Differential Equations

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May 8, 2022

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$$\int \delta(x)\varphi(x)\,dx.$$

7.8 (Fundamental solution of the Laplace equation). Let $d \geq 2$. The *Fundamental solution of the Laplace equation* is a function $\Phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that solves the boundary value problem

$$\begin{cases} -\Delta\Phi(x) = \delta(x) & \text{in } \mathbb{R}^d, \\ \quad \Phi(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

(a) The funcdamental solution is given by

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d}\frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}.$$

In particular, Φ and $\nabla\Phi$ are locally integrable on $\mathbb{R}^d$ but $\nabla^2\Phi$ is not.

(b) For $u \in C_0^2(\mathbb{R}^d)$,

$$u(x) = -\int \Phi(x-y)\Delta u(y)\,dy.$$

Proof. Note that $\nabla\Phi(y)\cdot\nabla u(x-y)$ is integrable in y. Then,

$$\begin{aligned}-\int \Phi(y)\Delta u(x-y)\,dy &= -\int \nabla\Phi(y)\cdot\nabla u(x-y)\,dy \\ &= -\lim_{\varepsilon\to\infty}\int_{|y|\geq\varepsilon}\nabla\Phi(y)\cdot\nabla u(x-y)\,dy \\ &= -\lim_{\varepsilon\to\infty}\int_{|y|=\varepsilon}\nabla\Phi(y)u(x-y)\cdot\nu\,dS.\end{aligned}$$

Since

$$\nabla\Phi(x) = -\frac{1}{\omega_d}\frac{x}{|x|^d}, \qquad \nu = \frac{x}{|x|},$$

we get

$$-\int \Phi(y)\Delta u(x-y)\,dy = \lim_{\varepsilon\to\infty}\frac{1}{\omega_d\varepsilon^{d-1}}\int_{|y|=\varepsilon} u(x-y)\,dS_y = u(x).$$

□

7.9 (Green's function of the Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$ for $d\geq 2$. *Green's function of the Poisson equation* is a function $G:\Omega^2\setminus\{(x,x)\in\Omega\}\to\mathbb{R}$ that solves the boundary value problem

$$\begin{cases}-\Delta_y G(x,y) = \delta(x-y) & \text{in } y\in\Omega\setminus\{x\},\\ \quad G(x,y) = 0 & \text{on } y\in\partial\Omega.\end{cases}$$

for each $x\in\Omega$.

Define $\phi:\Omega^2\to\mathbb{R}$ to be a function that solves the boundary value problem

$$\begin{cases}-\Delta_y\phi(x,y) = 0 & \text{in } y\in\Omega,\\ \quad \phi(x,y) = \Phi(x-y) & \text{on } y\in\partial\Omega.\end{cases}$$

for each $x\in\Omega$. Assume for the domain Ω that there exists a unique ϕ.

(a) Green's function is given by

$$G(x,y) = \Phi(x-y) - \phi(x,y),$$

where Φ is the fundamental solution of the Laplace equation. Physically, $y\mapsto -\phi(x,y)$ has a meaning of the electric potential generated by the induced surface charge of a grounded conductor provided a point charge is at x.

(b) The *Green representation formula* holds: for $u\in C^2(\Omega)\cap C(\overline{\Omega})$,

$$u(x) = -\int_\Omega G(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} u(y)\nabla_y G(x,y)\cdot\nu\,dS_y.$$

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