Complex Analysis

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Contents

Chapter 1

Holomorphic functions

1.1 Ho

1.1. Let $p \in \mathbb{C}[z]$ with $p(z) = \sum_{k=0}^{n} a_k z^k$.

(a) $|p(z)| \lesssim |z|^n$.

(b) There is $R > 0$ such that $|p(z)| \gtrsim |z|^n$ for $|z| \geq R$.

Proof. If we take $R > 0$ such that

$$\sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2},$$

then $|z| \geq R$ implies

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \\ &\geq |a_n||z|^n - \sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} |z|^n \\ &\geq \frac{|a_n|}{2} |z|^n. \end{aligned}$$

□

1.2. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain. Then, $\overline{f(z)} = f(\overline{z})$ if and only if $f(z) \in \mathbb{R}$ for $z \in \Omega \cap \mathbb{R}$.

1.2 The residue theorem

$$\int_0^{2\pi} \frac{dx}{1+a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \quad -1 < a < 1$$

1.3 (Semicircles). (a)

$$\int_0^\infty \frac{1}{1+x^2}\,dx =$$

(b)

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

(c)

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx =$$

(d)

$$\int_0^\infty \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1$$

1.4 (Computation of Fourier transforms)**.**

1.5 (Laplace transforms)**.**

1.6 (Gamma function)**.**

1.3 Argument principle

1.7. (a)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = i\text{winding number.}$$

(b)

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(\text{number of zeros} - \text{number of poles}).$$

1.8. Let f be a meromorphic function on Ω. Let γ be a curve...

(a) If $h : [0,1]\times\Omega \to \mathbb{C}$ is continuous and

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

(b) In particular, if $|g(z)| < |f(z)|$ on $z \in \gamma$, then

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_\gamma \frac{f'(z) + g'(z)}{f(z) + g(z)}\, dz.$$

1.9. Fundamental theorem of algebra, proof by the Liouville theorem, and proof by the Rouché theorem.