Abstract Harmonic Analysis

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Contents

Part I

Chapter 1

Locally compact groups

1.1

1.1 (Non-σ-finite measures)**.** Following technical issues are important

(a) The Fubini theorem

(b) The Radon-Nikodym theorem

(c) The dual space of L^1 space

1.2 (Existence of the Haar measure)**.**

1.3 (Left and right uniformities)**.**

1.4 (Modular functions)**.**

1.5 (Uniformly continuous functions)**.** G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$. $g \in C_c(G)$ is two-sided uniformly continuous.

1.6 (Convolution Hilbert algebra)**.** Let G be a locally compact group. Since G is a locally compact Hausdorff space and the left Haar measure is a faithful semi-finite lower semi-continuous weight on the commutative C*-algebra $C_0(G)$, we have a corresponding semi-cyclic representation $m : C_0(G) \to B(L^2(G))$ which is normally extended to a von Neumann algebra $L^\infty(G)$ with $m(L^\infty(G)) = m(C_0(G))''$, and $L^1(G)$ is identified with the predual $L^\infty(G)_*$.

By the left Haar measure, $C_c(G)$ has a natural non-commutative left Hilbert algebra structure

$$(f * g)(s) := \int f(t)g(t^{-1}s)\,dt, \qquad \langle f, g\rangle := \int \overline{g(s)}f(s)\,ds, \qquad f^\sharp(s) := \nabla(s^{-1})\overline{f(s^{-1})},$$

where ∇ is the modular function for G, and it induces the regular representation $\lambda : C_c(G) \to B(L^2(G))$. By the group structure of G, the Hilbert algbera $C_c(G)$ is also a commutative counital multiplier Hopf $*$-algebra

$$(fg)(s) := f(s)g(s), \qquad \Delta f(s,t) = f(st), \qquad f^*(s) := \overline{f(s)}, \qquad \kappa f(s) = f(s^{-1}).$$

We start from this structures.

They satisfy a compatibility condition $\langle fg, h\rangle = \langle f, g^*h\rangle$.

With the integral notation $\lambda(f) = \int \lambda_s f(s)\,ds$, we can write

From now on, we are going to exclude any measure theory and the theory of non-commutative L^p spaces. First, we have the completion $H =: L^2(G)$. Consider two representations

$$\lambda : (C_c(G), *, {}^\sharp) \to B(L^2(G)), \qquad m : (C_c(G), \cdot, {}^*) \to B(L^2(G)).$$

(a) λ is well-defined.

(b) m is well-defined.

Proof. The multiplication representation m is well-defined because for $f \in C_c(G)$ we have $f^*f \in C_c(G) \subset L^2(G)$ so

$$\|m(f)g\|^2 = \langle fg, fg\rangle = \langle f^*fg, g\rangle, \qquad g \in C_c(G).$$

□

1.2

We use the notation $L^p(G)$ for the non-commutative L^p-spaces constructed with the left Haar measure on G, which is a faithful semi-finite normal weight of $L^\infty(G)$. The predual of $L^\infty(G)$ can be identified with $L^1(G)$. The regular representation on $L^2(G)$ is the Gelfand-Naimark-Segal representation associated with the left Haar measure.

Density of $C_c(G)$?

1.7 (Convolution algebra)**.** Let G be a locally compact group. Then, $L^1(G)$ is a hermitian Banach $*$-algebra such that

$$(f * g)(x) := (f \otimes g)\Delta(x), \qquad f, g \in L^1(G),\ x \in L^\infty(G).$$

Importance of L^1 instead of C_c: representation equivalence and predual.

(a) $L^1(G)$ has a two-sided approximate unit in $C_c(G)$.

(b) $\alpha : G \to \mathrm{Aut}(L^1(G))$ is point-norm continuous.

(c) $\lambda : G \to U(L^2(G))$ and $\lambda : L^1(G) \to B(L^2(G))$ are strongly continuous.

(d) Convolution inequalities.

(e) Representation theory equivalence.

Proof. Let (U_α) be a directed set of open neighborhoods of the identity e of G. By the Urysohn lemma, there is $e_\alpha \in C_c(U)^+$ such that $\|e_\alpha\|_1 = 1$ for each α. We claim that e_α is a two-sided approximate unit for $L^1(G)$. Suppose $g \in C_c(G)$, which is two-sided uniformly continuous. For any $\varepsilon > 0$, take α_0 such that $\|g - \lambda_s g\| < \varepsilon$ and $\|g - \rho_s g\| < \varepsilon$ for all $s \in U_\alpha$ for $\alpha \succ \alpha_0$. Then, we have

$$\begin{aligned}\|e_\alpha * g - g\|_1 &= \int |e_\alpha * g(t) - g(t)|\,dt \le \iint e_\alpha(s)|g(s^{-1}t) - g(t)|\,ds\,dt\\ &= \int_{U_\alpha} e_\alpha(s)\|\lambda_s g - g\|_1\,ds < \varepsilon \int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|g * e_\alpha - g\|_1 &= \int |g * e_\alpha(s) - g(s)|\,ds \le \iint |g(t) - g(s)|e_\alpha(t^{-1}s)\,dt\,ds\\ &= \iint |g(t) - g(ts)|e_\alpha(s)\,dt\,ds = \int \|g - \rho_s g\|_1 e_\alpha(s)\,ds < \varepsilon \int e_\alpha(s)\,ds \le \varepsilon,\end{aligned}$$

and they imply $\lim_\alpha \|e_\alpha * g - g\|_1 = \lim_\alpha \|g * e_\alpha - g\|_1 = 0$. We can approximate $f \in L^1(G)$ with compactly supported continuous functions by the $\varepsilon/3$ argument. □

Note that we have

$$\begin{aligned}
|\langle\lambda(\xi)\eta,\zeta\rangle|^2 &= |\iint \xi(t)\eta(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\
&\le \iint |\xi(t)||\eta(t^{-1}s)|^2\,ds\,dt\cdot\iint |\xi(t)||\zeta(s)|^2\,ds\,dt \\
&= \|\xi\|_1^2\|\eta\|_2^2\|\zeta\|_2^2
\end{aligned}$$

and

$$\begin{aligned}
|\langle\rho(\xi)\eta,\zeta\rangle|^2 &= |\iint \eta(t)\xi(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\
&\le \iint |\xi(t^{-1}s)||\eta(t)|^2\,ds\,dt\cdot\iint |\xi(t^{-1}s)||\zeta(s)|^2\,ds\,dt \\
&= \|\xi\|_1\|F\xi\|_1\|\eta\|_2^2\|\zeta\|_2^2
\end{aligned}$$

imply

$$\|\lambda(\xi)\|_{2\to2}\le\|\xi\|_1,\qquad \|\rho(\xi)\|_{2\to2}\le\sqrt{\|\xi\|_1\|F\xi\|_1}.$$

The equalities do not hold, consider $\|\lambda(\xi)\|=\|\widehat{\xi}\|_\infty$ if $G=\mathbb{R}$.

1.8 (Riemann sum approximation)**.** $\lambda(\delta_s)=\lambda_s$, $\langle\delta_s^{\frac12},\delta_t^{\frac12}\rangle=\delta_{s,t}$

For $f\in L^1(G)$,

$$f=\int_G\delta_s f(s)\,ds,\qquad \lambda(f)=\int_G\lambda_s f(s)\,ds.$$

For $\xi\in L^2(G)$,

$$\xi=\int_G\delta_s^{\frac12}\xi(s)\,ds,\qquad \langle\xi,\eta\rangle=\iint_{G^2}\overline{\eta(t)}\xi(s)\langle\delta_s^{\frac12},\delta_t^{\frac12}\rangle\,ds\,dt.$$

1.3

1.9 (Regular representation)**.** Let G be a locally compact group. Associated to the Hilbert algebra $C_c(G)$, we have a standard form $(W_r^*(G),L^2(G),J,P)$, where $W_r^*(G):=\lambda(C_c(G))''\subset B(L^2(G))$ is called the *group von Neumann algebra* of G.

$$\begin{array}{ccc}
M(G) & \xrightarrow{\lambda} & W_r^*(G) \\
\uparrow & & \uparrow \\
L^1(G) & \xrightarrow{\lambda} & C_r^*(G).
\end{array}$$

(a)

Proof. □

1.10 (Fourier algebras)**.** Let G be a locally compact group. The *Fourier algebra* is the algebra $A(G)$ of *matrix coefficients* of the regular representation $\lambda:G\to U(L^2(G))$, that is, the linear span of functions $s\mapsto\langle\lambda(s)\xi,\xi\rangle$ for $\xi\in L^2(G)$. Since every normal state of $W_r^*(G)$ is a vector state in the regular representation, the Fourier algebra also can be defined as the image of the adjoint $\lambda^*:W_r^*(G)_*\to C_0(G)$.

$$\begin{array}{ccc}
A(G) & \longrightarrow & C_0(G) \\
\downarrow & & \downarrow \\
C_r^*(G)^* & \xrightarrow{\lambda^*} & L^\infty(G).
\end{array}$$

(a) $A(G)$ is a dense Banach subalgebra of $C_0(G)$ such that $A(G) \to W_r^*(G)_* : \eta^*\xi \mapsto \omega_{\xi,\eta}$ is an isometric isomorphism.

Proof. $\square$

1.11 (Fourier-Stieltjes algebras)**.** Let G be a locally compact group.

(a) On $B(G)_1$, the compact open topology is stronger than the weak* topology.

(b) On $B(G)_1$, the strict topology with respect to $A(G)$ is equivalent to the weak* topology.

Proof. $\square$

1.12 (Plancherel theorem)**.** With the left Haar measure on a Banach $*$-algebra $L^1(G)$ or $M(G)$, we want to construct a faithful semi-finite normal weight called the *Planceherel weight*, and describe the corresponding semi-cyclic representation and left Hilbert algebra for $C_r^*(G)$ and $W_r^*(G)$.

By analyze the decomposition of the canonical representation of $C_r^*(G)$ and $W_r^*(G)$ in $B(L^2(G))$....? Then, we can consider a unitary operator from $L^2(G)$ to the square integrable section space of a bundle on $\widehat{G}$...

Proof. $\square$

1.13 (Locally compact abelian groups)**.** Let G be a locally compact abelian group. Since every irreducible representation of a locally compact abelian group is one-dimensional, we introduce the notation $\langle s,p\rangle = p_{s^{-1}} \in \mathbb{T}$. The *Fourier transform* of an integrable function $f \in L^1(\widehat{G})$ is defined as

$$\mathcal{F}f(p) := \int_G \overline{\langle s,p\rangle} f(s)\,ds, \qquad p \in \widehat{G},$$

and the *Fourier-Stieltjes transform* of a finite complex measure $\mu \in M(G)$ is defined as

$$\mathcal{F}\mu(p) := \int_G \overline{\langle s,p\rangle}\,d\mu(s), \qquad p \in \widehat{G}.$$

(a) The compact open topology of $C(G)$ and the weak* topology of $L^\infty(G)$ coincide on $\widehat{G}$, which provides a locally compact abelian group.

(b) The canonical homomorphism $\Phi : G \to \widehat{\widehat{G}}$ defined such that $\Phi(s)(p) = \langle s,p\rangle$ for $s \in G$ and $p \in \widehat{G}$ is a topological isomorphism.

Proof. (b) Consider a commutative diagram of topological $*$-algebras

$$\begin{tikzcd}
M(G) \arrow[r] & W_r^*(G) \arrow[r, "(3)"] & L^\infty(\widehat{G}) \\
L^1(G) \arrow[r] \arrow[u] & C_r^*(G) \arrow[r, "(2)"] \arrow[u] & C_0(\widehat{G}) \arrow[u] \\
L^1(G) \arrow[r] \arrow[u, equal] & C^*(G) \arrow[r, "(1)"] \arrow[u] & C_0(\widehat{G}) \arrow[u, equal]
\end{tikzcd}$$

of injective densely valued $*$-homomorphisms. The bijectivity of (1) follows from the equivalence between representation theories of G and $C^*(G)$ and the Gelfand duality. The existence of (2) follows from the amenability of G. The isomorphism (3) is constructed by taking double commutant in the Plancherel isomorphism $B(L^2(G)) \to B(L^2(\widehat{G}))$. Note that the third and first rows are respectively the Fourier transform and Fourier-Stieltjes transform.

Putting $\widehat{G}$ instead of G on the third row and taking the dual for the first row, we have tow injective densely valued $*$-homomorphisms

$$L^1(\widehat{G}) \to C_0(\widehat{\widehat{G}}), \qquad L^1(\widehat{G}) \to C_0(G).$$

Then, the restriction map $C_0(\widehat{\widehat{G}}) \to C_0(G)$ along $\Phi : G \to \widehat{\widehat{G}}$ is obtained. The surjectivity is clear because it is a $*$-homomorphism between C*-algebras with dense range. Since $L^1(G)$ is dense in $C_0(\widehat{G})$ via Fourier transform, and $C_0(\widehat{G})$ is weakly * dense in $B(\widehat{G})$, so $M(G)$ is weakly* dense in $M(\widehat{\widehat{G}}) \cong B(\widehat{G})$, which means that $C_0(\widehat{\widehat{G}}) \to C_0(G)$ is injective. $\square$

1.14 (Absorption principle)**.** Let G be a locally compact group.

$$w :$$

The *structure operator* of G is an oeprator $w \in U(L^2(G \times G))$ defined such that $w\xi(s,t) := \xi(s,st)$, or $w \in L^\infty(G)\overline{\otimes}W_r^*(G)$ such that $\operatorname{Ad} w(\lambda_s \otimes \lambda_s) := \lambda_s \otimes 1$. If $w(x \otimes x)w^* = x \otimes 1$, then $x = \lambda_s$ for some $s \in G$.

(a) $\lambda \otimes u$ and $\lambda \otimes 1$ are unitarily equivalent. It is called the *Fell absorption principle*.

Proof. The Fell absorption principle states that the composition of equivariant operators

$$\begin{array}{ccccc} L^2(G) \otimes H & \xrightarrow{\Delta\otimes 1} & L^2(G) \otimes L^2(G) \otimes H & \xrightarrow{1\otimes ?} & L^2(G) \otimes H \\ \lambda \otimes 1 & \longmapsto & \lambda \otimes \lambda \otimes 1 & \longmapsto & \lambda \otimes u \end{array}$$

is unitary.

The structure operator is a special case of the Fell absorption operator

$$\begin{array}{ccccc} L^2(G) \otimes L^2(G) & \xrightarrow{\Delta\otimes 1} & L^2(G) \otimes L^2(G) \otimes L^2(G) & \xrightarrow{1\otimes ?} & L^2(G) \otimes L^2(G) \\ \lambda \otimes 1 & \longmapsto & \lambda \otimes \lambda \otimes 1 & \longmapsto & \lambda \otimes \lambda \end{array}$$

$\square$

Chapter 2

2.1 Spectral synthesis

Chapter 3

Part II

Topological quantum groups

Chapter 4

Bialgebras

4.1

Multiplier Hopf $*$-algebras

Algebraic quantum groups

idempotent ring assumption

4.2

4.1. A *counital coalgebra* is a vector space A over a field equipped with

(i) a unital homomorphism $\delta : A \to A \otimes A$ called the *comultiplication* such that $(\delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \delta)\Delta$,

(ii) a homomorphism $\varepsilon : A \to \mathbb{C}$ called the *counit* such that $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta$.

A *bialgebra* if comultiplication is an algebra homomorphism.

A *Hopf algebra* is a biunital bialgebra A over a field together with a linear map $S : A \to A$, called the *antipode*, satisfying

$$\nabla(S \otimes \mathrm{id})\Delta = \eta\varepsilon = \nabla(\mathrm{id} \otimes S)\Delta.$$

A morphism between Hopf algebras is a linear map preserving multiplication, unit, comultiplication, counit, and antipode.

The convolution algebra is a bialgebra for a monoid, and is a Hopf algebra for a group.

matrix coefficients, coordinate algebra. universal enveloping algebra. q-deformations of the coordinate Hopf algebras $\mathcal{O}(G)$ of a semi-simple complex Lie group, and the universal enveloping algebra $U(\mathfrak{g})$ of a semi-simple complex Lie algebra.

If A is a coalgebra and B is an algebra, then $\mathrm{Hom}_{\mathbb{C}}(A, B)$ becomes an algebra with convolution. If A is a coalgebra, then A^* is an algebra. If A is a bialgebra, then A is a bimodule over A^*.

Duality for finite-dimensional Hopf ($*$-)algebras. dual pairing

matrix coefficients for compact groups regular functions for affine algebraic groups

Chapter 5

Compact quantum groups

5.1 (Compact quantum groups)**.** A *compact quantum group* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a bisimplifiable C*-bialgebra $C(\mathbb{G})$. It is not in general a Hopf algebra.

$$C_0(G), \quad L^\infty(G), \qquad C^*(G), \quad C^*_r(F), \quad W^*_r(G)$$
$$A(G), B(G)$$

For a compact group G, $C(G)$ has a coalgebra structure induced from $C(G) \subset L^1(G)$.

5.2. A *compact algebraic quantum group* is a Hopf $*$-algebra with a positive integral. For a compact quantum group $\mathbb{G}$, the subspace $\mathbb{C}(\mathbb{G})$ spanned by the matrix coefficients of corepresentations is an algebraic quantum group.

A *locally compact quantum group* is a von Neumann bialgebra admitting left-invariant and right-invariant faithful semi-finite normal weights. A *reduced locally compact quantum group* is a C*-bialgebra such that 8.1.17.

Probably, a Hopf-von Neumann algebra in Enock-Schwartz is just a von Neumann bialgebra in Timmerman, a coinvolutive Hopf-von Neumann algebra in Enock-Schwartz is just a Hopf-von Neumann algebra in Timmerman. Since a locally compact quantum group has counit and antipode as unbounded operators, I do not know if I can say there is a Hopf algebra structure.

5.1 Kac algebras

Chapter 6

Locally compact quantum groups

6.1 Multiplicative unitaries

Part III

Representation categories

Chapter 7

Representations of compact groups

7.1 Peter-Weyl theorem

7.2 Tannaka-Krein duality

7.3 Mackey machine

Example of non-compact Lie groups, Wigner classification