Algebraic Topology

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Contents

Part I

Chapter 1

Convenient categories

1.1 Compactly generated weakly Hausdorff spaces

bicomplete cartesian closed monoidal category. Here, closed means that the right tensoring admits an adjoint called internal hom functor. braided and symmetric...?

A pointed space is a pair of a space and a 0-cell. The smash product is the categorical product in the category of pointed space.

Let Top be the bicomplete cartesian closed monoidal category of CGWH spaces.

1.2 CW complexes

1.3 Simplicial complexes

Chapter 2

Cohomology operations

2.1 Eilenberg-Steenrod axioms

cohomology cup product universal coefficient theorem Poincaré duality

2.2 Characteristic classes

A characteristic class is a natural transformation $\mathrm{Prin}_G = [-, BG] \to H^n(-, A)$ for some n. They are always can be given by the pullback of classes in $H^n(BG, A)$ by the Yoneda lemma.

1. $G = \mathrm{GL}(1, \mathbb{R})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Grp}$. $BG = G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$.
$$(\mathrm{Prin}_G, \otimes) \cong (H^1(-, \mathbb{Z}/2\mathbb{Z}), +).$$
2. $G = \mathrm{GL}(1, \mathbb{C})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Grp}$. $BG = G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.
$$(\mathrm{Prin}_G, \otimes) \cong (H^2(-, \mathbb{Z}), +).$$
3. $G = \mathrm{GL}(n, \mathbb{R})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Set}$. $BG = G_n(\mathbb{R}^\infty)$. By Thom and Gysin,
$$H^*(BG, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, \cdots, w_n].$$

Since there is a special class in $H^n(K(A, n))$ so that the inducing map provides an isomorphism $[X, K(A, n)] = H^n(X, A)$, we have $H^n(B\mathrm{GL}(n, \mathbb{C})) \to H^n(X, A)$.

Exercises

characteristic class of projective spaces

Chapter 3

Spectral sequences

3.1 Serre spectral sequence

(Lyndon-Hochschild-Serre)

3.2 Adams spectral sequence

Chapter 4

Fibrations

4.1 Fiber bundles

Locally trivial bundles

pullback bundles: universal property, functoriality, restriction, section prolongation

4.2 Obstruction theory

4.3 Hurewicz theorem

$H_\bullet(\Omega S_n)$ and $H_\bullet(U(n))$ Spin, $\mathrm{Spin}_{\mathbb{C}}$ structure

4.4 Model categories

4.1 (Model structures)**.** Let $\mathcal{C}$ be a category. A *model structure* on $\mathcal{C}$ is a triple $(W, \mathrm{Cof}, \mathrm{Fib})$ of classes of morphisms of $\mathcal{C}$ such that W satisfies the two-out-of-three law, and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are weak factorization systems.

A class W of morphisms of $\mathcal{C}$ is called a *weak equivalences*. A morphism in $\mathrm{Cof} \cap W$ is called a *acyclic cofibration*.

Serre model structure and Hurewicz model structure on Top.

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Chapter 5

K-theory

bott periodicity Hopf invariant

Chapter 6

Cobordisms

Chapter 7

Simplicial methods

Part II

Stable homotopy theory

Chapter 8

8.1 Homotopy groups of spheres

Freudenthal suspension theorem. Spanier-Whitehead category, does not contain reduced cohomology theory? Boardman's stable homotopy category, Lima's notion of spectra, Kan's semi-simplicial category, Whitehead's notion of spectra, and finally Adams's construction of stable homotopy category. Bousfield localization is a kind of a category of fractions and Adamas spectral sequence. It leads to the chromatic homotopy.

A commutative monoidal point-set model for the stable homotopy category? Coordinate-free spectra by May, but commutative and associative only up to homotopy. S-modules in EKMM97, symmetric spectra in HSS00, which are shown Quillen equivalent in Sch01. They give closed symmetric monoidal model categories of spectra and model categories of ring spectra. HPS97 axiomatize the stable homotopy theories.

8.1 (Freudenthal suspension theorem)**.** For each r, we have the suspension homomorphism

$$E_n : \pi_{n+r}(S^n) \to \pi_{n+r+1}(S^{n+1}), \qquad n \geq 0,$$

which are isomorphisms if $n > r+1$. The *stable homotopy groups of spheres* is $\lim_n \pi_{n+r}(S^n)$.

For example, it is known that $\pi_{n+1}(S^n) \cong \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. The computation $\pi_4(S^3)$ is a nice exercise which is done by Serre in his thesis.

Note that $\pi_{n+r}(S^n) = [S^{n+r}, S^n]$. Suspension $\Sigma := S^1 \wedge -$ is a functor, so it defines a function $S : [X,Y] \to [\Sigma X, \Sigma Y]$.

Spanier-Whitehead and Adama category of spectra. Triangulatedness of the homotopy category of a stable model category. a symmetric monoidal smash product and an internal function object.

8.2 Generalized homology theory

A *generalized reduced cohomology theory on pointed CW complexes* is a sequence of functors $\widetilde{E}_q : \mathbf{hCW}_* \to \mathbf{Ab}$ for $q \in \mathbb{Z}$ which is exact and additive, and satisfies the suspension axiom.

8.2. Let X and Y be pointed CW complexes.

(a) Suppose Y is $(n-1)$-connected with non-degenerate base point for some n. Then, $[X,Y] \to [\Sigma X, \Sigma Y]$ is surjective if $\dim X \leq 2n-1$, and bijective if $\dim X \leq 2n-2$.

8.3. A *spectrum* is a sequence $E := (E_n)_n$ of pointed spaces together with structure maps, either $\sigma_n : \Sigma E_n \to E_{n+1}$ or $\sigma'_n : E_n \to \Omega E_{n+1}$. We have

$$[X, E_n] \xrightarrow{\sigma'_n} [X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}].$$

8.4 (Properties of spectra). A spectrum $E = (E_n)_n$ is called an *Ω-spectrum* if $\sigma'_n : E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence. A *ring spectrum* is a spectrum together with a

(a) E is an Ω-spectrum if and only if $[-, E_n]$ defines a generalized reduced cohomology theory on based CW complexes.

Sphere spectra, Suspension spectra Eilenberg-MacLane spectra(ordinary cohomology theories), K-theory spectra(K-theories), Thom spectra(cobordism theories)

Let E^* be a (generalized) cohomology theory. Then, the computation of $\mathrm{Nat}([-, BO(n)], E^*) \cong E^*(BO(n))$ determines all characteristic classes of real vector bundles.

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem