Low Dimensional Topology

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Hyperbolization

4.1 Geometric structures

4.1 (Space forms). Let X be a simply connected homogeneous Riemannian manifold. A *space form* of X is an orbit space X/Γ of a subgroup $\Gamma \le \operatorname{Isom}(X)$ such that $X \to X/\Gamma$ is a covering. If M is a space form of X, we say M is *modelled on* X.

(a) A complete finite-volume Riemannian manifold is locally isometric to X is a space form of X.

(b) X/Γ is a space form of X if and only if Γ acts properly discontinuously and freely on X. (if X is locally compact?)

(c) X/Γ is a space form of X if and only if Γ is a discrete torsion-free subgroup of $\operatorname{Isom}(X)$. (if every periodic isometry of X has a fixed point)

(d) There is a following one-to-one correspondence:

$$\begin{array}{ccc} \dfrac{\{\text{space forms of } X\}}{\text{isometry}} & \xrightarrow{\sim} & \dfrac{\{\text{discrete torsion-free subgroups of } \operatorname{Isom}(X)\}}{\text{conjugacy}}, \\ X/\Gamma & \mapsto & \Gamma. \end{array}$$

Proof. □

4.2 (Model geometry). A *model geometry* is a simply connected homogeneous Riemannian manifold X that has a space form of finite-volume.

(a) There are three model geometries of dimension two, up to isometry.

(b) There are eight model geometries of dimension three, up to isometry.

Proof. □

4.3 ((G,X)-manifolds). Let X be a topological space. A *pseudogroup* Γ on X is a wide subgroupoid of $\operatorname{Homeo}(X)$ such that $U \mapsto \{g \in \Gamma : \operatorname{dom} g = U\}$ is a sheaf on the topology of X. Let Γ be a pseudogroup on X, and M be a topological space. A Γ-*atlas* on M is an atlas whose charts have X as the codomain and transition maps belong to Γ. A Γ-*structure* on M is defined as an equivalence class of Γ-atlases on M.

(a) $(G,X) := \{g|_U : g \in G,\ U \in \mathcal{T}\}$ is a pseudogroup on X for $G \le \operatorname{Homeo}(X)$.

(b) ... Note that G does not act on (G,X)-manifold M.

4.4 (Developing and holonomy). Complete (G,X)-manifolds

4.5 ((Isom(X), X)-manifolds). Let X be a simply connected homogeneous Riemannian manifold. We will show that a complete (G,X)-structure on a connected manifold corresponds to a geometric structure modelled on X. In this regard, we will call a (G,X)-structure that is not complete as *incomplete geometric structure*.

(a) M is a connected complete (Isom(X), X)-manifold if and only if M is a space form of X.

4.2 Poincaré polyhedron theorem

4.6 (Fundamental polyhedron).

4.7 (Side pairing). We want to develop a method for obtaining geometric 3-manifolds by gluing tetrahedra. Let X be a model geometry and X/Γ be a space form of X with finite volume.

4.8 (Elliptic cycle condition). Tesselation of $\mathbb{S}^2$ about verteices and edges, but edges are redundant.

4.9 (Parabolic cycle condition). Tesselation of $\mathbb{E}^2$ about ideal vertices.

4.3 Hyperbolic Dehn surgery

4.10 (Cusp and horoball).

4.11 (Thick-thin decomposition). Margulis constant.

4.12 (Thurston's hyperbolic Dehn surgery).

4.4 Mostow rigidity

Kleinian groups Several topological invariants: volume, trace fields, etc.

4.5 Orbifolds

Exercises

4.13 (Action of compact stabilizer). Let M be a connected smooth manifold. Recovering metric from action of compact stabilizers...? We want to establish the following one-to-one correspondence.

$$\left\{\begin{array}{c}\text{Homogeneous}\\ \text{Riemannian metrics}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{Maximal}\\ \text{homogeneous smooth group actions}\\ \text{of compact stabilizers}\end{array}\right\}.$$

(a) If g is a homogeneous Riemannian metric on M, the group action on M by Isom(M,g) is maximal among smooth group actions with compact stabilizers.

(b) If a smooth group action on M by G is maximal among smooth group actions with compact stabilizers, then there is a homogeneous Riemannian metric on M such that $G \cong$ Isom(M,g).

Proof. □

4.14 (Hyperbolization of punctured surfaces). Euler characteristic χ.

4.15 (Figure-eight knot complement). Find an ideal triangulation. Find the angle condition for the two ideal tetrahedra. Find the generators of Γ.

4.16 (Geometrically finiteness).

4.17 (Siegel theorem).

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Part III

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Seiberg-Witten theory

11.1. Let V be a real inner product space. In $\mathrm{Spin}(4) \subset \mathrm{Cl}^{\mathrm{ev}}(4) \cong \mathrm{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$, the spin group $\mathrm{Spin}(4)$ is isomorphic to $S^3 \times S^3$.

If V is even-dimensional, then $\mathrm{Cl}(V)$ has a unique finite-dimensional irreducible complex graded representation $\mathrm{Cl}(V) \to \mathrm{End}_{\mathbb{C}}(S(V))$ such that the complexification induces an isomorphism $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(S(V)) = S(V) \otimes S(V)^*$.

If V is odd-dimensional, then $\mathrm{Cl}(V)$ has two non-isomorphic finite-dimensional irreducible complex representations $\mathrm{Cl}(V) \to \mathrm{End}_{\mathbb{C}}(S_+(V \oplus \mathbb{R}))$ and $\mathrm{Cl}(V) \to \mathrm{End}_{\mathbb{C}}(S_-(V \oplus \mathbb{R}))$ obtained from the direct sum decomposition of $\mathrm{Cl}(V) = \mathrm{Cl}^0(V \oplus \mathbb{R}) \subset \mathrm{Cl}^0(V \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(S(V \oplus \mathbb{R}))$.

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