Operator Algebra Seminar Note II

Ikhan Choi

January 11, 2024

Contents

Acknowledgement

This note has been written based on the first-year graduate seminar presented at the University of Tokyo in the 2023 Autumn semester. Each seminar was delivered for 105 minutes. I gratefully acknowledge advice of Prof. Yasuyuki Kawahigashi and support of my colleagues Futaba Sato, Yusuke Suzuki.

1 October 18

1.1 Countably decomposable von Neumann algebras

Definition 1.1 (Countably decomposable von Neumann algebras)**.** Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal non-zero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is.

Proposition 1.2. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M is countably decomposable.*

(b) *M admits a faithful normal state.*

(c) *M admits a faithful normal non-degenerate representation with a cyclic and separating vector.*

(d) *The unit ball of M is metrizable in one of the following topologies: σ-strong*, σ-strong, strong*, strong.*

Proof. (a)$\iff$(b) Suppose M is countably decomposable. Let $\{\xi_i\} \subset H$ be a maximal family of unit vectors such that $\overline{M'\xi_i}$ are mutually orthogonal subspaces, taken by Zorn's lemma. If we let p_i be the projection on $\overline{M'\xi_1}$, then $p_i z p_i = z p_i$ for $z \in M'$ implies $p_i \in M'' = M$. By the assumption, the family $\{\xi_i\}$ is countable. Define a state ω of M such that

$$\omega(x) := \sum_{i=1}^{\infty} \omega_{2^{-i}\xi_i}(x), \qquad x \in M.$$

It converges due to $\|\omega_{2^{-i}\xi_i}\| = 2^{-i+1}$. It is normal since the sequence $(2^{-i}\xi_i)$ belongs to $\ell(\mathbb{N}, H)$, and it is faithful because $\omega(x^*x) = 0$ implies $x\xi_i = 0$ for all i, which deduces that $x = \sum_i x p_i = 0$.

Conversely, if ω is a faithful normal state, then for a mutually orthogonal family of non-zero projections $\{p_i\} \subset M$, we have

$$\{p_i\} = \bigcup_{n=1}^{\infty} \{p_i : \varphi(p_i) > n^{-1}\}$$

the countable union of finite sets. Thus M is countable decomposable.

(b)$\iff$(c) Let ω be a faithful normal state of M. Consider any faithful normal nondegenerate representation in which ω is a vector state so that the corresponding vector is a separating vector by the faithfullness of ω. Examples include the GNS representation of ω, and the composition with the diagonal map $B(H) \to B(\ell^2(\mathbb{N}, H))$. Then, $\overline{M\Omega}$ admits a cyclic and separating vector Ω of M. The converse is immediate, i.e. the vector state ω_Ω is a faithful normal state of M.

(a)$\iff$(d) Suppose M is countably decomposable and take $\{\xi_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ as we did. Define

$$d(x, y) := \sum_{i=1}^{\infty} 2^{-i} \|(x - y)\xi_i\|.$$

Clearly it generates a topology coarser than strong topology. It is also finer because if a bounded net x_α in M converges to zero in the metric d so that $x\xi_i \to 0$ for all i, then $H = \bigoplus_i M'\xi_i$ implies that for every $\xi \in H$ and $\varepsilon > 0$ we have $\|\xi - \sum_{k=1}^{n} z_k \xi_{i_k}\| < \varepsilon$ for some $z_k \in M'$ so that

$$\|x_\alpha \xi\| \le \|x_\alpha(\xi - \sum_{k=1}^{n} z_k \xi_{i_k})\| + \sum_{k=1}^{n} \|x_\alpha z_k \xi_{i_k}\| < \varepsilon + \sum_{k=1}^{n} \|z_k\| \|x_\alpha \xi_{i_k}\| \to \varepsilon.$$

Since on the bounded part the strong and σ-strong topologies coincide, the two topologies on the unit ball are metrizable. We can do similar for the strong* and the σ-strong* topologies.

Conversely, for a mutually orthogonal family of non-zero projections $\{p_i\}_{i \in I} \subset M$, since the net of finite partial sums $p_F := \sum_{i \in F} p_i$ is an non-decreasing net in the closed unit ball whose supremum is the identity of M, there is a convergent subsequence $p_{F_n} \uparrow 1$ by the metrizability, which implies $I = \bigcup_{n=1}^{\infty} F_n$, the countable union of finite sets. $\square$

Proposition 1.3. *For a von Neumann algebra M, the followings are all equivalent.*

(a) *M has the separable predual.*

(b) *M admits a faithful normal non-degenerate representation on a separable Hilbert space.*

(c) *M is countably decomposable and countably generated.*

(d) *The unit ball of M is metrizable in one of the following topologies: σ-weak, weak.*

1.2 Semi-cyclic representations

Definition 1.4 (Weights)**.** Let M be a von Neumann algebra. A *weight* is a function $\varphi : M^+ \to [0,\infty]$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(\lambda x) = \lambda\varphi(x), \qquad x,y \in M^+,\ \lambda \geq 0,$$

where we use the convention $0\cdot\infty = 0$.

For a weight, we will show that the following properties are all equivalent:

(i) completely additive,

(ii) order continuous,

(iii) σ-weakly lower semi-continuous,

(iv) supremum of normal positive linear functionals.

A weight satisfying one of them is called *normal*. Note that (iv)→(iii)→(ii)→(i) is clear.

Definition 1.5. Let φ be a weight on a von Neumann algebra M. Define a left ideal of M

$$\mathfrak{n} := \{x \in M : \varphi(x^*x) < \infty\},$$

and a hereditary $*$-subalgebra of M

$$\mathfrak{m} := \mathfrak{n}^*\mathfrak{n} = \{\sum_{i=1}^{n} y_i^* x_i : (x_i),(y_i) \in \mathfrak{n}^n\}.$$

Lemma 1.6. *If $x,y \in M$ satisfies $y^*y \leq x^*x$, then there is a unique $s \in B(H)$ such that $y = sx$ and $s = sp$, where p is the range projection of x, and $s \in M$.*

Proof. Suppose $\mathrm{id}_H \in M \subset B(H)$. The operator $s_0 : \overline{xH} \to \overline{yH} : x\xi \mapsto y\xi$ is well defined because

$$\|y\xi\|^2 = \langle y^*y\xi,\xi\rangle \leq \langle x^*x\xi,\xi\rangle = \|x\xi\|^2.$$

Let p be the range projection of x and let $s := s_0 p$. Then, $y\xi = sx\xi$ for all $\xi \in H$. If $y = s'x$ and $s' = s'p$, then

$$x^*(s-s')^*(s-s')x = (y-y)^*(y-y) = 0$$

implies

$$0 = p(s-s')^*(s-s')p = (s-s')^*(s-s').$$

Therefore, s is unique in $B(H)$. If $u \in M'$ is unitary, then usu^* satisfies the same property $y = usu^*x$ and $usu^* = usu^*p$, so $us = su$. Since the unitary span the whole C*-algebra, we have $s \in M'' = M$. □

Proposition 1.7. *Let φ be a weight on a von Neumann algebra M.*

(a) *Every element of $\mathfrak{m}^+$ can be written to be x^*x for some $x \in \mathfrak{n}$.*

(b) *Every element of $\mathfrak{m}$ can be written to be y^*x for some $x,y \in \mathfrak{n}$.*

Proof. (a) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}^+$ for some $x_i, y_i \in \mathfrak{n}$. The polarization writes

$$a = \frac{1}{4}\sum_{i=1}^n \sum_{k=0}^3 i^k |x_i + i^k y_i|^2$$

and $a^* = a$ implies

$$a = \frac{1}{2}\sum_{i=1}^n (|x_i + y_i|^2 - |x_i - y_i|^2) \le \frac{1}{2}\sum_{i=1}^n |x_i + y_i|^2$$

implies

$$\varphi(a) \le \frac{1}{2}\sum_{i=1}^n \varphi(|x_i + y_i|^2) < \infty.$$

Therefore, if $x := a^{\frac{1}{2}} \in \mathfrak{n}$, then $a = x^* x$.

(b) Let $a := \sum_{i=1}^n y_i^* x_i \in \mathfrak{m}$ for some $x_i, y_i \in \mathfrak{n}$. Let $x := (\sum_{i=1}^n x_i^* x_i)^{\frac{1}{2}} \in \mathfrak{n}$. Since $x_i^* x_i \le x^2$, we have $s_i \in M$ such that $x_i = s_i x$. If we let $y := \sum_{i=1}^n s_i^* y_i \in \mathfrak{n}$, then

$$a = \sum_{i=1}^n y_i^* x_i = \sum_{i=1}^n y_i^* s_i x = (\sum_{i=1}^n s_i^* y_i)x = y^* x. \qquad \square$$

Definition 1.8 (Semi-cyclic representations)**.** Let φ be a weight on a von Neumann algebra. Let H be the Hilbert space defined by the separation and completion of a sesquilinear form

$$\mathfrak{n} \times \mathfrak{n} \to \mathbb{C} : (x, y) \mapsto \varphi(y^* x)$$

and let $\Lambda : \mathfrak{n} \to H$ be the canonical image map. The pair (π, Λ) is called the *semi-cyclic representation* associated to φ.

Proposition 1.9 (Bounded Radon-Nikodym)**.** *Let φ be a weight on a von Neumann algebra and (π, Λ) be the associated semi-cyclic representation to φ. Consider a map*

$$\Theta : \mathfrak{m} \times \pi(M)' \to \mathbb{C} : (y^* x, z) \mapsto \langle z\Lambda(x), \Lambda(y)\rangle$$

and define

$$\theta : \mathfrak{m} \to \pi(M)'_*, \qquad \theta^* : \pi(M)' \to \mathfrak{m}^\#$$

such that $\Theta(y^* x, z) = \theta(y^* x)(z) = \theta^*(z)(y^* x)$ for $x, y \in \mathfrak{n}$ and $z \in \pi(M)'$.

(a) *Θ is a well-defined bilinear form.*

(b) *θ^* is bijective onto the space of linear functionals on $\mathfrak{m}$ whose absolute value is majorized by φ. In particular, if $l \in \mathfrak{m}^\#$ satisfies $|l| \le C\varphi$ for some $C > 0$, then there is $z \in \pi(M)'$ such that $\|z\| \le C$ and $l(y^* x) = \langle z\Lambda(x), \Lambda(y)\rangle$ for $x, y \in \mathfrak{n}$.*

Proof. (a) The linearity in the second argument is obvious. Fix $z \in \pi(M)'$. We first check the well-definedness on $\mathfrak{m}^+$. Let $x^* x = y^* y \in \mathfrak{m}^+$ for $x, y \in \mathfrak{n}$. Then, there is $s \in M$ such that $y = sx$ and $s = sp$, where p is the range projection of x, so

$$x^*(1 - s^* s)x = x^* x - y^* y = 0$$

implies

$$0 = p(1 - s^* s)p = p - s^* s$$

and $x = px = s^* sx = s^* y$. The well-definedness follows from

$$\Theta(x^* x, z) = \langle z\Lambda(x), \Lambda(x)\rangle = \langle \pi(s)z\pi(s^*)\Lambda(y), \Lambda(y)\rangle = \langle z\Lambda(ss^* y), \Lambda(y)\rangle = \Theta(y^* y, z).$$

The homogeneity is clear, so now we prove the addivitiy. Let $x^*x, y^*y \in \mathfrak{m}^+$ for some $x, y \in \mathfrak{n}$. Let $a := (x^*x + y^*y)^{\frac{1}{2}}$ and take $s, t \in M$ such that $x = sa$, $y = ta$, $s = sa$, and $t = ta$, where p is the range projection of a. Then,

$$a(1 - s^*s - t^*t)a = a^*a - x^*x - y^*y = 0$$

implies

$$p(1 - s^*s - t^*t)p = p - s^*s - t^*t.$$

It follows that

$$\begin{aligned}
\Theta(x^*x + y^*y, z) &= \langle z\Lambda(a), \Lambda(a)\rangle = \langle z\pi(p)\Lambda(a), \Lambda(a)\rangle \\
&= \langle z\pi(s^*s)\Lambda(a), \Lambda(a)\rangle + \langle z\pi(t^*t)\Lambda(a), \Lambda(a)\rangle \\
&= \langle z\Lambda(x), \Lambda(x)\rangle + \langle z\Lambda(y), \Lambda(y)\rangle \\
&= \Theta(x^*x, z) + \Theta(y^*y, z).
\end{aligned}$$

Now the $\Theta(\cdot, z)$ is linearly extendable to $\mathfrak{m}$.

(b) The linear map θ^* is injective since Λ has dense range. Take $z \in \pi(M)'$ and consider $\theta^*(z)$, which maps x^*x to $\langle z\Lambda(x), \Lambda(x)\rangle$ for $x \in \mathfrak{n}$. The image is majorized by φ as

$$|\langle z\Lambda(x), \Lambda(x)\rangle| \le \|z\|\|\Lambda(x)\|^2 = \|z\|\varphi(x^*x).$$

Conversely, let $l \in \mathfrak{m}^\#$ is a linear functional majorized by φ, i.e. there is a constant $C > 0$ such that

$$|l(x^*x)| \le C\varphi(x^*x), \qquad x \in \mathfrak{n}.$$

Define a sesquilinear form $\sigma : \mathfrak{n} \times \mathfrak{n} \to \mathbb{C}$ such that $\sigma(x, y) := l(y^*x)$. It is well-defined after separation of $\mathfrak{n}$ and is bounded by the Cauhy-Schwartz inequality

$$|\sigma(x, y)|^2 = |l(y^*x)|^2 \le \|l(x^*x)\|\|l(y^*y)\| \le \varphi(x^*x)\varphi(y^*y) = \|\Lambda(x)\|^2\|\Lambda(y)\|^2.$$

Therefore, σ defines a bounded linear operator $z \in \pi(M)'$ such that

$$\sigma(x, y) = \langle z\Lambda(x), \Lambda(y)\rangle,$$

exactly meaning $\theta^*(z)(y^*x) = l(y^*x)$ for $x, y \in \mathfrak{n}$. □

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, "\Lambda"] \arrow[dd, "|\cdot|^2"'] & H \arrow[d, "\omega"] \\
 & B(H)_* \arrow[d, "\mathrm{res}"] \\
\mathfrak{m}^+ \arrow[r, "\theta"] & \pi(M)'_*.
\end{tikzcd}$$

In particular, for $x \in \mathfrak{n}^+$ we have

$$\|\theta(x^2)\| = \|\omega_{\Lambda(x)}\| = \|\Lambda(x)\|^2 = \varphi(x^2).$$

Lemma 1.10. *Let For $z \in \mathfrak{m}^{sa}$, we have*

$$\inf\{\varphi(a) : z \le a \in \mathfrak{m}^+\} \le \|\theta(z)\|.$$

In particular, for $x, y \in \mathfrak{n}^+$ and for any $\varepsilon > 0$ there is $a \in \mathfrak{m}^+$ such that $x^2 - y^2 \le a$ and

$$\varphi(a) \le \|\theta(x^2 - y^2)\| + \varepsilon = \|\omega_{\Lambda(x)} - \omega_{\Lambda(y)}\| + \varepsilon.$$

Proof. Denote by $p(z)$ the left-hand side of the inequality. Then, we can check $p : \mathfrak{m}^{sa} \to \mathbb{R}_{\geq 0}$ is a semi-norm such that $p(z) = \varphi(z)$ for $z \geq 0$. (If we take $p(z) := \varphi(z^+)$, then it seems to be dangerous when checking the sublinearity. I could not find the counterexample.)

Fix any non-zero $z_0 \in \mathfrak{m}^{sa}$. By the Hahn-Banach extension, there is an algebraic real linear functional $l : \mathfrak{m}^{sa} \to \mathbb{R}$ such that

$$l(z_0) = p(z_0), \qquad |l(z)| \leq p(z), \qquad z \in \mathfrak{m}^{sa}.$$

Extend linearly l to be $l : \mathfrak{m} \to \mathbb{C}$. Since $|l(z)| \leq \varphi(z)$ for $z \in \mathfrak{m}^+$, by the bounded Radon-Nikodym theorem, we have a corresponding operator $a \in \pi(M)'_1$ such that $\theta^*(a) = l$, hence

$$p(z_0) = l(z_0) = \theta^*(a)(z_0) = \theta(z_0)(a) \leq \|\theta(z_0)\|.$$

Since $z_0 \in \mathfrak{m}^{sa}$ is aribtrary, we are done. □

1.3 σ-weak lower semi-continuity

Theorem 1.11. *Let M be a countably decomposable von Neumann algebra. Then, a completely additive weight on M is σ-weakly lower semi-continuous.*

Proof. We first prove a completely additive weight φ on M is normal if M is countably decomposable. Let x_α be a bounded increasing net in M^+ with $x := \sup_\alpha x_\alpha$. Take any faithful normal state ω of M and extract inductively an increasing subsequence x_n of x_α such that $\omega(x_n) \to \omega(x)$. Then, $\omega(x_n) \leq \omega(\sup_n x_n) \leq \omega(x)$ implies $\omega(\sup_n x_n) = \omega(x)$, and since ω is faithful we have $\sup_n x_n = x$. By the complete additivity, we have

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi(x_{n+1} - x_n) \leq \sup_n \varphi(x_n) \leq \sup_\alpha \varphi(x_\alpha) \leq \varphi(x),$$

hence φ is order continuous.

Now, let φ be an order continuous weight on M and let (π, Λ) be the associated semi-cyclic representation. In the spirit of the Krein-Šmulian theorem, the σ-weak lower semi-continuity is equivalent to the σ-weak closedness of the intersection with the ball

$$\begin{aligned} \varphi^{-1}([0,1])_1 &= \{x \in M^+ : \varphi(x) \leq 1,\ \|x\| \leq 1\} \\ &= \{x \in \mathfrak{m}^+ : \|\Lambda(x^{\frac{1}{2}})\| \leq 1,\ \|x^{\frac{1}{2}}\| \leq 1\}. \end{aligned}$$

Since that the σ-weak and σ-strong closedness of a convex set are equivalent and that the square root operation on M_1^+ is σ-strongly continuous, we are enough to show the set

$$(\varphi^{-1}([0,1])_1)^{\frac{1}{2}} = \{x \in \mathfrak{n}^+ : \|\Lambda(x)\| \leq 1,\ \|x\| \leq 1\}$$

is σ-weakly closed. This set, if we denote the graph of $\Lambda : \mathfrak{n} \to H$ by Γ_Λ, is exactly the image of the positive part of the unit ball

$$(\Gamma_\Lambda)_1^+ = \{(x, \Lambda(x)) \in \mathfrak{n}^+ \oplus_\infty H : \|\Lambda(x)\| \leq 1,\ \|x\| \leq 1\}$$

under the projection $M \oplus_\infty H \to M$. Observing $M \oplus_\infty H \cong (M_* \oplus_1 H)^*$, if we prove $(\Gamma_\Lambda)_1^+$ is weakly* closed, then we are done by its compactness.

Consider a linear functional $l : M \oplus_\infty H \to \mathbb{C}$ that is continuous with respect to $(\sigma s, \|\cdot\|)$. If we define $l_1 : M \to \mathbb{C}$ and $l_2 : H \to \mathbb{C}$ such that $l_1(x) := l(x, 0)$ and $l_2(\xi) = (0, \xi)$, then they satisfy $l(x, \xi) = l_1(x) + l_2(\xi)$, and are continuous in σ-strong and norm topologies, hence to σ-weak and weak topologies, respectively. Since a net (x_α, ξ_α) converges to (x, ξ) weakly* if and only if $x_\alpha \to x$ σ-weakly and $\xi_\alpha \to \xi$ weakly, l is weakly* continuous. Because $(\Gamma_\Lambda)_1^+$ is convex, we will now show that $(\Gamma_\Lambda)_1^+$ is closed with respect to $(M, \sigma s) \times (H, \|\cdot\|)$.

Note that the unit ball M_1 is metrizable in σ-strong topology since M is countably decomposable. Suppose a sequence $x_n \in \mathfrak{n}_1^+$ satisfies $x_n \to x$ σ-strongly and $\Lambda(x_n) \to \xi$ in H. Then, it suffices to show the following two statements: $x \in \mathfrak{n}_1^+$ and $\Lambda(x) = \xi$. We first observe that since $\Lambda(x_n)$ is Cauchy, so is $\omega_{\Lambda(x_n)}$ in $\pi(M)'_*$.

Consider for a while, a family of functions

$$f_a(t) := \frac{t}{1+at}, \qquad t \in (-a^{-1}, \infty),$$

parametrized by $a > 0$. They have several properties. At first, they are operator monotone. Next, they are σ-strongly continuous on a closed subset of its domain due to the boundedness of f_a, as we can see in the proof of the Kaplansky density theorem. Finally, for each $x \in M_+$, the increasing limit $f_a(x) \uparrow x$ in norm as $a \to 0$ implies that $\sup_a f_a(x) = x$.

First we show $x \in \mathfrak{n}_1^+$. It is clear that $x \in M_1^+$, so it is enough to show $\varphi(x^2) < \infty$. By taking a subsequence, we may assume $\|\omega_{\Lambda(x_{n+1})} - \omega_{\Lambda(x_n)}\| < \frac{1}{2^n}$. In order to dominate x_n with an monotone sequence, find $a_n \in \mathfrak{m}^+$ such that

$$x_{n+1}^2 - x_n^2 \le a_n, \qquad \varphi(a_n) < \frac{1}{2^n},$$

using the previous lemma. Then, we can write

$$x_{n+1}^2 \le x_1^2 + \sum_{k=1}^{n}(x_{k+1}^2 - x_k^2) \le x_1 + \sum_{k=1}^{n} a_k.$$

Here the right-hand side is non-decreasing but not a bounded sequence so we take f_a to get the σ-strong limit

$$f_a(x^2) \le \sup_n f_a(x_1^2 + \sum_{k=1}^{n} a_k).$$

Then, by the normality of φ, we have

$$\begin{aligned}\varphi(f_a(x^2)) &\le \sup_n \varphi(f_a(x_1^2 + \sum_{k=1}^{n} a_k)) \\ &\le \sup_n \varphi(x_1^2 + \sum_{k=1}^{n} a_k) \\ &= \varphi(x_1^2) + \sum_{k=1}^{\infty} \varphi(a_k) \\ &< \varphi(x_1^2) + 1 < \infty\end{aligned}$$

which implies by sending $a \to 0$ that $\varphi(x^2) < \infty$, whence $x \in \mathfrak{n}$.

Next we show $\Lambda(x) = \xi$. If we prove $\varphi((x_n - x)^2) \to 0$, then

$$\|\xi - \Lambda(x)\| \le \|\xi - \Lambda(x_n)\| + \|\Lambda(x_n) - \Lambda(x)\| = \|\xi - \Lambda(x_n)\| + \varphi((x_n - x)^2)^{\frac{1}{2}} \to 0$$

deduces the desired result. By taking a subsequence, since $\Lambda(x_n - x)$ is Cauchy, we may assume

$$\|\omega_{\Lambda(x_n - x)} - \omega_{\Lambda(x_{n+1} - x)}\| < \frac{1}{2^n}.$$

Let $b_n \in \mathfrak{m}^+$ such that

$$(x_n - x)^2 - (x_{n+1} - x)^2 \le b_n, \qquad \varphi(b_n) < \frac{1}{2^n}$$

As we did previously, we have

$$f_a((x_n - x)^2) \le f_a((x_{m+1} - x)^2) + f_a(\sum_{k=n}^{m} b_k) \to \sup_m f_a(\sum_{k=n}^{m} b_k)$$

as $m \to \infty$ and

$$\varphi(f_a((x_n - x)^2)) \leq \sup_m \varphi(f_a(\sum_{k=n}^{m} b_k)) \leq \sup_m \varphi(\sum_{k=n}^{m} b_k) < \frac{1}{2^{n-1}}.$$

Therefore,

$$\varphi((x_n - x)^2) \leq \frac{1}{2^{n-1}} \to 0. \qquad \square$$

Theorem 1.12. *Let M be an arbitrary von Neumann algebra. Then, a completely additive weight on M is σ-weakly lower semi-continuous.*

Proof. Let φ be a completely additive weight of M. Let Σ be the set of all countably decomposable projections of M and let $M_0 := \bigcup_{p \in \Sigma} pMp$. The equivalent condition for $x \in M$ to belong to M_0 is that the left and right support projections of x are countably decomposable. Since then the left support projection p and the right support projection q of x are Murray-von Neumann equivalent so that there is a $*$-isomorphism between pMp and qMq, the countable decomposability is equivalent for p and q. It implies that M_0 is an algebraic ideal of M. (Moreover, M_0 is σ-weakly sequentially closed in M since if a sequence $x_n \in M_0$ converges to $x \in M$ σ-weakly, then for $p_n \in \Sigma$ such that $x_n = p_n x_n p_n$, we have $p \in \Sigma$ with $p_n \leq p$ so that $x_n = p x_n p$ converges to $x = pxp$ σ-weakly. This fact is not needed in the proof.)

We first claim that $\varphi^{-1}([0,1])_1$ is relatively σ-weakly closed in M_0. Let $y \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0$ so that there is a net $y_\alpha \in \varphi^{-1}([0,1])_1$ converges σ-weakly to y, and there is $p \in \Sigma$ such that $pyp = y$. Note that the previous theorem states that $\varphi^{-1}([0,1]) \cap pMp$ is σ-weakly closed. Since $py_\alpha p$ is a net in $\varphi^{-1}([0,1])_1 \cap pMp$ that also converges σ-weakly to $pyp = y$, we have $y \in \varphi^{-1}([0,1])$. The claim proved.

We now claim that $\varphi^{-1}([0,1])_1$ is σ-weakly closed in M. Suppose a net $x_\alpha \in \varphi^{-1}([0,1])_1$ converges to $x \in M$ σ-weakly. Clearly $x \in M_1^+$. Let $\{p_i\}_{i \in I}$ be a maximal mutually orthogonal projections in Σ, and let $p_J := \sum_{i \in J} p_i$ for finite sets $J \subset I$ so that $\sup_J p_J = 1$. It clearly follows that for each α we have

$$x_\alpha^{\frac{1}{2}} p_J x_\alpha^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

Then, we can show easily with boundedness of x_α that

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w}.$$

Because $p_J \in M_0$ and M_0 is an ideal,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \overline{\varphi^{-1}([0,1])_1}^{\sigma w} \cap M_0.$$

By the above claim,

$$x^{\frac{1}{2}} p_J x^{\frac{1}{2}} \in \varphi^{-1}([0,1])_1.$$

By the complete additivity of φ, we finally obtain

$$x \in \varphi^{-1}([0,1])_1.$$

Therefore, $\varphi^{-1}([0,1])_1$ is σ-weakly closed. $\square$

1.4 Supremum of normal positive linear functionals

Endow a partial order on the set of all weights. Then, every set of monotonically increasing subadditive homogeneous functions $\varphi : M^+ \to [0,\infty]$ always have its supremum given by its pointwise supremum. Since if φ is the supremum of σ-weakly lower semi-continuous φ_i, then

$$\varphi^{-1}([0,1]) = \bigcap_i \varphi_i^{-1}([0,1])$$

implies the σ-weak lower semi-continuity of φ. Conversly, the following theorem holds.

Theorem 1.13. *Let M be a von Neumann algebra. Then, a σ-weakly lower semi-continuous monotonically increasing additivie homogeneous function $\varphi : M^+ \to [0,\infty]$ is given by the supremum of a set of normal positive linear functionals.*

Proof. Let $F := \varphi^{-1}([0,1])$. It is a hereditary closed convex subset of the real locally convex space $(M^{sa}, \sigma w)$. Denote by the superscript circle the real polar set. Since

$$F^{\circ+} = \{\omega \in M_*^+ : \omega \le \varphi\}, \qquad F^{\circ+\circ+} = \{x \in M^+ : \sup_{\omega \le \varphi,\ \omega \in M_*^+} \omega(x) \le 1\},$$

it is enough to show $F^{\circ+\circ+} = F$. The positive part of the real polar of F is generally written as

$$F^{\circ+} = F^\circ \cap M_*^+ = F^\circ \cap (-M^+)^\circ = (F \cup -M^+)^\circ = (F - M^+)^\circ.$$

Consider a sequence of inclusions

$$F \subset \overline{F} \subset \overline{(F - M^+)^+} \subset \overline{(F - M^+)}^+ \subset (F - M^+)^{\circ\circ+} = F^{\circ+\circ+}.$$

The first, second, and forth inclusions are in fact full because F is closed, hereditary, and convex. So we claim that the reverse of the third inclusion $\overline{(F - M^+)}^+ \subset \overline{(F - M^+)^+}$.

Let $x \in \overline{(F - M^+)}^+$. For arbitrary $a > 0$, it is enough to show $f_a(x) \in F - M^+$ because $x \ge 0$ implies $f_a(x) \ge 0$ and $f_a(x) \uparrow x$ as $a \to 0$. Let x_α be a net in $F - M^+$ that converges to x σ-strongly, which can be done by the convexity of $F - M^+$. Let y_α be a net in F such that $f_{a/2}(x_\alpha) \le y_\alpha$. Since $f_{a/2}(y_\alpha)$ is a bounded net, we may assume it is σ-weakly convergent. By the σ-strong continuity of f_a, the net $f_a(x_\alpha)$ converges to $f_a(x)$ σ-strongly, hence σ-weakly. Therefore, by the closedness of F,

$$f_a(x) = \lim_\alpha f_a(x_\alpha) \le \lim_\alpha f_{a/2}(y_\alpha) \in F,$$

so we conclude $f_a(x) \in F - M^+$. $\square$

2 November 10

2.1 Hilbert algebras

Definition 2.1 (Left Hilbert algebra)**.** A *left Hilbert algebra* is a $*$-algebra A together with an inner product such that the involution is closable on H and the square A^2 is dense in H, where $H := \overline{A}$. A left Hilbert algebra A has the following additional devices:

(i) a closable densely defined anti-linear operator $S : A \to H$, defined by the involution,

(ii) a faithful non-degenerate $*$-homomorphism $\lambda : A \to B(H)$, defined by the left multiplication.

The *associated von Neumann algebra* of a left Hilbert algebra A is defined as $M := \lambda(A)''$.

Definition 2.2 (Right Hilbert algebra)**.** Let A be a left Hilbert algebra. For $\eta \in H$, define:

(i) a linear functional $F\eta : A \to \mathbb{C}$ such that $F\eta(\xi) := \langle \eta, S\xi \rangle$ for $\xi \in A$,

(ii) a linear operator $\rho(\eta) : A \to H$ such that $\rho(\eta)\xi := \lambda(\xi)\eta$ for $\xi \in A$.

Define also:

$$D' := \{\eta \in H \mid F\eta \text{ is bounded}\}, \qquad B' := \{\eta \in H \mid \rho(\eta) \text{ is bounded}\}, \qquad A' := B' \cap D'.$$

Then, for $\eta \in D'$, we can identify $F\eta$ with a vector in H by the Riesz representation theorem, and for $\eta \in B'$, we can identify $\rho(\eta)$ with an element of $B(H)$.

Proposition 2.3. *Let A be a left Hilbert algebra.*

(a) *A' is a $*$-algebra such that $\eta^* := F\eta$ and $\eta\zeta := \rho(\zeta)\eta$.*

(b) *$\rho(A')A'$ is dense in H.*

(c) *A' is a right Hilbert algebra such that $\overline{A'} = H$.*

Proof. (a) Combining from (i) to (iv) in the below, the claim follows clearly:

(i) For $\eta \in D'$, we have $FF\eta = \eta$ in H by

$$FF\eta(\xi) = \langle F\eta, S\xi \rangle = \langle SS\xi, \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi \in A.$$

Therefore, if $\eta \in D'$, then $F\eta \in D'$.

(ii) For $\eta \in D'$, we have $\rho(F\eta) = \rho(\eta)^*$ on A by

$$\begin{aligned}\langle \rho(F\eta)\xi, \xi \rangle &= \langle \lambda(\xi)F\eta, \xi \rangle = \langle F\eta, \lambda(\xi)^*\xi \rangle = \langle S\lambda(\xi)^*\xi, \eta \rangle \\ &= \langle \lambda(\xi)^*\xi, \eta \rangle = \langle \xi, \lambda(\xi)\eta \rangle = \langle \xi, \rho(\eta)\xi \rangle = \langle \rho(\eta)^*\xi, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta \in A'$, then $F\eta \in B'$.

(iii) For $\eta, \zeta \in B'$, we have $F(\rho(\eta)^*\zeta) = \rho(\zeta)^*\eta$ in H by

$$\begin{aligned}\langle F(\rho(\eta)^*\zeta), \xi \rangle &= \langle S\xi, \rho(\eta)^*\zeta \rangle = \langle \rho(\eta)S\xi, \zeta \rangle = \langle \lambda(\xi)^*\eta, \zeta \rangle \\ &= \langle \eta, \lambda(\xi)\zeta \rangle = \langle \eta, \rho(\zeta)\xi \rangle = \langle \rho(\zeta)^*\eta, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta, \zeta \in B'$, then $\rho(\eta)^*\zeta \in D'$.

(iv) For $\eta \in B'$ and $\zeta \in H$, we have $\rho(\rho(\eta)^*\zeta) = \rho(\eta)^*\rho(\zeta)$ on A by

$$\begin{aligned}\langle \rho(\rho(\eta)^*\zeta)\xi, \xi \rangle &= \langle \lambda(\xi)\rho(\eta)^*\zeta, \xi \rangle = \langle \zeta, \rho(\eta)\lambda(\xi)^*\xi \rangle = \langle \zeta, \lambda(\lambda(\xi)^*\xi)\eta \rangle \\ &= \langle \zeta, \lambda((S\xi)\xi)\eta \rangle = \langle \zeta, \lambda(\xi)^*\lambda(\xi)\eta \rangle = \langle \lambda(\xi)\zeta, \lambda(\xi)\eta \rangle \\ &= \langle \rho(\zeta)\xi, \rho(\eta)\xi \rangle = \langle \rho(\eta)^*\rho(\zeta)\xi, \xi \rangle, \qquad \xi \in A.\end{aligned}$$

Therefore, if $\eta, \zeta \in B'$, then $\rho(\eta)\zeta \in B'$.

(b) Since D' is dense in H by the closability of S, it suffices to verify the inclusion $D' \subset \overline{\rho(A')A'}$. Let $\eta \in D'$. Since $\rho(\eta)$ has densely defined adjoint $\rho(F\eta)$, we may assume $\rho(\eta)$ to be closed and densely defined by taking closure, so we can write down the polar decomposition

$$\rho(\eta) = vh = kv, \qquad h := |\rho(\eta)|, \quad k := |\rho(\eta)^*|.$$

To control the unboundedness of $\rho(\eta)$, we introduce $f \in C_c((0,\infty))^+$ to cutoff $\rho(\eta)$. Let $\acute{f}(t) := tf(t)$ and $\grave{f}(t) := t^{-1}f(t)$. Now we have $f(k) \in \rho(B')$ since $f(k)$ is bounded and

$$f(k) = f(vhv^*) = vf(h)v^* = v\grave{f}(h)\rho(\eta)^* = \rho\left(v\grave{f}(h)F\eta\right).$$

We also have $f(k)\eta \in B'$ since

$$\rho(f(k)\eta) = f(k)\rho(\eta) = \acute{f}(k)v$$

is bounded. Applying the above arguments for $f^{\frac{1}{3}} \in C_c((0,\infty))$,

$$f(k)\eta = (f(k)^{\frac{1}{3}})^3\eta \in \rho(B')^*\rho(B')\rho(B')^*B'.$$

Because $\rho(B')^*B' \subset A'$ and $\rho(B')^*\rho(B) \subset \rho(A')$ by (iii) and (iv) in the part (a), we have $f(k)\eta \in \rho(A')A'$.

If we construct a non-decreasing net $f_\alpha \in C_c((0,\infty))$ such that $\sup_\alpha f_\alpha = 1_{(0,\infty)}$, then the strong limit implies

$$\lim_\alpha f_\alpha(k)\eta = 1_{(0,\infty)}(k)\eta = s(k)\eta = s_l(\rho(\eta))\eta.$$

Here we use the non-degeneracy of λ to verify η belongs to the closure of the range of $\rho(\eta)$, i.e. since M contains the identity operator on H, we have a net $\xi_\alpha \in A$ such that $\lambda(\xi_\alpha)$ converges to the identity strongly so that $\lambda(\xi_\alpha)\eta \to \eta$. It implies that $\eta \in \overline{\lambda(A)\eta} = \overline{\rho(\eta)A}$ and $s_l(\rho(\eta))\eta = \eta$. Therefore, $\eta = s_l(\rho(\eta))\eta \in \overline{\rho(A')A'}$.

(c) The involution $F : A' \to H$ is a closable densely defined anti-linear operator because A' is dense in H by (b) and the closability follows from the dense domain of its adjoint S. The right multiplication $\rho : A'^{\mathrm{op}} \to B(H)$ is a faithful non-degenerate $*$-homomorphism because $\rho(A')H$ is dense in H by (b) and the faithfulness follows from the non-degeneracy of λ. Therefore, A' is a right Hilbert algebra with $\overline{A}' = H$. □

Corollary 2.4. $\rho(A')' = M$.

Proof. One direction is clear, i.e. $\rho(A') \subset M'$ implies $\rho(A')'' \subset M'$. Conversely, let $y \in M'^+$. Since $\rho : A'^{\mathrm{op}} \to B(H)$ is non-degenerate, there is a net $\eta_\alpha \in A'$ such that $\rho(\eta_\alpha)$ converges to the identity σ-weakly. Then,

$$\rho(\eta_\alpha)^* y \rho(\eta_\alpha) = \rho(y^{\frac{1}{2}}\eta_\alpha)^*\rho(y^{\frac{1}{2}}\eta_\alpha) \in \rho(B')^*\rho(B') \subset \rho(A')$$

converges to y σ-weakly, hence $y \in \rho(A')''$. □

Definition 2.5 (Full Hilbert algebra)**.** Let A be a left Hilbert algebra. Symmetrically as above, starting from the right Hilbert algebra A', we can construct a left Hilbert algebra A''. We say A is *full* if $A = A''$.

Definition 2.6 (Modular operator and conjugation)**.** Let A be a left Hilbert algebra. Denote the polar decomposition of S by $S = J\Delta^{\frac{1}{2}}$. The unbounded operators Δ and J are called the *modular operator* and the *modular conjugation*.

Corollary 2.7. *From the polar decomposition theorem for unbounded (anti-)linear operators, we have*

(a) S *is injective with* $S = S^{-1}$ *and* $D = \operatorname{dom} S = \operatorname{dom} \Delta^{\frac{1}{2}}$.

(b) F *is injective with* $F = F^{-1}$ *and* $D' = \operatorname{dom} F = \operatorname{dom} \Delta^{-\frac{1}{2}}$.

(c) Δ *is an injective positive self-adjoint operator.*

(d) J *is a conjugation, i.e. an anti-linear isometric involution.*

(e) $S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$, $F = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J$, *and* $J\Delta J = \Delta^{-1}$.

2.2 Faithful semi-finite normal weights

Definition 2.8. Let φ be a weight on a von Neumann algebra M. We say φ is *faithful* if $\varphi(x^*x)=0$ implies $x=0$ for $x\in\mathfrak{n}$. We say φ is *semi-finite* if $\mathfrak{m}$ is σ-weakly dense in M. Recall that a weight φ on a von Neumann algebra M is normal if and only if it is obtained by the pointwise supremum of a set of normal positive linear functionals.

In the proofs of theorems of this section, the following diagram might be helpful:

$$\begin{tikzcd}
\mathfrak{m}:=\mathfrak{n}^*\mathfrak{n} \arrow[r, phantom, "\subset"] & \mathfrak{a}:=\mathfrak{n}\cap\mathfrak{n}^* \arrow[r, phantom, "\subset"] \arrow[d, shift left, "\Lambda"] & \mathfrak{n} \arrow[r, phantom, "\subset"] \arrow[d, shift left, "\Lambda"] & \pi(M) \arrow[r, phantom, "\subset"] & B(H) \\
& A \arrow[r, phantom, "\subset"] \arrow[u, shift left, "\lambda"] & B \arrow[r, phantom, "\subset"] \arrow[u, shift left, "\lambda"] & H. &
\end{tikzcd}$$

Recall that for a weight φ on a von Neumann algebra M and its semi-cyclic representation (π,Λ) of M we have $\varphi(x^*x)=\|\Lambda(x)\|^2$ for $x\in\mathfrak{n}$.

Theorem 2.9. *Let M be a von Neumann algebra. If A is a full left Hilbert algebra together with a faithful normal non-degenerate representation $\pi:M\to B(H)$ such that $\lambda(A)''=\pi(M)$, then*

$$\varphi(x^*x):=\begin{cases}\|\xi\|^2 & \text{if } \pi(x)=\lambda(\xi)\in\lambda(B),\\ \infty & \text{otherwise,}\end{cases}$$

is a faithful semi-finite normal weight on M.

Proof. We use the notation $x=\pi(x)$. We first check that the weight φ is well-defined. Let $x_1=\lambda(\xi_1),x_2=\lambda(\xi_2)\in\lambda(B)$ such that $x_1^*x_1=x_2^*x_2$. Since $x_1,x_2\in M$, we have a partial isometry $v\in M$ such that $x_2=vx_1$ and $v^*v=s_l(x_1)$, and it is not diffcult to see $\xi_2=v\xi_1$. As we know $s_l(x)\xi_1=\xi_1$,

$$\|\xi_2\|^2=\langle\xi_2,\xi_2\rangle=\langle v\xi_1,v\xi_1\rangle=\langle v^*v\xi_1,\xi_1\rangle=\langle\xi_1,\xi_1\rangle=\|\xi_1\|^2,$$

which proves the well-definedness.

With this weight φ, we can see

$$\mathfrak{n}=\lambda(B),\qquad \mathfrak{a}=\lambda(A),\qquad \mathfrak{m}=\lambda(B)^*\lambda(B).$$

The first one is by definition of φ, and the third one is by definition of $\mathfrak{m}$. Since A is full so that $A=B\cap D$, λ is injective, $\lambda(A)^*=\lambda(A)$, and $\lambda(D)^*=\lambda(D)$, we have $\lambda(A)=\lambda(B)\cap\lambda(D)=\lambda(B)^*\cap\lambda(D)$, which implies $\lambda(A)=\lambda(B)\cap\lambda(B)^*\cap\lambda(D)$. If $\xi_1,\xi_2\in B$ satisfy $\lambda(\xi_1)=\lambda(\xi_2)^*$, then

$$\begin{aligned}S\xi_1(\rho(\eta)^*\zeta)&=\langle F\rho(\eta)^*\zeta,\xi_1\rangle=\langle\rho(\zeta)^*\eta,\xi_1\rangle=\langle\eta,\rho(\zeta)\xi_1\rangle=\langle\eta,\lambda(\xi_1)\zeta\rangle\\&=\langle\lambda(\xi_2)\eta,\zeta\rangle=\langle\rho(\eta)\xi_2,\zeta\rangle=\langle\xi_2,\rho(\eta)^*\zeta\rangle,\qquad \eta,\zeta\in A'.\end{aligned}$$

We have $\xi_1\in D$ by the density of A'^2 in H, so $\lambda(B)\cap\lambda(B)^*\subset\lambda(D)$, hence the second equality follows.

From now in the rest of proof, we will always denote $y=\rho(\eta)$ and $z=\rho(\zeta)$ for $y,z\in\mathfrak{n}'$. The weight φ is clearly faithful, and semi-finiteness is because $x\in M$ is approximated by a net $\lambda(\xi_\alpha)x\lambda(\xi_\alpha)\in\lambda(B)^*\lambda(B)=\mathfrak{m}$, where $\lambda(\xi_\alpha)\in\lambda(B)$ converges σ-weakly to id_H. To verify the normality of φ, we will show

$$\varphi(x^*x)=\sup_{y\in\mathfrak{n}_1'}\omega_\eta(x^*x),\qquad x\in\mathfrak{n},$$

where $\mathfrak{n}':=\rho(B')$.

($\geq$) We may assume $x=\lambda(\xi)\in\mathfrak{n}=\lambda(B)$ so that $\varphi(x^*x)<\infty$. Since the unit ball $\mathfrak{n}_1'$ has a net y_α that converges to id_H strongly by the Kaplansky density theorem, we have an inequality

$$\omega_{\eta_\alpha}(x^*x)=\|x\eta_\alpha\|^2=\|\lambda(\xi)\eta_\alpha\|^2=\|\rho(\eta_\alpha)\xi\|^2=\|y_\alpha\xi\|^2\leq\|\xi\|^2=\varphi(x^*x),$$

in which the equality condition is attained at its limit.

($\le$) Suppose $x \in M$ is taken such that the right-hand side $\sup_{y\in\mathfrak{n}'_1} \omega_\eta(x^*x)$ is finite. If we show $x \in \mathfrak{n}$, then we are done from $\varphi(x^*x) < \infty$ by the previous argument. To motivate the strategy, consider the opposite weight

$$\varphi'(y^*y) := \begin{cases} \|\eta\|^2 & \text{if } y \in \rho(B'), \\ \infty & \text{otherwise,} \end{cases}$$

and the associated linear map

$$\theta'^* : M \to \mathfrak{m}'^\# : x^*x \mapsto (z^*y \mapsto \langle x^*x\eta, \zeta\rangle), \qquad y, z \in \mathfrak{n}',$$

where we can check $\mathfrak{m}' = \rho(B')^*\rho(B')$. The idea is to show a well-designed linear functional $l \in \mathfrak{m}'^\#$ such that $l = \theta'^*(x^*x)$ is contained in the image $\theta'^*(\mathfrak{m})$ using the assumption that the right-hand side is finite to verify $x \in \mathfrak{n}$.

Define a linear functional

$$l : \mathfrak{m}' \to \mathbb{C} : z^*y \mapsto \langle x^*x\eta, \zeta\rangle.$$

Then, by the assumption we have

$$\|l\| = \sup_{y\in\mathfrak{n}'_1} \langle x^*x\eta, \eta\rangle = \sup_{y\in\mathfrak{n}'_1} \omega_\eta(x^*x) < \infty,$$

and

$$|l(y)| \le \|l\| l(y^*y)^{\frac{1}{2}} = \|l\| \|x\eta\|, \qquad y \in \mathfrak{n}'$$

implies the well-definedness as well as boundedness of the linear functional $\overline{xH} \to \mathbb{C} : x\eta \mapsto l(y)$ for any $\eta \in H$, and it follows the existence of $\xi \in \overline{xH}$ such that

$$l(y) = \langle x\eta, \xi\rangle, \qquad y \in \mathfrak{n}'$$

by the Riesz representation theorem on $\overline{xH}$. We have $\lambda(\xi)\zeta \in \overline{xH}$ and

$$\begin{aligned} \langle x\eta, x\zeta\rangle &= l(z^*y) = \langle x\rho^{-1}(z^*y), \xi\rangle = \langle xz^*\eta, \xi\rangle \\ &= \langle z^*x\eta, \xi\rangle = \langle x\eta, z\xi\rangle = \langle x\eta, \rho(\zeta)\xi\rangle = \langle x\eta, \lambda(\xi)\zeta\rangle, \qquad y, z \in \mathfrak{n}', \end{aligned}$$

hence $x = \lambda(\xi)$. The vector ξ is left bounded by definition and $x = \lambda(\xi) \in \lambda(B) = \mathfrak{n}$. □

Theorem 2.10. *Let M be a von Neumann algebra. If φ is a faithful semi-finite normal weight on M and (π, Λ) is the associated semi-cyclic representation of M, then $A := \Lambda(\mathfrak{a})$ is a full left Hilbert algebra with*

$$\langle \Lambda(x_1), \Lambda(x_2)\rangle := \varphi(x_2^*x_1), \qquad \Lambda(x_1)\Lambda(x_2) := \Lambda(x_1x_2), \qquad \Lambda(x)^* := \Lambda(x^*),$$

such that $\lambda(A)'' = \pi(M)$.

Proof. We use the notation $\pi(x) = x$. It does not make any confusion because the semi-cyclic representation $\pi : M \to B(H)$ is always unital and is faithful due to the assumption that φ is faithful. We clearly see that A is a $*$-algebra and the left multiplication provides a $*$-homomorphism $\lambda : A \to B(H)$. By the construction of the semi-cyclic representation associated to φ, A is dense in H. We need to show the non-degeneracy of λ, the closability of the involution, and the fullness of A.

(non-degeneracy) Since φ is semi-finite, there is a net x_α in $(\mathfrak{n} \cap \mathfrak{n}^*)_1$ converges strongly to the identity of M by the Kaplansky density theorem. Then, it follows that λ is non-degenerate from

$$\lambda(\Lambda(x_\alpha))\Lambda(x) = \Lambda(x_\alpha)\Lambda(x) = \Lambda(x_\alpha x) = x_\alpha\Lambda(x) \to \Lambda(x), \qquad x \in \mathfrak{n} \cap \mathfrak{n}^*.$$

(closability) We need to prove the domain of the adjoint

$$D' := \{\eta \in H \mid A \to \mathbb{C} : \Lambda(x) \mapsto \langle \eta, \Lambda(x^*)\rangle \text{ is bounded}\}$$

is dense in H. Let

$$\mathcal{G} := \{\omega \in M_*^+ : (1+\varepsilon)\omega \leq \varphi \text{ for some } \varepsilon > 0\}.$$

Note that the normality of φ says that $\varphi(x^*x) = \sup_{\omega\in\mathcal{G}} \omega(x^*x)$ for any $x \in M$. For each $\omega \in \mathcal{G}$, by the bounded Radon-Nikodym theorem, there is $h_\omega \in M'^+$ such that $\|h_\omega\| < 1$ and

$$\omega(x^*x) = \langle h_\omega \Lambda(x), \Lambda(x)\rangle, \qquad x \in \mathfrak{n}.$$

Also, if we take a net $x_\alpha \in \mathfrak{n}_1$ that converges σ-strongly to the identity of M using the strong density of $\mathfrak{n}$ in M, the Kaplansky density, and the coincidence of strong and the σ-strong topologies on the bounded part, then we have a σ-weak limit $\lim_{\alpha,\beta} |x_\alpha - x_\beta|^2 = 0$ so that by the normality of ω we obtain

$$\lim_{\alpha,\beta} \|h_\omega^{\frac{1}{2}}\Lambda(x_\alpha) - h_\omega^{\frac{1}{2}}\Lambda(x_\beta)\|^2 = \lim_{\alpha,\beta} \omega(|x_\alpha - x_\beta|^2) = 0.$$

Thus, the vector $\Lambda_\omega := \lim_\alpha h_\omega^{\frac{1}{2}}\Lambda(x_\alpha)$ can be defined, and in particular, we have $h_\omega^{\frac{1}{2}}\Lambda(x) = x\Lambda_\omega$ for $x \in \mathfrak{n}$ and $\omega = \omega_{\Lambda_\omega}$.

If $\eta = h_{\omega_2}^{\frac{1}{2}} y\Lambda_{\omega_1}$ for some $y \in M'$ and $\omega_1, \omega_2 \in \mathcal{G}$, then

$$\begin{aligned}
|\langle \eta, \Lambda(x^*)\rangle| &= |\langle h_{\omega_2}^{\frac{1}{2}} y\Lambda_{\omega_1}, \Lambda(x^*)\rangle| = |\langle y\Lambda_{\omega_1}, h_{\omega_2}^{\frac{1}{2}}\Lambda(x^*)\rangle| = |\langle y\Lambda_{\omega_1}, x^*\Lambda_{\omega_2}\rangle| \\
&= |\langle yx\Lambda_\omega, \Lambda_{\omega_2}\rangle| = |\langle y h_{\omega_1}^{\frac{1}{2}}\Lambda(x), \Lambda_{\omega_2}\rangle| = |\langle \Lambda(x), h_{\omega_1}^{\frac{1}{2}} y^*\Lambda_{\omega_2}\rangle| \\
&\leq \|\Lambda(x)\|\|h_{\omega_1}^{\frac{1}{2}} y^*\Lambda_{\omega_2}\|, \qquad x \in \mathfrak{n}\cap\mathfrak{n}^*,
\end{aligned}$$

which deduces that $\eta \in D'$. Therefore, it suffices to show the following space is dense in H:

$$\{h_{\omega_2}^{\frac{1}{2}} y\Lambda_{\omega_1} : \omega_1, \omega_2 \in \mathcal{G},\ y \in M'\}.$$

Thanks to the normality of φ, we can write

$$\begin{aligned}
\langle \Lambda(x), \Lambda(x)\rangle &= \|\Lambda(x)\|^2 = \varphi(x^*x) = \sup_{\omega\in\mathcal{G}} \omega(x^*x) \\
&= \sup_{\omega\in\mathcal{G}} \|x\Lambda_\omega\|^2 = \sup_{\omega\in\mathcal{G}} \|h_\omega^{\frac{1}{2}}\Lambda(x)\|^2 = \sup_{\omega\in\mathcal{G}} \langle h_\omega\Lambda(x), \Lambda(x)\rangle, \qquad x \in \mathfrak{n}\cap\mathfrak{n}^*.
\end{aligned}$$

Because A in H, for any $\xi \in H$ and $\varepsilon > 0$ there is $x \in \mathfrak{n}\cap\mathfrak{n}^*$ such that $\|\xi - \Lambda(x)\| < \varepsilon$, so the inequality

$$\langle (1-h_\omega)\xi, \xi\rangle \leq \varepsilon(\|\xi\| + \|\Lambda(x)\|) + \langle (1-h_\omega)\Lambda(x), \Lambda(x)\rangle$$

deduces $\inf_{\omega\in\Phi} \langle (1-h_\omega)\xi, \xi\rangle = 0$ by limiting $\varepsilon \to 0$ and taking infinimum on $\omega \in \mathcal{G}$. In other words, for each $\xi \in H$ and $\varepsilon > 0$, we can find $\omega \in \mathcal{G}$ such that $\langle (1-h_\omega)\xi, \xi\rangle < \varepsilon$. At this point, we claim the set $\{h_\omega : \omega \in \mathcal{G}\}$ is upward directed. If the claim is true, then we can construct an increasing net ω_α in $\mathcal{G}$ such that h_{ω_α} converges weakly to the identity of M, and also strongly by the nature of increasing nets. To prove the claim, take $h_1 = h_{\omega_1}$ and $h_2 = h_{\omega_2}$ with $\omega_1, \omega_2 \in \mathcal{G}$. Introduce a operator monotone function $f(t) := t/(1+t)$ and its inverse $f^{-1}(t) = t/(1-t)$ to define

$$h_0 := f(f^{-1}(h_1) + f^{-1}(h_2)).$$

Then, we have $h_0 \geq h_1$, $h_0 \geq h_2$, and $\|h_0\| < 1$. Consider a linear functional

$$\omega_0 : \mathfrak{n} \to \mathbb{C} : x \mapsto \langle h_0\Lambda(x), \Lambda(x)\rangle.$$

Write

$$\begin{aligned}
\omega_0(x^*x) &\leq \langle f^{-1}(h_1)\Lambda(x), \Lambda(x)\rangle + \langle f^{-1}(h_2)\Lambda(x), \Lambda(x)\rangle \\
&\leq (1-\|h_1\|)^{-1}\langle h_1\Lambda(x), \Lambda(x)\rangle + (1-\|h_2\|)^{-1}\langle h_2\Lambda(x), \Lambda(x)\rangle \\
&= (1-\|h_1\|)^{-1}\omega_1(x^*x) + (1-\|h_2\|)^{-1}\omega_2(x^*x), \qquad x \in \mathfrak{n}.
\end{aligned}$$

Then, since ω_1 and ω_2 are normal, we can define $\Lambda_0 := \lim_\alpha h_0^{\frac{1}{2}}\Lambda(x_\alpha) \in H$ for a σ-strongly convergent net $x_\alpha \in \mathfrak{n}_1$ to the identity of M as we have taken above, and we have the vector functional $\omega_0 = \omega_{\Lambda_0}$. Henceforth, ω_0 is extended to a normal positive linear functional on the whole M, and finally the norm condition $\|h_0\| < 1$ tells us that $\omega_0 \in \mathcal{G}$, so the claim is true.

Now the problem is reduced to the density of $\{y\Lambda_\omega : \omega \in \mathcal{G},\ y \in M'\}$ in H. Let $p \in B(H)$ be the projection to the closure of this space. Then, $p\Lambda_\omega = \Lambda_\omega$ for every $\omega \in \mathcal{G}$. Since the space is left invariant under the action of the self-adjoint set M', we have $p \in M$. Then,

$$\varphi(1-p) = \sup_{\omega\in\mathcal{G}} \omega(1-p) = \sup_{\omega\in\mathcal{G}} \langle (1-p)\Lambda_\omega, \Lambda_\omega\rangle = 0$$

implies $p = 1$, hence the density.

(fullness) We have $\lambda(\Lambda(x)) = x$ for $x \in \mathfrak{a}$ since $\Lambda(\mathfrak{a}) = A$ is dense in H and

$$x_1\Lambda(x_2) = \Lambda(x_1x_2) = \Lambda(x_1)\Lambda(x_2) = \lambda(\Lambda(x_1))\Lambda(x_2), \qquad x_1, x_2 \in \mathfrak{n}\cap\mathfrak{n}^*.$$

Also we have for $\xi = \Lambda(x) \in A$ that

$$\Lambda(\lambda(\xi)) = \Lambda(\lambda(\Lambda(\xi))) = \Lambda(x) = \xi.$$

For $\xi \in B$ so that $\lambda(\xi) \in M$, since

$$\varphi(\lambda(\xi)^*\lambda(\xi)) = \|\Lambda(\lambda(\xi))\|^2 = \|\xi\|^2 < \infty,$$

we get $\lambda(B) \subset \mathfrak{n}$. Therefore, A is full by

$$\lambda(A'') = \lambda(B) \cap \lambda(B)^* \subset \mathfrak{a} = \lambda(A).$$

□

Corollary 2.11. *The operations giving a faithful semi-finite normal weight and a full left Hilbert algebra in the above two theorems are mutually inverses of each other.*

Proposition 2.12. *Every von Neumann algebra admits a faithful semi-finite normal weight.*

Proof. Let M be a von Neumann algebra and let $\{\omega_i\}_{i\in I}$ be a maximal family of normal states on M with orthogonal support projections $p_i := s(\omega_i)$. Here, the support projection $s(\omega)$ of a normal state ω is the minimal projection p such that $\omega(px) = \omega(x) = \omega(xp)$ for all $x \in M$. Since every countably decomposable projection p is a support of a normal state, a faithful normal state on pMp, we have $\sum_i p_i = 1$. Define a weight φ by

$$\varphi(x) := \sum_i \omega_i(x).$$

It is faithful because $\varphi(x) = 0$ with $x \geq 0$ means $\omega_i(x) = 0$ and $p_i x p_i = 0$ for all i, and it implies

$$x^{\frac{1}{2}} = x^{\frac{1}{2}}\sum_i p_i = \sum_i x^{\frac{1}{2}} p_i = 0.$$

It is normal because it is completely additive. It is semi-finite because $p_J \uparrow 1$ with $\varphi(p_J) < \infty$ as $J \to I$, where $p_J := \sum_{i\in J} p_i$ and J runs through finite subsets of I. □

2.3 Examples

Example 2.13 (Locally compact groups)**.** For a locally compact group G, the set $A := C_c(G)$ together with a left Haar measure on G has the following left Hilbert algebra structure

$$\langle \xi_1, \xi_2\rangle := \int \overline{\xi_2(s)}\xi_1(s)\,ds, \qquad (\xi_1\xi_2)(s) := \int_G \xi_1(t)\xi_2(t^{-1}s)\,dt, \qquad \xi^*(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}.$$

We have S, F, Δ, and J given by

$$S\xi(s) := \Delta(s^{-1})\overline{\xi(s^{-1})}, \qquad F\xi(s) = \overline{\xi(s^{-1})},$$

$$\Delta\xi(s) = \Delta(s)\xi(s), \qquad J\xi(s) = \Delta(s)^{-\frac{1}{2}}\overline{\xi(s^{-1})},$$

and they have the following norm formulas

$$\|S\xi\|_2 = \|\Delta^{\frac{1}{2}}\xi\|_2, \quad \|F\xi\|_2 = \|\Delta^{-\frac{1}{2}}\xi\|_2, \quad \|S\xi\|_1 = \|\xi\|_1, \quad \|F\xi\|_1 = \|\Delta^{-1}\xi\|_1.$$

The left von Neumann algebra $\lambda(A)''$ is called the *group von Neumann algebra*.

For a locally compact abelian group G, the corresponding f.n.s. weight is a suitably normalized Haar measure on the Pontryagin dual group $\widehat{G}$, called the Plancherel measure, not the Haar measure on the original group G. For a locally compact non-abelian group G, there is no characterization of the corresponding f.n.s. weight as a measure because the left Hilbert algebra $(C_c(G), *)$ is not commutative.

Example 2.14 (Locally compact abelian groups)**.** If G is a locally compact abelian group, then $A = \mathcal{F}^{-1}(L^2(\widehat{G}) \cap L^\infty(\widehat{G}))$ is a full Hilbert algebra, where $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$ is the Fourier transform, such that $B = A$, $D = H = L^2(G)$.

Example 2.15 (Measure spaces)**.** If (X, μ) is a σ-finite measure space, then $L^2(X) \cap L^\infty(X)$ is a full Hilbert algebra.

Example 2.16 (Cyclic separating vector)**.** Let M be a countably decomposable von Neumann algebra and ω be a faithful normal state. If we consider the associated cyclic representation of ω, then we have an action of M on H together with a cyclic separating vector $\Omega \in H$. Then, $A := M\Omega$ has the following left Hilbert algebra structure:

$$\langle x\Omega, y\Omega\rangle \text{ is defined as it is}, \qquad (x\Omega)(y\Omega) := xy\Omega, \qquad (x\Omega)^* := x^*\Omega.$$

There is no specific description of Δ and J in general, but it is known that $B = H$ and

$$A'' = D = \{c\Omega : c \in C(H) \text{ affiliated with } M \text{ such that } \Omega \in \operatorname{dom} c \cap \operatorname{dom} c^*\}.$$

3 December 20

3.1 Review on Pettis integral

Definition 3.1 (Properties of dual pairs). Let (X,F) be a dual pair. For example, if X is a topological vector space and F is a linear subspace of X^*, then (X,F) is a dual pair if and only if F is weakly* dense in X^*. Conversely, every dual pair (X,F) can be understood as (X,X^*) by endowing with $\sigma(X,F)$ on X.

We say (X,F) has the *Krein property* if the $\sigma(X,F)$-closed balanced convex hull of every $\sigma(X,F)$-compact subset of X is $\sigma(X,F)$-compact. We say (X,F) has the *Goldstine property* if X is $\beta(X,F_\beta)$-complete.

Remark. For a Banach space X, the choices $F=X^*$ and $F=X_*$, where X_* is a Banach predual of X if it exists, satisfy the two properties.

Proposition 3.2 (Dunford operator). *Let (Ω,μ) be a finite measure space. Let X be a topological vector space and F is a weakly* dense subspace of X^*. For a $\sigma(X,F)$-integrable map $x:\Omega\to X$, we can define a linear operator $\phi_x : L^\infty(\mu)\to F^\#$, defined such that*

$$\langle\phi_x(p),x^*\rangle := \int_\Omega p(s)\langle x(s),x^*\rangle\,d\mu(s),\qquad p\in L^\infty(\mu),\ x^*\in F.$$

The operator ϕ_x is called the Dunford operator, *and abusing the notation, we usually write as*

$$\phi_x(p)=\int_\Omega p(s)x(s)\,d\mu(s).$$

(a) *ϕ_x is weak*-$\sigma(F^\#,F)$-continuous and $\sigma(F^\#,F)$-compact.*

(b) *Suppose (X,F) has the Krein property. If x is $\sigma(X,F)$-compactly valued, then $\phi_x(L^\infty(\mu))\subset X$.*

(c) *Suppose (X,F) has the Krein and Goldstine property. Suppose Ω is a topological space with tight Borel μ. If x is $\sigma(X,F)$-bounded and $\sigma(X,F)$-continuous, then then $\phi_x(L^\infty(\mu))\subset X$.*

(d) *Suppose we have $\phi_x(L^\infty(\mu))\subset X$. Let Y be another topological vector space and G is a weakly* dense subspace of Y^*. If $T:X\to Y$ is a $\sigma(X,F)$-$\sigma(Y,G)$-continuous linear operator, then $T\phi_x=\phi_{T\circ x}$. In other words,*

$$T\int_\Omega p(s)x(s)\,d\mu(s)=\int_\Omega p(s)Tx(s)\,d\mu(s),\qquad p\in L^\infty(\mu).$$

Proof. (a) If $p_\alpha\in L^\infty(\mu)$ converges to zero weakly*. Then,

$$\langle\phi_x(p_\alpha),x^*\rangle=\int_\Omega p(s)\langle x(s),x^*\rangle\,d\mu(s)\to 0,\qquad x^*\in F$$

because x is $\sigma(X,F)$-integrable so that $(s\mapsto\langle x(s),x^*\rangle)\in L^1(\mu)$, so the continuity of ϕ_x. Then, the compactness of ϕ_x easily follows the weak* compactness of the unit ball of $L^\infty(\mu)$.

(b) Fix $p\in L^\infty(\mu)$ and let C be the $\sigma(X,F)$-closed balanced convex hull of $x(\Omega)\subset X$. Then C is $\sigma(X,F)$-compact by the Krein property. Since for every $x^*\in F$ we have

$$|\langle\phi_x(p),x^*\rangle|=|\int_\Omega p(s)\langle x(s),x^*\rangle\,d\mu(s)|\le\|\mu\|\|p\|_{L^\infty}\sup_{s\in\Omega}|\langle x(s),x^*\rangle|\lesssim\sup_{y\in C}|\langle y,x^*\rangle|,$$

the linear functional $\phi_x(p)$ on F is continuous with respect to the Mackey topology $\tau(F,X)$, which is a dual topology so that $\phi_x(p)$ can be naturally identified with a vector in $(F_\tau)^*=X$.

(c) Fix $p \in L^\infty(\mu)$ and let B be a $\sigma(X,F)$-bounded set containing the image $x(\Omega) \subset X$. By the tightness of μ, there is a sequence of compact sets $K_n \subset \Omega$ such that $\mu(\Omega \setminus K_n) < n^{-1}$. Since for every $x^* \in F$ we have

$$|\langle \phi_x(p) - \phi_{x|_{K_n}}(p), x^*\rangle| = |\int_{\Omega\setminus K_n} p(s)\langle x(s), x^*\rangle\, d\mu(s)| \le n^{-1}\|p\|_{L^\infty} \sup_{s\in\Omega} |\langle x(s), x^*\rangle| \lesssim n^{-1} \sup_{y\in B} |\langle y, x^*\rangle|,$$

which implies that $\phi_{x|_{K_n}}(p)$ converges to $\phi_x(p)$ in $\beta(F^\#, F_\beta)$. Since $\phi_{x|_{K_n}}(p) \in X$ by the part (b) and X is closed in $\beta(F^\#, F_\beta)$, we have $\phi_x(p) \in X$.

(d) By the continuity of T, the adjoint $T^* : G \to F$ is well-defined. The measurability of T and the existence of the adjoint T^* imply that the composition $T \circ x : \Omega \to Y$ is $\sigma(Y,G)$-integrable, so the Dunford operator $\phi_{T\circ x} : L^\infty(\mu) \to G^\#$ is well-defined. Then,

$$\begin{aligned}\langle T\phi_x(p), y^*\rangle = \langle \phi_x(p), T^*y^*\rangle &= \int_\Omega p(s)\langle x(x), T^*y^*\rangle\, d\mu(s)\\ &= \int_\Omega p(s)\langle Tx(s), y^*\rangle\, d\mu(s) = \langle \phi_{T\circ x}(p), y^*\rangle, \qquad p \in L^\infty(\mu),\ y^* \in G.\end{aligned}$$

In particular, $\phi_{T\circ x} : L^\infty(\mu) \to Y$. □

Corollary. *A weakly* holomorphic function $x : \Omega \subset \mathbb{C} \to X$ for a Banach space $X = (X_*)^*$ is strongly holomorphic.*

Proof. I guess it is true... □

3.2 One-parameter group of isometries

Here are our settings in this section: Let X be a Banach space and F is a weakly* dense subspace of X^* such that (X,F) satisfies the Krein and Goldstine property. Let G be a locally compact group. Let $\alpha : G \to \mathrm{Isom}(X) \subset B(X)$ be a $\sigma(X,F)$-continuous one-parameter group of $\sigma(X,F)$-continuous linear isometries.

As a remark, we note that if we choose $X = A$ a C*-algebra and $F = A^*$ its continuous dual, then an action $\alpha : G \to \mathrm{Aut}(A)$ which is continuous in the above sense is enhanced to the continuity with respect to the point-norm topology, because a weakly continuous semi-group on a Banach space is strongly continuous. (It can be shown by applying the uniform boundedness principle twice.)

Proposition 3.3. *There is a (faithful non-degenerate?) homomorphism $\pi_\alpha : M(G) \to B(X)$ defined by*

$$\pi_\alpha(\mu)x := \int \alpha_s(x)\, d\mu(s),$$

which is justified by the Pettis integral.

Proof. By (c) of the previous proposition, we can define

$$\pi_\alpha(\mu)x := \phi_x(1) = \int_G \alpha_s(x)\, d\mu(s).$$

For $\mu, \nu \in M(G)$,

$$\begin{aligned}\pi_\alpha(\mu * \nu)x &= \iint \alpha_{st}(x)\, d\mu(s)\, d\nu(t) = \iint \alpha_s(\alpha_t(x))\, d\nu(t)\, d\mu(s)\\ &= \int \alpha_s\Big(\int \alpha_t(x)\, d\nu(t)\Big)\, d\mu(s) = \pi_\alpha(\mu)\pi_\alpha(\nu)x, \qquad x \in M.\end{aligned}$$ □

Lemma 3.4. *For $\mu \in M(G)$, the linear map*

$$X \to X : x \mapsto \int \alpha_s(x)\, d\mu(s)$$

is $\sigma(X,F)$-$\sigma(X,F)$-continuous.

Proof. Consider the dual one-parameter group $\alpha^* : G \to \mathrm{Isom}(F)$, which is $\sigma(F,X)$-continuous group of $\sigma(F,X)$-$\sigma(F,X)$-continuous linear isometries. Since it satisfies the conditions in (c) of the proposition at the first with the dual pair (F,X), the Pettis integral

$$\int \alpha_s^*(x^*)\, d\mu(s)$$

is well-defined in F. Therefore, if a net x_i converges to zero in $\sigma(X,F)$, then for $x^* \in F$

$$\langle \int \alpha_s(x_i)\, d\mu(s), x^*\rangle = \int \langle \alpha_s(x_i), x^*\rangle\, d\mu(s) = \int \langle x_i, \alpha_s^*(x^*)\rangle\, d\mu(s) = \langle x_i, \int \alpha_s^*(x^*)\, d\mu(s)\rangle \to 0. \qquad \square$$

Theorem 3.5 (Analytic extensions)**.** *Let $G = \mathbb{R}$. Then, there is a family of densely defined closed operators $\{\alpha_z : z \in \mathbb{C}\}$ on X which extends the original α, such that*

(i) $\alpha_z\alpha_t = \alpha_{z+s} = \alpha_s\alpha_z$ *and* $\alpha_z\alpha_w \subset \alpha_{z+w}$ *for $s \in \mathbb{R}$ and $z, w \in \mathbb{C}$,*

(ii) $\alpha_z^{-1} = \alpha_{-z}$*,*

(iii) $\mathrm{dom}\,\alpha_z \subset \mathrm{dom}\,\alpha_w$ *if* $\mathrm{Im}\, z \ge \mathrm{Im}\, w \ge 0$*,*

(iv) $\bigcap_{z\in\mathbb{C}} \mathrm{dom}\,\alpha_z$ *is dense in X.*

Proof. Consider the set of regularized vectors

$$X_0 := \Big\{ \int_{\mathbb{R}} \frac{n}{\sqrt{\pi}} e^{-n^2 s^2} \alpha_s(x)\, ds : n \in \mathbb{N},\ x \in X \Big\}.$$

Now we define $\alpha_z : X_0 \to X$ for $z \in \mathbb{C}$ such that

$$\alpha_z\Big(\int_{\mathbb{R}} f(s)\alpha_s(x)\, ds\Big) := \int_{\mathbb{R}} f(s-z)\alpha_s(x)\, ds.$$

It satisfies some properties:

(a) It extends the original $\{\alpha_s : s \in \mathbb{R}\}$.

(b) For fixed $x \in X_0$, $z \mapsto \alpha_z(x)$ is $\sigma(X,F)$-entire.

(c) X_0 is $\sigma(X,F)$-dense in E, so α_z is densely defined for each $z \in \mathbb{C}$.

(d) α_z is closable for each $z \in \mathbb{C}$.

(a) is clear by coordinate change, and (b) follows from the Fubini and the Morera after taking arbitrary elements of E^*. (c) is by an approximate identity e_n of $L^1(\mathbb{R})$ has $x = \lim_{n\to\infty} \int_{\mathbb{R}} e_n(t)\alpha_s(x)\, ds$. For (d), we have the adjoint $(\alpha_z)_0^* \supset (\alpha_{-\bar{z}})_0$, which is densely defined. Now we have a family of closed densely defined operators $\{\alpha_z : z \in \mathbb{C}\}$ on E such that $\alpha_z\alpha_w \subset \alpha_{z+w}$ for all $z, w \in \mathbb{C}$. $\square$

Definition 3.6 (Entire elements)**.** entire elements from a $*$-subalgebra

Definition 3.7 (Tomita algebras)**.** Let A be a full left Hilbert algebra.

3.3 Tomita-Takesaki commutation theorem

In this section, we let A be a left Hilbert algebra. We will use the following notations freely:

$$H, M, S, \lambda, F, \rho, B, D, A', B', D', \Delta, J.$$

Also note that

$$\begin{tikzcd}[column sep=small]
\mathfrak{m} := \mathfrak{n}^*\mathfrak{n} & \subset & \mathfrak{a} := \mathfrak{n} \cap \mathfrak{n}^* \arrow[d, shift left, "\Lambda"] & \subset & \mathfrak{n} \arrow[d, shift left, "\Lambda"] & \subset & M \\
 & & A \arrow[u, shift left, "\lambda"] & \subset & B \arrow[u, shift left, "\lambda"] & \subset & H.
\end{tikzcd}$$

The goal of this section is to prove that there exists the following commutative "cube" diagram:

$$\begin{tikzcd}
 & A \arrow[rr, "\Delta^{it}"] \arrow[dl, leftrightarrow, "J"'] & & A \arrow[dl, leftrightarrow, "J"] \arrow[dd, "\lambda"] \\
A' \arrow[rr, "\Delta^{it}"'] \arrow[dd, "\rho"'] & & A' \arrow[dd, "\rho"] & \\
 & & & \lambda(A) \arrow[dl, leftrightarrow, "\mathrm{Ad}_J"] \\
\rho(A') \arrow[rr, "\mathrm{Ad}_{\Delta^{it}}"'] & & \rho(A') &
\end{tikzcd}.$$

Lemma. *For every $t \in \mathbb{R}$, the unitary operator Δ^{it} commutes with J, S, and F.*

Proof. It is enough to show $\Delta^{it}J = J\Delta^{it}$. By the relation $J\Delta J = \Delta^{-1}$, the anti-linearity of J, and the uniqueness of the bounded Borel functional calculus, we have the commutation. More precisely, if we let $f(s) := e^{it\log s}$ on $(0, \infty)$, then

$$\Delta^{-it} = f(\Delta^{-1}) = f(J\Delta J) = J\overline{f(\Delta)}J = J(\Delta^{it})^*J = J\Delta^{-it}J.$$

(Here we omit the detailed proof of $f(J\Delta J) = J\overline{f(\Delta^{-1})}J$.) □

Lemma. *$J : D' \to D$ and $\Delta^{it} : D \to D$.*

Proof. We have $J : D' \to D$ since $\eta \in D'$ implies that $SJ\eta = JF\eta$ is well-defined in H. We have $\Delta^{it} : D \to D$ for real t since $\xi \in D$ implies that $S\Delta^{it}\xi = \Delta^{it}S\xi$ is well-defined in H because $S\xi \in D$. □

We need two lemmas.

Lemma 3.8 (Fourier inversion of sech)**.** *Let $G = \mathbb{R}$ and fix $s \in \mathbb{R}$. Let H be a Hilbert space.*

(a) *We have a Pettis integral*

$$\int_{\mathbb{R}} (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}}) \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \alpha_t(x)\, dt = x, \qquad x \in \operatorname{dom}\alpha_{-\frac{i}{2}} \cap \operatorname{dom}\alpha_{\frac{i}{2}}.$$

(b) *If $\sigma : \mathbb{R} \to \mathrm{Aut}(B(H))$ such that $\sigma_t = \mathrm{Ad}_{\Delta^{it}}$ for a injective positive self-adjoint operator on H, then we have a σ-weak Pettis integral*

$$\int_{\mathbb{R}} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \alpha_t(x)\, dt = (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})^{-1}x, \qquad x \in B(H).$$

(c) *If $u : \mathbb{R} \to U(H)$ such that $u_t = \Delta^{-it}$ for a injective positive self-adjoint operator on H, then we have a Pettis integral*

$$\int_{\mathbb{R}} \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \alpha_t(\xi)\, dt = (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})^{-1}\xi, \qquad \xi \in H.$$

Remark. If we let $f(t) := (e^{\frac{t}{2}} + e^{-\frac{t}{2}})^{-1}$ and write $\alpha_t = e^{t\delta}$, then the equation in the lemma can be rewritten formally as the Fourier inversion

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it(-i\delta-s)}\widehat{f}(t)\,dt = f(-i\delta-s), \qquad s\in\mathbb{R}.$$

However, this Fourier calculus in general setting using an unbounded holomorphic functional calculus for unbounded operators acting on Banach spaces is impossible, because even for a fairly normal example (e.g. $\sigma_t = \mathrm{Ad}_{u_t^{-1}}$, u_t is given by the translation on $L^2(\mathbb{R})$) we have a counterexample having the entire spectrum of the analytic generator $\sigma(\sigma_{-i}) = \mathbb{C}$.

Proof. (a) We use the special fact that the function $\widehat{f}(t) := \sqrt{2\pi}(e^{\pi t} + e^{-\pi t})^{-1}$ has imaginary period i. Fix $s\in\mathbb{R}$ and $x \in \operatorname{dom}\alpha_{-\frac{i}{2}} \cap \operatorname{dom}\alpha_{\frac{i}{2}}$. Define a weakly* meromorphic vector function $g : \mathbb{C}\setminus i\mathbb{Z} \to X$ such that

$$g(z) := -i\sqrt{2\pi}\frac{e^{-isz}}{e^{\pi z} - e^{-\pi z}}\alpha_z(x).$$

It satisfies relations

$$g(t-\frac{i}{2}) = e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}}(e^{-ist}\widehat{f}(t)\alpha_t(x)), \qquad g(t+\frac{i}{2}) = -e^{\frac{s}{2}}\alpha_{\frac{i}{2}}(e^{-ist}\widehat{f}(t)\alpha_t(x))$$

and enjoys an estimate

$$\sup_{|r|\le\frac{1}{2}}\|g(t+ir)\| \le \sup_{|r|\le\frac{1}{2}}\sqrt{2\pi}\frac{e^{sr}}{|e^{\pi(t+ir)} - e^{-\pi(t+ir)}|}\|x\| = O(e^{-\pi|t|}), \qquad |t|\to\infty.$$

Then, by the residue theorem

$$\begin{aligned}\sqrt{2\pi}x = 2\pi\lim_{z\to 0} zg(z) &= \int_{-\infty}^{\infty} g(t-\frac{i}{2})\,dt - \int_{-\infty}^{\infty} g(t+\frac{i}{2})\,dt\\ &= \int_{-\infty}^{\infty}(e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})e^{-ist}\widehat{f}(t)\alpha_t(x)\,dt.\end{aligned}$$

Extend for $x\in X$ using the boundedness.

(b) Let $p := 1_{[-r,r]}(\Delta)$ for some $r > 1$ and let $x\in B(H)$. Then, α_z acts on $pB(H)p = B(pH)$ as bounded invertible operators $\alpha_z(x) = \Delta^{iz}x\Delta^{-iz}$, which is σ-weakly continuous, so

$$\begin{aligned}pxp &= \int_{\mathbb{R}}(e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\alpha_t(pxp)\,dt\\ &= \int_{\mathbb{R}}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}(e^{-\frac{s}{2}}\alpha_{t-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{t+\frac{i}{2}})(pxp)\,dt\\ &= (e^{-\frac{s}{2}}\alpha_{t-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{t+\frac{i}{2}})\int_{\mathbb{R}}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}(pxp)\,dt\\ &= (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})\int_{\mathbb{R}}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\alpha_t(pxp)\,dt.\end{aligned}$$

Since $pxp\to x$ as $r\to\infty$ σ-weakly, by the previous lemma, we can take the limit as sesquilinear forms on a dense subspace $\operatorname{dom}\Delta^{\frac{1}{2}}\cap\operatorname{dom}\Delta^{-\frac{1}{2}}\subset H$ to obtain an equation of bounded operators

$$x = (e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})\int_{\mathbb{R}}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\alpha_t(x)\,dt.$$

In particular, $(e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})$ is surjective. Since the injectivity follows from the part (a), we have

$$(e^{-\frac{s}{2}}\alpha_{-\frac{i}{2}} + e^{\frac{s}{2}}\alpha_{\frac{i}{2}})^{-1}x = \int_{\mathbb{R}}\frac{e^{-ist}}{e^{\pi t}+e^{-\pi t}}\alpha_t(x)\,dt.$$

(c) Similar to (b), but cut ξ off into $p\xi$. $\square$

Lemma 3.9. *Let A be a left Hilbert algebra. For $s \in \mathbb{R}$, we have $(e^{-s} + \Delta)^{-1} : A' \to A \cap D'$. In particular, $A \cap D'$ is dense in H.*

Proof. Let $\eta \in A'$ and $\xi := (e^{-s} + \Delta)^{-1}\eta$. Then, $\Delta\xi = \eta - e^{-s}\xi \in H$ implies $\xi \in \operatorname{dom}\Delta \subset \operatorname{dom}\Delta^{\frac{1}{2}} = D$, and $F\xi = e^{s}(F\eta - S\xi) \in H$ implies $\xi \in D'$. The only non-trivial fact is $\xi \in B$. Since $\xi \in D$, by the polar decomposition, we have

$$\lambda(\xi) = vh = kv, \qquad h := |\lambda(\xi)|, \quad k := |\lambda(\xi)^*|.$$

Let $f \in C_c((0,\infty))^+$. Since

$$\begin{aligned}
\langle f(h)S\xi, \zeta\rangle = \langle S\xi, f(h)\zeta\rangle &= \langle Ff(h)\zeta, \xi\rangle = \langle Fv^*\grave{f}(k)\lambda(\xi)\zeta, \xi\rangle = \langle F\lambda(v^*\grave{f}(k)\xi)\zeta, \xi\rangle \\
&= \langle F\rho(\zeta)v^*\grave{f}(k)\xi, \xi\rangle = \langle \rho(v^*\grave{f}(k)\xi)^*F\zeta, \xi\rangle = \langle F\zeta, \rho(v^*\grave{f}(k)\xi)\xi\rangle \\
&= \langle F\zeta, \lambda(\xi)v^*\grave{f}(k)\xi\rangle = \langle F\zeta, f(k)\xi\rangle = \langle Sf(k)\xi, \zeta\rangle, \qquad \xi \in D,\ \zeta \in D'
\end{aligned}$$

for every $f \in C_c((0,\infty))^+$, we have

$$\begin{aligned}
\|f(k)\eta\|^2 &= \|f(k)(e^{-s} + \Delta)\xi\|^2 \\
&= e^{-2s}\|f(k)\xi\|^2 + \|f(k)\Delta\xi\|^2 + 2e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\
&\geq 2e^{-s}\|f(k)\xi\|\|f(k)\Delta\xi\| + 2e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\
&\geq 4e^{-s}\operatorname{Re}\langle f(k)\xi, f(k)\Delta\xi\rangle \\
&= 4e^{-s}\operatorname{Re}\langle f(k)^2\xi, FS\xi\rangle \\
&= 4e^{-s}\operatorname{Re}\langle Sf(k)^2\xi, S\xi\rangle \\
&= 4e^{-s}\operatorname{Re}\langle f(h)^2S\xi, S\xi\rangle \\
&= 4e^{-s}\|f(h)S\xi\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|f(k)\eta\|^2 = \|\grave{f}(k)k\eta\|^2 &= \|\grave{f}(k)v\lambda(\xi)^*\eta\|^2 = \|v\grave{f}(h)\rho(\eta)S\xi\|^2 \\
&= \|\rho(\eta)v\grave{f}(h)S\xi\|^2 \leq \|\rho(\eta)\|^2\|\grave{f}(h)S\xi\|^2, \qquad \eta \in A',\ f \in C_c((0,\infty))^+,
\end{aligned}$$

which imply that $c := \frac{1}{2}e^{\frac{s}{2}}\|\rho(\eta)\|$ satisfies

$$\|f(h)S\xi\| \leq c\|\grave{f}(h)S\xi\|.$$

For arbitrary $\varepsilon > 0$, by considering a net $f_\alpha \uparrow 1_{(c+\varepsilon,\infty)}$ and defining $p_\varepsilon := 1_{[0,c+\varepsilon]}(h)$, we have $\grave{f} \leq (c+\varepsilon)^{-1}f$ and that

$$\|(1-p_\varepsilon)S\xi\| \leq \frac{c}{c+\varepsilon}\|(1-p_\varepsilon)S\xi\|,$$

which implies $p_\varepsilon S\xi = S\xi$ for all $\varepsilon > 0$, so $p_0 S\xi = S\xi$. Then,

$$\|\lambda(\xi)^*\zeta\| = \|p_0\lambda(\xi)^*\zeta\| = \|1_{[0,c]}(h)hv^*\zeta\| \leq c\|v^*\zeta\| \leq c\|\zeta\|, \qquad \zeta \in A'.$$

Therefore, $S\xi \in B$, which implies $S\xi \in A$ and $\xi \in A$.

For the density of $A \cap D'$, let $\zeta \in \operatorname{dom}\Delta$. Define a sequence $\eta_n \in A'$ such that $\lim_{n\to\infty}\eta_n = (1+\Delta)\zeta$. Then, since $(1+\Delta)^{-1}$ is bounded, we have $\zeta = \lim_{n\to\infty}(1+\Delta)^{-1}\eta_n \in \overline{A \cap D'}$. Since $\operatorname{dom}\Delta$ is dense in H, we are done. □

Theorem 3.10 (Tomita-Takesaki commutation theorem)**.** *Let A be a left Hilbert algebra. Then, for every*

$t \in \mathbb{R}$, *the following diagram commutes:*

$$\begin{tikzcd}
 & D \arrow[r, "u_t^{-1}=\Delta^{it}"] & D \arrow[d, "\lambda"] \\
A' \arrow[ur, "J"] \arrow[d, "\rho"'] & & C(H). \\
\rho(A') \arrow[r, "\sigma_t=\mathrm{Ad}_{\Delta^{it}}"'] & B(H) \arrow[ur, "\mathrm{Ad}_J"'] &
\end{tikzcd}$$

Proof. Fix $\eta \in A'$ and define

$$\begin{aligned}
\xi &:= (e^{-\frac{s}{2}} u_{-\frac{i}{2}} + e^{\frac{s}{2}} u_{\frac{i}{2}})^{-1} J\eta = (e^{-\frac{s}{2}} \Delta^{-\frac{1}{2}} + e^{\frac{s}{2}} \Delta^{\frac{1}{2}})^{-1} J\eta \\
&= (e^{-\frac{s}{2}} \Delta^{-\frac{1}{2}} + e^{\frac{s}{2}} \Delta^{\frac{1}{2}})^{-1} \Delta^{-\frac{1}{2}} F\eta = e^{-\frac{s}{2}} (e^{-s} + \Delta)^{-1} F\eta \in A \cap D'.
\end{aligned}$$

By the computations

$$\begin{aligned}
\langle \rho(F\xi)\zeta_1, \zeta_2 \rangle &= \langle \lambda(\zeta_1)F\xi, \zeta_2 \rangle = \langle F\xi, \lambda(\zeta_1)^*\zeta_2 \rangle = \langle S\lambda(\zeta_1)^*\zeta_2, \xi \rangle \\
&= \langle \lambda(\zeta_2)^*\zeta_1, \xi \rangle = \langle \rho(\zeta_1)S\zeta_2, \xi \rangle = \langle S\zeta_2, \rho(\zeta_1)^*\xi \rangle \\
&= \langle S\zeta_2, \lambda(\xi)F\zeta_1 \rangle = \langle F\lambda(\xi)F\zeta_1, \zeta_2 \rangle, \qquad\qquad \zeta_1 \in A \cap D',\ \zeta_2 \in A, \\
\langle \rho(S\xi)\zeta_1, \zeta_2 \rangle &= \langle \lambda(\zeta_1)S\xi, \zeta_2 \rangle = \langle S\xi, \lambda(\zeta_1)^*\zeta_2 \rangle = \langle S\xi, \rho(\zeta_2)S\zeta_1 \rangle \\
&= \langle \rho(\zeta_2)^*S\xi, S\zeta_1 \rangle = \langle \lambda(\xi)^*F\zeta_2, S\zeta_1 \rangle = \langle F\zeta_2, \lambda(\xi)S\zeta_1 \rangle \\
&= \langle S\lambda(\xi)S\zeta_1, \zeta_2 \rangle, \qquad\qquad \zeta_1 \in A \cap D',\ \zeta_2 \in D',
\end{aligned}$$

the domains of $\rho(F\xi)$ and $\rho(S\xi)$ contain $A \cap D'$ and we have

$$\begin{aligned}
\rho(\eta) &= \rho(J(e^{-\frac{s}{2}} \Delta^{-\frac{1}{2}} + e^{\frac{s}{2}} \Delta^{\frac{1}{2}})\xi) \\
&= e^{-\frac{s}{2}} \rho(F\xi) + e^{\frac{s}{2}} \rho(S\xi) \\
&= e^{-\frac{s}{2}} F\lambda(\xi)F + e^{\frac{s}{2}} S\lambda(\xi)S \\
&= e^{-\frac{s}{2}} \Delta^{\frac{1}{2}} J\lambda(\xi)J\Delta^{-\frac{1}{2}} + e^{\frac{s}{2}} \Delta^{-\frac{1}{2}} J\lambda(\xi)J\Delta^{\frac{1}{2}} \\
&= (e^{-\frac{s}{2}} \sigma_{-\frac{i}{2}} + e^{\frac{s}{2}} \sigma_{\frac{i}{2}}) \mathrm{Ad}_J \lambda(\xi)
\end{aligned}$$

as sesquilinear forms on $A \cap D'$. The conditions for ξ and ζ_1 to belong to $A \cap D'$ are necessary in the above computation. By the density of $A \cap D'$ in H, we have the bounded operators

$$\mathrm{Ad}_J (e^{-\frac{s}{2}} \sigma_{-\frac{i}{2}} + e^{\frac{s}{2}} \sigma_{\frac{i}{2}})^{-1} \rho(\eta) = \lambda((e^{-\frac{s}{2}} u_{-\frac{i}{2}} + e^{\frac{s}{2}} u_{\frac{i}{2}})^{-1} J\eta).$$

Then, we get

$$\mathrm{Ad}_J\Big(\int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t(\rho(\eta))\,dt\Big) = \lambda\Big(\int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} u_t(J\eta)\,dt\Big), \qquad s \in \mathbb{R},\ \eta \in A',$$

as bounded linear operators on H. For every $\zeta \in B'$, since $\mathrm{Ad}_J : B(H) \to B(H)$, $\cdot\zeta : B(H) \to H$, and

$\rho(\zeta) : H \to H$ are all continuous between weak* topologies, we have

$$\begin{aligned}
\int \frac{e^{ist}}{e^{\pi t} + e^{-\pi t}} \operatorname{Ad}_J \sigma_t(\rho(\eta))\zeta \, dt &= \operatorname{Ad}_J\Big(\int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \sigma_t(\rho(\eta)) \, dt \Big)\zeta \\
&= \lambda\Big(\int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} u_t(J\eta) \, dt \Big)\zeta \\
&= \rho(\zeta) \int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} u_t(J\eta) \, dt \\
&= \int \frac{e^{-ist}}{e^{\pi t} + e^{-\pi t}} \rho(\zeta) u_t(J\eta) \, dt \\
&= \int \frac{e^{ist}}{e^{\pi t} + e^{-\pi t}} \lambda(u_t^{-1}(J\eta))\zeta \, dt.
\end{aligned}$$

Then, by taking arbitrary bounded linear functionals of H on the above integral, and by the injectivity of the Fourier transform, we finally obtain $\operatorname{Ad}_J \circ \sigma_t \circ \rho = \lambda \circ u_t^{-1} \circ J$ on A'. $\square$

Corollary 3.11. *Let A be a full left Hilbert algebra. Then, for $t \in \mathbb{R}$, the following diagram is well-defined and commutes:*

$$\begin{tikzcd}
& A \arrow[rr, "\Delta^{it}"] & & A \arrow[dd, "\lambda"] \\
A' \arrow[ur, leftrightarrow, "J"] \arrow[rr, "\Delta^{it}"'] \arrow[dd, "\rho"'] & & A' \arrow[ur, leftrightarrow, "J"'] \arrow[dd, "\rho"] & \\
& & & \lambda(A) \\
\rho(A') \arrow[rr, "\operatorname{Ad}_{\Delta^{it}}"'] & & \rho(A') \arrow[ur, leftrightarrow, "\operatorname{Ad}_J"'] &
\end{tikzcd}.$$

In particular, we have

$$JA = A', \qquad \Delta^{it}A = A, \qquad JMJ = M', \qquad \Delta^{it} M \Delta^{-it} = M,$$

and J is an anti-homomorphism, Δ^{it} is a $*$-homomorphism

Proof. $\square$

4 January 17

4.1 Connes cocyle

4.2 Standard form

abelian group

Chapter X: crossed product duality,

Let M be a von Neumann algebra and G be a locally compact group. An *action* of G on M is a σ-weakly continuous group homomorphism $\alpha : G \to \mathrm{Aut}(M)$. The triple (M, G, α) is called a *W^*-dynamical system*. A *covariant representation* of a W*-dynamical system is a pair (π, u) of a normal representation $\pi : M \to B(H)$ and a strongly continuous unitary representation $u : G \to U(H)$ such that $\alpha_s(x) = u_s x u_s^*$.

If (H, P, J) is a standard form of M, then there is a unique covariant representation of (M, G, α), called the *standard covariant representation*.

Let (M, G, α) be a W*-dynamical system. A *α-(one)-cocycle* of a strongly continuous group homomorphism $u : G \to U(M)$ such that $u_{st} = u_s \alpha_s(u_t)$, and we denote by $Z^1(M, G, \alpha)$ the set of all α-cycles.

5 March 8

Type III