Smooth Surfaces

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Contents

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Part I

Smooth manifolds

Chapter 1

Calculus on Euclidean spaces

1.1 Inverse function theorem

slice lemma

1.1 (Constant rank theorem)**.** Let $f : M \to N$ be a smooth map whose differential has a locally constant rank k at a point $p \in M$. For each pair of local charts (U, φ) at p and (V, ψ) at $q := f(p)$, we introduce functions $a : \varphi(U) \to \mathbb{R}^k$ and $b : \psi(V) \to \mathbb{R}^{n-k}$ to write

$$\psi \circ f \circ \varphi^{-1}(x, y) = (a(x, y), b(x, y))$$

for $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{m-k}$.

(a) $a(0,0) = b(0,0) = 0$ and $\frac{\partial a}{\partial x}$ is invertible everywhere.

(b) $a(x, y) = x$.

(c) $a(x, y) = x$ and $b(x, y) = 0$.

Proof. (b) $\square$

1.2 (Preimage theorem)**.** Let $q \in N$ be a regular value of f.

Proof. Let $p \in f^{-1}(q)$ be any point.

$$\begin{tikzcd} f^{-1}(q) \cap U \arrow[d, "\varphi|_{f^{-1}(q)\cap U}"] \\ \ker \widetilde{f} \cap \varphi(U) \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"] & V \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, "\widetilde{f}"] & \psi(V) \end{tikzcd}$$

By the constant rank theorem, there are charts (U, φ) at p and (V, ψ) at q such that $\widetilde{f} := \psi \circ f \circ \varphi^{-1}$ is linear and the restriction

$$\varphi|_{f^{-1}(q)\cap U} : f^{-1}(q) \cap U \to \ker \widetilde{f} \cap \varphi(U)$$

is bijective. A restriction of a homeomorphism is a homeomorphism if and only if it is bijective, so it is a homeomorphism.

Since the transition maps for this atlas on $f^{-1}(q)$ are restrictions of transition maps for M onto linear subsapces, they are smooth. $\square$

Let $f : M \to N$ be an injective immersion. There exists unique smooth structure on $f(M)$ such that f and i are smooth.

Let $f : M \to N$ be an embedding. There exists unique smooth structure on $f(M)$ such that i are smooth.

1.3 (Constant rank theorem)**.** Let $f : M \to N$ be a smooth map such that its differential has a locally constant rank k at a point $p \in M$. Then, there is a pair of local charts (U, φ) at p and (V, ψ) at $f(p)$ such that

(a) $\varphi(p) = (0, 0) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$, $\psi \circ f(p) = (0, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$,

(b) $\psi \circ f \circ \varphi^{-1}(x, y) = (x, 0)$,

(c) $f(U) = f(M) \cap V$.

The follwoing diagram would be helpful.

$$\begin{tikzcd}
 & M \arrow[r, "f"] & N & \\
 & \cup & \cup & \\
 & U \arrow[d, "\varphi"] & V \arrow[d, "\psi"] & \\
\mathbb{R}^m \supset & \varphi(U) \arrow[r, "f_{(0)} = \psi \circ f \circ \varphi^{-1}"] & \psi(V) & \subset \mathbb{R}^n \\
 & \cup & \cup & \\
 & U_1 \arrow[d, "\varphi_1"] & V_1 \arrow[d, "\psi_1"] & \\
 & \varphi_1(U_1) \arrow[r, "f_{(1)} = \psi_1 \circ f_{(0)} \circ \varphi_1^{-1}"] & \psi_1(V_1) & \\
 & \cup & \cup & \\
 & U_2 \arrow[d, "\varphi_2"] & V_2 \arrow[d, "\psi_2"] & \\
 & \varphi_2(U_2) \arrow[r, "f_{(2)} = \psi_2 \circ f_{(1)} \circ \varphi_2^{-1}"] \arrow[uuur, "f_{(1)} \circ \varphi_2^{-1}"] & \psi_2(V_2) &
\end{tikzcd}$$

Proof. Let $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ be coordinate maps such that $p \in U$ and $df : TU \to TV$ has a constant rank k.

Step 1: First reparametrization. Consider the coordinate representation

$$f_{(0)} := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V).$$

It is smooth becuase of definition of smooth maps. Since $Df_{(0)}|_{\varphi(p)} : \mathbb{R}^m \to \mathbb{R}^n$ is a matrix of rank k, there is an invertible $k \times k$ minor submatrix. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ be permutation matrices that reorder the coordinates in such a way that the invertible $k \times k$ minor submatrix becomes the leading principal minor submatrix.

Define reparametrizations $\varphi_1 : U_1 \to \varphi_1(U_1)$ and $\psi_1 : V_1 \to \psi_1(V_1)$ as

$$\varphi_1(v) := A(v - \varphi(p)), \quad \psi_1(v) := B(v - \psi(f(p))),$$

where U_1 and V_1 are arbitrarily taken open subsets in $\varphi(U)$ and $\psi(V)$ respectively. We can check that they are invertible linear maps, hence diffeomorphisms.

Step 2: Second reparametrization (1). Let $f_{(1)}$ be the new coordinate representation

$$f_{(1)} := \psi_1 \circ f_{(0)} \circ \varphi_1^{-1} : \varphi_1(U_1) \to \psi_1(V_1).$$

It can be written as

$$f_{(1)}(x, y) = (a(x, y), b(x, y))$$

for $(x, y) \in \varphi_1(U_1) \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$ and for some $a : \varphi_1(U_1) \subset \mathbb{R}^m \to \mathbb{R}^k$. Then, we have

$$f_{(1)}(0, 0) = (0, 0) \quad \text{and} \quad \left.\frac{\partial a}{\partial x}\right|_{\varphi_1(\varphi(p))} \quad \text{is invertible.}$$

Define a reparamterization $\varphi_2 : \varphi_1(U) \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ as

$$\varphi_2(x, y) := (a(x, y), y).$$

Then,

$$D\varphi_2 = \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix}.$$

Since $D\varphi_2$ is smooth and $D\varphi_2|_{\varphi_1(\varphi(p))}$ is invertible, there exists an open set $U_2 \subset \varphi_1(U_1)$ such that the restriction $\varphi_2 : U_2 \to \varphi_2(U_2)$ on an open subset $U_2 \subset \varphi_1(U_1)$ is a diffeomorphism.

Step 3: Second reparametrization (2). Consider

$$f_{(1)} \circ \varphi_2^{-1} : \varphi_2(U_2) \to \psi_1(V_1).$$

Then,

$$D(f_{(1)} \circ \varphi_2^{-1}) = Df_{(1)} \circ D\varphi_2^{-1}$$
$$= \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{\partial a}{\partial x}\right)^{-1} & -\left(\frac{\partial a}{\partial x}\right)^{-1} \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & 0 \end{pmatrix}.$$

The last equality is because it should have rank k. Thus we have

$$f_{(1)} \circ \varphi_2^{-1}(x, y) = (x, c(x))$$

for all $(x, y) \in \varphi_2(U_2)$ and for some $c : \pi(\varphi_2(U_2)) \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$, where $\pi : \mathbb{R}^m \to \mathbb{R}^k$ is the canonical projection $(x, y) \mapsto x$.

Define a reparametrization $\psi_2 : V_2 \to \psi_2(V_2)$ by

$$\psi_2(x, z) := (x, z - c(x)),$$

where V_2 is arbitrary taken open subset of $\psi_1(V_1)$ such that $f_{(1)}(U_2) \subset V_2$. We can check ψ_2 is a diffeomorphism by computing the differential

$$D\psi_2 = \begin{pmatrix} \mathrm{id}_k & 0 \\ -\frac{\partial c}{\partial x} & \mathrm{id}_{n-k} \end{pmatrix}.$$

Step 4: Final verification. If we let

$$f_{(2)} := \psi_2 \circ f_{(1)} \circ \varphi_2^{-1} : \varphi_2(U_2) \to \psi_2(V_2),$$

then it satisfies

$$f_{(2)}(x, y) = (x, 0).$$

Define coordinate charts $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ around p and $f(p)$ such that

$$\widetilde{\varphi} := \varphi_2 \circ \varphi_1 \circ \varphi, \quad \widetilde{\psi} := \psi_2 \circ \psi_1 \circ \psi,$$

with domains

$$\widetilde{U} := \widetilde{\varphi}^{-1}(\varphi_2(U_2)), \quad \widetilde{V} := \widetilde{\psi}^{-1}\left((\pi(\operatorname{im} f_{(2)}) \times \mathbb{R}^{n-k}) \cap \psi_2(V_2)\right).$$

The new charts are compatible with old charts since the transitions $\varphi_2 \circ \varphi_1 : \varphi(\widetilde{U}) \to \widetilde{\varphi}(\widetilde{U})$ and $\psi_2 \circ \psi_1 : \psi(\widetilde{V}) \to \widetilde{\psi}(\widetilde{V})$ are diffeomorphisms. Then, the map f has the coordinate representation $f_{(2)} : (x, y) \mapsto (x, 0)$ on $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$.

We can check the last proposition as follows. Suppose $f(p) \in \widetilde{V}$. Then,

$$f_{(2)}(\widetilde{\varphi}(p)) = \widetilde{\psi}(f(p)) \in \pi(\operatorname{im} f_{(2)}) \times \mathbb{R}^{n-k} = \pi(\varphi_2(U_2)) \times \mathbb{R}^{n-k}.$$

Since $f_{(2)} = (\pi, 0)$, we have $f_{(2)}(\widetilde{\varphi}(p)) \in f_{(2)}(\varphi_2(U_2))$, so

$$f(p) = \widetilde{\psi}^{-1}(f_{(2)}(\widetilde{\varphi}(p))) \in \widetilde{\psi}^{-1}(f_{(2)}(\varphi_2(U_2))) = f(\widetilde{U}). \qquad \square$$

For the case that f is an either immersion or submersion, the constant rank theorem is sometimes refered as the local immersion theorem and the local submersion theorem respectively.

Corollary 1.1.1 (Immersion is a local embedding). *Let $f : M \to N$ be an immersion at $p \in M$. Then, there is a local chart (V, ψ) at $f(p)$ such that*

(a) *$W = f(M) \cap V$ is an embedded submanifold of V,*

(b) *there is a retract $V \to W$.*

Proof. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have

$$\varphi(p) = 0 \in \mathbb{R}^m, \quad \psi(f(p)) = (0,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad \text{and} \quad \psi \circ f \circ \varphi^{-1}(x) = (x, 0).$$

Let $W := f(M) \cap V$. Then, the injectivity of φ shows that

$$\psi(W) = \psi(f(U)) = \psi \circ f \circ \varphi^{-1}(\varphi(U)) = \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : x \in \varphi(U)\}$$

is an open subset of $\mathbb{R}^m$, so $(W, \psi|_W)$ is a chart at $f(p)$. □

Corollary 1.1.2 (Preimage theorem). *Let $f : M \to N$ be a submersion at $p \in M$. Then, there is a local chart (U, φ) at p such that*

(a) *$W = f^{-1}(f(p)) \cap U$ is an embedded submanifold of M,*

(b) *there is a retract $U \to W$.*

Proof. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have

$$\varphi(p) = (0,0) \in \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad \psi(f(p)) = 0 \in \mathbb{R}^n, \quad \text{and} \quad \psi \circ f \circ \varphi^{-1}(x, y) = x.$$

Let $W := f^{-1}(f(p)) \cap U$. Then, the injectivity of φ shows that

$$\begin{aligned} \varphi(W) &= \varphi(f^{-1}(f(p))) \cap \varphi(U) = (\psi \circ f \circ \varphi^{-1})^{-1}(\psi(f(p))) \cap \varphi(U) \\ &= \{(0, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : y \in \mathbb{R}^{m-n}\} \cap \varphi(U) \end{aligned}$$

is an open subset of $\mathbb{R}^{m-n}$, so $(W, \varphi|_W)$ is a chart at p. □

Corollary 1.1.3. *Let $f : M \to N$ be a smooth map and let $q \in N$. If f is a submersion at all points on $f^{-1}(q)$, then $f^{-1}(q)$ is an embedded submanifold of M.*

Chapter 2

Regular manifolds

2.1 Parameterizations and coordinates

2.1. *Regular parameterizations.* A *smooth parameterization* is a smooth map $\alpha : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ on a connected open subset Ω with $m \leq n$. The integer m is called its *dimension*. If α is a smooth parameterization such that the Fréchet derivative $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$ is injective for every $x \in \Omega$, then it is called an *immersion*. If α is an immersion and a homeomorphism onto its image, then it is called a *regular parameterization* or an *embedding*.

Let $\alpha : (0, 2\pi) \to \mathbb{R}^2$ and $\beta : \mathbb{R}^2 \to \mathbb{R}^3$ be smooth parameterizations given by

$$\alpha(t) := (\sin t, \sin 2t), \qquad \beta(x, y) := (x, y, x^2 + y^2).$$

(a) α is an immersion.

(b) α is not an embedding.

(c) β is an immersion.

(d) β is an embedding.

2.2. *Regular manifolds.* Let M be a subset of $\mathbb{R}^n$ with the subspace topology. The set M is called a *regular manifold* of dimension m if for each point $p \in M$ there is an open neighborhood U of p in M which is the image of an m-dimensional regular parameterization α. In this note we call this α a regular parameterization *for M at p*. Equivalently, M is a regular manifold if there is a collection of m-dimensional regular parameterizations $\{\alpha_i : \Omega_i \to \mathbb{R}^n\}_{i \in I}$ such that $\{\alpha_i(\Omega_i)\}_{i \in I}$ is an open cover of M.

The terms *regular curves* and *regular surfaces* refer to either the regular parameterizations or regular manifolds when their dimension is one and two, respectively.

(a) The image of a regular parameterization is a regular manifold.

(b) Every open subset of a regular manifold is a regular manifold.

(c) The sphere $S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ is a regular surface.

(d) The set $\{ (x, y) \in \mathbb{R}^2 : y^2 = x^3 + x^2 \}$ is not a regular curve.

(e) The set $\{ (x, y) \in \mathbb{R}^2 : y = |x| \}$ is not a regular curve.

2.3. *Coordinate maps.* Let M be a regular manifold in $\mathbb{R}^n$. A *coordinate map* is defined as the inverse map $\varphi : U = \alpha(\Omega) \to \varphi(U) = \Omega$ of a regular parameterization $\alpha : \Omega \to \mathbb{R}^n$ such that $\alpha(\Omega)$ is open in M.

Let $S := \{ (x, y) \in \mathbb{R}^2 : x > 0 \text{ or } y \neq 0 \}$. Then, S is a regular surface in $\mathbb{R}^2$.

(a) The map $(x, y) : S \to S$, $(x, y) \mapsto (x, y)$ is a coordinate map.

(b) The map

$$(r, \theta) : S \to (0, \infty) \times (-\pi, \pi),\ (x, y) \mapsto \left(\sqrt{x^2 + y^2},\ 2 \tan^{-1} \frac{y}{\sqrt{x^2 + y^2} + x} \right)$$

is a coordinate map, where $\tan^{-1}(t) := \int_0^t (1 + s^2)^{-1}\, ds$.

2.4. *Reparameterization.* A manifold can be analyzed with uncountably many coordiante charts, and an appropriate choice of charts allows explicit computations to be handy. A term *reparameterization* is nothing but a choice of another regular parameterization for the same curves or surfaces. Let M be an m-dimensional regular manifold in $\mathbb{R}^n$. Let $\alpha : \Omega_\alpha \to \mathbb{R}^n$ and $\beta : \Omega_\beta \to \mathbb{R}^n$ be m-dimensional regular parameterizations.

(a) If $\alpha(\Omega) \subset M$, then $\alpha(\Omega)$ is open in M.

2.2 Differentiations on regular manifolds

2.5. *Tangent spaces.* Let p be a point on an m-dimensional regular manifold M in $\mathbb{R}^n$. Take a regular parameterization α for M at p, and let $x := \alpha^{-1}(p)$ be the coordinates of p. The *tangent space* T_pM of M at p is defined as the image of $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$.

(a) T_pM is a m-dimensional vector subspace of $\mathbb{R}^n$.

(b) If $v \in T_pM$, then $\gamma(0) = p$ and $\gamma'(0) = v$ for a regular curve $\gamma : I \to M$.

(c) If $\gamma(0) = p$ and $\gamma'(0) = v$ for a regular curve $\gamma : I \to M$, then $v \in T_pM$. (Use the constant rank theorem.)

(d) The definition of T_pM is independent on the parameterization α.

2.6. Let M be a regular manifold. A *scalar field*, *smooth function*, or just a *function* is a function $f : M \to \mathbb{R}$ such that $f \circ \alpha : \Omega \to \mathbb{R}$ is smooth for any regular parameterization α for M.

(a)

A *vector field* is a map $X : M \to \mathbb{R}^n$ such that $X \circ \alpha : \Omega \to \mathbb{R}^n$ is smooth. A *tangent vector field* is a vector field $X : M \to \mathbb{R}^n$ such that $X|_p \in T_pM$. The set of tangent vector fields is often denoted by $\mathfrak{X}(M)$.

2.3 Linear algebra on tangent spaces

Chapter 3

Smooth manifolds

3.1 Smooth structures

3.1. *Locally Euclidean spaces.* continuous atlas

(a) For a regular manifold, a coordinate map is a chart.

3.2. *Smooth atlases.*

(a) Every continuous atlas is equivalent.

(b)

3.3. *Smooth structures.*

(a)

3.4. *Manifolds.* A *topological manifold* is a second-countable Hausdorff locally Euclidean space. A *smooth manifold* is a second-countable Hausdorff space with a smooth structure. The term *manifold* may refer to any of either a topological or a smooth manifold, which depends on the context of each reference. This note only concerns with smooth manifolds but will not omit the modifier "smooth".

(a)

We will provide four different definitions of tangent spaces:

(a) the space of equivalence classes of smooth curves,

(b) the space of tangent vectors embedded in an ambient space,

(c) the space of derivations on the ring of smooth functions,

(d) the dual space of algebraically defined cotangent spaces.

Reference

Remark. We can easily check that $T_p\mathbb{R}^n = \mathbb{R}^n$ for any $p \in \mathbb{R}^n$. The notation $T_p\mathbb{R}^n$ will be used to emphasize that a vector in $\mathbb{R}^n$ is geometrically recognized to cast from the point p. Since $T_p\mathbb{R}^n = \mathbb{R}^n = T_q\mathbb{R}^n$ for every pair of points $p, q \in \mathbb{R}^n$, summation and inner product of a vector in $T_p\mathbb{R}^n$ and a vector in $T_q\mathbb{R}^n$ make sense. This identification of tangent spaces are allowed *only for the case of linear spaces* such as $\mathbb{R}^n$. (In fact, the identification $T_p\mathbb{R}^n = \mathbb{R}^n$ is *natural* in categorical language.)

Remark. One way to view tangent spaces is to see them as domains and codomains of Fréchet derivatives. For open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$, the Fréchet derivative of a smooth map $F : U \to V$ at $x \in U$ is a linear transformation $dF|_x : T_xU \to T_{F(x)}V$. Since $T_xU = \mathbb{R}^m$ and $T_{F(x)}V = \mathbb{R}^n$, the original definition on Euclidean spaces agrees with it. In this reason, the Fréchet derivative dF is also called a *tangent map*, *pushforward*, or *differential* in differential geoemtry.

Notation. Let α be a parametrization for a regular curve or surface M. For derivatives of α, we will use the following notations:

$$\partial_t \alpha = \alpha', \quad \partial_x \alpha = \alpha_x, \quad \partial_i \alpha = \alpha_i.$$

The set $\{\alpha_i\}_i$ will be used to denote a basis of tangent space T_pM.

3.2 Differentiation by tangent vectors

Definition 3.2.1. Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization with $M = \operatorname{im}\alpha$.

(a) A *scalar field*, *smooth function*, or just a *function* is a function $f : M \to \mathbb{R}$ such that $f \circ \alpha : U \to \mathbb{R}$ is smooth.

(b) A *vector field* is a map $X : M \to \mathbb{R}^n$ such that $X \circ \alpha : U \to \mathbb{R}^n$ is smooth.

(c) A *tangent vector field* is a vector field $X : M \to \mathbb{R}^n$ such that $X|_p \in T_pM$.

The set of tangent vector fields is often denoted by $\mathfrak{X}(M)$.

Remark. In general, the word *vector fields* are basically assumed to be tangent. However, we will distinguish them in this note.

The following proposition proves that the smoothness of functions and vector fields does not depend on parametrizations.

Proposition 3.2.1. *Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ and $\beta : V \subset \mathbb{R}^m \to \mathbb{R}^n$ be parametrizations with same image $M = \operatorname{im}\alpha = \operatorname{im}\beta$. Then, the map $\beta^{-1} \circ \alpha : U \to V$ is smooth.*

Proof. □

Remark. The map $\beta^{-1} \circ \alpha$ is called the *transition map*.

Definition 3.2.2. Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization $M = \operatorname{im}\alpha$.

(a) The coordinate representation of a function $f : M \to \mathbb{R}$ is

$$f \circ \alpha : U \to \mathbb{R}.$$

(b) The (external) coordinate representation of a vector field $X : M \to \mathbb{R}^n$ is

$$X \circ \alpha : U \to \mathbb{R}^n.$$

(c) The coordinate representation of a tangent vector field $X : M \to \mathbb{R}^n$ is

$$(X^1 \circ \alpha, \cdots, X^m \circ \alpha) : U \to \mathbb{R}^m$$

where $X = \sum_i X^i \alpha_i$.

Definition 3.2.3. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. Let $v = \sum_i v^i \alpha_i|_p \in T_pM$ be a tangent vector at $p = \alpha(x)$. For a function $f : M \to \mathbb{R}$, its partial derivative is defined by

$$\partial_v f(p) := \sum_{i=1}^{m} v^i \partial_i (f \circ \alpha)(x) \in \mathbb{R}.$$

For a vector field $X : M \to \mathbb{R}^n$, its partial derivative is defined by

$$\partial_v X|_p := \sum_{i=1}^{m} v^i \partial_i (X \circ \alpha)(x) \in \mathbb{R}^n.$$

This definition is not dependent on parametrization α.

Proposition 3.2.2. *Let M be the image of a parametrization. Let X be a tangent vector field on M.*

(a) *If f is a function, then so is $\partial_X f$.*

(b) *If Y is a vector field, then so is $\partial_X Y$.*

(c) *If Y is a tangent vector field, then so is $\partial_X Y - \partial_Y X$.*

Proof. (1) and (2) are clear. For (3), if we let $X = \sum_i X^i \alpha_i$ and $Y = \sum_j Y^j \alpha_j$ for a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$, then

$$\begin{aligned}
\partial_X Y - \partial_Y X &= \partial_X(\textstyle\sum_j Y^j \alpha_j) - \partial_Y(\sum_i X^i \alpha_i) \\
&= \textstyle\sum_j [(\partial_X Y^j)\alpha_j + Y^j \partial_X \alpha_j] - \sum_i [(\partial_Y X^i)\alpha_i + X^i \partial_Y \alpha_i] \\
&= \textstyle\sum_j [(\partial_X Y^j)\alpha_j + Y^j \sum_i X^i \partial_i \alpha_j] - \sum_i [(\partial_Y X^i)\alpha_i + X^i \sum_j Y^j \partial_i \alpha_j] \\
&= \textstyle\sum_j (\partial_X Y^j)\alpha_j - \sum_i (\partial_Y X^i)\alpha_i \\
&= \textstyle\sum_i (\partial_X Y^i - \partial_Y X^i)\alpha_i.
\end{aligned}$$

□

Notation. Let M be the image of a parametrization α. For derivatives of functions on M by tangent vectors, we will use

$$\partial_{\alpha_i} f = \partial_i f, \quad \partial_{\alpha_t} f = \partial_t f = f', \quad \partial_{\alpha_x} f = \partial_x f = f_x.$$

For derivatives of vector fields on M by tangent vectors, we will use

$$\partial_{\alpha_i} X = \partial_i X, \quad \partial_{\alpha_t} X = \partial_t X = X', \quad \partial_{\alpha_x} X = \partial_x X = X_x.$$

We will *not* use f_i or X_i for $\partial_i f$ and $\partial_i X$ because it is confusig with coordinate representations, and *not* use the nabula symbol ∇_v in this sense because it will be devoted to another kind of derivatives introduced in Section 4.

Example 3.2.1. (a) Let α be an m-dimensional parametrization with $M = \operatorname{im}\alpha$. The value of $\partial_i \alpha = \alpha_i : M \to \mathbb{R}^3$ is always a tanget vector at each point $p = \alpha(x)$, and α_i becomes a vector field.

Let s be either a smooth function or vector field on α. Then, we can compute the directional derivative as

$$\partial_i s := \partial_i(s \circ \alpha) = \partial_t(s \circ \gamma)$$

by taking $\gamma(t) = \alpha(x + te_i)$, where e_i is the i-th standard basis vector for $\mathbb{R}^m$.

(b) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface given by

$$\alpha(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2} \right).$$

This map gives a parametrization for the sphere S^2 without the north pole $(0,0,1)$, and is called the *stereographic projection*. Let $f : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}$ be the height function of α defined by

$$f(p) := z$$

for $p = (x,y,z) \in S^2 \setminus \{(0,0,1)\}$. Its coordinate representation is

$$f \circ \alpha(x,y) = 1 - \frac{2}{1+x^2+y^2}.$$

Then, the directional derivative is

$$\partial_x f = \frac{\partial(f \circ \alpha)}{\partial x} = \frac{\partial}{\partial x}\left(1 - \frac{2}{1+x^2+y^2}\right) = \frac{4x}{(1+x^2+y^2)^2}.$$

Note that $\partial_x f \neq \partial_{(1,0,0)} z = 0$.

Part II

Local theory of curves and surfaces

Chapter 4

Curves

4.1 Parametrization

By definition, a regular curve has at least one parametrization. However, a given parametrization may not have useful properties, so we often take a new parametrization. The existence of a parametrization with certain properties is one of the main problems in differential geometry. Practically, the existence proof is usually done by constructing a *diffeomorphism* between open sets in $\mathbb{R}^m$; a bijective smooth map whose inverse is also smooth.

We introduce the arc-length reparametrization. It is the most general choice for the local study of curves.

Definition 4.1.1. A parametrization α of a regular curve is called a *unit speed curve* or an *arc-length parametrization* when it satisfies $\|\alpha'\| = 1$.

Theorem 4.1.1. *Every regular curve may be assumed to have unit speed. Precisely, for every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. By the definition of regular curves, we can take a parametrization $\beta : I_t \to \mathbb{R}^d$ for a given regular curve. We will construct an arc-length parametrization from β.

Define $\tau : I_t \to I_s$ such that

$$\tau(t) := \int_0^t \|\beta'(s)\| \, ds.$$

Since τ is smooth and $\tau' > 0$ everywhere so that τ is strictly increasing, the inverse $\tau^{-1} : I_s \to I_t$ is smooth by the inverse function theorem; τ is a diffeomorphism.

Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta'\left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}. \qquad \square$$

4.2 Frenet-Serret frame

The Frenet-Serret frame is a standard frame for a curve, and it is in particular effective when we assume the arc-length parametrization. It is defined for nondegenerate regular curves, i.e. nowhere straight curves. It provides with a useful orthonormal basis of $T_p\mathbb{R}^3 \supset T_p\gamma(I)$ for points p on a regular curve $\gamma : I \to \mathbb{R}^3$.

4.1. A regular curve $\gamma : I \to \mathbb{R}^3$ is called *non-degenerate* if the normalized tangent vector $\gamma'/\|\gamma'\|$ is never locally constant everywhere. In other words, γ is nowhere straight.

Definition 4.2.1 (Frenet-Serret frame)**.** Let α be a nondegenerate curve. The *tangent unit vector*, *normal unit vector*, *binormal unit vector* are $T_p\mathbb{R}^3$-valued vector fields on α defined by:

$$\mathrm{T}(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad \mathrm{N}(t) := \frac{\mathrm{T}'(t)}{\|\mathrm{T}'(t)\|}, \qquad \mathrm{B}(t) := \mathrm{T}(t) \times \mathrm{N}(t).$$

The set of vector fields $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$, which is called *Frenet-Serret frame*, forms an orthonormal basis of $T_p\mathbb{R}^3$ at each point p on α. The Frenet-Serret frame is uniquely determined up to sign as α changes.

We study the derivatives of the Frenet-Serret frame and their coordinate representations. In the coordinate representations on the Frenet-Serret frame, important geometric measurements such as curvatrue and torsion come out as coefficients.

Definition 4.2.2. Let α be a nondegenerate curve. The *curvature* and *torsion* are scalar fields on α defined by:

$$\kappa(t) := \frac{\langle \mathrm{T}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}, \quad \tau(t) := -\frac{\langle \mathrm{B}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}.$$

Note that $\kappa > 0$ cannot vanish by definition of nondegenerate curve. This definition is independent on α.

4.2. *Frenet-Serret formula.* Let γ be a non-degenerate regular curve. Then,

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \\ \mathrm{B}' \end{pmatrix} = \|\gamma'\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \\ \mathrm{B} \end{pmatrix}.$$

(a) $T' = \|\gamma'\|\kappa N$.

(b) $B' = -\|\gamma'\|\tau N$.

(c) $N' = -\|\gamma'\|\kappa T + \|\gamma'\|\tau B$.

Proof. Note that $\{T, N, B\}$ is an orthonormal basis.

(a) Two vectors T' and N are parallel by definition of N. By the definition of κ, we get $T' = \|\gamma'\|\kappa N$.

(b) Since $\langle T, B\rangle = 0$ and $\langle B, B\rangle = 1$ are constant, we have

$$\langle B', T\rangle = \langle B, T\rangle' - \langle B, T'\rangle = 0, \qquad \langle B', B\rangle = \tfrac{1}{2}\langle B, B\rangle' = 0.$$

By the definition of τ, we get $B' = -\|\alpha'\|\tau N$.

(c) Since

$$\begin{aligned}
\langle N', T\rangle &= -\langle N, T'\rangle = -\|\alpha'\|\kappa, \\
\langle N', N\rangle &= \tfrac{1}{2}\langle N, N\rangle' = 0, \\
\langle N', B\rangle &= -\langle N, B'\rangle = \|\alpha'\|\tau,
\end{aligned}$$

we have

$$N' = \|\alpha'\|(-\kappa T + \tau B). \qquad \square$$

Remark. Let $X(t)$ be the curve of orthogonal matrices $(T(t), N(t), B(t))^T$. Then, the Frenet-Serret formula reads

$$X'(t) = A(t)X(t)$$

for a matrix curve $A(t)$ that is completely determined by $\kappa(t)$ and $\tau(t)$, if we let us only consider arc-length parametrized curves. This is a typical form of an ODE system, so we can apply the Picard-Lindelöf theorem to get the following proposition: if we know $\kappa(t)$ and $\tau(t)$ for all time t, and if $T(0)$ and $N(0)$ are given so that an initial condition

$$X(0) = (T(0),\ N(0),\ T(0) \times N(0))$$

is established, then the solution $X(t)$ exists and uniquely determined in a short time range. Furthermore, if $\alpha(0)$ is given in addition, the integration

$$\alpha(t) = \alpha(0) + \int_0^t T(s)\, ds$$

provides a complete formula for unit speed parametrization α.

Remark. Skew-symmetry in the Frenet-Serret formula is not by chance. Let $\mathrm{X}(t) = (\mathrm{T}(t), \mathrm{N}(t), \mathrm{B}(t))^T$ and write $\mathrm{X}'(t) = A(t)\mathrm{X}(t)$ as we did in the above remark. Since $\mathrm{X}(t+h) = R_t(h)\mathrm{X}(t)$ for a family of special orthogonal matrices $\{R_t(h)\}_h$ with $R_t(0) = I$, we can describe $A(t)$ as

$$A(t) = \left.\frac{dR_t}{dh}\right|_{h=0}.$$

By differentiating the relation $R_t^T(h)R_t(h) = I$ with respect to h, we get to know that $A(t)$ is skew-symmetric for all t. In other words, the tangent space $T_I\mathrm{SO}(3)$ forms a skew symmetric matrix.

4.3 Computational problems

The following proposition gives the most effective and shortest way to compute the Frenet-Serret apparatus in general case. If we try to reparametrize the given curve into a unit speed curve or find κ by differentiating T, then we must encounter the normalizing term of the form $\sqrt{(-)^2 + (-)^2 + (-)^2}^{-1}$, and it must be painful when time is limited. The Frenet-Serret frame is useful in proofs of interesting propositions, but not a good choice for practical computation. Instead, a computation from derivatives of parametrization is highly recommended.

Proposition 4.3.1. *Let α be a nondegenerate curve. Then,*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}$$

and

$$\mathrm{T} = \frac{\alpha'}{\|\alpha'\|}, \qquad \mathrm{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \qquad \mathrm{N} = \mathrm{B} \times \mathrm{T}.$$

Proof. If we let $s = \|\alpha'\|$, then

$$\begin{aligned}
\alpha' &= s\mathrm{T}, \\
\alpha'' &= s'\mathrm{T} + s^2\kappa\mathrm{N}, \\
\alpha''' &= (s'' - s^3\kappa^2)\mathrm{T} + (3ss'\kappa + s^2\kappa')\mathrm{N} + (s^3\kappa\tau)\mathrm{B}.
\end{aligned}$$

Now the formulas are easily derived. □

4.4 General problems

We are interested in regular curves, not a particular parametrization. By the Theorem 2.1, we may always assume that a parametrization α has unit speed. Let α be a nondegenerate unit speed space curve, and let $\{\mathrm{T},\mathrm{N},\mathrm{B}\}$ be the Frenet-Serret frame for α.

Consider a diagram as follows:

$$\begin{array}{ccccc}
\langle \alpha,\mathrm{T}\rangle = ? & \longleftrightarrow & \langle \alpha,\mathrm{N}\rangle = ? & \longleftrightarrow & \langle \alpha,\mathrm{B}\rangle = ? \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\langle \alpha',\mathrm{T}\rangle = 1 & & \langle \alpha',\mathrm{N}\rangle = 0 & & \langle \alpha',\mathrm{B}\rangle = 0.
\end{array}$$

Here the arrows indicate which term we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t),\mathrm{T}(t)\rangle = f(t),$$

then we can obtain

$$\langle \alpha(t),\mathrm{N}(t)\rangle = \frac{f'(t)-1}{\kappa(t)}$$

by direct differentiation since we have known $\langle \alpha',\mathrm{T}\rangle$ but not $\langle \alpha,\mathrm{N}\rangle$. Further, we get

$$\langle \alpha(t),\mathrm{B}(t)\rangle = \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t)f(t)}{\tau(t)}$$

since we have known $\langle \alpha,\mathrm{T}\rangle$ and $\langle \alpha',\mathrm{N}\rangle$ but not $\langle \alpha,\mathrm{B}\rangle$. Thus, $\langle \alpha,\mathrm{T}\rangle = f$ implies

$$\alpha(t) = f(t)\cdot \mathrm{T} + \frac{f'(t)-1}{\kappa(t)}\cdot \mathrm{N} + \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t)f(t)}{\tau(t)}\cdot \mathrm{B},$$

when given $\tau(t)\neq 0$.

We suggest a strategy for space curve problems:

- Build and differentiate equations of the following form:

$$\langle\ \text{(interesting vector)},\ \text{(Frenet-Serret basis)}\ \rangle\ =\ \text{(some function)}.$$

- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of κ and τ by comparing with assumptions.

- Heuristically find a constant vector and show what you want directly.

Here we give example solutions of several selected problems. Always α denotes a reparametrized unit speed nondegenerate curve in $\mathbb{R}^3$.

If

$$f = \langle \alpha - p, \mathrm{T} \rangle, \quad g = \langle \alpha - p, \mathrm{N} \rangle, \quad h = \langle \alpha - p, \mathrm{B} \rangle,$$

then

$$f' = 1 + \kappa g, \quad g' = -\kappa f + \tau h, \quad h' = -\tau g.$$

4.3. A curve whose normal lines always pass through a fixed point lies in a circle.

Solution. Step 1: Formulate conditions. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and N are parallel so that we have

$$\langle \alpha - p, \mathrm{T} \rangle = 0, \qquad \langle \alpha - p, \mathrm{B} \rangle = 0.$$

Our goal is to show that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v \rangle = 0$.

Step 2: Collect information. Differentiate $\langle \alpha - p, \mathrm{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{B} \rangle = 0$ to get

$$\tau = 0.$$

Step 3: Complete proof. We can deduce that $\|\alpha - p\|$ is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p \rangle' = 2\langle \alpha - p, \mathrm{T} \rangle = 0.$$

Also, if we heuristically define a vector $v := \mathrm{B}$, then v is constant since

$$v' = -\tau \mathrm{N} = 0,$$

and clearly $\langle \alpha - p, v \rangle = 0$ □

4.4. A spherical curve of constant curvature lies in a circle.

Solution. Step 1: Formulate conditions. The condition that α lies on a sphere can be given as follows: for a constant point $p \in \mathbb{R}^3$,

$$\|\alpha - p\| = \mathrm{const}.$$

Also we have

$$\kappa = \text{const}\,.$$

Step 2: Collect information. Differentiate $\|\alpha - p\|^2 = \text{const}$ to get

$$\langle \alpha - p, \mathrm{T}\rangle = 0.$$

Differentiate $\langle \alpha - p, \mathrm{T}\rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N}\rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{N}\rangle = -1/\kappa = \text{const}$ to get

$$\tau\langle \alpha - p, \mathrm{B}\rangle = 0.$$

There are two ways to show that $\tau = 0$.

Method 1: Assume that there is t such that $\tau(t) \neq 0$. By the continuity of τ, we can deduce that τ is locally nonvanishing. In other words, we have $\langle \alpha - p, \mathrm{B}\rangle = 0$ on an open interval containing t. Differentiate $\langle \alpha - p, \mathrm{B}\rangle = 0$ at t to get $\langle \alpha - p, \mathrm{N}\rangle = 0$ near t, which is a contradiction. Therefore, $\tau = 0$ everywhere.

Method 2: Since $\langle \alpha - p, \mathrm{B}\rangle$ is continuous and

$$\langle \alpha - p, \mathrm{B}\rangle = \pm\sqrt{\|\alpha - p\|^2 - \langle \alpha - p, \mathrm{T}\rangle^2 - \langle \alpha - p, \mathrm{N}\rangle^2} = \pm\,\text{const},$$

we get $\langle \alpha - p, \mathrm{B}\rangle = \text{const}$. Differentiate to get $\tau\langle \alpha - p, \mathrm{N}\rangle = 0$. Finally we can deduce $\tau = 0$ since $\langle \alpha - p, \mathrm{N}\rangle \neq 0$.

Step 3: Complete proof. The zero torsion implies that the curve lies on a plane. A planar curve in a sphere is a circle. □

4.5. A curve such that $\tau/\kappa = (\kappa'/\tau\kappa^2)'$ lies on a sphere.

Solution. Step 1: Find the center heuristically. If we assume that α is on a sphere so that we have $\|\alpha - p\| = r$ for constants $p \in \mathbb{R}^3$ and $r > 0$, then by the routine differentiations give

$$\langle \alpha - p, \mathrm{T}\rangle = 0, \qquad \langle \alpha - p, \mathrm{N}\rangle = -\frac{1}{\kappa}, \qquad \langle \alpha - p, \mathrm{B}\rangle = -\left(\frac{1}{\kappa}\right)'\frac{1}{\tau},$$

that is,

$$\alpha - p = -\frac{1}{\kappa}\mathrm{N} - \left(\frac{1}{\kappa}\right)'\frac{1}{\tau}\mathrm{B}.$$

Step 2: Complete proof. Let us get started the proof. Define

$$p := \alpha + \frac{1}{\kappa}\mathrm{N} + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\mathrm{B}.$$

We can show that it is constant by differentiation. Also we can show that

$$\langle \alpha - p, \alpha - p \rangle$$

is constant by differentiation. So we are done. □

4.6. A curve with more than one Bertrand mates is a circular helix.

Solution. Step 1: Formulate conditions. Let β be a Bertrand mate of α so that we have

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \mathrm{N}_\beta = \pm \mathrm{N},$$

where λ is a function not vanishing somewhere and $\{\mathrm{T}_\beta, \mathrm{N}_\beta, \mathrm{B}_\beta\}$ denotes the Frenet-Serret frame of β. We can reformulate the conditions as follows:

Note that β is not unit speed.

Step 2: Collect information. Differentiate $\langle \beta - \alpha, \mathrm{N} \rangle = \lambda$ to get

$$\lambda = \mathrm{const} \neq 0.$$

Differentiate $\langle \beta - \alpha, \mathrm{T} \rangle = 0$ and $\langle \beta - \alpha, \mathrm{B} \rangle = 0$ to get

$$\langle \mathrm{T}_\beta, \mathrm{T} \rangle = \frac{1 - \lambda\kappa}{\|\beta'\|}, \qquad \langle \mathrm{T}_\beta, \mathrm{B} \rangle = \frac{\lambda\tau}{\|\beta'\|}.$$

Differentiate $\langle \mathrm{T}_\beta, \mathrm{T} \rangle$ and $\langle \mathrm{T}_\beta, \mathrm{B} \rangle$ to get

$$\frac{1 - \lambda\kappa}{\|\beta'\|} = \mathrm{const}, \qquad \frac{\lambda\tau}{\|\beta'\|} = \mathrm{const}.$$

Thus, there exists a constant μ such that

$$1 - \lambda\kappa = \mu\lambda\tau$$

if α is not planar so that $\tau \neq 0$.

We have shown that the torsion is either always zero or never zero at every point: $\lambda\tau/\|\beta'\| = \mathrm{const}$. The problem can be solved by dividing the cases, but in this solution we give only for the case that α is not planar; the other hand is not difficult.

Step 3: Complete proof. If

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \widetilde{\beta} = \alpha + \widetilde{\lambda}\mathrm{N}$$

are different Bertrand mates of α with $\lambda \neq \widetilde{\lambda}$, then (κ, τ) solves a two-dimensional linear system

$$\begin{aligned} \kappa + \mu\tau &= \lambda^{-1}, \\ \kappa + \widetilde{\mu}\tau &= \widetilde{\lambda}^{-1}. \end{aligned}$$

It is nonsingular since $\mu = \widetilde{\mu}$ implies $\lambda = \widetilde{\lambda}$, which means we can represent κ and τ in terms of constants $\lambda, \widetilde{\lambda}, \mu$, and $\widetilde{\mu}$. Therefore, κ and τ are constant. □

Here is a well-prepared problem set for exercises.

4.7 (Plane curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α lies on a plane,

(b) $\tau = 0$,

(c) the osculating plane constains a fixed point.

4.8 (Helices)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α is a helix,

(b) $\tau/\kappa = \mathrm{const}$,

(c) normal lines are parallel to a plane.

4.9 (Sphere curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α lies on a sphere,

(b) $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \mathrm{const}$,

(c) $\tau/\kappa = (\kappa'/\tau\kappa^2)'$,

(d) normal planes contain a fixed point.

4.10 (Bertrand mates)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(a) the curve α has a Bertrand mate,

(b) there are two constants $\lambda \neq 0, \mu$ such that $1/\lambda = \kappa + \mu\tau$.

Chapter 5

Surfaces

5.1 Reparametrization

Theorem 5.1.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 5.1.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 5.1.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 5.1.1. *Metric coefficients*

$$\begin{aligned} \langle \alpha_x, \alpha_x \rangle &=: g_{11} & \langle \alpha_x, \alpha_y \rangle &=: g_{12} \\ \langle \alpha_y, \alpha_x \rangle &=: g_{21} & \langle \alpha_y, \alpha_y \rangle &=: g_{22} \end{aligned}$$

Theorem 5.1.4 (Normal coordinates)**.** *...?*

5.2 Differentiation of tangent vectors

Definition 5.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x, y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x, y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 5.2.2 (Gauss formula, Γ_{ij}^k, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$ and $\{L_{ij}\}_{i,j=1}^2$ by the Gauss formula

$$\begin{aligned} \alpha_{xx} &=: \Gamma_{11}^1 \alpha_x + \Gamma_{11}^2 \alpha_y + L_{11}\nu, & \alpha_{xy} &=: \Gamma_{12}^1 \alpha_x + \Gamma_{12}^2 \alpha_y + L_{12}\nu, \\ \alpha_{yx} &=: \Gamma_{21}^1 \alpha_x + \Gamma_{21}^2 \alpha_y + L_{21}\nu, & \alpha_{yy} &=: \Gamma_{22}^1 \alpha_x + \Gamma_{22}^2 \alpha_y + L_{22}\nu. \end{aligned}$$

The *Christoffel symbols* refer to eight functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij} \nu. \end{aligned}$$

5.3 Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu \rangle = \frac{1}{2} \partial_X \langle \nu, \nu \rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 5.3.1. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 5.3.1. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y)\rangle = \langle X, -\partial_Y \nu\rangle = \langle \partial_Y X, \nu\rangle = \langle \partial_X Y, \nu\rangle = \langle \mathcal{S}(X), Y\rangle. \qquad \square$$

Theorem 5.3.2. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i \alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j \alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}^j_i X_i$. Then,

$$g_{ik} X^i \mathcal{S}^k_j Y^j = \langle X, \mathcal{S}(Y)\rangle = \langle \partial_X Y, \nu\rangle = X^i Y^j L_{ij}$$

implies $g_{ik}\, \mathcal{S}^k_j = L_{ij}$. $\square$

5.4 Computational problems

Definition 5.4.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$\begin{aligned} E &:= \langle \alpha_x, \alpha_x\rangle = g_{11}, & F &:= \langle \alpha_x, \alpha_y\rangle = g_{12}, & G &:= \langle \alpha_y, \alpha_y\rangle = g_{22}, \\ L &:= \langle \alpha_{xx}, \nu\rangle = L_{11}, & M &:= \langle \alpha_{xy}, \nu\rangle = L_{12}, & N &:= \langle \alpha_{yy}, \nu\rangle = L_{22}. \end{aligned}$$

Corollary 5.4.1. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\begin{aligned} \nu_x &= \frac{FM - GL}{EG - F^2}\alpha_x + \frac{FL - EM}{EG - F^2}\alpha_y, \\ \nu_y &= \frac{FN - GM}{EG - F^2}\alpha_x + \frac{FM - EN}{EG - F^2}\alpha_y. \end{aligned}$$

Theorem 5.4.2.
$$\Gamma^l_{ij} = \frac{1}{2} g^{kl}(g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2}(\log g)_x = \Gamma^1_{11}.$$

$$\nu_x \times \nu_y = K\sqrt{\det g}\ \nu.$$
$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$
$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det \begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

Theorem 5.4.3 (Gaussian curvature formula)**.**

(a) *In general,*
$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) *For orthogonal coordinates such that* $F \equiv 0$*,*
$$K = -\frac{1}{2\sqrt{\det g}} \left(\left(\frac{1}{\sqrt{\det g}} E_y\right)_y + \left(\frac{1}{\sqrt{\det g}} G_x\right)_x \right).$$

(c) *For* $f(x, y, z) = 0$*,*
$$K = -\frac{1}{|\nabla f|^4} \begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f) \end{vmatrix},$$
where ∇f *denotes the gradient* $\nabla f = (f_x, f_y, f_z)$*.*

(d) *(Beltrami-Enneper) If* τ *is the torsion of an asymptotic curve, then*
$$K = -\tau^2.$$

(e) *(Brioschi)* E, F, G *describes* K*.*

Proof.

(a) Clear.

(b) We have $GM = EM$ and

$$\nu_x = -\frac{L}{E}\alpha_x - \frac{M}{G}\alpha_y, \qquad \nu_y = -\frac{M}{E}\alpha_x - \frac{N}{G}\alpha_y.$$

$$\nu_x \times \nu_y = \frac{LN - M^2}{EG}\alpha_x \times \alpha_y$$

After curvature tensors...

□

Example 5.4.1. (a) (Monge's patch) For $(x, y, f(x, y))$,

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t) = (r(t), z(t))$ be a plane curve with $r(t) > 0$. Let

$$\alpha(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

be a parametrization of a surface of revolution.

Then,

$$\begin{aligned}
\alpha_\theta &= (-r(t)\sin\theta, r(t)\cos\theta, 0) \\
\alpha_t &= (r'(t)\cos\theta, r'(t)\sin\theta, z'(t)) \\
\nu &= \frac{1}{\sqrt{r'(t)^2 + z'(t)^2}}(z'(t)\cos\theta, z'(t)\sin\theta, -r'(t)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{\theta\theta} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0) \\
\alpha_{\theta t} &= (-r'(t)\sin\theta, -r'(t)\cos\theta, 0) \\
\alpha_{tt} &= (r''(t)\cos\theta, r''(t)\sin\theta, z''(t)).
\end{aligned}$$

Thus we have

$$E = r(t)^2, \quad F = 0, \quad G = r'(t)^2 + z'(t)^2,$$

and

$$L = -\frac{r(t)z'(t)}{\sqrt{r'(t)^2 + z'(t)^2}}, \quad M = 0, \quad N = \frac{r''(t)z'(t) - r'(t)z''(t)}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

5.5 General problems

Theorem 5.5.1. *Surfaces of the same constant Gaussian curvature are locally isomorphic.*

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 6

Geodesics

Part III

Riemannian manifolds

Chapter 7

Intrinsic geometry

We say a quantity on a surface is *intrinsic* if it is independent of how the surface is embedded in space.

Notations: Einstein summation convention, set of vector fields.

To n-dimensional.

7.1 Covariance and contravariance

7.2 Theorema Egregium

- Intrinsic: g_{ij}, Γ_{ij}^k, K, $R^l{}_{ijk}$;
- Not intrinsic: ν, L_{ij}, κ_i, H.

Isometry

Example 7.2.1. Let $\alpha : (-\log 2, \log 2) \times (0, 2\pi) \to \mathbb{R}^3$ and $\beta : (-\frac{3}{4}, \frac{3}{4}) \times (0, 2\pi) \to \mathbb{R}^3$ be regular surfaces given by

$$\alpha(x,\theta) = (\cosh x \cos\theta,\ \cosh x \sin\theta,\ x), \qquad \beta(r,z) = (r\cos z,\ r\sin z,\ z).$$

Their Riemannian metrics are

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix}_{(\alpha_x,\alpha_\theta)}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1+r^2 \end{pmatrix}_{(\beta_r,\beta_z)}.$$

Define a map $f : \operatorname{im}\alpha \to \operatorname{im}\beta$ by

$$f : \alpha(x,\theta) \mapsto \beta(\sinh x, \theta) = (r(x,\theta), z(x,\theta)).$$

The Jacobi matrix of f is computed

$$df|_{\alpha(x,\theta)} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}_{(\alpha_x,\alpha_\theta)\to(\beta_r,\beta_z)}.$$

Since f is a diffeomorphism and

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1+r^2 \end{pmatrix} \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix},$$

the map f is an isometry.

Chapter 8

Covariant derivatives

8.1 Orthogonal projection

We are going to think about "intrinsic" derivatives for tangent vectors. For coordinate independence, directional derivatives of a tangent vector field should be at least a tangent vector field, which is false for the obvious partial derivatives in the embedded surface setting; for example, T is a tangent vector, but $\mathrm{N} = \kappa \mathrm{T}'$ is not tangent.

Recall that the Gauss formula reads

$$\partial_i \alpha_j = \Gamma_{ij}^k \alpha_k + L_{ij} \nu$$

so that we have

$$\begin{aligned}
\partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\
&= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\
&= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij} \nu.
\end{aligned}$$

If we write $\nabla_X Y = \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k$, then it embodies the orthogonal projection of $\partial_X Y$ onto its tangent space, and we have

$$\partial_X Y = \nabla_X Y + \mathrm{II}(X, Y) \nu.$$

Definition 8.1.1. Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $\operatorname{im} \alpha = M$. Let $X = X^i \alpha_i$ and $Y = Y^j \alpha_j$ be tangent vector fields on M. The *covariant derivative* of Y along X is defined as the orthogonal projection of the partial derivative $\partial_X Y$ onto the tangent space:

$$\nabla_X Y := \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k.$$

Proposition 8.1.1. *Covariant derivatives are intrinsic. In other words, the above definition does not depend on the choice of parametrizations.*

Proof. Recall that the Christoffel symbols transform as follows:

$$X^i Y^j \Gamma_{ij}^k = X^a Y^b \left(\Gamma_{ab}^c + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} \right) \frac{\partial x^k}{\partial x^c}.$$

Thus, we have

$$\begin{aligned}
&\left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k \\
&\quad = X^a \frac{\partial}{\partial x^a} \left(Y^c \frac{\partial x^k}{\partial x^c} \right) \alpha_k + X^a Y^b \left(\frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} + \Gamma_{ab}^c \right) \frac{\partial x^k}{\partial x^c} \alpha_k \\
&\quad = X^a \frac{\partial Y^c}{\partial x^a} \alpha_c + X^a Y^b \left(\frac{\partial^2 x^k}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial x^k} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} \right) \alpha_c + X^a X^b \Gamma_{ab}^c \alpha_c \\
&\quad = \left(X^a \partial_a Y^c + X^a Y^b \Gamma_{ab}^c \right) \alpha_c
\end{aligned}$$

since

$$\frac{\partial^2 x^j}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial x^j} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^a} \left(\frac{\partial x^j}{\partial x^b} \frac{\partial x^c}{\partial x^j} \right) = \partial_a \delta_b^c = 0. \qquad \square$$

8.2 Connections

We will give a coordinate-free axiomatic definition of covariant derivatives and show that they coincide. By doing this, we obtain an alternative proof for the statement that covariant derivatives are intrinsic.

Definition 8.2.1 (Affine connection)**.** Let M be the image of a parametrization. An *affine connection* is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(a) $\nabla_{(-)} Y : \mathfrak{X}(M) \to \mathfrak{X}(M) : X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear;

(b) $\nabla_X(-) : \mathfrak{X}(M) \to \mathfrak{X}(M) : Y \mapsto \nabla_X Y$ is $\mathbb{R}$-linear;

(c) the Leibniz rule
$$\nabla_X(fY) = (\partial_X f)Y + f \nabla_X Y$$
is satisfied.

Definition 8.2.2 (Metric connection)**.** Let M be the image of a parametrization and $\langle , \rangle$ be a Riemannian metric on M. A *metric connection* is an affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

$$\partial_Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle.$$

Definition 8.2.3 (Levi-Civita connection)**.** Let M be the image of a parametrization and $\langle , \rangle$ be a Riemannian metric on M. A *Levi-Civita connection* is a metric connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:

$$\nabla_X Y - \nabla_Y X = \partial_X Y - \partial_Y X.$$

Theorem 8.2.1. *Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $M = \operatorname{im}\alpha$. Then, there is a unique Levi-Civita connection on M.*

Proof. (Uniqueness) Suppose ∇ is a Levi-Citiva connection on M.

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & \partial_X \langle Y, Z\rangle + \partial_Y \langle X, Z\rangle - \partial_Z \langle X, Y\rangle \\ & - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle. \end{aligned}$$

(Existence) $\square$

Our claim is that this definition is equivalent to the above coordinate dependent definition, the Levi-Civita connection, of the covariant derivative.

Proposition 8.2.2. *Let S be a regular surface embedded in $\mathbb{R}^3$. If we define Christoffel symbols as the Gauss formula, then*

$$\mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S) : (X^i \alpha_i, Y^j \alpha_j) \mapsto \left(X^i \partial_i Y^k + X^i Y^j \Gamma^k_{ij} \right) \alpha_k$$

defines a Levi-Civita connection.

8.3 Curvature tensor

Chapter 9

Parallel transport

Part IV

Global theory of curves and surfaces

Chapter 10

Global theory of curves

10.1 Isoperimetric inequality

10.2 Four vertex theorem

10.3 Ovals

Chapter 11

Global theory of surfaces

11.1 Minimal surfaces

11.2 Classification of compact surfaces

11.3 The Hilbert theorem

Chapter 12

Total curvatures

12.1 The Fary-Minor theorem

Fenchel's theorem

12.2 The Gauss-Bonnet theorem