Integrable Systems

Ikhan Choi
Lectured by Ralph Willox
University of Tokyo, Autumn 2023

February 1, 2024

1 Symmetric polynomials

Let $x = (x_i)_{i=1}^n$ be some auxiliary variables for some n. The *power sum symmetric polynomial* is defined by

$$p_k(x) := \sum_i x_i^k.$$

We define *flow variables*

$$t = (t_1, t_2, \cdots), \qquad t_k := k^{-1} p_k$$

The *complete homogeneous symmetric polynomial* is

$$h_k(x) := \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k}.$$

For Schur polynomial s_λ, there are various definitions, where λ is a Young diagram for partition of m.

Since every symmetric function is generated by power sum symmetric functions p_k, we can represent h_k and s_λ in terms of t. Furthermore, h_k has generating function representation

$$\sum_{k=0}^\infty h_k(t) z^k = \exp \sum_{k=1}^\infty t_k z^k.$$

For example,

$$h_1(t) = t_1, \qquad h_2(t) = t_2 + \frac{1}{2} t_1, \qquad h_3(t) = t_3 + t_1 t_2 + \frac{1}{6} t_1^3$$
$$s_{(1,1)}(t) = \frac{1}{2} t_1^2 - t_2, \qquad s_{(2)}(t) = t_2 + \frac{1}{2} t_1^2.$$

From now on, we will forget any of information for the variables x_i, and t_k will be the most fundamental variables.

Let V be a vector space over $\mathbb{C}$ with a fixed basis $\{e_i\}$. A basis of $\wedge^m V$ can be indexed by subset of $\{e_i\}$ of cardinality m. For such a subset l, we will write $l = (l_1, \cdots, l_m)$ with $l_1 \le \cdots \le l_m$. This kind of m-tuple is a *Maya diagram*. For a Maya diagram l, we can associate a Young diagram $\lambda = (\lambda_1, \cdots, \lambda_m)$ such that $l_j = \lambda_{m-j+1} + j - 1$. Zeros in the Young diagram will be omitted. For each Young diagram λ, we will use the notation

$$e_\lambda := e_{l_1} \wedge \cdots \wedge e_{l_m},$$

where $(l_1, \cdots, l_m)$ is the corresponding Maya digram of the Young diagram λ. Then, $\{e_\lambda\}$ forms a basis of $\wedge^m V$.

2 Plücker coordinates

For a positive integer m, the *Grassmann variety* $\mathrm{Gr}_m(V)$ is the algebraic variety of m-dimensional subspaces of V. The *Plücker embedding*

$$\psi : \mathrm{Gr}_m(V) \to \mathbb{P}(\Lambda^m V) : w = \mathrm{span}\{w_j\}_{j=1}^m \mapsto \wedge^m w = \mathrm{span}\{\wedge_{j=1}^m w_j\}$$

shows that the Grassmann variety is projective.

The *tautological vector bundle* T over the grassmann variety $\mathrm{Gr}_m(V)$ is defined as a subbundle of the trivial bundle $\mathrm{Gr}_m(V) \times V \to \mathrm{Gr}_m(V)$ such that

$$T := \{(w, v) \in \mathrm{Gr}_m(V) \times V : v \in w\}.$$

The rank of the tautological bundle T is m. The *determinant line bundle* Det over the grasmann variety $\mathrm{Gr}_m(V)$ is the top exterior power of the tautological vector bundle

$$\mathrm{Det} := \wedge^m T.$$

The tautological line bundle of the projective space $\mathbb{P}(\wedge^m V)$ is identified with $\mathcal{O}_{\mathbb{P}(\wedge^V)}(-1)$. The identity

$$\mathrm{Det}_w = \wedge^m w = \psi(w) = \mathcal{O}_{\mathbb{P}(\wedge^V)}(-1)_{\psi(w)}$$

on each fiber at w and $\psi(w)$ defines a bundle isomorphism $\mathrm{Det} \to \mathcal{O}_{\mathbb{P}(\wedge^m V)}(-1)$, so we have the isomorphic line bundles

$$\mathrm{Det} \cong \psi^* \mathcal{O}_{\mathbb{P}(\wedge^m V)}(-1).$$

Taking the inverses, we have

$$\mathrm{Det}^* \cong \psi^* \mathcal{O}_{\mathbb{P}(\wedge^m V)}(1).$$

The line bundle $\mathcal{O}_{\mathbb{P}(\wedge^m V)}(1)$ admits global sections spanned by the homogeneous polynomial of degree one, which are identified to the coordinate functions $e_\lambda^* : \wedge^m V \to \mathbb{C}$ defined such that $e_\lambda^*(e_{\lambda'}) = \delta_{\lambda,\lambda'}$, where λ and λ' are Young diagrams. For each Young diagram λ, define the *Plücker coordinate* $\pi_\lambda := \psi^* e_\lambda^*$ as a global section of the dual determinant line bundle Det^*.

Since $\mathrm{Gr}_m(V)$ is projective via the Plücker embedding ψ, there is a homogeneous ideal I such that the image of ψ has the homogeneous coordinate ring $\mathbb{C}[e_\lambda^*]_\lambda / I$. The *Plücker relations* are special generators of I, which are quadratic homogeneous polynomials.

3 Tau functions

Suppose $V = \mathbb{C}^\infty$. Consider an abelian group action γ of $\mathbb{C}^\infty$ on the Grassmann variety $\mathrm{Gr}_m(V)$ defined such that

$$\gamma(t) := \exp \sum_{k=1}^{\infty} t_k \Lambda^k, \qquad t = (t_1, t_2, \cdots) \in \mathbb{C}^\infty,$$

where $\Lambda : V \to V$ is a linear map satisfying $\Lambda e_i := e_{i+1}$, which is called the *shift matrix*.

Fix $w \in \mathrm{Gr}_m(V)$. The *τ-function* associated with the abelian group action and the initial point w is the function $\tau : \mathbb{C}^\infty \to \mathbb{C}$ defined by the first Plücker coordinate of the curve $\gamma(t)w$, i.e.

$$\tau(t) := \pi_{(0)}(\gamma(t)w).$$

We can also define

$$\tau_\lambda(t) := \pi_\lambda(\gamma(t)w).$$

(This is the τ-function given in the problem.) Then, we have the Schur function expansion

$$\tau(t) = \sum_\lambda \pi_\lambda(w) s_\lambda(t),$$

and

$$\tau_\lambda(t) = s_\lambda(\widetilde{\partial}_t)\tau(t), \qquad \widetilde{\partial}_t = (k^{-1}\partial_{t_k})_{k=1}^\infty.$$

4 KP equation

Let $x = t_1$, $y = t_2$, and $t = t_3$. The equation

$$\tau_{(0)}\tau_{(2,2)} - \tau_{(1)}\tau_{(2,1)} + \tau_{(2)}\tau_{(1,1)} = 0$$

is deduced from a Plücker relation

$$(e_0^* \wedge e_1^*)(e_2^* \wedge e_3^*) - (e_0^* \wedge e_2^*)(e_1^* \wedge e_3^*) + (e_0^* \wedge e_3^*)(e_1^* \wedge e_2^*) = 0.$$

Since

$$\begin{aligned}
\tau_{(0)} &= \tau \\
\tau_{(1)} &= \tau_x \\
\tau_{(1,1)} &= \frac{\tau_{2x} - \tau_y}{2} \\
\tau_{(2)} &= \frac{\tau_{2x} + \tau_y}{2} \\
\tau_{(2,1)} &= \frac{\tau_{3x} - \tau_{xy}}{2} \\
\tau_{(2,2)} &= \frac{\tau_{2x,y} + \tau_{2y} - 2\tau_{tx}}{2},
\end{aligned}$$

Let

$$u := \partial_x^2 \log \tau, \qquad v := \partial_x \partial_y \log \tau.$$