Pseudodifferential Operators

Ikhan Choi
Lectured by Kenichi Ito
The University of Tokyo, Spring 2023

April 25, 2023

Contents

1 Day 1: April 11

Notation

$$D_j = (-1)\partial_j$$

$$\xi^\alpha \mathcal{F}u = \mathcal{F}D^\alpha u, \quad \xi^\alpha \mathcal{F}^* u = \mathcal{F}^*(-D)^\alpha u, \quad D^\alpha \mathcal{F}u = \mathcal{F}(-x)^\alpha u$$

Let

$$A = \sum a_\alpha(x)D^\alpha, \quad a(x,\xi) = \sum a_\alpha(x)\xi^\alpha.$$

Then,

$$Au(x) = \mathcal{F}^* M_{a(x,\xi)} \mathcal{F}u(x)$$

$$= (2\pi)^{-1} \int e^{ix\xi} a(x,\xi) \int e^{-iy\xi} u(y)\, dy\, d\xi$$

$$= (2\pi)^{-1} \iint e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi.$$

If a has a polynomial growth in ξ, then the integrand $e^{i(x-y)\xi} a(x,\xi)u(y)$ is not integrable in (y,ξ), so we need to justify it as an oscillatory integral.

Japanese bracket, originated by Kitada or Kumano-go (akumade setsu)

$$\langle x + y \rangle^{-2} \leq 4\langle x \rangle^2 \langle y \rangle^{-2}$$

$$\langle x^2 + x \rangle \asymp \langle x^2 \rangle \asymp \langle x \rangle^2$$

Here we define the amplitude functions as

$$|\partial^\alpha a(x)| \lesssim \langle x \rangle^{m+\delta|\alpha|}, \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^d$$

Example (Justification of a qudaratic oscillation). Let Q be a nondegenerate real quadratic form, then for $a \in A_\delta^m(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$,

$$I_Q(a) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} e^{i\frac{1}{2}Q(x)} \chi(\varepsilon x) a(x)\, dx$$

exists. The term $e^{i\frac{1}{2}Q(x)}$ oscillates fast where $|x| \gg 1$, the term $\chi(\varepsilon x)$ becomes flatten as $\varepsilon \to 0$. When we do integrate by parts, we integrate the oscillating term, and differentiate the cutoff and amplitude. If we differentiate the amplitude, the integrability is enhanced.

Proof. We compute for $Q = I$ as an example. Since

$$De^{i\frac{1}{2}x^2} = xe^{i\frac{1}{2}x^2},$$

we have

$$(1 + x \cdot D)e^{i\frac{1}{2}x^2} = (1 + x^2)e^{i\frac{1}{2}x^2}.$$

Define a differential operator L such that

$${}^t L := \frac{1}{1+x^2} + \frac{x}{1+x^2} \cdot D = \langle x \rangle^{-2} + \langle x \rangle^{-2} x \cdot D,$$

that is,

$$L = \frac{1}{1+x^2} - D \cdot \frac{x}{1+x^2} - \frac{x}{1+x^2} \cdot D$$

$$= \frac{1}{1+x^2} + i\left(\frac{d}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right) - \frac{x}{1+x^2} \cdot D$$

$$= (1 + (d+2)i)\langle x \rangle^{-2} - 2\langle x \rangle^{-4} - \langle x \rangle^{-2} x \cdot D.$$

Then $^t L$ fixes $e^{i\frac{1}{2}x^2}$, so

$$\int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2}\chi(\varepsilon x)a(x)\,dx = \int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2}L^k[\chi(\varepsilon x)a(x)]\,dx$$

for any $k \geq 0$. Since

$$L = c_0(x) + c_j(x)\partial_j, \qquad c_0 \in A_{-1}^{-2}, \quad c_j \in A_{-1}^{-1},$$

we have

$$|L^k[\chi(\varepsilon x)a(x)]| \lesssim |L^k[a(x)]| \lesssim |a|_k\langle x\rangle^{m-\min\{1-\delta,2\}k}$$

bounded $\varepsilon > 0$.

Then,

$$|L^k[\chi(\varepsilon x)a(x)]| \xrightarrow{\varepsilon \to 0} L^k[a(x)] \quad \text{pointwise.}$$

$\square$

2 Day 2: April 18

- Lemma 1.3: Coordinate changes and integration by parts work. Also we can check even if we do coordiante change and differeniation (of oscillating term), we also have oscillatory integral.

- Theorem 1.4: For amplitude functions with $\delta < 1$, an operator *defined* by the multiplier

$$e^{i\frac{1}{2}Q(D)} : \mathcal{S}' \to \mathcal{S}'$$

  have an explicit expression.

- Theorem 1.5: The above multiplier also can be defined by the Taylor expansion. This kind of theorems may be called a expansion formula (I think). The last part of the proof holds from the integral by parts.

- Corollary 1.7: We want to have an extension with a parameter. The parameter h is called s semiclassical parameter. As $h \to 0$, the oscillation goes rapid. The name stationary phase is implied by the origin zero is the only critical points of the phase function.

- Lemma 1.6: Here we introduce a sequence of Schwarz functions which converges in $\mathcal{S}'$

$$e^{-\varepsilon x^2}e^{i\frac{1}{2}Q(x)} \xrightarrow{\varepsilon \to 0} e^{i\frac{1}{2}Q(x)}.$$

  Between the second row and the third row in the aligned equations, we have used

$$\mathcal{F}(e^{i\frac{1}{2}Q(x)}e^{-\varepsilon x(^tP^{-1}P^{-1})x})(P^{-1}\eta) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P^{-1}\eta}e^{i\frac{1}{2}Q(x)}e^{-\varepsilon x(^tP^{-1}P^{-1})x}\,dx.$$

3 Day 3: April 25

- x ! x xi 0 .
- x : tempered distribution . tempered distribution sense .
- Some references only use $S^m = S_{1,0}^m$.
- Remakr 1.
- homogeneous polynomial euler relation(xi xi), a xi
- smoothing operator operator
- Theorem 2.1.

$$\langle\xi\rangle^{-2}\langle D_y\rangle^2 e^{i(x-y)\xi} = e^{i(x-y)\xi}$$