Contents

1 Nets of measurable functions

1.1. (a)

If f_α is continuous, then f is lower semi-continuous. We use the inner regularity of the measure on the open set $f^{-1}(j2^{-n}, \infty)$.

2 Potential from a source

Theorem. *Let $d \geq 3$. A distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity if and only if there is a distribution $\rho \in \mathcal{D}'(\mathbb{R}^d)$ such that $u = \Phi * \rho$ and $\operatorname{supp}(\rho) \subset \{0\}$, where Φ denotes the fundamental solution of Laplace's equation.*

Proof. ($\Rightarrow$) Define a distribution ρ by

$$\langle \rho, \varphi \rangle := -\langle u, \Delta\varphi \rangle$$

for $\varphi \in C_c^\infty(\mathbb{R}^d)$. In other words, $\rho = -\Delta u$ in distributional sense. Then, ρ has the support contained in $\{0\}$ because if $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ then

$$\langle \rho, \varphi \rangle = -\langle u, \Delta\varphi \rangle = -\int u(x)\Delta\varphi(x)\,dx = -\int \Delta u(x)\varphi(x)\,dx = 0.$$

Therefore, we only need to verify $u = \Phi * \rho$ to complete the proof.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Be cautious that the argument

$$\langle \Phi * \rho, \varphi \rangle = \langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle = \langle u, \varphi \rangle$$

fails to provide a proof because the function $\Phi * \rho$ is not compactly supported so that we cannot deduce $\langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$, and here we use the condition that u vanishes at infinity to justify the equality. Define a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{5}{4} \\ 0 & \text{if } |x| \geq \frac{7}{4} \end{cases}.$$

If we denote $\chi_r(x) := \chi(\frac{x}{r})$, then we have

$$\langle \rho, (\Phi\chi_r) * \varphi \rangle = -\langle u, \Delta((\Phi\chi_r) * \varphi) \rangle$$

by the definition of ρ. We have the limit of the left-hand side

$$\lim_{r\to\infty} \langle \rho, (\Phi\chi_r) * \varphi \rangle = \langle \rho, \Phi * \varphi \rangle$$

because

$$\begin{aligned} \operatorname{supp}((\Phi(1-\chi_r) * \varphi) &\subset \operatorname{supp}(\Phi(1-\chi_r)) + \operatorname{supp}(\varphi) \\ &\subset \mathbb{R}^d \setminus B(0,2R) + B(0,R) = \mathbb{R}^d \setminus B(0,R) \end{aligned}$$

for all $r > 2R$ so that the supports of $\Phi(1-\chi_r)*\varphi$ and ρ are disjoint, where we define $R := \sup_{x\in\operatorname{supp}(\varphi)} |x|$. However, the right-hand limit

$$-\lim_{r\to\infty} \langle u, \Delta((\Phi\chi_r) * \varphi) \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$$

is not a trivial result.

Assuming $\chi(x) = \chi(-x)$ without loss of generality, we have

$$\langle u, \Delta(\Phi(1-\chi_r) * \varphi) \rangle = \langle u * \Delta(\Phi(1-\chi_r)), \varphi \rangle.$$

Because

$$\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big] = 0$$

for $|y| < R$ and $x \in \operatorname{supp}(\varphi)$ if $r > 2R$, we can write

$$\langle u * \Delta(\Phi(1-\chi_r)), \varphi \rangle = \int \varphi(x) \int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\,dy\,dx.$$

We compute

$$\begin{aligned}\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big] &= 2\nabla\Phi(x-y)\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r})-\Phi(x-y)\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r})\\ &= -\frac{2}{\omega_d}\frac{x-y}{|x-y|^d}\cdot\frac{1}{r}\nabla\chi(\tfrac{x-y}{r})-\frac{1}{(d-2)\omega_d}\frac{1}{|x-y|^{d-2}}\frac{1}{r^2}\Delta\chi(\tfrac{x-y}{r}).\end{aligned}$$

Then, since $\frac{5}{4}r\le|x-y|\le\frac{7}{4}r$ if $\nabla\chi(\frac{x-y}{r})\neq0$ and $\Delta\chi(\frac{x-y}{r})\neq0$, we obtain

$$\Big|\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\Big|\le C\frac{1}{r^d}\psi(\frac{x-y}{r})$$

for some constant $C>0$, where

$$\psi(y):=|\nabla\chi(y)|+|\Delta\chi(y)|.$$

For each $x\in\operatorname{supp}(\varphi)$, since we have $\frac{5}{4}r\le|x-y|\le\frac{7}{4}r$ implies $r\le|y|\le2r$ if $r>4R$, it follows that

$$\begin{aligned}|\int u(y)\Delta_y\big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\big]\,dy| &\le C\int|u(y)\frac{1}{r^d}\psi(\frac{x-y}{r})|\,dy\\ &\le C\max_{r\le|y|\le2r}u(y)\end{aligned}$$

converges to zero as $r\to\infty$. By the bounded convergence theorem, we can deduce

$$\lim_{r\to\infty}\int\varphi(x)\int u(y)\Delta_y\Big[\Phi(x-y)\big(1-\chi(\tfrac{x-y}{r})\big)\Big]\,dy\,dx=0,$$

so we are done.

($\Leftarrow$) Let $\varphi\in C_c^\infty(\mathbb{R}^d\setminus\{0\})$. Since

$$\langle\Phi*\rho,\Delta\varphi\rangle=\langle\rho,\Phi*(\Delta\varphi)\rangle=\langle\rho,\varphi\rangle=0,$$

the distribution $\Phi*\rho$ on $\mathbb{R}^d\setminus\{0\}$ is weakly harmonic, and by Weyl's lemma for distributions, it is a smooth harmonic function on $\mathbb{R}^d\setminus\{0\}$.

Since ρ is supported at zero, we have a positive integer k and constants a_α such that

$$|\langle\rho,\varphi\rangle|\le\sum_{|a|\le k}|a_\alpha D^\alpha\varphi(0)|$$

for $\varphi\in C^\infty(\mathbb{R}^d)$. Then, for non-zero $x\in\mathbb{R}^d$, by taking a cutoff function $\chi\in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(y)=\begin{cases}1 & \text{if } |y-x|\le\frac{1}{3}|x|\\ 0 & \text{if } |y|\le\frac{1}{3}|x|\end{cases},$$

we have

$$|\Phi*\rho(x)|=|(\Phi\chi)*\rho(x)|=|\langle\rho(x-y),\Phi(y)\chi(y)\rangle_y|\le\sum_{|a|\le k}|a_\alpha D^\alpha\Phi(x)|=O(r^{2-d})$$

as $r\to\infty$. Therefore, $\Phi*\rho$ vanishes at infinity. $\square$

Lemma. *Let ρ be a distribution on $\mathbb{R}^d$ such that* $\operatorname{supp}(\rho)\subset\{0\}$. *Then, there is a constant coefficient partial differential operator $P(D)$ such that $\rho=P(D)\delta$.*

Corollary. *Let $d\ge3$. If a distribution $u\in\mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d\setminus\{0\}$ and vanishes at infinity, then there are an integer $k\ge0$ and constants a_α such that*

$$u(x)=\sum_{|a|\le k}a_\alpha D^\alpha\Phi(x)$$

for $x\neq0$, where Φ denotes the fundamental solution of Laplace's equation.

3 Unified error analysis

3.1 Approximation of Banach spaces

We follow closely Temam for the abstract error analysis. The word "approximation" in here can be replaced into "discretization".

Definition 3.1 (Approximation)**.** Let X be a Banach space. An *approximation* of X is an indexed family X_h of finite-dimensional normed spaces, with a *prolongation operator* $p_h \in B(X_h, X)$ and a *restriction operator* $r_h : X \to X_h$). The operator $\tau_h := p_h r_h$ is called the *truncation operator*.

$$\begin{tikzcd} X \arrow[d, shift right, "r_h"'] \\ X_h \arrow[u, shift right, bend right, "p_h"'] \end{tikzcd}$$

Definition 3.2 (Errors)**.** Let X_h be an approximation of a Banach space X. For $x \in X$ and $x_h \in X_h$, the quantities $E(x_h, x) := \|p_h x_h - x\|$ and $DE(x_h, x) := \|x_h - r_h x\|_h$ are called the *error* and the *discrete error* between x and x_h. The quantity $TE(x) := \|x - \tau_h x\|$ is called the *truncation error*.

Definition 3.3 (Stable and convergent approximations)**.** We say an approximation X_h is

(a) *stable* if $\|p_h\| + \|r_h\| \lesssim 1$,

(b) *convergent* if $\|x - \tau_h x\| \to 0$.

Lemma 3.1. *Let X_h be an approximation of a Banach space X. If X_h is stable and convergent, then the discrete convergence implies the strong convergence.*

Proof.
$$\|p_h x_h - x\| \le \|p_h x_h - p_h r_h x\| + \|p_h r_h x - x\| \le \|p_h\|\|x_h - r_h x\| + \|\tau_h x - x\|. \qquad \square$$

3.2 Approxiamation of problems

A *well-posed problem* is an operator $L : \mathcal{X} \to \mathcal{Y}$ such that there is a continuous operator $L^{-1} : Y \to X$ satisfying $LL^{-1} = \mathrm{id}_Y$, where $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are embeddings. Say, consider the spaces $\mathcal{X}$ and $\mathcal{Y}$ as space of distributions. We will always assume $L : X \to Y$ is a right invertible(i.e. well-posed) linear operator between Banach spaces.

Definition 3.4 (Approximation)**.** Let L be a well-posed linear problem. An *approximation* of L is an indexed family $L_h \in L(X_h, Y_h)$ of invertible linear operators, where X_h and Y_h are stable and convergent approximations of X and Y.

Note that we never use the prolongation operator for Y_h, but the restriction operator r_h for Y_h is taken to be routine. We also do not need to assume in fact the stability of r_h. The approximation X_h of X is where we should take subtly, and the art of numerical analysis begins with the choice of X_h. The following diagram does not commute, but *approximately* commute.

$$\begin{tikzcd} X \arrow[r, "L"] \arrow[d, dashed, shift right, bend right, "r_h"'] & Y \arrow[d, "r_h"] \\ X_h \arrow[r, "L_h"] \arrow[u, dashed, shift right, bend right, "p_h"'] & Y_h \end{tikzcd}$$

Given an approximated problem L_h, we will always write $x_h := L_h^{-1} r_h L x$ the *approximate solution* for each $x \in X$.

Definition 3.5. Let L_h be an approximation of a well-posed linear problem L. We say L_h is

(a) *stable* if $\|L_h^{-1}\| \lesssim 1$,

(b) *consistent* if $CE = \|r_h Lx - L_h r_h x\|_h \to 0$,

(c) *convergent* if $DE = \|x_h - r_h x\|_h \to 0$.

Theorem 3.2 (Lax equivalence). *Let L_h be an approximation of a well-posed linear problem L. If L_h is consistent, then it is stable if and only if it is convergent.*

Proof. ($\Rightarrow$) It is clear from

$$DE = \|x_h - r_h x\|_h \le \|L_h^{-1}\| \|r_h Lx - L_h r_h x\|_h = \|L_h^{-1}\| \cdot CE.$$

Conversely, suppose L_h is convergent. For any $y_h \in Y_h$, since $r_h : Y \to Y_h$ and L are surjective, there is x such that $y_h = r_h Lx$ so that $L_h^{-1} y_h = x_h$ is convergent since

$$DE = \|x_h - r_h x\|_h \le \|r_h\|_{X \to X_h} \cdot SE.$$

By the uniform boundedness principle, L_h^{-1} is uniformly bounded. □

Example 3.1.

3.3 Consistency analysis

The Taylor's theorem method for finite difference scheme.

3.4 Stability analysis

We must bound $\|L_h^{-1}\|$.

The von Neumann stability analysis for finite difference scheme.

3.5 Applications

Example 3.2. Consider

$$\begin{cases} u'(x) - u(x) = f(x) & \text{in } x \in (0,1), \\ u(0) = c. \end{cases}$$

Let $X := C^1([0,1])$, $Y := C([0,1]) \times \mathbb{R}$, and $Au(x) := (u'(x) - u(x), u(0))$. Then it is well-posed since there is $E : Y \to X$ defined by

$$E(f,c)(x) := c + \int_0^x e^{-y} f(y)\, dy$$

satisfies

Example 3.3. Consider

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } x \in (0,1)^2, \\ u(x) = 0 & \text{on } x \in \partial(0,1)^2. \end{cases}$$

Let $X =$, $Y =$, Au

Example 3.4. Consider

$$\begin{cases} \partial u(t,x) = \Delta u(t,x) & \text{in } (t,x) \in (0,\infty) \times (0,1), \\ u(0,x) = f(x) & \text{on } x \in [0,1], \\ u(t,0) = 0 & \text{on } t \in [0,\infty), \\ u(t,1) = 0 & \text{on } t \in [0,\infty), \end{cases}$$

Let $X =$, $Y =$, Au

u_j^n, $t = t_0 + nk$, $x = x_0 + jh$

4 Kinetic theory

4.1 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 4.1 (Velocity averaging). *Let L be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. Then,*

$$\| \int u\varphi \, dv \|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|^{1/2}_{L^2_{t,x,v}} \|Lu\|^{1/2}_{L^2_{t,x,v}}$$

for $\varphi \in C^\infty_c(\mathbb{R}^n_v)$,

Proof. Let $m(t,x) = \int u\varphi \, dv$. By Fourier transform with respect to t and x, we have

$$\widehat{u}(\tau,\xi,v) = \frac{1}{i} \frac{\widehat{Lu}(\tau,\xi,v)}{\tau + v\cdot\xi}$$

and

$$\widehat{m}(\tau,\xi) = \int \widehat{u}(\tau,\xi,v)\varphi(v)\, dv.$$

Fixing τ, ξ, decompose the integral and use Hölder's inequality to get

$$\begin{aligned} |\widehat{m}(\tau,\xi)| &\le \int_{|\tau+v\cdot\xi|<\alpha} |\widehat{u}\varphi| \, dv + \int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\widehat{Lu}\varphi|}{|\tau+v\cdot\xi|} \, dv \\ &\le \|\widehat{u}\|^{1/2}_{L^2_v} \Big(\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2 \, dv\Big)^{1/2} + \|\widehat{Lu}\|^{1/2}_{L^2_v} \Big(\int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|^2} \, dv\Big)^{1/2}, \end{aligned}$$

where $\alpha > 0$ is an arbitrary constant that will be determined later. Let

$$I_s(\tau,\xi,\alpha) := \int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2 \, dv, \qquad I_n(\tau,\xi,\alpha) := \int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|} \, dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}, \qquad I_n \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

Define coordinates (v_1, v_2) on $\mathbb{R}_v$ as follows:

$$v_1 := \frac{\tau+v\cdot\xi}{|\xi|} \ \in \mathbb{R}\,, \qquad v_2 := v - \frac{v\cdot\xi}{|\xi|^2}\xi \ \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v_1 - \frac{\tau}{|\xi|})^2 + |v_2|^2 \qquad \text{and} \qquad \int dv = \iint dv_2 \, dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| \le R$. If $\frac{|\tau|-\alpha}{|\xi|} > R$, then the region of integration vanishes so that $I_s = 0$. If $|\tau| \le \alpha + R|\xi|$, then

$$\begin{aligned} I_s &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \int_{|v_2|^2 \le R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 \, dv_1 \\ &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1|\le R} \int_{|v_2|\le R} dv_2 \, dv_1 \\ &\lesssim \min\{\frac{2\alpha}{|\xi|}, R\} \cdot R^{n-1} \\ &\simeq \frac{1}{\sqrt{1+(\frac{|\xi|}{\alpha})^2}} \\ &\lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}. \end{aligned}$$

For the second integral, suppose that φ is supported on $|v|<R$ so that $|v_1-\frac{\tau}{|\xi|}|, |v_2|<R$. Then,

$$
\begin{aligned}
I_n &\lesssim \int_{|v_1|\geq \frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<R} \int_{|v_2|<R} \frac{1}{v_1^2|\xi|^2}\, dv_2\, dv_1 \\
&\simeq \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-R\}\leq v_1 < \frac{|\tau|}{|\xi|}+R} \frac{1}{v_1^2|\xi|^2}\, dv_1 \\
&\simeq \frac{1}{|\xi|^2}\Big(\frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-R\}} - \frac{1}{\frac{|\tau|}{|\xi|}+R}\Big).
\end{aligned}
$$

If $\frac{|\tau|}{|\xi|}-R>\frac{\alpha}{|\xi|}$, then

$$
I_n \lesssim \frac{2R}{\tau^2-(R|\xi|)^2} < \frac{2R}{\alpha(|\tau|+R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.
$$

If $|\tau|\leq \alpha+R|\xi|$, then

$$
I_n \lesssim \frac{1}{|\xi|}\frac{(|\tau|+R|\xi|)-\alpha}{\alpha(|\tau|+R|\xi|)} \leq \frac{2R}{\alpha(|\tau|+R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.
$$

To sum up, we have

$$
|\widehat{m}(\tau,\xi)| \lesssim \frac{1}{(\tau^2+|\xi|^2)^{1/4}}\Big(\sqrt{\alpha}\cdot\|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}}\cdot\|\widehat{Lu}\|_{L_v^2}^{1/2}\Big).
$$

Letting $\alpha=\sqrt{\|\widehat{Lu}\|_{L_v^2}/\|\widehat{u}\|_{L_v^2}}$ and squaring,

$$
(\tau^2+|\xi|^2)^{1/2}|\widehat{m}(\tau,\xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2}\|\widehat{Lu}\|_{L_v^2}^{1/2}.
$$

Therefore, the integration on $\mathbb{R}_\tau\times\mathbb{R}_\xi^n$ and Plancheral's theorem gives

$$
\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2}\|Lu\|_{L_{t,x,v}^2}^{1/2}.
$$

$\square$

Corollary 4.2. *Let $\mathcal{F}$ be a family of functions on $\mathbb{R}_t\times\mathbb{R}_x^n\times\mathbb{R}_v^n$. If $\mathcal{F}$ and $L\mathcal{F}$ are bounded in $L_{t,x,v}^2$, then $\int\mathcal{F}\varphi\, dv$ is bounded in $H_{t,x}^{1/2}$.*

Theorem 4.3. *Let $\mathcal{F}$ be a family of functions on $I_t\times\mathbb{R}_x^n\times\mathbb{R}_v^n$. If $\mathcal{F}$ is weakly relatively compact and $L\mathcal{F}$ is bounded in $L_{t,x,v}^1$, then $\int\mathcal{F}\varphi\, dv$ is relatively compact in $L_{t,x}^1$.*

5 Sturm-Liouville theory

5.1 Self-adjointness

Let $I=[a,b]$ and

$$
\begin{gathered}
L=-\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right],\\
0\leq p(x)\in C^\infty(I),\quad q(x)\in C^\infty(I),\quad 0<w(x)\in C^\infty(I).
\end{gathered}
$$

We expect L to be self-adjoint. In this regard, our interest is ellimination of the difference term

$$
\langle f, Lg\rangle - \langle Lf, g\rangle = p(f'g-fg')|_a^b.
$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\dfrac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\dfrac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\dfrac{d}{dx}\left((1-x^2)\dfrac{d}{dx}\right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[\dfrac{d}{dx}\left((1-x^2)\dfrac{d}{dx}\right) - \dfrac{m^2}{1-x^2}\right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2}\left[\dfrac{d}{dx}\left(e^{-x^2}\dfrac{d}{dx}\right)\right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

5.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that p does not vanish on the boundary of I that we should cancel $f'g - fg'|_a^b$. View the Sturm-Liouville operator L as a non-densely defined operator on the space $C^\infty(I)$ with inner product $\langle f, g\rangle = \int_I fgw$ with domain

$$V = \{u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0,\ \beta_0 u(b) + \beta_1 u'(b) = 0\},$$

the subspace for the *separated* Robin boundary condition.

Proposition 5.1. *The operator $L : V \to C^\infty(I)$ is self-adjoint when $C^\infty(I)$ has the inner product $\langle f, g\rangle = \int_I fgw$.*

We are interested in the eigenvalue problem of $L : V \to C^\infty(I)$ on V. Fortunately, if we choose a constant $z \in \mathbb{C} \setminus \mathbb{R}$, then $(L - z)^{-1} : C^\infty(I) \to V$ is well-defined.

Proposition 5.2. *If z is not an eigenvalue of L, then $L - z : V \to C^\infty(I)$ is bijective.*

Proof. The injectivity follows from the definition of eigenvalues. We may assume that L is injective by translation $q \mapsto q - \lambda$.

Suppose $f \in C^\infty(I)$. The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$-pu'' - p'u' - qu = fw,$$
$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Let u_a, u_b be the unique solutions of the corresponding homogeneous equation with initial conditions

$$u_a(a) = -\alpha_1, \quad u_a'(a) = \alpha_0, \quad u_b(b) = -\beta_1, \quad u_b'(b) = \beta_0.$$

Then we can define $L^{-1} : C^\infty([0,1]) \to D(L)$ by

$$L^{-1}f(x) := u_a(x)\int_x^b \frac{u_b}{W[u_a, u_b]}\frac{f}{(-p)}w + u_b(x)\int_a^x \frac{u_a}{W[u_a, u_b]}\frac{f}{(-p)}w,$$

where $W[u_a, u_b] := u_a u_b' - u_b u_a'$ denotes the Wronskian. This formula is derived from variation of parameters: we can compute c_a and c_b from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u_a' & u_b' \end{pmatrix} \begin{pmatrix} c_a' \\ c_b' \end{pmatrix} \quad \Longrightarrow \quad L(c_a u_a + c_b u_b) = f.$$

Then, we can check that

$$L^{-1} L u = u$$

for $u \in D(L)$ by computation, which implies L is surjective. □

5.3 Legendre's equation

The Legendre equation is

$$(1 - x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx}\left((1 - x^2)\frac{d}{dx}\right).$$

Since $p(\pm 1) = 0$, the operator $L : C^\infty([-1, 1]) \to C^\infty([-1, 1])$ is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

l	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2}\log\frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x\log\frac{1+x}{1-x}, \ \cdots \in L^2(-1, 1) \setminus C^\infty([-1, 1])$$

as eigenvectors of L, then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x)\big(Q_0'(x)f(x) - Q_0(x)f'(x)\big)\big|_{-1}^{1} \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for $f \in C^\infty([-1, 1])$.

5.4 Bessel's equation

The Bessel equation is

$$x^2 u'' + xu' + (k^2 x^2 - \nu^2)u = 0, \quad \text{on } (0, \infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \nu^2\frac{1}{x}\right].$$

6 Peetre's theorem

Lemma 6.1. *Suppose a linear operator $L : C_c^\infty(M) \to C_c^\infty(M)$ satisfies*

$$\operatorname{supp}(Lu) \subset \operatorname{supp}(u) \quad \textit{for} \quad u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m, we can find $u \in C_c^\infty(U \setminus \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C. We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x.

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing \quad \text{for} \quad i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m:

$$\begin{aligned}
\sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\
&\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\
&< \infty.
\end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

□

7 Characteristic curve

Algorithm:

(a) Establish the associated vector field by substituting $u \mapsto y$.

(b) Find the integral curve.

(c) Eliminate the auxiliary variables to get an algebraic equation.

(d) Verify the computed solution is in fact the real solution.

Proposition 7.1. *Suppose that there exists a smooth solution $u : \Omega \to \mathbb{R}_y$ of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, (t,x) \in \Omega \subset \mathbb{R}_{t\geq 0} \times \mathbb{R}_x, \\ \quad u(0,x) = x, at\ x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t,x)$.

Let $\gamma : I \to \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\widetilde{\gamma} : \widetilde{I} \to M$ as the maximal integral curve of the vector field

$$\widetilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\widetilde{\gamma}(0) = \gamma(0)$. Since X and $\widetilde{X}$ coincide on M, the curve $\widetilde{\gamma}$ is also an integral curve of X with $\widetilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\widetilde{I} \subset I$ and $\gamma(\theta) = \widetilde{\gamma}(\theta)$ for all $\theta \in \widetilde{I}$.

Since M is closed in E, the open interval $\widetilde{I} = \gamma^{-1}(M)$ is closed in I, hence $\widetilde{I} = I$ by the connectedness of I. □

Definition 7.1. The projection of the integral curve γ onto Ω is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X. Once we find a necessary condition of the form of algebraic equation, we can demostrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u, we can solve the ODE: let $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$ be the integral curve of X such that $\gamma(0) = (0, \xi, \xi)$. Then, the system of ODEs

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi \end{aligned}$$

is solved as

$$t(\theta) = \theta, \qquad y(\theta) = \xi, \qquad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t,x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega = \{ (t,x) : tx > -\tfrac{1}{4} \}.$$

7.1 Wave equation

$$\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad t, x > 0, \\
&u(0,x) = g(x), \qquad u(0,x) = h(x), \qquad u_x(t,0) = \alpha(t).
\end{aligned}$$

Define $v := u_t - cu_x$. Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0,x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t,x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0,x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t,0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter $\xi > 0$,

$$\begin{aligned}
&\frac{dt}{d\theta} = 1, \qquad \frac{dx}{d\theta} = -c, \qquad \frac{dy}{d\theta} = -v(t,x), \\
&t(0) = 0, \qquad x(0) = \xi, \qquad y(0) = g(\xi)
\end{aligned}$$

is solved as

$$t(\theta) = \theta, \qquad x(\theta) = -c\theta + \xi, \qquad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta')\, d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned}
u(t,x) &= g(\xi) - \int_0^\theta v(s, \xi - cs)\, ds \\
&= g(x + ct) \\
&= \frac{3g(x+ct) - g(x-ct)}{2} - \int_0^t h(x + c(t - 2s))\, ds
\end{aligned}$$

7.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

(a) Suppose $u(0,x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(b) Suppose $u(0,x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(c) Suppose

$$u(0,x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1, \\ 1, & 1 \le x \end{cases} .$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and "paste" the solution together.

8 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0,1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every seperable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem -> continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable stict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by $\mathcal{T}_\beta$ coincides with $\mathcal{T}_\alpha$)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- Th integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x} : D' \to D'$ is continuous. They commute (Schwarz theorem holds).
- $D \to S \to L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

9 Ultrafilter

Definition 9.1. An *ultrafilter* is a synonym for maximal filter. If we sat $\mathcal{U}$ is an *ultrafilter on a set* A, then it means $\mathcal{U}$ is a maximal filter as a directed subset of $\mathcal{P}(A)$.

existence of ultrafilter.

Theorem 9.1. *Let $\mathcal{U}$ be an ultrafilter on a set A and X be a compact space. For a function $f : A \to X$, the limit $\mathcal{U}$-$\lim f$ always exists.*

Theorem 9.2. *Let $X = \prod_{\alpha \in A} X_\alpha$ be a product space of compact spaces X_α. A net $f : \mathcal{D} \to X$ has a convergent subnet.*

Proof 1. Use Tychonoff. Compactness and net compactness are equivalent. □

Proof 2. It is a proof without Tychonoff. Let $\mathcal{U}$ be a ultrafilter on a set $\mathcal{D}$ contatining all $\uparrow d$. Define a directed set $\mathcal{E} = \{(d,U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$ as $(d,U) \succ (d',U')$ for $U \subset U'$. Let $f : \mathcal{E} \to X$ be a subnet of $f : \mathcal{D} \to X$ defined by $f_{(d,U)} = f_d$.

By the previous theorem, $\mathcal{U}\text{-}\lim \pi_\alpha f_d \in X_\alpha$ exsits for each α. Define $f \in X$ such that $\pi_\alpha f = \mathcal{U}\text{-}\lim \pi_\alpha f_d$. Let $G = \prod_\alpha G_\alpha \subset X$ be any open neighborhood of f. Then, $\pi_\alpha f \in G_\alpha$ and we have $G_\alpha = X_\alpha$ except finite. For α, we can take $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$ by definition of convergence with ultrafilter Since $U_\alpha = \mathcal{D}$ except finites, we can take an upper bound $U_0 \in \mathcal{U}$ of $\{U_\alpha\}_\alpha$. Then, by taking any $d_0 \in U_0$, we have $f_{(d,U)} \in G$ for every $(d, U) \succ (d_0, U_0)$. This means $f = \lim_{\mathcal{E}} f_{(d,U)}$, so we can say $\lim_{\mathcal{E}} f_{(d,U)}$ exists. □

10 Selected analysis problems

10.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \le x - x_0 \le x \le 2^k$, $x - x_0 \in A_k$.

$$|A_k| \ge \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\ge \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\
&\ge \sum_{k=1}^{N} (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\
&= \sum_{k=1}^{N} \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&> \sum_{k=1}^{N} \frac{2^k}{2^{k+2}} \frac{1}{7k} \\
&= \frac{1}{28} \sum_{k=1}^{N} \frac{1}{k} \\
&\to \infty.
\end{aligned}$$

□

10.2. If $|xf'(x)| \le M$ and $\frac{1}{x}\int_0^x f(y)\,dy \to L$, then $f(x) \to L$ as $x \to \infty$.

Solution. It is a kind of Tauberian theorems. Since for each fixed $\varepsilon > 0$ we have

$$\begin{aligned} |f(x) - \frac{1}{\varepsilon x}\int_{(1-\varepsilon)x}^{x} f(y)\,dy| &\le \frac{1}{\varepsilon x}\int_{(1-\varepsilon)x}^{x} |f(x)-f(y)|\,dy \\ &\le \frac{M}{\varepsilon x}\int_{(1-\varepsilon)x}^{x} \frac{x-y}{y}\,dy \\ &= M(\tfrac{1}{\varepsilon}\log\tfrac{1}{1-\varepsilon} - 1) = O(\varepsilon) \end{aligned}$$

by the mean value theorem and

$$\frac{1}{\varepsilon x}\int_{(1-\varepsilon)x}^{x} f(y)\,dy = \frac{1}{\varepsilon x}\int_0^x f(y)\,dy - \frac{1}{\varepsilon x}\int_0^{(1-\varepsilon)x} f(y)\,dy \to \frac{1}{\varepsilon}L - \frac{1-\varepsilon}{\varepsilon}L = L$$

as $x \to \infty$, we get

$$\limsup_{x\to\infty} |f(x) - L| = O(\varepsilon),$$

so we are done. $\square$

10.3. Let $f_n : [0,1] \to [0,1]$ be a sequence of functions such that $|f_n(x)-f_n(y)| \le |x-y|$ whenever $|x-y| \ge \frac{1}{n}$ for each $n \ge 1$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap [0,1] \to [0,1]$ pointwisely.

Let $n \ge 4$. Then, for $x \in [0,1]$ there is $z \in [0,1]$ such that $|x-z| = \frac{2}{n}$ so that

$$|f_n(x)-f_n(z)| \le |x-z| = \frac{2}{n}.$$

Whenever $y \in [0,1]$ satisfies $|x-y| \le \frac{1}{n}$, then we have $|y-z| \ge |x-z|-|x-y| \ge \frac{1}{n}$, so we get

$$|f_n(y)-f_n(z)| \le |y-z| \le |y-x|+|x-z| \le \frac{3}{n}.$$

Combining the two inequalities, we obtain

$$|x-y| \le \frac{1}{n} \quad \Longrightarrow \quad |f_n(x)-f_n(y)| \le \frac{5}{n} \tag{1}$$

for $n \ge 4$.

Let $\varepsilon > 0$ and suppose $|x-y| \le \frac{\varepsilon}{5}$. For every $n \ge \max\{\frac{10}{\varepsilon}, 4\}$, since $|x-y| \le \frac{1}{n}$ implies by the inequality (1) that

$$|f_n(x)-f_n(y)| \le \frac{5}{n} \le \frac{\varepsilon}{2},$$

and since $|x-y| > \frac{1}{n}$ implies by the condition in the problem that

$$|f_n(x)-f_n(y)| \le |x-y| \le \frac{\varepsilon}{5} < \frac{\varepsilon}{2},$$

we have

$$|x-y| \le \frac{\varepsilon}{5} \quad \Longrightarrow \quad |f_n(x)-f_n(y)| \le \frac{\varepsilon}{2} \tag{2}$$

for all $n \ge \max\{\frac{10}{\varepsilon}, 4\}$.

For $\varepsilon > 0$, take $\delta := \varepsilon/5$ and fix x and y in $\mathbb{Q} \cap [0,1]$ satisfying $|x-y| < \delta$. Then, we have

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)| \\ &\le |f(x)-f_n(x)| + \frac{\varepsilon}{2} + |f_n(y)-f(y)|\end{aligned}$$

for all $n \ge \max\{\frac{10}{\varepsilon}, 4\}$, and by limiting $n \to \infty$,

$$|f(x)-f(y)| \le 0 + \frac{\varepsilon}{2} + 0 < \varepsilon.$$

Therefore, f is uniformly continuous on $\mathbb{Q} \cap [0,1]$ so that it has a unique continuous extension on the whole $[0,1]$. Let it denoted by the same notation f.

Finally, we are going to show $f_n \to f$ uniformly on $[0,1]$. By the uniform continuity of f, for each $\varepsilon > 0$ we have $\delta > 0$ such that

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x)-f(y)| < \frac{\varepsilon}{2}. \tag{3}$$

Take a finite subset $F \in \mathbb{Q} \cap [0,1]$, such that for every x there is y satisfying $|x-y| < \min\{\frac{\varepsilon}{5}, \delta\}$. Then, by (2) and (3), we have an inequality

$$\begin{aligned}|f_n(x)-f(x)| &\le |f_n(x)-f_n(y)| + |f_n(y)-f(y)| + |f(y)-f(x)| \\ &< \frac{\varepsilon}{2} + \max_{z \in F} |f_n(z)-f(z)| + \frac{\varepsilon}{2}\end{aligned}$$

for all $n \ge \max\{\frac{10}{\varepsilon}, 4\}$. Therefore, by taking supremum for x and limiting $n \to \infty$ on it we have

$$\limsup_{n \to \infty} \|f_n - f\| \le \varepsilon,$$

so we are done because ε is arbitrary. □

11 Physics problem

11.1 Resonance

Let m, b, k, A, ω_d be positive real constants. Consider an underdamped oscillator with sinusoidal diving force described as

$$mx'' + bx' + kx = A\sin\omega_d t, \quad x(0) = x_0,\ x'(0) = 0.$$

There are some observations:

(a) The underdamping condition means $b^2 - 4mk < 0$ so that the roots of characteristic equation are imaginary.

(b) The positivity of m, b implies the real part of solution that will be denoted by $-\beta = -\frac{b}{2m}$ is negative; it shows exponential decay of solutions.

(c) Introducing the natural frequency $\omega_n = \sqrt{k/m}$, we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A\sin\omega t.$$

(d) The complementary solution is computed as

$$x_c(t) = x_0\, e^{-\beta t}\cos\sqrt{\beta^2 - \omega_n^2}\,t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t\to\infty} x_c(t) = 0.$$

(e) The condition $\beta > \omega_n$ is equivalent to that the oscillator is underdamped.

(f) Let m, k be fixed. Then, the solution x_c decays most fastly when b satisfied $b^2 = 4mk$, equivalently, $\beta = \omega_n$.

(g) When $\omega_d = \omega_n$ such that the amplitude of particular solution diverges.