Mathematical Logic

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Contents

Part I

Set theory

Chapter 1

First-order logic

1.1 First-order logic

1.1 (Meta-mathematics)**.** Most of modern usual mathematics is based on the informal ZC^-, the Zermelo-Fraenkel set theory with choice but removed the axiom of regularity and the axiom schema of replacement. This informal theory is undefinable in a mathematical rigorous sense but appreciated as the bottom assumption, believing in our innately shared and experimentally established ability of reasoning. It can be regarded as a kind of statement of faith for the assumption on which every informal finite reasoning in usual mathematics relies. Now then, we can study formal theories on logical systems and ZFC itself as an interesting example of first-order formal theories via the informal ZC^-; we can prove (meta-)theorems on the formal ZFC.

1.2 (Languages)**.** In general, a *language* is a subset $\mathcal{L}$ of the Kleene star Σ^* of a set Σ called an *alphabet*. Recall that the *Kleene star* Σ^* of a given set Σ is another name of the free monoid generated by the set. An element of a given alphabet is called a *symbol* or a *letter*, and an element of a language is called a *formula*. A language is usually given by an alphabet together with a finite set of formula-formation rules, called a grammar. To stress that each formula satisfies the formation rule, a formula of a language given by a grammar is also called a *well-formed formula*.

1.3 (Signatures)**.** A *signature*, also called a *lexicon* or a *vocabulary*, is a datum consisting of

(i) a non-empty countable set $\mathcal{S}$ of *sorts*,

(ii) a non-empty set of symbols partitioned into the following two disjoint sets:

  (ii.i) a set $\mathcal{F}$ of *function symbols* together with functions $\mathrm{dom} : \mathcal{F} \to \mathcal{S}^*$ and $\mathrm{cod} : \mathcal{F} \to \mathcal{S}$ called the *domain function*, and *codomain function*,

  (ii.ii) a set $\mathcal{P}$ of *predicate* or *relation symbols* together with a function $\mathrm{ar} : \mathcal{P} \to \mathcal{S}^*$ called the *arity function*.

A function symbol $f \in \mathcal{F}$ is called a *constant symbol* if $\mathrm{dom}(f) = 0$, and a predicate symbol $p \in \mathcal{P}$ is called a *truth value* if $\mathrm{ar}(p) = 0$. If $\mathcal{S}$ has only one or two elements, then the signature is called *single-sorted* or *double-sorted* respectively. For instance, the signatures for groups and rings are single-sorted, and the signatures for modules and vector spaces are double-sorted.

1.4 (Syntax of first-order logic)**.** A *first-order language* is a language $\mathcal{L}$, determined by a signature $(\mathcal{S}, \mathcal{F}, \mathcal{P})$, with an alphabet and a grammar defined as follows. The alphabet is given by the disjoint union of the following sets:

(i) the set of *logical symbols*, which is the disjoint union

$$(\bigcup_{s\in\mathcal{S}}\mathcal{V}_s)\cup\{=\}\cup\{\neg,\wedge,\vee,\rightarrow,\longleftrightarrow\}\cup\{\forall,\exists\},$$

where $\mathcal{V}_s$ is any countably infinite set of *variables* of sort s,

(ii) the set of *non-logical symbols*, which is the disjoint union $\mathcal{F}\cup\mathcal{P}$.

The grammar is given as follows:

(i) the set $\mathcal{T}_s$ of *terms* of a sort s are defined recursively such that

(i.i) $v\in\mathcal{T}_s$ for $v\in\mathcal{V}_s$,

(i.ii) $f(\overline{t})\in\mathcal{T}_s$ for $f\in\mathcal{F}_{\overline{s}\to s}$ and $\overline{t}\in\mathcal{T}_{\overline{s}}$

(ii) the set $\mathcal{L}$ of *formulas* are defined such that

(iii.i) $=tt'\in\mathcal{L}$ for $t,t'\in\mathcal{T}_s$,

(iii.ii) $p(\overline{t})\in\mathcal{L}$ for $p\in\mathcal{P}_{\overline{s}}$ and a finite sequence $\overline{t}\in\mathcal{T}_{\overline{s}}$,

(iii.iii) $\neg\varphi,\wedge\varphi\psi,\vee\varphi\psi,\rightarrow\varphi\psi,\longleftrightarrow\varphi\psi\in\mathcal{L}$ for $\varphi,\psi\in\mathcal{L}$,

(iii.iv) $\forall v\varphi,\exists v\varphi\in\mathcal{L}$ for $v\in\mathcal{V}$ and $\varphi\in\mathcal{L}$.

Because a first-order language is totally determined by a signature, some authors simply define or identify a first-order language with a signature.

(notation $\overline{s}$, $\overline{t}$, $\mathcal{V}_s$, $\mathcal{V}_{\overline{s}}$, $\mathcal{F}_{\overline{s}\to s}$, $\mathcal{P}_s$, paranthese, $\varphi(v)$)

Polish notation and readability.

free variables and substitution

A *sentence* or a *closed formula* is a formula over $\mathcal{L}$ not having any free variables. A *theory* over a first-order language $\mathcal{L}$ is simply a set consisting of some sentences over $\mathcal{L}$, which we designate each sen to be true. Let $\mathcal{L}$ be a first-order language with the signature $(\mathcal{S},\mathcal{F},\mathcal{P})$.

(a) In our definition, we always have $|\mathcal{L}|=\aleph_0+|\mathcal{F}|+|\mathcal{P}|$.

(b)

1.5 (Semantics of first-order logic). Let $\mathcal{L}$ be a first-order language with the signature $(\mathcal{S},\mathcal{F},\mathcal{P})$. A *structure* M over $\mathcal{L}$ is a family of non-empty sets $\{M_s\}_{s\in\mathcal{S}}$ together with functions $f_M:M_{\overline{s}}\to M_s$ and relations $p_M\subset M_{\overline{s}}$ assgined to each symbol of $f\in\mathcal{F}_{\overline{s}\to s}$ and $p\in\mathcal{P}_{\overline{s}}$. The datum of the functions $f_M:M_{\overline{s}}\to M_s$ and relations $p_M\subset M_{\overline{s}}$ is called an *interpretations* in M of the function symbols f and predicate symbols p. Given an $\mathcal{L}$-structure M, letting a variable be an identity function and applying the composition of the interpretations of function symbols repeatedly, we can also interpret a term $t\in\mathcal{T}_s$ of free variables $\overline{v}\in\mathcal{V}_{\overline{s}}$ as a function $t_M:M_{\overline{s}}\to M_s$.

Let M be an $\mathcal{L}$-structure. For an $\mathcal{L}$-formula φ of free variables $\overline{v}\in\mathcal{V}_{\overline{s}}$, and for elements $\overline{a}\in M_{\overline{s}}$, we say M *satisfies* $\varphi(\overline{a}/\overline{v})$ and write $M\vDash\varphi(\overline{a}/\overline{v})$ if

(i) $t_M(\overline{a}/\overline{v})=t'_M(\overline{a}/\overline{v})$, when φ is $t=t'$ for $t,t'\in\mathcal{T}_s$ of free variables $\overline{v}$,

(ii) $\overline{t}_M(\overline{a}/\overline{v})\in p_M$, when φ is $p(\overline{t})$ for $p\in\mathcal{P}_{\overline{s}}$ and $\overline{t}\in\mathcal{T}_{\overline{s}}$ of free variables $\overline{v}$,

(iii) $M\nvDash\psi(\overline{a}/\overline{v})$, when φ is $\neg\psi$, where ψ is an $\mathcal{L}$-formula of free variables $\overline{v}$,

(iv) $M\vDash\psi(\overline{a}/\overline{v})$ and $M\vDash\psi'(\overline{x})$, when φ is $\psi\wedge\psi'$, where ψ and ψ' are $\mathcal{L}$-formulas of free variables $\overline{v}$,

(v) $M\vDash\psi(\overline{a}'/\overline{v}',a/v,\overline{a}''/\overline{v}'')$ for some $a\in M_s$ and $(\overline{a}',\overline{a}'')=\overline{a}$, when φ is $\exists v\psi$, where ψ is an $\mathcal{L}$-formula of free variables $(\overline{v}',v,\overline{v}'')$ with $(\overline{v}',\overline{v}'')=\overline{v}$.

1.6 (Deduction in first-order logic).

Let M be a structure over a first-order language $\mathcal{L}$. A relation p_M on $M_{\overline{s}}$ is called *definable* if there is an $\mathcal{L}$-formula φ and a constant $\overline{c}\in M_{\overline{s}'}$ such that $p=\{\overline{a}\in M_{\overline{s}}:\varphi(\overline{a}/\overline{v},\overline{c}/\overline{w})\}$.

1.2 Compactness and completeness

proofs: finite, sound, decidable.

A first-order language $\mathcal{L}$ is called *recursive* or *decidable* if there is an algorithm that,

recursively enumerable if there is an algorithm that, when given an $\mathcal{L}$-sentence φ as input, halts accepting if $T \vdash \varphi$, and does not halt if $T \nvdash \varphi$.

1.7 (Henkin constructions)**.** Let $\mathcal{L}$ be a single-sorted first-order theory, and let $\mathcal{T}$ be an $\mathcal{L}$-theory.

The completeness theorem clearly implies the compactness theorem. We give a proof of the compactness theorem that does not use any proof system.

We will say an $\mathcal{L}$-theory $\mathcal{T}$ has *witnesses* if for every $\mathcal{L}$-formula φ with a single free variable $v \in \mathcal{L}$ there is a constant symbol $c \in \mathcal{L}$ such that $\mathcal{T}_0 \vDash (\exists v \varphi(v)) \to \varphi(c)$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$. Note that the witness property of $\mathcal{T}$ implies that the set $\mathcal{C} := \mathcal{F}_0$ of constant symbols is non-empty.

(a) If $\mathcal{T}$ is finitely satisfiable and maximal with witnesses, then it is satisfiable by a model of a cardinality $\kappa \le |\mathcal{C}|$.

(b) witnessing extension of languages

(c) maximal extension of theories (extension of theories preserves the property of having witnesses)

(d) If $\mathcal{T}$ is finitely satisfiable, then it is satisfiable by a model of any cardiality $\kappa \ge |\mathcal{L}|$.

Proof. (a) To construct a model, we need to define a set M with interpretations $f : M^n \to M$ and $p \subset M^n$ for each $f \in \mathcal{F}_n$ and $p \in \mathcal{P}_n$, where $n \in \mathbb{N}$ is fixed. Define the underlying set $M := \mathcal{C}/\sim$, where an equivalence relation on $\mathcal{C}$ is defined such that $c \sim d$ if and only if $\mathcal{T}_0 \vDash c = d$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$. For each $c \in \mathcal{C}$, we have a trivial interpretation $c_M := [c]$ defined by the equivalence class of c in M.

For each $p \in \mathcal{P}_n$, we interpret it by

$$p_M := \{\bar{c}_M \in M^n : p(\bar{c}) \in \mathcal{T}\}.$$

To check the well-definedness, letting $\bar{c}_M = \bar{c}'_M$ and assuming $\bar{c}_M \in p_M$, we show $\bar{c}'_M \in p_M$. Take representatives $\bar{c}, \bar{c}' \in \mathcal{C}^n$ such that $p(\bar{c}) \in \mathcal{T}$ and $\mathcal{T}_0 \vDash \bar{c} = \bar{c}'$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$. If $p(\bar{c}') \notin \mathcal{T}$, then we have $\neg p(\bar{c}') \in \mathcal{T}$ by the maximality of $\mathcal{T}$ and a contradiction $\{p(c), \neg p(c')\} \cup \mathcal{T}_0 \vDash \{\bar{c} = \bar{c}', \bar{c} \ne \bar{c}'\}$ to the finite satisfiability of $\mathcal{T}$, so we have $p(\bar{c}') \in \mathcal{T}$ and $\bar{c}'_M \in p_M$.

For each $f \in \mathcal{F}_n$ with $n \ge 1$, we interpret it by

$$f_M := \{(\bar{c}_M, c_M) \in M^{n+1} : f(\bar{c}) = c \in \mathcal{T}\}.$$

First, we claim that for $\bar{c}_M \in M^n$ there is $c_M \in M$ such that $(\bar{c}_M, c_M) \in f_M$. If we choose the representative $\bar{c} \in \mathcal{C}^n$ for $\bar{c}_M$, then since $\varnothing \vDash \exists v(f(\bar{c}) = v)$ by definition of interpretation of function symbols, and since there is $c \in \mathcal{C}$ such that $\mathcal{T}_0 \vDash \exists v(f(\bar{c}) = v) \to f(\bar{c}) = c$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ by the witness property of $\mathcal{T}$, we have $\mathcal{T}_0 \vDash f(\bar{c}) = c$. If $f(\bar{c}) = c \notin \mathcal{T}$, then we have $f(\bar{c}) \ne c \in \mathcal{T}$ by the maximality and a contradiction $\{f(\bar{c}) \ne c\} \cup \mathcal{T}_0 \vDash \{f(\bar{c}) \ne c, f(\bar{c}) = c\}$ to the finite satisfiability of $\mathcal{T}$, so we get $f(\bar{c}) = c \in T$ and $(\bar{c}_M, c_M) \in f_M$. Second, we claim that $(\bar{c}_M, c_M) \in f_M$ and $(\bar{c}_M, c'_M) \in f_M$ imply $c_M = c'_M$. Choose the representatives $(\bar{c}, c) \in \mathcal{C}^{n+1}$ such that $f(\bar{c}) = c \in \mathcal{T}$ and $(\bar{c}', c') \in \mathcal{C}^{n+1}$ such that $f(\bar{c}') = c' \in \mathcal{T}$ and $\bar{c}_M = \bar{c}'_M$. Since $\mathcal{T}_0 \vDash \bar{c} = \bar{c}'$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ by definition of $\sim$, we have $\{f(\bar{c}) = c, f(\bar{c}') = c'\} \cup \mathcal{T}_0 \vDash c = c'$, so $c_M = c'_M$ by definition of $\sim$ again. By the above two claims, the interpretation $f_M : M^n \to M$ is a well-defined function.

Now we have an $\mathcal{L}$-structure M with $|M| \le |\mathcal{C}|$, so it is enough to prove $M \vDash \varphi$ if $\varphi \in \mathcal{T}$.

For any term t of free variables $\bar{v} \in \mathcal{V}^n$ and for any $(\bar{a}, a) \in M^{n+1}$, we will show that $t_M(\bar{a}/\bar{v}) = a$ if and only if there exists $(\bar{c}, c) \in \mathcal{C}^{n+1}$ such that $t(\bar{c}) = c \in \mathcal{T}$.

Now we prove $M \vDash \varphi$ for $\varphi \in \mathcal{T}$ by the induction on the logical complexity. We need to divide the cases into the five situations.

If $\varphi \in \mathcal{T}$ is $t(\bar{c}) = t'(\bar{c})$

If $\varphi \in \mathcal{T}$ is $p(t(\bar{c}))$

If $\varphi \in \mathcal{T}$ is $\neg\psi$,

If $\varphi \in \mathcal{T}$ is $\psi \wedge \psi'$,

If $\varphi \in \mathcal{T}$ is $\exists v\psi$, where ψ is an $\mathcal{L}$-formula of a single free variable v, then there is $c \in \mathcal{C}$ such that $\mathcal{T}_0 \vDash \varphi \to \psi(c)$ for a finite subset $\mathcal{T}_0 \subset \mathcal{T}$ because $\mathcal{T}$ has witnesses. If $\psi(c) \notin \mathcal{T}$, then we have $\neg\psi(c) \in \mathcal{T}$ by the maximality and a contradiction $\{\neg\psi(c), \varphi\} \cup \mathcal{T}_0 \vDash \{\neg\psi(c), \psi(c)\}$ to the finite satisfiability of $\mathcal{T}$, so $\psi(c) \in \mathcal{T}$. By the inductive assumption, we have $M \vDash \psi(c)$. Since $M \vDash \varphi$ is equivalent to the existence of $a \in M$ such that $M \vDash \psi(a/v)$, which is true by taking $a := c_M$. $\square$

1.3 Peano arithmetic

1.8 (Peano arithmetic)**.** Let $\mathcal{L}$ be a single-sorted first-order language with the set of non-logical symbols $\{+, \cdot, s, 0\}$, where $+$ and $\cdot$ are binary function symbols, s is a unary function symbol, and 0 is a constant. The *first-order Peano arithmetic* is a theory PA over $\mathcal{L}$ language consisting of the following sentences:

(i) $\forall x(s(x) \neq 0)$,

(ii) $\forall x \exists y(x \neq 0 \to s(y) = x)$,

(ii′) $\forall x \forall y(s(x) = s(y) \to x = y)$,

(iii) $\forall x(x + 0 = x)$,

(iv) $\forall x \forall y(x + s(y) = s(x + y))$,

(v) $\forall x(x \cdot 0 = 0)$,

(vi) $\forall x \forall y(x \cdot s(y) = x \cdot y + x)$,

(vii) $\forall \bar{t} \forall x \forall y((\varphi(\bar{t}, 0) \wedge (\varphi(\bar{t}, x) \to \varphi(\bar{t}, s(x)))) \to \varphi(\bar{t}, y))$ for each formula $\varphi \in \mathcal{L}$.

A model of the first-order Peano arithmetic is also called a model of the *natural numbers*. We can ask the classification of models for the natural numbers.

The standard model of the natural numbers cannot be characterized by PA, since definable formulas $\varphi \in \mathcal{L}$ are at most countable.

1.4 Zermel-Fraenkel axioms with choice

1.9 (Zermelo-Frankel set theory with choice)**.** Let $\mathcal{L}$ be a single-sorted first-order language with a single non-logical symbol $\in$ that is a binary relation. The *Zermelo-Fraenkel set theory with choice* is a theory ZFC over $\mathcal{L}$ consisting of the following sentences:

(i) $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$, (extensionality)

(ii) $\forall x(x \neq \varnothing \to \exists y(y \in x \wedge y \cap x = \varnothing))$, (regularity)

(iii) $\forall \bar{t} \forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \varphi(\bar{t}, z)))$ for each formula $\varphi \in \mathcal{L}$, (comprehension)

(iv) $\forall \bar{t} \forall x(\forall z(z \in x \to \exists! w \varphi(\bar{t}, z, w)) \to \exists y \forall z(z \in x \to \exists w(w \in y \wedge \varphi(\bar{t}, z, w))))$ for each formula $\varphi \in \mathcal{L}$, (replacement)

(v) $\forall x \forall y \exists z(x \in z \wedge y \in z)$, (pairing)

(vi) $\forall x \exists y \forall z \forall w(w \in z \wedge z \in x \to w \in y)$, (union)

(vii) $\forall x \exists y \forall z (z \subset x \to z \in y)$ (power set)

(viii) $\exists x (\varnothing \in x \land \forall y (y \in x \to (s(y) \in x)))$ (infinity)

(ix) $\forall x (\forall z \forall w (z \in x \land w \in x \land z \neq w \to z \cap w = \varnothing) \to \exists y \forall z (z \in x \to (|y \cap z| = 1)))$ (choice)

Fraenkel added the replacement axiom.

1.10 (Classes)**.**

1.11. An *ordinal* or a *von Neumann ordinal* is a set that is strictly well-ordered with respect to the set membership, and transitive.

A countable ordinal is corresponded to trees with finite leaves.

An ordinal is called a *successor ordinal* if it can be written as $\omega + 1$ for some ordinal ω. It an ordinal is not a successor, then it is called a *limit ordinal*.

Chapter 2

Consistency

Chapter 3

Large cardinals

Part II

Model theory

Chapter 4

satisfiable theory complete theory κ-categorical theory

For a first-order theory T, the completeness theorem says that a sentence is provable in T if and only if it is true in every model of T. The Gödel incompleteness theorem states that there exists a sentence in the first-order Peano arithmetic which is true in the standard model but false in another model.

4.1 (Henkin construction)**.** Fix a signature Σ. We say a theory T has the *witness property* if for every formula φ with a single free variable there exists a constant c such that $T \vDash (\exists x\ \varphi(x)) \to \varphi(c)$. If a theory is maximal, finitely satisfible, and has the witness property, then it has a model M such that $|M| \leq \kappa$ for every cardinal $\kappa \geq |\mathrm{Const}(\Sigma)|$.

4.2 (Vaught test)**.**

4.3 (Löwenheim-Skolem theorem)**.**

Part III

Proof theory

Chapter 5

Proof calculi

Hilbert calculus

Gentzen natural calculus

Gentzen sequent calculus

Part IV

Recursion theory

Chapter 6

Let D be a countably infinite set.

We say a function is *computable* if there is an informal algorithm

We say a relation is *decidable* if there is an informal algorithm

For a relation R on a set A of arity n. We say R is Δ_0 if there is a Δ_0 formula φ with n free variables such that $R = \{\overline{a} : A \models \varphi(\overline{a})\}$.

Δ_0: $\forall x\ x \in y \to \varphi$ and $\exists x\ x \in y \land \varphi$..

definition of Turing machines.

A *Turing machine* is a triple of a finite set S of states, a finite set Σ of symbols, and a transition function $\delta : S \times \Sigma \to S \times \Sigma \times \{L, R\}$.