Noncommutative Algebraic Geometry

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1 Algebras

- 1987: Artin-Schelter, regular algebra.
- 1990: Artin-Tate-Bergh, three dimensional, geometrically classified.
- 1994: Artin-Zhang, noncommutative scheme, categorical perspective.

1.1

Let k be an algebraically closed field of characteristic zero. Examples of k-algebras include the free algebra $T := k\langle x_1, \cdots, x_n\rangle$, which is noncommutative for $n \geq 2$. It consists of linear combinations of monomials, and there are 2^n monomials of degree n in T, and T is k-isomorphic to the tensor algebra constructed from n-dimensional vector space k^n. Note that $(x+y)^2 = x^2 + xy + yx + y^2$ in T. An algebra R is finitely generated if and only if $R \cong T/I$ for some n and some ideal I of R. If $n \geq 2$, then T is not right noetherian, $I = \sum_{i=0}^{\infty} x^i yR$ is a right ideal which is not finitely generated for exmaple(not easy to show finitely generatedness). Is $k\langle x, y\rangle/(yx, y^2)$ noetherian? It is known that it is left notherian, but not right noetherian.

1.2

Let R be a ring and let $\sigma \in \mathrm{Aut}(R)$. An additive map $\delta : R \to R$ is called a *σ-derivation* if $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for $a, b \in R$. We define a ring $R[x; \sigma, \delta]$, called the *Ore extension*, as an additive group $R[x]$ together with multiplication defined by

$$xa := \sigma(a)x + \delta(a), \qquad a \in R.$$

Example 1.1.

(a) We can compute

$$\begin{aligned}(ax+b)(cx+d) &= axcx + axd + bcx + bd \\ &= a(\sigma(c)x + \delta(c))x + a(\sigma(d)x + \delta(d)) + bcx + bd \\ &= a\sigma(c)x^2 + (a\delta(c) + a\sigma(d) + bc)x + a\delta(d) + bd.\end{aligned}$$

(b) We have $R[x; \mathrm{id}_R, 0] \cong R[x]$ as rings.

(c) If $\sigma(f(x)) := f(\alpha x)$ for some non-zero $\alpha \in k$, then $k[x][y; \sigma, 0] \cong k\langle x, y\rangle/(\alpha xy - yx)$ since $yx = \sigma(x)y - \delta(x) = \alpha xy$.

(d) If $\delta(f(x)) := f'(x)$, then $k[x][y;\mathrm{id}_{k[x]},\delta] \cong k\langle x,y\rangle/(xy-yx+1)$, called the *Weyl algebra*, since $yx = \sigma(x)y + \delta(x) = xy + 1$.

(e) How can we find a k-automorphism σ of $k[x]$ and a σ-derivation δ such that $k\langle x,y\rangle/(xy-yx+x^2) \cong k[x][y;\sigma,\delta]$? What should $\delta(x^i)$ be? One answer is $\sigma = \mathrm{id}_{k[x]}$ and $\delta(f(x)) = x^2 f'(x)$.

Theorem 1.2. *Let R be a ring and $S := R[x;\sigma,\delta]$ be an Ore extension.*

(a) *If R is right noetherian, then so is S.*

(b) *If R is a domain, then so is S.*

(c) *If R is of finite global dimension, then so is S.*

As examples, we have $k\langle x,y\rangle/(\alpha xy - yx)$ and $\dim k\langle x,y\rangle/(xy-yx+1)$ are noetherian domains of global dimensions 2 and 1, respectively. There is a result that left and right global dimensions coincide when R is two-sided noetherian.

1.3

Theorem 1.3. *If R is a k-algebra and $a_1,\cdots,a_n \in R$, then there is a unique k-algebra homomorphism $\varphi : k\langle x_1,\cdots,x_n\rangle \to R$ such that $\varphi(x_i) = a_i$. If a k-algebra homomorphism $\varphi : S \to R$ satisfies $\varphi(I) = 0$ for an ideal I of S, then it factors through S/I.*

With the above theorem we can construct an k-algebra isomorphism $k[x] \cong k\langle x,y\rangle/(x^2-y)$. As an another example, for $\operatorname{char} k \neq 2$, then

$$k\langle x,y\rangle/(x^2+y^2, xy+yx) = k\langle x+y, x-y\rangle/((x+y)^2,(x-y)^2) \cong k\langle x,y\rangle/(x^2+y^2).$$

1.4

We now consider grading, a direct sum decomposition over a monoid. The free k-algebra $T = k\langle x_1,\cdots,x_n\rangle$ is $\mathbb{N}$-graded by degree. Let $A = \bigoplus A_i$ be a graded ring. We can define homogeneous ideals of A, and the quotient can be written as $A/I \cong \bigoplus A_i/I_i$, where $I_i := I \cap A_i$. Also, graded homomorphisms between graded rings or graded modules are able to be introduced. Let I and J be homogeneous ideal of $T_n := k\langle x_1,\cdots,x_n\rangle$ and $T_m := k\langle y_1,\cdots,y_m\rangle$ such that $J_0 = J_1 = 0$. Then, a graded algebra homomorphism $\varphi : T_n \to T_m$ is uniquely determined by $\varphi(x_i) = a_{ij}y_j$ for $(a_{ij}) \in M_{nm}(k)$. Let GrAut(A) be the group of graded algebra automorphisms of A. Then,

$$\mathrm{GrAut}(T_n) \cong \mathrm{GrAut}(k[x_1,\cdots,x_n]) \cong \mathrm{GL}(n,k),$$

and if I is a homogeneous ideal of T_n such that $I_0 = I_1 = 0$, then $\mathrm{GrAut}(T_n/I)$ is a subgroup of $\mathrm{GL}(n,k)$. For example, we have

$$\mathrm{GrAut}(k\langle x,y\rangle/(x^2)) \cong \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a,d \in k^\times\right\}$$

and for $\alpha \neq \pm 1$ we have

$$\mathrm{GrAut}(k\langle x,y\rangle/(\alpha xy - yx)) \cong \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a,d \in k^\times\right\}$$

since $\alpha\varphi(x)\varphi(y) - \varphi(y)\varphi(x) = (\alpha-1)(acx^2 + bdy^2) + (\alpha^2-1)bcxy$.

Fix $\theta \in \mathrm{GrAut}(A)$. Define an algebra $A^\theta := A$ as sets and multiplication $a * b := a\theta^i(b)$ on A^θ for $a \in A_i$ and $b \in A$. It is called the *twist* of A by θ, and it is also graded. For example, if we let $A = k[x,y]$, then

$$\text{If } \theta = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \text{ then } \quad A^\theta \cong k\langle x,y\rangle/(\alpha xy - yx)$$

and

$$\text{If } \theta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \text{ then} \quad A^\theta \cong k\langle x, y\rangle/(xy - yx + x^2).$$

Note that $\varphi(xy - yx) = (ad - bc)(xy - yx)$ if $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Theorem 1.4. *Let A be a graded ring and $\theta \in \mathrm{GrAut}(A)$.*

(a) *If A is right noetherian, then so is A^θ.*

(b) *If A is a domain, then so is A^θ.*

(c) *If A is of finite global dimension, then so is A^θ.*

2 Quantum polynomial algebras

2.1

Today, let $A := k\langle x_1, \cdots, x_n\rangle / I$ be a finitely generated graded algebra such that I is a homogeneous ideal satisfying $I_0 = I_1 = 0$, i.e. I is an admissible ideal. Let M be a graded right A-module, $M_{\geq n} := \bigoplus_{i \geq n} M_i$ be a graded submodule of M, and $M(n)$ be a graded module such that $M(n) := M$ as a set but $M(n)_i := M_{n+i}$. With this notation, $\mathfrak{m} := A_{\geq 1}$ is the unique maximal homogeneous ideal of A. A *free* graded right A-module is a graded right A-module of the form $\oplus_s A(n_s)$. A finitely generated graded right A-module is free if and only if projective. A function $\varphi : A(l) \to A(m)$ is a graded right A-module homomorphism if and only if $\varphi = a\cdot$ for some $a \in A_{m-l}$. Therefore, between free right A-modules, $\varphi : \oplus A(l_s) \to \oplus A(m_t)$ is a graded right A-module homomorphism if and only if $\varphi = (a_{st})\cdot$, for some $a_{st} \in A_{m_t - l_s}$. A free resolution

$$\cdots \to F^2 \to F^1 \to F^0 \to M \to 0$$

is called *minimal* if the map $\varphi_i : F^i \to F^{i-1}$ is given by the left multiplication of a matrix whose entries are in A_1. We can define the projective dimension of a module as the minimal length of free resolution, and the global dimension of A as the supremum of the projective dimension of graded right A-modules.

Lemma 2.1. $\mathrm{gldim} A = \mathrm{pd}(k)$.

For example, $A = k\langle x, y\rangle$, then $k = A/(xA + yA)$, so $\mathrm{pd}(k) = 1$, hence $\mathrm{gldim} A = 1$, and in generally $\mathrm{gldim} A = 1$ for $I = 0$.

2.2

Let M be a finitely generated graded right A-module. Suppose further M is locally finite, i.e. $\dim_k M_i < \infty$ for each i. Then,

$$H_M(t) := \sum_{i \in \mathbb{Z}} (\dim_k M_i) t^i \in \mathbb{Z}[[t, t^{-1}]]$$

is called the *Hilbert series* of M. For example, letting $M = A$,

$$H_{k[x_1, \cdots, x_n]}(t) = \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} t^i = (1-t)^{-n},$$

and

$$H_{k\langle x_1, \cdots, x_n\rangle}(t) = \sum_{i=0}^{\infty} n^i t^i = (1-nt)^{-1}.$$

Lemma 2.2. *Let M be a finitely generated graded right A-module.*

(a) $H_{M^{\oplus r}}(t) = r H_M(t)$.

(b) $H_{M(n)}(t) = t^{-n} H_M(t)$.

(c) *If* $0 \to M^r \to \cdots \to M^1 \to M^0 \to 0$ *is exact, then* $\sum_{i=0}^{r} (-1)^i H_{M_i}(t) = 0$.

For example for (c), consider

$$0 \to A(-1)^{\oplus 2} \to A \to k \to 0.$$

Then, we can check $H_A(t) = (1-2t)^{-1}$ from

$$0 = H_k(t) - H_A(t) + H_{A(-1)^{\oplus 2}}(t) = 1 - H_A(t) + 2t H_A(t).$$

2.3

Definition 2.3 (Artin-Schelter)**.** We say A is a d-dimensional *quantum polynomial algebra*(QPA) if $\mathrm{gldim} A = d < \infty$, $H_A(t) = (1-t)^{-d}$, and $\mathrm{Ext}^i_A(k,A) = \delta_{di} \cdot k(d)$. The last condition is called the *Gorenstein condition*.

If a QPA is commutative, then it is isomorphic to the polynomial algebra. The above two conditions are equivalent to have the minimal free resolution of the graded right A-module k

$$0 \to A(-d) \to \oplus A(-d+1) \to \cdots \to \oplus A(-1) \to A \to k \to 0,$$

where $\phi^i : \oplus A(-i) \to \oplus A(-i+1)$ is the left multiplication of a matrix whose components are in A_1. The Gorenstein condition is equivalent to the transpose

$$0 \leftarrow k(d) \leftarrow \oplus A(d) \leftarrow \cdots \leftarrow \oplus A(1) \leftarrow A \leftarrow 0$$

is a minimal free resolution of left A-module $k(d)$, where the arrows are right multiplications of matrices whose compoenents are in A_1. Ranks of each free modules must be determined by the Hilbert series.

For example, $A = k\langle x, y\rangle/(\alpha xy - yx)$ is a 2-dimensional QPA for all non-zero $\alpha \in k$. The classification up to dimension two is easy:

Lemma 2.4. *Let A be a QPA over an algebraically closed field k.*

(a) $\mathrm{gldim} A = 0$ *iff* $A \cong k$,

(b) $\mathrm{gldim} A = 1$ *iff* $A \cong k[x]$,

(c) $\mathrm{gldim} A = 2$ *iff* $A \cong k[x, y]^\theta$ *for some* $\theta \in \mathrm{GL}(2, k)$.

2.4

We can describe three-dimensional QPAs are classified in terms of derivation quotient algebras.

Definition 2.5. Let $V = k^n$ and let

$$\varphi : V^{\otimes m} \to V^{\otimes m} : v_1 \otimes \cdots \otimes v_m \mapsto v_2 \otimes \cdots \otimes v_1.$$

We say $w \in V^{\otimes m}$ is called a *superpotential*(SP) if $\varphi(w) = w$, and a *twisted superpotential*(TSP) if $(\sigma \otimes \mathrm{id}^{\otimes(m-1)})\varphi(w) = w$ for all $\sigma \in \mathrm{GL}(V)$.

Example 2.6. Let $V = kx + ky$, and $w = \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \in V^{\otimes 2}$. Then, w is SP iff $\beta = \gamma$ and $SP^2(V) = kx^2 + k(xy + yx) + ky^2 \subset V^{\otimes 2}$.

Definition 2.7. For $\dim_k V = n$ and $w \in V^{\otimes m}$, we can define $\partial_i w, w\partial_i \in V^{\otimes(m-1)}$ such that $w = \sum x_i \otimes \partial_i w = \sum w\partial_i \otimes x_i$. *Derivation quotient algebras* are

$$D_l(w) := k\langle x_1, \cdots, x_n\rangle/(\partial_1 w, \cdots, \partial_n w), \qquad D_r(w) := k\langle x_1, \cdots, x_n\rangle/(w\partial_1, \cdots, w\partial_n).$$

Lemma 2.8.

(a) *w is SP iff $\partial_i w = w\partial_i$.*

(b) *w is TSP iff $D_l(w) = D_r(w) =: D(w)$ (ideals quotiented are same as sets.)*

Example 2.9. If $V = kx + ky$, and $w = \alpha x^2 + \beta xy + \gamma yx + \delta y^2 \in V^{\otimes 2}$, then

$$\partial_x w = \alpha x + \beta y, \qquad w\partial_x = \alpha x + \gamma y.$$

Theorem 2.10.

(a) *If ω is TSP with $m = n = 3$, then $D(w)$ is a three-dimensional QPA.*

(b) *The converse holds.*

Example 2.11 (Sklyanin algebra). For $\alpha, \beta, \gamma \in k$,

$$w = \alpha(xyz + yzx + zxy) + \beta(xzy + yxz + zyx) + \gamma(x^3 + y^3 + z^3)$$

is a superpotential. $D(w)$ is called the Sklyanin algebra. We can construct with $M = \begin{pmatrix} \gamma x & \beta z & \alpha y \\ \alpha z & \gamma y & \beta x \\ \beta y & \alpha x & \gamma z \end{pmatrix}$ the minimal free resolutions of k and $k(3)$.

There is $\theta \in \mathrm{GrAut}(k\langle x, y\rangle/(\alpha xy - yx))$ such that

$$(k\langle x, y\rangle/(\alpha xy - yx))^\theta \cong k\langle x, y\rangle/(xy - yx + x^2)$$

if and only if $\alpha = 1$. We can see this for $\alpha = -1$ by computing GrAut. Note that

$$(k\langle x, y\rangle/(\alpha xy - yx))^\theta \cong k\langle x, y\rangle/(\alpha\theta(x)y - \theta(y)x)$$

If $\alpha \neq \pm 1$....?

3

Artin-Tate-van den Bergh classification of 3-dimensional QPA.

Point varieties.

4

4.1

Definition 4.1. A *noncommutative scheme* is a pair $X = (\operatorname{Mod} X, \mathcal{O}_X)$ of an abelian category $\operatorname{Mod} X$ and an object $\mathcal{O}_X$ in it. A *morphism* between noncommutative schemes X and Y is an adjoint of pair of functors $f_* : \operatorname{Mod} X \to \operatorname{Mod} Y$ and $f^* : \operatorname{Mod} Y \to \operatorname{Mod} X$ such that $f^*\mathcal{O}_Y = \mathcal{O}_X$.

For a scheme X, then X can be considered as a noncommutative scheme by the pair of the category of quasi-coherent sheaves and the structure sheaf.

Consider the noncommutative affine schemes. For a ring R, we define its noncommutative spectrum as $\operatorname{Spec}_{nc} R := (\operatorname{Mod} R, R)$. Note that for a ring homomorphism $\varphi : R \to S$, S can be seen as R-S-bimodule. Here the morphism $f : \operatorname{Spec}_{nc} R \to \operatorname{Spec}_{nc} S$ can be given as the pair of

$$f^* : \operatorname{Mod} R \to \operatorname{Mod} S : M \mapsto M \otimes_R S, \qquad f_* : \operatorname{Mod} S \to \operatorname{Mod} R : N \mapsto \operatorname{Hom}_S(S, N),$$

which we can check they are adjoint and $f^*S = R$. In general, an equivalence of the category of modules does not imply the isomorphism between rings. However, if two noncommutative schemes are isomorphic by $f : \operatorname{Spec}_{nc} R \to \operatorname{Spec}_{nc} S$, then

$$R = \operatorname{End}_R(R) \cong \operatorname{End}_S(f^*(R)) = \operatorname{End}_S(S) = S.$$

4.2

For the rest of today, a graded ring is an $\mathbb{N}$-graded ring. Let A be a graded ring. Let $\operatorname{GrMod} A$ be the category of graded right A-modules. A graded right A-module M is called a torsion module if mA is right bounded for all $m \in M$. We define $\operatorname{Tails} A := \operatorname{GrMod} A / \operatorname{Tors} A$. For $M \in \operatorname{GrMod} A$, we will use somtimes $\mathcal{M} = \pi(M) \in \operatorname{Tails} A$ to denote the image of M under the projection functor $\pi : \operatorname{GrMod} A \to \operatorname{Tails} A$.

Theorem 4.2 (Serre)**.** *Let A be a commutative graded ring finitely generated in degree one. Then,* $\operatorname{Tails} A \cong \operatorname{Mod}(\operatorname{Proj} A) : \pi(A) \mapsto \mathcal{O}_{\operatorname{Proj} A}$.

Definition 4.3. Let A be a graded ring. We define $\operatorname{Proj}_{nc}(A) := (\operatorname{Tails} A, \pi(A))$. For A an n-dimensional QPA, $\operatorname{Proj}_{nc} A$ is called the quantum $\mathbb{P}^{n-1}$.

If A is right noetherian graded ring, then we define $\operatorname{grmod} A$ as teh category of finitely generated modules. It is an abelian category. We may also define $\operatorname{tors}(A) := \operatorname{Tors}(A) \cap \operatorname{grmod}(A)$ and tails $:=$ grmod / tors.

Theorem 4.4. *Let A be a connected graded right coherent algebra. Then,*

$$\operatorname{Hom}_{\operatorname{tails} A}(\mathcal{M}, \mathcal{N}) \cong \lim_{n\to\infty} \operatorname{Hom}_A(M_{\geq n}, N).$$

In particular, for finitely generated *M and N, $\pi(M) \cong \pi(N)$ if there is n such that $M_{\geq n} \cong N_{\geq n}$.*

4.3

Morita theory asks $\operatorname{Mod} R \cong \operatorname{Mod} R'$, and Artin-Zhang theory asks $\operatorname{Mod} A \cong \operatorname{Mod} A'$.

Theorem 4.5. *An abelian category $\mathcal{C}$ is equivalent to* $\operatorname{Mod} R$ *if there is $\mathcal{O} \in \operatorname{Mod} R$ such that*

(i) $\operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, -)$ *preserves small coproducts,*

(ii) every $M \in \mathcal{C}$ admits an epi $\oplus\mathcal{O} \to M$,

(iii) for epi $M \to N$, $\operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, M) \to \operatorname{Hom}_{\mathcal{C}}(\mathcal{O}, N)$ *is epi,*

and $\mathcal{C}$ has small coproducts. In particular, the equivalence is given as $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O},-) : \mathcal{C} \to \mathrm{Mod}\, R$*, where* $R := \mathrm{End}_{\mathcal{C}}(\mathcal{O})$*, and in this case,* $\mathrm{Spec}_{nc} R \cong (\mathcal{C},\mathcal{O})$*. The object $\mathcal{O}$ is called the compact projective generator.*

compact? small coproducts?

Corollary 4.6. $\mathrm{Mod}\, R \cong \mathrm{Mod}\, R'$ *iff there is a finitely generated projective generator* $P \in \mathrm{Mod}\, R$ *such that* $R' = \mathrm{End}_R(P)$.

Example 4.7. Let R be a ring. R and $M_n(R)$ are Morita equivalent since R^n is a finitely generated projective generator for $\mathrm{Mod}\, R$, but their schemes are not isomorphic in general.

Definition 4.8 (Twisting systems)**.** Let A be a graded ring. A twisting system is a family $\{\theta_i\}_{i\in\mathbb{Z}}$ such that $\theta_i : A \to A$ is a graded abelian group isomorphisms such that $\theta_i(a\theta_j(b)) = \theta_i(a)\theta_{i+j}(b)$. We can define twists $A^{\{\theta_i\}}$ and $M^{\{\theta_i\}}$.

Theorem 4.9 (Zhang)**.**

(a) $\mathrm{GrMod} A \cong \mathrm{GrMod} A^{\{\theta_i\}}$. *In particular,* $\mathrm{Proj}_{nc}(A) \cong \mathrm{Proj}_{nc}(A^{\{\theta_i\}})$.

(b) *If A, A' are finitely generated in degree one, and if* $\mathrm{GrMod} A \cong \mathrm{GrMod} A'$*, then there is a twisting system $\{\theta_i\}$ such that* $A' \cong A^{\{\theta_i\}}$.

Let $A_\alpha := k\langle x,y\rangle/(\alpha xy - yx)$ and $A_J := k\langle x,y\rangle/(xy - yx + x^2)$. Although we have seen that there is no twist $\theta \in \mathrm{GrAut} A_\alpha$ such that $A_\alpha^\theta \cong A_J$, but there is a twisting system $A_\alpha^{\{\theta_i\}} \cong A_J$. We do not exactly know how to construct $\{\theta_i\}$ in general, but we can give the concrete computation in this case:

$$\theta_i := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-i} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{i}$$

since $A_\alpha = k[x,y]^{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)}$ and $A_J = k[x,y]^{\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)}$. Generalizing this, there is a theorem that if $\{\theta_i\}$ and $\{\theta_i'\}$ are twisting systems of A and $A^{\{\theta_i\}}$ then there is a twisting system $\{\theta_i''\}$ such that $A^{\{\theta_i''\}} \cong (A^{\{\theta_i\}})^{\{\theta_i'\}}$.

4.4

Consider an abelian category $\mathcal{C}$, an object $\mathcal{O}$ in $\mathcal{C}$, and an autoequivalence s of $\mathcal{C}$. We call the triple as an algebraic triple here. Then,

$$B(\mathcal{C},\mathcal{O},s) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^i\mathcal{O}).$$

For example, if X is a projective scheme, and $\mathcal{L} \in \mathrm{Pic}$ is very ample, then we have a graded ring

$$B := B(\mathrm{Mod}\, X, \mathcal{O}_X, -\otimes_X \mathcal{L}) = \bigoplus \Gamma(X, \mathcal{L}^{\otimes i}),$$

gives $X = \mathrm{Proj}\, B$. In other words, B is the homogeneous coordinate ring of X.

Definition 4.10. For an algebraic triple $(\mathcal{C},\mathcal{O},s)$, we say $(\mathcal{O},s)$ is ample for $\mathcal{C}$ if

(i) for every $M \in \mathcal{C}$, there is $\{p_i\}_{i=1}^m$ with $\oplus_i s^{-p_i}\mathcal{O} \twoheadrightarrow M$.

(ii) for every epi $\mathcal{M} \to \mathcal{N}$ in $\mathcal{C}$, there is an integer n_0 such that $\mathrm{Hom}_{\mathcal{C}}(s^{-n}\mathcal{O},\mathcal{M}) \to \mathrm{Hom}_{\mathcal{C}}(s^{-n}\mathcal{O},\mathcal{N})$ is epi for $n \geq n_0$.

Theorem 4.11 (Artin-Zhang)**.** *If $(\mathcal{O},s)$ is ample for $\mathcal{C}$, then* $\mathcal{C} \cong \mathrm{tails}\, B(\mathcal{C},\mathcal{O},s) : \mathcal{O} \mapsto B$*, i.e.* $(\mathcal{C},\mathcal{O}) \cong \mathrm{Proj}_{nc} B$.

Corollary 4.12 (Veronese algebra)**.** *If A is finitely generated in degree one, then* $\mathrm{Proj}_{nc} A^{(r)} \cong \mathrm{Proj}_{nc} A$.

For examples, $A = k[x,y]$ with $\mathrm{Proj} A = \mathbb{P}^1$ and $A^{(2)} = k[x^2, xy, y^2] \cong k[s,t,u]/(su - t^2)$, we can check manually $\mathrm{Proj}\, k[s,t,u]/(su - t^2) \cong \mathbb{P}^1$.

For a graded algebra A and B, we can define the Segre product

$$A \circ B := \bigoplus_i A_i \otimes B_i.$$

If A, B are commutative and f.g. in degree one, then $\mathrm{Proj}(A \circ B) \cong \mathrm{Proj} A \times \mathrm{Proj} B$.

Example 4.13. We show that for $A := k\langle x,y\rangle/(x^2y - yx^2, xy^2 - y^2x)$ that $\mathrm{Proj}_{nc} A \cong \mathbb{P}^1 \times \mathbb{P}^1$. Note that the second Veronese algebra is

$$A^{(2)} = k\langle x^2, xy, yx, y^2\rangle/(x^2y - yx^2, xy^2 - y^2x).$$

By observing eight relations

$$x(x^2y - yx^2) = 0, \quad (x^2y - yx^2)x = 0, \quad y(x^2y - yx^2) = 0, \; \cdots \; (xy^2 - y^2x)y = 0,$$

and by letting $s = x^2$, $t = xy$, $u = yx$, $v = y^2$, we can conclude the existence of a surjection

$$B := k[s,t,u,v]/(sv - tu) \twoheadrightarrow A^{(2)}.$$

The monomaial of $(A^{(2)})_i$ can be reduced to an element of the form $y^a(xy)^b x^c$ up to scalar multiple, where $a + 2b + c = 2i$, so $\dim_k (A^{(2)})_i = (i+1)^2$. For B, if we count the dimension of the space spanned by monomials without tu, then we can see $\dim_k B_i = 2 \cdot \binom{i+1}{2} - \binom{i+1}{1} = (i+1)^2$. Thus $H_{A^{(2)}}(t) = H_B(t)$ implies $A^{(2)} \cong B$

On the other hands, we also have for the Segre product

$$B = k[s,t,u,v]/(sv - tu) \twoheadrightarrow k[x_1, y_1] \circ k[x_2, y_2] =: C.$$

We can easily see that $H_C(t) = \sum_i (i+1)^2 t^i$, so $B \cong C$. Consequently,

$$\mathrm{Proj}_{nc} A \cong \mathrm{Proj}_{nc} A^{(2)} \cong \mathrm{Proj} A^{(2)} \cong \mathrm{Proj}\, C \cong \mathrm{Proj}\, k[x,y] \times \mathrm{Proj}\, k[x,y] \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Noncommutative algebraic geometry

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November 28

1. Show that $A=k\langle x,y\rangle/(yx,y^2)$ is left noetherian but not right noetherian.

Solution. Let J be a left ideal of $k\langle x,y\rangle$ containing (yx,y^2), generated by $\{f_i\}_i\subset k\langle x,y\rangle$ as a left $k\langle x,y\rangle$-module. Since $k\langle x,y\rangle=k[x]+k[x]y+(yx,y^2)$, there are $g_i,h_i\in k[x]$ such that $f_i(x,y)\equiv g_i(x)+h_i(x)y$ modulo (yx,y^2) for each i. Then, we have

$$J=\sum_i k\langle x,y\rangle f_i=\sum_i(k\langle x,y\rangle g_i+k\langle x,y\rangle h_iy)=\sum_i(k[x]g_i+k[x]g_i(0)+k[x]h_iy)=J_1+J_2y,$$

where J_1 and J_2 are ideals of $k[x]$ generated by $\{g_i,g_i(0)\}$ and $\{h_i\}$ respectively. Because $k[x]$ is noetherian, the ideals J_1 and J_2 are finitely generated over $k[x]$, so J is finitely generated over $k\langle x,y\rangle$. Thus, A is left noetherian.

Let $J_n:=\sum_{i=0}^n x^iyk+(yx,y^2)$. Since $x^iyk\langle x,y\rangle=x^iyk+(yx,y^2)$, we can see that J_n is an increasing sequence of right ideals of $k\langle x,y\rangle$ containing (yx,y^2) for all n. Because the sequence J_n does not terminate, A is not right noetherian. □

2. Compute the Hilbert series of $A=k\langle x,y\rangle/(x^2)$.

Solution. Note that $A_{i+2}=A_{i+1}y+A_iyx$ and $A_{i+1}y\cap A_iyx=0$ imply that the dimensions of the homogeneous modules satisfy the recurrence relation of Fibonacci sequence: $\dim_k A_{i+2}=\dim_k A_{i+1}+\dim_k A_i$ for all $i\ge 0$. Since $\dim_k A_1=2$, $\dim_k A_0=1$, and $\dim_k A_i=0$ for $i<0$, we have with the generating function that

$$H_A(t)=1+2t+3t^2+5t^3+8t^4+\cdots=\boxed{\frac{1+t}{1-t-t^2}}.$$ □

6. Compute the point variety of $A=k\langle x,y\rangle/(yx)$.

Solution. Note that $\Gamma_1=\mathbb{P}^1$. Suppose $((a_1,b_1),\cdots,(a_n,b_n))\in\Gamma_n$ for $n\ge 2$. If $a_n=0$, then since $b_n\ne 0$ we have $((a_1,b_1),\cdots,(a_{n-1},b_{n-1}))\in\Gamma_{n-1}$ because

$$g((a_1,b_1),\cdots,(a_{n-1},b_{n-1}))b_n=f((a_1,b_1),\cdots,(a_n,b_n))=0$$

implies $g((a_1,b_1),\cdots,(a_{n-1},b_{n-1}))=0$ for all $g\in(yx)_{n-1}$, where $f:=gy$ belongs to $(yx)_n$. If $a_n\ne 0$, then since for each $1\le i<n$ we have $a_i\ne 0$ or $b_i\ne 0$, so $c_1c_2\cdots c_{n-1}a_n\ne 0$, where $c_i\in\{a_i,b_i\}$. If $c_i=b_i$ for some i, then implies that there is $1\le i<n$ such that $b_ia_{i+1}\ne 0$, which leads a contradiction to the definitioin of Γ_n, so $c_i=a_i\ne 0$ and $b_i=0$ for all $1\le i<n$. Consequently, we have $\Gamma_n\subset(\Gamma_{n-1}\times\{0\})\cup(\{\infty\}^{\times(n-1)}\times\mathbb{P}^1)$.

Conversely, if $((a_1,b_1),\cdots,(a_{n-1},b_{n-1}))\in\Gamma_{n-1}$ and $a_n=0$, then every monomial $f\in(yx)_n$ satisfies

$$f((a_1,b_1),\cdots,(a_n,b_n))=g((a_1,b_1),\cdots,(a_n,b_n))c_n=0,\qquad c_n\in\{a_n,b_n\},$$

so $((a_1,b_1),\cdots,(a_n,b_n))\in\Gamma_n$, and if $b_1=\cdots b_{n-1}=0$, then every monomial $f\in(yx)_n$ satisfies $f((a_1,b_1),\cdots,(a_n,b_n))=0$ clearly. Thus the inverse inclusion holds so that $\Gamma_n=(\Gamma_{n-1}\times\{0\})\cup(\{\infty\}^{\times(n-1)}\times\mathbb{P}^1)$ for all $n\ge 2$.

Therefore, the point variety of A is

$$\Gamma_A=\varprojlim_N\Gamma_N=\varprojlim_N\bigcup_{n=0}^N\{\infty\}^n\times\mathbb{P}^1\times\{0\}^{N-n-1}=\boxed{\bigcup_{n=0}^\infty\{\infty\}^n\times\mathbb{P}^1\times\{0\}^\infty}.$$ □