

Sheaves and Bundles

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Part I

Bundles

Chapter 1

Fiber bundles

1.1 Principal bundles

1.1 (Locally trivial bundles). A *fiber bundle* is a map $p : E \rightarrow B$ such that $p^{-1}(b)$ is homeomorphic to F for each $b \in B$, where E, B, F are topological spaces called the *total space*, *base space*, and *fiber*. We say a fiber bundle ξ is *locally trivial* if it admits an *atlas* $\{\varphi_i\}$, a family of homeomorphisms $\varphi_i : p^{-1}(U_i) \rightarrow U_i \times F$ which indexed by an open cover $\{U_i\}$ of B such that

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times F \\ & \searrow p \quad \swarrow \text{pr}_1 & \\ & U_i & \end{array}$$

commutes. In this note, we are only concerned with locally trivial bundles and every fiber bundle will be assumed to be locally trivial.

A *bundle map* between fiber bundles $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ is a map of pairs $(\tilde{u}, u) : (E_1, B_1) \rightarrow (E_2, B_2)$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{u}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{u} & B_2 \end{array}$$

commutes.

(a) p is surjective and open.

1.2 (Structure groups). Let F be a left G -space for a topological group G . A fiber bundle $p : E \rightarrow B$ with fiber F is said to have a *structure group* G if it admits a *G-atlas*, an atlas $\{\varphi_i\}$ that has a set $\{g_{ij}\}$ of maps $g_{ij} : U_i \cap U_j \rightarrow G$ such that the transition maps $\varphi_j \varphi_i^{-1}$ are given by

$$\varphi_j \varphi_i^{-1}(b, f) = (b, g_{ij}(b)f), \quad b \in U_i \cap U_j, f \in F.$$

A *G-bundle* with fiber F is a fiber bundle $p : E \rightarrow B$ together with a *G-structure*, a maximal G -atlas.

A *G-bundle map* is a bundle map $(\tilde{u}, u) : (E_1, B_1) \rightarrow (E_2, B_2)$ between G -bundles such that there is a set $\{h_{ij}\}$ of maps $h_{ij} : U_{1,i} \cap u^{-1}(U_{2,j}) \rightarrow G$ such that

$$\varphi_{2,j} \tilde{u} \varphi_{1,i}^{-1}(b, f) = (u(b), h_{ij}(b)f), \quad b \in U_{1,i} \cap u^{-1}(U_{2,j}), f \in F.$$

(a) If F is a locally connected locally compact Hausdorff space, then every fiber bundle with fiber F has the structure group $\text{Homeo}(F)$ with respect to the compact-open topology.

- (b) A G -bundle map (\tilde{u}, u) is an isomorphism if and only if u is a homeomorphism.
- (c) A bundle map $(\tilde{u}, \text{id}_B) : (E_1, B) \rightarrow (E_2, B)$ is a G -bundle map if and only if there is a set $\{h_i\}$ of maps $h_i : U_i \rightarrow G$ such that

$$\varphi_{2,i} \tilde{u} \varphi_{1,i}^{-1}(b, f) = (b, h_i(b)f), \quad b \in U_i, f \in F.$$

Proof. (a)

(b) (\Rightarrow) Clear.

(\Leftarrow) The total map \tilde{u} is continuous bijection because u is a bijection, so it suffices to show \tilde{u}^{-1} is continuous. Fix $U_i \subset B$ and $U'_j \subset B'$. By substitution of $b' := u(b)$, $f' := h_{ij}(b)f$, we can write

$$\varphi_i \circ \tilde{u}^{-1} \circ \varphi_{j'}^{-1}(b', f') = (u^{-1}(b'), h_{ij'}(u^{-1}(b'))^{-1} f').$$

Since the local trivializations, the inverse operation of G , and the inverse u^{-1} are all continuous, \tilde{u}^{-1} is also continuous. \square

1.3 (Fiber bundle construction theorem). Let $\{U_i\}_i$ be an open cover of a topological space B , and let G be a topological group. Let $Z^1(\{U_i\}, G)$ be the set of every Čech 1-cocycle on $\{U_i\}$ with coefficients in G , a set $\{g_{ij}\}$ of maps $g_{ij} : U_i \cap U_j \rightarrow G$ satisfying the cocycle condition:

$$g_{ik}(b) = g_{jk}(b)g_{ij}(b), \quad b \in U_i \cap U_j \cap U_k.$$

Also let $C^0(\{U_i\}, G)$ be the set of every Čech 0-cochain on $\{U_i\}$ with coefficients in G , a collection $\{h_i\}$ of maps $h_i : U_i \rightarrow G$ of maps without any conditions.

The first Čech cohomology $\check{H}^1(\{U_i\}, G)$ of $\{U_i\}$ with coefficients in G is defined to be the orbit space of an action $\check{C}^0(\{U_i\}, G) \curvearrowright \check{Z}^1(\{U_i\}, G)$ defined as

$$(\{h_i\}\{g_{ij}\})_{ij}(b) := h_j(b)g_{ij}(b)h_i(b)^{-1}, \quad b \in U_i \cap U_j.$$

We define the first Čech cohomology of B with coefficients in G as the direct limit of sets

$$\check{H}^1(B, G) := \varinjlim_{\{U_i\}} \check{H}^1(\{U_i\}, G).$$

Let F be a left G -space, and let $\text{Bun}_F(B)$ be the set of isomorphism classes of G -bundles over B with fiber F .

- (a) $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$ is well-defined.
- (b) $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$ is surjective.
- (c) $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G/\ker \sigma)$ is injective, where $\sigma : G \rightarrow \text{Homeo}(F)$.

Proof. (a) Suppose $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$ are isomorphic G -bundles with fiber F , and $\tilde{u} : E_1 \rightarrow E_2$ is a G -bundle isomorphism. By considering the refinement, we can find an open cover $\{U_i\}$ of B on which E_1 and E_2 are simultaneously locally trivialized.

(b) Let F be a left G -space and let $\{g_{ij}\} \in \check{Z}^1(B, G)$ that is defined on an open cover $\{U_i\}$. Define

$$E := \left(\coprod_i (U_i \times F) \right) / \sim,$$

where \sim is an equivalence relation generated by

$$(i, b, f) \sim (j, b, g_{ij}(b)f), \quad b \in U_i \cap U_j, f \in F.$$

Also define $p : E \rightarrow B : [i, b, f] \mapsto b$ and $\varphi_i^{-1} : U_i \times F \rightarrow p^{-1}(U_i) : (b, f) \mapsto [i, b, f]$, which are clearly continuous and surjective without the cocycle condition.

We first claim that φ_i^{-1} is injective. Suppose $\varphi_i^{-1}(b, f) = \varphi_i^{-1}(b', f')$. Since $(i, b, y) \sim (i, b', y')$, we have $b = b'$ and there is a sequence of open sets U_{i_0}, \dots, U_{i_n} in $\{U_i\}$ such that $i_0 = i_n = i$ and

$$f' = g_{i_{n-1}i_n}(b)g_{i_{n-2}i_{n-1}}(b) \cdots g_{i_0i_1}(b)f.$$

By applying the cocycle condition inductively, we obtain $f = f'$, which implies the injectivity of φ_i^{-1} .

Next we claim that φ_i^{-1} is open. The map φ_i^{-1} factors through $\coprod_i (U_i \times F)$ such that

$$\varphi_i^{-1} : U_i \times F \rightarrow \coprod_i (U_i \times F) \xrightarrow{\pi} p^{-1}(U_i).$$

Since the canonical inclusion to disjoint union is open, it suffices to show the quotient map $\pi : \coprod_i (U_i \times F) \rightarrow E$ is open. Let $V \subset \coprod_i (U_i \times F)$ be open. Observe that

$$\pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open for each pair of i and j because it is exactly same as the inverse image of the open set $V \cap (U_i \times F)$ under the map

$$(U_i \cap U_j) \times F \subset U_j \times F \rightarrow U_i \times F : (b, f) \mapsto (b, g_{ij}(b)f).$$

Here we used the cocycle condition of $\{g_{ij}\}$. Therefore,

$$\pi^{-1}\pi(V) = \bigcup_{i,j} \pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open, hence the open π .

The transition maps of the G -atlas $\{\varphi_i\}$ coincides with the cocycle $\{g_{ij}\}$ by the cocycle condition. \square

1.4 (Principal bundles). Let G be a topological group, and X be a left *principal homogeneous G -space*, i.e. a free and transitive left G -space such that the shear map $G \times X \rightarrow X \times X : (g, x) \mapsto (gx, x)$ is a homeomorphism.

A *principal G -bundle* is a G -bundle $p : P \rightarrow B$ with fiber X , often together with a fiber-preserving continuous right action $\rho : P \times G \rightarrow P$ such that each chart $\varphi_i : p^{-1}(U_i) \rightarrow U_i \times X$ induces a principal homogeneous right action on $\{b\} \times X \subset U_i \times X$ which commutes with the left action. The right action ρ is called the *principal right action* or (*global*) *gauge transformation*. Note that for each $b \in B$ the fiber $\{b\} \times X$ has commuting left and right actions, but the fiber $p^{-1}(b)$ can admit only the principal right action.

The category of principal G -bundles over B is denoted by $\mathbf{Prin}_G(B)$, and the morphisms are usually defined as right G -equivariant maps with respect to the principal right actions. Then, we may consider the forgetful functor $\mathbf{Prin}_G(B) \rightarrow \mathbf{Bun}_X(B)$.

- (a) $\mathbf{Prin}_G(B) \rightarrow \mathbf{Bun}_X(B)$ is fully faithful, i.e. a bundle map $u : P \rightarrow P'$ over B is a G -bundle map if and only if it is a right G -equivariant map.
- (b) $\mathbf{Prin}_G(B) \rightarrow \mathbf{Bun}_X(B)$ is surjective, i.e. every G -bundle with fiber X has a principal right action.
- (c) A principal bundle is trivial if it has a global section.

Proof. (a) (\Rightarrow) Let $u : P \rightarrow P'$ be a G -bundle map over B so that there is a set $\{h_i : U_i \rightarrow G\}_i$ of maps such that

$$\varphi_i \circ u \circ \varphi_i^{-1}(b, x) = (b, h_i(b)x), \quad b \in U_i, x \in X.$$

If we write $\rho_s : P \rightarrow P : e \mapsto \rho(e, s)$ for $s \in G$, then the induced right action $\varphi_i \circ \rho_s \circ \varphi_i^{-1}$ commutes with the left action $\varphi_i \circ u \circ \varphi_i^{-1}$ on $U_i \times X$. Now for every $e \in P_1$, we have

$$\begin{aligned} \rho_s \circ u(e) &= \varphi_i^{-1} \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= \varphi_i^{-1} \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= u \circ \rho_s(e), \end{aligned}$$

therefore u is right G -equivariant.

(\Leftarrow) let $u : P \rightarrow P'$ be a right G -equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \rightarrow G$ such that

$$(b, g_i(b)x_0) := \varphi_i \circ u \circ \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xrightarrow{\varphi_i \circ u \circ \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The continuity of the last map is due to the assumption that the map $(g, x) \mapsto (gx, x)$ is a homeomorphism.

Then, for every $(b, x) \in U_i \times X$ there is a unique $s \in G$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned} \varphi_i \circ u \circ \varphi_i^{-1}(b, x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\ &= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b, x_0) \\ &= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b, x_0) \\ &= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b, x_0) \\ &= (\varphi_i \circ \rho_s \circ \varphi_i^{-1})g_i(b)(b, x_0) \\ &= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\ &= g_i(b)(b, x) \\ &= (b, g_i(b)x). \end{aligned}$$

Hence, u is a G -bundle map.

(b) Fix $x_0 \in X$ and consider the homeomorphism $G \rightarrow X : g \mapsto gx_0$. Define a right action

$$X \times G \rightarrow X : (gx_0, s) \mapsto gx_0s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X .

Define $\rho : P \times G \rightarrow P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b , the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge transformation.

(c) (\Rightarrow) Clear.

(\Leftarrow) Let $s : B \rightarrow E$ be a global section and define

$$\tilde{u} : B \times X \rightarrow E : (b, gx_0) \mapsto s(b)g$$

for any fixed $x_0 \in X$. Then, the continuous map (\tilde{f}, id_B) preserves fibers and defines a right G -equivariant isomorphism. \square

1.5 (Quotient principal bundles).

1.6 (Reduction of structure groups). Let H be a closed subgroup of G . Then, there is a map $\check{H}^1(B, H) \rightarrow \check{H}^1(B, G)$, which is neither in general injective nor surjective. If a G -bundle ξ is contained in the image of $\check{H}^1(B, H)$ through the correspondence $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$, then we may give a H -bundle structure on ξ .

A reduction of G to H is a H -structure on a principal G -bundle.

1.2 Classifying spaces

Let G be a topological group. Let $\text{Prin}_G(B)$ be the set of isomorphism classes of principal G -bundles over a topological B . Then, we have a contravariant functor

$$\text{Prin}_G : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{Set}.$$

On paracompact spaces:

1. The functor Prin_G is homotopy invariant.
2. The functor Prin_G is representable.
3. The universal elements can be computed using contractibility.

1.7 (Homotopy properties). Let $p : E \rightarrow B$ be a vector bundle

- (a) If $p : E \rightarrow B \times [0, \frac{1}{2}]$ and $p' : E' \rightarrow B \times [\frac{1}{2}, 1]$ are trivial, then
- (b) If $f, g : B' \rightarrow B$ are homotopic, then $f^*\xi \cong g^*\xi$.

1.3 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

1.8 (Vector bundles). Let $p : E \rightarrow B$ and $p' : E' \rightarrow B$ be vector bundles.

- (a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \leq n \leq \infty$. If $f, g : B \rightarrow G_k(\mathbb{R}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^{2n})$ is the natural inclusion.

1.9. Riemannian and Hermitian metrics spin structures

Exercises

1.10. Let G be a topological group, and X be a free right G -space.

- (a) If the action is proper and the projection $X \rightarrow X/G$ admits local sections, then $X \rightarrow X/G$ is a principal G -bundle.

1.11 (Clutching functions).

1.12. Suppose $F \rightarrow E \rightarrow B$ is a principal

- (a) If X is contractible, then $X \rightarrow$

1.13 (Group quotients). Sufficient conditions for principal bundles. Let G be a Lie group and, X be a free right smooth G -manifold.

- (a) If G is compact, then $X \rightarrow X/G$ is a principal G -bundle. (Gleason)
- (b) The irrational slope provides a counterexample if G is not compact.

- (c) Suppose X is a Lie group. If G is a closed subgroup of X , then $X/G \rightarrow X/G$ is a principal G -bundle. (Samelson) In particular, if M is a transitive left smooth X -manifold such that G is the isotropy group, then $X \rightarrow M$ is a principal G -bundle.

1.14 (Homogeneous spaces). They are all principal bundles.

$$\begin{aligned} O(n-k) &\rightarrow O(n) \rightarrow V_k(\mathbb{R}^n), & U(n-k) &\rightarrow U(n) \rightarrow V_k(\mathbb{C}^n), \\ O(n-k) \times O(k) &\rightarrow O(n) \rightarrow G_k(\mathbb{R}^n), & U(n-k) \times U(k) &\rightarrow U(n) \rightarrow G_k(\mathbb{C}^n), \\ T(n) \cap O(n) &\rightarrow O(n) \rightarrow F(\mathbb{R}^n), & T(n) \cap U(n) &\rightarrow U(n) \rightarrow F(\mathbb{C}^n), \\ & & T(n) &\rightarrow GL(n, \mathbb{C}) \rightarrow F(\mathbb{C}^n), \end{aligned}$$

where $T(n)$ is the group of invertible upper triangular matrices.

$$SO(n) \rightarrow SO^+(1, n) \rightarrow \mathbb{H}^n, \quad PSO(2) \rightarrow PSL(2, \mathbb{R}) \rightarrow \mathbb{H}^2, \quad ?? \rightarrow PSL(2, \mathbb{C}) \rightarrow \mathbb{H}^3,$$

where $PSL(2, \mathbb{R}) \cong SO(1, 2)^+$ is the modular group and $PSL(2, \mathbb{C}) \cong SO(1, 3)^+$ is the restricted Lorentz group, also called the Möbius group.

1.15 (Hopf fibration). A principal S^1 -bundle $S^1 \rightarrow S^3 \rightarrow S^2$, where we see S^1 as the circle group. The Hopf fibrations are used in describing universal principal bundles off orthogonal or unitary groups. We have principal bundles:

- (a) The quaternionic construction gives $S^3 \rightarrow S^7 \rightarrow S^4$ and the octonionic construction gives $S^7 \rightarrow S^{15} \rightarrow S^8$. Adams' theorem.
- (b) $O(k) \rightarrow V_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$. In particular, $\mathbb{Z}/2\mathbb{Z} \rightarrow S^n \rightarrow \mathbb{RP}^n$ for $k = 1$.
- (c) $U(k) \rightarrow V_k(\mathbb{C}^n) \rightarrow G_k(\mathbb{C}^n)$. In particular, $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ for $k = 1$.

Hopf fibration(real, complex, quaternionic, but not octonionic)

In the category of smooth manifolds, if f diffeomorphic, then \tilde{f} diffeomorphic.

1.16 (Associated bundles).

$$\text{Prin}_G(B) \xrightarrow{\sim} \text{Bun}_X(B) \xrightarrow{\sim} \check{H}^1(B, G) \hookrightarrow \text{Bun}_F(B)$$

can be given in a more simple way.

Part II

Sheaves

Chapter 2

General sheaf theory

2.1 (Étale bundles). Let X be a topological space. An *étale bundle* over a X is a local homeomorphism $p : E \rightarrow X$, where E is a topological space. An étale bundle over X is also often called a *sheaf* on X , but we let this term be reserved for a different but equivalent notion to the étale bundles, which will be introduced later. Étale bundles over X form a category, with morphisms defined as continuous maps $\varphi : E_1 \rightarrow E_2$ satisfying $p_1 = \varphi p_2$ for two sheaves $p_i : E_i \rightarrow X$ with $i \in \{1, 2\}$.

germs and stalks, section, basis of étale space

- (a) A subset $F \subset E$ defines a subsheaf if and only if F is open in E .
- (b) A covering space is nothing but a locally constant sheaf of sets.
- (c)

Proof.

□

2.2 (Presheaves). Let X be a topological space. A *presheaf* (of sets) on X is a contravariant functor \mathcal{F} from the category of open sets of X with inclusions as its morphisms to the category of sets. Presheaves on X form a category with natural transformations as its morphisms.

We construct a functor from the category of presheaves on X to the category of étale bundles over X , sometimes called the *étale space construction* or the *sheafification*. For a presheaf \mathcal{F} on X , define

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U), \quad U \text{ open in } X, \quad E(\mathcal{F}) := \bigsqcup_{x \in X} \mathcal{F}_x$$

and let $p : E(\mathcal{F}) \rightarrow X$ be such that $p(e) := x$ for $e \in \mathcal{F}_x$. The set \mathcal{F}_x and its element are called the *stalk* and a *germ* at x respectively, and the set $E(\mathcal{F})$ is called the *étale space* of \mathcal{F} .

- (a) There exists a unique natural topology on $E(\mathcal{F})$ such that $p : E \rightarrow X$ is a local homeomorphism.
- (b) There exists a unique natural function $\mathcal{F}(U) \rightarrow \Gamma(U, E(\mathcal{F}))$ such that .. for open subsets $U \subset X$.

Proof. (a) We endow a topology on E generated by a base $\{s(U) : s \in \mathcal{F}(U), U \text{ open}\}$.

(b) For $x \in U$ and $s \in \mathcal{F}(U)$, we define $s_x \in p^{-1}(x)$ by the image of $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ at s , and it defines a morphism of presheaves $\mathcal{F} \rightarrow \Gamma(E(\mathcal{F}))$. □

2.3 (Sheaves). Let X be a topological space. A *sheaf* (of sets) on X is a presheaf on X that satisfies the following two conditions:

- (i) (Locality)
- (ii) (Gluing)

The category of sheaves on X is defined to be the full subcategory of the category of presheaves on X .

- (a) For an étale bundle $p : E \rightarrow X$ over X , the functor $\mathcal{F} : U \mapsto \Gamma(U, E)$ defines a sheaf on X .
- (b) The étale space construction defines an equivalence of the category of sheaves on X and the category of étale bundles over X .

2.4 (Morphism of sheaves). epic and monic. The description for $\mathcal{F}(U)$ and the description for \mathcal{F}_X .

2.5 (Operations on sheaves). inverse image functor(restriction, pullback) direct image functor(push-forward) Whitney sum, constant sheaf, subsheaf
étale space descriptions

- (a)

2.6 (Sheaves of rings). Let X be a topological space.

A *ringed space* is a pair (X, \mathcal{O}_X) of a topological space X and a sheaf \mathcal{O}_X of rings on X .

2.7 (Sheaves of module). Let X be a topological space.

A sheaf \mathcal{F}_X of \mathcal{O}_X -modules is called *locally finite* or *finite type* if there is an open cover $\{U_i\}$ of X together with surjective ring homomorphisms $\mathcal{O}_X(U_i) \twoheadrightarrow \mathcal{F}_X(U_i)$ for all i . The section space $\Gamma(U, \mathcal{O}_X)$ will be also denoted by $\mathcal{O}_X(U)$.

- (a) The category of sheaves of modules over \mathcal{O}_X is abelian.

2.8 (Coherent sheaves). Let (X, \mathcal{O}_X) be a ringed space. Consider a quasi-coherent sheaf with an exact sequence of \mathcal{O}_X -modules

$$\mathcal{O}_U^q \rightarrow \mathcal{O}_U^p \rightarrow \mathcal{F}_U \rightarrow 0.$$

The *generating system* is the basis of \mathcal{O}_X^p , and the *relation sheaf* is the kernel of $\mathcal{O}_X^p \rightarrow \mathcal{F}_X$. We say \mathcal{F}_X is *coherent* if \mathcal{F}_X is of finite type and the relation sheaf is also of finite type.

- (a) If \mathcal{O}_X is itself a coherent module, then every locally finitely presented \mathcal{O}_X -module is coherent.

2.9 (Yoga of coherent sheaves).

- (a) extension principle?
- (b) If a ring \mathcal{O} has a split epi $\mathcal{O} \rightarrow \mathcal{O}'$ to a coherent \mathcal{O}' , then \mathcal{O} is coherent.

Proof. Consider the following diagram in which every row is exact and K, K_1, K_2 are kernels:

$$\begin{array}{ccccccc} K & \longrightarrow & \mathcal{O}^p & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \parallel & & \downarrow \delta & & \\ K_1 & \longrightarrow & \mathcal{O}^p & \longrightarrow & \mathcal{O}' & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \\ K_2 & \longrightarrow & \mathcal{O}'^p & \longrightarrow & \mathcal{O}' & \longrightarrow & 0. \end{array}$$

Then, K_2 is finitely generated by the coherence of \mathcal{O}' , K_1 is finitely generated by the Schanuel lemma, and K is finitely generated by the snake lemma. \square