Smooth Manifolds

Ikhan Choi

August 29, 2022

Contents

I Smooth manifolds 3

1 Smooth structures 4
1.1 Local coordinate systems . . . 4
1.2 Tangent spaces . . . 5
1.3 Differentials . . . 5

2 Tensor fields 7
2.1 Vector fields . . . 7
2.2 Tensor fields of higher order . . . 8
2.3 Differential forms . . . 8
2.4 Lie derivatives . . . 8

3 Submanifolds 9
3.1 Constant rank theorem . . . 9
3.2 Embeddings . . . 11
3.3 Distributions . . . 11

II Riemannian manifolds 12

4 Intrinsic geometry 13
4.1 Covariance and contravariance . . . 13
4.2 Theorema Egregium . . . 13

5 Covariant derivatives 14
5.1 Orthogonal projection . . . 14
5.2 Connection . . . 15
5.3 Curvature tensor . . . 15

6 Parallel transport 16

III Local theory of curves and surfaces 17

7 Local theory of curves 18
7.1 Parametrization . . . 18
7.2 Frenet-Serret frame . . . 18
7.3 Computational problems . . . 20
7.4 General problems . . . 20

8 Local theory of surfaces **25**
8.1 Reparametrization 25
8.2 Differentiation of tangent vectors 25
8.3 Differentiation of normal vector 26
8.4 Computational problems 27
8.5 General problems 28

9 Geodesics **30**

IV Global theory of curves and surfaces **31**

10 Global theory of curves **32**
10.1 Isoperimetric inequality 32
10.2 Four vertex theorem 32
10.3 Ovals 32

11 Global theory of surfaces **33**
11.1 Minimal surfaces 33
11.2 Classification of compact surfaces 33
11.3 The Hilbert theorem 33

12 Total curvatures **34**
12.1 The Fary-Minor theorem 34
12.2 The Gauss-Bonnet theorem 34

Part I

Smooth manifolds

Chapter 1

Smooth structures

1.1 Local coordinate systems

1.1 (Local coordinates)**.** Let M be a topological space and $p \in M$ a point. Consider a fixed positive integer m. An m-dimensional (local) *coordinate system*, or (local) *chart*, at p is a pair (U, φ) consisting of an open neighborhood U of p and a topological embedding $\varphi : U \to \mathbb{R}^m$. The embedding φ is called a *coordinate map*, and each component of φ with respect to a basis of $\mathbb{R}^m$ is called a *coordinate function*.

An m-dimensional *atlas* on M is an indexed family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ of m-dimensional local charts such that every point is contained in some U_α, that is, $\{U_\alpha\}_\alpha$ is a cover of M. In geography, an atlas means a book of maps of Earth. A term *locally Euclidean space* is sometimes used to refer a topological space M together with an m-dimensional atlas.

(a) Let $U = \{(x, y) \in \mathbb{R}^2 : x \neq 0 \text{ or } y > 0\}$. For two functions $r, \theta : U \to \mathbb{R}$ defined by

$$r(x, y) := \sqrt{x^2 + y^2}, \quad \theta(x, y) := 2\tan^{-1} \frac{y}{x + \sqrt{x^2 + y^2}},$$

the map

$$U \to \mathbb{R}^2 : (x, y) \mapsto (r(x, y), \theta(x, y))$$

is a coordinate map, where $\tan^{-1}(t) := \int_0^t (1 + s^2)^{-1}\, ds$.

1.2 (Smooth atlases)**.** Let M be a topological space and m a positive integer. A *smooth atlas* on M is an atlas $\mathcal{A}$ on M such that every *transition map*

$$\tau_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth for all $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{A}$. Let $\mathcal{A}$ be a smooth atlas on M. Then, we can define the *smoothness* of a function $f : M \to \mathbb{R}$ with respect to $\mathcal{A}$ as follows: we say f is smooth if its *coordinate representation*

$$f \circ \varphi^{-1} : \varphi_\alpha(U) \to \mathbb{R}$$

is smooth for all $(U, \varphi) \in \mathcal{A}$.

Two smooth atlas $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. A *smooth structure* on M is a maximal smooth atlas $\mathcal{A}$; there is no smooth atlas $\mathcal{A}'$ that contains $\mathcal{A}$ properly.

(a) For a given smooth atlas, every transition map is a diffeomorphism.

(b) If two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, then a function $f : M \to \mathbb{R}$ is smooth with respect to $\mathcal{A}_1$ if and only if it is smooth with respect to $\mathcal{A}_2$.

(c) There is a one-to-one correspondence between smooth structures and equivalence classes of smooth atlases. Therefore, we can describe a smooth structure by giving a particular smooth atlas.

1.3 (Manifolds)**.** A *topological manifold* is defined as a second-countable and Hausdorff space together with a maximal atlas, and a *smooth manifold* is defined as a second-countable and Hausdorff space together with a smooth structure. The term *manifold* may refer to any of either a topological or a smooth manifold, which depends on contexts of each reference.

(a) The long line admits a smooth structure, and it is Hausdorff but not second countable.

(b) The line with two origins admits a smooth structure, and it is second countable but not Hausdorff.

1.4 (Partition of unity)**.**

1.5 (Smooth maps and diffeomorphisms)**.** scalar functions, scalar fields

1.6 (Embedded manifolds)**.** a *embedded manifold* or a *regular manifold*. *parametrization*

If $\alpha : U \to \mathbb{R}^n$ is a topological embedding, then we can endow with a unique smooth structure on $\operatorname{im}\alpha$ such that α is smooth.(?)

(a) The image of a regular parameterization is an embedded manifold.

(b) Every open subset of a embedded manifold is a embedded manifold.

(c) Monge patch.

(d) The sphere $S^2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ is a regular surface.

(e) The set $\{ (x,y) \in \mathbb{R}^2 : y^2 = x^3 + x^2 \}$ is not a regular curve.

(f) The set $\{ (x,y) \in \mathbb{R}^2 : y = |x| \}$ is not a regular curve.

1.2 Tangent spaces

1.7 (Tangent spaces of embedded manifolds)**.** Let M be an m-dimensional embedded manifold in $\mathbb{R}^n$. For a point $p \in M$, take a parameterization α for M at p, and let $x := \alpha^{-1}(p)$ be the coordinates of p. The *tangent space* T_pM of M at p is defined as the image of $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$.

(a) T_pM is a m-dimensional vector subspace of $\mathbb{R}^n$ with a basis $\{\partial_i \alpha(x)\}_{i=1}^m$.

(b) If $v \in T_pM$, then we have a smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(c) If we have a smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, then $v \in T_pM$.

(d) The definition of T_pM is independent on the parameterization α.

1.8 (Tangent spaces as equivalence classes of curves)**.**

1.9 (Tangent spaces as derivations)**.**

the space of derivations on the ring of smooth functions, the dual space of algebraically defined cotangent spaces.

1.3 Differentials

Exercises

1.10 (Smooth structure on spheres)**.** Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface given by

$$\alpha(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2} \right).$$

This map gives a parametrization for the sphere S^2 without the north pole $(0,0,1)$, and is called the *stereographic projection*. Let $f : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}$ be the height function of α defined by

$$f(p) := z$$

for $p = (x, y, z) \in S^2 \setminus \{(0,0,1)\}$. Its coordinate representation is

$$f \circ \alpha(x, y) = 1 - \frac{2}{1 + x^2 + y^2}.$$

Then, the directional derivative is

$$\partial_x f = \frac{\partial(f \circ \alpha)}{\partial x} = \frac{\partial}{\partial x}\left(1 - \frac{2}{1 + x^2 + y^2}\right) = \frac{4x}{(1 + x^2 + y^2)^2}.$$

Note that $\partial_x f \neq \partial_{(1,0,0)} z = 0$.

(a) The minimal cardinality of a smooth atlas on S^n is two.

1.11 (Smooth structure on projective spaces).

1.12 (Stiefel and Grassmann varieties).

1.13 (Parallelization of spheres).

1.14 (Tagent space of matrix groups). Jacobi formula

1.15 (Recovery of compact smooth manifolds). Let M be a compact smooth manifold. C^∞ functor is a fully faithful contravariant functor.

(a) Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Chapter 2

Tensor fields

2.1 Vector fields

2.1 (Vector fields)**.** Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization with $M = \operatorname{im}\alpha$. A *vector field* is a map $X : M \to \mathbb{R}^n$ such that $X \circ \alpha : U \to \mathbb{R}^n$ is smooth. A *tangent vector field* is a vector field $X : M \to \mathbb{R}^n$ such that $X|_p \in T_pM$. The set of tangent vector fields is often denoted by $\mathfrak{X}(M)$.

2.2. Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization $M = \operatorname{im}\alpha$.

(a) The coordinate representation of a function $f : M \to \mathbb{R}$ is

$$f \circ \alpha : U \to \mathbb{R}.$$

(b) The (external) coordinate representation of a vector field $X : M \to \mathbb{R}^n$ is

$$X \circ \alpha : U \to \mathbb{R}^n.$$

(c) The coordinate representation of a tangent vector field $X : M \to \mathbb{R}^n$ is

$$(X^1 \circ \alpha, \cdots, X^m \circ \alpha) : U \to \mathbb{R}^m$$

where $X = \sum_i X^i \alpha_i$.

2.3. Let α be an m-dimensional parametrization with $M = \operatorname{im}\alpha$. The value of $\partial_i \alpha = \alpha_i : M \to \mathbb{R}^3$ is always a tanget vector at each point $p = \alpha(x)$, and α_i becomes a vector field.

Let s be either a smooth function or vector field on α. Then, we can compute the directional derivative as

$$\partial_i s := \partial_i(s \circ \alpha) = \partial_t(s \circ \gamma)$$

by taking $\gamma(t) = \alpha(x + te_i)$, where e_i is the i-th standard basis vector for $\mathbb{R}^m$.

2.4. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. Let $v = \sum_i v^i \alpha_i|_p \in T_pM$ be a tangent vector at $p = \alpha(x)$. For a function $f : M \to \mathbb{R}$, its partial derivative is defined by

$$\partial_v f(p) := \sum_{i=1}^m v^i \partial_i (f \circ \alpha)(x) \in \mathbb{R}.$$

For a vector field $X : M \to \mathbb{R}^n$, its partial derivative is defined by

$$\partial_v X|_p := \sum_{i=1}^m v^i \partial_i (X \circ \alpha)(x) \in \mathbb{R}^n.$$

This definition is not dependent on parametrization α.

2.5. Let M be the image of a parametrization. Let X be a tangent vector field on M.

(a) If f is a function, then so is $\partial_X f$.

(b) If Y is a vector field, then so is $\partial_X Y$.

(c) If Y is a tangent vector field, then so is $\partial_X Y - \partial_Y X$.

Proof. (a) and (b) are clear. For (c), if we let $X = \sum_i X^i \alpha_i$ and $Y = \sum_j Y^j \alpha_j$ for a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$, then

$$\begin{aligned}
\partial_X Y - \partial_Y X &= \partial_X(\textstyle\sum_j Y^j \alpha_j) - \partial_Y(\sum_i X^i \alpha_i) \\
&= \textstyle\sum_j [(\partial_X Y^j)\alpha_j + Y^j \partial_X \alpha_j] - \sum_i [(\partial_Y X^i)\alpha_i + X^i \partial_Y \alpha_i] \\
&= \textstyle\sum_j [(\partial_X Y^j)\alpha_j + Y^j \sum_i X^i \partial_i \alpha_j] - \sum_i [(\partial_Y X^i)\alpha_i + X^i \sum_j Y^j \partial_i \alpha_j] \\
&= \textstyle\sum_j (\partial_X Y^j)\alpha_j - \sum_i (\partial_Y X^i)\alpha_i \\
&= \textstyle\sum_i (\partial_X Y^i - \partial_Y X^i)\alpha_i.
\end{aligned}$$

□

2.6. Let M be the image of a parametrization α. For derivatives of functions on M by tangent vectors, we will use

$$\partial_{\alpha_i} f = \partial_i f, \quad \partial_{\alpha_t} f = \partial_t f = f', \quad \partial_{\alpha_x} f = \partial_x f = f_x.$$

For derivatives of vector fields on M by tangent vectors, we will use

$$\partial_{\alpha_i} X = \partial_i X, \quad \partial_{\alpha_t} X = \partial_t X = X', \quad \partial_{\alpha_x} X = \partial_x X = X_x.$$

We will *not* use f_i or X_i for $\partial_i f$ and $\partial_i X$ because it is confusig with coordinate representations, and *not* use the nabula symbol ∇_v in this sense because it will be devoted to another kind of derivatives introduced in Section 4.

2.2 Tensor fields of higher order

tensor bundle tensor fields,

2.3 Differential forms

forms, exterior structures, pullback, interior product

2.4 Lie derivatives

2.7 (Integral curves).

Exercises

2.8 (Orientation).

Chapter 3

Submanifolds

3.1 Constant rank theorem

3.1 (Constant rank theorem)**.** Let M and N be smooth manifolds of dimensions m and n, and $f : M \to N$ a smooth map. Let $p \in M$ and $q \in N$ such that $f(p) = q$. For each pair of local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$, we can introduce functions $a : \varphi(U) \to \mathbb{R}^k$ and $b : \varphi(U) \to \mathbb{R}^{n-k}$ such that the coordinate representation $\widetilde{f} : \varphi(U) \to \psi(V)$ of f is written as

$$\widetilde{f}(x, y) := \psi \circ f \circ \varphi^{-1}(x, y) = (a(x, y), b(x, y))$$

for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{m-k}$ with $(x, y) \in \varphi(U)$. Then, the differential df on U is represented by its Jacobian matrix

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix}.$$

Suppose the differential of f has a locally constant rank k at p.

(a) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and $\partial a/\partial x$ is a $k \times k$ invertible matrix everywhere.

(b) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & 0 \end{pmatrix}.$$

(c) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and $\widetilde{f}(x, y) = (x, 0)$.

Proof. (a) Let (U, φ) and (V, ψ) be local charts at p and q such that $f(U) \subset V$ and the Jacobian matrix $D\widetilde{f}|_{(x,y)}$ is of rank k for every $(x, y) \in \varphi(U)$. For each $(x, y) \in \varphi(U)$, the matrix $D\widetilde{f}|_{(x,y)}$ has an invertible $k \times k$ minor submatrix. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ be permutation matrices that reorder the coordinates in such a way that the invertible $k \times k$ minor submatrix becomes the leading principal minor submatrix.

Define reparametrizations $\varphi' := A \circ \varphi : U \to A(\varphi(U))$ and $\psi' := B \circ \psi : V \to B(\psi(V))$. Then, they are clearly local charts and

$$D(\psi' \circ f \circ \varphi'^{-1}) = D(B \circ \psi \circ f \circ \varphi^{-1} \circ A^{-1}) = B \circ D\widetilde{f} \circ A^{-1}$$

has an invertible leading principal minor submatrix of dimension $k \times k$ at every $(x,y) \in \varphi(U)$.

(b) Let (U,φ) and (V,ψ) be local charts at p and q satisfying the conditions given in the part (a). Consider a map $F : \varphi(U) \to \mathbb{R}^m$ defined by

$$F(x,y) := (a(x,y), y).$$

Then, since

$$DF|_{(x,y)} = \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix}$$

is smooth and invertible everywhere on $\varphi(U)$, there exists an open neighborhood $\varphi(U') \subset \varphi(U)$ of $\varphi(p)$ such that the restriction $F : \varphi(U') \to F(\varphi(U'))$ is a diffeomorphism by the inverse function theorem.

Define a reparamterization $\varphi' := F \circ \varphi : U' \to F(\varphi(U'))$. Then, it is clearly a local chart and

$$\begin{aligned} D(\psi \circ f \circ \varphi'^{-1}) &= D(\psi \circ f \circ \varphi^{-1} \circ F^{-1}) = D\widetilde{f} \circ (DF)^{-1} \\ &= \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial a}{\partial x}\right)^{-1} & -\left(\frac{\partial a}{\partial x}\right)^{-1} \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & 0 \end{pmatrix}. \end{aligned}$$

The last equality holds because the transpose of this matrix has rank k, and the conditions are satisfied with the local charts (U',φ') and (V,ψ).

(c) Let (U,φ) and (V,ψ) be local charts at p and q satisfying the conditions given in the part (b). Then, we have $\widetilde{f}(x,y) = (x, b(x))$ for all $(x,y) \in \varphi(U)$. Consider a map $G : \psi(V) \to \mathbb{R}^n$ defined by

$$G(x,z) := (x, z - b(x)).$$

Then, since

$$DG|_{(x,z)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ -\frac{\partial b}{\partial x} & \mathrm{id}_{n-k} \end{pmatrix}$$

is smooth and invertible everywhere on $\psi(V)$, there exists an open neighborhood $\psi(V') \subset \psi(V)$ of $\psi(q)$ such that the restriction $G : \psi(V') \to G(\psi(V'))$ is a diffeomorphism by the inverse function theorem.

Define a reparamterization $\psi' := G \circ \psi : V' \to G(\psi(V'))$. Then, it is clearly a local chart and

$$\begin{aligned} D(\psi' \circ f \circ \varphi^{-1}) &= D(G \circ \psi \circ f \circ \varphi^{-1}) = DG \circ D\widetilde{f} \\ &= \begin{pmatrix} \mathrm{id}_k & 0 \\ -\frac{\partial b}{\partial x} & \mathrm{id}_{n-k} \end{pmatrix} \begin{pmatrix} \mathrm{id}_k & 0 \\ \frac{\partial b}{\partial x} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, the conditions are satisfied with the local charts (U,φ) and (V',ψ').

(d) Let (U,φ) and (V,ψ) be local charts at p and q satisfying the conditions given in the part (c). Then, by translating constants for these local coordinate systems, we obtain $\widetilde{f}(x,y) = (x,0)$. □

3.2 (Preimage theorem). Let M and N are smooth manifolds of dimensions m and n. Let $f : M \to N$ be a smooth map. A *critical point* is a point $p \in M$ such that $df|_p$ is not surjective, and a *critical value* is a point $q \in N$ such that $f(p) = q$ for some critical point p. If $q \in N$ is not a critical value, then it is called a *regular value*.

Suppose $q \in N$ is a regular value of f, and $p \in M$ be any points satisfying $f(p) = q$. We will show that $f^{-1}(q)$ is an embedded submanifold of M. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have local charts (U,φ) and (V,ψ) at p and q such that

$$\varphi(p) = (0,0) \in \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad \psi(q) = 0 \in \mathbb{R}^n, \quad \text{and} \quad \widetilde{f}(x,y) = x.$$

(a) $(U \cap f^{-1}(q), \varphi|_{U\cap f^{-1}(q)})$ is an $(m-n)$-dimensional chart at p on $f^{-1}(q)$.

(b) The charts of the form $(U \cap f^{-1}(q), \varphi|_{U\cap f^{-1}(q)})$ defines a smooth atlas.

(c) The inclusion is an embedding.

Proof. (a) Note that every open subset of $U \subset f^{-1}(q)$ is of the form $W \cap f^{-1}(q)$ for an open set $W \subset U$. Since $\varphi(W)$ is open in $\mathbb{R}^m$ for any open $W \subset U$,

$$\begin{aligned}\varphi(W \cap f^{-1}(q)) &= \varphi(W) \cap \varphi(f^{-1}(q))\\ &= \varphi(W) \cap \widetilde{f}^{-1}(\psi(q))\\ &= \varphi(W) \cap \widetilde{f}^{-1}(0)\\ &= \varphi(W) \cap (\{0\} \times \mathbb{R}^{m-n})\end{aligned}$$

is open in $\{0\} \times \mathbb{R}^{m-n}$. It means that the restriction of φ on $U \cap f^{-1}(q)$ is an injective open map, so it is a topological embedding into the Euclidean space $\{0\} \times \mathbb{R}^{m-n}$.

□

3.2 Embeddings

3.3 (Immersion is a local embedding)**.** Let $f : M \to N$ be an immersion at $p \in M$. Then, there is a local chart (V, ψ) at $f(p)$ such that

(a) $W = f(M) \cap V$ is an embedded submanifold of V,

(b) there is a retract $V \to W$.

Proof. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have

$$\varphi(p) = 0 \in \mathbb{R}^m, \quad \psi(f(p)) = (0,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad \text{and} \quad \widetilde{f}(x) = (x,0).$$

Let $W := f(M) \cap V$. Then, the injectivity of φ shows that

$$\psi(W) = \psi(f(U)) = \psi \circ f \circ \varphi^{-1}(\varphi(U)) = \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : x \in \varphi(U)\}$$

is an open subset of $\mathbb{R}^m$, so $(W, \psi|_W)$ is a chart at $f(p)$.

Transition maps are smooth?

The inclusion is a smooth embedding? □

3.4 (Extension of smooth functions)**.** from an embedded manifold.

Let $f : M \to N$ be an injective immersion. There exists unique smooth structure on $f(M)$ such that f and i are smooth.

Let $f : M \to N$ be an embedding. There exists unique smooth structure on $f(M)$ such that i are smooth.

3.3 Distributions

3.5 (Foliation)**.**

Part II

Riemannian manifolds

Chapter 4

Intrinsic geometry

We say a quantity on a surface is *intrinsic* if it is independent of how the surface is embedded in space.

Notations: Einstein summation convention, set of vector fields.

To n-dimensional.

4.1 Covariance and contravariance

4.2 Theorema Egregium

- Intrinsic: g_{ij}, Γ_{ij}^k, K, $R^l{}_{ijk}$;
- Not intrinsic: ν, L_{ij}, κ_i, H.

Isometry

Example 4.2.1. Let $\alpha : (-\log 2, \log 2)\times(0, 2\pi) \to \mathbb{R}^3$ and $\beta : (-\frac{3}{4}, \frac{3}{4})\times(0, 2\pi) \to \mathbb{R}^3$ be regular surfaces given by

$$\alpha(x,\theta) = (\cosh x \cos\theta,\ \cosh x \sin\theta,\ x), \qquad \beta(r,z) = (r\cos z,\ r\sin z,\ z).$$

Their Riemannian metrics are

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix}_{(\alpha_x,\alpha_\theta)}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1+r^2 \end{pmatrix}_{(\beta_r,\beta_z)}.$$

Define a map $f : \operatorname{im}\alpha \to \operatorname{im}\beta$ by

$$f : \alpha(x,\theta) \mapsto \beta(\sinh x, \theta) = (r(x,\theta), z(x,\theta)).$$

The Jacobi matrix of f is computed

$$df|_{\alpha(x,\theta)} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}_{(\alpha_x,\alpha_\theta)\to(\beta_r,\beta_z)}.$$

Since f is a diffeomorphism and

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1+r^2 \end{pmatrix} \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix},$$

the map f is an isometry.

Chapter 5

Covariant derivatives

5.1 Orthogonal projection

We are going to think about "intrinsic" derivatives for tangent vectors. For coordinate independence, directional derivatives of a tangent vector field should be at least a tangent vector field, which is false for the obvious partial derivatives in the embedded surface setting; for example, T is a tangent vector, but $\mathrm{N} = \kappa \mathrm{T}'$ is not tangent.

Recall that the Gauss formula reads

$$\partial_i \alpha_j = \Gamma_{ij}^k \alpha_k + L_{ij} \nu$$

so that we have

$$\begin{aligned}
\partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\
&= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\
&= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij} \nu.
\end{aligned}$$

If we write $\nabla_X Y = \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k$, then it embodies the orthogonal projection of $\partial_X Y$ onto its tangent space, and we have

$$\partial_X Y = \nabla_X Y + \mathrm{II}(X, Y) \nu.$$

Definition 5.1.1. Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $\operatorname{im} \alpha = M$. Let $X = X^i \alpha_i$ and $Y = Y^j \alpha_j$ be tangent vector fields on M. The *covariant derivative* of Y along X is defined as the orthogonal projection of the partial derivative $\partial_X Y$ onto the tangent space:

$$\nabla_X Y := \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k.$$

Proposition 5.1.1. *Covariant derivatives are intrinsic. In other words, the above definition does not depend on the choice of parametrizations.*

Proof. Recall that the Christoffel symbols transform as follows:

$$X^i Y^j \Gamma_{ij}^k = X^a Y^b \left(\Gamma_{ab}^c + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} \right) \frac{\partial x^k}{\partial x^c}.$$

Thus, we have

$$\begin{aligned}
&\left(X^i\partial_i Y^k + X^i Y^j \Gamma_{ij}^k\right)\alpha_k \\
&= X^a \frac{\partial}{\partial x^a}\left(Y^c \frac{\partial x^k}{\partial x^c}\right)\alpha_k + X^a Y^b \left(\frac{\partial x^i}{\partial x^a}\frac{\partial x^j}{\partial x^b}\frac{\partial^2 x^c}{\partial x^i \partial x^j} + \Gamma_{ab}^c\right)\frac{\partial x^k}{\partial x^c}\alpha_k \\
&= X^a \frac{\partial Y^c}{\partial x^a}\alpha_c + X^a Y^b \left(\frac{\partial^2 x^k}{\partial x^a \partial x^b}\frac{\partial x^c}{\partial x^k} + \frac{\partial x^i}{\partial x^a}\frac{\partial x^j}{\partial x^b}\frac{\partial^2 x^c}{\partial x^i \partial x^j}\right)\alpha_c + X^a X^b \Gamma_{ab}^c \alpha_c \\
&= \left(X^a \partial_a Y^c + X^a Y^b \Gamma_{ab}^c\right)\alpha_c
\end{aligned}$$

since

$$\frac{\partial^2 x^j}{\partial x^a \partial x^b}\frac{\partial x^c}{\partial x^j} + \frac{\partial x^i}{\partial x^a}\frac{\partial x^j}{\partial x^b}\frac{\partial^2 x^c}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^a}\left(\frac{\partial x^j}{\partial x^b}\frac{\partial x^c}{\partial x^j}\right) = \partial_a \delta_b^c = 0. \qquad \square$$

5.2 Connection

5.1 (Affine connection)**.** Let M be a smooth manifold An *affine connection* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$$

such that

(i) $C^\infty(M)$-linear in the first argument X,

(ii) the *Leibniz rule*

$$\nabla_X(fY) = XfY + f\nabla_X Y$$

for $f \in C^\infty(M)$ in the second argument Y is satisfied.

5.2 (Levi-Civita connection)**.** Let M be a Riemannian manifold. A *metric connection* is an affine connection ∇ such that $\nabla g = 0$. A *Levi-Civita connection* is a metric connection ∇ such that $\nabla T = 0$.

(a) ∇ is a metric connection if and only if $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$.

(b) ∇ is a Levi-Civita connection if and only if $\nabla_X Y - \nabla_Y X = [X, Y]$.

(c) There exists a unique Levi-Civita connection on M.

Proof. (Uniqueness) Suppose ∇ is a Levi-Citiva connection on M.

$$\begin{aligned}
2\langle \nabla_X Y, Z\rangle = \partial_X \langle Y, Z\rangle + \partial_Y \langle X, Z\rangle - \partial_Z \langle X, Y\rangle \\
- \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle.
\end{aligned}$$

(Existence) $\square$

5.3. Let S be a regular surface embedded in $\mathbb{R}^3$. If we define Christoffel symbols as the Gauss formula, then

$$\mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S) : (X^i \alpha_i, Y^j \alpha_j) \mapsto \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k\right)\alpha_k$$

defines a Levi-Civita connection.

5.4 (Connection form)**.**

5.3 Curvature tensor

Chapter 6

Parallel transport

Part III

Local theory of curves and surfaces

Chapter 7

Local theory of curves

7.1 Parametrization

By definition, a regular curve has at least one parametrization. However, a given parametrization may not have useful properties, so we often take a new parametrization. The existence of a parametrization with certain properties is one of the main problems in differential geometry. Practically, the existence proof is usually done by constructing a *diffeomorphism* between open sets in $\mathbb{R}^m$; a bijective smooth map whose inverse is also smooth.

We introduce the arc-length reparametrization. It is the most general choice for the local study of curves.

Definition 7.1.1. A parametrization α of a regular curve is called a *unit speed curve* or an *arc-length parametrization* when it satisfies $\|\alpha'\| = 1$.

Theorem 7.1.1. *Every regular curve may be assumed to have unit speed. Precisely, for every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. By the definition of regular curves, we can take a parametrization $\beta : I_t \to \mathbb{R}^d$ for a given regular curve. We will construct an arc-length parametrization from β.

Define $\tau : I_t \to I_s$ such that

$$\tau(t) := \int_0^t \|\beta'(s)\|\, ds.$$

Since τ is smooth and $\tau' > 0$ everywhere so that τ is strictly increasing, the inverse $\tau^{-1} : I_s \to I_t$ is smooth by the inverse function theorem; τ is a diffeomorphism. Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta'\left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}.$$

□

7.2 Frenet-Serret frame

The Frenet-Serret frame is a standard frame for a curve, and it is in particular effective when we assume the arc-length parametrization. It is defined for nondegenerate regular curves, i.e. nowhere straight curves. It provides with a useful orthonormal basis of $T_p\mathbb{R}^3 \supset T_p\gamma(I)$ for points p on a regular curve $\gamma : I \to \mathbb{R}^3$.

7.1. A regular curve $\gamma : I \to \mathbb{R}^3$ is called *non-degenerate* if the normalized tangent vector $\gamma'/\|\gamma'\|$ is never locally constant everywhere. In other words, γ is nowhere straight.

Definition 7.2.1 (Frenet-Serret frame)**.** Let α be a nondegenerate curve. The *tangent unit vector*, *normal unit vector*, *binormal unit vector* are $T_p\mathbb{R}^3$-valued vector fields on α defined by:

$$\mathrm{T}(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad \mathrm{N}(t) := \frac{\mathrm{T}'(t)}{\|\mathrm{T}'(t)\|}, \qquad \mathrm{B}(t) := \mathrm{T}(t) \times \mathrm{N}(t).$$

The set of vector fields $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$, which is called *Frenet-Serret frame*, forms an orthonormal basis of $T_p\mathbb{R}^3$ at each point p on α. The Frenet-Serret frame is uniquely determined up to sign as α changes.

We study the derivatives of the Frenet-Serret frame and their coordinate representations. In the coordinate representations on the Frenet-Serret frame, important geometric measurements such as curvatrue and torsion come out as coefficients.

Definition 7.2.2. Let α be a nondegenerate curve. The *curvature* and *torsion* are scalar fields on α defined by:

$$\kappa(t) := \frac{\langle \mathrm{T}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}, \quad \tau(t) := -\frac{\langle \mathrm{B}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}.$$

Note that $\kappa > 0$ cannot vanish by definition of nondegenerate curve. This definition is independent on α.

7.2. *Frenet-Serret formula.* Let γ be a non-degenerate regular curve. Then,

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \\ \mathrm{B}' \end{pmatrix} = \|\gamma'\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \\ \mathrm{B} \end{pmatrix}.$$

(a) $\mathrm{T}' = \|\gamma'\|\kappa\mathrm{N}$.

(b) $\mathrm{B}' = -\|\gamma'\|\tau\mathrm{N}$.

(c) $\mathrm{N}' = -\|\gamma'\|\kappa\mathrm{T} + \|\gamma'\|\tau\mathrm{B}$.

Proof. Note that $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$ is an orthonormal basis.

(a) Two vectors T' and N are parallel by definition of N. By the definition of κ, we get $\mathrm{T}' = \|\gamma'\|\kappa\mathrm{N}$.

(b) Since $\langle \mathrm{T}, \mathrm{B}\rangle = 0$ and $\langle \mathrm{B}, \mathrm{B}\rangle = 1$ are constant, we have

$$\langle \mathrm{B}', \mathrm{T}\rangle = \langle \mathrm{B}, \mathrm{T}\rangle' - \langle \mathrm{B}, \mathrm{T}'\rangle = 0, \qquad \langle \mathrm{B}', \mathrm{B}\rangle = \tfrac{1}{2}\langle \mathrm{B}, \mathrm{B}\rangle' = 0.$$

By the definition of τ, we get $\mathrm{B}' = -\|\alpha'\|\tau\mathrm{N}$.

(c) Since

$$\begin{aligned}
\langle \mathrm{N}', \mathrm{T}\rangle &= -\langle \mathrm{N}, \mathrm{T}'\rangle = -\|\alpha'\|\kappa, \\
\langle \mathrm{N}', \mathrm{N}\rangle &= \tfrac{1}{2}\langle \mathrm{N}, \mathrm{N}\rangle' = 0, \\
\langle \mathrm{N}', \mathrm{B}\rangle &= -\langle \mathrm{N}, \mathrm{B}'\rangle = \|\alpha'\|\tau,
\end{aligned}$$

we have

$$\mathrm{N}' = \|\alpha'\|(-\kappa\mathrm{T} + \tau\mathrm{B}).$$

□

Remark. Let $\mathrm{X}(t)$ be the curve of orthogonal matrices $(\mathrm{T}(t), \mathrm{N}(t), \mathrm{B}(t))^T$. Then, the Frenet-Serret formula reads

$$\mathrm{X}'(t) = A(t)\mathrm{X}(t)$$

for a matrix curve $A(t)$ that is completely determined by $\kappa(t)$ and $\tau(t)$, if we let us only consider arc-length parametrized curves. This is a typical form of an ODE system, so we can apply the Picard-Lindelöf theorem to get the following proposition: if we know $\kappa(t)$ and $\tau(t)$ for all time t, and if $\mathrm{T}(0)$ and $\mathrm{N}(0)$ are given so that an initial condition

$$\mathrm{X}(0) = (\mathrm{T}(0), \mathrm{N}(0), \mathrm{T}(0) \times \mathrm{N}(0))$$

is established, then the solution $X(t)$ exists and uniquely determined in a short time range. Furthermore, if $\alpha(0)$ is given in addition, the integration

$$\alpha(t) = \alpha(0) + \int_0^t T(s)\, ds$$

provides a complete formula for unit speed parametrization α.

Remark. Skew-symmetry in the Frenet-Serret formula is not by chance. Let $X(t) = (T(t), N(t), B(t))^T$ and write $X'(t) = A(t)X(t)$ as we did in the above remark. Since $X(t+h) = R_t(h)X(t)$ for a family of special orthogonal matrices $\{R_t(h)\}_h$ with $R_t(0) = I$, we can describe $A(t)$ as

$$A(t) = \left.\frac{dR_t}{dh}\right|_{h=0}.$$

By differentiating the relation $R_t^T(h)R_t(h) = I$ with respect to h, we get to know that $A(t)$ is skew-symmetric for all t. In other words, the tangent space $T_I SO(3)$ forms a skew symmetric matrix.

7.3 Computational problems

The following proposition gives the most effective and shortest way to compute the Frenet-Serret apparatus in general case. If we try to reparametrize the given curve into a unit speed curve or find κ by differentiating T, then we must encounter the normalizing term of the form $\sqrt{(-)^2 + (-)^2 + (-)^2}^{\,-1}$, and it must be painful when time is limited. The Frenet-Serret frame is useful in proofs of interesting propositions, but not a good choice for practical computation. Instead, a computation from derivatives of parametrization is highly recommended.

Proposition 7.3.1. *Let α be a nondegenerate curve. Then,*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}$$

and

$$T = \frac{\alpha'}{\|\alpha'\|}, \qquad B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \qquad N = B \times T.$$

Proof. If we let $s = \|\alpha'\|$, then

$$\begin{aligned} \alpha' &= sT, \\ \alpha'' &= s'T + s^2\kappa N, \\ \alpha''' &= (s'' - s^3\kappa^2)T + (3ss'\kappa + s^2\kappa')N + (s^3\kappa\tau)B. \end{aligned}$$

Now the formulas are easily derived. □

7.4 General problems

We are interested in regular curves, not a particular parametrization. By the Theorem 2.1, we may always assume that a parametrization α has unit speed. Let α be a nondegenerate unit speed space curve, and let $\{T, N, B\}$ be the Frenet-Serret frame for α.

Consider a diagram as follows:

$$\begin{array}{ccccc} \langle \alpha, T\rangle = ? & \longleftrightarrow & \langle \alpha, N\rangle = ? & \longleftrightarrow & \langle \alpha, B\rangle = ? \\ \downarrow & & \downarrow & & \downarrow \\ \langle \alpha', T\rangle = 1 & & \langle \alpha', N\rangle = 0 & & \langle \alpha', B\rangle = 0. \end{array}$$

Here the arrows indicate which term we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t), \mathrm{T}(t)\rangle = f(t),$$

then we can obtain

$$\langle \alpha(t), \mathrm{N}(t)\rangle = \frac{f'(t)-1}{\kappa(t)}$$

by direct differentiation since we have known $\langle \alpha', \mathrm{T}\rangle$ but not $\langle \alpha, \mathrm{N}\rangle$. Further, we get

$$\langle \alpha(t), \mathrm{B}(t)\rangle = \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t)f(t)}{\tau(t)}$$

since we have known $\langle \alpha, \mathrm{T}\rangle$ and $\langle \alpha', \mathrm{N}\rangle$ but not $\langle \alpha, \mathrm{B}\rangle$. Thus, $\langle \alpha, \mathrm{T}\rangle = f$ implies

$$\alpha(t) = f(t)\cdot \mathrm{T} + \frac{f'(t)-1}{\kappa(t)}\cdot \mathrm{N} + \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t)f(t)}{\tau(t)}\cdot \mathrm{B},$$

when given $\tau(t) \neq 0$.

We suggest a strategy for space curve problems:

- Build and differentiate equations of the following form:

$$\langle \text{ (interesting vector), (Frenet-Serret basis) } \rangle \;=\; \text{(some function)}.$$

- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of κ and τ by comparing with assumptions.

- Heuristically find a constant vector and show what you want directly.

Here we give example solutions of several selected problems. Always α denotes a reparametrized unit speed nondegenerate curve in $\mathbb{R}^3$.

If

$$f = \langle \alpha - p, \mathrm{T}\rangle, \quad g = \langle \alpha - p, \mathrm{N}\rangle, \quad h = \langle \alpha - p, \mathrm{B}\rangle,$$

then

$$f' = 1 + \kappa g, \quad g' = -\kappa f + \tau h, \quad h' = -\tau g.$$

7.3. A curve whose normal lines always pass through a fixed point lies in a circle.

Solution. Step 1: Formulate conditions. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and N are parallel so that we have

$$\langle \alpha - p, \mathrm{T}\rangle = 0, \qquad \langle \alpha - p, \mathrm{B}\rangle = 0.$$

Our goal is to show that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v\rangle = 0$.

Step 2: Collect information. Differentiate $\langle \alpha - p, \mathrm{T}\rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N}\rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{B}\rangle = 0$ to get

$$\tau = 0.$$

Step 3: Complete proof. We can deduce that $\|\alpha - p\|$ is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p\rangle' = 2\langle \alpha - p, \mathrm{T}\rangle = 0.$$

Also, if we heuristically define a vector $v := \mathrm{B}$, then v is constant since

$$v' = -\tau \mathrm{N} = 0,$$

and clearly $\langle \alpha - p, v\rangle = 0$ □

7.4. A spherical curve of constant curvature lies in a circle.

Solution. Step 1: Formulate conditions. The condition that α lies on a sphere can be given as follows: for a constant point $p \in \mathbb{R}^3$,

$$\|\alpha - p\| = \text{const}.$$

Also we have

$$\kappa = \text{const}.$$

Step 2: Collect information. Differentiate $\|\alpha - p\|^2 = \text{const}$ to get

$$\langle \alpha - p, \mathrm{T} \rangle = 0.$$

Differentiate $\langle \alpha - p, \mathrm{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{N} \rangle = -1/\kappa = \text{const}$ to get

$$\tau \langle \alpha - p, \mathrm{B} \rangle = 0.$$

There are two ways to show that $\tau = 0$.

Method 1: Assume that there is t such that $\tau(t) \neq 0$. By the continuity of τ, we can deduce that τ is locally nonvanishing. In other words, we have $\langle \alpha - p, \mathrm{B} \rangle = 0$ on an open interval containing t. Differentiate $\langle \alpha - p, \mathrm{B} \rangle = 0$ at t to get $\langle \alpha - p, \mathrm{N} \rangle = 0$ near t, which is a contradiction. Therefore, $\tau = 0$ everywhere.

Method 2: Since $\langle \alpha - p, \mathrm{B} \rangle$ is continuous and

$$\langle \alpha - p, \mathrm{B} \rangle = \pm\sqrt{\|\alpha - p\|^2 - \langle \alpha - p, \mathrm{T} \rangle^2 - \langle \alpha - p, \mathrm{N} \rangle^2} = \pm \text{const},$$

we get $\langle \alpha - p, \mathrm{B} \rangle = \text{const}$. Differentiate to get $\tau \langle \alpha - p, \mathrm{N} \rangle = 0$. Finally we can deduce $\tau = 0$ since $\langle \alpha - p, \mathrm{N} \rangle \neq 0$.

Step 3: Complete proof. The zero torsion implies that the curve lies on a plane. A planar curve in a sphere is a circle. □

7.5. A curve such that $\tau/\kappa = (\kappa'/\tau\kappa^2)'$ lies on a sphere.

Solution. Step 1: Find the center heuristically. If we assume that α is on a sphere so that we have $\|\alpha - p\| = r$ for constants $p \in \mathbb{R}^3$ and $r > 0$, then by the routine differentiations give

$$\langle \alpha - p, \mathrm{T} \rangle = 0, \qquad \langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}, \qquad \langle \alpha - p, \mathrm{B} \rangle = -\left(\frac{1}{\kappa}\right)' \frac{1}{\tau},$$

that is,

$$\alpha - p = -\frac{1}{\kappa}\mathrm{N} - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\mathrm{B}.$$

Step 2: Complete proof. Let us get started the proof. Define

$$p := \alpha + \frac{1}{\kappa}\mathrm{N} + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\mathrm{B}.$$

We can show that it is constant by differentiation. Also we can show that

$$\langle \alpha - p, \alpha - p \rangle$$

is constant by differentiation. So we are done. □

7.6. A curve with more than one Bertrand mates is a circular helix.

Solution. Step 1: Formulate conditions. Let β be a Bertrand mate of α so that we have

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \mathrm{N}_\beta = \pm \mathrm{N},$$

where λ is a function not vanishing somewhere and $\{\mathrm{T}_\beta, \mathrm{N}_\beta, \mathrm{B}_\beta\}$ denotes the Frenet-Serret frame of β. We can reformulate the conditions as follows:

Note that β is not unit speed.

Step 2: Collect information. Differentiate $\langle \beta - \alpha, \mathrm{N} \rangle = \lambda$ to get

$$\lambda = \text{const} \neq 0.$$

Differentiate $\langle \beta - \alpha, \mathrm{T} \rangle = 0$ and $\langle \beta - \alpha, \mathrm{B} \rangle = 0$ to get

$$\langle \mathrm{T}_\beta, \mathrm{T} \rangle = \frac{1 - \lambda\kappa}{\|\beta'\|}, \qquad \langle \mathrm{T}_\beta, \mathrm{B} \rangle = \frac{\lambda\tau}{\|\beta'\|}.$$

Differentiate $\langle \mathrm{T}_\beta, \mathrm{T} \rangle$ and $\langle \mathrm{T}_\beta, \mathrm{B} \rangle$ to get

$$\frac{1 - \lambda\kappa}{\|\beta'\|} = \text{const}, \qquad \frac{\lambda\tau}{\|\beta'\|} = \text{const}.$$

Thus, there exists a constant μ such that

$$1 - \lambda\kappa = \mu\lambda\tau$$

if α is not planar so that $\tau \neq 0$.

We have shown that the torsion is either always zero or never zero at every point: $\lambda\tau/\|\beta'\| = \text{const}$. The problem can be solved by dividing the cases, but in this solution we give only for the case that α is not planar; the other hand is not difficult.

Step 3: Complete proof. If

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \widetilde{\beta} = \alpha + \widetilde{\lambda} \mathrm{N}$$

are different Bertrand mates of α with $\lambda \neq \widetilde{\lambda}$, then (κ, τ) solves a two-dimensional linear system

$$\begin{aligned} \kappa + \mu\tau &= \lambda^{-1}, \\ \kappa + \widetilde{\mu}\tau &= \widetilde{\lambda}^{-1}. \end{aligned}$$

It is nonsingular since $\mu = \widetilde{\mu}$ implies $\lambda = \widetilde{\lambda}$, which means we can represent κ and τ in terms of constants $\lambda, \widetilde{\lambda}, \mu$, and $\widetilde{\mu}$. Therefore, κ and τ are constant. $\square$

Here is a well-prepared problem set for exercises.

7.7 (Plane curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α lies on a plane,

(b) $\tau = 0$,

(c) the osculating plane constains a fixed point.

7.8 (Helices)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α is a helix,

(b) $\tau/\kappa = \text{const}$,

(c) normal lines are parallel to a plane.

7.9 (Sphere curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α lies on a sphere,

(b) $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$,

(c) $\tau/\kappa = (\kappa'/\tau\kappa^2)'$,

(d) normal planes contain a fixed point.

7.10 (Bertrand mates)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α has a Bertrand mate,

(b) there are two constants $\lambda \neq 0, \mu$ such that $1/\lambda = \kappa + \mu\tau$.

Chapter 8

Local theory of surfaces

8.1 Reparametrization

Theorem 8.1.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 8.1.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 8.1.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 8.1.1. *Metric coefficients*

$$\begin{aligned} \langle \alpha_x, \alpha_x \rangle &=: g_{11} & \langle \alpha_x, \alpha_y \rangle &=: g_{12} \\ \langle \alpha_y, \alpha_x \rangle &=: g_{21} & \langle \alpha_y, \alpha_y \rangle &=: g_{22} \end{aligned}$$

Theorem 8.1.4 (Normal coordinates)**.** *...?*

8.2 Differentiation of tangent vectors

Definition 8.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x, y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x, y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 8.2.2 (Gauss formula, Γ_{ij}^k, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$ and $\{L_{ij}\}_{i,j=1}^2$ by the Gauss formula

$$\alpha_{xx} =: \Gamma_{11}^1 \alpha_x + \Gamma_{11}^2 \alpha_y + L_{11}\,\nu, \qquad \alpha_{xy} =: \Gamma_{12}^1 \alpha_x + \Gamma_{12}^2 \alpha_y + L_{12}\,\nu,$$
$$\alpha_{yx} =: \Gamma_{21}^1 \alpha_x + \Gamma_{21}^2 \alpha_y + L_{21}\,\nu, \qquad \alpha_{yy} =: \Gamma_{22}^1 \alpha_x + \Gamma_{22}^2 \alpha_y + L_{22}\,\nu.$$

The *Christoffel symbols* refer to eight functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned}\partial_X Y &= X^i \partial_i (Y^j \alpha_j)\\ &= X^i (\partial_i Y^k)\alpha_k + X^i Y^j \partial_i \alpha_j\\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k\right)\alpha_k + X^i Y^j L_{ij}\,\nu.\end{aligned}$$

8.3 Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu\rangle = \frac{1}{2}\partial_X \langle \nu, \nu\rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 8.3.1. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 8.3.1. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y)\rangle = \langle X, -\partial_Y \nu\rangle = \langle \partial_Y X, \nu\rangle = \langle \partial_X Y, \nu\rangle = \langle \mathcal{S}(X), Y\rangle. \qquad \square$$

Theorem 8.3.2. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i \alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j \alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1\\ \mathcal{S}(Y)^2\end{pmatrix} = \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix} X^1\\ X^2\end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}_i^j X_i$. Then,

$$g_{ik} X^i \mathcal{S}_j^k Y^j = \langle X, \mathcal{S}(Y)\rangle = \langle \partial_X Y, \nu\rangle = X^i Y^j L_{ij}$$

implies $g_{ik}\,\mathcal{S}_j^k = L_{ij}$. $\square$

8.4 Computational problems

Definition 8.4.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$\begin{aligned} E &:= \langle \alpha_x, \alpha_x \rangle = g_{11}, & F &:= \langle \alpha_x, \alpha_y \rangle = g_{12}, & G &:= \langle \alpha_y, \alpha_y \rangle = g_{22}, \\ L &:= \langle \alpha_{xx}, \nu \rangle = L_{11}, & M &:= \langle \alpha_{xy}, \nu \rangle = L_{12}, & N &:= \langle \alpha_{yy}, \nu \rangle = L_{22}. \end{aligned}$$

Corollary 8.4.1. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\begin{aligned} \nu_x &= \frac{FM - GL}{EG - F^2} \alpha_x + \frac{FL - EM}{EG - F^2} \alpha_y, \\ \nu_y &= \frac{FN - GM}{EG - F^2} \alpha_x + \frac{FM - EN}{EG - F^2} \alpha_y. \end{aligned}$$

Theorem 8.4.2.

$$\Gamma_{ij}^l = \frac{1}{2} g^{kl} (g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2} (\log g)_x = \Gamma_{11}^1.$$

$$\nu_x \times \nu_y = K \sqrt{\det g}\ \nu.$$

$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$

$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det \begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

Theorem 8.4.3 (Gaussian curvature formula)**.**

(a) *In general,*

$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) *For orthogonal coordinates such that $F \equiv 0$,*

$$K = -\frac{1}{2\sqrt{\det g}} \left(\left(\frac{1}{\sqrt{\det g}} E_y\right)_y + \left(\frac{1}{\sqrt{\det g}} G_x\right)_x \right).$$

(c) *For $f(x, y, z) = 0$,*

$$K = -\frac{1}{|\nabla f|^4} \begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f) \end{vmatrix},$$

where ∇f denotes the gradient $\nabla f = (f_x, f_y, f_z)$.

(d) *(Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then*

$$K = -\tau^2.$$

(e) *(Brioschi) E, F, G describes K.*

Proof.

(a) Clear.

(b) We have $GM = EM$ and

$$\nu_x = -\frac{L}{E} \alpha_x - \frac{M}{G} \alpha_y, \qquad \nu_y = -\frac{M}{E} \alpha_x - \frac{N}{G} \alpha_y.$$

$$\nu_x \times \nu_y = \frac{LN - M^2}{EG} \alpha_x \times \alpha_y$$

After curvature tensors...

□

Example 8.4.1. (a) (Monge's patch) For $(x, y, f(x, y))$,

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t) = (r(t), z(t))$ be a plane curve with $r(t) > 0$. Let

$$\alpha(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

be a parametrization of a surface of revolution.

Then,

$$\begin{aligned}
\alpha_\theta &= (-r(t)\sin\theta, r(t)\cos\theta, 0)\\
\alpha_t &= (r'(t)\cos\theta, r'(t)\sin\theta, z'(t))\\
\nu &= \frac{1}{\sqrt{r'(t)^2 + z'(t)^2}}(z'(t)\cos\theta, z'(t)\sin\theta, -r'(t)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{\theta\theta} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0)\\
\alpha_{\theta t} &= (-r'(t)\sin\theta, -r'(t)\cos\theta, 0)\\
\alpha_{tt} &= (r''(t)\cos\theta, r''(t)\sin\theta, z''(t)).
\end{aligned}$$

Thus we have

$$E = r(t)^2, \quad F = 0, \quad G = r'(t)^2 + z'(t)^2,$$

and

$$L = -\frac{r(t)z'(t)}{\sqrt{r'(t)^2 + z'(t)^2}}, \quad M = 0, \quad N = \frac{r''(t)z'(t) - r'(t)z''(t)}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

8.5 General problems

Theorem 8.5.1. *Surfaces of the same constant Gaussian curvature are locally isomorphic.*

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle\alpha_r, \alpha_t\rangle \\ \langle\alpha_t, \alpha_r\rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r, t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 9

Geodesics

Part IV

Global theory of curves and surfaces

Chapter 10

Global theory of curves

10.1 Isoperimetric inequality

10.2 Four vertex theorem

10.3 Ovals

Chapter 11

Global theory of surfaces

11.1 Minimal surfaces

11.2 Classification of compact surfaces

11.3 The Hilbert theorem

Chapter 12

Total curvatures

12.1 The Fary-Minor theorem

Fenchel's theorem

12.2 The Gauss-Bonnet theorem