

Algebraic Topology

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Contents

I	2
1 Convenient categories	3
1.1 Compactly generated weakly Hausdorff spaces	3
1.2 CW complexes	3
1.3 Simplicial complexes	3
2 Cohomology operations	4
2.1 Eilenberg-Steenrod axioms	4
3 Spectral sequences	5
3.1 Serre spectral sequence	5
3.2 Adams spectral sequence	5
4 Fiber bundles	6
4.1 Principal bundles	6
4.2 Vector bundles	10
4.3 Classifying spaces	10
4.4 Characteristic classes	11
5	13
5.1 Obstruction theory	13
5.2 Hurewicz theorem	13
6 Simplicial methods	14
II Stable homotopy theory	15
7	16
7.1 Homotopy groups of spheres	16
III Generalized cohomology theories	17
8	18
9 K-theory	19
10 Cobordisms	20

Part I

Chapter 1

Convenient categories

1.1 Compactly generated weakly Hausdorff spaces

bicomplete cartesian closed monoidal category. Here, closed means that the right tensoring admits an adjoint called internal hom functor. braided and symmetric...?

A pointed space is a pair of a space and a 0-cell. The smash product is the categorical product in the category of pointed space.

Let \mathbf{Top} be the bicomplete cartesian closed monoidal category of CGWH spaces.

1.2 CW complexes

1.3 Simplicial complexes

Chapter 2

Cohomology operations

2.1 Eilenberg-Steenrod axioms

cohomology cup product universal coefficient theorem Poincaré duality

Exercises

characteristic class of projective spaces

Chapter 3

Spectral sequences

3.1 Serre spectral sequence

(Lyndon-Hochschild-Serre)

3.2 Adams spectral sequence

Chapter 4

Fiber bundles

4.1 Principal bundles

4.1 (Locally trivial fiber bundles). Let $p : E \rightarrow B$ be a map and F a space. An *atlas* or *local trivialization* for p with respect to F is a family $\{\varphi_\alpha\}$ of homeomorphisms $\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ satisfying $p = \text{pr}_{U_\alpha} \varphi_\alpha$ as maps $p^{-1}(U_\alpha) \rightarrow U_\alpha$, indexed by an open cover $\{U_\alpha\}$ of B :

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ & \searrow p \quad \hookrightarrow & \swarrow \\ & U_\alpha & \end{array} .$$

A *locally trivial fiber bundle*, or just a *fiber bundle*, is a map $p : E \rightarrow B$ together with an equivalence class of atlases $\{\varphi_\alpha\}$ for p with respect to F . The spaces F, E, B are called the *fiber space*, *total space*, and *base space* of the fiber bundle p . A *bundle map* between fiber bundles $p_i : E_i \rightarrow B_i$ with fiber space F_i for $i \in \{1, 2\}$ is a map of pairs $(u, f) : (E_1, B_1) \rightarrow (E_2, B_2)$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{u} & E_2 \\ p_1 \downarrow & \hookrightarrow & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2. \end{array}$$

The category $\text{Bun}_F(B)$ is defined such that objects are fiber bundles over a base space B with fiber space F , and morphisms are commutative squares.

(a) p is surjective and open.

4.2 (Structure groups). Let $p : E \rightarrow B$ be a fiber bundle whose fiber space F is an effective left G -space for a topological group G . An atlas $\{\varphi_\alpha\}$ for p is called a *G-atlas* if there is a family $\{g_{\alpha\beta}\}$, which is unique if it exists by effectiveness of F , of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ such that

$$\varphi_\beta \varphi_\alpha^{-1}(b, v) = (b, g_{\alpha\beta}(b)v), \quad b \in U_\alpha \cap U_\beta, v \in F.$$

A *G-bundle* is a fiber bundle p whose fiber space is an effective left G -space together with an equivalence class of G -atlases $\{\varphi_\alpha\}$ for p .

A bundle map $(u, f) : (E_1, B_1) \rightarrow (E_2, B_2)$ between G -bundles $p_i : E_i \rightarrow B_i$ with a common fiber space F for $i \in \{1, 2\}$ is called a *G-bundle map* if there is a family $\{h_{\alpha\beta}\}$, which is unique if it exists also by effectiveness of F , of maps $h_{\alpha\beta} : U_{1\alpha} \cap f^{-1}(U_{2\beta}) \rightarrow G$ such that

$$\varphi_{2\beta} u \varphi_{1\alpha}^{-1}(b, v) = (f(b), h_{\alpha\beta}(b)v), \quad b \in U_{1\alpha} \cap f^{-1}(U_{2\beta}), v \in F,$$

where $\{\varphi_{i\alpha}\}$ is a G -atlas for p_i . Note that the definition of G -bundle maps does not depend on the choice of G -atlases of p_i .

- (a) A G -bundle map (u, f) is an isomorphism if and only if u is a homeomorphism. In particular, the category of G -bundles over B with fiber space F is a groupoid.
- (b) A bundle map $(u, \text{id}_B) : (E_1, B) \rightarrow (E_2, B)$ is a G -bundle map if and only if there is a family $\{h_\alpha\}$ of maps $h_\alpha : U_\alpha \rightarrow G$ such that

$$\varphi_{2\alpha} u \varphi_{1\alpha}^{-1}(b, v) = (b, h_\alpha(b)v), \quad b \in U_\alpha, v \in F,$$

where $\{\varphi_{i\alpha}\}$ is a G -atlas for p_i defined on a common open cover $\{U_\alpha\}$ of B .

Proof.

□

4.3 (Fiber bundle construction theorem). Let $\{U_\alpha\}$ be an open cover of a space B , and G a topological group. A Čech 1-cocycle on $\{U_\alpha\}$ with coefficients in G is a family $\{g_{\alpha\beta}\}$ of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ satisfying the cocycle condition:

$$g_{\alpha\gamma}(b) = g_{\beta\gamma}(b)g_{\alpha\beta}(b), \quad b \in U_\alpha \cap U_\beta \cap U_\gamma.$$

A Čech 0-cochain on $\{U_\alpha\}$ with coefficients in G is a family $\{h_\alpha\}$ of maps $h_\alpha : U_\alpha \rightarrow G$ of maps without any conditions. The set of Čech 1-cocycles and Čech 0-cochains on $\{U_\alpha\}$ with coefficients in G are denoted by $\check{Z}^1(\{U_\alpha\}, G)$ and $\check{C}^0(\{U_\alpha\}, G)$ respectively. The first Čech cohomology $\check{H}^1(\{U_\alpha\}, G)$ of $\{U_\alpha\}$ with coefficients in G is defined to be the orbit space of an action of $\check{C}^0(\{U_\alpha\}, G)$ on $\check{Z}^1(\{U_\alpha\}, G)$ defined as

$$(\{h_\alpha\}\{g_{\alpha\beta}\})_{\alpha\beta}(b) := h_\beta(b)g_{\alpha\beta}(b)h_\alpha(b)^{-1}, \quad b \in U_\alpha \cap U_\beta.$$

We define the first Čech cohomology of B with coefficients in G as the direct limit of sets

$$\check{H}^1(B, G) := \varinjlim_{\{U_\alpha\}} \check{H}^1(\{U_\alpha\}, G).$$

Let F be an effective left G -space. Let $\text{Bun}_F(B)$ be the set of all isomorphism classes of G -bundles over B with fiber space F .

- (a) $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$ is well-defined and surjective.
- (b) $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$ is injective.

Proof. (a) Suppose $p_i : E_i \rightarrow B$ are isomorphic G -bundles with fiber space F , and $u : E_1 \rightarrow E_2$ is a G -bundle isomorphism. Considering the refinement, we fix an open cover $\{U_\alpha\}$ of B on which E_i are simultaneously locally trivialized.

Let $\{g_{\alpha\beta}\} \in \check{Z}^1(\{U_\alpha\}, G)$. Define

$$E := \left(\bigsqcup_{\alpha} (U_\alpha \times F) \right) / \sim,$$

where \sim is an equivalence relation generated by

$$(b, v)_\alpha \sim (b, g_{\alpha\beta}(b)v)_\beta, \quad b \in U_\alpha \cap U_\beta, v \in F.$$

Define $p : E \rightarrow B$ and $\varphi_\alpha^{-1} : U_\alpha \times F \rightarrow p^{-1}(U_\alpha)$ such that

$$p([(b, v)_\alpha]) := b, \quad \varphi_\alpha^{-1}(b, v) := [(b, v)_\alpha], \quad b \in U_\alpha, v \in F.$$

They are clearly continuous and surjective. We need to show φ_α^{-1} is a homeomorphism.

We first claim that φ_α^{-1} is injective. Assume $\varphi_\alpha^{-1}(b, v) = \varphi_\alpha^{-1}(b', v')$, which means $(b, v)_\alpha \sim (b', v')_\alpha$. Then, we have $b = b'$ and there is a finite sequence $(\alpha_i)_{i=0}^n$ such that $\alpha_0 = \alpha_n = \alpha$ and

$$v' = g_{\alpha_{n-1}\alpha_n}(b)g_{\alpha_{n-2}\alpha_{n-1}}(b)\cdots g_{\alpha_0\alpha_1}(b)v.$$

Applying the cocycle condition inductively, we obtain $v = v'$, which implies the injectivity of φ_α^{-1} .

Next we claim that φ_α^{-1} is open. The map φ_α^{-1} is given by the composition

$$\varphi_\alpha^{-1} : U_\alpha \times F \xrightarrow{\iota} \bigsqcup_\alpha U_\alpha \times F \xrightarrow{\pi} E,$$

where ι and π are the canonical inclusion and the canonical projection. Since ι is clearly open, it suffices to show π is open. Suppose $V \subset \bigsqcup_\alpha U_\alpha \times F$ is open so that we have $V = \bigsqcup_\alpha V_\alpha \times F$ for open subsets $V_\alpha \subset U_\alpha$. Observe that for each β we have $(b, v)_\beta \in \pi^{-1}\pi(V_\alpha \times F)$ if and only if $(b, v)_\beta \sim (b', v')_\alpha$ for some $(b', v') \in V_\alpha \times F$. It is equivalent to that $b = b'$ and

$$v' = g_{\alpha_{n-1}\alpha_n}(b)g_{\alpha_{n-2}\alpha_{n-1}}(b)\cdots g_{\alpha_0\alpha_1}(b)v,$$

where $(\alpha_i)_{i=0}^n$ is a finite sequence such that $\alpha_0 = \beta$ and $\alpha_n = \alpha$. Applying the cocycle condition inductively, we can see that it is just $g_{\alpha\beta}(b)v' = v$. Thus, the set $\pi^{-1}\pi(V_\alpha \times F) \cap (U_\beta \times F)$ is exactly the same as the inverse image of the open set $V_\alpha \times F$ under the map

$$(U_\alpha \cap U_\beta) \times F \rightarrow U_\alpha \times F : (b, v) \mapsto (b, g_{\alpha\beta}(b)v).$$

It concludes that

$$\pi^{-1}\pi(V) = \bigcup_{\alpha, \beta} \pi^{-1}\pi(V_\alpha \times F) \cap (U_\beta \times F)$$

is open, so $\pi(V)$ is open. Therefore, π is open.

Finally, it obviously follows that the transition maps of the G -atlas $\{\varphi_\alpha\}$ constructed as above coincides with the cocycle $\{g_{\alpha\beta}\}$ up to 0-cochain by the cocycle condition, hence the surjectivity.

(b) □

4.4 (Principal bundles). Let G be a topological group. A *principal G -bundle* is a fiber bundle $\pi : P \rightarrow B$ together with a continuous right action of G on P such that $\pi^{-1}(b)$ is a right principal homogeneous G -space for each $b \in B$. The right action ρ is called the *principal right action* or (*global*) *gauge transformation*. Having right G -equivariant maps as morphisms, let $\text{Prin}_G(B)$ be the isomorphism classes of principal G -bundles over B .

Let F be an effective left G -space. For a principal G -bundle $\pi : P \rightarrow B$, define $p : E \rightarrow B$ by

$$E := P \times_G F = P \times F / \sim, \quad p([u, v]) := \pi(u),$$

where the equivalence relation \sim is generated by

$$(ug, v) \sim (u, gv), \quad u \in P, g \in G, v \in F.$$

The G -bundle p with fiber space F is called the *associated bundle* to the principal bundle π . The associated bundles gives rise to a natural function $\text{Prin}_G(B) \rightarrow \text{Bun}_F(B)$.

- (a) $\text{Prin}_G(B) \rightarrow \text{Bun}_F(B)$ is well-defined.
- (b) $\text{Prin}_G(B) \rightarrow \text{Bun}_F(B)$ is injective.
- (c) $\text{Prin}_G(B) \rightarrow \text{Bun}_F(B)$ is surjective.
- (d) A principal bundle is trivial if it has a global section.

Proof. (a) Let $u : P \rightarrow P'$ be a right G -equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \rightarrow G$ such that

$$(b, g_i(b)x_0) := \varphi_i \circ u \circ \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xrightarrow{\varphi_i \circ u \circ \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The continuity of the last map is due to the assumption that the map $(g, x) \mapsto (gx, x)$ is a homeomorphism.

Then, for every $(b, x) \in U_i \times X$ there is a unique $s \in G$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned} \varphi_i \circ u \circ \varphi_i^{-1}(b, x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\ &= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b, x_0) \\ &= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b, x_0) \\ &= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b, x_0) \\ &= (\varphi_i \circ \rho_s \circ \varphi_i^{-1})g_i(b)(b, x_0) \\ &= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\ &= g_i(b)(b, x) \\ &= (b, g_i(b)x). \end{aligned}$$

Hence, u is a G -bundle map.

(b) Let $u : P \rightarrow P'$ be a G -bundle map over B so that there is a set $\{h_i : U_i \rightarrow G\}_i$ of maps such that

$$\varphi_i \circ u \circ \varphi_i^{-1}(b, x) = (b, h_i(b)x), \quad b \in U_i, x \in X.$$

If we write $\rho_s : P \rightarrow P : e \mapsto \rho(e, s)$ for $s \in G$, then the induced right action $\varphi_i \circ \rho_s \circ \varphi_i^{-1}$ commutes with the left action $\varphi_i \circ u \circ \varphi_i^{-1}$ on $U_i \times X$. Now for every $e \in P_1$, we have

$$\begin{aligned} \rho_s \circ u(e) &= \varphi_i^{-1} \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= \varphi_i^{-1} \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= u \circ \rho_s(e), \end{aligned}$$

therefore u is right G -equivariant.

(c) Fix $x_0 \in X$ and consider the homeomorphism $G \rightarrow X : g \mapsto gx_0$. Define a right action

$$X \times G \rightarrow X : (gx_0, s) \mapsto gx_0s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X .

Define $\rho : P \times G \rightarrow P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b , the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge transformation.

(d) (\Rightarrow) Clear.

(\Leftarrow) Let $s : B \rightarrow E$ be a global section and define

$$\tilde{u} : B \times X \rightarrow E : (b, gx_0) \mapsto s(b)g$$

for any fixed $x_0 \in X$. Then, the continuous map (\tilde{f}, id_B) preserves fibers and defines a right G -equivariant isomorphism. \square

4.5 (Quotient principal bundles).

4.6 (Reduction of structure groups). Let H be a closed subgroup of G . Then, there is a function $\check{H}^1(B, H) \rightarrow \check{H}^1(B, G)$, which is neither in general injective nor surjective. If a G -bundle p is contained in the image of $\check{H}^1(B, H)$ through the correspondence $\text{Bun}_F(B) \rightarrow \check{H}^1(B, G)$, then we may give a H -bundle structure on p .

A reduction of G to H is a H -structure on a principal G -bundle.

4.2 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

$$\text{Bun}_{\text{GL}(n, \mathbb{R})}(B) \rightarrow \text{Vect}_n^{\mathbb{R}}(B)$$

is a faithful essentially surjective functor. If we drop the non-invertible morphisms in $\text{Vect}_n(B)$, then the functor becomes an equivalence.

$$\text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow \text{GL}(n, \mathbb{R})$$

4.7 (Vector bundles). Let $p : E \rightarrow B$ and $p' : E' \rightarrow B$ be vector bundles.

- (a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \leq n \leq \infty$. If $f, g : B \rightarrow G_k(\mathbb{R}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^{2n})$ is the natural inclusion.

4.8. Riemannian and Hermitian metrics spin structures

4.3 Classifying spaces

In this section(?), our goal is to construct BG and prove the natural isomorphism $\text{Prin}_G \rightarrow [-, BG]$.

pullback bundles: universal property, functoriality, restriction, section prolongation

4.9 (Pullback bundles). Let $p : E \rightarrow B$ be a G -bundle with fiber space F .

- (a) If A is paracompact and $f_0, f_1 : A \rightarrow B$ are homotopic, then G -bundles f_0^*p and f_1^*p are isomorphic.

Proof. If $p : E \rightarrow B \times [0, \frac{1}{2}]$ and $p' : E' \rightarrow B \times [\frac{1}{2}, 1]$ are trivial, then

If $f, g : B' \rightarrow B$ are homotopic, then $f^*\xi \cong g^*\xi$. □

4.10 (Universal principal bundles). Representability of

$$\text{Prin}_G : \text{Top}^{\text{op}} \rightarrow \text{Set}.$$

Milnor construction.

4.11.

If $EG \rightarrow BG$ is universal and BG is paracompact, then BG is unique up to homotopy.

If EG is contractible and BG is paracompact, then $EG \rightarrow BG$ is universal.

4.4 Characteristic classes

4.12. Let G be a topological group. Let A be an abelian group and n a positive integer. A *characteristic class* is a cohomology class on the classifying space, i.e. an element of $H^n(BG, A)$. A characteristic class $c \in H^n(BG, A)$ gives rise to a natural transformation $c : \text{Prin}_G \rightarrow H^n(-, A)$ via the Brown representation and the Yoneda lemma. Explicitly, for each $X \in \text{Top}$, we have $c : \text{Prin}_G(X) \rightarrow H^n(X, A) : E \mapsto f^*c$, where $E \cong f^*(EG)$. Using the Thom isomorphism and Gysin sequences, we can compute the cohomology of classifying spaces, which makes the classification of principal bundles

Suppose we have $BG = K(A, n)$. Take $c \in H^n(BG, A)$ which corresponds to the identity of $\text{End}(A)$ in the isomorphism

$$H^n(BG, A) = H^n(K(A, n), A) \cong \text{Hom}(H_n(K(A, n), \mathbb{Z}), A) = \text{Hom}(\pi_n(K(A, n)), A) = \text{Hom}(A, A) = \text{End}(A),$$

which follows from the Hurewicz theorem. Then, we can show that c defines a natural isomorphism $c : \text{Prin}_G \rightarrow H^n(-, A)$. In this case, we can say c is the *characteristic class* for G -bundles.

1. Real line bundles are classified by the first Stiefel-Whitney class $w_1 \in H^1(BG, A) \cong Aw_1$, where

$$G := \text{GL}(1, \mathbb{R}) = \mathbb{Z}/2\mathbb{Z}, \quad A := \mathbb{Z}/2\mathbb{Z}, \quad BG = \mathbb{RP}^\infty = K(A, 1).$$

2. Complex line bundles are classified by the first Chern class $c_1 \in H^2(BG, A) \cong Ac_1$, where

$$G := \text{GL}(1, \mathbb{C}) = \mathbb{T}, \quad A := \mathbb{Z}, \quad BG = \mathbb{CP}^\infty = K(A, 2).$$

3. Real vector bundles.

$$G := \text{GL}(r, \mathbb{R}), \quad A := \mathbb{Z}/2\mathbb{Z}, \quad BG = \text{Gr}_r(\mathbb{R}^\infty).$$

By Thom and Gysin, we can compute the cohomology ring

$$H^*(BG, A) \cong A[w_1, \dots, w_r].$$

4.13 (Thom isomorphism). Let E be a real vector bundle of rank r over a paracompact arc-connected space B . Let A be a principal ideal domain. Then, there is unique $t(E) \in H^r(E, E_0, A)$ whose image under $H^r(E, E_0, A) \rightarrow H^r(E_b, (E_b)_0, A) \cong A$ is the unit for each $b \in B$. With this t , we have an isomorphism

$$\cdot \smile t(E) : H^i(E) \rightarrow H^{i+r}(E, E_0), \quad i \geq 0.$$

The Euler class can be defined from Thom class.

Exercises

4.14. Let G be a topological group, and X be a free right G -space.

- (a) If the action is proper and the projection $X \rightarrow X/G$ admits local sections, then $X \rightarrow X/G$ is a principal G -bundle.

4.15 (Clutching functions).

4.16. Suppose $F \rightarrow E \rightarrow B$ is a principal

- (a) If X is contractible, then $X \rightarrow$

4.17 (Group quotients). Sufficient conditions for principal bundles. Let G be a Lie group and M be a free right smooth G -manifold.

- (a) If G is compact, then $M \rightarrow M/G$ is a principal G -bundle. (Gleason)
- (b) The irrational slope provides a counterexample if G is not compact.
- (c) Suppose M is a Lie group. If G is a closed subgroup of M , then $M \rightarrow M/G$ is a principal G -bundle. (Samelson) In particular, if N is a transitive left smooth M -manifold such that G is the isotropy group, then $M \rightarrow N$ is a principal G -bundle.

Proof. (a) We need to check the local triviality from smoothness and the properness from compactness. □

4.18 (Homogeneous spaces). They are all principal bundles.

$$\begin{array}{ll}
 O(n-k) \rightarrow O(n) \rightarrow V_k(\mathbb{R}^n), & U(n-k) \rightarrow U(n) \rightarrow V_k(\mathbb{C}^n), \\
 O(n-k) \times O(k) \rightarrow O(n) \rightarrow G_k(\mathbb{R}^n), & U(n-k) \times U(k) \rightarrow U(n) \rightarrow G_k(\mathbb{C}^n), \\
 T(n) \cap O(n) \rightarrow O(n) \rightarrow F(\mathbb{R}^n), & T(n) \cap U(n) \rightarrow U(n) \rightarrow F(\mathbb{C}^n), \\
 & T(n) \rightarrow GL(n, \mathbb{C}) \rightarrow F(\mathbb{C}^n),
 \end{array}$$

where $T(n)$ is the group of invertible upper triangular matrices.

$$SO(n) \rightarrow SO^+(1, n) \rightarrow \mathbb{H}^n, \quad PSO(2) \rightarrow PSL(2, \mathbb{R}) \rightarrow \mathbb{H}^2, \quad ?? \rightarrow PSL(2, \mathbb{C}) \rightarrow \mathbb{H}^3,$$

where $PSL(2, \mathbb{R}) \cong SO(1, 2)^+$ is the modular group and $PSL(2, \mathbb{C}) \cong SO(1, 3)^+$ is the restricted Lorentz group, also called the Möbius group.

4.19 (Hopf fibration). A principal S^1 -bundle $S^1 \rightarrow S^3 \rightarrow S^2$, where we see S^1 as the circle group. The Hopf fibrations are used in describing universal principal bundles off orthogonal or unitary groups. We have principal bundles:

- (a) The quaternionic construction gives $S^3 \rightarrow S^7 \rightarrow S^4$ and the octonionic construction gives $S^7 \rightarrow S^{15} \rightarrow S^8$. Adams' theorem.
- (b) $O(k) \rightarrow V_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$. In particular, $\mathbb{Z}/2\mathbb{Z} \rightarrow S^n \rightarrow \mathbb{RP}^n$ for $k = 1$.
- (c) $U(k) \rightarrow V_k(\mathbb{C}^n) \rightarrow G_k(\mathbb{C}^n)$. In particular, $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ for $k = 1$.

Hopf fibration(real, complex, quaternionic, but not octonionic)

In the category of smooth manifolds, if f diffeomorphic, then \tilde{f} diffeomorphic.

- 4.20.** (a) $\mathbb{R} \rightarrow S^1$ for \mathbb{Z} .
 (b) $S^\infty \rightarrow \mathbb{CP}^\infty$ for $U(1)$.
 (c) $? \rightarrow (S^1)^{\vee n}$ for F_n .
 (d) $S^\infty \rightarrow \mathbb{RP}^\infty$ for $\mathbb{Z}/2\mathbb{Z}$.
 (e) $V_n(\mathbb{R}^\infty) \rightarrow Gr_n(\mathbb{R}^\infty)$ for $O(n)$.

Chapter 5

5.1 Obstruction theory

5.2 Hurewicz theorem

$H_*(\Omega S_n)$ and $H_*(U(n))$ Spin, $\text{Spin}_\mathbb{C}$ structure

Chapter 6

Simplicial methods

Part II

Stable homotopy theory

Chapter 7

7.1 Homotopy groups of spheres

Freudenthal suspension theorem. Spanier-Whitehead category, does not contain reduced cohomology theory? Boardman's stable homotopy category, Lima's notion of spectra, Kan's semi-simplicial category, Whitehead's notion of spectra, and finally Adams's construction of stable homotopy category. Bousfield localization is a kind of a category of fractions and Adams spectral sequence. It leads to the chromatic homotopy.

A commutative monoidal point-set model for the stable homotopy category? Coordinate-free spectra by May, but commutative and associative only up to homotopy. S -modules in EKMM97, symmetric spectra in HSS00, which are shown Quillen equivalent in Sch01. They give closed symmetric monoidal model categories of spectra and model categories of ring spectra. HPS97 axiomatize the stable homotopy theories.

7.1 (Freudenthal suspension theorem). For each r , we have the suspension homomorphism

$$E_n : \pi_{n+r}(S^n) \rightarrow \pi_{n+r+1}(S^{n+1}), \quad n \geq 0,$$

which are isomorphisms if $n > r + 1$. The *stable homotopy groups of spheres* is $\lim_n \pi_{n+r}(S^n)$.

For example, it is known that $\pi_{n+1}(S^n) \cong \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. The computation $\pi_4(S^3)$ is a nice exercise which is done by Serre in his thesis.

Note that $\pi_{n+r}(S^n) = [S^{n+r}, S^n]$. Suspension $\Sigma := S^1 \wedge -$ is a functor, so it defines a function $S : [X, Y] \rightarrow [\Sigma X, \Sigma Y]$.

Spanier-Whitehead and Adams category of spectra. Triangulatedness of the homotopy category of a stable model category. a symmetric monoidal smash product and an internal function object.

Part III

Generalized cohomology theories

Chapter 8

8.1. Note that $\mathbf{hSpc} = \mathbf{hKan} = \mathbf{hCW}$ is triangulated.

8.2 (Generalized cohomology theories). A *generalized cohomology theory*, or simply a *cohomology theory*, is defined as a homotopy invariant product-preserving contravariant functor $E^\bullet : \mathbf{Spc}_*^{\text{op}} \rightarrow \mathbf{grAb}$, such that

- (i) a cofiber sequence is mapped to an exact sequence, (half-exact)
- (ii) a natural isomorphism $\sigma : E^{\bullet+1} \circ \Sigma \rightarrow E^\bullet$ exists, called the *suspension isomorphism*. (long exact)

A cohomology theory is called *multiplicative* if...

$$E^1(\Sigma X) = E^0(X).$$

$$K_0(\Sigma A) = K_1(A).$$

Two motivations for spectra:

- representation of cohomology theories
- suspension stabilization

8.3. Let X and Y be pointed CW complexes.

- (a) Suppose Y is $(n-1)$ -connected with non-degenerate base point for some n . Then, $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is surjective if $\dim X \leq 2n-1$, and bijective if $\dim X \leq 2n-2$.

8.4. A *spectrum* is a sequence $E := (E_n)_n$ of pointed spaces together with structure maps, either $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$ or $\sigma'_n : E_n \rightarrow \Omega E_{n+1}$. We have

$$[X, E_n] \xrightarrow{\sigma'_n} [X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}].$$

8.5 (Properties of spectra). A spectrum $E = (E_n)_n$ is called an Ω -*spectrum* if $\sigma'_n : E_n \rightarrow \Omega E_{n+1}$ is a weak homotopy equivalence. A *ring spectrum* is a spectrum together with a

- (a) E is an Ω -spectrum if and only if $[-, E_n]$ defines a generalized reduced cohomology theory on based CW complexes.

Sphere spectra, Suspension spectra Eilenberg-MacLane spectra (ordinary cohomology theories), K-theory spectra (K-theories), Thom spectra (cobordism theories)

Let E^* be a (generalized) cohomology theory. Then, the computation of $\text{Nat}([- , BO(n)], E^*) \cong E^*(BO(n))$ determines all characteristic classes of real vector bundles.

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem

Chapter 9

K-theory

9.1 (K-theory of locally compact Hausdorff spaces). compactly supported? one-point compactification? representability?

What relations do we have?

Chapter 10

Cobordisms