Foundations of Calculus

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Contents

Part I

Sequences

Chapter 1

Convergence

1.1 Control of the error

1.2 Approximate sequences

1.3 Boundedness of sequences

Exercises

1.1. Every real sequence $(a_n)_{n=1}^{\infty}$ has a monotonic subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.

Chapter 2

Series

2.1 Convergence tests

2.1 (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_k B_k - A_{k-1}B_{k-1}$$

$$\sum_{m<k\le n} A_k b_k = A_n B_n - A_m B_m - \sum_{m<k\le n} a_k B_{k-1}.$$

2.2 (Dirichlet test).

2.2 Divide and conquer

2.3 Absolute convergence

2.3 (Unconditional convergence).

2.4 (Mertens' theorem). If $\sum_{k=0}^{\infty} a_k$ converges to A absolutely and $\sum_{k=0}^{\infty} b_k$ converges to B, then their Cauchy product $\sum_{k=0}^{\infty} c_k$ with $c_k := \sum_{l=0}^{k} a_l b_{k-l}$ converges to AB.

Proof. Let

$$A_n := \sum_{k=0}^{n} a_k,\ B_n := \sum_{k=0}^{n} b_k,\ \text{and } C_n := \sum_{k=0}^{n} c_k.$$

Consider the regions

$$T_n := \{(k,l) \in \mathbb{Z}^2_{\ge 0} : k + l \le n\}, \qquad R_m : \{(k,l) \in \mathbb{Z}^2_{\ge 0} : k \le m\}.$$

Write

$$AB - C_n = \sum_{k\le m}\sum_{l>n-k} a_k b_l + \sum_{k>m}\sum_{l\ge 0} a_k b_l - \sum_{m<k\le n}\sum_{l\le n-k} a_k b_l$$
$$= \sum_{k\le m} a_k(B - B_{n-k}) + \sum_{k>m} a_k B - \sum_{m<k\le n} a_k B_{n-k}.$$

The first term

$$|\sum_{k\le m} a_k(B-B_{n-k})| \le (\max_k |a_k|)(\sum_{l\ge n-m} |B-B_l|)$$

converges to zero as $n\to\infty$ for fixed m, the second term

$$|\sum_{k>m} a_k B| \le |A - A_m||B|$$

converges to zero as $m\to\infty$ for any n, and finally the third term

$$|\sum_{m<k\le n} a_k B_{n-k}| \le (\sum_{k>m} |a_k|)(\max_l |B_l|)$$

converges to zero as $m\to\infty$ for any n.

Fix m such that the second and third terms are bounded by arbitrary $\frac{\varepsilon}{2}>0$ so that

$$|C_n - AB| \le |\sum_{k\le m} a_k(B-B_{n-k})| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Then, by taking $n\to\infty$, we obtain

$$\limsup_{n\to\infty} |C_n - AB| \le \varepsilon.$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty} C_n = AB. \qquad \square$$

Exercises

2.5. If $a_n \to 0$, then $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$.

2.6. If $a_n \ge 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ also diverges.

2.7. If $a_n \downarrow 0$ and $S_n \le 1 + n a_n$, then $S_n \le 1$.

Chapter 3

Metrics and norms

3.1 Normed spaces

banach space introduction

3.2 Open sets and closed sets

convergence, limit point

3.3 Compact sets

3.4 Connected sets

Exercises

Part II

Real functions

Chapter 4

Continuous functions

4.1 Intermediate and extreme value theorems

4.2 Uniform continuity

4.3 Uniform convergence

Exercises

4.1. The set of local minima of a convex real function is connected.

4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The equation $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.

4.3. A continuous function that takes on no value more than twice takes on some value exactly once.

4.4. Let f be a function that has the intermediate value property. If the preimage of every singleton is closed, then f is continuous.

4.5. * If a sequence of real functions $f_n\colon [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$, then the sequence has a uniformly convergent subsequence.

Chapter 5

Differentiable functions

5.1 Monotonicty and convexity

5.2 Mean value theorem

Darboux

5.3 Taylor's theorem

5.4 Differentiable class

completeness

Exercises

5.1. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} f'(x) = b$, then $a = 0$.

5.2. Let f be a real C^2 function with $f(0) = 0$ and $f''(0) \neq 0$. Defined a function ξ such that $f(x) = xf'(\xi(x))$ with $|\xi| \le |x|$, we have $\xi'(0) = 1/2$.

5.3. Let f be a C^2 function such that $f(0) = f(1) = 0$. We have $\|f\| \le \frac{1}{8}\|f''\|$.

5.4. A smooth function such that for each x there is n having the nth derivative vanish is a polynomial.

5.5. If a real C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.

5.6. Let a real function f be differentiable. For $a < a' < b < b'$ there exist $a < c < b$ and $a' < c' < b'$ such that $f(b)-f(a) = f'(c)(b-a)$ and $f(b')-f(a') = f'(c')(b'-a')$.

5.7. Let f be a differentiable function on the unit closed interval. If $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

5.8. Let f be a differentiable function on the unit closed interval. If $f(0) = 0$ there is c such that $cf(c) = (1-c)f'(c)$.

Chapter 6

Analytic functions

6.1 Convergence of power series

uniform convergence and absolute convergence, abel theorem? differentiation convergence of radius sum, product, composition, reciprocal? closed under uniform convergence

6.2 Complex analytic functions

complex domain (real analytic iff its domain contains real line) convergence of radius, revisited identity theorem

6.3 Special functions

hypergeometric, bessel, gamma, zeta

Exercises

Part III

Integration

Chapter 7

Riemann integration

7.1 Riemann integral

tagged partition

7.2 Henstock-Kurzweil intergral

bounded compact support <-> lebesgue

7.3 Improper integral

7.4 Fundamental theorem of calculus for continuous functions

Exercises

7.1. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

7.2. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

Chapter 8

Integrable functions

Chapter 9

Part IV

Multivariable Calculus

Chapter 10

Frećhet derivatives

10.1 Inverse function theorem

Chapter 11

Differential forms

Chapter 12