Functional Analysis

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Part I

Topological vector spaces

Chapter 1

Locally convex spaces

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1.4 (Seminorms)**.**

$$\bigcap_{i=1}^{m}\{: p() < 1\}$$

Equivalent conditions on the continuity of seminorms

Proof. □

boundedness by seminorms, normability

1.3 Continuous linear functionals

1.5. Let $\overline{x^*} = (x_1^*, \cdots, x_n^*) \in X^{*n}$. $\overline{x^*} : X \to \mathbb{F}^n$. If $x^* \in X^*$ vanishes on $\bigcap_{i=1}^n \ker x_i^*$, then x^* is a linear combination of $\{x_i^*\}$.

1.6 (Hahn-Banach extension)**.** Let X be a real vector space. Suppose V is a linear subspace of X and $l : V \to \mathbb{R}$ is a linear functional dominated by a sublinear functional $q : X \to \mathbb{R}$, that is, $l(v) \le q(v)$ for all $v \in V$.

(a) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l.

(b) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l and is dominated by q if $\dim X/V = 1$.

(c) There is a linear functional $\widetilde{l} : X \to \mathbb{R}$ that extends l and is dominated by q.

Proof. (a) It can be done by the Hamel basis.

(b) Let $e \in X \setminus V$ so that every vector $x \in X$ can be uniquely written as $x = v + te$ with $v \in V$ and $t \in \mathbb{R}$. For $v_1, v_2 \in V$,

$$l(v_1) + l(v_2) = l(v_1 + v_2) \le q(v_1 + v_2) \le q(v_1 - e) + q(v_2 + e)$$

implies

$$l(v_1) - q(v_1 - e) \le -l(v_2) + q(v_2 + e).$$

Define a linear functional $\widetilde{l} : X \to \mathbb{R}$ such that $\widetilde{l}(v) = v$ and

$$l(v) - q(v - e) \le \widetilde{l}(e) \le -l(v) + q(v + e)$$

for all $v \in V$. Since

$$\widetilde{l}(v + te) = l(v) + t\widetilde{l}(e) \le l(v) + t(-l(t^{-1}v) + q(t^{-1}v + e)) = q(v + te)$$

if $t \ge 0$ and

$$\widetilde{l}(v + te) = l(v) + t\widetilde{l}(e) \le l(v) + t(l(-t^{-1}v) - q(-t^{-1}v - e)) = q(v + te)$$

if $t \le 0$, we have $\widetilde{l}(x) \in q(x)$ for all $x \in X$.

(c) With X and q, Consider a partially ordered set

$$\{(\widetilde{V}, \widetilde{l}) \mid V \le \widetilde{V} \le X,\ \widetilde{l} : \widetilde{V} \to \mathbb{R} \text{ is a linear extension of } l \text{ dominated by } q\}$$

such that $(V_1, l_1) \prec (V_2, l_2)$ if and only if $V_1 \le V_2$ and $l_2|_{V_1} = l_1$. The nonemptyness and the chain condition is easily satisfied, so it has a maximal element $(\widetilde{V}, \widetilde{l})$ by the Zorn lemma. By the part (b), we have $\widetilde{V} = X$. □

1.7 (Complex linear functionals)**.** Let X be a complex vector space. Consider a map

$$\begin{array}{ccc} \{\mathbb{C}\text{-linear functionals on } X\} & \to & \{\mathbb{R}\text{-linear functionals on } X\} \\ l & \mapsto & \operatorname{Re} l. \end{array}$$

Let p be a seminorm on X and l a complex linear functional on X.

(a) The above map is bijective.

(b) $|l(x)| \le p(x)$ if and only if $|\operatorname{Re} l(x)| \le p(x)$.

Proof. (b) There is λ such that $|\lambda| = 1$ and $l(\lambda x) \ge 0$. Then,

$$|l(x)| = |\lambda^{-1} l(\lambda x)| = l(\lambda x) = \operatorname{Re} l(\lambda x) \le p(\lambda x) = p(x).$$

□

1.8 (Hahn-Banach separation)**.**

Exercises

1.9 (Topology of compact convergence)**.**

Chapter 2

Barreled spaces

2.1 Uniform boundedness principle

2.1 (Barreled spaces). Let X be a topological vector space. A *barrel* is an absorbing, balanced, convex, and closed subset of X. A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

2.2 (Uniform boundedness principle). Let X and Y be topological vector spaces. Let $\mathcal{F}$ be a family of continuous linear operator from X to Y. Suppose $\bigcup_{T\in\mathcal{F}} Tx$ is bounded for each $x \in D$, where $D \subset X$.

(a) If D is dense in X, then $\bigcap_{T\in\mathcal{F}} T^{-1}\overline{U}$ is absorbing.

(b) If X is barreled, then $\mathcal{F}$ is equicontinuous.

2.2 Baire category theorem

2.3 (Baire spaces). A topological space is called a *Baire space* if the countable intersection of open dense subsets is always dense.

(a) If a topological vector space is Baire, then it is barreled.

(b) A Baire space is second category in itself.

(c) A topological group that is second category in itself is Baire.

2.4 (Absorbing sets). Let X be a topological vector space that is Baire. A subset $U \subset X$ is said to be *absorbing* if for every $x \in X$ there is a sufficiently large $t > 0$ such that $x \in tU$. Let $U \subset X$.

(a) If U is closed and absorbing, then U has a non-empty open subset.

(b) If U is closed and absorbing, then $U - U$ is a neighborhood of zero.

(c) If U is closed, convex, and absorbing, then U is a neighborhood of zero.

2.5 (Baire category theorem). The Baire category theorem proves many exmples of topological vector space are Baire, in particular barreled.

(a) A complete metric space is Baire.

(b) A locally compact Hausdorff space is Baire.

2.3 Open mapping theorem

2.6 (Open mapping theorem)**.** Let X be a F-space and Y a barreled space. Suppose $T : X \to Y$ is a continuous and surjective linear operator.

(a) $\overline{TU}$ is a neighborhood of zero.

(b) TU is a neighborhood of zero.

Proof. (a) Let U' be a neighborhood of zero such that $U \supset U' - U'$. Because T is surjective, the set $\overline{TU'}$ is a closed absorbing set, so it contains a non-empty open subset, since Y is barreled. Thus, $\overline{TU} \supset \overline{TU'} - \overline{TU'}$ is a neighborhood of zero.

(b) We claim $\overline{TU_{2^{-1}}} \subset TU_1$. Take $y_1 \in \overline{TU_{2^{-1}}}$.

Assume $y_n \in \overline{TU_{2^{-n}}}$. Since $\overline{TU_{2^{-(n+1)}}}$ is a neighborhood of zero, we have

$$(y_n + \overline{TU_{2^{-(n+1)}}}) \cap TU_{2^{-n}} \neq \varnothing.$$

Then, there is $x_n \in U_{2^{-n}}$ such that $Tx_n \in y_n + \overline{TU_{2^{-(n+1)}}}$. Define

$$y_{n+1} := y_n - Tx_n.$$

Then, $\sum_{n=1}^{\infty} x_n$ clearly converges to $x \in U_1$. Therefore,

$$Tx = \sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} (y_n - y_{n+1}) = y_1. \qquad \square$$

Exercises

2.7. Let (T_n) be a sequence in $B(X,Y)$. If T_n coverges strongly then $\|T_n\|$ is bounded by the uniform boundedness principle.

2.8. There is a closed absorbing set in $\ell^2(\mathbb{Z}_{\geq 0})$ that is not a neighborhood of zero;

$$\overline{B}(0,1) \setminus \bigcup_{i=2}^{\infty} B(i^{-1}e_i, i^{-2})$$

is a counterexample.

2.9. There is no metric d on $C([0,1])$ such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ pointwise as $n \to \infty$ for every sequence f_n. Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

2.10. We show that there is no projection from ℓ^∞ onto c_0.

2.11 (Schur property)**.** ℓ^1

2.12. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

2.13 (Daugavet property)**.** (a) The real Banach space $C([0,1])$ satisfies the Daugavet property.

Proof. Let T be a finite rank operator on $C([0,1])$, and e_i be a basis of $\operatorname{im} T$. Then, for some measures μ_i,

$$Tf(t) = \sum_{i=1}^{n} \int_0^1 f \, d\mu_i e_i(t).$$

Let $M := \max \|e_i\|$.

Take f_0 such that $\|f_0\| = 1$ and $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$. Reversing the sign of f_0 if necessary, take an open interval Δ such that $Tf_0(t) \geq \|T\| - \frac{\varepsilon}{2}$ and $|\mu_i|(\Delta) \leq \frac{\varepsilon}{4nM}$ for all i. Define f_1 such that $f_0 = f_1$ on Δ^c, $f_1(t_0) = 1$ for some $t_0 \in \Delta$, and $\|f_1\| = 1$. Then, $\|Tf_1 - Tf_0\| \leq \frac{\varepsilon}{2}$ shows $Tf_1 \geq \|T\| - \varepsilon$ on Δ. Therefore,

$$\|1 + T\| \geq \|f_1 + Tf_1\| \geq f_1(t_0) + Tf_1(t_0) \leq 1 + \|T\| - \varepsilon.$$

□

Problems

2.14. Let T be an invertible linear operator on a normed space. Then, $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

Chapter 3

Weak topologies

3.1 Dual spaces

3.1 (Bidual).

3.2. Let X be a locally convex space. The *weak topology* is the topology w on X defined by the family of seminorms $\{x \mapsto |\langle x, \xi\rangle|\}_{\xi \in X^*}$. The *weak* topology* is the topology w^* on X^* defined by the family of seminorms $\{\xi \mapsto |\langle x, \xi\rangle|\}_{x \in X}$. Let $J : X \to X^{**}$ be the canonical embedding.

(a) (X, w) and (X^*, w^*) are locally convex.

(b) $(X, w)^* = X^*$.

(c) $(X^*, w^*)^* = X$. Every locally convex space is a dual of a locally convex space.

Proof. (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show $(X^*, w^*)^* \subset X$. If $u \in (X^*, w^*)^*$, then there are $x_1, \cdots, x_m \in X$ such that

$$|\langle u, \xi\rangle| \le \sum_{i=1}^m |\langle x_i, \xi\rangle|$$

for all $\xi \in X^*$. If we let $\ker \vec{x} := \bigcap_{i=1}^m \ker x_i$, then it is a closed subspace of X^* such that $\ker \vec{x} \subset \ker u$, so we have $u \in \operatorname{span} \vec{x} \subset X$. □

3.3. closure and weak closure of convex subsets

Proof. Hahn-Banach □

3.4 (Polar).

boundedness, incompleteness

3.5 (Weak convergence by dense set). Let X be a Banach space, D^* a subset of X^*, and $\overline{D^*}$ the norm closure of D^*. For example, if X has a predual $X_* \subset X^*$ and D^* is dense in X_*, then $\sigma(X, \overline{D^*})$ is the weak* topology.

(a) There is a squence $x_n \in X$ converges to zero in $\sigma(X, D^*)$ but not in $\sigma(X, \overline{D^*})$.

(b) A bounded sequence $x_n \in X$ converges to zero in $\sigma(X, \overline{D^*})$ if in $\sigma(X, D^*)$.

Proof. (b) Let $\xi \in \overline{D^*}$ and choose $\eta \in D^*$ such that $\|\xi - \eta\| < \varepsilon$. Then,

$$|\langle x_n, \xi\rangle| \le \|x_n\|\|\xi - \eta\| + |\langle x_n, \eta\rangle| \lesssim \varepsilon + |\langle x_n, \eta\rangle| \to \varepsilon.$$

□

3.2 Weak compactness

3.6 (Banach-Alaoglu theorem)**.**

Proof. Consider
$$B_{X^*} \to \prod_{x\in X} \|x\|B : l \mapsto (l(x))_{x\in X}.$$
Since it is an embedding into a compact space, it suffices to show the closedness of image: for $l(x) := \lim_\alpha l_\alpha(x)$, we have
$$\|l(x)\| \le \|l(x) - l_\alpha(x)\| + \|x\| \xrightarrow{\alpha\to\infty} \|x\|,$$
so l is contained in the range. □

3.7 (Eberlein-Šmulian theorem)**.**

3.8 (James' theorem)**.**

3.3 Weak density

Bishop-Phelps theorem

3.9 (Goldstine's theorem)**.** Let X be a Banach space and $J : X \to X^{**}$ the canonical embedding. Our claim is that $\overline{B}$ is weak*-dense in $\overline{B}_{X^{**}}$. Let $x_0^{**} \in X^{**}$ with $\|x_0^{**}\| \le 1$, and let
$$\bigcap_{i=1}^m \{\, x^{**} \in X^{**} : |\langle x^{**} - x_0^{**}, x_i^*\rangle| < \varepsilon \,\}$$
be an open weak*-neighborhood of zero in X^{**} with $\|x_i^*\| \le 1$ and $\varepsilon > 0$. Let
$$S := \bigcap_{i=1}^m \{\, x \in X : \langle x, x_i^*\rangle = \langle x_0^{**}, x_i^*\rangle \,\}.$$

(a) S is not empty.

(b) $S \cap (1+\varepsilon)\overline{B}_X$ is not empty for all $\varepsilon > 0$.

(c) $\overline{B}_X$ is weak*-dense in $\overline{B}_{X^{**}}$

Proof. (a)

(b) From the part (a), we have $x \in S$. Suppose S does not intersect $(1+\varepsilon)\overline{B}_X$. By the Hahn-Banach theorem, there is $y^* \in X^*$ such that
$$y^*|_{S-x} = 0, \quad \langle x, y^*\rangle > 1+\varepsilon, \quad \text{and} \quad \|y^*\| = 1.$$
Since $S - x = \bigcap_{i=1}^m \ker x_i^*$, the linear functional y^* is a linear combination of $x_1^*, \cdots, x_m^*$, so we have
$$1+\varepsilon < \langle x, y^*\rangle = \langle x_0^{**}, y^*\rangle \le \|x_0^{**}\|\|y^*\| \le 1.$$

(c) Take $\varepsilon > 0$ such that $\varepsilon \max_{1\le i\le m} \|x_i^*\| < 1$. By the part (b), there is $y \in X$ such that $\|y\| \le 1+\varepsilon$ and $\langle y, x_i^*\rangle = \langle x^{**}, x_i^*\rangle$. If we let $x := (1+\varepsilon)^{-1}y$, then $x \in \overline{B}_X$ so that
$$|\langle x - x_0^{**}, x_i^*\rangle| = |\langle x - y, x_i^*\rangle| = |\langle \varepsilon x, x_i^*\rangle| \le \varepsilon\|x\|\|x_i^*\| < \varepsilon$$
for all i. □

3.4 Krein-Milman theorem

Choquet theory

3.5 Polar topologies

Mackey-Arens

Exercises

3.10 (James' space)**.** not reflexive but isometrically isomorphic to bidual

3.11 (Predual correspondence)**.** Let X be a Banach space. Let

$$\{(Y,\varphi) \mid \varphi : X \to Y^* \text{ is an isometric isorphism}\}$$

and

$$\{Z \le X^* \mid \overline{B_X} \text{ is compact Hausdorff in } (X, \sigma(X,Z))\}.$$

$$(Y,\varphi) \mapsto \operatorname{im} \varphi^*|_{J(Y)}$$

(a) The map is well-defined.

(b) The map is surjective. (by Goldstein)

(c) The map is injective up to isomorphism for Y.

3.12. Let X be a closed subspace of a Banach space Y and

$$i : X \to Y$$

the inclusion. Suppose X and Y have preduals X_* and Y_* respectively. Let

$$j := i^*|_{Y_*} : Y_* \to Z \subset X^*,$$

where $Z := i^*(Y_*)^-$. Then we can show

$$j^* : Z^* \subset X^{**} \to Y$$

coincides with i on $X \cap Z^*$. From the existence of X_* we have $X^{**} \to X$, which is restricted to define a map $k : Z^* \to X$.

$$\begin{tikzcd} & X \arrow[r, "i"] & Y \\ X^{**} \arrow[ur] \arrow[r] & Z^* \arrow[u, "k"] \arrow[ur, "j"'] & \end{tikzcd}$$

We can show k is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

3.13 (Mazur's lemma)**.**

Part II

Banach spaces

Chapter 4

Operators on Banach spaces

4.1 Bounded operators

4.1 (Bounded belowness in Banach spaces)**.** Let $T \in B(X,Y)$ for Banach spaces X and Y. The following statements are equivalent:

(a) T is bounded below.

(b) T is injective and has closed range.

(c) T is a topological isomorphism onto its image.

4.2 (Bounded belowness in Hilbert spaces)**.** Let $T \in B(H,K)$ for Hilbert spaces H and K. The following statements are equivalent:

(a) T is bounded below.

(b) T is left invertible.

(c) T^* is right invertible.

(d) T^*T is invertible.

4.3 (Injectivity and surjectivity of adjoint)**.** Let $T \in B(X,Y)$ for Banach spaces X and Y.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is bounded below.

4.2 Compact operators

$K(X,Y)$ is closed in $B(X,Y)$. $K(X)$ is an ideal of $B(X)$. adjoint is $K(X,Y) \to K(Y^*,X^*)$. integral operators are compact. riesz operator, quasi-nilpotent operator.

4.3 Fredholm operators

4.4. A bounded linear operator $T : X \to Y$ between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

(a) A Fredholm operator T has closed range.

Proof. (a) Let C be a finite dimensional subsapce of Y such that $\operatorname{im} T \oplus C = Y$. Let $\widetilde{T} : X/\ker T \to Y$ be the induced operator of T. Define $S : (X/\ker T) \oplus C \to Y$ such that $S(x + \ker T, c) := \widetilde{T}(x + \ker T) + c$. Then, S is an topological isomorhpism between Banach spaces by the open mapping theorem, so $S(X/\ker T \oplus \{0\}) = \operatorname{im} \widetilde{T} = \operatorname{im} T$ is closed. □

4.5 (Atkinson's theorem)**.** An operator $T \in B(X, Y)$ is Fredholm if and only if there is $S \in B(Y, X)$ such that $TS - I$ and $ST - I$ is finite rank.

4.6 (Fredholm index)**.** locally constant, in particular, continuous. composition makes the addition of indices.

4.4 Nuclear operators

tensor products

Exercises

4.7 (Completely continuous operators)**.** On reflexive spaces, completely continuous operators are same with compact operators.

4.8 (Dunford-Pettis property)**.** A Banach space X is said to have the *Dunford-Pettis property* if all weakly compact operators $T : X \to Y$ to any Banach space Y is completely continuous.

(a) X has the Dunford-Pettis property if and only if for every sequences $x_n \in X$ and $x_n^* \in X^*$ that converge to x and x^* weakly we have $x_n^*(x_n) \to x^*(x)$.

(b) $C(\Omega)$ for a compact Hausdorff space Ω has the Dunford-Pettis property.

(c) $L^1(\Omega)$ for a probability space Ω has the Dunford-Pettis property.

(d) Infinite dimensional reflexive Banach space does not have the Dunfor-Pettis property.

Problems

1. If $T \in B(L^2([0,1]))$ is a compact operator, then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|Tf\|_{L^2} \le \varepsilon \|f\|_{L^2} + C_\varepsilon \|f\|_{L^1}.$$

Proof. 1. Suppose there is $\varepsilon > 0$ such that we have sequence $f_n \in L^2$ satisfying $\|f_n\|_2 = 1$ and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of T, there is a subsequence Tf_{n_k} converges to $g \neq 0$ in L^2. Then, $\|f_{n_k}\|_1 \to 0$ implies $f_{n_k} \to 0$ weakly in L^2, hence also for Tf_{n_k}. It means $g = 0$, which contradicts to the assumption. □

Chapter 5

Geometry of Banach spaces

5.1 Tensor products

5.2 Approximation property

dual is Banach. Basis problem, Mazur' duck.

Exercises

Chapter 6

Part III

Spectral theory

Chapter 7

Operators on Hilbert spaces

7.1 Hilbert spaces

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

7.1. (a) A Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

7.2 (Riesz representation theorem)**.** Let H be a Hilbert space over a field $\mathbb{F}$, which is either $\mathbb{R}$ of $\mathbb{C}$.

We use the bilinear form $\langle -,- \rangle : X \times X^* \to \mathbb{F}$ of canonical duality. *Dirac* notation $\langle -|- \rangle$ for the inner product of a complex Hilbert spaces such that $\langle x, y \rangle = \langle y|x \rangle$. The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each $x^* \in H^*$, there is a unique $x \in H$ such that $\langle y, x^* \rangle = \langle y, x \rangle$ for every $y \in H$.

(b) $H \to H^* : x \mapsto \langle -, x \rangle$ is a natural linear and anti-linear isomorphism if $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$, respectively.

Let H be a separable Hilbert space. Find a positive sequence a_n such that every sequence x_n of unit vectors of H satisfying $|\langle x_i, x_j \rangle| \le a_j$ for all $i < j$ converges weakly to zero.

7.3 (Normal operators)**.** For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. Suppose T is normal.

(a) $\ker T = \ker T^*$.

(b) T is bounded below if and only if T is invertible.

(c) If T is surjective, then T is invertible.

7.4 (Invariant and Reducing subsapces)**.** Let K be a closed subspace of H.

(a) K is reducing for T if and only if K is invariant for T and T^*.

(b) K is reducing for T if and only if $TP = PT$, where P is the orthogonal projection on K.

7.2 Spectral theorem for normal operators

There is an orthonormal basis $E \subset H$ such that

$$T = \sum_{e \in E} \lambda_e |e\rangle\langle e|.$$

7.5 (Spectral measure)**.** Let $(\Omega, \mathcal{M})$ be a measurable space and H a Hilbert space. A *projection-valued measure* on Ω for H is a map $E : \mathcal{M} \to B(H)$ with $E(\varnothing) = 0$ such that $E(A)$ is a projection for every $A \in \mathcal{M}$, and one of the following equivalent conditions hold:

(i) the set function $E_{\xi,\eta} : \mathcal{M} \to \mathbb{C} : A \mapsto \langle E(A)\xi, \eta \rangle$ is a complex measure on Ω for each $\xi, \eta \in H$.

(ii) the countable additivity holds, i.e. $E(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} E(A_i)$ in the strong operator topology of $B(H)$ for $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$.

(a) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{M}$.

7.6. Let $T \in B(H)$ be a normal operator. Then, there exists a projection-valued measure E on $\sigma(T)$ for H such that

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

This spectral measure E is also called the *resolution of the identity*.

Let E be the spectral measure of a normal operator $T \in B(H)$. If we choose $\xi \in E(B(\lambda, n^{-1}))H$, then since $E(B(\lambda, n^{-1})^c)\xi = 0$, or since $E(B(\lambda, n^{-1}))\xi = \xi$, we have

$$\begin{aligned}
\|(\lambda - T)\xi\|^2 &= \int |\lambda - z|^2 \, d\langle E(z)\xi, \xi \rangle \\
&= \int_{B(\lambda,n^{-1})} |\lambda - z|^2 \, d\langle E(z)\xi, \xi \rangle \\
&\le n^{-2} \int_{B(\lambda,n^{-1})} d\langle E(z)\xi, \xi \rangle \\
&\le n^{-2} \int d\langle E(z)\xi, \xi \rangle \\
&= n^{-2}\|\xi\|^2.
\end{aligned}$$

7.7 (Spectral representation)**.** Let T be a bounded normal operator on a Hilbert space H. Then, the unital C*-algebra $C^*(T)$ generated by T is $*$-isomorphic to $C(\sigma(T))$, and it has a canonical faithful representation $\pi : C(\sigma(T)) \to B(H)$. Decompose $\pi = \bigoplus_\alpha \pi_\alpha$ to cyclic representations $\pi_\alpha : C(\sigma(T)) \to B(H_\alpha)$ with cyclic unit vectors ψ_α. Each vector state ψ_α induces a probability measure μ_α on $\sigma(T)$. It is called the spectral measure associated to the cyclic vector ψ_α. We can check that the GNS-representation of μ_α is unitarily equivalent to π_α. The direct sum $C(\sigma(T)) \to \bigoplus_\alpha B(L^2(\mu_\alpha))$ can be defined.

The bounded normal operator T is always unitarily equivalent to the multiplication operator on a finite measure space. However, in order for T to be unitarily equivalent to the multiplication operator by the identity function of $\mathbb{C}$, T must be multiplicity free, equivalently, T must have a cyclic vector.

On a C*-algebra $\mathcal{A}$, each state ω defines a von Neumann algebra $\pi_\omega(\mathcal{A})''$, which is the start of measure theory.

Two bounded normal operators are unitarily equivalent if and only if the sequence of multiplicity measure classes are identical.

Two spectral theorems: Multiplication operator form(unitary equivalence), Projection-valued measure form(functional calculus).

7.3 Decomposition of spectrum

$$\sigma = \sigma_p \sqcup \sigma_c \sqcup \sigma_r = \overline{\sigma_{pp}} \cup \sigma_{ac} \sigma_{sc} = \sigma_d \sqcup \sigma_{ess,5}.$$

7.4 Trace class and Hilbert-Schmidt operators

7.8 (Trace class operators)**.** An operator $K \in B(H)$ is

$$K = \sum_i \lambda_i \theta_{\xi_i,\eta_i}, \qquad \mathrm{Tr}(\cdot K) = \sum_i \lambda_i \omega_{\xi_i,\eta_i},$$

for $(\lambda_i) \in \ell^1$ and orthonormal sequences $(\xi_i)_i$ and $(\eta_i)_i$

7.5 Operator topologies

7.9. Let f be a linear functional on $B(H)$ for a Hilbert space H. Then, TFAE:

(a) f is weakly continuous,

(b) f is strongly continuous,

(c) $f = \sum_{i=1}^n \omega_{\xi_i,\eta_i}$ for some $\xi_i, \eta_i \in H$.

Proof. (2) $\Rightarrow$ (3) is the only nontrivial implication. By the definition of the strong operator topology, there exists $v \in \mathcal{H}^n$ such that

$$|f(T)| \le \|T^{\oplus n} v\|.$$

The functional $f : \mathcal{A} \to \mathbb{C}$ factors through $\mathcal{H}^n$ such that

$$\mathcal{A} \to v\mathcal{H}^n \to \mathbb{C}.$$

□

7.10 (Trace class operators)**.**

Exercises

7.11 (Strong* operator topology)**.** Let H be a Hilbert space. We provides some conditions for a strongly convergent sequence to converge strongly*. Let $(T_\alpha)_\alpha \subset B(H)$ and suppose $T_\alpha \to T$ strongly.

7.12 (Strict topology)**.** Let H be a Hilbert space. Let $(T_\alpha)_\alpha \subset B(H)$ and $K \in K(H)$.

(a) The strong* topology and the strict topology agree on bounded sets of $B(H)$.

7.13 (Unitary group)**.** Let H be a Hilbert space.

(a) The weak topology and the strict topology agree on $U(H)$.

(b) $U(H)$ is neither complete nor locally compact.

Chapter 8

Unbounded operators

8.1

8.1 (Closed operators)**.**

8.2 (Adjoint operators)**.** Let $T : X \to Y$ be an unbounded linear operator between Banach spaces. Define an unbounded operator $T^* : Y^* \to (\operatorname{dom} T)^*$ by

$$\operatorname{dom} T^* := \{y^* \in Y^* \mid \operatorname{dom} T \to \mathbb{C} : x \mapsto \langle Tx, y^* \rangle \text{ is bounded}\},$$

$$\langle x, T^* y^* \rangle := \langle Tx, y^* \rangle, \qquad x \in \operatorname{dom} T,\ y^* \in \operatorname{dom} T^*.$$

Suppose T is densely defined so that we can write $T^* : Y^* \to X^*$.

(a) If $T \subset S$, then $S^* \subset T^*$.

(b) T^* is closed.

(c) T^* is densely defined if an only if T is closable.

(d) If T is closable, then $\overline{T} = T^{**}$. (Only on Hilbert spaces?)

(e) If T is closable, then $T^* = \overline{T}^*$. Since T^* is a priori closed, we will use the notation $\overline{T}^* := \overline{T}^*$.

Let $L : H \to H$ be a densely defined operator. Here is a routine to find a closure.

1. Compute $\operatorname{dom} L^*$ and check it is dense to show L is closable.
2. Compute $\operatorname{dom} L^{**}$ to find the closure of L.
3. Do work with our densely defined closed operator $\overline{L} = L^{**}$.

8.3. Let $T : X \to Y$ be a densely defined closed operator between Banach spaces.

(a) T^* is injective if and only if T has dense range.

(b) T^* is surjective if and only if T is bounded below.

Proof. (b) Suppose T is bounded below. Fix $x^* \in X^*$. Since T is bounded below, x^* defines a bounded linear functional on $\operatorname{dom} T$ with respect to $\|x\| + \|Tx\|$, which is embedded in Y as a closed subspace. By the Hahn-Banach extension, we have an element $y^* \in Y^*$ such that $\langle Tx, y^* \rangle = \langle x, x^* \rangle$ for all $x \in X$, which is contained in $\operatorname{dom} T^*$ by the definition of $\operatorname{dom} T^*$. This implies that T is surjective because $T^* y^* = x^*$.

Conversely, suppose T^* is surjective. Let $F := \{x \in \operatorname{dom} T : \|Tx\| \le 1\}$. Since for every $x^* \in X^*$ we have for some $y^* \in \operatorname{dom} T^*$ such that

$$\sup_{x \in F} |\langle x, x^* \rangle| = \sup_{x \in F} |\langle x, T^* y^* \rangle| = \sup_{x \in F} |\langle Tx, y^* \rangle| \le \|y^*\|.$$

By the uniform boundedness principle, we have $\sup_{x \in F}(\|x\| + \|Tx\|)$ is bounded, we are done. $\square$

8.4 (Symmetric operators). An unbounded operator $T : H \to H$ is called *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad x, y \in \operatorname{dom} T.$$

(a) A symmetric operator is always closable and its closure is also symmetric.

(b) If T is symmetric, then $T \subset T^*$. If T is densely defined, then the converse holds.

8.5 (Symmetric extensions).

(a) If T is symmetric, then every symmetric extension is a restriction of T^*.

(b) If T is symmetric, then it has a maximal symmetric extension. Note that T^* is not symmetric in general.

(c) A maximal symmetric operator is closed since the closure of a .

(d) A self-adjoint operator is maximal.

(e) A densely defined symmetric operator is essentially self-adjoint if and only if it has a unique self adjoint extension.

(f) A densely defined symmetric operator may have no or many self-adjoint extensions.

8.6 (Cayley transform).

8.2 Spectral theorem

A self-adjoint operator must be a densely defined and closed.

8.7. For a densely defined closed operator $T : H \to H$, $\sigma(T^*) = \overline{\sigma(T)}$.

8.8. Let $T : H \to H$ be a

(a)

Kato-Rellich theorem
analytic vector theorem

Chapter 9

Toeplitz operators

Part IV

Operator algebras

Chapter 10

Banach algebras

10.1 Spectral theory of unital Banach algebras

10.1 (Banach algebras)**.** For a Banach algebra $\mathcal{A}$ with multiplicative unit, there is a complete renorming such that $\|1\| = 1$, i.e. we can always assume $\|1\| = 1$.

Let $\mathcal{A}$ be a unital Banach algebra.

(a) If $\|a\| < 1$, then $1 - a$ is invertible. So $\mathcal{A}^\times$ is open.

(b) $\mathcal{A}^\times \to \mathcal{A}^\times : a \mapsto a^{-1}$ is continuous.

(c) $\mathcal{A}^\times \to \mathcal{A}^\times : a \mapsto a^{-1}$ is differentiable.

Proof. (a) We can show

$$(1-a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

If a is invertible, then $a + h = a(1 + a^{-1}h)$ has the inverse $(1 + a^{-1}h)^{-1}a^{-1}$ if h is sufficiently small such that $\|a^{-1}h\| < 1$.

(b) Clear from

$$b^{-1} - a^{-1} = b^{-1}(a-b)a^{-1}.$$

(c)

$$\begin{aligned}\frac{\|b^{-1} - a^{-1} - (-a^{-1}(b-a)a^{-1})\|}{\|b-a\|} &= \frac{\|(a^{-1} - b^{-1})(b-a)a^{-1}\|}{\|b-a\|} \\ &\le \|a^{-1} - b^{-1}\|\|a^{-1}\| \xrightarrow{b \to a} 0.\end{aligned}$$

□

10.2 (Spectrum and resolvent)**.** Let a be an element of a unital Banach algebra $\mathcal{A}$. The *spectrum* of a in $\mathcal{A}$ is defined to be the set

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } \mathcal{A}\},$$

and the *resolvent* of a in $\mathcal{A}$ is defined to be its complement $\rho_{\mathcal{A}}(a) := \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$. We can now define the *resolvent map* $R : \rho_{\mathcal{A}}(a) \to \mathcal{A}$ such that

$$R(\lambda) = R(\lambda; a) := (\lambda - a)^{-1}.$$

If $\mathcal{A}$ is clear in its context, we omit it to just write $\sigma(a)$ and $\rho(a)$.

(a) $\sigma(a)$ is compact.

(b) $\sigma(a)$ is non-empty.

(c) If $\mathcal{A}$ is a division ring, then $\mathcal{A} \cong \mathbb{C}$. This result is called the *Gelfand-Mazur theorem*.

Proof. (a) If $|\lambda| > \|a\|$, then $\lambda - a$ is always invertible, so the spectrum is bounded. Closedness follows from the fact that the set of invertibles is open.

(b) Suppose the spectrum $\sigma(a) = \varnothing$ so that the resolvent function $R : \mathbb{C} \to \mathcal{A}$ is well-defined on the entire $\mathbb{C}$. Note that $a \neq 0$. Since R is continuous and since

$$\|(\lambda - a)^{-1}\| = \|\lambda^{-1}(1 - \lambda^{-1}a)^{-1}\| = \left\|\lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}a)^k\right\| < (2\|a\|)^{-1}\sum_{k=0}^{\infty}2^{-k} = \|a\|^{-1}$$

on $\{\lambda \in \mathbb{C} : |\lambda| > 2\|a\|\}$, the function R is bounded. Also, for every $l \in \mathcal{A}^*$ we have that the function $\mathbb{C} \to \mathbb{C} : \lambda \mapsto \langle R(\lambda), l\rangle$ is holomorphic since $a \mapsto a^{-1}$ is differentiable.

Therefore, by the Liouville theorem, the bounded entire function $\lambda \mapsto \langle R(\lambda), l\rangle$ is constant for all $l \in \mathcal{A}^*$. Because $\mathcal{A}^*$ separates points of $\mathcal{A}$, the function R is constant, which implies $a \in \mathbb{C}$ and contradicts to $\sigma(a) = \varnothing$.

(c) For any $a \in \mathcal{A}$, by the part (b), there must be λ such that $\lambda - a$ is not invertible. In a division ring, zero is the only non-invertible element, so $\lambda = a$. $\square$

10.3 (Spectral radius)**.** Let a be an element of a unital Banach algebra $\mathcal{A}$. The *spectral radius* of a in $\mathcal{A}$ is defined to be

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$

(a) $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) $\limsup_n \|a^n\|^{\frac{1}{n}} \le r(a)$, i.e. $r(a) = \lim_n \|a^n\|^{\frac{1}{n}}$.

Proof. (a) Since $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1}$ exists if $|\lambda| > \|a\|$, we have $r(a) \le \|a\|$ for all $a \in \mathcal{A}$. For every $\lambda \in \sigma(a)$ and every integer $n \ge 1$ we have

$$|\lambda|^n = |\lambda^n| \le r(a^n) \le \|a^n\|,$$

and it proves $r(a) \le \inf_n \|a^n\|^{\frac{1}{n}}$.

(b) Consider a holomorphic function

$$f : \{\lambda \in \mathbb{C} : |\lambda| > r(a)\} \to \mathbb{C} : \lambda \mapsto \langle R(\lambda), l\rangle$$

for each $l \in \mathcal{A}^*$. Since on a smaller domain $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$, the function f can be given by

$$f(\lambda) = \left\langle \lambda^{-1}\sum_{k=0}^{\infty}(\lambda^{-1}a)^k, l\right\rangle,$$

we can determine the coefficients of the Laurent series of f at infinity as

$$f(\lambda) = \sum_{k=0}^{\infty}\langle a^k, l\rangle\lambda^{-k-1}$$

on $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$.

Take λ such that $|\lambda| > r(a)$. Then, the sequence $(a^k\lambda^{-k-1})_k \in \mathcal{A}$ is weakly bounded, hence is normly bounded by the uniform boundedness principle. Let $\|a^n\| \le C_\lambda|\lambda^{n+1}|$ for all $n \ge 1$. Then,

$$\limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}} \le \limsup_{n\to\infty} C_\lambda^{\frac{1}{n}}|\lambda^{n+1}|^{\frac{1}{n}} = |\lambda|.$$

If we limit $|\lambda| \downarrow r(a)$, we are done. $\square$

10.4 (Spectrum in closed subalgebras)**.** For fixed element, smaller the ambient algebra, less "holes" in the spectrum. Let $\mathcal{B} \subset \mathcal{A}$ be a closed subalgebra containing $1_{\mathcal{A}}$. Note that $\mathcal{B}$ may be unital even for $1_{\mathcal{A}} \notin \mathcal{B}$.

(a) $\mathcal{B}^\times$ is clopen in $\mathcal{A}^\times \cap \mathcal{B}$.

10.2 Ideals

10.5 (Ideals)**.** (a) If I is a left ideal, then $\mathcal{A}/I$ is a left $\mathcal{A}$-module.

10.6 (Modular left ideals)**.** A left ideal I is called *modular* if there is $e \in \mathcal{A}$ such that $a - ae \in I$ for all $a \in \mathcal{A}$. The element e is called a *right modular unit* for I.

(a) I is modular if and only if $\mathcal{A}/I$ is unital(?).

(b) A proper modular left ideal is contained in a maximal left ideal.

(c) I is a maximal modular left ideal if and only if I is a modular maximal left ideal.

(d) There is a non-modular maximal ideal in the disk algebra.

10.7 (Closed ideals)**.** (a) closure of proper left ideal is proper left.

(b) maximal modular left ideal is closed.

10.8 (Unitization)**.** Let $\mathcal{A}$ be an algebra. Recall that we always assume algebras are associative. Consider an embedding $\mathcal{A} \to B(\mathcal{A}) : a \mapsto L_a$, where $L_a(b) = ab$. Define

$$\widetilde{\mathcal{A}} := \{ L_a + \lambda \operatorname{id}_{B(\mathcal{A})} : a \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital $\mathcal{A}$.

(a) If $\mathcal{A}$ is normed, then $\widetilde{\mathcal{A}}$ is a normed algebra such that there is an isometric embedding $\mathcal{A} \to \widetilde{\mathcal{A}}$.

(b) If $\mathcal{A}$ is Banach, then $\widetilde{\mathcal{A}}$ is a Banach algebra.

(c) $\mathcal{A} \oplus \mathbb{C}$ is topologically isomorphic to $\widetilde{\mathcal{A}}$ as normed spaces.

Proof. (a) The space of bounded operators $B(\mathcal{A})$ is a normd algebra. Then, $\widetilde{\mathcal{A}}$ is a normed $*$-algebra with induced norm

$$\|L_a + \lambda \operatorname{id}_{B(\mathcal{A})}\| = \sup_{b \in \mathcal{A}} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then, $\mathcal{A}$ is a normed $*$-subalgebra of $\widetilde{\mathcal{A}}$ because the norm and involution of $\mathcal{A}$ agree with $\widetilde{\mathcal{A}}$.

(b) Suppose (x_n, λ_n) is Cauchy in $\widetilde{\mathcal{A}}$. Since $\mathcal{A}$ is complete so that it is closed in $\widetilde{\mathcal{A}}$, we can induce a norm on the quotient $\widetilde{\mathcal{A}}/\mathcal{A}$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|x\| \leq \|(x,\lambda)\| + |\lambda|$ shows that x_n is Cauchy in $\mathcal{A}$.

Since a finite dimensional normed space is always Banach and $\mathcal{A}$ is Banach, λ_n and x_n converge. Finally, the inequality $\|(x,\lambda)\| \leq \|x\| + |\lambda|$ implies that (x_n, λ_n) converges.

(c) Check the topology on $\mathcal{A} \oplus \mathbb{C}$ in detail... □

unitization, homomorphisms, category(direct sum, product, etc.)
$B(\mathbb{C}^n)$ is simple, but $B(X)$ is not simple.

10.3 Holomorphic functional calculus

10.9. Let a be an element of a unital Banach algebra $\mathcal{A}$. Let f be a holomorphic function on a neighborhood U of $\sigma(a)$. Let C be a positively oriented smooth simple closed curve in U enclosing $\sigma(a)$. Define $f(a) \in \mathcal{A}^{**}$ as the Dunford integral

$$\langle f(a), l\rangle := \int_C f(\lambda)\langle R(\lambda), l\rangle \, d\lambda, \qquad l \in \mathcal{A}^*.$$

Let $\mathrm{Hol}(\sigma(a))$ be the space of all holomorphic functions on a neighborhood of $\sigma(a)$ endowed with the topology of compact convergence. Note that $\mathrm{Hol}(\sigma(a))$ is not Banach. We define the *holomorphic functional calculus* by the map

$$\mathrm{Hol}(\sigma(a)) \to \mathcal{A} : f \mapsto f(a).$$

It is also called the Riesz or the Riesz-Dunford functional calculus.

(a) $f(a) \in \mathcal{A}$, i.e. $f(a)$ is given by the Pettis integral.

(b) $f(a)$ is independent of the choice of C.

(c) The functional calculus is an algebra homomorphism.

(d) The functional calculus is bounded.

(e) injective.

(f) unital, $\mathrm{id}_{\mathbb{C}} \mapsto a$.

(g) spectral mapping.

(h) power series.

Proof. (a)

□

lin map > alg hom > star hom > cts

10.4 Gelfand theory

Banach algebra of single generator semisimplicity and symmetricity

10.10 (Spectrum of a Banach algebra)**.** Let $\mathcal{A}$ be a commutative Banach algbera. A *character* of $\mathcal{A}$ is a non-zero algebra homomorphism $\varphi : \mathcal{A} \to \mathbb{C}$. Denote by $\sigma(\mathcal{A})$ the set of all characters of $\mathcal{A}$. We will show that all characters are bounded. Then, endow with the weak* topology on $\sigma(\mathcal{A})$ from the inclusion $\sigma(\mathcal{A}) \subset \mathcal{A}^*$. We call this space as the *spectrum* of $\mathcal{A}$. Let $\varphi \in \sigma(\mathcal{A})$.

(a) $\|\varphi\| = 1$.

(b) If $\mathcal{A}$ is unital, then $\sigma(\mathcal{A})$ is compact and Hausdorff.

(c) Even if $\mathcal{A}$ is non-unital, $\sigma(\mathcal{A})$ is locally compact and Hausdorff.

10.11 (Gelfand transform)**.** Let $\mathcal{A}$ be a commutative Banach algebra.

$$\Gamma : \mathcal{A} \to C_0(\sigma(\mathcal{A})).$$

(a) $\Gamma(\mathcal{A})$ separates points.

(b) Γ has closed range if

(c) Γ is injective if

(d) Γ is isometric if $r(a) = \|a\|$ for all $a \in \mathcal{A}$.

Exercises

10.12 (Basic properties of spectrum)**.** Let $\mathcal{A}$ be a unital algebra.

(a) $\sigma(ab)\setminus\{0\}=\sigma(ba)\setminus\{0\}$.

(b) If $\sigma(a)$ is non-empty, then $\sigma(p(a))=p(\sigma(a))$.

Proof. (a) Intuitively, the inverse of $1-ab$ is $c=1+ab+abab+\cdots$. Then, $1+bca=1+ba+baba+\cdots$ is the inverse of $1-ba$. □

$C_b(\Omega)$ $\ell^\infty(S)$ $L^\infty(\Omega)$ $B_b(\Omega)$ $A(\mathbb{D})$ $B(X)$

10.13. In $C(\mathbb{R})$, the modular ideals correspond to compact sets.

10.14 (Disk algebra)**.** (a) Every continuous homomorphism is an evaluation.

10.15 (Polynomial convexity)**.** (See Conway)

10.16 (Inclusion relation on spectra)**.** (a) $\sigma(a+b)\subset\sigma(a)+\sigma(b)$ and $\sigma(ab)\subset\sigma(a)\sigma(b)$ for unital cases.

(b) $\sigma(a^{-1})=\sigma(a)^{-1}$ for unital cases.

(c) $r(a)^n=r(a^n)$.

10.17 (Spectral radius function)**.** (a) upper semi-continuous

10.18 (Vector-valued complex function theory)**.** Let Ω be an open subset of $\mathbb{C}$ and X a Banach space. For a vector-valued function $f:\Omega\to X$, we say f is *differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}(f(\lambda)-f(\lambda_0))$$

exists in X for every $\lambda\in\Omega$, and *weakly differentiable* if the limit

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{-1}\langle f(\lambda)-f(\lambda_0),x^*\rangle$$

exists in $\mathbb{C}$ for each $x^*\in X^*$ and every $\lambda\in\Omega$. Then, the followings are all equivalent.

(a) f is differentiable.

(b) f is weakly differentiable.

(c) For each $\lambda_0\in\Omega$, there is a sequence $(x_k)_{k=0}^\infty$ such that we have the power series expansion

$$f(\lambda)=\sum_{k=0}^\infty(\lambda-\lambda_0)^kx_k,$$

where the series on the right hand side converges absolutely and uniformly on any closed ball in Ω centered at λ_0.

10.19 (Exponential of an operator)**.**

Chapter 11

C*-algebras

11.1 C* identity

11.1 (Involutive Banach algebras)**.** Banach $*$-algebra: $\|a^*\| = \|a\|$.

11.2 (C*-identity)**.** A normed $*$-algebra $\mathcal{A}$ is called a *C*-algebra* if

(a) $\mathcal{A}$ is Banach,

(b) $\mathcal{A}$ satisfies the C*-identity: $\|x^*x\| = \|x\|^2$.

11.3 (Unitization of C*-algebras)**.**

$$(L_a + \lambda \operatorname{id}_{B(\mathcal{A})})^* = L_{a^*} + \overline{\lambda} \operatorname{id}_{B(\mathcal{A})}.$$

Proof. The C*-identity easily follows from the following inequality:

$$\begin{aligned}
\|(x,\lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\
&= \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\| \\
&= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y)\| \\
&\le \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y\| \\
&= \|(x,\lambda)^*(x,\lambda)\|.
\end{aligned}$$

□

11.2 Continuous functional calculus

11.4 (Gelfand-Naimark representation for C*-algebras)**.** For a commutative unital C*-algebra $\mathcal{A}$, consider the Gelfand transform $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$.

(a) Γ is a $*$-homomorphism.

(b) Γ is an isometry.

(c) Γ is a $*$-isomorphism.

Proof. (a)

(b) Note that we have

$$\|\Gamma a\| = \sup_{\varphi \in \sigma(\mathcal{A})} |\Gamma a(\varphi)| = \sup_{\varphi \in \sigma(\mathcal{A})} |\varphi(a)| = r(a)$$

for all $a \in \mathcal{A}$. If we assume a is self-adjoint, then since $\|a\|^2 = \|a^*a\| = \|a^2\|$, the spectral radius coincides with the norm by the Beurling formula for spectral radius in Banach algebras:

$$\|\Gamma a\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Hence we have for all $a \in \mathcal{A}$ that

$$\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|(\Gamma a)^*\Gamma a\| = \|\Gamma a\|^2.$$

(c) By the part (a) and (b), the image $\Gamma(\mathcal{A})$ is a closed unital $*$-subalgebra of $C(\sigma(\mathcal{A}))$, and it separates points by definition. Then, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$ by the Stone-Weierstrass theorem, which implies $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A})$. □

11.5 (Generators of a C*-algebra)**.** joint spectrum.

11.6 (Continuous functional calculus)**.** Let $\mathcal{A}$ be a C*-algebra, and $a \in \mathcal{A}$ a normal element. Then, we have an isometric $*$-homomorphism

$$C(\sigma(a)) \to \mathcal{A}$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

(a) $\mathrm{id} \mapsto a$.

(b) $(f+g)(a) = f(a) + g(a)$ and $(fg)(a)$.

(c) $(f \circ g)(a) = f(g(a))$.

11.7 (Normal elements)**.** Let a be an element of a unital C*-algebra $\mathcal{A}$. We say a is *normal*, *unitary*, and *self-adjoint* if $a^*a = aa^*$, $a^*a = aa^* = e$, and $a^* = a$ respectively. For normality and self-adjointness, the definitions can be extended to non-unital C*-algebras.

(a) If a is normal, then a is unitary if and only if $\sigma(a) \subset \mathbb{T}$.

(b) If a is normal, then a is self-adjoint if and only if $\sigma(a) \subset \mathbb{R}$.

Proof. (a)

(b) We may assume $\mathcal{A}$ is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \in \mathcal{A},$$

and the inverse of e^{ia} is e^{-ia}. Since the involution $^* : \mathcal{A} \to \mathcal{A}$ is continuous, we can check e^{ia} is unitary by

$$(e^{ia})^* = \sum_{n=1}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia}.$$

For every $\varphi \in \sigma(\mathcal{A})$, then by the part (a) the equality

$$e^{-\operatorname{Im}\varphi(a)} = |e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$$

proves $\varphi(a) \in \mathbb{R}$, hence $\sigma(a) \subset \mathbb{R}$. □

11.8 ($*$-homomorphism)**.** Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism between C*-algerbas.

(a) φ is determined by self-adjoint elements.

(b) φ is contractive.

(c) The iamge of φ is closed.

(d) The induced map $\mathcal{A}/\ker\varphi \to \mathcal{B}$ is an isometry.

11.3 Positive elements

11.9 (Positive elements)**.** Let a, b be elements of a C*-algebra $\mathcal{A}$. We say a is *positive* and write $a \geq 0$ if it is normal and $\sigma(a) \subset \mathbb{R}_{\geq 0}$. If we define a relation $a \leq b$ as $b - a \geq 0$, then we can see that it is a partial order on $\mathcal{A}$.

(a) $a \geq 0$ if and only if $\|\lambda - a\| \leq \lambda$ for some $\lambda \geq \|a\|$.

(b) If $a \geq 0$ and $\sigma(b) \subset \mathbb{R}_{\geq 0}$, then $\sigma(a + b) \subset \mathbb{R}_{\geq 0}$.

(c) If $a^*a \leq 0$, then $a = 0$.

(d) $a \geq 0$ if and only if $a = b^*b$ for some $b \in \mathcal{A}$.

Proof. Let $b := a^*a$. If we consider the case $\mathcal{A} = B(H)$, then one of the properties characterizing b is that $\langle b\xi, \xi\rangle \geq 0$ for all $\xi \in H$. Let $b = b_+ - b_-$. Then,

$$0 \leq \langle bb_-\xi, b_-\xi\rangle = -\langle b_-^3\xi, \xi\rangle \leq 0$$

implies $b_- = 0$. Now we assume $\mathcal{A}$ is defined axiomatically. Then,

$$(ab_-)^*(ab_-) = b_-bb_- = -b_-^3 \leq 0$$

implies $(ab_-)(ab_-)^* \leq 0$ and

$$0 \leq (ab_-)^*(ab_-) + (ab_-)(ab_-)^* \leq 0.$$

Thus we have $ab_- = 0$ and $b_-^3 = 0$.

□

11.10 (Operator monotone operations)**.** (a) If $0 \leq a \leq b$, then $a^{-1} \geq b^{-1}$.

(b) If $a \leq b$, then $cac^* \leq cbc^*$.

11.11 (Positive linear functionals)**.**

11.12 (Approximate identity)**.** separable Let e_α be an approximate identity of $\mathcal{A}$.

(a) For a positive linear functional ω, we have $\lim_\alpha \omega(e_\alpha) = \|\omega\|$.

(b)

(c) separable.

11.4 Representations of C*-algebras

11.13 (Representation of C*-algebras)**.** Let $\mathcal{A}$ be a C*-algebra. A *representation* of $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$ for a Hilbert space H. We say a representation $\pi : \mathcal{A} \to B(H)$ is *non-degenerate* if $\pi(\mathcal{A})H$ is dense in H, *cyclic* if there is $\psi \in H$ such that $\mathcal{A}\psi$ is dense in H, and *irreducible* if there is no proper closed subspace $K \subset H$ such that $\pi(\mathcal{A})K \subset K$.

(a) The following statements are equivalent:

  (i) π is non-degenerate.

  (ii) For each $\xi \in H$ there is $a \in \mathcal{A}$ such that $\pi(a)\xi \neq 0$.

  (iii) $\pi(e_\alpha) \to \mathrm{id}_H$ strongly for an approximate identity e_α of $\mathcal{A}$.

(b) The following statements are equivalent:

(i) π is irreducible

(ii) $\pi(\mathcal{A})' = \mathbb{C}\,\mathrm{id}_H$.

(iii) $\pi(\mathcal{A})$ is strongly dense in $B(H)$.

(iv) Every non-zero vector is cyclic.

11.14 (Gelfand-Naimark-Segal representation)**.** Let $\mathcal{A}$ be a C*-algebra, and ρ be a state on $\mathcal{A}$. The *left kernel* of ρ is defined to be

$$L_\rho := \{a \in \mathcal{A} : \rho(a^*a) = 0\}.$$

(a) L_ρ is a left ideal of $\mathcal{A}$.

(b) $\langle a + L, b + L\rangle := \rho(b^*a)$ is an inner product on $\mathcal{A}/L_\rho$.

(c) There is a unique representation $\pi_\rho : \mathcal{A} \to B(H_\rho)$ such that $\pi_\rho(a)(b + L) := ab + L$ for $a, b \in \mathcal{A}$.

(d) $\pi_\rho : \mathcal{A} \to B(H_\rho)$ is a cyclic representation.

11.15 (Left ideals)**.**

11.16 (Primitive ideals)**.**

11.17 (Hull-kernel topology)**.**

Exercises

11.18 (Operator monotone square)**.** Let $\mathcal{A}$ be a C*-algebra in which the square function is operator monotone, that is, $0 \le a \le b$ implies $a^2 \le b^2$ for any positive elements a and b in $\mathcal{A}$. We are going to show that $\mathcal{A}$ is necessarily commutative. Let a and b denote arbitrary positive elements of $\mathcal{A}$.

(a) Show that $ab + ba \ge 0$.

(b) Let $ab = c + id$ where c and d are self adjoints. Show that $d^2 \le c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \le c^2$. Show that $c^2d^2 + d^2c^2 - 2\lambda d^4 \ge 0$.

(d) Show that $\lambda(cd + dc)^2 \le (c^2 - d^2)^2$.

(e) Show that $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \le c^2$ and deduce $d = 0$.

(f) Extend the result for general exponent: $\mathcal{A}$ is commitative if $f(x) = x^\beta$ is operator monotone for $\beta > 1$.

11.19 (States on unitization)**.** Let $\mathcal{A}$ and $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be a C*-algebra and its unitization respectively. Let $\widetilde{\rho} = \rho \oplus \lambda$ be a bounded linear functional on $\widetilde{\mathcal{A}}$, where $\rho \in \mathcal{A}^*$ and $\lambda \in \mathbb{C}^* = \mathbb{C}$.

(a) $\widetilde{\rho}$ is positive if and only if $\lambda \ge 0$ and $0 \le \rho \le \lambda$.

(b) $\widetilde{\rho}$ is a state if and only if $\lambda = 1$ and ρ is positive with $\|\rho\| \le 1$.

(c) $\widetilde{\rho}$ is a pure state if and only if $\lambda = 1$ and ρ is either a pure state or zero.

11.20 (Representations of $C_0(\Omega)$)**.** Let $\mathcal{A} = C_0(\Omega)$ and μ be a state on $\mathcal{A}$, a regular Borel probability measure on Ω.

(a) The left kernel of μ is $L_\mu = \{f \in \mathcal{A} : f|_{\operatorname{supp}\mu} = 0\}$.

(b) The quotient is $\mathcal{A}/L_\mu \cong C(\operatorname{supp}\mu)$ so that $H_\mu = L^2(\operatorname{supp}\mu, \mu)$.

(c) The canonical cyclic vector is the unity function.

11.21 (Representations of $K(H)$)**.**

11.22 (Approximate eigenvectors)**.**

11.23 (Kadison transitivity theorem)**.**

11.24 (Hereditary C*-algebras)**.**

Problems

*1. A C*-algebra is commutative if and only if a function $f(x) = x(1+x)^{-1}$ is operator subadditive.

Chapter 12

Von Neumann algebras

12.1 Borel functional calculus

12.1 (Von Neumann algebras)**.** A C*-algebra $\mathcal{A}$ is called a *von Neumann algebra* if there is a isometric $*$-homomorphism $\mathcal{A} \to B(H)$ for a Hilbert space H whose image is closed in the weak operator topology.

12.2 (Vigier theorem)**.** Increasing bounded net is convergent in strong operator topology. The boundedness is important because we have to construct a bounded sesquilinear form using the monotone convergence in $\mathbb{R}$.

12.3 (Borel functional calculus)**.** Let $\mathcal{A}$ be a von Neumann algebra.

$$B^\infty(\sigma(a)) \to \mathcal{A}.$$

(a) The Borel functional calculus is in general not injective.

(b) If we endow the topology of pointwise convergence on $B^\infty(\sigma(a))$ and the strong operator topology on $\mathcal{A}$, then the Borel functional calculus is continuous.

(c) not isometric, even if it is injective.

(d) Every von Neumann algebra is the closed span of projections.

12.4. (b) By the bounded convergence theorem.

(d) This is because $\sigma(a) \subset \mathbb{C}$ is compact so that it is separable and metrizable; every bounded measurable function is a pointwise limit of simple functions.

12.2 Density theorems

12.5 (Bicommutant theorem)**.** Let $\mathcal{A}$ be a non-degenerate C*-subalgebra of $B(H)$.

(a) $\mathcal{A}'$ and $\mathcal{A}''$ are weakly closed.

(b) For $a \in \mathcal{A}''$ and $\xi \in H$, there is a sequence $a_n \in \mathcal{A}$ such that $a_n(\xi) \to a(\xi)$.

(c) For $a \in \mathcal{A}''$ and $\xi_1, \cdots, \xi_m \in H$, there is a sequence $a_n \in \mathcal{A}$ such that $a_n(\xi_i) \to a(\xi_i)$ for all i.

(d) $\mathcal{A}$ is von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.

Proof. (b) Let $K := \overline{\mathcal{A}\xi}$ be the cyclic subspace of ξ in H and p its orthogonal projection. We claim $a\xi \in K$. For every $b \in \mathcal{A}$, we have $bK \subset K$ because the multiplication by b is continuous on H, and $b^*K \subset K$ because $\mathcal{A}$ is self-adjoint. It means that K reduces all $b \in \mathcal{A}$, and then $bp = pb$ implies $ap = pa$, so K also reduces a. Therefore, $aK \subset K$ proves $a\xi = \lim_\alpha e_\alpha a\xi \in K$, where e_α is an approximate identity of $\mathcal{A}$.

(e) Since $\overline{\mathcal{A}}^w$ is closed convex, $\overline{\mathcal{A}}^s = \overline{\mathcal{A}}^w$. Also, $\mathcal{A}''$ is weakly closed, $\overline{\mathcal{A}}^s \subset \mathcal{A}''$. □

12.6 (Kaplansky density theorem)**.**

12.3 Envelpoing von Neumann algebra

12.7 (Sherman-Takeda theorem)**.**

12.8 (Conditional expectations)**.** Let $\mathcal{B}$ be a closed subalgebra of a C*-algebra $\mathcal{A}$ with $1_\mathcal{A} \in \mathcal{B}$. A *conditional expectation* is a positive $\mathcal{B}$-bimodule map $\varphi : \mathcal{A} \to \mathcal{B}$.

(a)

(b) Tomiyama theorem: contractive idempotent linear map $\mathcal{A} \to \mathcal{B}$ is a conditional expectation.

12.9 (Sakai theorem)**.**

central projections

Exercises

12.10 (Extremally disconnected space)**.** $\sigma(B^\infty(\Omega))$ is extremally disconnected.

resolution of identity normal operator theories: multiplicity, invariant subspaces L^∞ representation