Noncommutative geometry

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2.1 Atiyah-Singer index

2.1. Let X be an n-dimensional compact smooth manifold. Let E and F be smooth complex vector bundles on X. A map $P : \Gamma(E) \to \Gamma(F)$ is called a *linear partial differential operator* if it is locally a polynomial in the operators ∂_i with smooth matrix-valued coefficients. A linear partial differential operator P is called *elliptic* if its symbol is invertible for all $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

(a) An elliptic operator $D : \Gamma(E) \to \Gamma(F)$ is Fredholm, and defines an element of the K-homology group $K_0(X)$ and an element of $K^0(T^*X, (T^*X)_0)$.

For locally compact Hausdorff spaces, is the K-theory with compact supports equal to the representable K-theory?

The Thom space of a vector bundle E is just the one-point compactification E_+ of the total space E if the base space is compact.

An elliptic operator $D : \Gamma E \to \Gamma F$ defines a section $\sigma(D) \in \mathrm{Hom}(\pi^*E, \pi^*F)$ of a vector bundle over T^*X, and it gives an element of the K-theory of the Thom space $K(DT^*X, ST^*X) = K(T^*X, T^*X_0)$. Note that it is equal to $K(T^*X)$ if X is compact. The analytic index is a map

$$K(T^*X, T^*X_0) \to \mathbb{Z}.$$

For an embedding $X \hookrightarrow Y = \mathbb{R}^{n+m}$, the topological index map is defined as the composition

$$K(T^*X) \to K(T^*N) \to K(T^*Y) \cong K(\mathbb{R}^{2(n+m)}) \cong \mathbb{Z},$$

where the first map is the Thom isomoprhism established because T^*N can be given a complex vector bundle structure, and the second map is the induced map of the quotient map $(T^*Y_+, *) \to (T^*N_+, *)$.

2.2 Dirac operators

2.2 (Unbounded Kasparov modules)**.** Let A and B be C*-algebras. An *unbounded Kasparov module* is a super-correspondence E from A to B together with a dense $*$-subalgebra $A^\infty \subset A$ and an odd self-adjoint regular operator D on E such that $[D, A^\infty] \subset B(E)$ and $(D+i)^{-1}A^\infty \subset K(E)$.

On a oriented Riemannian manifold, we have the Hodge-Dirac operator $D := d + d^*$ and the Laplace-de Rham operator D^*D.

2.3 Fredholm theory of Mishchenko and Fomenko

Chapter 3

Quantum metric spaces

Chapter 4

Coarse geometry

4.1 Quantum Hall effect

A wave function is a section ψ of a $\mathrm{U}(1)$-line bundle $\mathcal{L} \to M$, more generally, a section ψ of a Hermitian G-vector bundle $\mathcal{V} \to M$. Galilean invariant momentum operator? We fix a connection ∇ on $\mathcal{V}$, which can be forgotten if M is contractible and no magnetic fields are considered. A $\mathrm{U}(N)$-gauge fixing is just a choice of a local field of orthonormal frames on the intersection of two charts of M, making $\mathcal{V}$ locally trivially $\mathbb{C}^N$ with the standard basis. After gauge fixing, the covariant derivative is represented locally as $\nabla_j \psi = (\partial_j - iA_j)\psi$, where $A = \sum_j A_j dx^j$ is a $\mathbb{R} = -i\mathfrak{u}(1)$-valued one-form.

The connection ∇ is independent If M has non-trivial holonomy (the Aharonov-Bohm flux), then it comes into play.

On $M = \mathbb{R}^2$ with a connection ∇ such that the curvature form is $\mathcal{F}^\nabla = b \cdot \mathrm{vol}_M$ for some $b \in \mathbb{R} \setminus \{0\}$, then the *Landau operator*, which is a free Hamiltonian defined by the connection Laplacian or the Bochner Laplacian $H := \nabla^* \nabla = -\operatorname{tr}(\nabla^2) \geq 0$, its spectrum is known to be $\sigma(H) = (2\mathbb{N} + 1)|b|$. This discreteness of the spectrum is called the *Landau quantization*.

We consider a spin manifold M with the spinor bundle $\mathcal{S}$, and the sections of $\mathcal{S} \otimes \mathcal{V} \to M$ represent the space of wave functions. curvature of a spin connection the Laplacians

Chapter 5

Infinite dimensional manifolds

5.1

Loop spaces, Loop groups