Algebraic Geometry

Ikhan Choi

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Contents

Part I

Chapter 1

Schemes

1.1 (Affine schemes)**.** Let A be a ring. Every ring will be commutative and unital if not mentioned. The spectrum $\operatorname{Spec} A$ of A is defined as the partially ordered set of all prime ideals of A. It is topologized by the Zariski topology in which a subset of $\operatorname{Spec} A$ is closed if and only if it is given by the zero set $\operatorname{Spec} A/\mathfrak{a} = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{a} \subset \mathfrak{p}\}$ of some ideal $\mathfrak{a} \subset A$. It also admits a canonical structure sheaf $\mathcal{O}_{\operatorname{Spec} A} : \operatorname{Open}(\operatorname{Spec} A)^{\mathrm{op}} \to \mathrm{CRing}$ of rings characterized by

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) := A_f = A[f^{-1}], \qquad D(f) := (\operatorname{Spec} A/(f))^c = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}, \qquad f \in A.$$

In conclusion, a ring A defines a locally ringed space $\operatorname{Spec} A$.

(a) There is a one-to-one correspondence between Zariski closed sets and radical ideals, and a Zariski closed subset is an upper set.

(b) An ideal $\mathfrak{a}$ of A is proper if and only if the zero set $\operatorname{Spec} A/\mathfrak{a}$ is non-empty.

1.2 (Schemes)**.** A *scheme* is a locally ringed space such that affine open subsets form a basis. In fact, the existence of an affine open cover is enough.

A *generic point* of a topological space is a point whose closure is the whole space. A *closed point* of a topological space is a point which is closed. specialization and generalization. Closed points of an affine scheme are exactly maximal ideals.

$$\operatorname{Spec} \mathbb{Z} = \{(p) : p \in \mathbb{Z} \text{ prime}\} \cup \{(0)\}.$$
$$\operatorname{Spec} \mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}} = \{(x-a) : a \in \mathbb{R}\} \cup \{(f) : f \in \mathbb{R}[x] \text{ irreducible quadratic}\} \cup \{(0)\}.$$
$$\operatorname{Spec} \mathbb{Q}[x] = \mathbb{A}^1_{\mathbb{Q}}$$
$$\operatorname{Spec} \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p} = \{(f) : f \in \mathbb{F}_p[x] \text{ irreducible}\} \cup \{(0)\}.$$
$$\operatorname{Spec} \mathbb{C}[x,y] = \mathbb{A}^2_{\mathbb{C}} = \{(x-a, y-b) : (a,b) \in \mathbb{C}^2\} \cup \{(f) : f \in \mathbb{C}[x,y] \text{ irreducible}\} \cup \{(0)\}.$$

Nulstellensatz states that the set of closed points of the affine scheme $\mathbb{A}^n$ over an algebraically closed field k is exactly k^n. It connects the theory of classical algebraic geometry to scheme theory. Zariski lemma, somtimes called the Nullstellensatz, states that for a field k the residue field of a maximal ideal of $k[x_1, \cdots, x_n]$ is a finite extension of k. In other words, for a field extension K/k, K is finitely generated as k-modules if K is finitely generated as k-algebras.

1.3 (Functor of points)**.** The *functor of points* of a scheme X is a functor $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set} : \operatorname{Spec} A \mapsto [\operatorname{Spec} A, X]$ or $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set} : T \mapsto [T, X]$. A *rational point* of X over a ring A is a morphism $\operatorname{Spec} A \to X$ of schemes.

Conversely, a functor $\mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$ is representable by scheme if and only if it is a sheaf on the site Aff and it has an open cover by affine schemes.

1.4 (Quotients and localizations)**.**

For an ideal $\mathfrak{a} \subset A$, the spectrum of the quotient $\operatorname{Spec} A/\mathfrak{a}$ gives a closed subset of $\operatorname{Spec} A$. For an element $f \in A$, the localization is $A_f = \{1, f, f^2, \cdots\}^{-1}A$, and the spectrum $\operatorname{Spec} A_f$ gives a distinguished open subset of $\operatorname{Spec} A$ with complement $\operatorname{Spec} A/(f)$, which generate a topological base when f runs through A. For a prime ideal $\mathfrak{p} \subset A$, the localization $A_\mathfrak{p} = (A \setminus \mathfrak{p})^{-1}A$ is a local ring, and the spectrum $\operatorname{Spec} A_\mathfrak{p}$ gives the set of prime ideals $\operatorname{Spec} A$ contained in $\mathfrak{p}$.

$$\operatorname{Spec} \mathbb{C}[x]_x = \operatorname{Spec} \mathbb{C}[x] \setminus \operatorname{Spec} \mathbb{C}[x]/(x) = \{(x-a) : a \in \mathbb{C} \setminus \{0\}\} \cup \{(0)\}$$

$$\operatorname{Spec} \mathbb{C}[x]_{(x)} = \{(x)\} \cup \{(0)\}.$$

$$\operatorname{Spec} \mathbb{C}[x,y]_{(x)} = \{(x, y-b) : b \in \mathbb{C}\} \cup \{(x)\} \cup \{(0)\}$$

$$\operatorname{Spec} \mathbb{Z}[x] \text{ over } \operatorname{Spec} \mathbb{Z}$$

1.5 (Integral schemes)**.** Let X be a scheme. We say X is *reduced* if every stalk is reduced, that is, it has no non-zero nilpotents, i.e. "a function is zero if it is zero at every point". We say X is *irreducible* if every two open subsets intersects. It is an algebro-geometric analogue of connectedness. We say X is *integral* if it is non-empty and every non-empty affine open subset is isomorphic to the spectrum of an integral domain.

(a) A scheme is integral if and only if it is reduced and irreducible.

(b) An integral scheme has a unique generic point η.

(c) The stalk $\mathcal{O}_{X,\eta}$ at the generic point is naturally identified with the field $K(A)$ of fractions, where $\operatorname{Spec} A$ is any non-empty affine open subset of an integral scheme X. So, we can define "rational functions" on integral schemes.

1.6 (Separated schemes)**.** *quasi-separated* if the intersection of any two quasi-compact open subsets is quasi-compact.

1.7 (Schemes of finite type)**.** Let X be a scheme. We say X is *quasi-compact* if it the Zariski topology is compact, *locally noetherian* if it is covered by the spectrum of noetherian rings, and *locally of finite type* (over a ring A) if it is covered by the spectrum of finitely generated algebras (over A). A *notherian* scheme is a quasi-compact locally notherian scheme, and a scheme of *finite type* is a quasi-compact scheme of locally finite type.

(a) A notherian scheme is automatically quasi-separated.

(b) A noetherian scheme is integral if and only if it is non-empty connected and every stalk is an integral domain.

(c) A scheme of finite type over a noetherian ring is noetherian.

1.8 (Normal and factorial schemes)**.**

1.1 Constructions for schemes

1.9 (Projective schemes)**.** We say a variety is *projective* if it is isomorphic to a closed subvariety of $\mathbb{P}^n$ for some n.

For a fixed a base ring A, let S be a $\mathbb{Z}_{\geq 0}$-graded ring such that $S_0 = A$, and define the *irrelavent ideal* $S_+ := \bigoplus_{i \geq 1} S_i$ of S. The *Proj construction* of S is a scheme $\operatorname{Proj} S$ constructed as follows. The set $\operatorname{Proj} S$ consists of all homogeneous prime ideals of S not containing S_+, the topology is determined by setting $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{a} \subset \mathfrak{p}\}$ as closed sets where $\mathfrak{a}$ runs through the homogeneous ideals of S, and the structure sheaf defined such that $\mathcal{O}_{\operatorname{Proj} S}(D(f)) := S_{((f))}$ for homogeneous $f \in S_+$, where $S_{(\mathfrak{p})} := (S_{\mathfrak{p}})_0$ denotes the zeroth graded piece of localized $\mathbb{Z}$-graded rings $S_{\mathfrak{p}}$, and the set $D(f) := \operatorname{Proj} S \setminus V(f)$ is called a *standard open* of $\operatorname{Proj} S$, which can be shown to be affine.

There is a canonical $\mathbb{Z}$-graded $\mathcal{O}_{\operatorname{Proj} S}$-modules, of which the graded pieces $\mathcal{O}(i)$ are line bundles called the *Serre twisting sheaves*.

A quasi-projective scheme X over A is of finite type of A. If A is furthermore noetherian, then X is noetherian.

Chapter 2

Morphisms

2.1

smooth, finite type, proper, regular, dominant, unramified, flat, complete intersection closed immersion

direct image, inverse image

Chapter 3

Quasi-coherent sheaves

Part II

Birational geometry

Chapter 4

Curves

In general, over an algebraically closed field, a *variety* refers to an integral separated scheme of finite type. If the underlying field is not algebraically closed, the definition slightly differs depending on the references. We define a *curve* as a 1-dimensional variety, and we want to classify smooth complete curves over an algebraically closed field k. I think the followings are equivalent to smooth complete curves:

- Hartshorne: integral scheme of dimension 1 which is proper and regular.
- Vakil: integral scheme of dimension 1 which is projective and regular.

Representations for morphisms when varieties are embedded in a projective space.

4.1 Preliminaries

Invariants

- genus: $p_a(X) = p_g(X) = h^1(\mathcal{O}_X)$
- Weil vs Cartier divisor groups: $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$

The moduli stack $\mathcal{M}_g$ of each genus.

Computation tools

- $|D| \leftrightarrow PH^0(X, \mathcal{L}(D))$ so that $|D|$ is identified as a projective space
- $\Omega_X \cong \omega_X$
- Riemann-Roch theorem: $l(D) - l(K - D) = \deg D + 1 - g$
- Hurwitz theorem: $2g(X) - 2 = \deg f \cdot (2g(Y) - 2) + \deg R$

birational iff isomorphic A morphism $f : X \to Y$ induces a field extension $\mathcal{K}(X)/\mathcal{K}(Y)$.

4.2 Lower genus

elliptic: invariants, moduli space, structures hyperelliptic: non-hyperelliptic: canonical embedding

4.3 Classification by genus and moduli spaces

Deligne-Mumford: $\mathcal{M}_g$ for $g \geq 2$ is an irreducible quasi-projective variety of dimension $3g - 3$.

4.4 Classification by degree in $\mathbb{P}^3$

A divisor D is called *very ample* if $\mathcal{L}(D) \cong \mathcal{O}(1)$ in some closed immersion into a projective space. A divisor D is called *ample* if $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for sufficiently large n, for each coherent sheaf $\mathcal{F}$. A *linear system* is a projective subspace of some complete linear system $|D| \cong \mathbb{P}^{l(D)-1}$, the set of all effective divisors linearly equivalent to D, which is identified to a projective space. The *base locus* of a linear system $\mathfrak{d}$ is the set $\bigcap_{D\in\delta} \operatorname{supp} D$. It is known that $|D|$ is base point free if and only if $\mathcal{L}(D)$ is generated by global sections, and a linear system is base point free if and only if some embedding....?

Any choice of a finite system of non-simultaneously vanishing global sections of a globally generated line bundle defines a morphism to a projective space. If the line bundle is very ample, then the morphism is an embedding.

chow variety or hilbert scheme

Chapter 5

Surfaces

Kodaira-Enriques

Fano three-folds

Moduli stack...?