Algebraic Topology

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November 5, 2022

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Part I

Homology

Chapter 1

Homology groups

1.1 Singular homology

1.2 Simplicial homology

1.3 Cellular homology

1.4 Eilenberg-Steenrod axioms

Chapter 2

Cohomology groups

cup product Universal coefficient theorem

2.1 Poincaré duality

Chapter 3

Part II

Homotopy

Chapter 4

Fundamental groups

4.1 Path lifting proerty

4.2 Van Kampen theorem

4.3 Covering spaces

Chapter 5

Higher homotopy groups

Chapter 6

Part III

Fiber bundles

Chapter 7

Principal bundles

7.1 Category of bundles

7.1 (Pullback and restricted bundles)**.**

7.2 (Product of bundles)**.** the *Fiber product* or the *Whitney sum*

7.2 Classifying spaces

7.3 Čech cohomology

7.3. Let $\{U_\alpha\}_\alpha$ be an open cover of a topological space X. We say a presheaf $\mathcal{F}$ on X is sheaf if the sequence

$$\mathcal{F}(U) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is an equalizer.

7.4. Let $\mathcal{F}$ be a preseah of groups and let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an ordered open cover of a toplogical space X.

$$\prod_\alpha \mathcal{F}(U_\alpha) \overset{\mathrm{res}_\alpha}{\underset{\mathrm{res}_\beta}{\rightrightarrows}} \prod_{\alpha<\beta} \mathcal{F}(U_\alpha \cap U_\beta) \overset{\mathrm{res}_\alpha}{\underset{\mathrm{res}_\gamma}{\overset{-\,\mathrm{res}_\beta}{\rightrightarrows}}} \prod_{\alpha<\beta<\gamma} \mathcal{F}(U_\alpha \cap U_\beta \cap U_\gamma)$$

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

$$f = \{f_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})\}_\alpha \in \prod_\alpha \mathcal{F}(U_\alpha) = C^0(\mathcal{U}, \mathcal{F})$$

7.5. Let G be a sheaf of groups. Then, we have a natural one-to-one correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles} \end{array} \right\} \xrightarrow{\sim} H^1(B, G).$$

Proof. (Injectivity) We show the correspondence is left invertible. Let $p : E \to B$ be a principal G-bundle. Define $p' : E' \to B$ by

$$E' := \left(\bigsqcup_\alpha \{\alpha\} \times U_\alpha \times \mathbb{R}^k \right) / \sim,$$

where the equivalence relation $\sim$ denotes $(\alpha, b, g_{\alpha\beta}(v)) \sim (\beta, b, v)$ with $b \in U_\alpha \cap U_\beta$ for some α, β, and define $p' : E' \to B$, which is well-defined bundle.

We first check that p' is a principal G-bundle.

(Surjectivity) $\square$

7.4 Vector bundles

7.6 (Vector space structure on total spaces)**.**

7.7 (Vector bundle maps)**.** Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be vector bundles.

(a) A vector bundle map u over B is vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

7.8 (Tautological bundles)**.**

7.9 (Homotopy properteis)**.** Let $p : E \to B$ be a vector bundle

If $p_1 : E_1 \to B \times [0, \frac{1}{2}]$ and $p_2 : E_2 \to B \times [\frac{1}{2}, 1]$ are trivial, then

(a)

7.10 (Principal bundle over the general linear group)**.**

Chapter 8

Characteristic classes

Chapter 9

Spectral sequences Serre, Lyndon-Hochschild-Serre, Adams

Stable homotopy theory

Part IV

K-theory