor}_1^R(H_n(C_\bullet),M) \to B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet) \otimes_R M \to 0.$$

Since Z_n is flat so that $\mathrm{Tor}_1^R(Z_n,M) = 0$, we have

$$\operatorname{coker} f_n = H_n(C_\bullet) \otimes_R M, \quad \ker f_n = \mathrm{Tor}_1^R(H_n(C_\bullet),M).$$

Therefore, we have an exact sequence

$$0 \to H_n(C_\bullet) \otimes_R M \to H_n(C_\bullet \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C_\bullet),M) \to 0.$$

Universal coefficient theorem states that if R is a PID, then the Künneth formula splits non-canonically. □

Chapter 2

$$\begin{array}{ccccccc}
K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \big\downarrow\!\!\!\!\downarrow & & \big\downarrow & & \\
K' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 0
\end{array}$$

(a) If $A \to A'$ is monic, then $K \to K'$ is monic.

(b) If $B \to B'$ is monic, then $K \to K'$ is epic.

Chapter 3

Cohomology of algberas

3.1 Group cohomology

The category of G-modules can be identified with the category of $\mathbb{Z}[G]$-modules, which is abelian.

Let M be a G-module. The *invariant submodule* of M is denoted by M^G. Sending M to M^G yields a functor Grp $\to$ Ab, which is left exact but not right exact in general. Then we can consider the right derived functor to define cohomology groups. Let us do this concretely.

Let M be a G-module. Define $C^n(G,M)$ be the abelian group of all functions $G^n \to M$. The coboundary homomorphism $d : C^n(G,M) \to C^{n+1}(G,M)$ is defined such that

$$d\varphi(g_1,\cdots,g_{n+1}) := g_1\varphi(g_2,\cdots,g_{n+1})+\sum_{i=1}^{n}(-1)^i\varphi(g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+2},\cdots,g_{n+1})+(-1)^{n+1}\varphi(g_1,\cdots,g_n).$$

$$H^0(G,M) = M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M).$$

For $x \in C^0(G,M) = M$, $dx(g) = gx - x$. For $\varphi \in C^1(G,M)$, $d\varphi(g,h) = g\varphi(h) - \varphi(gh) + \varphi(g)$.

Part II

Derived categories

Chapter 4

Derived categories

4.1 Differential graded categories

4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and suppose $\mathcal{A}$ has enough injectives, that is, every object $A \in \mathcal{A}$ admits a monomorphism $A \to I$ for an injective object I. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a left-exact functor.

derived category of differential graded category.

4.2 Triangulated categories

4.2 (Triangulated categories)**.** A *triangulated category* is an additive functor $\mathcal{D}$ together with a translation functor $\mathcal{D} \to \mathcal{D} : X \mapsto X[1]$, which is an equivalence of categories, and a collection of distinguished triangles

Part III

Homotopical algebra

Chapter 5

Model categories

5.1 (Model structures)**.** Let $\mathcal{C}$ be a category. Following the definition of Hovey, a *model structure* on $\mathcal{C}$ is a three subcategories of $\mathcal{C}$ called *weak equivalences*, *cofibrations*, and *fibrations* such that

(i) the weak equivalences satisfy the two-out-of-three law,

(ii) cofibrations and acyclic fibrations form a functorial weak factorization system,

(iii) acyclic cofibrations and fibrations form a functorial weak factorization system.

We denote by $\mathcal{W}$ the subcategory of weak equivalences is denoted by.

(a) retract closedness

(b)

Serre model structure and Hurewicz model structure on Top.

Chapter 6

Infinity categories

6.1 Simplicial sets

Two representative examples: nerves and Kan complexes

infinity categories as simplicially enriched categories

6.1 (Nerves)**.** For an ordinary category as a nerve, two morphisms are homotopic only if they are identical.

6.2 (Kan complexes)**.** A geometric model for infinity groupoids. In a Kan complex, including Sing of a topological space, every morphism is invertible up to homotopy.

Infinity groupoids are usually considered as "spaces".

6.3 (Dold-Kan correspondence)**.**

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{\mathbb{Z}[\cdot]} \mathrm{sAb} \xrightarrow{C_\bullet \text{ or } N_\bullet} \mathrm{Ch}(\mathbb{Z}) \xrightarrow{H_n} \mathrm{Ab}$$

Two descriptions for normalized Moore complexes:

$$0 \to N_\bullet(A) \to C_\bullet(A) \to D_\bullet(A) \to 0.$$

Eilenberg-Maclane functor $K : \mathrm{Ch}(\mathbb{Z}) \to \mathrm{sAb}$ as the right adjoint for the functor $N_\bullet$.

6.2 Kan complexes

The *infinity category of spaces*, denoted by Spc, is defined as the homotopy-coherent nerve of the category Kan of Kan complexes.

6.3 Stable infinity categories

examples of stable infinity category: the infinity category of spectra, the dervied category of an abelian category

6.4. A *stable infinity category* is an infinity category such that

(i) there is a zero object,

(ii) every morphism admits a fiber and cofiber,

(iii) a triangle is a fiber sequence if and only if it is a cofiber sequence.

It is known that its homotopy category is tricngulated.

6.5 (Triangulated categories).

6.6 (Differential graded category).