Harmonic Analysis

Ikhan Choi

July 19, 2022

Contents

Part I

Fourier analysis

Chapter 1

Fourier series

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

Riemann-Lebesgue Lp extension

Gaussian function computation: differential equation method, contour integral method inversion theorem Plancherel

2.2 Tempered distributions

Exercises

Chapter 3

Part II

Singular integral operators

Chapter 4

Caldéron-Zygmund theory

4.1 Hilbert transform

4.2 Calderón-Zygmund operators of convolution type

4.1 (Calderón-Zygmund decomposition of sets)**.** Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf = \sup_n E_n|f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|}\int_{Q_x} |f| > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \le \frac{1}{\lambda}\|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\mathbb{R}^d\setminus\Omega)} \le \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|}\int_{Q_x} |f| \le 2^d\lambda.$$

4.2 (Calderón-Zygmund decomposition of functions)**.** Let

$$g(x) := \begin{cases} |f(x)| & ,x \notin \Omega \\ \frac{1}{|Q_x|}\int_{Q_x}|f| & ,x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \le 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof. □

4.3 (Calderón-Zygmund operators of convolution type)**.** Let $T : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that

$$Tf(x) = \int K(x-y)f(y)\,dy$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \operatorname{supp} f$. If T is L^2-bounded

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$$

and satisfies the *Hörmander condition*

$$\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx \lesssim 1,$$

then it is called a *Calderón-Zygmund* operator.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \le \frac{4}{\lambda^2}\|Tg\|_{L^2(\Omega)}^2 \le \frac{4C}{\lambda^2}\|g\|_{L^2(\Omega)}^2 \le \frac{4C}{\lambda^2}\|g\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \le \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega^*} |Tb(x)|\,dx \le \frac{2}{\lambda}\sum_i \int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\operatorname{supp} b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y)b_i(y)\,dy = \int_{Q_i} [K(x-y)-K(x)]b_i(y)\,dy,$$

and

$$\int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d\setminus Q_i^*} |K(x-y)-K(x)|\,dx\,dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.3 L^2-boundedness of truncated integrals

4.4 Calderón-Zygmund operators of non-convolution type

standard kernels

Exercises

4.4 (Gradient size condition)**.** Let $|\nabla K(x)| \lesssim \frac{1}{|x|^{d+1}}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Part III

Oscillatory integral operators

Chapter 7

Stationary phase

Chapter 8

Restriction and Kekeya problems

Chapter 9

Dispersive equations

Part IV

Pseudo-differential operators

Chapter 10

10.1

$S^m_{\rho,\delta}$

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \lesssim \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}.$$

Let a be a symbol on $M = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$. Then, the associated ΨDO is

$$T_a\psi(x) := \frac{1}{(2\pi)^d} \iint e^{i\langle x-y,\xi\rangle} a(x,\xi)\psi(y)\,dy\,d\xi.$$

For parameters $0 \le \lambda \le 1$ and $h > 0$, let

$$\widehat{a}\psi(x) := \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle} a((1-\lambda)x + \lambda y, \xi)\psi(y)\,dy\,d\xi.$$

For example, regardless of h and λ,

$$\widehat{\xi}\psi(x) = \frac{h}{i}\psi'(x)$$

and

$$\widehat{H}\psi(x) = -h^2\Delta\psi(x) + V(x)\psi(x),$$

where $V : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ and $H : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ such that

$$H(x,\xi) := |\xi|^2 + V(x).$$

$$\frac{d}{dt}a(t) = \{a(t), H\} = X_H a(t)$$

$$\frac{d}{dt}\widehat{a}(t) = \frac{d}{dt}e^{\frac{i}{h}t\widehat{H}}\widehat{a}e^{-\frac{i}{h}t\widehat{H}} = -\frac{i}{h}[\widehat{a}(t), \widehat{H}]$$

Chapter 11

Semiclassical analysis

11.1 Quantization

11.1 (Composition of Weyl quantization).

Chapter 12

Microlocal analysis