

# Measure Theory

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# **Part I**

# Chapter 1

## Measures

### 1.1 Measures

**1.1 ( $\sigma$ -algebras).** Let  $X$  be a set. A  $\sigma$ -algebra of sets on  $X$  is a collection  $\mathcal{A} \subset \mathcal{P}(X)$  which is closed under countable unions and complements.

- (a) generated by a set.
- (b) countable and cocountable sets
- (c) Borel

**1.2 (Measures).** A *measurable space* or a *Borel space* is a pair  $(X, \mathcal{A})$  of a set  $X$  and a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ . Each element of  $\mathcal{A}$  is called *measurable*. We often omit  $\mathcal{A}$  to just write  $X$  for  $(X, \mathcal{A})$  if there is no confusion.

Let  $(X, \mathcal{A})$  be a measurable space. A *measure* on  $(X, \mathcal{A})$  is a set function  $\mu : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  that is *countably additive*: we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i), \quad (E_i)_{i=1}^{\infty} \subset \mathcal{A}.$$

Here the squared cup notation reads the disjoint union. A *measure space* is a triple  $(X, \mathcal{A}, \mu)$ , where  $\mu$  is a measure on  $(X, \mathcal{A})$ . Let  $\mu$  be a measure on  $X$ .

- (a)  $\mu$  is monotone: for  $E, F \in \mathcal{A}$  if  $E \subset F$  then  $\mu(E) \leq \mu(F)$ .
- (b)  $\mu$  is countably subadditive: for
- (c)  $\mu$  is continuous from below:
- (d)  $\mu$  is continuous from above:

**1.3 (Complete measures).** Let  $(X, \mathcal{A}, \mu)$  be a measure space. A *null set* is a measurable set  $N$  satisfying  $\mu(N) = 0$ , and a *full set* is a measurable set whose complement is a null set.

A *complete measure* is a measure such that every subset of a null set is measurable.

For a predicate  $P$  of points  $x \in X$ , we say  $P$  is true *almost everywhere* or *a.e.* on  $X$  if there is a full set  $F \subset X$  such that  $P(x)$  is true for all  $x \in F$ .

### 1.2 Carathéodory extension

**1.4 (Outer measures).** Let  $X$  be a set. An *outer measure* on  $X$  is a set function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty] : \emptyset \mapsto 0$  which is monotone and countably subadditive.

(i)  $\mu^*$  is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \leq \mu^*(S_2), \quad S_1, S_2 \in \mathcal{P}(X),$$

(ii)  $\mu^*$  is *countably subadditive*: we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \sum_{i=1}^{\infty} \mu^*(S_i), \quad (S_i)_{i=1}^{\infty} \subset \mathcal{P}(X).$$

Comparing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty] : \emptyset \mapsto 0$  is an outer measure if and only if  $\mu^*$  is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \leq \sum_{i=1}^{\infty} \mu^*(S_i), \quad S \in \mathcal{P}(X), (S_i)_{i=1}^{\infty} \subset \mathcal{P}(X).$$

(b) For any  $\emptyset \in \mathcal{A}_0 \subset \mathcal{P}(X)$ , let  $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty] : \emptyset \mapsto 0$  be a set function. We can associate an outer measure  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  by defining as

$$\mu^*(S) := \inf \left\{ \sum_{i=1}^{\infty} \mu_0(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i, B_i \in \mathcal{A}_0 \right\},$$

where we use the convention  $\inf \emptyset = \infty$ .

*Proof.*

□

**1.5** (Carathéodory measurable sets). Let  $\mu^*$  be an outer measure on a set  $X$ . We want to construct a measure by restriction of  $\mu^*$  on a properly defined  $\sigma$ -algebra. A subset  $E \subset X$  is called *Carathéodory measurable* relative to  $\mu^*$  if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every  $S \in \mathcal{P}(X)$ . Let  $\mathcal{A} \subset \mathcal{P}(X)$  be the collection of all Carathéodory measurable subsets relative to  $\mu^*$ .

- (a)  $\mathcal{A}$  is an algebra and  $\mu^*$  is finitely additive on  $\mathcal{A}$ .
- (b)  $\mathcal{A}$  is a  $\sigma$ -algebra and  $\mu^*$  is countably additive on  $\mathcal{A}$ . That is,  $\mu := \mu^*|_{\mathcal{A}}$  is a measure.
- (c) The measure  $\mu$  is complete.

*Proof.*

□

**1.6** (Carathéodory extension theorem). The Carathéodory extension is a construction method for a measure extending a given set function  $\mu_0$  on  $\mathcal{A}_0 \subset \mathcal{P}(X)$  for a set  $X$ . The idea is to restrict the outer measure  $\mu^*$  associated to  $\mu_0$  in order to obtain a measure  $\mu$ . We want to find a sufficient condition for  $\mu$  to be a measure on a  $\sigma$ -algebra containing  $\mathcal{A}_0$ .

Let  $\emptyset \in \mathcal{A}_0 \subset \mathcal{P}(X)$ , and let  $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$  be a set function with  $\mu_0(\emptyset) = 0$ . Let  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  be the associated outer measure of  $\mu_0$ , and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  the measure defined by the restriction of  $\mu^*$  on Carathéodory measurable subsets.

(a)  $\mu^*$  extends  $\mu_0$  if  $\mu_0$  satisfies the monotone countable subadditivity: we have

$$A \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \mu_0(A) \leq \sum_{i=1}^{\infty} \mu_0(B_i), \quad A \in \mathcal{A}_0, (B_i)_{i=1}^{\infty} \subset \mathcal{A}_0$$

- (b)  $\mu$  extends  $\mu_0$  if  $\mu_0$  satisfies the following property in addition: for  $B, A \in \mathcal{A}_0$  and any  $\varepsilon > 0$ , there are  $(C_j)_{j=1}^\infty, (D_j)_{j=1}^\infty \subset \mathcal{A}_0$  such that

$$B \cap A \subset \bigcup_{j=1}^\infty C_j, \quad B \setminus A \subset \bigcup_{j=1}^\infty D_j, \quad \sum_{j=1}^\infty (\mu_0(C_j) + \mu_0(D_j)) < \mu_0(B) + \varepsilon.$$

*Proof.* (a) Fix  $A \in \mathcal{A}_0$ . Clearly  $\mu^*(A) \leq \mu_0(A)$ . For the opposite direction, we may assume  $\mu^*(A) < \infty$ . By the finiteness of  $\mu^*(A)$ , for any  $\varepsilon > 0$  we have  $(B_i)_{i=1}^\infty \subset \mathcal{A}_0$  such that  $A \subset \bigcup_{i=1}^\infty B_i$  and

$$\sum_{i=1}^\infty \mu_0(B_i) < \mu^*(A) + \varepsilon.$$

Therefore we have  $\mu_0(A) < \mu^*(A) + \varepsilon$  by the assumption, and we get  $\mu_0(A) \leq \mu^*(A)$  by limiting  $\varepsilon \rightarrow 0$ .

(b) Fix  $A \in \mathcal{A}_0$ . It is enough to check the inequality  $\mu^*(S \cap A) + \mu^*(S \setminus A) \leq \mu^*(S)$  for  $S \in \mathcal{P}(X)$  with  $\mu^*(S) < \infty$ . By the finiteness of  $\mu^*(S)$ , we have  $(B_i)_{i=1}^\infty \subset \mathcal{B}$  such that  $S \subset \bigcup_{i=1}^\infty B_i$ . From the condition, we have  $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$  and  $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$  satisfying

$$\begin{aligned} \mu^*(S \cap A) + \mu^*(S \setminus A) &\leq \mu^*\left(\bigcup_{j=1}^\infty (B_i \cap A)\right) + \mu^*\left(\bigcup_{j=1}^\infty (B_i \setminus A)\right) \\ &\leq \sum_{i,j=1}^\infty (\mu_0(C_{i,j}) + \mu_0(D_{i,j})) \\ &\leq \sum_{i=1}^\infty (\mu_0(B_i) + 2^{-i} \varepsilon) \\ &< \mu^*(S) + \varepsilon. \end{aligned}$$

Therefore,  $A$  is Carathéodory measurable relative to  $\mu^*$ , so the domain of  $\mu$  contains the domain of  $\mu_0$ . The values coincide by the part (a).  $\square$

**1.7 (Uniqueness of extension of measures).** The Carathéodory extension also provides a uniqueness result for measure extensions. Let  $\rho : \mathcal{B} \rightarrow [0, \infty] : \emptyset \mapsto 0$  be a set function, where  $\emptyset \in \mathcal{B} \subset \mathcal{P}(X)$  for a set  $X$ . We say  $\rho$  is  $\sigma$ -finite if there is a cover  $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$  of  $X$  such that  $\rho(B_i) < \infty$  for each  $i$ .

Let  $\mathcal{A}$  be a  $\sigma$ -algebra containing  $\mathcal{B}$ . Let  $\mu$  be a measure on  $\mathcal{A}$ , which extends  $\rho$ , given by the restriction of the outer measure  $\mu^*$  associated to  $\rho$ . Let  $\nu$  be another measure on  $\mathcal{A}$  which extends  $\rho$ . Let  $E \in \mathcal{A}$  and  $\{E_i\}_{i=1}^\infty \subset \mathcal{A}$ .

- (a)  $\nu(E) \leq \mu(E)$ .
- (b)  $\nu(E_i) = \mu(E_i)$  implies  $\nu\left(\bigcup_{i=1}^\infty E_i\right) = \mu\left(\bigcup_{i=1}^\infty E_i\right)$ .
- (c)  $\nu(E) = \mu(E)$  for  $\mu(E) < \infty$ .
- (d)  $\nu(E) = \mu(E)$  for  $\mu(E) = \infty$ , if  $\rho$  is  $\sigma$ -finite

*Proof.* (a) We may assume  $\mu(E) < \infty$ . By the definition of the outer measure, there is  $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$  such that  $E \subset \bigcup_{i=1}^\infty B_i$ . Also, whenever  $E \subset \bigcup_{i=1}^\infty B_i$  we have

$$\nu(E) \leq \nu\left(\bigcup_{i=1}^\infty B_i\right) \leq \sum_{i=1}^\infty \nu(B_i) = \sum_{i=1}^\infty \rho(B_i) = \sum_{i=1}^\infty \mu(B_i),$$

hence  $\nu(E) \leq \mu(E)$ .

(b) In the light of the inclusion-exclusion principle, we have

$$\mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j) - \mu(E_i \cap E_j) \leq \nu(E_i) + \nu(E_j) - \nu(E_i \cap E_j) = \nu(E_i \cup E_j),$$

so that  $\mu(E_i \cup E_j) = \nu(E_i \cap E_j)$ . Applying it inductively, we have for every  $n$  that

$$\mu\left(\bigcup_{i=1}^n B_i\right) = \nu\left(\bigcup_{i=1}^n B_i\right),$$

and by limiting  $n \rightarrow \infty$  the continuity from below gives

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \nu\left(\bigcup_{i=1}^{\infty} B_i\right).$$

(c) Because  $\mu(E) < \infty$ , for any  $\varepsilon > 0$  we have a sequence  $(B_i)_{i=1}^{\infty} \subset \mathcal{B}$  such that  $E \subset \bigcup_{i=1}^{\infty} B_i$  and

$$\sum_{i=1}^{\infty} \rho(B_i) < \mu(E) + \varepsilon.$$

Applying the part (b) Then, we have

$$\mu(E) \leq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \nu\left(\bigcup_{i=1}^{\infty} B_i\right) = \nu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) + \nu(E)$$

and

$$\nu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) \leq \mu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) - \mu(E) \leq \sum_{i=1}^{\infty} \mu(B_i) - \mu(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu(E) < \varepsilon,$$

we get  $\mu(E) < \nu(E) + \varepsilon$  and  $\mu(E) \leq \nu(E)$  by limiting  $\varepsilon \rightarrow 0$ .

(d) Let  $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$  be a cover of  $X$  such that  $\rho(B_i) < \infty$ . Define  $E_1 := B_1$  and  $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$  for  $n \geq 2$  so that  $\{E_i\}_{i=1}^{\infty}$  is a pairwise disjoint cover of  $X$  with

$$\mu(E \cap E_i) \leq \mu(E_i) \leq \mu(B_i) = \rho(B_i) < \infty$$

for each  $i$ , so we have by the part (c) that

$$\nu(E) = \sum_{i=1}^{\infty} \nu(E \cap E_i) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \mu(E).$$

□

## 1.3 Measures on the real line

1.8 (Borel  $\sigma$ -algebra).

1.9 (Distribution functions).

1.10 (Helly selection theorem).

1.11 (Vitali set).

## Exercises

1.12 (Boolean algebras and rings).

1.13 (Cardinalities). infinite  $\sigma$ -algebra is  $\geq \mathfrak{c}$ .

1.14 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let  $\mathcal{A} \subset \mathcal{P}(X)$  such that  $\emptyset \in \mathcal{A}$ . We say that  $\mathcal{A}$  is a semi-ring if it is closed under finite intersections, and each relative complement is a finite union of elements of  $\mathcal{A}$ . We say that  $\mathcal{A}$  is a semi-algebra

Let  $\mathcal{A}$  be a semi-ring of sets over  $X$ . Suppose a set function  $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$  satisfies



(i)  $\rho$  is *disjointly countably subadditive*: we have

$$\rho\left(\bigsqcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \rho(A_i)$$

for  $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$ ,

(ii)  $\rho$  is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for  $A_1, A_2 \in \mathcal{A}$ .

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

**1.15** (Monotone class lemma). A collection  $\mathcal{C} \subset \mathcal{P}(X)$  is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let  $H$  be a vector space closed under bounded monotone convergence. If  $\text{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$  then  $B^\infty(\sigma(\mathcal{A})) \subset H$ .

**1.16** (Steinhaus theorem). Let  $\lambda$  denote the Lebesgue measure on  $\mathbb{R}$  and let  $E \subset \mathbb{R}$  be a Lebesgue measurable set with  $\lambda(E) > 0$ .

(a) For any  $0 < \alpha < 1$ , there is an interval  $I = (a, b)$  such that  $\lambda(E \cap I) > \alpha\lambda(I)$ .

(b)  $E - E = \{x - y : x, y \in E\}$  contains an open interval containing zero.

*Proof.* (a) We may assume  $\lambda(E) < \infty$ . Since  $\lambda$  is outer measure and  $\lambda(E) \neq 0$ , we have an open subset  $U$  of  $\mathbb{R}$  such that  $\lambda(U) < \alpha^{-1}\lambda(E)$ . Because  $U$  is a countable disjoint union of open intervals  $U = \bigsqcup_{i=1}^{\infty} (a_i, b_i)$ , we have

$$\sum_{i=1}^{\infty} \lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1} \sum_{i=1}^{\infty} \lambda(E \cap (a_i, b_i)).$$

Therefore, there is  $i$  such that  $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$ . □

## Problems

- \*1. Every Lebesgue measurable set in  $\mathbb{R}$  of positive measure contains an arbitrarily long arithmetic progression.

## Chapter 2

# Measurable functions

### 2.1 Simple functions

**2.1** (Measurability of pointwise limits). Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

*Proof.* Let  $f(x) = \lim_{n \rightarrow \infty} s_n(x)$ .

□

### 2.2 Almost everywhere convergence

**2.2** (Almost everywhere convergence). Let  $(X, \mu)$  be a measure space and let  $f_n : X \rightarrow \overline{\mathbb{R}}$  and  $f : X \rightarrow \overline{\mathbb{R}}$  be measurable functions. The set of convergence of the sequence  $f_n$  is defined as the set

$$\{x \in X : \lim_{n \rightarrow \infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say  $f_n$  converges to  $f$  *almost everywhere* with respect to  $\mu$  if the set of divergence is a null set in  $\mu$ . We simply write

$$f_n \rightarrow f \text{ a.e.}$$

if  $f_n$  converges to  $f$  almost everywhere, and we frequently omit the measure  $\mu$  if it has no confusion.

(a) If  $\mu$  is complete and, if  $f_n \rightarrow f$  a.e., then  $f$  is measurable.

**2.3** (Borel-Cantelli lemma). Let  $(X, \mu)$  be a measure space and let  $f_n : X \rightarrow \overline{\mathbb{R}}$  and  $f : X \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon > 0} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a)  $f_n \rightarrow f$  a.e. if and only if for each  $\varepsilon > 0$  we have

$$\mu(\{x : \limsup_{n \rightarrow \infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b)  $f_n \rightarrow f$  a.e. if and only if for each  $\varepsilon > 0$  we have

$$\mu(\limsup_{n \rightarrow \infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(c)  $f_n \rightarrow f$  a.e. if for each  $\varepsilon > 0$  we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \infty.$$

*Proof.* (b) The set of divergence of the sequence  $f_n$  is given by

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \frac{1}{m}\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} (X \setminus E_n^m).$$

(c) Since

$$\mu\left(\bigcup_{i=1}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \leq \sum_{i=1}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned} \mu(\limsup_{n \rightarrow \infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) &= \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0. \end{aligned} \quad \square$$

**2.4 (Convergence in measure).** Let  $(X, \mu)$  be a measure space and let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions. We say  $f_n$  converges to a measurable function  $f : X \rightarrow \overline{\mathbb{R}}$  in measure if for each  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(a) If  $f_n \rightarrow f$  in measure, then there is a subsequence  $f_{n_k}$  such that  $f_{n_k} \rightarrow f$  a.e.

(b) If every subsequence  $f_{n_k}$  of  $f_n$  has a further subsequence  $f_{n_{k_j}}$  such that  $f_{n_{k_j}} \rightarrow f$  a.e., then  $f_n \rightarrow f$  in measure.

*Proof.* (a) Since for each positive integer  $k$  we have  $\mu(\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}) \rightarrow 0$  as  $n \rightarrow \infty$ , there exists  $n_k$  such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\}) = 0.$$

Then, for each  $\varepsilon > 0$ ,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \varepsilon\} &= \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \varepsilon\} \\ &\subset \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \frac{1}{k}\} \\ &= \limsup_{k \rightarrow \infty} \{x : |f_{n_k}(x) - f(x)| \geq \frac{1}{k}\} \end{aligned}$$

implies the limit superior of the tail events is a null set, hence  $f_{n_k} \rightarrow f$  a.e.

(b) □

**2.5 (Egorov theorem).** Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient " $\varepsilon/2^m$ " argument, occasionally used throughout measure theory to construct a measurable set having a special property.

Let  $(X, \mu)$  be a finite measure space and let  $f_n : X \rightarrow \overline{\mathbb{R}}$  be a sequence of measurable functions such that  $f_n \rightarrow f$  a.e. For each positive integer  $m$ , which indexes the tolerance  $1/m$ , consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty} \{x : |f_i(x) - f(x)| < \frac{1}{m}\}.$$

- (a)  $E_n^m$  converges to a full set for each  $m$ .
- (b) For every  $\varepsilon > 0$  there is a measurable  $K \subset X$  such that  $\mu(X \setminus K) < \varepsilon$  and for each  $m$  there is finite  $n$  satisfying  $K \subset E_n^m$ .
- (c) For every  $\varepsilon > 0$  there is a measurable  $K \subset X$  such that  $\mu(X \setminus K) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $K$ .

*Proof.* (a) Recall that the a.e. convergence  $f_n \rightarrow f$  means that for every fixed  $m$  the intersection

$$\bigcap_{n=1}^{\infty} (X \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \frac{1}{m}\}$$

is a null set. Since  $\mu(X) < \infty$ , it is equivalent to  $E_n^m$  converges to a full set for each  $m$  by the continuity from above.

(b) For each  $m$ , we can find  $n_m$  such that

$$\mu(X \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(X \setminus K) = \mu\left(\bigcup_{m=1}^{\infty} (X \setminus E_{n_m}^m)\right) \leq \sum_{m=1}^{\infty} \mu(X \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix  $m > 0$ . Since  $n \geq n_m$  implies  $K \subset E_{n_m}^m \subset E_n^m$ , we have

$$n \geq n_m \Rightarrow \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \quad \square$$

## Exercises

**2.6 (Cauchy's functional equation).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Cauchy's functional equation refers to the equation  $f(x + y) = f(x) + f(y)$ , satisfied for all  $x, y \in \mathbb{R}$ . Suppose  $f$  satisfies the Cauchy functional equation. We ask if  $f$  is linear, that is  $f(x) = ax$  for all  $x \in \mathbb{R}$ , where  $a := f(1)$ .

- (a)  $f(x) = ax$  for all  $x \in \mathbb{Q}$ , but there is a nonlinear solution of Cauchy's functional equation.
- (b) If  $f$  is continuous at a point, then  $f$  is linear.
- (c) If  $f$  is Lebesgue measurable, then  $f$  is linear.

**2.7 (Pointwise approximation by simple functions).** Let  $(X, \mu)$  be a measure space and  $X$  a metric space with Borel measurable structure. By a *simple function* we mean a measurable function  $s : X \rightarrow X$  of finite image.

- (a) For each open set  $U \subset X$  there is a sequence of open sets  $U_i$  such that  $U = \bigcup_i U_i$  and  $\overline{U_i} \subset U$ . Let  $f : X \rightarrow X$  be any function.
- (b) If  $f$  is the pointwise limit of a sequence of measurable functions, then  $f$  is measurable.
- (c) If  $f$  is measurable, then  $f$  is the pointwise limit of a sequence of simple functions, if  $X$  is separable.
- \*(d) The pointwise limit of a net of simple functions may not be measurable.

*Proof.* (b) Suppose a sequence  $(f_n)_n$  of measurable functions converges pointwisely to a function  $f$ . For fixed open  $U \subset X$  we claim

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} \liminf_{n \rightarrow \infty} f_n^{-1}(U_i).$$

If it is true, then  $f^{-1}(U)$  is the countable set operation of measurable sets  $f_n^{-1}(U_i)$ . Let  $U_i$  be the sequence associated to  $U$  taken by the part (a).

( $\subset$ ) If  $\omega \in f^{-1}(U)$ , then for some  $i$  we have  $f(\omega) \in U_i$ , so  $f_n(\omega)$  is eventually in  $U_i$ , thus we have  $\omega \in \liminf_{n \rightarrow \infty} f_n^{-1}(U_i)$ .

( $\supset$ ) If  $\omega \in \liminf_{n \rightarrow \infty} f_n^{-1}(U_i)$  for some  $i$ , then  $f_n(\omega)$  is eventually in  $U_i$ , so  $f(\omega) \in \overline{U_i} \subset U$ , thus we have  $\omega \in f^{-1}(U)$ .

(c) Suppose there is a increasing sequence of finite tagged partitions  $\mathcal{P}_n \subset \mathcal{B}$  satisfying the following property: for each open-neighborhood pair  $(x, U)$  there is  $n$  and  $i$  such that  $P_{n,i} \in \mathcal{P}_n$  and  $x \in P_{n,i} \subset U$ . We denote the tags by  $t_{n,i} \in P_{n,i}$  for each  $P_{n,i} \in \mathcal{P}_n$ . Define

$$s_n(\omega) := t_{n,i} \quad \text{for } f(\omega) \in P_{n,i}.$$

To show  $s_n(\omega) \rightarrow f(\omega)$ , fix an open  $f(\omega) \in U \subset X$ . Then, there is  $n_0$  such that there is a sequence  $(P_{n,i_n})_{n=n_0}^{\infty}$  satisfying  $P_{n,i_n} \in \mathcal{P}_n$  and  $f(\omega) \in P_{n,i_n} \subset U$ . Then, for all  $n \geq n_0$ , we have for  $f(\omega) \in P_{n,i_n}$  that  $s_n(\omega) = t_{n,i_n} \in P_{n,i_n} \subset U$ .

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem. □

## Chapter 3

# Lebesgue integral

### 3.1 Monotone convergence theorem

**3.1** (Lebesgue integral of non-negative functions). Let  $(X, \mu)$  be a measure space. Let  $f : X \rightarrow \mathbb{R}_{\geq 0}$  be a measurable function. The *Lebesgue integral* of  $f$  is defined by

$$\int f \, d\mu := \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, s \text{ simple} \right\}$$

**3.2** (Monotone convergence theorem). Let  $(X, \mu)$  be a measure space. Let  $f_n$  and  $f$  be measurable functions  $X \rightarrow \mathbb{R}_{\geq 0}$ .

- (a)  $E \mapsto \int_E f \, d\mu$  is a measure if  $f$  is simple.
- (b)  $E \mapsto \int_E f \, d\mu$  is a measure even if  $f$  is not simple.
- (c) If  $f_n \uparrow f$  a.e., then  $\int f_n \rightarrow \int f$ .

*Proof.* (a) Clear from the linearity of the integral for simple functions.

(b) For  $E_n \uparrow E$ , we want to show the continuity from below,  $\int_{E_n} f \rightarrow \int_E f$ . Take  $\varepsilon > 0$ . We introduce a continuous bijection  $\beta : [0, \infty] \rightarrow [0, 1] : t \mapsto t/(1+t)$  to avoid dividing the cases for infinity. By the definition of the Lebesgue integral, we have a simple function  $s$  such that  $0 \leq s \leq f$  and

$$\beta\left(\int_E f\right) - \beta\left(\int_E s\right) < \varepsilon,$$

whether or not  $\int_E f$  diverges. Then,

$$\begin{aligned} \beta\left(\int_E f\right) - \beta\left(\int_{E_n} f\right) &= [\beta\left(\int_E f\right) - \beta\left(\int_E s\right)] + [\beta\left(\int_E s\right) - \beta\left(\int_{E_n} s\right)] + [\beta\left(\int_{E_n} s\right) - \beta\left(\int_{E_n} f\right)] \\ &< \varepsilon + [\beta\left(\int_E s\right) - \beta\left(\int_{E_n} s\right)] + 0 \\ &\xrightarrow{n \rightarrow \infty} \varepsilon \end{aligned}$$

by the part (a). We are done by letting  $\varepsilon \rightarrow 0$ .

(c) Define  $E_n := \{x : f(x) < (1+\varepsilon)f_n(x)\}$ , which converges to a full set because  $f_n \rightarrow f$  a.e. Since  $f$  is a measure, we can choose  $N$  such that

$$\beta\left(\int_E f\right) - \beta\left(\int_{E_N} f\right) < \varepsilon.$$

With this  $N$ , we have

$$\beta\left(\int_{E_N} f\right) - \beta\left(\int_{E_N} f_n\right) \leq \frac{\int_{E_N} f - \int_{E_N} f_n}{(1 + \int_{E_N} f)(1 + \int_{E_N} f_n)} < \varepsilon, \quad n \geq N.$$

Then, we have for  $n \geq N$  that

$$\begin{aligned}\beta(\int_E f) - \beta(\int_E f_n) &= [\beta(\int_E f) - \beta(\int_{E_N} f)] + [\beta(\int_{E_N} f) - \beta(\int_{E_N} f_n)] + [\beta(\int_{E_N} f_n) - \beta(\int_E f_n)] \\ &< 0 + \varepsilon + \varepsilon,\end{aligned}$$

so we are done by letting  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ .  $\square$

**3.3** (Corollaries of monotone convergence theorem). Fatou's lemma, linearity of the integral,  $f \geq 0$  and  $\int f = 0$  imply  $f = 0$  a.e.

## 3.2 Dominated convergence theorem

**3.4** (Lebesgue integral of complex-valued functions).

**3.5** (Bounded convergence theorem). Semifinite measures

(a)

$$\sup_{g \leq f} \int g \, d\mu = \int f \, d\mu$$

where  $g$  runs through bounded measurable functions.

(b)

## 3.3 Product measures

**3.6** (Fubini-Tonelli theorem). Lebesgue measure on Euclidean spaces

## Exercises

**3.7** (Convergence of one-parameter family).

If  $\|f_n\|_{L^2([0,1])} \leq C$  and  $f_n \rightarrow f$  almost everywhere, then  $f_n \rightarrow f$  weakly.

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_0^1 n^3 x^2 (1-x)^n \, dx &= 2 \neq 0 = \int_0^1 \lim_{n \rightarrow \infty} n^3 x^2 (1-x)^n \, dx. \\ \lim_{n \rightarrow \infty} \int_0^\infty n^2 e^{-nx} \, dx &= \infty \neq 0 = \int_0^\infty \lim_{n \rightarrow \infty} n^2 e^{-nx} \, dx.\end{aligned}$$

## **Part II**



## Chapter 4

# Signed measures

### 4.1 Radon-Nikodym theorem

An integrable function as a measure  $\sigma$ -finite measures

# Chapter 5

## Borel measures

### 5.1 Continuous functions on metric spaces

Urysohn and Tietze.

**5.1** (Regular Borel measures on metric spaces). Let  $\mu$  be a Borel measure on a metric space  $X$ . We say  $\mu$  is *outer regular* if

$$\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\},$$

and say  $\mu$  is *inner regular* if

$$\mu(E) = \sup\{\mu(F) : F \subset E, F \text{ closed}\},$$

for every Borel subset  $E \subset X$ . If  $\mu$  is both outer and inner regular, we say  $\mu$  is *regular*.

- (a) Let  $E$  be  $\sigma$ -finite. Then,  $E$  is  $\mu$ -regular if and only if for any  $\varepsilon > 0$  there are open  $U$  and closed  $F$  such that  $F \subset E \subset U$  and  $\mu(U \setminus F) < \varepsilon$ .
- (b) If  $\mu$  is  $\sigma$ -finite, then the set of  $\mu$ -regular subsets is a  $\sigma$ -algebra. (may be extended?)
- (c) Every closed set is  $G_\delta$ .
- (d) Every finite Borel measure on  $X$  is regular.

*Proof.*

□

**5.2** (Luzin's theorem). Let  $\mu$  be a regular Borel measure on a metric space  $X$ . Let  $f : X \rightarrow \mathbb{R}$  be a Borel measurable function. Two proofs: direct and Egoroff.

- (a) If  $E \subset X$  is  $\sigma$ -finite, then there is a continuous  $g$  blabla
- (b) If  $f$  vanishes outside a  $\sigma$ -finite set, then for any  $\varepsilon > 0$  there is a closed set  $F \subset X$  such that  $f|_F : F \rightarrow \mathbb{R}$  is continuous and  $\mu(X \setminus F) < \varepsilon$ .
- (c) If  $f$  vanishes outside a  $\sigma$ -finite set, then for any  $\varepsilon > 0$  there is a closed set  $F \subset X$  and continuous  $g : X \rightarrow \mathbb{R}$  such that  $f|_F = g|_F$  and  $\mu(X \setminus F) < \varepsilon$ .
- (d) If  $f$  is further bounded, then  $g$  also can be taken to be bounded.

*Proof.* (a) Let  $\varepsilon > 0$  and suppose  $E \subset X$  is measurable with  $\mu(E) < \infty$ . Since  $E$  is  $\sigma$ -finite, we have open  $U$  and closed  $F$  such that  $F \subset E \subset U$  and  $\mu(U \setminus F) < \varepsilon/2$ . By the Urysohn lemma, there is a continuous function  $g : X \rightarrow [0, 1]$  such that  $g|_{U^c} = 0$  and  $g|_F = 1$ . Then,

$$\int |\mathbf{1}_E - g| d\mu = \int_{U \setminus F} |\mathbf{1}_E - g| d\mu \leq 2\mu(U \setminus F) < \varepsilon.$$

(b) Since  $\mathbb{R}$  is second countable, we have a base  $(V_n)_{n=1}^\infty$  of  $\mathbb{R}$ . Since  $\mu$  is  $\sigma$ -finite, for each  $n$  we can take open  $U_n$  and closed  $F_n$  such that

$$F_n \subset f^{-1}(V_n) \subset U_n$$

and  $\mu(U_n \setminus F_n) < \varepsilon/2^n$ . Define  $F := (\bigcup_{n=1}^\infty (U_n \setminus F_n))^c$  so that  $\mu(X \setminus F) < \varepsilon$  and  $F$  is closed. Then,

$$\begin{aligned} U_n \cap F &= U_n \cap ((U_n^c \cup F_n) \cap F) \\ &= (U_n \cap (U_n^c \cup F_n)) \cap F \\ &= (\emptyset \cup (U_n \cap F_n)) \cap F \\ &\subset F_n \cap F \end{aligned}$$

proves  $f^{-1}(V_n)$  is open in  $F$  for every  $n$ , hence the continuity of  $f|_F$ . (In fact, we require that  $X$  to be just a topological space.)

(b') We can alternatively use the part (a) and the Egoroff theorem. By the part (a), we can construct a sequence  $(f_n)$  of continuous functions  $X \rightarrow \mathbb{R}$  such that  $f_n \rightarrow f$  in  $L^1$ . By taking a subsequence, we may assume  $f_n \rightarrow f$  pointwise. Assuming  $\mu$  is finite, by the Egorov theorem, there is a measurable  $A \subset X$  such that  $f_n \rightarrow f$  uniformly on  $A$  and  $\mu(X \setminus A) < \varepsilon/2$ . Since  $\mu$  is inner regular, we have closed  $F \subset A$  such that  $\mu(A \setminus F) < \varepsilon/2$ , so that we have  $\mu(X \setminus F) < \varepsilon$ . Then,  $f$  is continuous on  $A$ , and of course on  $F$ .

□

## 5.2 Locally compact spaces

compact closed set not containing infy

open open not containing infy

closed closed set containing infy

for a measure that “vanishes at infy” = tight  
two definitions of inner regularity is equivalent.

inner regular on compact sets  $\rightarrow$  inner regular on closed sets  
inner regular on compact sets + sigma finite  $\rightarrow$  tight

### 5.3 (One-point compactification).

### 5.4 (Regular Borel measures on locally compact metric spaces). sss

(a)  $C_c(X)$  is dense in  $L^p(\mu)$  for  $1 \leq p < \infty$ .

(b) If  $\mu$  is  $\sigma$ -finite, then for any  $\varepsilon > 0$  there is compact  $K \subset X$  and continuous  $g : X \rightarrow \mathbb{R}$  such that  $f|_K = g|_K$  and  $\mu(X \setminus K) < \varepsilon$ .

**5.5 (Tightness and inner regularity).** We have a similar but confusing concept called tightness; we say a Borel measure  $\mu$  on a topological space  $X$  is *tight* if for any  $\varepsilon > 0$  there is a compact  $K \subset X$  such that  $\mu(X \setminus K) < \varepsilon$ .

History of Bourbaki's text.

(a)

### 5.3 Riesz-Markov-Kakutani representation theorem

5.6 (Riesz-Markov-Kakutani representation theorem for  $C_0$ ). Let  $X$  be a locally compact metric space. We want to establish the following one-to-one correspondence:

$$\begin{array}{ccc} \{\text{finite Borel measures on } X\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_0(X)\} \\ \mu & \mapsto & (f \mapsto \int f d\mu). \end{array}$$

Let  $I$  a positive linear functional on  $C_0(X)$ . Let  $\mathcal{T}$  be the set of all open subsets of  $X$  and  $\rho : \mathcal{T} \rightarrow [0, \infty]$  a set function such that

$$\rho(U) := \sup \{I(f) : f \in C_c(U, [0, 1])\}$$

for open  $U$ . Let  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  be the associated outer measure defined from  $\rho$ , and  $\mu := \mu^*|_{\mathcal{A}}$  the Carathéodory measure, where  $\mathcal{A}$  is the  $\sigma$ -algebra of Carathéodory measurable subsets relative to  $\mu^*$ , and  $\mathcal{B}$  is the Borel  $\sigma$ -algebra of  $X$ .

- (a)  $\mu^*|_{\mathcal{T}} = \rho$ .
- (b)  $\mathcal{B} \subset \mathcal{A}$ .
- (c)  $I(f) = \int f d\mu$  for  $f \in C(X)$ , i.e. the map given above is surjective.
- (d) The map given above is injective.

*Proof.* (a) It suffices to show that  $\rho$  satisfies monotonically countably subadditive. Take an open set  $U$  and a countable open cover  $\{U_i\}_{i=1}^{\infty}$  of  $U$ . Take any  $f \in C_c(U, [0, 1])$  and let  $K := \text{supp } f$ . Since  $K$  is compact, there is a finite subcover  $\{U_j\}_{j=1}^n$  of  $K$ , and since  $K$  is paracompact Hausdorff, there is a partition of unity  $\{\chi_j\}_j$  on  $K$  subordinate to the open cover  $\{U_j \cap K\}_j$ . Note that  $\text{supp } \chi_j \subset U_j \cap K$  for each  $j$ .

The set  $\text{supp}(f \chi_j)$  is closed in  $K$  so the compactness, and we also have the inclusion  $\text{supp}(f \chi_j) \subset \text{supp } \chi_j \subset U_j$ . For every  $0 < a \leq 1$ , since  $(f \chi_j)^{-1}((a, 1])$  is open in the interior of  $K$  and  $(f \chi_j)^{-1}([a, 1])$  is closed in  $K$ ,  $f \chi_j$  is continuous on  $U_j$ . Now we have checked  $f \chi_j \in C_c(U_j, [0, 1])$ .

Then, because  $I$  is linear so that it preserves finite sum, we have

$$I(f) = I\left(\sum_{j=1}^n f \chi_j\right) = \sum_{j=1}^n I(f \chi_j) \leq \sum_{j=1}^n \rho(U_j) \leq \sum_{i=1}^{\infty} \rho(U_i).$$

Since  $f$  is arbitrary, we get  $\rho(U) \leq \sum_{i=1}^{\infty} \rho(U_i)$ .

(b) It suffices to show  $\mathcal{T} \subset \mathcal{A}$ . Clearly  $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \setminus U)$  for any measurable  $E$  and open  $U$ . For the opposite direction, take  $\varepsilon > 0$ . Note that we may assume  $\mu^*(E) < \infty$ . There are open  $U_i$  such that  $E \subset \bigcup_{i=1}^{\infty} U_i$  and

$$\mu^*(E) + \frac{\varepsilon}{3} > \sum_{i=1}^{\infty} \rho(U_i).$$

Take  $f_i \in C_c(U_i \cap U, [0, 1])$  such that

$$\rho(U_i \cap U) - \frac{1}{3} \cdot \frac{\varepsilon}{2^i} < I(f_i),$$

and take  $g_i \in C_c(U_i \setminus \text{supp } f_i, [0, 1])$  such that

$$\rho(U_i \setminus \text{supp } f_i) - \frac{1}{3} \cdot \frac{\varepsilon}{2^i} < I(g_i).$$

Then, since  $f_i + g_i \in C_c(U_i, [0, 1])$ , we have

$$\begin{aligned} \rho(U_i) &\geq I(f_i + g_i) > \rho(U_i \cap U) + \rho(U_i \setminus \text{supp } f_i) - \frac{2}{3} \cdot \frac{\varepsilon}{2^i} \\ &\geq \rho(U_i \cap U) + \rho(U_i \setminus U) - \frac{2}{3} \cdot \frac{\varepsilon}{2^i}. \end{aligned}$$

It implies

$$\mu^*(E) + \varepsilon > \sum_{i=1}^{\infty} \rho(U_i \cap U) + \sum_{i=1}^{\infty} \rho(U_i \setminus U) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

because  $E \cap U \subset \bigcup_{i=1}^{\infty} U_i \cap U$  and  $E \setminus U \subset \bigcup_{i=1}^{\infty} U_i \setminus U$ .

(c) Note that we have

$$\rho(U) = \sup_{f \in C_c(U, [0, 1])} I(f), \quad \mu(E) = \inf_{\substack{E \subset U \\ U \text{ open}}} \rho(U).$$

We first claim that for  $g \in C_c(X, [0, 1])$ , if  $K$  and  $K'$  are compact sets such that  $g|_K = 1$  and  $g|_{K'^c} = 0$  respectively, then we have

$$\mu(K) \leq I(g) \leq \mu(K').$$

The one inequality directly follows from

$$I(g) \leq \inf_{K' \subset U} \rho(U) = \mu(K').$$

For the other, take sufficiently small  $\varepsilon > 0$  such that  $U := g^{-1}((1 - \varepsilon, 1])$  satisfies  $K \subset U \subset \text{supp } g$ . For any  $h \in C_c(U, [0, 1])$ , the inequality  $(1 - \varepsilon)h \leq g$  implies  $I(h) \leq (1 - \varepsilon)^{-1}I(g)$ , so

$$\mu(K) \leq \rho(U) \leq I(h) \leq (1 - \varepsilon)^{-1}I(g).$$

By limiting  $\varepsilon \rightarrow 0$ , we get  $\mu(K) \leq I(g)$ , the claim proved.

Since  $C_c(X)$  is the linear span of  $C_c(X, [0, 1])$ , it is enough to show  $I(f) = \int f d\mu$  for  $f \in C_c(X, [0, 1])$ . For a fixed positive integer  $n$  and for each index  $1 \leq i \leq n$ , let  $K_i := f^{-1}([i/n, 1])$  and define

$$f_i(x) := \begin{cases} 0 & \text{if } x \in K_{i-1}^c, \\ f(x) - \frac{i-1}{n} & \text{if } x \in K_{i-1} \setminus K_i, \\ \frac{1}{n} & \text{if } x \in K_i, \end{cases}$$

where  $K_0 := \text{supp } f$ . Note that  $nf_i \in C_c(X, [0, 1])$  and  $f = \sum_{i=1}^n f_i$ . For  $1 \leq i \leq n$  we have  $\mu(K_i) < \infty$  because  $K_i$  is compact subsets contained in a locally compact Hausdorff space  $U := f^{-1}((0, 1])$ , but  $\mu(K_0)$  is possibly infinite. By the previous claim and the property of integral, we have

$$\frac{\mu(K_i)}{n} \leq I(f_i), \quad \frac{\mu(K_i)}{n} \leq \int f_i d\mu$$

for  $1 \leq i \leq n$  and

$$I(f_i) \leq \frac{\mu(K_{i-1})}{n}, \quad \int f_i d\mu \leq \frac{\mu(K_{i-1})}{n}$$

for  $2 \leq i \leq n$ . Then, using the above inequalities and  $\mu(K_n) \geq 0$ , we have

$$I(f) \leq I(f_1) + \int f d\mu \quad \text{and} \quad \int f d\mu \leq \int f_1 d\mu + I(f).$$

Note that  $f_1 = \min\{f, 1/n\}$  is a sequence of functions indexed by  $n$ . By the monotone convergence theorem,  $\int f_1 d\mu \rightarrow 0$  as  $n \rightarrow \infty$ . We now show  $I(f_1)$  converges to zero.

(d) Let  $\mu$  and  $\nu$  be finite Borel measures on  $X$  such that

$$\int g d\mu = \int g d\nu$$

for all  $g \in C(X)$ . Let  $E$  be any measurable set. Since  $\mu + \nu$  is a finite Borel measure, it is regular, and by the Luzin theorem, we have a closed set  $F$  and  $g \in C(X)$  with  $0 \leq g \leq 1$  such that  $\mathbf{1}_E|_F = g|_F$  and  $(\mu + \nu)(X \setminus F) < \varepsilon/2$ . Then,

$$\begin{aligned} |\mu(E) - \nu(E)| &= \left| \int \mathbf{1}_E d\mu - \int \mathbf{1}_E d\nu \right| \\ &\leq \int_{X \setminus F} |\mathbf{1}_E - g| d\mu + \int_{X \setminus F} |g - \mathbf{1}_E| d\nu \\ &\leq 2\mu(X \setminus F) + 2\nu(X \setminus F) < \varepsilon. \end{aligned}$$

By limiting  $\varepsilon \rightarrow 0$ , we have  $\mu(E) = \nu(E)$ . □

5.7 (Dual of continuous function spaces).

## 5.4 Hausdorff measures

### Exercises

## Chapter 6

# Lebesgue spaces

### 6.1 $L^p$ spaces

6.1 (Hölder inequality).

*Proof.*

$$\int f g \leq C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take  $C$  such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \left( \int |f|^p \right)^{\frac{1}{p}} \left( \int |g|^q \right)^{\frac{1}{q}}.$$

Note that we can show that  $1 \leq 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \leq 2$  and the minimum is attained only if  $p = q = 2$ , so this method does not provide the sharpest constant.  $\square$

### 6.2 $L^1$ spaces

6.2 (Convolution?).

6.3 (Approximate identity?).

6.4 (Continuity of translation?).

### 6.3 $L^2$ spaces

### 6.4 $L^\infty$ spaces

## **Part III**

# **Distribution theory**



## **Chapter 7**

# **Test functions**

## **Chapter 8**

# **Distributions**

## Chapter 9

# Linear operators

### 9.1 Boundedness

Translation and multiplication operators

9.1 (Bitranspose extension).

### 9.2 Kernels

9.2 (Schur test).

9.3 (Young's inequality of integral operators).

### 9.3 Convolution

9.4 (Approximation of identity). Fejér, Poisson, box?

9.5 (Summability methods).

## **Part IV**

# **Fundamental theorem of calculus**

## Chapter 10

# Absolute continuity

The space of weakly differentiable functions with respect to all variables  $= W_{\text{loc}}^{1,1}$ .

**10.1** (Product rule for weakly differentiable functions). We want to show that if  $u$ ,  $v$ , and  $uv$  are weakly differentiable with respect to  $x_i$ , then  $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$ .

(a) If  $u$  is weakly differentiable with respect to  $x_i$  and  $v \in C^1$ , then  $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$ .

**10.2** (Interchange of differentiation and integration). Let  $f : X_x \times X_y \rightarrow \mathbb{R}$  be such that  $\partial_{x_i}f$  is well-defined. Suppose  $f$  and  $\partial_{x_i}f$  are locally integrable in  $x$  and integrable  $y$ .

Then,

$$\partial_{x_i} \int f(x, y) dy = \int \partial_{x_i} f(x, y) dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if  $F$  is  $C^2$  then the *classical* partial differentiation commute, and if  $F$  is not  $C^2$  then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.

### 10.1 Absolutely continuity

(a)  $f$  is  $\text{Lip}_{\text{loc}}$  iff  $f'$  is  $L_{\text{loc}}^\infty$

(b)  $f$  is  $\text{AC}_{\text{loc}}$  iff  $f'$  is  $L_{\text{loc}}^1$

(a)  $f$  is  $\text{Lip}$  iff  $f'$  is  $L^\infty$

(b)  $f$  is  $\text{AC}$  iff  $f'$  is  $L^1$

(c)  $f$  is  $\text{BV}$  iff  $f'$  is a finite regular Borel measure

**10.3** (Absolute continuous measures).

**10.4** (Absolute continuous functions).

**10.5** (Functions of bounded variation).

### 10.2 Interpolations

weak  $L_p$ , marcinkiewicz

**Definition 10.2.1.** Let  $f$  be a measurable function on a measure space  $(X, \mu)$ . The *distribution function*  $\lambda_f : [0, \infty) \rightarrow [0, \infty)$  is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}) = \mu(|f| > \alpha).$$

Do not use  $\mu(\{x : |f(x)| \geq \alpha\})$ . The strict inequality implies the *lower semi-continuity* of  $\lambda_f$ .

For  $p > 0$ ,

$$\begin{aligned} \|f\|_{L^p}^p &= \int |f(x)|^p d\mu(x) \\ &= \int \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\ &= \int_0^\infty \int_{|f(x)| > \alpha} p\alpha^{p-1} d\mu(x) d\alpha \\ &= p \int_0^\infty \left[ \alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}} \right]^p \frac{d\alpha}{\alpha}. \end{aligned}$$

**Definition 10.2.2.**

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty \left[ \alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}} \right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0 < \alpha < \infty} \left[ \alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}} \right].$$

**Theorem 10.2.3.** For  $p \geq 1$  we have  $\|f\|_{p,\infty} \leq \|f\|_p$ .

*Proof.* By the Chebyshev inequality,

$$\sup_{0 < \alpha < \infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha) d\alpha = \|f\|_{L^p}^p.$$

□

**10.6 (Marcinkiewicz interpolation).** Let  $X$  be a  $\sigma$ -finite measure space and  $Y$  be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator  $T : L^{p_0}(X) + L^{p_1}(X) \rightarrow M(Y)$  has two weak-type estimates

$$\|T\|_{L^{p_0}(X) \rightarrow L^{p_0,\infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X) \rightarrow L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X) \rightarrow L^p(Y)} < \infty.$$

*Proof.* Let  $f \in L^p(X)$  and denote  $f_h = \chi_{|f| > \alpha} f$  and  $f_l = \chi_{|f| \leq \alpha} f$ . It is easy to show  $f_h \in L^{p_0}$  and  $f_l \in L^{p_1}$ . Then,

$$\begin{aligned} \|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha) \frac{d\alpha}{\alpha} \\ &\lesssim \int \alpha^p \cdot \mu(|Tf_h| > \alpha) \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha) \frac{d\alpha}{\alpha} \\ &\leq \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0,\infty}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{p_1}} \|Tf_l\|_{L^{p_1,\infty}}^{p_1} \frac{d\alpha}{\alpha} \\ &\lesssim \int \alpha^{p-p_0} \|f_h\|_{L^{p_0}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{L^{p_1}}^{p_1} \frac{d\alpha}{\alpha} \\ &\sim \|f\|_p^p. \end{aligned}$$

by (1) Fubini, (2) Sublinearity, (3) Chebyshev, (4) Boundedness, (5) Fubini.

□

**10.7** (Hadamard's three line lemma). Let  $f$  be a bounded holomorphic function on vertical unit strip  $\{z : 0 < \operatorname{Re} z < 1\}$  which is continuously extended to the boundary. Then, for  $0 < \theta < 1$  we have

$$\|f\|_{L^\infty(\operatorname{Re}=\theta)} \leq \|f\|_{L^\infty(\operatorname{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\operatorname{Re}=1)}^\theta.$$

*Proof.* Fix  $n$  and define

$$g_n(z) := \frac{f(z)}{\|f\|_{L^\infty(\operatorname{Re}=0)}^{1-z} \|f\|_{L^\infty(\operatorname{Re}=1)}^z} e^{-\frac{z(1-z)}{n}}.$$

Then,

$$|g_n(z)| \leq e^{-\frac{(\operatorname{Im} z)^2}{n}}$$

for  $z$  in the strip. By the maximum principle,

$$|f(z)| \leq \|f\|_{L^\infty(\operatorname{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\operatorname{Re}=1)}^\theta e^{\frac{y^2}{n}}.$$

Letting  $n \rightarrow \infty$ , we are done. □

**10.8** (Riesz-Thorin interpolation). Let  $X, Y$  be  $\sigma$ -finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta) \frac{1}{p_0} + \theta \frac{1}{p_1}, \quad \frac{1}{q_\theta} = (1-\theta) \frac{1}{q_0} + \theta \frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \rightarrow q_\theta} \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta.$$

*Proof.* Note that

$$\|T\|_{p_\theta \rightarrow q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f, g} \frac{|\langle Tf, g \rangle|}{\|f\|_{p_\theta} \|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z \rangle = \int \overline{g_z(y)} Tf_z(y) dy,$$

where  $f_z$  and  $g_z$  are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z) + \frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have  $f_\theta = f$  and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for  $\operatorname{Re} z = x$ .

Then,

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0} \|f_z\|_{p_0} \|g_z\|_{q'_0} = \|T\|_{p_0 \rightarrow q_0} \|f\|_{p_\theta}^{p_\theta/p_0} \|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for  $\operatorname{Re} z = 0$ , and

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_1 \rightarrow q_1} \|f_z\|_{p_1} \|g_z\|_{q'_1} = \|T\|_{p_1 \rightarrow q_1} \|f\|_{p_\theta}^{p_\theta/p_1} \|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for  $\operatorname{Re} z = 1$ . By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta \|f\|_{p_\theta} \|g\|_{q'_\theta}$$

for  $\operatorname{Re} z = \theta$ . Putting  $z = \theta$  in the last inequality, we get the desired result. □

# Chapter 11

## Lebesgue differentiation theorem

### 11.1 Hardy-Littlewood maximal function

Let  $T_m$  be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m, \varepsilon: |\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

**11.1 (Hardy-Littlewood maximal function).** The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

**11.2 (Weak type estimate).**

$$\|Mf\|_{1,\infty} \leq 3^d \|f\|_{L^1(X)}.$$

(a) Proof by covering lemma.

*Proof.* (a) By the inner regularity of  $\mu$ , there is a compact subset  $K$  of  $\{|Mf| > \lambda\}$  such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every  $x \in K$ , since  $|Mf(x)| > \lambda$ , we can choose an open ball  $B_x$  such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover  $\{B_i\}_i$  of  $K$ . Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection  $\{B_k\}_k$  such that

$$K \subset \bigcup_i B_i \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \leq \sum_k 3^d \mu(B_k) \leq \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \leq \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of  $B_k$ 's. □



**11.3** (Radially bounded approximate identity). If an approximate identity  $K_n$  is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every  $n$  and  $x$ , hence has a weak type estimate.

**11.4** (Almost everywhere convergence of operators). Suppose  $T_m$  is a sequence of linear operators such that the maximal function  $T^*f$  is dominated by  $Mf$ . If  $f \in L^1(X)$  and  $T_m g \rightarrow g$  pointwise for  $g \in C(X)$ , then  $T_m f \rightarrow f$  a.e.

*Proof.* Take  $\varepsilon > 0$  and  $g \in C(X)$  such that  $\|f - g\|_{L^1(X)} < \varepsilon$ . Since  $T_m g(x) \rightarrow g(x)$  pointwise, we have

$$\begin{aligned} & \mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\ & \leq \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \frac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \frac{\lambda}{2}\}) \\ & \leq \mu(\{x : M(f - g)(x) > \frac{\lambda}{2}\}) + \frac{2}{\lambda} \|f - g\|_{L^1(X)} \\ & \lesssim \frac{1}{\lambda} \varepsilon \end{aligned}$$

for every  $\lambda > 0$ . Limiting  $\varepsilon \rightarrow 0$ , we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every  $\lambda > 0$ , hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

**Definition 11.1.1.**

$$f^*(x) := \lim_{r \rightarrow 0^+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| dy.$$

**Theorem 11.1.2** (Lebesgue differentiation).  $f^* = 0$  a.e.

*Proof.* Note that  $f^* \leq Mf + |f|$  implies

$$\|f^*\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that  $g^* = 0$  for  $g \in C_c$ . Approximate using  $f^* = (f - g)^*$ .

□

## Exercises

**11.5** (Doubling measure).