Pseudodifferential Operators

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Contents

1 Day 1: April 11

Notation

$D_j = (-1)\partial_j$

$$\xi^\alpha \mathcal{F}u = \mathcal{F}D^\alpha u, \quad \xi^\alpha \mathcal{F}^* u = \mathcal{F}^*(-D)^\alpha u, \quad D^\alpha \mathcal{F}u = \mathcal{F}(-x)^\alpha u$$

Let

$$A = \sum a_\alpha(x) D^\alpha, \quad a(x,\xi) = \sum a_\alpha(x)\xi^\alpha.$$

Then,

$$\begin{aligned} Au(x) = \mathcal{F}^* M_{a(x,\xi)} \mathcal{F}u(x) \\ &= (2\pi)^{-1} \int e^{ix\xi} a(x,\xi) \int e^{-iy\xi} u(y)\, dy\, d\xi \\ &= (2\pi)^{-1} \iint e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi. \end{aligned}$$

If a has a polynomial growth in ξ, then the integrand $e^{i(x-y)\xi}a(x,\xi)u(y)$ is not integrable in (y,ξ), so we need to justify it as an oscillatory integral.

Japanese bracket, originated by Kitada or Kumano-go (akumade setsu)

$$\langle x+y\rangle^{-2} \le 4\langle x\rangle^2 \langle y\rangle^{-2}$$

$$\langle x^2 + x\rangle \asymp \langle x^2\rangle \asymp \langle x\rangle^2$$

Here we define the amplitude functions as

$$|\partial^\alpha a(x)| \lesssim \langle x\rangle^{m+\delta|\alpha|}, \quad \forall \alpha \in \mathbb{Z}^d_{\ge 0}$$

Example (Justification of a qudaratic oscillation). Let Q be a nondegenerate real quadratic form, then for $a \in A^m_\delta(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$,

$$I_Q(a) := \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} e^{i\frac{1}{2}Q(x)} \chi(\varepsilon x) a(x)\, dx$$

exists. The term $e^{i\frac{1}{2}Q(x)}$ oscillates fast where $|x| \gg 1$, the term $\chi(\varepsilon x)$ becomes flatten as $\varepsilon \to 0$. When we do integrate by parts, we integrate the oscillating term, and differentiate the cutoff and amplitude. If we differentiate the amplitude, the integrability is enhanced.

Proof. We compute for $Q = I$ as an example. Since

$$De^{i\frac{1}{2}x^2} = xe^{i\frac{1}{2}x^2},$$

we have

$$(1 + x\cdot D)e^{i\frac{1}{2}x^2} = (1+x^2)e^{i\frac{1}{2}x^2}.$$

Define a differential operator L such that

$${}^tL := \frac{1}{1+x^2} + \frac{x}{1+x^2}\cdot D = \langle x\rangle^{-2} + \langle x\rangle^{-2} x\cdot D,$$

that is,

$$\begin{aligned} L &= \frac{1}{1+x^2} - D\cdot \frac{x}{1+x^2} - \frac{x}{1+x^2}\cdot D \\ &= \frac{1}{1+x^2} + i\left(\frac{d}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right) - \frac{x}{1+x^2}\cdot D \\ &= (1+(d+2)i)\langle x\rangle^{-2} - 2\langle x\rangle^{-4} - \langle x\rangle^{-2} x\cdot D. \end{aligned}$$

Then ${}^tL$ fixes $e^{i\frac{1}{2}x^2}$, so

$$\int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2}\chi(\varepsilon x)a(x)\,dx = \int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2} L^k[\chi(\varepsilon x)a(x)]\,dx$$

for any $k \geq 0$. Since

$$L = c_0(x) + c_j(x)\partial_j, \qquad c_0 \in A^{-2}_{-1}, \quad c_j \in A^{-1}_{-1},$$

we have

$$|L^k[\chi(\varepsilon x)a(x)]| \lesssim |L^k[a(x)]| \lesssim |a|_k \langle x\rangle^{m-\min\{1-\delta,2\}k}$$

bounded $\varepsilon > 0$.

Then,

$$|L^k[\chi(\varepsilon x)a(x)]| \xrightarrow{\varepsilon\to 0} L^k[a(x)] \quad \text{pointwise.}$$

□

2 Day 2: April 18

- Lemma 1.3: Coordinate changes and integration by parts work. Also we can check even if we do coordiante change and differeniation (of oscillating term), we also have oscillatory integral.
- Theorem 1.4: For amplitude functions with $\delta < 1$, an operator *defined* by the multiplier

  $$e^{i\frac{1}{2}Q(D)} : \mathcal{S}' \to \mathcal{S}'$$

  have an explicit expression.
- Theorem 1.5: The above multiplier also can be defined by the Taylor expansion. This kind of theorems may be called a expansion formula (I think). The last part of the proof holds from the integral by parts.
- Corollary 1.7: We want to have an extension with a parameter. The parameter h is called s semiclassical parameter. As $h \to 0$, the oscillation goes rapid. The name stationary phase is implied by the origin zero is the only critical points of the phase function.
- Lemma 1.6: Here we introduce a sequence of Schwarz functions which converges in $\mathcal{S}'$

  $$e^{-\varepsilon x^2} e^{i\frac{1}{2}Q(x)} \xrightarrow{\varepsilon\to 0} e^{i\frac{1}{2}Q(x)}.$$

  Between the second row and the third row in the aligned equations, we have used

  $$\mathcal{F}(e^{i\frac{1}{2}Q(x)}e^{-\varepsilon x({}^tP^{-1}P^{-1})x})(P^{-1}\eta) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P^{-1}\eta} e^{i\frac{1}{2}Q(x)} e^{-\varepsilon x({}^tP^{-1}P^{-1})x}\,dx.$$

3 Day 3: April 25

- x ! x xi 0 .
- x : tempered distribution . tempered distribution sense .
- Some references only use $S^m = S^m_{1,0}$.
- Remakr 1.
- homogeneous polynomial euler relation(xi xi), a xi
- smoothing operator operator
- Theorem 2.1.

$$\langle \xi \rangle^{-2} \langle D_y \rangle^2 e^{i(x-y)\xi} = e^{i(x-y)\xi}$$

- Theorem 2.2. . rho delta m . Borel ( ) .
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4 Day 4: May 2

For a fixed cutoff $\chi \in \mathcal{S}$,

$$\partial^\alpha(\chi(\varepsilon x)a(x)) = \begin{cases} \partial^\alpha a(x), & \text{if } |x| < \varepsilon^{-1} \\ \partial^\alpha a(x) + O_\alpha(\varepsilon), & \text{if } \varepsilon^{-1} \le |x| < 2\varepsilon^{-1} \\ 0, & \text{if } |x| \ge 2\varepsilon^{-1} \end{cases}.$$

and

$$\partial^\alpha([1-\chi(\varepsilon x)]a(x)) = \begin{cases} 0, & \text{if } |x| < \varepsilon^{-1} \\ \partial^\alpha a(x) + O_\alpha(\varepsilon), & \text{if } \varepsilon^{-1} \le |x| < 2\varepsilon^{-1} \\ \partial^\alpha a(x), & \text{if } |x| \ge 2\varepsilon^{-1} \end{cases}.$$

We have if

$$|\partial_\xi^\beta a(\xi)| \lesssim \langle \xi \rangle^{m-\rho|\beta|},$$

then

$$|\partial_\xi^\beta(\chi(\varepsilon\xi)a(\xi))| \lesssim \langle \xi \rangle^{m-\rho|\beta|} + \varepsilon \quad \text{with} \quad \langle \xi \rangle \lesssim \varepsilon^{-1},$$

and

$$|\partial_\xi^\beta([1-\chi(\varepsilon\xi)]a(\xi))| \lesssim \langle \xi \rangle^{m-\rho|\beta|} + \varepsilon \quad \text{with} \quad \varepsilon^{-1} \lesssim \langle \xi \rangle,$$

Proof. Suppose we have a sequence $(a_j)_{j=0}^\infty$ of functions such that $a_j \in S^{m_j}_{\rho,\delta}(\mathbb{R}^{2d})$ with $m_j \downarrow -\infty$. Then, there exists $a \in S^{m_0}_{\rho,\delta}$ such that the asymptotic expansion formula holds:

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{\rho,\delta}.$$

In addition, we have a uniqueness result and a support description.

Note that

$$a(x,\xi) = \sum_{j=0}^\infty a_j(x,\xi)$$

will not converges, so we introduce a cutoff function χ to define

$$a(x,\xi) := \sum_{j=0}^\infty [1-\chi(\varepsilon_j\xi)]a_j(x,\xi)$$

with $\varepsilon_j \downarrow 0$ so that the summation is locally finite. Then, the error of the expansion formula is decomposed into

$$a - \sum_{j=0}^{k-1} a_j = -\sum_{j=0}^{k-1} \chi(\varepsilon_j\xi)a_j(x,\xi) + \sum_{j=k}^\infty [1-\chi(\varepsilon_j\xi)]a_j(x,\xi).$$

$$|\partial_x^\alpha\partial_\xi^\beta([1-\chi(\varepsilon_j\xi)]a_j(x,\xi))| \lesssim \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|} + \varepsilon_j \quad \text{with} \quad \varepsilon_j^{-1} \lesssim \langle \xi \rangle$$

The first term vanishes

□

- formal adjoint: transpose
-
- y eta , xi . $(y,\eta) = 0$ stationary χ_1 0 .