Contents

Preliminaries

1 Calculus

2 Linear algebra

3 Set theory and number systems

Part I

Analysis

1 Foundation of Calculus

Sequences

1.1. Show that for a nonnegative sequence a_n if $\sum a_n$ diverges then $\sum \frac{a_n}{1+a_n}$ also diverges.

1.2. Show that every real sequence has a monotonic subsequence that converges to the limit superior of the original sequence.

1.3. Show that if a decreasing nonnegative sequence a_n converges to 0 and satisfies $S_n \leq 1 + na_n$ then S_n is bounded by 1.

Functions

1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that f is identically zero if $f'(x) = f(x)^2$ for all x.

1.5. Show that if both limits of a real function and its derivative at infinity exist then the function vanishes at infinity.

1.6. Let f be a real C^2 function with $f''(c) \neq 0$. Defined a function ξ such that $f(x) - f(c) = f'(\xi(x))(x-c)$ with $|\xi - c| \leq |x - c|$, show that $\xi'(c) = 1/2$.

1.7. Let f be a C^2 function such that $f(0) = f(1) = 0$. Show that $\|f\| \leq \frac{1}{8}\|f''\|$.

1.8. Show that the set of local minima of a convex real function is connected.

1.9. Show that a smooth function such that for each x there is n having the nth derivative vanish is a polynomial.

1.10. Show that if a real C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.

1.11. Let a real function f be differentiable. For $a < a' < b < b'$ show that there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b') - f(a') = f'(c')(b' - a')$.

1.12. * Show that if a sequence of real functions $f_n \colon [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \le |x-y|$ whenever $|x-y| \ge \frac{1}{n}$, then the sequence has a uniformly convergent subsequence.

1.13. Let f be a differentiable function on the unit closed interval. Show that if $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

1.14. Let f be a differentiable function on the unit closed interval. Show that if $f(0) = 0$ there is c such that $cf(c) = (1-c)f'(c)$.

1.15. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.

1.16. Show that a continuous function that takes on no value more than twice takes on some value exactly once.

1.17. Let f be a function that has the intermediate value property. Show that if the preimage of every singleton is closed, then f is continuous.

Integration

1.18. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

1.19. Show that if $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

Multivariable calculus

2 Lebesgue theory

Measure theory

2.1. * Show that a measurable subset of $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression.

3 General topology

3.1. Show that if $A^\circ \subset B$ and B is closed, then $(A \cup B)^\circ \subset B$.

4 Harmonic analysis

5 Complex analysis

5.1. Show that if a holomorphic function has positive real parts on the open unit disk then $|f'(0)| < 2\operatorname{Re} f(0)$.

5.2. Show that if at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.

5.3. Show that if a holomorphic function on a domain containing the closed unit disk is injective on the unit circle then so is on the disk.

5.4. Show that for a holomorphic function f and every z_0 in the domain there are $z_1 \neq z_2$ such that $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$.

5.5. For two linearly independent entire functions, show that one cannot dominate the other.

5.6. Show that uniform limit of injective holomorphic function is either constant or injective.

5.7. Suppose the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions (f_n) converges has a limit point. Show that (f_n) compactly converges.

6 Functional analysis

Topological vector spaces

6.1. Let T be an invertible linear operator on a normed space. Show that $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

Weak topologies

Spectral theory

Operator algebras

6.2. * Let $f(x) = x(1+x)^{-1}$ be a function on $\mathbb{R}_{\geq 0}$. Show that a C*-algebra $\mathcal{A}$ is commutative if and only if f is operator subadditive in $\mathcal{A}$.

7 Proability theory

7.1. Find the probability that arbitrarily chosen positive integers are coprime.

8 Differential equations

9 Linear partial differential equations

9.1. * Describe the range of the operator $T : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ defined by $Tf = \Phi * f$ for $d \geq 3$, where Φ is the fundamental solution of Laplace's equation.

Part II

Algebra

1 Algebraic structures

Groups

1.1. Show that a finite symmetric group has two generators.

1.2. Show that a group of order $2p$ for a prime p has exactly two isomorphic types.

1.3. Show that a group G is abelian if $|G| = p^2$ for a prime p.

1.4. Show that a group G is abelian if it has a surjective cube map.

1.5. Let G be a finite group of order n and p the smallest prime divisor of n. Show that a subgroup of G of index p is normal in G.

1.6. Find all n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic.

1.7. Show that a nontrivial normalizer of a p-group meets its center out of identity.

1.8. Show that a proper subgroup of a finite p-group is a proper subgroup of its normalizer. In particular, every finite p-group is nilpotent.

1.9. Show that a finite group G satisfying $\sum_{g\in G} \operatorname{ord}(g) \leq 2n$ is abelian.

1.10. Show that the order of a group with trivial automorphism group is either 1 or 2.

1.11. Find all homomorphic images of A_4 up to isomorphism.

1.12. Show that in a group of order 105 is a single Sylow p-subgroup for $p = 5, 7$.

1.13. Show that the number of Sylow p-subgroups of $\mathrm{SL}_3(\mathbb{F}_p)$ is $(p^2+p+1)(p+1)$.

Rings

1.14. Show that a finite integral domain is a field.

1.15. Show that every ring of order p^2 for a prime p is commutative.

1.16. Show that a semiring with multiplicative identity and cancellative addtion has commutative addition.

1.17. Show that the complement of a saturated monoid in a commutative ring is a union of prime ideals.

Vector spaces

1.18. Show that normal nilpotent matrix equals zero.

1.19. Show that two matrices AB and BA have same nonzero eigenvalues whose both multiplicities are coincide...

1.20. Show that if A is a square matrix whose characteristic polynomial is minimal then a matrix commuting A is a polynomial in A.

1.21. Show that the order of 2×2 integer matrices divide 12 if it is finite.

1.22. Let X be a square matrix. Show that there is another matrix Y such that $X + Y$ is invertible.

1.23. Show that a determinant-preserving linear map is rank-preserving.

2 Galois theory

2.1. Show that the Galois group of a quintic over $\mathbb{Q}$ having exactly three real roots is isomorphic to S_5.

2.2. Let $v \notin \mathbb{Q}$ be an algebraic element over $\mathbb{Q}$ and $F/\mathbb{Q}$ be a maximal algebraic extension not containing v. Show that every finite extension of F is cyclic.

3 Category theory

4 Commutative algebra

5 Representation theory

6 Homological algebra

7 Discrete mathematics

8 Number theory

8.1. Show that there is no integral solution of the equation $x^7 + 7 = y^2$.

8.2. Show that if $(x^2+y^2+z^2)/(xy+yz+zx)$ is an integer, then it is not divided by 3.

8.3. Show that there is no non-trivial integral solution of $x^4-y^4=z^2$.

9 Algebraic number theory

Part III
Geometry and Topology

1 Classical geometry

2 Smooth surfaces

3 Differential topology

3.1. Show that the tangent space of the unitary group at the identity is identified with the space of skew Hermitian matrices.

3.2. Prove the Jacobi formula for matrix.

3.3. Show that S^3 and T^2 are parallelizable.

3.4. Show that $\mathbb{R}P^n = S^n/Z_2$ is orientable if and only if n is odd.

4 Geometric analysis

5 Algebraic curves

6 Algebraic geometry

7 Complex geometry

8 Surface topology

9 Algebraic topology

10 Geometric topology