# 複素解析学I演習2023年（チョイ）

**問 1** (フックス群としてのモジュラー群). 複素数体 $\mathbb{C}$ の部分集合 $A$ に対して、成分 $a,b,c,d$ が $A$ の元で $ad-bc=1$ を満たす一次分数変換 $f(z)=(az+b)/(cz+d)$ の集合を $\mathrm{PSL}(2,A)$ と書く．特に $\mathrm{PSL}(2,\mathbb{Z})$ を**モジュラー群**と呼ぶ．上半平面 $\mathbb{H}:=\{z\in\mathbb{C}:\mathrm{Im}\,z>0\}$ の部分集合 $D:=\{z\in\mathbb{H}:|z|>1,\ |\mathrm{Re}\,z|<\frac{1}{2}\}$ を定義する．

(1) $\mathrm{PSL}(2,\mathbb{R})$ の元 $f$ は全単射写像 $\mathbb{H}\to\mathbb{H}$ を定義することを示せ．

(2) $\mathrm{PSL}(2,\mathbb{Z})$ は $S(z):=-1/z$ と $T(z):=z+1$ によって生成されることを示せ．つまり、全ての元が $S^{\pm1}$ と $T^{\pm1}$ の有限回の合成として表れることを示せ．

(3) 集合 $D$ は $\mathrm{PSL}(2,\mathbb{Z})$ の**基本領域**であることを示せ．つまり、次の二つが成り立つことを示せ：

  (a) 任意の点 $z\in\mathbb{H}$ に対して $f(z)\in\overline{D}$ を満たす $f\in\mathrm{PSL}(2,\mathbb{Z})$ が少なくとも一つ存在する．

  (b) 任意の点 $z\in\mathbb{H}$ に対して $f(z)\in D$ を満たす $f\in\mathrm{PSL}(2,\mathbb{Z})$ が多くとも一つしか存在しない．

(4) $\mathrm{PSL}(2,\mathbb{Z})$ は $\mathbb{H}$ に**真性不連続に作用**することを示せ．つまり、任意の点 $z\in\mathbb{H}$ に対して軌道 $\{f(z):f\in\mathrm{PSL}(2,\mathbb{Z})\}$ が離散集合であることを示せ．

**問 2** (カラテオドリ級関数集合の極点). 開単位円板上で定義された正則関数 $f$ が $f(0)=1$ を満たすとする．もし任意の $|z|<1$ を満たす複素数 $z$ に対して $\mathrm{Re}\,f(z)>0$ ならば、$f$ を**カラテオドリ級**の関数という．関数 $f$ が冪級数展開 $f(z)=1+2\sum_{k=1}^{\infty}c_kz^k$ を持つとする．

(1) 正の整数 $k$ と実数 $0<r<1$ に対して次の式を示せ：

$$c_kr^k=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,f(re^{i\theta})e^{-ik\theta}\,d\theta.$$

(2) 次の二つの条件が同値であることを示せ：

  (a) 関数 $f$ がカラテオドリ級である．

  (b) 任意の正の整数 $n$ に対して点 $(c_1,\cdots,c_n)\in\mathbb{C}^n$ は $\theta\in[0,2\pi)$ によって媒介変数表示された曲線 $(e^{-i\theta},\cdots,e^{-in\theta})\in\mathbb{C}^n$ の凸包絡の元である．

**問 3** (アールフォルス・清水標数). 複素平面上の有理型関数 $f$ を考える．次のように $r\geq 0$ に対する関数 $A(\cdot,f)$ を定義する：

$$A(r,f):=\frac{1}{\pi}\int_{\sqrt{x^2+y^2}\leq r}f^{\#}(x+iy)^2\,dx\,dy,\qquad \text{ただし、}f^{\#}(z):=\frac{|f'(z)|}{1+|f(z)|^2},\quad z\in\mathbb{C}.$$

関数 $f^{\#}$ を $f$ の**球面導関数**と呼ぶ．

(1) 任意の点 $(x,y)\in\mathbb{R}^2$ に対して、

$$\frac{1}{\pi}f^{\#}(x+iy)^2=\frac{\partial Q}{\partial x}(x,y)-\frac{\partial P}{\partial y}(x,y)$$

を満たす実平面 $\mathbb{R}^2$ 上の実関数 $P$ と $Q$ を求め、関数 $K(x,y):=1+|f(x+iy)|^2$ を用いて表せ．

(2) グリーンの定理と偏角の原理を用いて $r \geq 0$ に対して次の式が成り立つことを示せ：

$$\int_0^r A(t,f)\frac{dt}{t} = \int_0^r n(t,f)\frac{dt}{t} + \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta - \log\sqrt{1+|f(0)|^2}.$$

ただし、$n(r,f)$ は閉円板 $\overline{B(0,r)}$ 内にある重複度を込めて数えた $f$ の極の数である．左辺の関数を $f$ の**アールフォルス・清水標数**と呼ぶ.

(3) 球面導関数 $f^{\#}$ が有界ならば、ある定数 $C>0$ が存在して、全ての $z \in \mathbb{C}$ に対して $|f(z)| \leq Ce^{|z|^2}$ であることを示せ．特に、$f$ は $\mathbb{C}$ 全体上正則である.

**問 4** (四分円上のディリクレ問題)．領域 $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0,\ y>0\}$ 上に定義された調和関数 $v \in C^2(\Omega,\mathbb{R})$ が次の境界値条件を満たすとする：各点 $(x_0,y_0) \in \partial\Omega$ に対して

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \begin{cases} 1 & \text{if } y_0>0, \\ 0 & \text{if } y_0=0 \text{ and } 0<x_0<1. \end{cases}$$

(1) シュワルツの鏡像の原理を用いて $v$ は領域 $\widetilde{\Omega} := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0\}$ 上の調和関数 $\tilde{v} \in C^2(\widetilde{\Omega},\mathbb{R})$ に拡張されることを示せ.

(2) 適切な等角変換とポアソン積分を用いて $v$ を求めよ.

*Solution of 1.* (1) Let $f(z)=(az+b)/(cz+d)$ with $a,b,c,d\in\mathbb{R}$ such that $ad-bd=1$. Since it has the inverse transform $z\mapsto(dz-b)/(-cz+a)$ that is also an element of $\mathrm{PSL}(2,\mathbb{R})$, it is enough to show the well-definedness $f(z)\in\mathbb{H}$ for $z\in\mathbb{H}$. Let $z=x+iy\in\mathbb{H}$ with $y>0$. Then,

$$\operatorname{Im} f(z)=\operatorname{Im}\frac{ax+b+iay}{cx+d+icy}=\frac{ay(cx+d)-(ax+b)cy}{(cx+d)^2+(cy)^2}=\frac{y}{(cx+d)^2+(cy)^2}>0,$$

so $f(z)\in\mathbb{H}$.

(2) Let $f(z)=(az+b)/(cz+d)$ with $a,b,c,d\in\mathbb{Z}$ such that $ad-bd=1$. Consider the following two kinds of moves of $f$:

- When $|a|<|c|$, we take
$$Sf(z)=\frac{-cz-d}{az+b}.$$
- When $|a|\ge|c|>0$, with $q,r\in\mathbb{Z}$ such that $a=qc+r$ and $0\le r<|c|$, we take
$$T^{-q}f(z)=\frac{rz+b-qd}{cz+d}.$$

By repeating the two moves alternately, we arrive at $c=0$ in finitely many steps because $|c|$ strictly decreases. Then, since $ad-bc=1$, we may assume $a=d=1$ so that $(az+b)/(cz+d)=z+b=T^b(z)$.

(3) (a) Let $z_0\in\mathbb{H}$. We may assume $\operatorname{Re} z_0\in[-\frac12,\frac12)$ by taking $T^q$ on $z_0$ for appropriate $q\in\mathbb{Z}$. Define a sequence $z_n\in\mathbb{H}$ inductively by

$$z_n:=T^{-\lfloor \operatorname{Re} S(z_{n-1})+\frac12\rfloor}S(z_{n-1}),\qquad n\ge1.$$

Then, one can show $\operatorname{Re} z_n\in[-\frac12,\frac12)$ for all $n$. Since

$$\operatorname{Im} z_n=\operatorname{Im} S(z_{n-1})=\frac{\operatorname{Im} z_{n-1}}{(\operatorname{Re} z_{n-1})^2+(\operatorname{Im} z_{n-1})^2}\ge g(\operatorname{Im} z_{n-1}),$$

where $g(y):=4y/(1+4y^2)$, and since $g^n(y)\uparrow\frac{\sqrt3}{2}$ for $0<y<\frac{\sqrt3}{2}$ as $n\to\infty$, there is $n$ such that

$$-\frac12\le\operatorname{Re} z_n<\frac12,\qquad \operatorname{Im} z_n>\frac{\sqrt3}{4}.$$

If $|z_n|\ge1$, then we are done, so assume $|z_n|<1$. Now we have three possibilities: $|z_n-1|<1$, $|z_n+1|<1$, or $\min\{|z_n-1|,|z_n+1|\}\ge1$. For each case, we can check that $T^{-1}Sz_n$, $TSz_n$, $Sz_n$ is contained in $\overline{D}$, respectively.

(b) For $z\in D$, let $w=(az+b)/(cz+d)\in D$ with $a,b,c,d\in\mathbb{Z}$ such that $ad-bd=1$. It suffices to show $c=0$. Suppose $c\ne0$. Note that $|z-n|>1$ and $|w-n|>1$ for every integer $n$ since $z,w\in D$. Write

$$1<|w-n|=\left|\frac{az+b}{cz+d}-n\right|\le\left|\frac{az+b}{cz+d}-\frac{a}{c}\right|+\left|n-\frac{a}{c}\right|=\left|\frac{1}{c(cz+d)}\right|+\left|n-\frac{a}{c}\right|,\qquad n\in\mathbb{Z}.$$

If $|c|\ge2$, then by taking $n$ such that $|n-(a/c)|\le\frac12$, the estimate $|c(cz+d)|\ge|c|^2\operatorname{Im} z>2\sqrt3$ leads a contradiction to the above inequality. If $|c|=1$, then since $a/c$ is an integer, by letting $n=a/c$, we have a contradiction $|c(cz+d)|=|z+cd|>1$ from the assumption $z\in D$. Thus, $c=0$, and we are done.

(4) Suppose the orbit $\{f(z):f\in\mathrm{PSL}(2,\mathbb{Z})\}$ is not discrete. Then, there is $z_0\in\mathbb{H}$ and a sequence $f_n\in\mathrm{PSL}(2,\mathbb{Z})$ such that $f_n(z)\ne z_0$ for all $n$ and $f_n(z)\to z_0$ as $n\to\infty$. We may assume $z,z_0\in\overline{D}$ by the part (a) of (3). Consider

$$P:=\{I,T,TS,ST^{-1}S=TST,ST^{-1},S,ST,STS=T^{-1}ST^{-1},T^{-1}S,T^{-1}\}\subset\mathrm{PSL}(2,\mathbb{Z}).$$

Then, we can check that $\bigcup_{f\in P} f(\overline{D})$ contains an open neighborhood $U$ of $\overline{D}$. For every $n$ that is large enough, from $f_n(\overline{D}) \cap U \neq \varnothing$, it follows that $f_n(D)$ intersects $U \subset \bigcup_{f\in P} f(\overline{D})$, that is, there is $f_0 \in P$ such that $f_n(D) \cap f_0(\overline{D}) \neq \varnothing$, and easily $f_n(D) \cap f_0(D) \neq \varnothing$, because $f(D)$ is open and $f(\overline{D})$ is closed for any $f \in \mathrm{PSL}(2,\mathbb{Z})$. By the part (b) of (3), we can conclude that $f_n$ belongs eventually to $P$ as $n \to \infty$. Since $P$ is a finite set, $f_n(z)$ cannot converge to $z_0$ unless $f_n(z) = z_0$ for sufficiently large $n$, therefore the orbit is discrete. $\square$

*Remark.* A discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ is called a *Fuchsian group* and a *Kleinian group* respectively. It is known that a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is discrete if and only if it properly discontinuously acts on $\mathbb{H}$. There is a more generalized theorem used for verifying a group generated by several elements of $\mathrm{PSL}(2,\mathbb{R})$ is Fuchsian, the *Poincare polygon theorem*. It states that if there is a polygon in $\mathbb{H}$ satisfying two conditions called a side pairing condition and elliptic cycle condition is realized as a fundamental domain, so the group acts on $\mathbb{H}$ properly discontinuously. $\square$

*Solution of 2.* (1) Suppose $k > 0$ first. The Cauchy integral formula writes

$$2c_k k! = \frac{\partial^k f}{\partial z^k}(0) = \frac{k!}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{k+1}}\,dz = \frac{k!}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})}{(re^{i\theta})^k}\,d\theta,$$

and it implies

$$2c_k r^k = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}\,d\theta.$$

Since $f(z)z^k$ is analytic, the Cauchy theorem can be applied to get

$$0 = \frac{1}{2\pi i}\int_{|z|=r} f(z)z^k\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})r^k e^{ik\theta}\,d\theta,$$

and it implies

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\overline{f(re^{i\theta})}e^{-ik\theta}\,d\theta.$$

By combining the above two equations, we obtain the formula. For $k = 0$, applying the Cauchy theorem for $f$, we have

$$c_0 = f(0) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\,d\theta.$$

Alternatively, we can show the same result using the orthogonal relation of complex exponential functions. An easy computation shows the identity

$$\begin{aligned}
\operatorname{Re} f(re^{i\theta}) &= \frac{1}{2}[f(re^{i\theta}) + \overline{f(re^{i\theta})}]\\
&= \frac{1}{2}\left[\left(1+\sum_{k=1}^{\infty}2c_k(re^{i\theta})^k\right) + \overline{\left(1+\sum_{k=1}^{\infty}2c_k(re^{i\theta})^k\right)}\right]\\
&= \frac{1}{2}\left[\left(1+\sum_{k=1}^{\infty}2c_k r^k e^{ik\theta}\right) + \left(1+\sum_{k=1}^{\infty}2\overline{c_k}r^k e^{-ik\theta}\right)\right]\\
&= \sum_{k=-\infty}^{\infty} c_k r^{|k|}e^{ik\theta}.
\end{aligned}$$

From the uniform convergence of the power series on the compact set $\{z : |z| \le (r+1)/2\}$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\frac{1}{2\pi}\int_0^{2\pi} e^{il\theta}e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\delta_{kl} = c_k r^{|k|}.$$

(2) (b)$\Rightarrow$(a) Denote by $K_n$ the convex hull of the curve $\theta \mapsto (e^{-i\theta},\cdots,e^{-in\theta}) \in \mathbb{C}^n$. Suppose first that $(c_1,\cdots,c_n) \in K_n$. For each $n$, there exists a finite sequence of pairs $(\lambda_{n,j},\theta_{n,j})_j$ having the following convex combination

$$(c_1,\cdots,c_n) = \sum_j \lambda_{n,j}(e^{-i\theta_{n,j}},\cdots,e^{-in\theta_{n,j}})$$

with coefficients $\lambda_{n,j} \ge 0$ such that $\sum_j \lambda_{n,j} = 1$. Define

$$f_n(z) := \sum_j \lambda_{n,j}\frac{e^{i\theta_{n,j}}+z}{e^{i\theta_{n,j}}-z},$$

which has positive real part on $|z| < 1$ because $\operatorname{Re}(e^{i\theta_{n,j}}+z)/(e^{i\theta_{n,j}}-z) > 0$ for $|z| < 1$. Then,

$$f_n(z) = \sum_j \lambda_{n,j}(1+\sum_{k=1}^{\infty}2e^{-ik\theta_{n,j}}z^k) = 1+\sum_{k=1}^{n}2c_k z^k + \sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k$$

implies

$$|f_n(z)-f(z)| = \left|\sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k - \sum_{k=n+1}^{\infty} 2c_k z^k\right|$$
$$\leq \sum_{k=n+1}^{\infty}\left|\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)-2c_k\right||z|^k \leq \sum_{k=n+1}^{\infty} 4|z|^k$$

converges to zero for $|z|<1$. Therefore, $f$ has a non-negative real part on the open unit disk. The non-negativity can be strengthened to positivity by the open mapping theorem, so $f$ belongs to the Carathéodory class.

(a)⇒(b) Conversely, suppose that $f$ is in the Carathéodory class. Let $(\gamma_1,\cdots,\gamma_n)$ be any point on the surface $\partial K_n$ of $K_n$ and $S$ any supporting hyperplane of $K_n$ tangent at $(\gamma_1,\cdots,\gamma_n)$. Let $(u_1,\cdots,u_n)\in\mathbb{C}^n$ be the outward unit normal vector of the supporting hyperplane $S$. Note that this outward unit normal vector is uniquely determined for each hyperplane $S$ with respect to the real inner product structure on the $2n$-dimensional real vector space $\mathbb{C}^n$ given by

$$\langle(z_1,\cdots,z_n),(w_1,\cdots,w_n)\rangle = \sum_{k=1}^{n}(\operatorname{Re}z_k\operatorname{Re}w_k+\operatorname{Im}z_k\operatorname{Im}w_k) = \operatorname{Re}\sum_{k=1}^{n} z_k\overline{w}_k.$$

Then, we know that $\sum_{k=1}^{n}|u_k|^2=1$ and the maximum

$$M := \max_{(x_1,\cdots,x_n)\in K_n}\operatorname{Re}\sum_{k=1}^{n}x_k\overline{u}_k > 0$$

is attained at $(\gamma_1,\cdots,\gamma_n)$. Our goal is now to verify the bound

$$\operatorname{Re}\sum_{k=1}^{n}c_k\overline{u}_k \leq M$$

from the assumption that $f$ is of Carathéodory class. Once the bound is obtained, then it means that $(c_1,\cdots,c_n)$ is contained in the same side as $K_n$ of arbitrary hyperplanes tangent to $K_n$, so we finally conclude $(c_1,\cdots,c_n)\in K_n$.

Since for any $\theta\in[0,2\pi)$ the point $(e^{-i\theta},\cdots,e^{-in\theta})$ is in $K_n$, we have

$$\operatorname{Re}\sum_{k=1}^{n}e^{-ik\theta}\overline{u}_k \leq M.$$

For $\varepsilon>0$, we have

$$\operatorname{Re}\sum_{k=1}^{n}\frac{1}{r^k}e^{-ik\theta}\overline{u}_k \leq M+\varepsilon$$

for any $0<r<1$ sufficiently close to 1, thus we can write

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^{n}c_k\overline{u}_k &= \operatorname{Re}\sum_{k=1}^{n}\frac{1}{2\pi r^k}\int_0^{2\pi}\operatorname{Re}f(re^{i\theta})e^{-ik\theta}\overline{u}_k\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\theta})\operatorname{Re}\sum_{k=1}^{n}\frac{1}{r^k}e^{-ik\theta}\overline{u}_k\,d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}f(re^{i\theta})\,d\theta\cdot(M+\varepsilon)\\ &= \operatorname{Re}f(0)(M+\varepsilon) = M+\varepsilon\end{aligned}$$

thanks to the part (1) and the positivity of $\operatorname{Re}f$, and by limiting $r\to 1$ from left we get the bound we want. □

*Solution of 3.* (1) Write $f = u + iv$ for real-valued $u$ and $v$. Since

$$d\,(P\,dx + Q\,dy) = \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx \wedge dy = \frac{1}{\pi} f^{\#2}\,dx \wedge dy,$$

and since

$$\begin{aligned}
\frac{1}{\pi} f^{\#2}\,dx \wedge dy &= \frac{u_x v_y - u_y v_x}{\pi(1+u^2+v^2)^2}\,dx \wedge dy = \frac{du \wedge dv}{\pi(1+u^2+v^2)^2} \\
&= d\left(-\frac{v}{2\pi(1+u^2+v^2)}\,du + \frac{u}{2\pi(1+u^2+v^2)}\,dv\right) \\
&= d\left(-\frac{v}{2\pi(1+u^2+v^2)}(u_x\,dx + u_y\,dy) + \frac{u}{2\pi(1+u^2+v^2)}(v_x\,dx + v_y\,dy)\right) \\
&= d\left(-\frac{vu_x - uv_x}{2\pi(1+u^2+v^2)}\,dx + \frac{uv_y - vu_y}{2\pi(1+u^2+v^2)}\,dy\right) \\
&= d\left(-\frac{uu_y + vv_y}{2\pi(1+u^2+v^2)}\,dx + \frac{uu_x + vv_x}{2\pi(1+u^2+v^2)}\,dy\right),
\end{aligned}$$

we can check the following satisfy the equation of the problem:

$$P = -\frac{K_y}{4\pi K}, \qquad Q = \frac{K_x}{4\pi K}.$$

(2) Since the equation holds for $r = 0$, it suffices to show the differentiated equation

$$A(r,f) = n(r,f) + \frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log\sqrt{K(r,\theta)}\,d\theta$$

for almost every $r > 0$, where $K(r,\theta) = 1 + |f(re^{i\theta})|^2$. In particular, we will prove this equation for every $r$ such that $f$ does not have a pole $a$ with $|a| = r$. Fix such $r$ and let $\{a_i\}_{i=1}^n$ be poles of $f$ in the region $|z| < r$ with multiplicities $m_i$ for each $a_i$. Since

$$\begin{aligned}
P\,dx + Q\,dy &= \frac{1}{2\pi}\frac{-K_y\,dx + K_x\,dy}{2K} = \frac{1}{2\pi i}\frac{-iK_y\,dx + K_x\,idy}{2K} \\
&= \frac{1}{2\pi i}\frac{(K_x - iK_y)(dx + idy)}{2K} - \frac{1}{2\pi i}\frac{K_x\,dx + K_y\,dy}{2K} \\
&= \frac{1}{2\pi i}\frac{uu_x + vv_x - iuu_y - ivv_y}{1+u^2+v^2}\,dz - \frac{1}{2\pi i}\frac{dK}{2K} \\
&= \frac{1}{2\pi i}\frac{(u_x + iv_x)(u - iv)}{1+u^2+v^2}\,dz - \frac{1}{2\pi i}\frac{d\log K}{2} \\
&= \frac{1}{2\pi i}\frac{f'(z)}{f(z)}\frac{|f(z)|^2}{1+|f(z)|^2}\,dz - \frac{1}{2\pi i}\frac{d\log K}{2},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log\sqrt{K(r,\theta)}\,d\theta &= \frac{r}{2\pi}\int_0^{2\pi} \frac{K_r}{2K}\,d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{uu_r + vv_r}{K}\,d\theta \\
&= \frac{r}{2\pi}\int_0^{2\pi} \frac{u(\cos\theta u_x + \sin\theta u_y) + v(\cos\theta v_x + \sin\theta v_y)}{K}\,d\theta \\
&= \frac{r}{2\pi}\int_0^{2\pi} \frac{\mathrm{Re}[(\cos\theta + i\sin\theta)(u_x + iv_x)(u - iv)]}{K}\,d\theta \\
&= \mathrm{Re}\,\frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{\theta} f'\overline{f}}{1+|f|^2}\,d\theta = \mathrm{Re}\,\frac{1}{2\pi i}\int_{|z|=r} \frac{f'\overline{f}}{1+|f|^2}\,dz \\
&= \mathrm{Re}\int_{|z|=r} (P\,dx + Q\,dy),
\end{aligned}$$

and by the argument principle and $|f(z)| \to \infty$ near the pole $z \to a_i$,

$$\int_{|z-a_i|=\varepsilon} (P\,dx + Q\,dy) = \frac{1}{2\pi i} \int_{|z-a_i|=\varepsilon} \frac{f'(z)}{f(z)} \frac{|f(z)|^2}{1+|f(z)|^2}\,dz$$
$$= -m_i - \frac{1}{2\pi i} \int_{|z-a_i|=\varepsilon} \frac{f'(z)}{f(z)} \frac{1}{1+|f(z)|^2}\,dz \to -m_i$$

as $\varepsilon \to 0$. Then, the Green theorem is applied to have

$$\begin{aligned} A(r,f) &= \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|z|\le r,\ \min_i |z-a_i|\ge\varepsilon} f^{\#}(x+iy)^2\,dx\,dy \\ &= \int_{|z|=r} (P\,dx + Q\,dy) - \lim_{\varepsilon\to 0} \sum_{i=1}^{n} \int_{|z-a_i|=\varepsilon} (P\,dx + Q\,dy) \\ &= \frac{r}{2\pi} \frac{d}{dr} \int_0^{2\pi} \log\sqrt{K(r,\theta)}\,d\theta + i\,\mathrm{Im} \int_{|z|=r} (P\,dx + Q\,dy) + \sum_{i=1}^{n} m_i. \end{aligned}$$

Because $\sum_{i=1}^{n} m_i = n(r,f)$ by definition, and seeing the real part, we obtain the desired equation.

(3) Since every Taylor coefficient of the logarithm is real, we have

$$\mathrm{Re}\log f(z) = \frac{1}{2}(\log f(z) + \log\overline{f(z)}) = \log|f(z)|.$$

Take $a \in \mathbb{C}$ and let $r := 2|a|$. By the Schwarz integral formula,

$$\begin{aligned} \log|f(a)| = \mathrm{Re}\log f(a) &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\,\frac{re^{i\theta}+a}{re^{i\theta}-a}\,\mathrm{Re}\log f(re^{i\theta})\,d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{re^{i\theta}+a}{re^{i\theta}-a}\right| \log|f(re^{i\theta})|\,d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} 3\log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta \\ &\le 3\int_0^r A(t,f)\frac{dt}{t} \le 3\int_0^r M^2 t^2 \frac{dt}{t} = 6M^2|a|^2, \end{aligned}$$

so $C := e^{6M^2}$ proves the theorem, where $M$ is a bound of the spherical derivative $f^{\#}$. $\square$

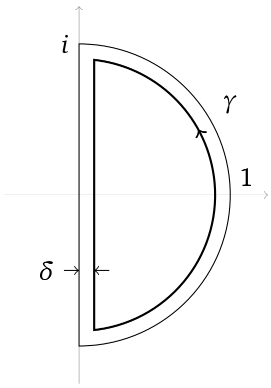
*Solution of 4.* (1) Identify $\Omega$ and $\widetilde{\Omega}$ as subsets of $\mathbb{C}$ by letting $(x,y)=x+iy$. Consider a harmonic conjugate $-u$ of $v$ on $\Omega$ such that a function $f(x+iy):=u(x,y)+iv(x,y)$ is holomorphic on $\Omega$. If we define

$$\widetilde{f}(z):=\begin{cases} f(z) & \text{if } \operatorname{Im} z\geq 0, \\ \overline{f(\overline{z})} & \text{if } \operatorname{Im} z<0, \end{cases} \qquad z\in\widetilde{\Omega},$$

then $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}\setminus(0,1)$, and is also continuous on the whole $\widetilde{\Omega}$ because of the boundary condition of $v$ on the real axis. We claim that $\widetilde{f}$ is in fact holomorphic on $\widetilde{\Omega}$. If the claim is true, then $\widetilde{v}:=\operatorname{Im}\widetilde{f}$ is the desired extension of $v$, which satisfies in addition that for $(x_0,y_0)\in\partial\widetilde{\Omega}$ we have

$$\lim_{(x,y)\to(x_0,y_0)}\widetilde{v}(x,y)=\begin{cases} 1 & \text{if } y_0>0, \\ -1 & \text{if } y_0<0. \end{cases}$$

Let $\gamma$ be a contour defined for sufficiently small $\delta>0$ as the following figure:




Denote by $\widetilde{\Omega}_\delta:=\{a\in\widetilde{\Omega}:\min_{z_0\in\partial\widetilde{\Omega}}|z_0-a|>\delta\}$ the interior of $\gamma$. Define a function $\widetilde{g}$ on $\widetilde{\Omega}_\delta$ such that

$$\widetilde{g}(a):=\frac{1}{2\pi i}\int_\gamma\frac{\widetilde{f}(z)}{z-a}\,dz, \qquad a\in\widetilde{\Omega}_\delta.$$

Note that the integrand is continuous on the contour $\gamma$, and $\widetilde{g}$ is holomorphic on $\widetilde{\Omega}_\delta$ by the Morera theorem, because for every affine triangle $\sigma$ in the interior of $\gamma$ we have

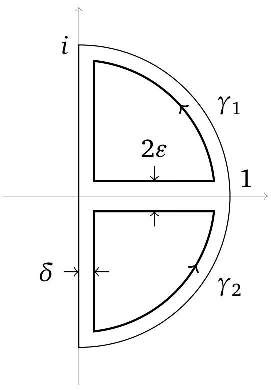
$$\int_\sigma\widetilde{g}(a)\,dz=\int_\sigma\frac{1}{2\pi i}\int_\gamma\frac{\widetilde{f}(z)}{z-a}\,dz\,da=\frac{1}{2\pi i}\int_\gamma\left[\int_\sigma\frac{\widetilde{f}(z)}{z-a}\,da\right]dz=0$$

by the Fubini theorem and the Cauchy theorem for $\sigma$.

Moreover, for $a\in\widetilde{\Omega}_\delta\cap\Omega$ we have

$$\widetilde{g}(a)=\lim_{\varepsilon\to0}\left[\frac{1}{2\pi i}\int_{\gamma_1}\frac{\widetilde{f}(z)}{z-a}\,dz+\frac{1}{2\pi i}\int_{\gamma_1}\frac{\widetilde{f}(z)}{z-a}\,dz\right]=\widetilde{f}(a)+0=\widetilde{f}(a),$$

where $\gamma_1$ and $\gamma_2$ are contours given as the folllowing figure for $\varepsilon>0$:



The same result holds also for $a \in \widetilde{\Omega}_\delta \setminus \overline{\Omega}$, so we can conclude $\widetilde{g}(a) = \widetilde{f}(a)$ on $a \in \widetilde{\Omega}_\delta \setminus (0,1)$, and by the contintuity of $\widetilde{f}$ and $\widetilde{g}$, we finally have $\widetilde{f} = \widetilde{g}$ so that $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}_\delta$. Since the above arguments make sense for every $\delta > 0$ small enough, the union $\widetilde{\Omega} = \bigcup_{\delta>0} \widetilde{\Omega}_\delta$ implies that the function $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}$.

(2) The domain $\widetilde{\Omega}$ is conformally mapped onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by

$$\varphi : \widetilde{\Omega} \to \mathbb{H} : z \mapsto \left(\frac{z+i}{iz+1}\right)^2.$$

Note that $\varphi(\Omega) = \{z \in \mathbb{H} : |z| > 1\}$.

We can compute for $(x,y) \in \widetilde{\Omega}$

$$|\varphi(x+iy)|^2 = \left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right)^2, \qquad \operatorname{Im}\varphi(x+iy) = \frac{4x(1-x^2-y^2)}{(x^2+(y-1)^2)^2}.$$

Define a function $V : \mathbb{H} \to \mathbb{R}$ such that $V := \widetilde{v} \circ \varphi^{-1}$. Then, $V$ is a harmonic function satisfying the boundary condition

$$\lim_{(x,y)\to(x_0,0)} V(x,y) = \begin{cases} -1 & \text{if } |x_0| < 1, \\ 1 & \text{if } |x_0| > 1. \end{cases}$$

For $(x,y) \in \varphi(\Omega)$ so that $x^2+y^2 > 1$ the Poisson kernel gives that

$$\begin{aligned} \frac{1-V(x,y)}{2} &= \frac{1}{\pi}\int_{-1}^{1} \frac{y}{(x-t)^2+y^2}\,dt \\ &= \frac{1}{\pi}\left(\tan^{-1}\frac{1-x}{y} + \tan^{-1}\frac{1+x}{y}\right) \\ &= \frac{1}{\pi}\tan^{-1}\frac{2y}{x^2+y^2-1}, \end{aligned}$$

so

$$V(x,y) = \frac{2}{\pi}\tan^{-1}\frac{x^2+y^2-1}{2y}.$$

Thus we have for $(x,y) \in \Omega$

$$v(x,y) = V(\varphi(x+iy)) = \frac{2}{\pi}\tan^{-1}\frac{y(1+x^2+y^2)}{x(1-x^2-y^2)}. \qquad \square$$