Three-dimensional Topology

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1 Day 1: April 11

Plan:

1. Fundamental groups of manifolds
2. Examples and constructions
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5. Haken manifolds
6. Seifert manifolds
7. JSJ composition
8. Geometrization
9. Virtually special theorems

References:

1. J. Hempel, 3-manifolds
2. W. Jaco, Lectures on three-manifold topology
3. B. Martelli, An introduction to geometric topology
4. Morimoto, An introduction to three-dimensional manifolds (Japanses)

Grading: Submit a report for any three among the exercises given in the lecture (ITC-LMS Kadai). Cancellation of class: 5/2, 7/11(maybe)
Convention:

- manifold = connected compact orientable smooth manifold
- surface = connected compact orientable smooth 2-dimensional manifold
- tub nbd, isotopy extension, transversality, triangulation, $\cdots$

1. Fundamental group

1.1. Fundamental groups of higher dimensional manifolds

Proposition 1.1. *Let π be a finitely presented group. Then, for every $d \geq 4$ there is a d-manifold X such that $\pi_1(X) \cong \pi$.*

Proof. Let $\pi = \langle x_1, \cdots, x_n \mid r_1, \cdots r_m \rangle$ be given. If $Y = (S^1 \times S^{d-1})^{\# n}$, then $\pi_1(Y) \cong \langle x_1, \cdots x_n \rangle$ by the van Kampen theorem. Let

$$Z = Y \setminus (\coprod_{i=1}^{m} \nu(l_i)),$$

where $l_i \subset Y$ is embedded loops representing r_i and ν denotes the open tubular neighborhood. Then, $\partial Z = \coprod_{i=1}^{m} l_i \times S^{d-2}$. Since $d \geq 4$, any loops and disks can be pushed off $l_1 \cdots, l_n$, we have an isomorphism $\pi_1(Z) \to \pi_1(Y)$. Then, if we let

$$X = Z \cup_\partial (\coprod_{i=1}^{m} D^2 \times S^{d-2}),$$

then $l_i \times * = \partial(D^2 \times *)$, we have $\pi_1(X) \cong \pi_1(Y)/\langle [l_1], \cdots, [l_m] \rangle \cong \pi$. □

1.2. Surfaces and their groups

Theorem 1.2 (Radó, Whitehead). *Every topological surface admits a unique smooth and PL structure.*

Theorem 1.3. *Every surface is diffeomorphic to only one of $\Sigma_{g,b}$, where $\Sigma_{g,b} = (T^2)^{\#g}\#(D^2)^{\#b}$.*

Corollary 1.4. *$S = S^2, T^2, D^2$ are prime, that is, $S = S_1\#S_2$ implies $S_i \approx S^2$ for $i = 1$ or $i = 2$.*

Remark 1.5. If the orientability is reduced out, then $\mathbb{RP}^2$ is prime. Also note that $T^2\#\mathbb{RP}^2 \approx (\mathbb{RP}^2)^{\#3}$.

Theorem 1.6 (Uniformization). *For every surface $S \neq D^2$, its interior admits a complete Riemannian metric of constant curvature*

$$\begin{cases} 1, & \chi(S) > 0 \\ 0, & \chi(S) = 0 \\ -1, & \chi(S) < 0 \end{cases}$$

with universal covering S^2, $\mathbb{R}^2$, $\mathbb{H}^2$, respectively.

The hyperbolic plane is $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, and $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$.

Proposition 1.7. *If a surface S has $\chi(S) < 0$, then there is a discrete group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ such that $S \approx \mathbb{H}^2/\Gamma$. In particular, $\pi_1(S)$ is isomorphic to Γ.*

We have

$$\pi_1(\Sigma_{g,b}) \cong F_{2g+b-1} \quad \text{and} \quad \pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1], \cdots, [a_g, b_g]\rangle.$$

Proposition 1.8. *$\pi_1(\Sigma_g)$ is torsion free.*

Exercise 1. Prove Proposition 1.7.

Theorem 1.9 (Newman). *$\pi_1(\Sigma_{g,b})$ is linear over $\mathbb{Z}$, that is, is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$. For example, we can check $F_n \hookrightarrow F_2 \hookrightarrow \mathrm{SL}_2(\mathbb{Z})$ according to the pingpong lemma.*

Over $\mathbb{R}$, we may embed $\pi_1(S) \hookrightarrow \mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{SO}^+_{1,2}(\mathbb{R}) < \mathrm{GL}_3(\mathbb{R})$ if $\chi(S) < 0$,

Definition. A group π is called residually finite(RF) if for every $1 \neq \gamma \in \pi$ there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \neq 1$. A subgroup $\pi' < \pi$ is called separable if there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \notin \varphi(\pi')$. In particular, π is residually finite if the trivial subgroup is separable in π. A group π is called locally extended residually finite(LERF) if every finitely generated subgroup of π is separable.

Theorem 1.10 (Mal'cev). *Every finitely generated linear group over a field is residually finite.*

Over $\mathbb{Z}$, if $1 \neq (a_{ij}) \in \pi < \mathrm{GL}_n(\mathbb{Z})$ given, then for $m > \max_{i,j} |a_{ij}|$ if we let $\varphi_m : \pi \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, then $\varphi_m((a_{ij})) \neq 1$.

Theorem 1.11 (Scott). *$\pi_1(\Sigma_{g,b})$ is LERF.*

2 Day 2: April 18

Examples and constructions of 3-manifolds

Theorem 2.1 (Moise). *Every topological 3-manifold(not neccesarily compact, connected, orientable) admits a unique smooth and PL structure.*

2.1 Spherical manifolds

Recall

$$\begin{aligned} S^3 :&= \{x \in \mathbb{R}^4 : |x| = 1\} \\ &= \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \\ &= \{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}. \end{aligned}$$

Lens spaces

Let $p, q \in \mathbb{Z}$, $p > 0$, $(p,q) = 1$. Then, $\mathbb{Z}/p\mathbb{Z} = \langle \zeta = \exp(2\pi\sqrt{-1}/p) \rangle$ acts on S^3 such that $\zeta \cdot (z,w) = (\zeta z, \zeta^q w)$. Then, the Lens spaces are defined as

$$L(p,q) := S^3/(\mathbb{Z}/p\mathbb{Z}) \quad \text{with} \quad \pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}.$$

For example, $L(1,1) = S^3$ and $L(2,1) = \mathbb{RP}^3$.

Theorem 2.2 (Reidemeister)**.**

(a) $L(p,q) \simeq L(p,q')$ *(homotopy equiv) if and only if there is* $a \in \mathbb{Z}$ *such that* $qq' \equiv \pm a^2 \pmod{p}$.

(b) $L(p,q) \approx L(p,q')$ *(diffeo) if and only if* $q' \equiv \pm q^{\pm 1} \pmod{p}$.

For example, $L(7,1) \simeq L(7,2)$ since $1 \cdot 2 \equiv 3^2 \pmod{7}$, but $L(7,1) \approx L(7,2)$ since $2 \not\equiv \pm 1 \pmod{7}$.

Sketch of ($\Leftarrow$). Direct construction. (a) With the linking form $H_1(L) \times H_1(L) \to \mathbb{Q}/\mathbb{Z}$. (b) Reidemeister torsion. $\square$

General quotients

A spherical manifold is the orbit space S^3/Γ, where Γ is a finite subgroup of SO(4) and $\Gamma \curvearrowright S^3$ freely.

Example. With an action $\langle -1, i, j, k \rangle \curvearrowright S^3$, we obtain the prism manifold.

Example. With an action of the binary icosahedral group $\Gamma = \mathbb{Z}/2\mathbb{Z} \rtimes A_5$ on S^3, we obtain the Poincaré sphere. We have $H_*(S^3/\Gamma) \cong H_*(S^3)$. If we take 3/10 turn instead of 1/10, we have the Seifert-Weber space.

2.2 Fibered manifolds

Twisted bundles

$$N_{g,b} = (\mathbb{RP}^2)^{\#g} \# (D^2)^{\#b}.$$

Let D be a polygon with oriented sides $a_1, a_1', a_2, a_2', \cdots, a_g, a_g'$.

$$N_g \widetilde{\times} [0,1] := D \times [0,1]/\sim,$$

where $(x,t) \sim (x', 1-t)$ for $x \in a_i$, $x' \in a_i'$, $t \in [0,1]$ with $[x] = [x'] \in N_g$, and it is orientable.

$$N_g \widetilde{\times} S^1 := N_g \widetilde{\times} [0,1]/(x,0) \sim (x,1),\ x \in N_g$$

$$N_{g,b} \widetilde{\times} S^1 := N_g \widetilde{\times} S^1 \setminus \nu(b \text{ fibers}).$$

Exercise 2. Show the following:

(a) $\mathbb{RP}^2 \widetilde{[0,1]} \approx \mathbb{RP}^3 \setminus$ (open ball).

(b) $\mathbb{RP}^2 \widetilde{\times} S^1 \approx \mathbb{RP}^3 \# \mathbb{RP}^3$.

(c) $N_{1,1} \widetilde{\times} S^1 \approx N_2 \widetilde{\times} [0,1]$.

Mapping tori

The mapping class group is

$$\mathcal{M}_{g,b} := \mathrm{Diff}^+(\Sigma_{g,b}, \partial\Sigma_{g,b})/\text{isotopy relative to } \partial.$$

Theorem 2.3 (Dehn, Lickorish)**.** *$\mathcal{M}_{g,b}$ is finitely generated by Dehn twists.*

For examples, $\mathcal{M}_0 = \mathcal{M}_{0,1} = 1$ by the Alexander trick, and $\mathcal{M}_{0,2} \cong \mathbb{Z}$, $\mathcal{M}_1 = \mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})$.
Let $\varphi \in \mathcal{M}_{g,b}$. Then, a mapping torus is defined by

$$M_\varphi := \Sigma_{g,b} \times [0,1]/(\varphi(x), 0) \sim (x, 1).$$

2.3 Heegaard decomposition

A manifold with a boundary

$$H_g := D^3 \cup (D^2 \times [0,1])^{\sqcup g}$$

is called the handle body with genus g. Then, $\pi_1(H_g) \cong F_g$ and $\partial H_g \approx \Sigma_g$. Let $\varphi : \partial H_g \to \partial H_g$ be an orientation-preserving diffeomorphism, i.e. an element of the mapping class group $\mathcal{M}_g$. If a 3-manifold M satisfies

$$M \approx H_g \approx H_g,$$

then the right-hand side is called the Heegaard decomposition(splitting) of M.

Proposition 2.4. *Every closed 3-manifold admitting a Heegaard decomposition of genus 0 is diffeomorphic to S^3.*

Proof. $\mathcal{M}_0 = 1$. □

Proposition 2.5. *Every closed 3-manifold admitting a Heegaard decomposition of genus 1 is diffeomorphic to S^3, $S^2\times$, or $L(p,q)$.*

Exercise 3. Prove the above proposition.

Theorem 2.6. *Every closed 3-manifold M admits a Heegaard decomposition along some Σ_g.*

Proof. Pick a triangulation T of M. Then, $H := \overline{\nu(T^{(1)})}$ is a handlebody. Then, $H' := M \setminus \mathrm{Int}H$ is also a handlebody. Since M is orientable, so are H and H', thus we are done. □

There is another proof using Morse theory.

Corollary 2.7. *The fundamental group of every closed 3-manifold admits a finite presentation of deficiency 0, i.e. the number of generators is equal to the number of relations.*

Proof. Apply the van Kampen theorem to

$$M = H_g \cup H_g = H_g \cup (D^2 \times [-\varepsilon, \varepsilon])^{\sqcup g} \cup D^3.$$ □

2.4 Dehn surgery

Let L be a link. The link exterior is the set $E_L = S^3 \setminus \nu(L)$.

Proposition 2.8. *Let M be a 3-manifold, and $T \subset \partial M$ a torus component. Let $h : \partial(D^2 \times S^1) \to T$ be a diffeomorphism. Then, $M \cup_h (D^2 \times S^1)$ is determined only by $\pm[h(\partial D^2 \times *)] \in H_1(T)$.*

Proof. Write

$$M \cup_h (D^2 \times S^1) = M \cup_h (D^2 \times (-\varepsilon, \varepsilon)) \cup D^3.$$ □

For a knot K, there are two generators μ and λ, called the meridian and the longitude, of $H_1(\partial E_K)$ such that $\ker(H_1(\partial E_K) \to H_1(E_K) = \mathbb{Z}\mu)$ is generated by λ.

Exercise 4. Show that $L(p,q)$ with $p \neq 0$ and $S^2 \times S^1$ are obtained by the p/q-Dehn surgery along the unknot.

Theorem 2.9 (Lickorish-Wallace)**.** *Every closed 3-manifold can be obtained by an (integral)-Dehn surgery along some link in* S^3.

Sketch. Heegaard decomposition and $\mathcal{M}_g = \langle \text{Dehn twists} \rangle$. Each Dehn twist realizes the Dehn surgery steps. □

3 Day 3: April 25

Prime decomposition

3.1 Alexander's theorem

Theorem 3.1. *Every (smooth) embedding $S^2 \subset \mathbb{R}^3$ bounds some (smooth) embedding $D^3 \subset \mathbb{R}^3$.*

Remark. The above theorem does not hold in the category of topological spaces. Alexander's horned sphere is one of the counterexamples.

If $\mathbb{R}^d$ for $d \geq 5$, then more complicated result such as h-cobordism theorem must be used to obtain the same conclusion.

Sketch. Isotope such a sphere S so that the coordinate $z : S \to \mathbb{R}$ is a Morse function. Assume that for all $p \neq q \in \mathrm{Crit}(z)$, then $z(p) \neq z(q)$. We use induction on (m, n), where m is the number of saddles and

$$n := \min\{\#\pi_0(S \cap z^{-1}(r)) : r \text{ is a regular value s.t. } z(p) < r < z(q) \text{ for some saddles } p, q\}.$$

Note that $\#(\text{minima of } z) - m + \#(\text{maxima of } z) = \chi(S) = 2$.

For the case $m = 0$ so that there are only one minimum and maximum, then we can construct a ball by applying the Jordan-Schönflies theorem to each level.

For the case $m = 1$, then only four types appear: a jelly bean, a red blood cell, and their upside down versions. Apply the Jordan Schönflies again.

For the case $m \geq 2$, let r be a regular value realizing the value of n. Let D be union of the closure of the interior of the innermost circles of $S \cap z^{-1}(r)$. Replace S by $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{-\varepsilon, \varepsilon\})$. Then, each connected component has lower (m, n) so that it bounds a ball. Attaching all balls bounded by the components with balls $D \times [-\varepsilon, \varepsilon]$, S also bounds a ball. □

3.2 Irreducible manifolds

The connected sum is defined as

$$M\#N := (M \setminus (\text{open ball})) \cup_\partial (N \setminus (\text{open ball})).$$

Proposition 3.2.

(a) $M\#N \approx N\#M$.

(b) $(M_1\#M_2)\#M_3 \approx M_1\#(M_2\#M_3)$.

(c) $M\#S^3 \approx M$.

We say a manifold M is *prime* if $M = N_1\#N_2$ implies $N_1 \approx S^3$ or $N_2 \approx S^3$. We say a 3-manifold M is *irreducible* if every embedding $S^2 \subset M$ bounds some embedding $D^3 \subset M$. In other words, in a prime manifold every separating sphere bounds a ball, in an irreducible manifold every sphere bounds a ball.

Corollary 3.3. *Every irreducible 3-manifold is prime.*

Corollary 3.4. *By Theorem 3.1, S^3 is irreducible.*

Theorem 3.5.

(a) *$S^2 \times S^1$ is prime, but is not irreducible.*

(b) *Every closed prime 3-manifold which is not irreducible is diffeomorphic to $S^2 \times S^1$.*

Proof. (a) A sphere $S^2 \times *$ cannot bound any $D^3 \subset S^2 \times S^1$ because $[S^2 \times *] \neq 0 \in H_2(S^2 \times S^1)$, so $S^2 \times S^1$ is not irreducible.

Suppose $S^2 \times S^1 = N_1 \# N_2$. Since $\pi_1(N_1) * \pi_1(N_2) \cong \mathbb{Z}$, one of $\pi_1(N_1)$ or $\pi_1(N_1)$ is trivial. Assume $\pi_1(N_1)$ is trivial and let $B := N_1 \setminus$(open ball). Since B is also simply connected, it lifts diffeomorphically into the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. Because $S^2 \times \mathbb{R} \approx \mathbb{R}^3 \setminus \{0\}$, we have an embedding $B \subset \mathbb{R}^3$. Because $\partial B \approx S^2$, by Theorem 3.1 we have $B \approx D^3$, so $N_1 \approx S^3$.

(b) If every sphere in M is separating, then it has to be irreducible since M is prime, so such M contains a nonseparating sphere S. Let γ be an arc connecting the inside and the outside of $\partial \nu(S)$. If we let $M' := \overline{\nu(S)} \cup \overline{\nu(\gamma)}$, then $\partial M' \approx S^2 \# (S^1 \times I) \# S^2 \approx S^2$ is a separating sphere. Since $M \setminus \mathrm{Int} M' \approx D^3$ because M' is not simply connected and M is prime, M is diffeomorphic to $S^2 \times S^1$. $\square$

Proposition 3.6. *If a covering space $\widetilde{M}$ of M is irreducible, then so is M.*

Exercise 4. Prove Proposition 3.6.

Remark 3.7. The converse of Proposition 3.6 is also known to be true.

3.3 Normal surfaces

Fix a 3-manifold M and its triangulation T. A (possibly disconnected) subsurface $S \subset M$ is called a *normla surface* with respect to T if S is a union of *normal disks*, defined as seven types of disks in a given tetrahedron: four triangles and three quadrilaterals.

Proposition 3.8. *Every (possibly disconnected) subsurface $S \subset M$ becomes a normal surface with respect to T by isotopies and the following operations:*

(i) Replace S by $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{\pm\varepsilon\})$ for a disk D satisfying $D \cap (S \cup \partial M) = \partial D$.

(ii) Remove a components of S contained in a ball in M.

Proof. Isotope S so that $S \pitchfork T$. It is sufficient to realize the following:

(a) For every tetrahedron $\Delta \subset T$, $S \cap \Delta = \coprod$(disks).

(b) For every disk D in (a) and for every edge $e \in T^{(1)}$, $\#(D \cap e) \leq 1$.

(c) For every triangle $\tau \subset T^{(2)}$, $S \cap \tau = \coprod$(arcs).

For (a), if there is a non-disk component of $S \cap \Delta$, perform (i) along innermost loops in $S \cap \partial \Delta$. Then, perform (ii) for closed components of S.

For (b), if there is a disk component of $S \cap \Delta$ which intersects an edge more than twice, inner most pair of two points connected by an arc in D, push the arc outside Δ with an ambient isotopy. (To be continued) $\square$

3.4 Prime decomposition theorem