Abstract Harmonic Analysis

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Part I

Chapter 1

Locally compact groups

1.1 Haar measures

1.1 (Non-σ-finite measures). Following technical issues are important

(a) The Fubini theorem

(b) The Radon-Nikodym theorem

(c) The dual space of L^1 space

1.2 (Existence of the Haar measure).

1.3 (Left and right uniformities).

1.4 (Modular functions).

1.5 (Uniformly continuous functions). G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$. $g \in C_c(G)$ is two-sided uniformly continuous.

1.2 Group algebras

1.6 (Convolution inequalities).

Justification of the following?

$$\lambda(f) = \int f(s)\lambda_s\, ds.$$

1.7 (Convolution action). Let G be a locally compact group.

(a) $L^1(G)$ has a two-sided approximate unit.

(b) $\alpha : G \to \mathrm{Aut}(L^1(G))$ is point-norm continuous.

(c) $\lambda : G \to U(L^2(G))$ and $\lambda : L^1(G) \to B(L^2(G))$ are strongly continuous.

Proof. Let (U_α) be a directed set of open neighborhoods of the identity e of G. By Urysohn lemma, there is $e_\alpha \in C_c(U)^+$ such that $\|e_\alpha\|_1 = 1$ for each α. We claim that e_α is a left approximate unit for $L^1(G)$.

Suppose $g \in C_c(G)$, which is two-sided uniformly continuous. For any $\varepsilon > 0$, take α_0 such that $\|g - \lambda_s g\| < \varepsilon$ and $\|g - \rho_s g\| < \varepsilon$ for all $s \in U_\alpha$ for $\alpha \succ \alpha_0$. Then, we have

$$\begin{aligned}\|e_\alpha * g - g\|_1 &= \int |e_\alpha * g(t) - g(t)|\, dt \le \iint e_\alpha(s)|g(s^{-1}t) - g(t)|\, ds\, dt \\ &= \int e_\alpha(s)\|\lambda_s g - g\|_1\, ds < \varepsilon \int e_\alpha(s)\, ds \le \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|g*e_\alpha-g\|_1&=\int|g*e_\alpha(s)-g(s)|\,ds\le\iint|g(t)-g(s)|e_\alpha(t^{-1}s)\,dt\,ds\\&=\iint|g(t)-g(ts)|e_\alpha(s)\,dt\,ds=\int\|g-\rho_sg\|_1e_\alpha(s)\,ds<\varepsilon\int e_\alpha(s)\,ds\le\varepsilon,\end{aligned}$$

and they imply $\lim_\alpha\|e_\alpha*g-g\|_1=\lim_\alpha\|g*e_\alpha-g\|_1=0$. For general $f\in L^1(G)$, by taking $g\in C_c(G)$ such that $\|f-g\|_1<\varepsilon$, we have

$$\|e_\alpha*f-f\|_1\le\|e_\alpha*(f-g)\|_1$$

□

Note that we have

$$\begin{aligned}|\langle\lambda(\xi)\eta,\zeta\rangle|^2&=|\iint\xi(t)\eta(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2\\&\le\iint|\xi(t)||\eta(t^{-1}s)|^2\,ds\,dt\cdot\iint|\xi(t)||\zeta(s)|^2\,ds\,dt\\&=\|\xi\|_1^2\|\eta\|_2^2\|\zeta\|_2^2\end{aligned}$$

and

$$\begin{aligned}|\langle\rho(\xi)\eta,\zeta\rangle|^2&=|\iint\eta(t)\xi(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2\\&\le\iint|\xi(t^{-1}s)||\eta(t)|^2\,ds\,dt\cdot\iint|\xi(t^{-1}s)||\zeta(s)|^2\,ds\,dt\\&=\|\xi\|_1\|F\xi\|_1\|\eta\|_2^2\|\zeta\|_2^2\end{aligned}$$

imply

$$\|\lambda(\xi)\|_{2\to2}\le\|\xi\|_1,\qquad\|\rho(\xi)\|_{2\to2}\le\sqrt{\|\xi\|_1\|F\xi\|_1}.$$

The equalities do not hold, consider $\|\lambda(\xi)\|=\|\widehat{\xi}\|_\infty$ if $G=\mathbb{R}$.

1.8 (Group algebras)**.**

1.9 (Fell absorption principle)**.**

1.10 (Fourier algebra)**.**

1.11 (Fourier-Stieltjes algebra)**.** positive definite functions, Bochner theorem

1.12 (GNS construction for locally compact groups)**.** Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

$$\begin{tikzcd}
G \arrow[r] & M(G) & & \\
L_1(G) \arrow[ur, hook] \arrow[r, hook] \arrow[d, dashed, "*"] & C^*(G) \arrow[r, two heads] \arrow[d, dashed, "*"] & C_r^*(G) \arrow[r, hook] \arrow[d, dashed, "*"] & L(G) \arrow[d, dashed, "*\text{ with }\sigma w"] \\
L^\infty(G) & B(G) \arrow[l, hook'] & C_r^*(G)^* \arrow[l, hook'] & A(G) \arrow[l, hook'] \\
& C_0(G) \arrow[ul, hook] & &
\end{tikzcd}$$

1.3 Pontryagin duality

1.13 (Dual group).

1.14 (Fourier inversion theorem).

1.15 (Plancherel's theorem).

1.4 Structure theorems

1.5 Spectral synthesis

Chapter 2

Representation theory

2.1 (Schur's lemma)**.**

2.2 (Operator-valued Fourier transform)**.**

Since it is not easy to introduce the quantum dual of G for now, we cannot discuss $L^1(G)$ as the Fourier algebra, the predual of the quantum group von Neumann algebra. ($A(G) = L(G)_* = L^1(\widehat{G})$ and also is the closed linear span of matrix coefficients of the left regular representation.)

Chapter 3

Compact groups

3.1 Peter-Weyl theorem

3.2 Tannaka-Krein duality

3.3 Example of compact Lie groups

Chapter 4

Mackey machine

4.1 Example of non-compact Lie groups

Wigner classification

Chapter 5

Kac algebras

Part II

Topological quantum groups

Chapter 6

Compact quantum groups

Chapter 7

Locally compact quantum groups

7.1 Multiplicative unitaries

Part III

Tensor categories