General Topology

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4.1 (Comparison of metrics)**.** Let d_1 and d_2 be metrics on a set X. We say that d_1 is *stronger than* d_2 (equivalently, d_2 is *weaker than* d_1) or d_1 *refines* d_2 if for any $x \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon),$$

where B_1 and B_2 refer to balls within the metrics d_1 and d_2 respectively.

(a) This refinement relation is a preorder.

(b) d_1 is stronger than d_2 if and only if every sequence that converges to $x \in X$ in d_1 converges to x in d_2.

(c) d_1 is stronger than d_2 if and only if the identity map $\mathrm{id} : (X, d_1) \to (X, d_2)$ is continuous.

Proof. (a) It is enough to show the transitivity. Suppose there are three metric d_1, d_2, and d_3 on a set X such that d_1 is stronger than d_2 and d_2 is stronger than d_3. For $i = 1, 2, 3$, let B_i be a notation for the balls defined with the metric d_i.

Take $x \in X$ and $\varepsilon > 0$ arbitrarily. Then, we can find $\varepsilon' > 0$ such that

$$B_2(x, \varepsilon') \subset B_3(x, \varepsilon).$$

Also, we can find $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon').$$

Therefore, we have $B_1(x,\delta) \subset B_3(x,\varepsilon)$ which implies that d_1 refines d_3.

(b) ($\Rightarrow$) Let $\{x_n\}_n$ be a sequence in X that converges to x in d_1. By the assumption, for an arbitrary ball $B_2(x,\varepsilon) = \{y : d_2(x,y) < \varepsilon\}$, there is $\delta > 0$ such that

$$B_1(x,\delta) \subset B_2(x,\varepsilon),$$

where $B_1(x,\delta) = \{y : d_1(x,y) < \delta\}$. Since $\{x_n\}_n$ converges to x in d_1, there is an integer n_0 such that

$$n > n_0 \implies x_n \in B_1(x,\delta).$$

Combining them, we obtain an integer n_0 such that

$$n > n_0 \implies x_n \in B_2(x,\varepsilon).$$

It means $\{x_n\}$ converges to x in the metric d_2.

($\Leftarrow$) We prove it by contradiction. Assume that for some point $x \in X$ we can find $\varepsilon_0 > 0$ such that there is no $\delta > 0$ satisfying $B_1(x,\delta) \subset B_2(x,\varepsilon_0)$. In other words, at the point x, the difference set $B_1(x,\delta) \setminus B_2(x,\varepsilon_0)$ is not empty for every $\delta > 0$. Thus, we can choose x_n to be a point such that

$$x_n \in B_1\left(x, \tfrac{1}{n}\right) \setminus B_2(x,\varepsilon_0)$$

for each positive integer n by putting $\delta = \frac{1}{n}$.

We claim $\{x_n\}_n$ converges to x in d_1 but not in d_2. For $\varepsilon > 0$, if we let $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$ so that we have $\frac{1}{n_0} \le \varepsilon$, then

$$n > n_0 \implies x_n \in B_1\left(x, \tfrac{1}{n}\right) \subset B_1(x,\varepsilon).$$

So $\{x_n\}_n$ converges to x in d_1. However in d_2, for $\varepsilon = \varepsilon_0$, we can find such n_0 like d_1 since

$$x_n \notin B_2(x,\varepsilon_0)$$

for every n. Therefore, $\{x_n\}$ does not converges to x in d_2. □

4.2 (Equivalence of metrics)**.** Let d_1 and d_2 be metrics on a set X. They are said to be *(topologically) equivalent* if they refines each other.

(a) d_1 and d_2 are equivalent if for each x in X there exist two constants C_1 and C_2 such that $d_2(x,y) \le C_1 d_1(x,y)$ and $d_1(x,y) \le C_2 d_2(x,y)$ for all y in X.

(b) d_1 and d_2 are equivalent if $d_2 = f \circ d_1$ for a monotonically increasing $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ that is continuous at 0.

Proof. (a) Let d_1 and d_2 be metrics on a set X. Suppose for each point x there exists a constant C which may depend on x such that $d_2(x,y) \leq Cd_1(x,y)$ for all $y \in Y$. We will show d_1 is stronger than d_2.

(b) For any ball $B_1(x,\varepsilon)$, we have a smaller ball

$$B_2(x, f(\varepsilon)) \subset B_1(x,\varepsilon)$$

since $f(d(x,y)) < f(\varepsilon)$ implies $d(x,y) < \varepsilon$. Conversely, take an arbitrary ball $B_2(x,\varepsilon)$. Since f is continuous at 0, we can find $\delta > 0$ such that

$$d(x,y) < \delta \quad \Longrightarrow \quad f(d(x,y)) < \varepsilon,$$

which implies $B_1(x,\delta) \subset B_2(x,\varepsilon)$. □

A metric can be viewed as a function that takes a sequence as input and returns whether the sequence converges or diverges. That is, a metric acts like a criterion which decides convergence of sequences. Take note on the fact that the sequence of real numbers defined by $x_n = \frac{1}{n}$ converges in standard metric but diverges in discrete metric. Like this example, even for the same sequence on a same set, the convergence depends on the attached metrics. What we are interested in is comparison of metrics and to find a proper relation structure. If a sequence converges in a metric d_2 but diverges in another metric d_1, we would say d_1 has stronger rules to decide the convergence. Refinement relation formalizes the idea.

Unlike metrics, there exist two different topologies that have same sequential convergence data. For example, a sequence in an uncountable set with cocountable topology converges to a point if and only if it is eventually at the point, which is same with discrete topology. This means the informations of sequence convergence are not sufficient to uniquely characterize a topology. Instead, convergence data of generalized sequences also called nets, recover the whole topology. For topologies having a property called the first countability, it is enough to consider only usual sequences in spite of nets. What we did in this subsection is not useless because topology induced from metric is a typical example of first countable topologies. These kinds of problems will be profoundly treated in Chapter 3.

One can ask some results for the equivalence of metrics characterized by a same set of continuous functions. However, they are generally difficult problems: is it possible to recover the base space from a continuous function space or a path space?

4.3 Sum of metrics

Topologies are occasionally described by not a single but several metrics. It provides a useful method to construct a metric or topology, which can be applied to a quite wide range of applications. Specifically, in a conventional way, metrics can be summed or taken maximum to make another metric out of olds. The following proposition can be easily generalized to an arbitrary finite number of metrics by mathematical induction.

Proposition 4.3.1. *Let d_1 and d_2 be metrics on a set X. For a sequecne $\{x_n\}_n$ in X, the following statements are all equivalent:*

(a) *it converges to x in both d_1 and d_2,*

(b) *it converges to x in $d_1 + d_2$,*

(c) *it converges to x in $\max\{d_1, d_2\}$.*

In particular, the metrics $d_1 + d_2$ and $\max\{d_1, d_2\}$ are equivalent.

Proof. We skip to prove $d_1 + d_2$ and $\max\{d_1, d_2\}$ are metrics.

(b) or (c)⇒(a) The inequalities $d_i \le d_1 + d_2$ and $d_i \le \max\{d_1, d_2\}$ imply the desired results.

(a)⇒(b) For $\varepsilon > 0$, we may find positive integers n_1 and n_2 such that $n > n_1$ and $n > n_2$ imply $d_1(x_n, x) < \frac{\varepsilon}{2}$ and $d_2(x_n, x) < \frac{\varepsilon}{2}$ respectively. If we define $n_0 := \max\{n_1, n_2\}$, then

$$n > n_0 \quad \Longrightarrow \quad d_1(x_n, x) + d_2(x_n, x) < \varepsilon.$$

(a)⇒(c) Take n_0 as we did previously. Then,

$$n > n_0 \quad \Longrightarrow \quad \max\{d_1(x_n, x), d_2(x_n, x)\} < \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

Remark. In general, for any norm $\|\cdot\|$ on $\mathbb{R}^2$, the function $\|(d_1, d_2)\|$ defines another equivalent metric.

There is also a method for combining not only finite family of metrics, but also infinite family of metrics. Since the sum of infinitely many positive numbers may diverges to infinity, we cannot sum the metrics directly. The strategy is to “bound” the metrics. We call a metric bounded when the range of metric function is bounded.

Proposition 4.3.2. *Every metric possesses an equivalent bounded metric.*

Proof. Let d be a metric on a set. Let f be a bounded, monotonically increasing, and subadditive function on $\mathbb{R}_{\geq 0}$ that is continuous at 0 and satisfies $f^{-1}(0) = \{0\}$. The mostly used examples are

$$f(x) = \frac{x}{1+x} \quad \text{and} \quad f(x) = \min\{x, 1\}.$$

Then, $f \circ d$ is a bounded metric equivalent to d by Example 1.4. □

Definition 4.3.1. Let d be a metric on a set X. A *standard bounded metric* means either metric

$$\min\{d, 1\} \quad \text{or} \quad \frac{d}{d+1},$$

and we will denote it by $\widehat{d}$.

Proposition 4.3.3. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a countable family of metrics on a set X. For a sequence $\{x_n\}_n$ in X, the following statements are all equivalent:*

(a) *it converges in d_i for every i,*

(b) *it converges in a metric*

$$d(x,y) := \sum_{i\in\mathbb{N}} 2^{-i}\widehat{d}_i(x,y),$$

(c) *it converges in a metric*

$$d'(x,y) := \sup_{i\in\mathbb{N}} i^{-1}\widehat{d}_i(x,y).$$

In particular, the metrics d and d' are equivalent.

Proof. The functions d and d' in (b) and (c) are well-defined by the monotone convergence theorem and the least upper bound property. We skip checking for them to satisfy the triangle inequality and be metrics.

(b) or (c)⇒(a) We have ineuqalities $\widehat{d}_i \leq 2^i d$ and $\widehat{d}_i \leq i d'$ for each i, so convergence in d or d' implies the convergence in each $\widehat{d}_i$. The equivalence of $\widehat{d}_i$ and d_i implies the desired result.

(a)⇒(b) Suppose a sequence $\{x_n\}_n$ converges to a point x in d_i for every index i. Take an arbitrary small ball $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad \Longrightarrow \quad \widehat{d}_i(x_n, x) < \frac{\varepsilon}{2}.$$

Define $k := \lceil 1 - \log_2 \varepsilon \rceil$ so that we have $2^{-k} \leq \frac{\varepsilon}{2}$. With this k, define

$$n_0 := \max_{1 \leq i \leq k} n_i.$$

If $n > n_0$, then

$$\begin{aligned} d(x_n, x) &= \sum_{i=1}^{k} 2^{-i} \widehat{d}_i(x_n, x) + \sum_{i=k+1}^{\infty} 2^{-i} \widehat{d}_i(x_n, x) \\ &< \sum_{i=1}^{k} 2^{-i} \frac{\varepsilon}{2} + \sum_{i=k+1}^{\infty} 2^{-i} \\ &< \frac{\varepsilon}{2} + 2^{-k} \leq \varepsilon, \end{aligned}$$

so x_n converges to x in the metric d.

(a)⇒(c) Suppose a sequence $\{x_n\}_n$ converges to a point x in each d_i, and take an arbitrary small ball $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad \Longrightarrow \quad \widehat{d}_i(x_n, x) < \varepsilon.$$

Define $k := \lceil \frac{1}{\varepsilon} \rceil$ so that we have $k^{-1} \leq \varepsilon$. With this k, define

$$n_0 := \max_{1 \leq i \leq k} n_i.$$

If $n > n_0$, then

$$i^{-1} \widehat{d}_i(x, y) \leq \widehat{d}_i(x, y) < \varepsilon \quad \text{for} \quad i \leq k$$

and

$$i^{-1} \widehat{d}_i(x, y) \leq i^{-1} < k^{-1} \leq \varepsilon \quad \text{for} \quad i > k$$

imply $d(x_n, x) < \varepsilon$, which means that x_n converges to x in the metric d. □

Combination of uncountably many metrics does not result in a single metric, but a topology which cannot be induced from a metric in general. It will be discussed in the rest of the note.

Remark. A metric

$$d''(x, y) = \sup_{i \in \mathbb{N}} d_i(x, y)$$

is not used because the convergence in this metric is a stronger condition than the convergence with respect to each metric d_i. In other words, this metric generates a finer(stronger) topology than the topology generated by subbase of balls. For example, the topology on $\mathbb{R}^{\mathbb{N}}$ generated by this metric defined with the projection pseudometrics is exactly what we often call the box topology.

We can also form a metric by summation of generalized metrics, called pseudometrics, which is defined by missing the nondegeneracy condition from the original definition of metric.

Definition 4.3.2. A function $\rho : X \times X \to \mathbb{R}_{\geq 0}$ is called a *pseudometric* if

(a) $\rho(x,x) = 0$ for all $x \in X$,

(b) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$, (symmetry)

(c) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$. (triangle inequality)

For pseudometrics, it is possible to duplicate every definition we studied in metric spaces: convergence of a sequence, continuity between a set endowed with a pseudometric, refinement and equivalence relations, and countable sum of bounded pseudometrics to make a new pseudometric. Furthermore, every statement for metrics can be generalized to pseudometrics since we have not actually used the condition that $d(x,y) = 0$ implies $x = y$. In fact, we have a flaw that the limit of a convergent sequence may not be unique within a pseudometric.

Example 4.3.1. Let $\rho(x,y) = \rho((x_1,x_2),(y_1,y_2)) = |x_1 - y_1|$ be a pseudometric on $\mathbb{R}^2$. Consider a sequence $\{(\frac{1}{n}, 0)\}_n$. Since $(0,c)$ satisfies

$$\rho((\tfrac{1}{n}, 0), (0,c)) = \tfrac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

for any real number c, the sequence converges to (x_1, x_2) if and only if $x_1 = 0$.

Although sequences may have several limits in each pseudometric, the sum of a family of pseudometrics can allow the sequences to have at most one limit, only if the sum satisfies the axioms of a metric.

Definition 4.3.3. A family of pseudometrics $\{\rho_\alpha\}_\alpha$ on a set X is said to *separate points* if the condition

$$\rho_\alpha(x,y) = 0 \quad \text{for all } \alpha$$

implies $x = y$.

Proposition 4.3.4. (a) *A finite family of pseudometrics $\{\rho_i\}_{i=1}^N$ separates points if and only if the pseudometric $\rho := \sum_{i=1}^N \rho_i$ is a metric.*

(b) *A countable family of pseudometrics $\{\rho_i\}_{i\in\mathbb{N}}$ separates points if and only if the pseudometric defined by*

$$\rho := \sum_{i\in\mathbb{N}} 2^{-i}\widetilde{\rho}_i \quad \text{or} \quad \sup_{i\in\mathbb{N}} i^{-1}\widetilde{\rho}_i,$$

where $\widetilde{\rho}_i$ is either $\min\{\rho_i, 1\}$ or $\rho_i/(\rho_i + 1)$, is a metric.

Exercises

4.3 (Discrete metrics)**.** Let X be a set, and define a metric as

$$d(x,y) := \begin{cases} 0 & ,x = y \\ 1 & ,x \neq y \end{cases}.$$

This metric is called *discrete*.

(a) The discrete metric is a strongest metric on X.

(b)

(c) A sequence x_n converges to a point x in a discrete metric space if and only if there is n_0 such that $n \geq n_0$ implies $x_n = x$.

4.4. (a) Let d be a metric on a set X. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $f^{-1}(0) = \{0\}$. If f is monotonically increasing and subadditive, then $f \circ d$ satisfies the triangle inequality, hence is another metric on X. Note that a function f is called *subadditive* if $f(x+y) \leq f(x) + f(y)$ for all x, y in the domain.

(b) Let $G = (V, E)$ be a connected graph. Define $d : V \times V \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the distance of two vertices; the length of shortest path connecting two vertices. Then, (V, d) is a metric space.

(c) Let $\mathcal{P}(X)$ be the power set of a finite set X. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the cardinality of the symmetric difference; $d(A, B) := |(A-B) \cup (B-A)|$. Then $(\mathcal{P}(X), d)$ is a metric space.

(d) Let C be the set of all compact subsets of $\mathbb{R}^d$. Recall that a subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded. Then, $d : C \times C \to \mathbb{R}_{\geq 0}$ defined by

$$d(A, B) := \max\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \ \sup_{b \in B} \inf_{a \in A} \|a - b\| \}$$

is a metric on C. It is a little special case of *Hausdorff metric*.

4.5 (Kuratowski embedding)**.** While every subset of a normed space is a metric space, we have a converse statement that every metric space is in fact realized as a subset of a normed space. Let X be a metric space, and denote by $C_b(X)$ the space of continuous and bounded real-valued functions on X with uniform norm given by

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Fix a point $p \in X$, which will serve as the origin.

(a) Show that a map $\phi : X \to C_b(X)$ such that

$$[\phi(x)](t) = d(x,t) - d(p,t)$$

is well-defined.

(b) Show that the map ϕ is an isometry; $d(x,y) = \|\phi(x) - \phi(y)\|$.

4.6 (Equivalence of norms in finite dimension). Let V be a vector space of dimension d over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Fix a basis $\{e_i\}_{i=1}^d$ on V and let $x = \sum_{i=1}^d x_i e_i$ denote an arbitrary element of V. We will prove all norms are equivalent to the standard Euclidean norm defined for this fixed basis:

$$\|x\|_2 := (\sum_{i=1}^{d} |x_i|^2)^{\frac{1}{2}}.$$

With this standard norm any theorems studied in elementary analysis including the Bolzano-Weierstrass theorem are allowed to be applied. Take a norm $\|\cdot\|$ on V which may differ to $\|\cdot\|_2$.

(a) Find a constant C_2 such that $\|x\| \le C_2 \|x\|_2$ for all $x \in V$.

(b) Show that if no constant C satisfies $\|x\|_2 \le C\|x\|$, then there exists a sequence $\{x_n\}_n$ such that $\|x_n\|_2 = 1$ and $\|x_n\| < \frac{1}{n}$.

(c) Show that there exists a constant C satisfies $\|x\|_2 \le C\|x\|$.

Chapter 5

Uniform spaces

5.1 Defintions of uniformity

order, metric,

Exercises

Chapter 6

Cauchy spaces

6.1 Completeness

6.2 Completion

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6.1. Even for a first countable uniform space, completeness and sequential completeness are not equivalent.

Part III

Topological properties

Chapter 7

Compactness

Chapter 8

Connectedness

Chapter 9

Separation Axioms

9.1 Regular spaces

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Dieudonne's theorem: paracompact Hausdorff is normal.

Part IV

Continuous Function Spaces

Chapter 10

Compact-open topology

Topologies on $C_c(X)$ for LCH X: weaker to stronger

(a) Topology of compact convergence: $\overline{C_c(X)} = \overline{C_{\mathrm{loc}}(X)} = C_{\mathrm{loc}}(X)$.

(b) Topology of uniform convergence: $\overline{C_c(X)} = C_0(X)$, $C_c(X)^* = M(X)$.

(c) Inductive topology: $\overline{C_c(X)} := \overline{\operatorname{colim}_{U \Subset X} C_c(U)} = C_c(X)$.

The space $C_{\mathrm{loc}}(X)$ is defined to be $C(X)$ as a set endowed with the topology of compact convergence.

Chapter 11

Approximation of continuous functions

11.1 Arzela-Ascoli theorem

11.2 Stone-Weierstrass theorem

11.1 (Bernstein polynomial)**.** We want to show $\mathbb{R}[x]$ is dense in $C([0,1],\mathbb{R})$. Let $f \in C([0,1],\mathbb{R})$ and define *Berstein polynomials* $B_n(f) \in \mathbb{R}[x]$ for each n such that

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

(a) $B_n(f)$ uniformly converges to f on $[0,1]$.

(b) There is a sequence $p_n \in \mathbb{R}[x]$ with $p_n(0) = 0$ uniformly convergent to $x \mapsto |x|$ on $[-1,1]$.

Proof. (b) Let

$$B_n(x) := \sum_{k=0}^{n} \left|1 - \frac{2k}{n}\right| \binom{n}{k} (1-2x)^k (2x-1)^{n-k}.$$

Since $B_n(x) \to |x|$ uniformly on $[-1,1]$ and $B_n(0) \to 0$, we have $B_n(x) - B_n(0) \to |x|$ uniformly on $[-1,1]$. $\square$

11.2 (Taylor series of square root)**.** We want to show the absolute value is approximated by polynomials in $C([-1,1],\mathbb{R})$ in another way. Let

$$f_n(x) := \sum_{k=0}^{n} a_k (x-1)^k$$

be the partial sum of the Taylor series of the square root function $\sqrt{x}$ at $x = 1$.

(a) By Abel's theorem, f_n uniformly converges to $\sqrt{x}$ on $[0,1]$

(b) There is a sequence $p_n \in \mathbb{R}[x]$ with $p_n(0) = 0$ uniformly convergent to $x \mapsto |x|$ on $[-1,1]$.

11.3 (Proof of Stone-Weierstrass theorem)**.** Let X be a compact Hausdorff space and $S \subset C(X,\mathbb{R})$. We say that S *separates points* if for every distinct x and y in X there is $f \in S$ such that $f(x) \neq f(y)$, and that S *vanishes nowhere* if for every x in X there is $f \in S$ such that $f(x) \neq 0$.

Let $\mathcal{A} = \overline{S\mathbb{R}[S]}$ be the real Banach subalgebra of $C(X,\mathbb{R})$ generated by S.

(a) $\mathcal{A}$ is a lattice.

(b) $\mathcal{A}$ is equal to $C(X,\mathbb{R})$

Locally compact version, complex version

11.4. Some examples

(a) $z\mathbb{R}[z]$ is dense in $C([1,2],\mathbb{R})$.

(b) $\mathbb{C}[z]$ is dense in $C([0,1],\mathbb{C})$.

(c) $z\mathbb{C}[z,\bar{z}]$ is dense in $C(\mathbb{T},\mathbb{C})$.

Chapter 12

Topological measures

12.1 Baire and Borel measures

locally finite Baire measures can define a linear functional on C_c.

12.2 Regular Borel measures

On which spaces are all Borel measures regular or Radon or etc?

Radon measures vs regular Borel measures.

Lusin theorem for normal spaces.

Lusin theorem for locally compact Hausdorff spaces.

Difference between tightness and inner regularity.

12.3 Riesz-Markov-Kakutani representation theorem

Consider

$$\text{regBorel} \to \text{Borel} \to \text{Baire} \to C_c^{*+}.$$

Note that the surjectivity of Borel$\to C_c^{*+}$ is valid for EVERY topological space! Regularity of the constructed measure comes from the results of section 12.2.

12.1 (Existence of Borel measures)**.** Let $(\Omega, \mathcal{T})$ be a topological space and I a positive linear functional on $C_c(\Omega)$. Define a set function $\rho : \mathcal{T} \to [0, \infty]$ such that

$$\rho(U) := \sup\{\, I(f) : f \in C_c(U),\ 0 \le f \le 1\,\}$$

for $U \in \mathcal{T}$. Define the associated outer measure $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$ such that

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^{\infty} \rho(U_i) : E \subset \bigcup_{i=1}^{\infty} U_i,\ U_i \in \mathcal{T} \right\}$$

for $E \in \mathcal{P}(\Omega)$. Let $\mathcal{M}$ be the σ-algebra of Carathéodory measurable subsets relative to μ^*, and $\mathcal{B}$ the Borel σ-algebra.

(a) $\mu^*|_{\mathcal{T}} = \rho$.

(b) $\mathcal{T} \subset \mathcal{M}$.

(c) $I(f) = \int f \, d\mu$ for $f \in C_c(\Omega)$.

Proof. (a) If we show the countable subadditivity of ρ, then we have

$$\mu^*(E) = \inf\{ \rho(U) : E \subset U \in \mathcal{T} \},$$

so we are done.

Let $\{U_i\}_{i=1}^{\infty}$ is a countable open cover of an open set U. Take any $f \in C_c(U)$ with $0 \le f \le 1$. There is a finite subcover $\{U_j\}_{j=1}^{n}$ of $\operatorname{supp} f$. Let χ_j a partition of unity subordinate to $\{U_j\}_j$. Then,

$$I(f) = I\left(\sum_{j=1}^{n} f \chi_j \right) = \sum_{j=1}^{n} I(f \chi_j) \le \sum_{j=1}^{n} \rho(U_j) \le \sum_{i=1}^{\infty} \rho(U_i).$$

(If this sum is infinite, then the second equality may fail, so the compactness of the support is necessary. In particular, the existence of measure in the C_b theory is not guaranteed.)

(b) Clearly $\mu^*(E) \le \mu^*(E \cap U) + \mu^*(E \setminus U)$. It is enough to show only for $\mu^*(E) < \infty$. Let $\varepsilon > 0$. There are $U_i \in \mathcal{T}$ such that $E \subset V := \bigcup_{i=1}^{\infty} U_i$ and

$$\mu^*(E) + \frac{\varepsilon}{3} > \sum_{i=1}^{\infty} \rho(U_i) \ge \rho(V).$$

(We used the countable subadditivity of ρ here, but it is in fact not required.) Take $f \in C_c(V \cap U)$ such that $0 \le f \le 1$ and

$$\rho(V \cap U) - \frac{\varepsilon}{3} < I(f).$$

Take $g \in C_c(V \setminus \operatorname{supp} f)$ such that $0 \le g \le 1$ and

$$\rho(V \setminus \operatorname{supp} f) - \frac{\varepsilon}{3} < I(g).$$

Then, since $f+g \in C_c(V)$ and $0 \le f+g \le 1$,

$$\rho(V) \ge I(f+g) > \rho(V \cap U) + \rho(V \setminus \operatorname{supp} f) - \frac{2}{3}\varepsilon.$$

Therefore,

$$\mu^*(E) + \varepsilon > \rho(V \cap U) + \rho(V \setminus \operatorname{supp} f) \ge \mu^*(V \cap U) + \mu^*(V \setminus U)$$

(c) Want to show $f|_C = 1$ and $f \ge 0$ imply $\mu(C) \le I(f)$. Let $U = f^{-1}((1-\varepsilon, \infty))$. Then, $C \subset U$.

For any $g \in C_b(\Omega)$ with $0 \le g \le 1$ and $g|_{U^c} = 0$, $(1-\varepsilon)g \le f$ implies $I(g) \le (1-\varepsilon)^{-1} I(f)$, so $\mu(U) \le (1-\varepsilon)^{-1} I(f)$. Done. □

12.2 (Uniqueness of regular Borel measures)**.** On $C_c(\Omega)$ is the most general case, then C_b case are covered. Injectivity heavily relies on the regularity of measures and the Lusin theorem.

(a) Injective if Ω is locally compact Hausdorff.

(b) Injective if Ω is normal.

12.3 (Dual of continuous function spaces)**.**

12.4

Lemma 12.4.1. *Let μ be a Borel measure on a LCH X. Then, μ is inner regular on open sets iff*

$$\mu(U) = \|\mu\|_{C_c(U)^*}$$

for every open U in X.

Proof. ($\Leftarrow$) ($\ge$) For $f \in C_c(U)$, we have

$$\left|\int f\, d\mu\right| = \left|\int_U f\, d\mu\right| \le \mu(U)\,\|f\|.$$

($\le$) Since μ is inner regular on U, there is a compact set $K \subset U$ such that $\mu(U) - \mu(K) < \varepsilon$ (for the case $\mu(U) = \infty$, we can deal with separately). We can find a nonnegative function $f \in C_c(U)$ with $f|_K \equiv 1$ and $f \le 1$ by the construction of Urysohn. Then, for all $\varepsilon > 0$ we have

$$\mu(U) < \mu(K) + \varepsilon \le \int f\, d\mu + \varepsilon \le \|\mu\|_{C_c^*(U)} + \varepsilon.$$

($\Rightarrow$) Let $f \in C_c(U)$ be a function such that $\|f\| = 1$ and

$$\mu(U) - \varepsilon < \int f \, d\mu.$$

Let $K = \operatorname{supp}(f)$. Then

$$\mu(K) \geq \int f > \mu(U) - \varepsilon.$$

□

Proposition 12.4.2. *A Radon measure is inner regular on all σ-finite Borel sets. (Folland's)*

Proof. First we approximate Borel sets of finite measure, with compact sets. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U - E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned} \mu(K - V) &\geq \mu(K) - \mu(V) \\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U - E) + \frac{\varepsilon}{2}\right) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^{\infty} K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^{\infty} \mu(K_n) > \sum_{n=1}^{\infty} \left(\mu(E_n) - \frac{\varepsilon}{2^n}\right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. □

Theorem 12.4.3. *If every open set in X is σ-compact(i.e. Borel sets and Baire sets coincide), then every locally finite Borel measure is regular.*

Proposition 12.4.4. *In a second countable space, every open set is σ-compact(i.e. Borel sets and Baire sets coincide).*

Two corollaries are presented as follows:

$$\begin{tikzcd}
\begin{array}{c}\text{locally finite}\\ \text{Borel regular}\end{array} \arrow[r] & \text{Radon} \arrow[l, dashed, bend right, "X \text{ is } \sigma\text{-compact}"'] \arrow[r] & \begin{array}{c}\text{locally finite}\\ \text{Borel}\end{array} \arrow[ll, dashed, bend left, "X \text{ is second countable}"]
\end{tikzcd}$$

12.4. Let X be compact. A positive linear functional ρ on $C(X)$ is bounded with norm $\rho(1)$.

Proof. Since $0 \le \rho(\|f\| \pm f) = \|f\|\rho(1) \pm \rho(f)$, we have $|\rho(f)| \le \rho(1)\|f\|$. □

12.5. Let X be a locally compact Hausdorff space.

(a) The Baire σ-algebra is generated by compact G_δ sets.

(b) If X is second countable, then every Baire set is Borel.

Solution. (b) (A second countable locally compact space is σ-compact.

Since X is σ-compact and Hausdorff, every closed set is a countable union of compact sets, so the Borel σ-algebra on X is generated by compact sets.)

Since locally compact Hausdorff space is regular, the Urysohn metrization implies X is metrizable, and every closed sets in metrizable space is G_δ set. □

12.4.1 The Riesz-Kakutani theorem for positive linear functionals

12.6. Let X be compact. There is a map from the set of finite Baire measures to the set of positive linear functionals on $C(X)$.

Solution. A function in $C(X)$ is Baire measurable and bounded. Thus the integration is well-defined. □

12.7. Let X be compact. There is a map from the set of positive linear functionals on $C(X)$ to the set of finite regular Borel measures.

Solution. i. and ii. and iii. of Theorem 7.2. □

12.8. Let X be compact. Let ρ be a positive linear functional on $C(X)$. Let ν be the regular Borel measure associated to ρ. Then, $\rho(f) = \int f \, d\nu$.

Solution. iv. of Theorem 7.2. □

12.9. Let X be compact. Let ν be a finite regular Borel measure. Let ν' be the regular Borel measure associated to the positive linear functional $f \mapsto \int f \, d\nu$. Then, $\nu = \nu'$ on Borel sets.

Solution. Theorem 7.8. □

The two results above establish the correspondence between positive linear functionals and regular Borel measures. The following is an additional topic: Borel extension of Baire measures.

12.10. Let X be compact. Let μ be a finite Baire measure. Let ν be the regular Borel measure associated to the positive linear functional $f \mapsto \int f \, d\mu$. Then, $\mu = \nu$ on Baire sets.

Solution. Let μ, ν be finite Baire measures. Enough to show if $\int f \, d\mu = \int f \, d\nu$ then $\mu = \nu$ according to the preceding two results.

Enough to show the regularity of Baire measures. □

- A second countable locally compact space is σ-compact.
- A σ-compact locally compact space is paracompact.
- A second countable regular space is paracompact.
- A locally compact Hausdorff space is regular.

semiring σ-finiteness implies the uniqueness