27 June 2023

Analysis VIII/Linear Differential Equations
Final Report Problems

Solve at least 3 problems from the following, and submit it to a report box. If you would like to submit it electrically for some reason, let me know by e-mail.
Deadline: 25 July 2023
(If you find errors in the problems, correct them suitably.)

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[1] Let

$$\Omega = \{\zeta \in \mathbb{C}^d;\ |\operatorname{Im}\zeta| < \epsilon|\operatorname{Re}\zeta|\}, \quad \epsilon > 0,$$

and assume that $a\colon \Omega \to \mathbb{C}$ is holomophic, and that there exists $C > 0$ such that for any $\zeta \in \Omega$

$$|a(\zeta)| \le C(1 + |\zeta|^2)^{m/2}.$$

For any $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi(\xi) = 1$ in a neighborhood of $\xi = 0$ set

$$b(x, \xi) = (1 - \chi(\xi))a(\xi).$$

Then show $b \in S^m(\mathbb{R}^{2d})$.

[2] Let $a \in S^0_{0,0}(\mathbb{R}^{2d})$.

1. Verify formally or rigorously, whichever, the identity

$$\mathcal{F} a^{\mathrm{W}}(x, D_x)\mathcal{F}^* = a^{\mathrm{W}}(-D_\xi, \xi)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d).$$

2. For any $t \in \mathbb{R}$ define the *free Schrödinger propagator* as

$$\mathrm{e}^{\mathrm{i}t\Delta/2} = \mathcal{F}^* \mathrm{e}^{-\mathrm{i}t\xi^2/2}\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d).$$

   Then verify formally or rigorously, whichever, the identity

$$\mathrm{e}^{-\mathrm{i}t\Delta/2} a^{\mathrm{W}}(x, D)\mathrm{e}^{\mathrm{i}t\Delta/2} = a^{\mathrm{W}}(x + tD, D).$$

[3]

1. Deduce the elliptic a priori estimate from the Gårding inequality.

2. Deduce the Gårding inequality from the sharp Gårding inequality.

[4] Compute the wave front sets of the following distributions.

1. The Dirac delta funcion δ on $\mathbb{R}^d$.
2. $\delta(x') \otimes 1(x'')$ for $(x', x'') \in \mathbb{R}^p \times \mathbb{R}^q$.
3. $\delta_{\mathbb{S}^{d-1}}$ on $\mathbb{R}^d$.
4. $(x + \mathrm{i}0)^{-1}$ on $\mathbb{R}$.
5. The characteristic function χ_Γ of an angular domain
$$\Gamma = \big\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ r > 0,\ \theta \in (0, \alpha)\big\}, \quad \alpha \in (0, 2\pi).$$

[5] Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $0 \le \delta < \rho \le 1$, and assume $a(x, D)$ is local.

1. Show, if $m < -d$, then $a \equiv 0$.
2. Show, for any $\alpha \in \mathbb{N}_0^d$, $(\partial_\xi^\alpha a)(x, D)$ is also local.
3. Show $a(x, D)$ is a partial differential operator.

[6] We consider a differential operator on $\mathbb{R}^2$:
$$a(x, D) = D_1 + \mathrm{i}b(x_1)D_2.$$
We assume $b \in C^\infty(\mathbb{R}; \mathbb{R})$, and
$$\pm b(x_1) > 0 \ \text{ for } \ \pm x_1 > 0,$$
respectively, and set
$$\psi(x) = B(x_1) - \mathrm{i}x_2 - (B(x_1) - \mathrm{i}x_2)^2; \quad B(x_1) = \int_0^{x_1} b(y)\,\mathrm{d}y.$$

1. Show for some neighborhood $U \subset \mathbb{R}^2$ of the origin and $c, C > 0$
$$c(B(x_1) + x_2^2) \le \operatorname{Re}\psi(x) \le C(B(x_1) + x_2^2) \ \text{ in } U.$$
2. Show that for any neighborhood $V \subset \mathbb{R}^2$ of the origin, $s, t \in \mathbb{R}$ and $c > 0$ there exists $v \in C_0^\infty(V)$ such that
$$\|a^*(x, D)v\|_{H^s} \le c\|v\|_{H^t}.$$
In particular, $a(x, D)$ is not locally solvable at the origin.

(*Hint.* Take $\chi \in C_0^\infty(V)$ with $\chi = 1$ in a neighborhood of the origin, and set
$$v_\mu(x) = \chi(x)\mathrm{e}^{-\mu\psi(x)}, \quad \mu \ge 1.$$
To estimate $\|v_\mu\|_{H^t}$ compute (v_μ, ϕ_μ) for some $\phi_\mu(x) = \phi(\mu x)$, $\phi \in C_0^\infty(V)$.)