Measure Theory

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Part I

Measures

Chapter 1

Measurable spaces

1.1 Measurable algebras

1.1 (Boolean σ-algebras)**.** Let X be a set. A *σ-algebra* of sets on X is a collection $\mathcal{A} \subset \mathcal{P}(X)$ which is closed under countable unions and complements.

(a) generated by a set.

(b) countable and cocountable sets

(c) Borel

1.2 (Measurable spaces)**.** A *measurable space* or a *Borel space* is a pair $(X, \mathcal{A})$ of a set X and a σ-algebra $\mathcal{A}$ on X. Each element of $\mathcal{A}$ is called *measurable*. We often omit $\mathcal{A}$ to just write X for $(X, \mathcal{A})$ if there is no confusion.

1.2 Localizability

1.3 Standard Borel spaces

Chapter 2

Measure spaces

2.1 (Measure spaces)**.** Let $(X, \mathcal{A})$ be a measurable space. A *measure* on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i), \qquad (E_i)_{i=1}^{\infty} \subset \mathcal{A}.$$

Here the squared cup notation reads the disjoint union. A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where μ is a measure on $(X, \mathcal{A})$. Let μ be a measure on X.

(a) μ is monotone: for $E, F \in \mathcal{A}$ if $E \subset F$ then $\mu(E) \leq \mu(F)$.

(b) μ is countably subadditive: for

(c) μ is continuous from below:

(d) μ is continuous from above:

2.2 (Complete measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. A *null set* is a measurable set N satisfying $\mu(N) = 0$, and a *full set* is a measurable set whose complement is a null set.

A *complete measure* is a measure such that every subset of a null set is measurable.

For a predicate P of points $x \in X$, we say P is true *almost everywhere* or *a.e.* on X if there is a full set $F \subset X$ such that $P(x)$ is true for all $x \in F$.

2.1 Carathéodory extension

2.3 (Outer measures)**.** Let X be a set. An *outer measure* on X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty] : \varnothing \mapsto 0$ which is monotone and countably subadditive.

(i) μ^* is *monotone*: we have

$$S_1 \subset S_2 \quad \Rightarrow \quad \mu^*(S_1) \leq \mu^*(S_2), \qquad S_1, S_2 \in \mathcal{P}(X),$$

(ii) μ^* is *countably subadditive*: we have

$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \leq \sum_{i=1}^{\infty} \mu^*(S_i), \qquad (S_i)_{i=1}^{\infty} \subset \mathcal{P}(X).$$

Compairing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(X) \to [0,\infty] : \varnothing \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^\infty S_i \quad \Rightarrow \quad \mu^*(S) \le \sum_{i=1}^\infty \mu^*(S_i), \qquad S \in \mathcal{P}(X),\ (S_i)_{i=1}^\infty \subset \mathcal{P}(X).$$

(b) For any $\varnothing \in \mathcal{A}_0 \subset \mathcal{P}(X)$, let $\mu_0 : \mathcal{A}_0 \to [0,\infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^\infty \mu_0(B_i) : S \subset \bigcup_{i=1}^\infty B_i,\ B_i \in \mathcal{A}_0 \right\},$$

where we use the convention $\inf\varnothing = \infty$.

Proof. $\square$

2.4 (Carathéodory measurable sets)**.** Let μ^* be an outer measure on a set X. We want to construct a measure by restriction of μ^* on a properly defined σ-algebra. A subset $E \subset X$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(X)$. Let $\mathcal{A} \subset \mathcal{P}(X)$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{A}$ is an algebra and μ^* is finitely additive on $\mathcal{A}$.

(b) $\mathcal{A}$ is a σ-algebra and μ^* is countably additive on $\mathcal{A}$. That is, $\mu := \mu^*|_{\mathcal{A}}$ is a measure.

(c) The measure μ is complete.

Proof. $\square$

2.5 (Carathéodory extension theorem)**.** The Carathéodory extension is a construction method for a measure extending a given set function μ_0 on $\mathcal{A}_0 \subset \mathcal{P}(X)$ for a set X. The idea is to restrict the outer measure μ^* associated to μ_0 in order to obtain a measure μ. We want to find a sufficient condition for μ to be a measure on a σ-algebra containing $\mathcal{A}_0$.

Let $\varnothing \in \mathcal{A}_0 \subset \mathcal{P}(X)$, and let $\mu_0 : \mathcal{A}_0 \to [0,\infty]$ be a set function with $\mu_0(\varnothing) = 0$. Let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be the associated outer measure of μ_0, and $\mu : \mathcal{A} \to [0,\infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) μ^* extends μ_0 if μ_0 satisfies the monotone countable subadditivity: we have

$$A \subset \bigcup_{i=1}^\infty B_i \quad \Rightarrow \quad \mu_0(A) \le \sum_{i=1}^\infty \mu_0(B_i), \qquad A \in \mathcal{A}_0,\ (B_i)_{i=1}^\infty \subset \mathcal{A}_0$$

(b) μ extends μ_0 if μ_0 satisfies the following property in addition: for $B, A \in \mathcal{A}_0$ and any $\varepsilon > 0$, there are $(C_j)_{j=1}^\infty, (D_j)_{j=1}^\infty \subset \mathcal{A}_0$ such that

$$B \cap A \subset \bigcup_{j=1}^\infty C_j, \quad B \setminus A \subset \bigcup_{j=1}^\infty D_j, \quad \sum_{j=1}^\infty (\mu_0(C_j) + \mu_0(D_j)) < \mu_0(B) + \varepsilon.$$

Proof. (a) Fix $A \in \mathcal{A}_0$. Clearly $\mu^*(A) \le \mu_0(A)$. For the opposite direction, we may assume $\mu^*(A) < \infty$. By the finiteness of $\mu^*(A)$, for any $\varepsilon > 0$ we have $(B_i)_{i=1}^\infty \subset \mathcal{A}_0$ such that $A \subset \bigcup_{i=1}^\infty B_i$ and

$$\sum_{i=1}^\infty \mu_0(B_i) < \mu^*(A) + \varepsilon.$$

Therefore we have $\mu_0(A) < \mu^*(A) + \varepsilon$ by the assumption, and we get $\mu_0(A) \le \mu^*(A)$ by limiting $\varepsilon \to 0$.

(b) Fix $A \in \mathcal{A}_0$. It is enough to check the inequality $\mu^*(S \cap A) + \mu^*(S \setminus A) \le \mu^*(S)$ for $S \in \mathcal{P}(X)$ with $\mu^*(S) < \infty$. By the finiteness of $\mu^*(S)$, we have $(B_i)_{i=1}^\infty \subset \mathcal{B}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. From the condition, we have $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$ satisfying

$$\begin{aligned}\mu^*(S \cap A) + \mu^*(S \setminus A) &\le \mu^*\Big(\bigcup_{j=1}^\infty (B_i \cap A)\Big) + \mu^*\Big(\bigcup_{j=1}^\infty (B_i \setminus A)\Big) \\ &\le \sum_{i,j=1}^\infty (\mu_0(C_{i,j}) + \mu_0(D_{i,j})) \\ &\le \sum_{i=1}^\infty (\mu_0(B_i) + 2^{-i}\varepsilon) \\ &< \mu^*(S) + \varepsilon.\end{aligned}$$

Therefore, A is Carathéodory measurable relative to μ^*, so the domain of μ contains the domain of μ_0. The values coincide by the part (a). □

2.6 (Uniqueness of extension of measures)**.** The Carathéodory extension also provides a uniqueness result for measure extensions. Let $\rho : \mathcal{B} \to [0, \infty] : \varnothing \mapsto 0$ be a set function, where $\varnothing \in \mathcal{B} \subset \mathcal{P}(X)$ for a set X. We say ρ is *σ-finite* if there is a cover $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ of X such that $\rho(B_i) < \infty$ for each i.

Let $\mathcal{A}$ be a σ-algebra containing $\mathcal{B}$. Let μ be a measure on $\mathcal{A}$, which extends ρ, given by the restriction of the outer measure μ^* associated to ρ. Let ν be another measure on $\mathcal{A}$ which extends ρ. Let $E \in \mathcal{A}$ and $\{E_i\}_{i=1}^\infty \subset \mathcal{A}$.

(a) $\nu(E) \le \mu(E)$.

(b) $\nu(E_i) = \mu(E_i)$ implies $\nu\Big(\bigcup_{i=1}^\infty E_i\Big) = \mu\Big(\bigcup_{i=1}^\infty E_i\Big)$.

(c) $\nu(E) = \mu(E)$ for $\mu(E) < \infty$.

(d) $\nu(E) = \mu(E)$ for $\mu(E) = \infty$, if ρ is σ-finite

Proof. (a) We may assume $\mu(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Also, whenever $E \subset \bigcup_{i=1}^\infty B_i$ we have

$$\nu(E) \le \nu\Big(\bigcup_{i=1}^\infty B_i\Big) \le \sum_{i=1}^\infty \nu(B_i) = \sum_{i=1}^\infty \rho(B_i) = \sum_{i=1}^\infty \mu(B_i),$$

hence $\nu(E) \le \mu(E)$.

(b) In the light of the inclusion-exclusion principle, we have

$$\mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j) - \mu(E_i \cap E_j) \le \nu(E_i) + \nu(E_j) - \nu(E_i \cap E_j) = \nu(E_i \cup E_j),$$

so that $\mu(E_i \cup E_j) = \nu(E_i \cap E_j)$. Applying it inductively, we have for every n that

$$\mu\Big(\bigcup_{i=1}^n B_i\Big) = \nu\Big(\bigcup_{i=1}^n B_i\Big),$$

and by limiting $n \to \infty$ the continuity from below gives

$$\mu\Big(\bigcup_{i=1}^\infty B_i\Big) = \nu\Big(\bigcup_{i=1}^\infty B_i\Big).$$

(c) Because $\mu(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^\infty \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^\infty B_i$ and

$$\sum_{i=1}^\infty \rho(B_i) < \mu(E) + \varepsilon.$$

Applying the part (b) Then, we have

$$\mu(E) \le \mu\Big(\bigcup_{i=1}^\infty B_i\Big) = \nu\Big(\bigcup_{i=1}^\infty B_i\Big) = \nu\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) + \nu(E)$$

and

$$\nu\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) \le \mu\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) = \mu\Big(\bigcup_{i=1}^\infty B_i\Big) - \mu(E) \le \sum_{i=1}^\infty \mu(B_i) - \mu(E) = \sum_{i=1}^\infty \rho(B_i) - \mu(E) < \varepsilon,$$

we get $\mu(E) < \nu(E) + \varepsilon$ and $\mu(E) \le \nu(E)$ by limiting $\varepsilon \to 0$.

(d) Let $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ be a cover of X such that $\rho(B_i) < \infty$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \ge 2$ so that $\{E_i\}_{i=1}^\infty$ is a pairwise disjoint cover of X with

$$\mu(E \cap E_i) \le \mu(E_i) \le \mu(B_i) = \rho(B_i) < \infty$$

for each i, so we have by the part (c) that

$$\nu(E) = \sum_{i=1}^\infty \nu(E \cap E_i) = \sum_{i=1}^\infty \mu(E \cap E_i) = \mu(E). \qquad \square$$

2.2 Measures on Euclidean spaces

2.7 (Borel σ-algebra).

2.8 (Distribution functions).

2.9 (Helly selection theorem).

2.10 (Vitali set).

2.3 Hausdorff measures

Hausdorff measure, surface measure, Brunn-Minkowski inequality

Exercises

2.11 (Boolean algebras and rings).

2.12 (Cardinalities). infinite σ-algebra is $\ge \mathfrak{c}$.

2.13 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A} \subset \mathcal{P}(X)$ such that $\varnothing \in \mathcal{A}$. We say that $\mathcal{A}$ is a semi-ring if it is closed under finite intersections, and each relative complement is a finite union of elements of $\mathcal{A}$. We say that $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over X. Suppose a set function $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^\infty A_i\Big) \le \sum_{i=1}^\infty \rho(A_i)$$

for $(A_i)_{i=1}^\infty \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

2.14 (Monotone class lemma)**.** A collection $\mathcal{C} \subset \mathcal{P}(X)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let H be a vector space closed under bounded monotone convergence. If $\operatorname{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^\infty(\sigma(\mathcal{A})) \subset H$.

2.15 (Steinhaus theorem)**.** Let λ denote the Lebesgue measure on $\mathbb{R}$ and let $\mathbb{E} \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.

(a) For any $0 < \alpha < 1$, there is an interval $I = (a, b)$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.

(b) $E - E = \{x - y : x, y \in E\}$ contains an open interval containing zero.

Proof. (a) We may assum $\lambda(E) < \infty$. Since λ is outer measure and $\lambda(E) \neq 0$, we have an open subset U of $\mathbb{R}$ such that $\lambda(U) < \alpha^{-1}\lambda(E)$. Because U is a countable disjoint union of open intervals $U = \bigsqcup_{i=1}^\infty (a_i, b_i)$, we have

$$\sum_{i=1}^\infty \lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1}\sum_{i=1}^n \lambda(E \cap (a_i, b_i)).$$

Therefore, there is i such that $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$. □

Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

Chapter 3

Lebesgue integral

3.1 Measurable functions

simple function approximations, convergence in measure

3.1 (Measurability of pointwise limits)**.** Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

□

3.2 (Almost everywhere convergence)**.** Let (X, μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence f_n is defined as the set

$$\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say f_n converges to f *alomst everywhere* with respect to μ if the set of divergence is a null set in μ. We simply write

$$f_n \to f \text{ a.e.}$$

if f_n converges to f almost everywhere, and we frequently omit the measure μ if it has no confusion.

(a) If μ is complete and, if $f_n \to f$ a.e., then f is measurable.

3.3 (Borel-Cantelli lemma)**.** Let (X, μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\{x : \limsup_{n\to\infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n\to\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(c) $f_n \to f$ a.e. if for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \infty.$$

Proof. (b) The set of divergence of the sequence f_n is given by

$$\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \tfrac{1}{m}\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} (X \setminus E_n^m).$$

(c) Since

$$\mu\Big(\bigcup_{i=1}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \leq \sum_{i=1}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned}
\mu(\limsup_{n\to\infty} \{x : |f_n(x) - f(x)| \geq \varepsilon\}) &= \mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&= \lim_{n\to\infty} \mu\Big(\bigcup_{i=n}^{\infty} \{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0.
\end{aligned}$$

□

3.4 (Convergence in measure). Let (X, μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. We say f_n converges to a measurable function $f : X \to \overline{\mathbb{R}}$ *in measure* if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(a) If $f_n \to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

(b) If every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}} \to f$ a.e., then $f_n \to f$ in measure.

Proof. (a) Since for each positive integer k we have $\mu(\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}) \to 0$ as $n \to \infty$, there exists n_k such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k\to\infty} \{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) = 0.$$

Then, for each $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{k\to\infty} \{x : |f_{n_k}(x) - f(x)| \geq \varepsilon\} &= \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \varepsilon\} \\
&\subset \bigcap_{k=\lceil \varepsilon^{-1} \rceil}^{\infty} \bigcup_{j=k}^{\infty} \{x : |f_{n_j}(x) - f(x)| \geq \tfrac{1}{k}\} \\
&= \limsup_{k\to\infty} \{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}
\end{aligned}$$

implies the limit superior of the tail events is a null set, hence $f_{n_k} \to f$ a.e.

(b) □

3.5 (Egorov theorem)**.** Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient "$\varepsilon/2^m$ argument", occasionally used throughout measure theory to construct a measurable set having a special property.

Let (X,μ) be a finite measure space and let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that $f_n \to f$ a.e. For each positive integer m, which indexes the tolerance $1/m$, consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty}\{x : |f_i(x) - f(x)| < \tfrac{1}{m}\}.$$

(a) E_n^m converges to a full set for each m.

(b) For every $\varepsilon > 0$ there is a measurable $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and for each m there is finite n satisfying $K \subset E_n^m$.

(c) For every $\varepsilon > 0$ there is a measurable $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on K.

Proof. (a) Recall that the a.e. convergence $f_n \to f$ means that for every fixed m the intersection

$$\bigcap_{n=1}^{\infty}(X \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \tfrac{1}{m}\}$$

is a null set. Since $\mu(X) < \infty$, it is equivalent to E_n^m converges to a full set for each m by the continuity from above.

(b) For each m, we can find n_m such that

$$\mu(X \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(X \setminus K) = \mu\Big(\bigcup_{m=1}^{\infty}(X \setminus E_{n_m}^m)\Big) \leq \sum_{m=1}^{\infty}\mu(X \setminus E_{n_m}^m) < \sum_{m=1}^{\infty}\frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix $m > 0$. Since $n \geq n_m$ implies $K \subset E_{n_m}^m \subset E_n^m$, we have

$$n \geq n_m \quad \Rightarrow \quad \sup_{x \in K}|f_n(x) - f(x)| < \frac{1}{m}.$$

□

3.2 Convergence theorems

3.6 (Lebesgue integral of non-negative functions)**.** Let (X,μ) be a measure space. Let $f : X \to [0,\infty)$ be a measurable function. The *Lebesgue integral* of f is defined by

$$\int f\,d\mu := \sup\left\{\int s\,d\mu : 0 \leq s \leq f,\ s \text{ simple}\right\}$$

3.7 (Monotone convergence theorem)**.** Let (X,μ) be a measure space. Let (f_n) be a non-decreasing sequence of measurable functions $X \to [0,\infty)$.

(a) $E \mapsto \int_E f\,d\mu$ is a measure.

(b) $\int \sup_n f_n\,d\mu = \sup_n \int f_n\,d\mu$.

Proof. (a) The map $E \mapsto \int_E f\, d\mu$ is a measure if f is simple, from the linearity of the integral for simple functions. For $E_n \uparrow E$, we want to show the continuity from below, $\int_{E_n} f \to \int_E f$. Take $\varepsilon > 0$. We introduce a continuous bijection $\beta : [0,\infty] \to [0,1] : t \mapsto t/(1+t)$ to avoid dividing the cases for infinity. By the definition of the Lebesgue integral, we have a simple function s such that $0 \le s \le f$ and

$$\beta(\textstyle\int_E f) - \beta(\int_E s) < \varepsilon,$$

whether or not $\int_E f$ diverges. Then,

$$\begin{aligned}\beta(\textstyle\int_E f) - \beta(\int_{E_n} f) &= [\beta(\textstyle\int_E f) - \beta(\int_E s)] + [\beta(\int_E s) - \beta(\int_{E_n} s)] + [\beta(\int_{E_n} s) - \beta(\int_{E_n} f)] \\ &< \varepsilon + [\beta(\textstyle\int_E s) - \beta(\int_{E_n} s)] + 0 \xrightarrow{n\to\infty} \varepsilon.\end{aligned}$$

We are done by letting $\varepsilon \to 0$.

(b) For any $\varepsilon > 0$ let $E_n := \{x : f(x) < (1+\varepsilon) f_n(x)\}$, which converges to a full set because $f_n \to f$ a.e. Since f is a measure, we can choose N such that

$$\beta(\textstyle\int_E f) - \beta(\int_{E_N} f) < \varepsilon.$$

With this N, we have

$$\beta(\textstyle\int_{E_N} f) \le \beta((1+\varepsilon)\int_{E_N} f_n) \le (1+\varepsilon)\beta(\int_{E_N} f_n) \le \beta(\int_{E_N} f_n) + \varepsilon, \qquad n \ge N.$$

Then, we have for $n \ge N$ that

$$\begin{aligned}\beta(\textstyle\int_E f) - \beta(\int_E f_n) &= [\beta(\textstyle\int_E f) - \beta(\int_{E_N} f)] + [\beta(\int_{E_N} f) - \beta(\int_{E_N} f_n)] + [\beta(\int_{E_N} f_n) - \beta(\int_E f_n)] \\ &< \varepsilon + \varepsilon + 0,\end{aligned}$$

so we are done by letting $n \to \infty$ and $\varepsilon \to 0$. □

3.8 (Corollaries of monotone convergence theorem)**.** Fatou's lemma, linearity of the integral, $f \ge 0$ and $\int f = 0$ imply $f = 0$ a.e.

3.9 (Lebesgue integral of complex-valued functions)**.**

3.10 (Bounded convergence theorem)**.** Semifinite measures

(a)
$$\sup_{g \le f} \int g\, d\mu = \int f\, d\mu$$
where g runs through bounded measurable functions.

(b)

3.3 Product measures

3.11 (Fubini-Tonelli theorem)**.** Lebesgue measure on Euclidean spaces

Lipschitz and differentiable transformations

3.4 Integrals on Euclidean spaces

Exercises

3.12 (Cauchy's functional equation)**.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x+y)=f(x)+f(y)$, satisfied for all $x,y\in\mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x)=ax$ for all $x\in\mathbb{R}$, where $a:=f(1)$.

(a) $f(x)=ax$ for all $x\in\mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If f is conitnuous at a point, then f is linear.

(c) If f is Lebesgue measurable, then f is linear.

3.13 (Pointwise approximation by simple functions)**.** Let (X,μ) be a measure space and X a metric space with Borel measurable structure. By a *simple function* we mean a measurable function $s:X\to X$ of finite image.

(a) For each open set $U\subset X$ there is a sequence of open sets U_i such that $U=\bigcup_i U_i$ and $\overline{U}_i\subset U$. Let $f:X\to X$ be any function.

(b) If f is the pointwise limit of a sequence of measurable functions, then f is measurable.

(c) If f is measurable, then f is the pointwise limit of a sequence of simple functions, if X is separable.

*(d) The pointwise limit of a net of simple functions may not be measurable.

Proof. (b) Suppose a sequence $(f_n)_n$ of measurable functions converges pointwisely to a function f. For fixed open $U\subset X$ we claim

$$f^{-1}(U)=\bigcup_{i=1}^\infty \liminf_{n\to\infty} f_n^{-1}(U_i).$$

If it is true, then $f^{-1}(U)$ is the countable set operation of measurable sets $f_n^{-1}(U_i)$. Let U_i be the sequence associated to U taken by the part (a).

($\subset$) If $\omega\in f^{-1}(U)$, then for some i we have $f(\omega)\in U_i$, so $f_n(\omega)$ is eventually in U_i, thus we have $\omega\in\liminf_{n\to\infty}f_n^{-1}(U_i)$.

($\supset$) If $\omega\in\liminf_{n\to\infty}f_n^{-1}(U_i)$ for some i, then $f_n(\omega)$ is eventually in U_i, so $f(\omega)\in\overline{U}_i\subset U$, thus we have $\omega\in f^{-1}(U)$.

(c) Suppose there is a increasing sequence of finite tagged partitions $\mathcal{P}_n\subset\mathcal{B}$ satisfying the following property: for each open-neighborhood pair (x,U) there is n and i such that $P_{n,i}\in\mathcal{P}_n$ and $x\in P_{n,i}\subset U$. We denote the tags by $t_{n,i}\in P_{n,i}$ for each $P_{n,i}\in\mathcal{P}_n$. Define

$$s_n(\omega):=t_{n,i}\quad\text{for}\quad f(\omega)\in P_{n,i}.$$

To show $s_n(\omega)\to f(\omega)$, fix an open $f(\omega)\in U\subset X$. Then, there is n_0 such that there is a sequence $(P_{n,i_n})_{n=n_0}^\infty$ satisfying $P_{n,i_n}\in\mathcal{P}_n$ and $f(\omega)\in P_{n,i_n}\subset U$. Then, for all $n\ge n_0$, we have for $f(\omega)\in P_{n,i_n}$ that $s_n(\omega)=t_{n,i_n}\in P_{n,i_n}\subset U$.

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem. $\square$

3.14 (Convergence of one-parameter family)**.**

If $\|f_n\|_{L^2([0,1])}\le C$ and $f_n\to f$ almost everywhere, then $f_n\to f$ weakly.

$$\lim_{n\to\infty}\int_0^1 n^3x^2(1-x)^n\,dx=2\neq 0=\int_0^1\lim_{n\to\infty}n^3x^2(1-x)^n\,dx.$$
$$\lim_{n\to\infty}\int_0^\infty n^2e^{-nx}\,dx=\infty\neq 0=\int_0^\infty\lim_{n\to\infty}n^2e^{-nx}\,dx.$$

Part II

Function spaces

Chapter 4

Lebesgue spaces

4.1

4.1 (Hölder inequality).

Proof.
$$\int fg \le C^p \int \frac{|f|^p}{p} + \frac{1}{C^q}\int \frac{|g|^q}{q}$$
Take C such that
$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q}\int \frac{|g|^q}{q}.$$
Then,
$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q}\int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}}q^{-\frac{1}{q}}\Big(\int |f|^p\Big)^{\frac{1}{p}}\Big(\int |g|^p\Big)^{\frac{1}{q}}.$$
Note that we can show that $1 \le 2p^{-\frac{1}{p}}q^{-\frac{1}{q}} \le 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. □

4.2 Convolutions

4.2 (Convolution?).

4.3 (Approximate identity?).

4.4 (Continuity of translation?).

4.3 Interpolations

Lorentz spaces Weak L^p spaces

Definition 4.3.1. Let f be a measurable function on a measure space (X,μ). The *distribution function* $\lambda_f : [0,\infty) \to [0,\infty)$ is defined as:
$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}) = \mu(|f| > \alpha).$$

Do not use $\mu(\{x : |f(x)| \ge \alpha\})$. The strict inequality implies the *lower semi-continuity* of λ_f.

For $p > 0$,

$$\begin{aligned}
\|f\|_{L^p}^p &= \int |f(x)|^p \, d\mu(x) \\
&= \int \int_0^{|f(x)|} p\alpha^{p-1} \, d\alpha \, d\mu(x) \\
&= \int_0^\infty \int_{|f(x)|>\alpha} p\alpha^{p-1} \, d\mu(x) \, d\alpha \\
&= p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^p \frac{d\alpha}{\alpha}.
\end{aligned}$$

Definition 4.3.2.

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0<\alpha<\infty} \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right].$$

Theorem 4.3.3. *For $p \geq 1$ we have $\|f\|_{p,\infty} \leq \|f\|_p$.*

Proof. By the Chebyshev inequality,

$$\sup_{0<\alpha<\infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha) \, d\alpha = \|f\|_{L^p}^p.$$

□

4.5 (Marcinkiewicz interpolation)**.** Let X be a σ-finite measure space and Y be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator $T : L^{p_0}(X) + L^{p_1}(X) \to M(Y)$ has two weak-type estimates

$$\|T\|_{L^{p_0}(X)\to L^{p_0,\infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X)\to L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X)\to L^p(Y)} < \infty.$$

Proof. Let $f \in L^p(X)$ and denote $f_h = \chi_{|f|>\alpha} f$ and $f_l = \chi_{|f|\leq\alpha} f$. It is easy to show $f_h \in L^{p_0}$ and $f_l \in L^{p_1}$. Then,

$$\begin{aligned}
\|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha) \frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^p \cdot \mu(|Tf_h| > \alpha) \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha) \frac{d\alpha}{\alpha} \\
&\leq \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0,\infty}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}} \|Tf_l\|_{L^{p_1,\infty}}^{p_1} \frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^{p-p_0} \|f_h\|_{p_0}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{p_1}^{p_1} \frac{d\alpha}{\alpha} \\
&\sim \|f\|_p^p.
\end{aligned}$$

by (1) Fubini, (2) Sublinearlity, (3) Chebyshev, (4) Boundedness, (5) Fubini. □

4.6 (Hadamard's three line lemma)**.** Let f be a bounded holomorphic function on vertical unit strip $\{z : 0 < \mathrm{Re}\, z < 1\}$ which is continuously extended to the boundary. Then, for $0 < \theta < 1$ we have

$$\|f\|_{L^\infty(\mathrm{Re}=\theta)} \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta}.$$

Proof. Fix n and define

$$g_n(z) := \frac{f(z)}{\|f\|_{L^\infty(\mathrm{Re}=0)}^{1-z} \|f\|_{L^\infty(\mathrm{Re}=1)}^{z}} e^{-\frac{z(1-z)}{n}}.$$

Then,

$$|g_n(z)| \le e^{-\frac{(\mathrm{Im}\, z)^2}{n}}$$

for z in the strip. By the maximum principle,

$$|f(z)| \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta} e^{\frac{y^2}{n}}.$$

Letting $n \to \infty$, we are done. □

4.7 (Riesz-Thorin interpolation)**.** Let X, Y be σ-finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \qquad \frac{1}{q_\theta} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \to q_\theta} \le \|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta}.$$

Proof. Note that

$$\|T\|_{p_\theta \to q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g\rangle|}{\|f\|_{p_\theta}\|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z\rangle = \int \overline{g_z(y)}\, Tf_z(y)\, dy,$$

where f_z and g_z are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z) + \frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have $f_\theta = f$ and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for $\mathrm{Re}\, z = x$.

Then,

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0} \|f_z\|_{p_0} \|g_z\|_{q'_0} = \|T\|_{p_0 \to q_0} \|f\|_{p_\theta}^{p_\theta/p_0} \|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for $\mathrm{Re}\, z = 0$, and

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_1 \to q_1} \|f_z\|_{p_1} \|g_z\|_{q'_1} = \|T\|_{p_1 \to q_1} \|f\|_{p_\theta}^{p_\theta/p_1} \|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for $\mathrm{Re}\, z = 1$. By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta} \|f\|_{p_\theta} \|g\|_{q'_\theta}$$

for $\mathrm{Re}\, z = \theta$. Putting $z = \theta$ in the last inequality, we get the desired result. □

Chapter 5

Topological measures

5.1 Borel measures

5.2 Locally compact spaces

5.1 (One-point compactification).

5.3 Locally finite measures

5.2 (Regular Borel measures on locally compact metric spaces). sss

(a) $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

(b) If μ is σ-finite, then for any $\varepsilon > 0$ there is compact $K \subset X$ and continuous $g : X \to \mathbb{R}$ such that $f|_K = g|_K$ and $\mu(X \setminus K) < \varepsilon$.

5.3 (Tightness and inner regularity). (a)

5.4 (Regular Borel measures on metric spaces). Let μ be a Borel measure on a metric space X. We say μ is *outer regular* if

$$\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\},$$

and say μ is *inner regular* if

$$\mu(E) = \sup\{\mu(F) : F \subset E,\, F \text{ closed}\},$$

for every Borel subset $E \subset X$. If μ is both outer and inner regular, we say μ is *regular*.

(a) Let E be σ-finite. Then, E is μ-regular if and only if for any $\varepsilon > 0$ there are open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.

(b) If μ is σ-finite, then the set of μ-regular subsets is a σ-algebra. (may be extended?)

(c) Every closed set is G_δ.

(d) Every finite Borel measure on X is regular.

Proof. □

5.5 (Luzin's theorem). Let μ be a regular Borel measure on a metric space X. Let $f : X \to \mathbb{R}$ be a Borel measurable function. Two proofs: direct and Egoroff.

(a) If $E \subset X$ is σ-finite, then there is a continuous g blabla

(b) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset X$ such that $f|_F : F \to \mathbb{R}$ is continuous and $\mu(X \setminus F) < \varepsilon$.

(c) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset X$ and continuous $g : X \to \mathbb{R}$ such that $f|_F = g|_F$ and $\mu(X \setminus F) < \varepsilon$.

(d) If f is further bounded, then g also can be taken to be bounded.

Proof. (a) Let $\varepsilon > 0$ and suppose $E \subset X$ is measurable with $\mu(E) < \infty$. Since E is σ-finite, we have open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon/2$. By the Urysohn lemma, there is a continuous function $g : X \to [0, 1]$ such that $g|_{U^c} = 0$ and $g|_F = 1$. Then,

$$\int |\mathbf{1}_E - g|\, d\mu = \int_{U \setminus F} |\mathbf{1}_E - g|\, d\mu \leq 2\mu(U \setminus F) < \varepsilon.$$

(b) Since $\mathbb{R}$ is second countable, we have a base $(V_n)_{n=1}^{\infty}$ of $\mathbb{R}$. Since μ is σ-finite, for each n we can take open U_n and closed F_n such that

$$F_n \subset f^{-1}(V_n) \subset U_n$$

and $\mu(U_n \setminus F_n) < \varepsilon/2^n$. Define $F := \left(\bigcup_{n=1}^{\infty}(U_n \setminus F_n)\right)^c$ so that $\mu(X \setminus F) < \varepsilon$ and F is closed. Then,

$$\begin{aligned} U_n \cap F &= U_n \cap ((U_n^c \cup F_n) \cap F) \\ &= (U_n \cap (U_n^c \cup F_n)) \cap F \\ &= (\varnothing \cup (U_n \cap F_n)) \cap F \\ &\subset F_n \cap F \end{aligned}$$

proves $f^{-1}(V_n)$ is open in F for every n, hence the continuity of $f|_F$. (In fact, we require that X to be just a topological space.)

(b') We can alternatively use the part (a) and the Egoroff theorem. By the part (a), we can construct a sequence (f_n) of continuous functions $X \to \mathbb{R}$ such that $f_n \to f$ in L^1. By taking a subsequence, we may assume $f_n \to f$ pointwise. Assuming μ is finite, by the Egorov theorem, there is a measurable $A \subset X$ such that $f_n \to f$ uniformly on A and $\mu(X \setminus A) < \varepsilon/2$. Since μ is inner regular, we have closed $F \subset A$ such that $\mu(A \setminus F) < \varepsilon/2$, so that we have $\mu(X \setminus F) < \varepsilon$. Then, f is continuous on A, and of course on F.

□

Proposition 5.3.1. *A σ-finite Radon measure is regular.*

Proof. First we approximate Borel sets of finite measure, with compact sets. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U - E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned} \mu(K - V) &\geq \mu(K) - \mu(V) \\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U - E) + \frac{\varepsilon}{2}\right) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^{\infty} K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^{\infty} \mu(K_n) > \sum_{n=1}^{\infty} \left(\mu(E_n) - \frac{\varepsilon}{2^n} \right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. □

5.4 Continuous functions in L^p spaces

Approximate identity density

Chapter 6

Dual spaces

6.1 Dual of Lebesgue spaces

Radon-Nikodym theorem

An integrable function as a measure σ-finite measures

6.2 Riesz-Markov-Kakutani representation theorem

locally finite tight measure.

6.1 (Radon measures)**.** Let X be a locally compact metric space. A *Radon measure* is a Borel measure μ on X such that

(i) μ is outer regular for every Borel set: $\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\}$ for Borel $E \subset X$,

(ii) μ is inner regular for every open set: $\mu(U) = \sup\{\mu(K) : K \subset U,\, K \text{ compact}\}$ for open $U \subset X$,

(iii) μ is locally finite.

(a) A σ-finite Radon measure is regular.

(b) If every open subset of X is σ-compact, then a locally finite Borel measure is Radon.

(c) $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

6.2 (Riesz-Markov-Kakutani representation theorem for $C_0(X)$)**.** Let X be a locally compact metric space. We want to establish the following one-to-one correspondence:

$$\begin{array}{ccc} \{\text{finite Radon measures on } X\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_0(X)\} \\ \mu & \mapsto & (f \mapsto \int f\, d\mu). \end{array}$$

Let I a positive linear functional on $C_0(X)$. Let $\mathcal{T}$ be the set of all open subsets of X and $\mu_0 : \mathcal{T} \to [0,\infty]$ a set function defined such that

$$\mu_0(U) := \sup\{I(f) : f \in C_c(U,[0,1])\}, \qquad U \in \mathcal{T}.$$

Let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be the associated outer measure defined by

$$\mu^*(S) := \inf\left\{\sum_{i=1}^{\infty} \mu_0(U_i) : S \subset \bigcup_{i=1}^{\infty} U_i,\ U_i \in \mathcal{T}\right\}, \qquad S \in \mathcal{P}(X),$$

and let $\mu := \mu^*|_{\mathcal{A}}$ be the restriction, where $\mathcal{A}$ is the σ-algebra of Carathéodory measurable subsets relative to μ^*.

(a) μ^* extends μ_0.

(b) μ extends μ_0.

(c) μ is a finite Radon measure.

(d) The correspondence is surjective.

(e) The correspondence is injective.

Proof. (a) It suffices to show that μ_0 satisfies monotonically countably subadditive. For an open set U and a countable open cover $\{U_i\}_{i=1}^{\infty}$ of U we claim that $\rho(U) \le \sum_{i=1}^{\infty} \rho(U_i)$.

Take any $f \in C_c(U,[0,1])$ and find a finite subcover $\{U_{i_k}\}_{k=1}^{n}$ of $\{U_i\}$ together with a partition of unitiy $\{\chi_{i_k}\}$ subordinate to the open cover $\{U_{i_k} \cap \operatorname{supp} f\}_k$. Now we have $f\chi_{i_k} \in C_c(U_{i_k},[0,1])$ for each k, because then I is linear so that it preserves finite sum, we have

$$I(f) = \sum_{k=1}^{n} I(f\chi_{i_k}) \le \sum_{k=1}^{n} \mu_0(U_{i_k}) \le \sum_{i=1}^{\infty} \mu_0(U_i).$$

Since f is arbitrary, we are done.

(b) We claim $\mathcal{T} \subset \mathcal{A}$. It suffices to show $\mu^*(E \cap U) + \mu^*(E \setminus U) \le \mu^*(E)$ for any measurable E and open U. Take $\varepsilon > 0$. Since we may assume $\mu^*(E) < \infty$, there is a countable open cover $\{U_i\}_{i=1}^{\infty}$ of E such that

$$\sum_{i=1}^{\infty} \mu_0(U_i) < \mu^*(E) + \frac{\varepsilon}{3}.$$

Take $f_i \in C_c(U_i \cap U,[0,1])$ such that

$$\mu_0(U_i \cap U) < I(f_i) + \frac{1}{3} \cdot \frac{\varepsilon}{2^i},$$

and take $g_i \in C_c(U_i \setminus \operatorname{supp} f_i,[0,1])$ such that

$$\mu_0(U_i \setminus \operatorname{supp} f_i) < I(g_i) + \frac{1}{3} \cdot \frac{\varepsilon}{2^i}.$$

Then, since $f_i + g_i \in C_c(U_i,[0,1])$, we have

$$\begin{aligned}
\mu^*(E \cap U) + \mu^*(E \setminus U) &\le \sum_{i=1}^{\infty} \mu_0(U_i \cap U) + \sum_{i=1}^{\infty} \mu_0(U_i \setminus U) \\
&< \sum_{i=1}^{\infty} I(f_i + g_i) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \sum_{i=1}^{\infty} \mu_0(U_i) + \frac{2}{3}\varepsilon \\
&\le \mu^*(E) + \varepsilon.
\end{aligned}$$

Limiting $\varepsilon \to 0$, we get the desired inequality.

(c) Since μ is a countably additive and $\mathcal{T}$ is closed under union, we can rewrite

$$\mu^*(S) = \inf\{\mu_0(U) : S \subset U \in \mathcal{T}\}, \qquad S \in \mathcal{P}(X),$$

hence μ is outer regular. Here now we claim for $f \in C_c(X,[0,1])$ and $0 < a < 1$ that

$$a\mu(f^{-1}((a,1])) \le I(f) \le \mu(\operatorname{supp} f).$$

If it is true, then the right inequality implies the inner regularity, and the left inequality together with the Urysohn lemma implies the local finiteness.

The right inequality directly follows from the definition of μ_0 and the outer regularity

$$I(f) \le \inf\{\mu_0(U) : \operatorname{supp} f \subset U \in \mathcal{T}\} = \mu(\operatorname{supp} f).$$

For the left, if $h \in C_c(f^{-1}((a,1]),[0,1])$, then the inequality $ah \le f$ implies

$$a\mu(f^{-1}((a,1])) = a\mu_0(f^{-1}((a,1])) \le aI(h) \le I(f).$$

(d) We will show $I(f) = \int f\,d\mu$ for $f \in C_c(X)$. Since $C_c(X)$ is the linear span of $C_c(X,[0,1])$, we may assume $f \in C_c(X,[0,1])$. For a fixed positive integer n and for each index $1 \le i \le n$, let $K_i := f^{-1}([i/n,1])$ and define

$$f_i(x) := \begin{cases} \frac{1}{n} & \text{if } x \in K_i, \\ f(x) - \frac{i-1}{n} & \text{if } x \in K_{i-1} \setminus K_i, \\ 0 & \text{if } x \in X \setminus K_{i-1}, \end{cases}$$

where $K_0 := \operatorname{supp} f$. Note that $f_i \in C_c(X,[0,n^{-1}])$ and $f = \sum_{i=1}^n f_i$. For $1 \le i \le n$ we have $\mu(K_i) < \infty$ because K_i is compact subsets contained in a locally compact Hausdorff space $U := f^{-1}((0,1])$. By the previous claim and the property of integral, we have

$$\frac{\mu(K_i)}{n} \le I(f_i), \qquad \frac{\mu(K_i)}{n} \le \int f_i\,d\mu, \qquad 1 \le i \le n$$

and

$$I(f_i) \le \frac{\mu(K_{i-1})}{n}, \qquad \int f_i\,d\mu \le \frac{\mu(K_{i-1})}{n}, \qquad 2 \le i \le n.$$

Then, using the above inequalities and $\mu(K_n) \ge 0$, we have

$$I(f) \le I(f_1) + \int f\,d\mu \quad \text{and} \quad \int f\,d\mu \le \int f_1\,d\mu + I(f).$$

Note that $f_1 = \min\{f, n^{-1}\}$ is a sequence of functions indexed by n. By the monotone convergence theorem, $\int f_1\,d\mu \to 0$ as $n \to \infty$. We now show $I(f_1)$ converges to zero. If we let $U := f^{-1}((0,1])$, then U is locally compact and $f_1 \in C_0(U) \subset C_c(X)$, and since a positive linear functional on $C_0(U)$ is bounded, we have $I(f_1) \le n^{-1}\|I\| \to 0$ as $n \to \infty$. ($\mu(K_0)$ is possibly infinite if X is not locally compact so that μ is not locally finite.)

(e) Let μ and ν be finite Radon measures on X such that

$$\int g\,d\mu = \int g\,d\nu$$

for all $g \in C(X)$. Let E be any measurable set. Since $\mu + \nu$ is a finite Radon measure, and by the Luzin theorem, we have a closed set F and $g \in C(X)$ with $0 \le g \le 1$ such that $\mathbf{1}_E|_F = g|_F$ and $(\mu+\nu)(X \setminus F) < \varepsilon/2$. Then,

$$\begin{aligned} |\mu(E) - \nu(E)| &= |\int \mathbf{1}_E\,d\mu - \int \mathbf{1}_E\,d\nu| \\ &\le \int_{X\setminus F} |\mathbf{1}_E - g|\,d\mu + \int_{X\setminus F} |g - \mathbf{1}_E|\,d\nu \\ &\le 2\mu(X \setminus F) + 2\nu(X \setminus F) < \varepsilon. \end{aligned}$$

By limiting $\varepsilon \to 0$, we have $\mu(E) = \nu(E)$. □

6.3 (Dual of continuous function spaces)**.**

Fremlin

Note that the inner regularity by Folland or Rudin is in fact the tightness, the inner regularity with respect to compact sets.

- A Fremlin-Radon measure is tight.
- A σ-finite Folland-Radon measure on a locally compact Hausdorff space is tight. Moreover, Folland-Radon and Fremlin-Radon coincides on σ-compact locally compact Hausdorff spaces.
- A locally finite Borel measure on a locally compact Hausdorff and second countable space is tight.
- A locally compact Hausdorff and second countable space is Polish.
- A tight measure on a topological space is always inner regular with respect to closed sets, and the converse is true on where???

Definitions

- A measurable algebra is called *localizable* if the essential union exists even for uncountable family of measurable sets.
- A *localizble measure* is a semi-finite measure on a localizable measurable algebra.
- A *strictly localizable measure* or *decomposable measure* is a measure which admits a partition $\{F_i\}$ of X, called the decomposition, such that F_i are finite measurable and $E \cap F_i \in \Sigma$ for all F_i implies $E \in \Sigma$ and $\mu(E) = \sum_{i \in J} \mu(E \cap F_i)$.
- A *locally determined measure* is a semi-finite measure such that $E \cap F \in \Sigma$ for any $F \in \Sigma$ of finite measure implies $E \in \Sigma$.(I think it is more natural to say a enhanced measurable space is locally determined by a semi-finite measure)

Locally finite measures

- A σ-finite measure is strictly localizable.
- A strictly localizable measure is localizable and locally determined.
- A tight measure on a topological space is τ-additive.
- A locally finite measure on a topological space is finite on compact sets.
- A locally finite measure on a Lindelöf space is σ-finite.
- A locally finite and tight measure is effectively locally finite.
- A effectively locally finite(non-negligible set has an open set of finite measure whose intersection with it is non-negligible) measure on a topological space is semi-finite.
-

Radon and quasi-Radon measures: A *quasi-Radon measure* on a Hausdorff space is a measure which is complete, locally determined, τ-additive, inner regular with respect to closed sets, and effectively locally finite. A *Radon measure* on a Hausdorff space is a measure which is complete, locally determined, locally finite, and tight. By the completeness condition, it is not Borel in general.

- 415A A quasi-Radon measure is strictly localizble.
- 416C For a locally finite quasi-Radon measure μ, μ is Radon iff
- 416F A Borel measure on a Hausdorff space has a Radon extension if and only if it is locally finite and tight, and in this case the extension is unique.
- 416G A locally finite quasi-Radon measure is Radon.

Riesz-Markov-Kakutani 436J and 436K

Proof. First we can show I is smooth(I think it is equivalent to normality). Since X is locally compact, it is the coarsest topology for which C_c is continuous, i.e. Baire=Borel. Also, C_c is truncated Riesz subspace of $\mathbb{R}^X$. So 436H implies there is a quasi-Radon measure μ such that $I(f) = \int f \, d\mu$ for $f \in C_c$, which is clearly locally finite. By 416G, μ is Radon. □

6.3 Dual of continuous function spaces

signed measure Hahn, Jordan decomposition

Part III

Distribution theory

Chapter 7

Test functions

Chapter 8

Distributions

Chapter 9

Linear operators

9.1 Boundedness

Translation and multiplication operators

9.1 (Bitranspose extension)**.**

9.2 Kernels

9.2 (Schur test)**.**

9.3 (Young's inequality of integral operators)**.**

9.3 Convolution

9.4 (Approximation of identity)**.** Fejér, Poisson, box?

9.5 (Summability methods)**.**

Part IV

Fundamental theorem of calculus

Chapter 10

10.1 Absolutely continuous functions

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\mathrm{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : X_x \times X_y \to \mathbb{R}$ be such that $\partial_{x_i}f$ is well-defined. Suppose f and $\partial_{x_i}f$ are locally integrable in x and integrable y.

Then,

$$\partial_{x_i}\int f(x,y)\,dy = \int \partial_{x_i}f(x,y)\,dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if F is C^2 then the *classical* partial differentiation commute, and if F is not C^2 then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}

(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞

(b) f is AC iff f' is L^1

(c) f is BV iff f' is a finite regular Borel measure

10.3 (Absolute continuous measures)**.**

10.4 (Absolute continuous functions)**.**

10.2 Functions of bounded variation

Chapter 11

Lebesgue differentiation theorem

11.1 Hardy-Littlewood maximal function

Let T_m be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m,\ \varepsilon:|\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

11.1 (Hardy-Littlewood maximal function)**.** The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

11.2 (Weak type estimate)**.**

$$\|Mf\|_{1,\infty} \le 3^d \|f\|_{L^1(X)}.$$

(a) Proof by covering lemma.

Proof. (a) By the inner regularity of μ, there is a compact subset K of $\{|Mf| > \lambda\}$ such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every $x \in K$, since $|Mf(x)| > \lambda$, we can choose an open ball B_x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover $\{B_i\}_i$ of K. Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection $\{B_k\}_k$ such that

$$K \subset \bigcup_i Bi \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \le \sum_k 3^d \mu(B_k) \le \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \le \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of B_k's. $\square$

11.3 (Radially bounded approximate identity)**.** If an approximate identity K_n is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every n and x, hence has a weak type estimate.

11.4 (Almost everywhere convergence of operators)**.** Suppose is T_m is a sequence of linear operators such that the maximal function T^*f is dominated by Mf. If $f \in L^1(X)$ and $T_m g \to g$ pointwise for $g \in C(X)$, then $T_m f \to f$ a.e.

Proof. Take $\varepsilon > 0$ and $g \in C(X)$ such that $\|f - g\|_{L^1(X)} < \varepsilon$. Since $T_m g(x) \to g(x)$ pointwise, we have

$$\begin{aligned}
&\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\
&\quad \leq \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \tfrac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \tfrac{\lambda}{2}\}) \\
&\quad \leq \mu(\{x : M(f-g)(x) > \tfrac{\lambda}{2}\}) + \frac{2}{\lambda}\|f - g\|_{L^1(X)} \\
&\quad \lesssim \frac{1}{\lambda}\varepsilon
\end{aligned}$$

for every $\lambda > 0$. Limiting $\varepsilon \to 0$, we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every $\lambda > 0$, hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

Definition 11.1.1.

$$f^*(x) := \lim_{r \to 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\, dy.$$

Theorem 11.1.2 (Lebesgue differentiation)**.** $f^* = 0$ *a.e.*

Proof. Note that $f^* \leq Mf + |f|$ implies

$$\|f^*\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that $g^* = 0$ for $g \in C_c$. Approximate using $f^* = (f - g)^*$. □

Exercises

11.5 (Doubling measure)**.**