Algebraic Topology

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7.1 (Structure groups). Let G be a topological group and F be a left G-space, and $p : E \to B$ be a fiber bundle with fiber F. We say an atlas $\{\varphi_i : p^{-1}(U_i) \to U_i \times F\}_i$ is a *G-atlas* if there is a set $\{g_{ij} : U_i \cap U_j \to G\}_{i,j}$ of maps such that the transition maps are given by

$$\varphi_j \circ \varphi_i^{-1}(b,f) = (b, g_{ij}(b)f), \qquad b \in U_i \cap U_j,\ f \in F.$$

A *G-bundle* with fiber F is a fiber bundle $p : E \to B$ that admits a G-atlas. In this case the group G is called the *structure group* of the fiber bundle. A *G-bundle map* is a bundle map $(\widetilde{u}, u) : (E,B) \to (E',B')$ with a set $\{h_{ij'} : U_i \cap u^{-1}(U'_{j'}) \to G\}_{i,j'}$ such that

$$\varphi'_{j'} \circ \widetilde{u} \circ \varphi_i^{-1}(b,f) = (u(b), h_{ij'}(b)f), \qquad b \in U_i \cap u^{-1}(U'_{j'}),\ f \in F.$$

If $B = B'$, a G-bundle map over B is a G-bundle map $(\widetilde{u}, u)$ such that $u = \mathrm{id}_B$. We denote by $\mathbf{Bun}_F(B)$ the category of G-bundles over B with fiber F.

(a) If F is a locally compact and locally connected Hausdorff space, then every fiber bundle with fiber F is a $\mathrm{Homeo}(F)$-bundle, where $\mathrm{Homeo}(F)$ is the group of autohomeomorphism group with compact-open topology.

(b) A G-bundle map $(\widetilde{u}, u)$ is an isomorphism if and only if u is a homeomorphism.

(c) A bundle map $(\widetilde{u}, \mathrm{id}_B) : (E,B) \to (E',B)$ is a G-bundle map if and only if there is a set $\{h_i : U_i \to G\}_i$ such that

$$\varphi'_i \circ \widetilde{u} \circ \varphi_i^{-1}(b,f) = (b, h_i(b)f), \qquad b \in U_i,\ f \in F,$$

where $\{U_i\}$ is an open cover over which both E and E' are trivialized.

Proof. (a)

(b) ($\Rightarrow$) Clear.

($\Leftarrow$) The total map $\widetilde{u}$ is continuous bijection because u is a bijection, so it suffices to show $\widetilde{u}^{-1}$ is continuous. Fix $U_i \subset B$ and $U'_{j'} \subset B'$. By substitution of $b' := u(b)$, $f' := h_{ij'}(b)f$, we can write

$$\varphi_i \circ \widetilde{u}^{-1} \circ \varphi'^{-1}_{j'}(b', f') = (u^{-1}(b'), h_{ij'}(u^{-1}(b'))^{-1} f').$$

Since the local trivializations, the inverse operation of G, and the inverse u^{-1} are all continuous, $\widetilde{u}^{-1}$ is also continuous. $\square$

7.2 (Fiber bundle construction theorem)**.** Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 1-cocyle* on $\mathcal{U}$ with coefficients in G is a set $\{g_{ij} : U_i \cap U_j \to G\}_{i,j}$ of maps such that the following *cocycle condition* holds:

$$g_{ik}(b) = g_{jk}(b)g_{ij}(b), \qquad b \in U_i \cap U_j \cap U_k.$$

The set of Čech 1-cocycles on $\mathcal{U}$ with coeffients in G is denoted by $\check{Z}^1(\mathcal{U}, G)$.

Let $g \in \check{Z}^1(\mathcal{U}, G)$ be a Čech 1-cocycle on $\mathcal{U}$. We will construct a G-bundle with fiber F for any left G-space F, which is trivialized over $\mathcal{U}$ in which the transition maps are given by $\{g_{ij}\}$. Define

$$E := \left(\coprod_i (U_i \times F) \right) / \sim,$$

where $\sim$ is an equivalence relation generated by

$$(b, f, i) \sim (b, g_{ij}(b)f, j), \qquad b \in U_i \cap U_j,\ f \in F.$$

Also define $p : E \to B : [b, f, i] \mapsto b$ and $\varphi_i^{-1} : U_i \times F \to p^{-1}(U_i) : (b, f) \mapsto [b, f, i]$, which are clearly continuous and surjective even without the cocycle condition.

(a) φ_i^{-1} is injective.

(b) φ_i^{-1} is open.

(c) The transition maps of the local trivialization $\{\varphi_i\}$ coincides with the cocycle $\{g_{ij}\}$.

Proof. (a) Suppose $\varphi_i^{-1}(b, f) = \varphi_i^{-1}(b', f')$. Since $(b, y, i) \sim (b', y', i)$, we have $b = b'$ and there is a sequence

$$f' = g_{i_{n-1}i_n}(b)g_{i_{n-2}i_{n-1}}(b) \cdots g_{i_0i_1}(b)f,$$

where $i_0 = i_n = i$. By applying the cocycle condition inductively, we obtain $f = f'$, which implies the injectivity of φ_i^{-1}.

(b) The map φ_i^{-1} factors through $\coprod_i (U_i \times F)$ such that

$$\varphi_i^{-1} : U_i \times F \to \coprod_i (U_i \times F) \xrightarrow{\pi} p^{-1}(U_i).$$

Since the canonical inclusion to disjoint union is open, it suffices to show the quotient map $\pi : \coprod_i (U_i \times F) \to E$ is open. Let $V \subset \coprod_i (U_i \times F)$ be open. Observe that

$$\pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open for each pair of i and j because it is exactly same as the inverse image of the open set $V \cap (U_i \times F)$ under the map

$$(U_i \cap U_j) \times F \subset U_j \times F \to U_i \times F : (b, f) \mapsto (b, g_{ij}(b)f).$$

Here we used the cocycle condition of $\{g_{ij}\}$. Therefore,

$$\pi^{-1}\pi(V) = \bigcup_{i,j} \pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open, hence the open π.

(c) Clear by the cocycle condition. $\square$

7.3 (Cohomologous transitions)**.** Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 0-cochain* on $\mathcal{U}$ with coefficients in G is a set $\{h_i : U_i \to G\}_i$ of maps. The group of Čech 0-cochains on $\mathcal{U}$ with coefficients in G is denoted by $\check{C}^0(\mathcal{U}, G)$.

The *first Čech cohomology group* of $\mathcal{U}$ with coefficients G is the orbit space of an action on $\check{Z}^1(\mathcal{U},G)$ by $\check{C}^0(\mathcal{U},G)$ defined as follows:

$$(hg)_{ij}(b) := h_j(b)g_{ij}(b)h_i(b)^{-1}, \qquad b \in U_i \cap U_j,$$

which is denoted by $\check{H}^1(\mathcal{U},G)$. We define the *first Čech cohomology group* of B with coefficients in G as the direct limit

$$\check{H}^1(B,G) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U},G).$$

Let F be a left G-space, and let $\mathrm{Bun}_F(B)$ be the set of isomorphism classes of G-bundles over B with fiber F.

(a) $\mathrm{Bun}_F(B) \to \check{H}^1(B,G)$ is well-defined.

(b) $\mathrm{Bun}_F(B) \to \check{H}^1(B,G)$ is surjective.

(c) $\mathrm{Bun}_F(B) \to \check{H}^1(B,G)$ is injective if F is faithful.

Proof. (a) Suppose $p : E_1 \to B$ and $p' : E' \to B$ be isomorphic G-bundles with fiber F. Let $u : E \to E'$ be a G-bundle isomorphism. By considering the refinement, we can find an open cover $\mathcal{U} = \{U_i\}_i$ of B on which E and E' are simultaneously locally trivialized.

$$\{g_{ij} : U_i \cap U_j \to G\}.$$

(b)

(c) □

7.4 (Principal bundles)**.** Let G be a topological group, and X be a left *principal homogeneous G-space*, i.e. a free and transitive left G-space such that the shear map $G \times X \to X \times X : (g,x) \mapsto (gx,x)$ is a homeomorphism.

A *principal G-bundle* is a G-bundle $p : P \to B$ with fiber X, often together with a fiber-preserving continuous right action $\rho : P \times G \to P$ such that each chart $\varphi_i : p^{-1}(U_i) \to U_i \times X$ induces a principal homogeneous right action on $\{b\} \times X \subset U_i \times X$ which commutes with the left action. The right action ρ is called the *principal right action* or *(global) gauge transformation*. Note that for each $b \in B$ the *fiber* $\{b\} \times X$ has commuting left and right actions, but the *fiber* $p^{-1}(b)$ can admit only the principal right action.

The category of principal G-bundles over B is denoted by $\mathbf{Prin}_G(B)$, and the morphisms are usually defined as right G-equivariant maps with respect to the pricipal right actions. Then, we may consider the forgetful functor $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$.

(a) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is fully faithful, i.e. a bundle map $u : P \to P'$ over B is a G-bundle map if and only if it is a right G-equivariant map.

(b) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is surjective, i.e. every G-bundle with fiber X has a principal right action.

(c) A principal bundle is trivial if it has a global section.

Proof. (a) Let $u : P \to P'$ be a G-bundle map over B so that there is a set $\{h_i : U_i \to G\}_i$ of maps such that

$$\varphi_i \circ u \circ \varphi_i^{-1}(b,x) = (b, h_i(b)x), \qquad b \in U_i,\ x \in X.$$

If we write $\rho_s : P \to P : e \mapsto \rho(e,s)$ for $s \in G$, then the induced right action $\varphi_i \circ \rho_s \circ \varphi_i^{-1}$ commutes with the left action $\varphi_i \circ u \circ \varphi_i^{-1}$ on $U_i \times X$. Now for every $e \in P_1$, we have

$$\begin{aligned}\rho_s \circ u(e) &= \varphi_i^{-1} \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(e)\\ &= \varphi_i^{-1} \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ \varphi_i(e)\\ &= u \circ \rho_s(e),\end{aligned}$$

therefore u is right G-equivariant.

Conversely, let $u : P \to P'$ be a right G-equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \to G$ such that

$$(b, g_i(b)x_0) := \varphi_i \circ u \circ \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xmapsto{\varphi_i \circ u \circ \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The continuity of the last map is due to the assumption that the map $(g, x) \mapsto (gx, x)$ is a homeomorphism.

Then, for every $(b, x) \in U_i \times X$ there is a unique $s \in G$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned}
\varphi_i \circ u \circ \varphi_i^{-1}(b, x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b, x_0) \\
&= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) g_i(b)(b, x_0) \\
&= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= g_i(b)(b, x) \\
&= (b, g_i(b)x).
\end{aligned}$$

Hence, u is a G-bundle map.

(b) Fix $x_0 \in X$ and consider the homeomorphism $G \to X : g \to gx_0$. Define a right action

$$X \times G \to X : (gx_0, s) \mapsto gx_0 s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X.

Define $\rho : P \times G \to P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b, the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge tranformation.

(c)

$\square$

7.5 (Associated bundles)**.**

$$\mathrm{Prin}_G(B) \xrightarrow{\sim} \mathrm{Bun}_X(B) \xrightarrow{\sim} \check{H}^1(B, G) \hookrightarrow \mathrm{Bun}_F(B)$$

can be given in a more simple way.

7.2 Classifying spaces

Let $\mathrm{Prin}_G(B)$ be the set of isomorphism classes of principal G-bundles. Then, we have a contravariant functor

$$\mathrm{Prin}_G : \mathbf{hTop}_{\mathrm{para}} \to \mathbf{Set}$$

such that there is a natural isomorphism between contravariant functors

$$[-, BG] \to \mathrm{Prin}_G.$$

7.6 (Homotopy properteis). Let $p : E \to B$ be a vector bundle

(a) If $p : E \to B \times [0, \frac{1}{2}]$ and $p' : E' \to B \times [\frac{1}{2}, 1]$ are trivial, then

(b) If $f, g : B' \to B$ are homotopic, then $f^*\xi \cong g^*\xi$.

7.7 (Finite type).

7.3 Reduction of structure groups

7.4 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

7.8 (Vector bundles). Let $p : E \to B$ and $p : E' \to B$ be vector bundles.

(a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \le n \le \infty$. If $f, g : B \to G_k(\mathbb{F}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{F}^n) \to G_k(\mathbb{F}^{2n})$ is the natural inclusion.

7.9. Riemannian and Hermitian metrics

Exercises

group quotient gives a principal G-bundle.

Hopf fibration(real, complex, quaternionic, but not octonianic)

Chapter 8

Characteristic classes

Chapter 9

K-theory

bott periodicity Hopf invariant

Part IV

Stable homotopy theory

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem