Measure Theory

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Contents

Part I

Measures

Chapter 1

Measure spaces

1.1 Measurable spaces

1.1 (Measurable spaces).

1.2 Measure spaces

1.2 (Definition of measures). Let $(\Omega, \mathcal{M})$ be a measurable space. A *measure* on $\mathcal{M}$ is a set function $\mu : \mathcal{M} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i)$$

for $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$. Here the squared cup notation reads the disjoint union.

1.3 (Continuity of measures).

1.4 (Pushforward measures).

1.5 (Complete measures).

1.3 Carathéodory extension

1.6 (Outer measures). Let Ω be a set. An *outer measure* on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \to [0, \infty] : \varnothing \mapsto 0$ such that

(i) μ^* is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \le \mu^*(S_2)$$

for $S_1, S_2 \in \mathcal{P}(\Omega)$,

(ii) μ^* is *countably subadditive*: we have

$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

Compairing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(\Omega) \to [0,\infty] : \varnothing \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \le \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $S \in \mathcal{P}(\Omega)$ and $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

(b) For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^{\infty} \rho(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{A} \right\},$$

where we use the convention $\inf \varnothing = \infty$.

Proof. □

1.7 (Carathéodory measurability)**.** Let μ^* be an outer measure on a set Ω. We want to construct a measure by restriction of μ^* on a properly defined σ-algebra. A subset $E \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(\Omega)$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0,\infty]$ is complete.

Proof. □

1.8 (Carathéodory extension theorem)**.** The Carathéodory extension is a construction method for a measure extending a given set function ρ. The idea is to restrict the outer measure μ^* associated to ρ in order to obtain a measure μ. We want to find a sufficient condition for μ to be a measure on a σ-algebra containing $\mathcal{A}$.

For $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0,\infty] : \varnothing \mapsto 0$ be a set function. Let $\mu^* : \mathcal{P}(\Omega) \to [0,\infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0,\infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) We have $\mu^*|_{\mathcal{A}} = \rho$ if ρ satisfies the monotone countable subadditivity:

$$A \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \rho(A) \le \sum_{i=1}^{\infty} \rho(B_i)$$

for $A \in \mathcal{A}$ and $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$.

(b) We have $\mathcal{A} \subset \mathcal{M}$ if ρ satisfies the following property: for every $B, A \in \mathcal{A}$, and for any $\varepsilon > 0$, there are $\{C_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty} \subset \mathcal{A}$ such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} C_j \quad \text{and} \quad B \setminus A \subset \bigcup_{j=1}^{\infty} D_j,$$

and

$$\rho(B) + \varepsilon > \sum_{j=1}^{\infty} \rho(C_j) + \sum_{j=1}^{\infty} \rho(D_j).$$

Proof. (a) Clearly $\mu^*(A) \le \rho(A)$ for $A \in \mathcal{A}$. We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{B_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(B_i) \ge \rho(A).$$

Limiting $\varepsilon \to 0$, we get $\mu^*(A) \ge \rho(A)$.

(b) Let $S \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. It is enough to check the inequality $\mu^*(S) \ge \mu^*(S \cap A) + \mu^*(S \setminus A)$ for S with $\mu^*(S) < \infty$, so we may assume there is a countable family $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. Then, we have $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$ satisfying

$$\mu^*(S) + \varepsilon > \sum_{i=1}^\infty (\rho(B_i) + \frac{\varepsilon}{2^{i+1}}) > \sum_{i,j=1}^\infty \rho(C_{i,j}) + \sum_{i,j=1}^\infty \rho(D_{i,j}) \ge \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Therefore, A is Carathéodory measurable relative to μ^*. □

1.9 (Uniqueness of extension of measures)**.** The existence of the Carathéodory extension provides a uniqueness theorem for the extension of measures. The important property here is σ*-finiteness*: for $\varnothing \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ be a set function. Then, we say ρ is σ-finite if there is a countable cover $(B_i)_{i=1}^\infty \subset \mathcal{A}$ of Ω such that $\rho(B_i) < \infty$ for each i.

Let μ^* be the outer measure associated to ρ. Let $\mathcal{M}$ be a σ-algebra such that the restriction $\mu^*|_{\mathcal{M}} : \mathcal{M} \to [0, \infty]$ is a measure, and $\mu : \mathcal{M} \to [0, \infty]$ be any measure. Suppose further that $\mu^*(A) = \rho(A) = \mu(A)$ for all $A \in \mathcal{A}$. Let $E \in \mathcal{M}$.

(a) $\mu(E) \le \mu^*(E)$.

(b) If $E_1, E_2 \in \mathcal{M}$ satisfy $\mu(E_1) = \mu^*(E_1)$ and $\mu(E_2) = \mu^*(E_2)$, then $\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2)$.

(c) $\mu(E) = \mu^*(E)$ if $\mu^*(E) < \infty$.

(d) If ρ is σ-finite, then $\mu(E) = \mu^*(E)$ for $\mu^*(E) = \infty$.

Proof. (a) If $\mu^*(E) = \infty$, then $\mu(E) \le \mu^*(E)$ trivially. Suppose $\mu^*(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Also, we have

$$\mu(E) \le \mu\Big(\bigcup_{i=1}^\infty B_i\Big) \le \sum_{i=1}^\infty \mu(B_i) = \sum_{i=1}^\infty \rho(B_i)$$

whenever $E \subset \bigcup_{i=1}^\infty B_i$, so $\mu(E) \le \mu^*(E)$.

(b) In the light of the inclusion-exclusion principle,

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2) - \mu^*(E_1 \cap E_2) \le \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \mu(E_1 \cup E_2)$$

proves the identity we want.

(c) Because $\mu^*(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(E) + \varepsilon > \sum_{i=1}^\infty \rho(B_i).$$

Applying the part (b) inductively, we have for every n that

$$\mu\Big(\bigcup_{i=1}^n B_i\Big) = \mu^*\Big(\bigcup_{i=1}^n B_i\Big),$$

and by limiting $n \to \infty$ the continuity from below gives

$$\mu\Big(\bigcup_{i=1}^\infty B_i\Big) = \mu^*\Big(\bigcup_{i=1}^\infty B_i\Big).$$

Then, we have

$$\mu^*(E) \le \mu^*\Big(\bigcup_{i=1}^\infty B_i\Big) = \mu\Big(\bigcup_{i=1}^\infty B_i\Big) = \mu\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) + \mu(E)$$

and

$$\mu\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) \le \mu^*\Big(\bigcup_{i=1}^\infty B_i \setminus E\Big) = \mu^*\Big(\bigcup_{i=1}^\infty B_i\Big) - \mu^*(E) \le \sum_{i=1}^\infty \mu^*(B_i) - \mu^*(E) = \sum_{i=1}^\infty \rho(B_i) - \mu^*(E) < \varepsilon,$$

we get $\mu^*(E) < \mu(E) + \varepsilon$ and $\mu^*(E) \le \mu(E)$ by limiting $\varepsilon \to 0$.

(d) Let $(B_i)_{i=1}^\infty \subset \mathcal{A}$ be such that $\rho(B_i) < \infty$ and $\Omega = \bigcup_{i=1}^\infty B_i$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \ge 2$. Then, $(E_i)_{i=1}^\infty$ is a pairwise disjoint cover of Ω with

$$\mu^*(E \cap E_i) \le \mu^*(E_i) \le \mu^*(B_i) = \rho(B_i) < \infty$$

for each i, so we have by the part (c) that

$$\mu(E) = \sum_{i=1}^\infty \mu(E \cap E_i) = \sum_{i=1}^\infty \mu^*(E \cap E_i) = \mu^*(E). \qquad \square$$

Exercises

1.10 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A}$ be a collection of subsets of a set Ω such that $\varnothing \in \mathcal{A}$. We say $\mathcal{A}$ is a semi-ring if it is closed under finite intersection, and the complement is a finite union of elements of $\mathcal{A}$. We say $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over Ω. Suppose a set function $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^\infty A_i\Big) \le \sum_{i=1}^\infty \rho(A_i)$$

for $(A_i)_{i=1}^\infty \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

1.11 (Monotone class lemma). A collection $\mathcal{C} \subset \mathcal{P}(\Omega)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let H be a vector space closed under bounded monotone convergence. If $\operatorname{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^\infty(\sigma(\mathcal{A})) \subset H$.

Chapter 2

Measures on the real line

2.1 (Borel σ-algebra).

2.2 (Distribution functions).

2.3 (Helly selection theorem).

2.4 (Non-Lebesgue measurable set).

Exercises

2.5 (Steinhaus theorem). Let λ denote the Lebesgue measure on $\mathbb{R}$ and let $\mathbb{E} \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.

(a) For any $0 < \alpha < 1$, there is an interval $I = (a, b)$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.

(b) $E - E = \{x - y : x, y \in E\}$ contains an open interval containing zero.

Proof. (a) We may assum $\lambda(E) < \infty$. Since λ is outer measure and $\lambda(E) \neq 0$, we have an open subset U of $\mathbb{R}$ such that $\lambda(U) < \alpha^{-1}\lambda(E)$. Because U is a countable disjoint union of open intervals $U = \bigsqcup_{i=1}^{\infty}(a_i, b_i)$, we have

$$\sum_{i=1}^{\infty}\lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1}\sum_{i=1}^{n}\lambda(E \cap (a_i, b_i)).$$

Therefore, there is i such that $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$. □

Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

Chapter 3

Measurable functions

3.1 Simple functions

3.1 (Measurability of pointwise limits)**.** Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

$\square$

3.2 Almost everywhere convergence

3.2 (Almost everywhere convergence)**.** Let (Ω,μ) be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence f_n is defined as the set

$$\{x \in \Omega : \lim_{n\to\infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say f_n converges to f *alomst everywhere* with respect to μ if the set of divergence is a null set in μ. We simply write

$$f_n \to f \text{ a.e.}$$

if f_n converges to f almost everywhere, and we frequently omit the measure μ if it has no confusion.

(a) If μ is complete and, if $f_n \to f$ a.e., then f is measurable.

3.3 (Borel-Cantelli lemma)**.** Let (Ω,μ) be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon>0}\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\{x : \limsup_{n\to\infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(c) $f_n \to f$ a.e. if for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \infty.$$

Proof. (b) The set of divergence of the sequence f_n is given by

$$\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \tfrac{1}{m}\} = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}(\Omega \setminus E_n^m).$$

(c) Since

$$\mu\Big(\bigcup_{i=1}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \leq \sum_{i=1}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned}
\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}) &= \mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&= \lim_{n\to\infty} \mu\Big(\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \geq \varepsilon\}\Big) \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(\{x : |f_i(x) - f(x)| \geq \varepsilon\}) = 0.
\end{aligned}$$

□

3.4 (Convergence in measure). Let (Ω, μ) be a measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions. We say f_n converges to a measurable function $f : \Omega \to \overline{\mathbb{R}}$ *in measure* if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(a) If $f_n \to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

(b) If every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}} \to f$ a.e., then $f_n \to f$ in measure.

Proof. (a) Since for each positive integer k we have $\mu(\{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}) \to 0$ as $n \to \infty$, there exists n_k such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}) = 0.$$

Then, for each $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \varepsilon\} &= \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \geq \varepsilon\} \\
&\subset \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \geq \tfrac{1}{k}\} \\
&= \limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{k}\}
\end{aligned}$$

implies the limit superior of the tail events is a null set, hence $f_{n_k} \to f$ a.e.

(b) □

3.5 (Egorov theorem)**.** Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient "$\varepsilon/2^m$ argument", occasionally used throughout measure theory to construct a measurable set having a special property.

Let (Ω, μ) be a finite measure space and let $f_n : \Omega \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that $f_n \to f$ a.e. For each positive integer m, which indexes the tolerance $1/m$, consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty} \{x : |f_i(x) - f(x)| < \tfrac{1}{m}\}.$$

(a) E_n^m converges to a full set for each m.

(b) For every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and for each m there is finite n satisfying $K \subset E_n^m$.

(c) For every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on K.

Proof. (a) Recall that the a.e. convergence $f_n \to f$ means that for every fixed m the intersection

$$\bigcap_{n=1}^{\infty} (\Omega \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \tfrac{1}{m}\}$$

is a null set. Since $\mu(\Omega) < \infty$, it is equivalent to E_n^m converges to a full set for each m by the continuity from above.

(b) For each m, we can find n_m such that

$$\mu(\Omega \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(\Omega \setminus K) = \mu\Big(\bigcup_{m=1}^{\infty} (\Omega \setminus E_{n_m}^m)\Big) \leq \sum_{m=1}^{\infty} \mu(\Omega \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix $m > 0$. Since $n \geq n_m$ implies $K \subset E_{n_m}^m \subset E_n^m$, we have

$$n \geq n_m \quad \Rightarrow \quad \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \qquad \square$$

Exercises

3.6 (Cauchy's functional equation)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x + y) = f(x) + f(y)$, satisfied for all $x, y \in \mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x) = ax$ for all $x \in \mathbb{R}$, where $a := f(1)$.

(a) $f(x) = ax$ for all $x \in \mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If f is conitnuous at a point, then f is linear.

(c) If f is Lebesgue measurable, then f is linear.

Part II

Lebesgue integral

Chapter 4

Convergence theorems

4.1 Definition of Lebesgue integral

4.2 Convergence theorems

4.1 (Monotone convergence theorem).

4.3 Radon-Nikodym theorem

An integrable function as a measure σ-finite measures

Exercises

4.2 (Convergence of one-parameter family).

Chapter 5

Product measures

5.1 Fubini-Tonelli theorem

5.2 Lebesgue measure on Euclidean spaces

Chapter 6

Measures on metric spaces

6.1 Continuous functions on metric spaces

Urysohn and Tietze.

6.1 (Regular Borel measures on metric spaces)**.** Let μ be a Borel measure on a metric space Ω. We say μ is *outer regular* if

$$\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\},$$

and say μ is *inner regular* if

$$\mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ closed}\},$$

for every Borel subset $E \subset \Omega$. If μ is both outer and inner regular, we say μ is *regular*.

(a) Let E be σ-finite. Then, E is μ-regular if and only if for any $\varepsilon > 0$ there are open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.

(b) If μ is σ-finite, then the set of μ-regular subsets is a σ-algebra. (may be extended?)

(c) Every closed set is G_δ.

(d) Every finite Borel measure on Ω is regular.

Proof. □

6.2 (Luzin's theorem)**.** Let μ be a regular Borel measure on a metric space Ω. Let $f : \Omega \to \mathbb{R}$ be a Borel measurable function. Two proofs: direct and Egoroff.

(a) If $E \subset \Omega$ is σ-finite, then there is a continuous g blabla

(b) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset \Omega$ such that $f|_F : F \to \mathbb{R}$ is continuous and $\mu(\Omega \setminus F) < \varepsilon$.

(c) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset \Omega$ and continuous $g : \Omega \to \mathbb{R}$ such that $f|_F = g|_F$ and $\mu(\Omega \setminus F) < \varepsilon$.

(d) If f is further bounded, then g also can be taken to be bounded.

Proof. (a) Let $\varepsilon > 0$ and suppose $E \subset \Omega$ is measurable with $\mu(E) < \infty$. Since E is σ-finite, we have open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon/2$. By the Urysohn lemma, there is a continuous function $g : \Omega \to [0,1]$ such that $g|_{U^c} = 0$ and $g|_F = 1$. Then,

$$\int |\mathbf{1}_E - g|\,d\mu = \int_{U\setminus F} |\mathbf{1}_E - g|\,d\mu \le 2\mu(U \setminus F) < \varepsilon.$$

(b) Since $\mathbb{R}$ is second countable, we have a base $(V_n)_{n=1}^\infty$ of $\mathbb{R}$. Since μ is σ-finite, for each n we can take open U_n and closed F_n such that

$$F_n \subset f^{-1}(V_n) \subset U_n$$

and $\mu(U_n \setminus F_n) < \varepsilon/2^n$. Define $F := \left(\bigcup_{n=1}^\infty (U_n \setminus F_n)\right)^c$ so that $\mu(\Omega \setminus F) < \varepsilon$ and F is closed. Then,

$$\begin{aligned} U_n \cap F &= U_n \cap ((U_n^c \cup F_n) \cap F) \\ &= (U_n \cap (U_n^c \cup F_n)) \cap F \\ &= (\varnothing \cup (U_n \cap F_n)) \cap F \\ &\subset F_n \cap F \end{aligned}$$

proves $f^{-1}(V_n)$ is open in F for every n, hence the continuity of $f|_F$. (In fact, we require that X to be just a topological space.)

(b') We can alternatively use the part (a) and the Egoroff theorem. By the part (a), we can construct a sequence (f_n) of continuous functions $X \to \mathbb{R}$ such that $f_n \to f$ in L^1. By taking a subsequence, we may assume $f_n \to f$ pointwise. Assuming μ is finite, by the Egorov theorem, there is a measurable $A \subset X$ such that $f_n \to f$ uniformly on A and $\mu(X \setminus A) < \varepsilon/2$. Since μ is inner regular, we have closed $F \subset A$ such that $\mu(A \setminus F) < \varepsilon/2$, so that we have $\mu(X \setminus F) < \varepsilon$. Then, f is continuous on A, and of course on F.

□

6.2 Locally compact metric spaces

compact closed set not containing infty
open open not containing infty
closed closed set containing infty

for a measure that "vanishes at infty" = tight
two definitions of inner regularity is equivalent.

inner regular on compact sets -> inner regular on closed sets
inner regular on compact sets + sigma finite -> tight

6.3 (One-point compactification).

6.4 (Regular Borel measures on locally compact metric spaces). sss

(a) $C_c(\Omega)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

(b) If μ is σ-finite, then for any $\varepsilon > 0$ there is compact $K \subset \Omega$ and continuous $g : \Omega \to \mathbb{R}$ such that $f|_K = g|_K$ and $\mu(\Omega \setminus K) < \varepsilon$.

6.5 (Tightness and inner regularity). We have a similar but confusing concept called tightness; we say a Borel measure μ on a topological space X is *tight* if for any $\varepsilon > 0$ there is a compact $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$.

History of Bourbaki's text.

(a)

6.3 Riesz-Markov-Kakutani representation theorem

6.6 (Riesz-Markov-Kakutani representation theorem for C_0)**.** Let Ω be a locally compact metric space. We want to establish the following one-to-one correspondence:

$$\begin{array}{ccc} \{\text{finite Borel measures on } \Omega\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_0(\Omega)\} \\ \mu & \mapsto & (f \mapsto \int f\,d\mu). \end{array}$$

Let I a positive linear functional on $C_0(\Omega)$. Let $\mathcal{T}$ be the set of all open subsets of Ω and $\rho : \mathcal{T} \to [0,\infty]$ a set function such that

$$\rho(U) := \sup\{\, I(f) : f \in C_c(U,[0,1])\,\}$$

for open U. Let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be the associated outer measure defined from ρ, and $\mu := \mu^*|_{\mathcal{M}}$ the Carathéodory measure, where $\mathcal{M}$ is the σ-algebra of Carathéodory measurable subsets relative to μ^*, and $\mathcal{B}$ is the Borel σ-algebra of Ω.

(a) $\mu^*|_{\mathcal{T}} = \rho$.

(b) $\mathcal{B} \subset \mathcal{M}$.

(c) $I(f) = \int f\,d\mu$ for $f \in C(\Omega)$.

(d) The map $\mu \mapsto (f \mapsto \int f\,d\mu)$ is injective.

Proof. (a) It suffices to show that ρ satisfies monotonically countably subadditive. Take an open set U and a countable open cover $\{U_i\}_{i=1}^\infty$ of U. Take any $f \in C_c(U,[0,1])$ and let $K := \operatorname{supp} f$. Since K is compact, there is a finite subcover $\{U_j\}_{j=1}^n$ of K, and since K is paracompact Hausdorff, there is a partition of unitiy $\{\chi_j\}_j$ on K subordinate to the open cover $\{U_j \cap K\}_j$. Note that $\operatorname{supp}\chi_j \subset U_j \cap K$ for each j.

The set $\operatorname{supp}(f\chi_j)$ is closed in K so the compactness, and we also have the inclusion $\operatorname{supp}(f\chi_j) \subset \operatorname{supp}\chi_j \subset U_j$. For every $0 < a \le 1$, since $(f\chi_j)^{-1}((a,1])$ is open in the interior of K and $(f\chi_j)^{-1}([a,1])$ is closed in K, $f\chi_j$ is continuous on U_j. Now we have checked $f\chi_j \in C_c(U_j,[0,1])$.

Then, because I is linear so that it preserves finite sum, we have

$$I(f) = I\left(\sum_{j=1}^n f\chi_j\right) = \sum_{j=1}^n I(f\chi_j) \le \sum_{j=1}^n \rho(U_j) \le \sum_{i=1}^\infty \rho(U_i).$$

Since f is arbitrary, we get $\rho(U) \le \sum_{i=1}^\infty \rho(U_i)$.

(b) It suffices to show $\mathcal{T} \subset \mathcal{M}$. Clearly $\mu^*(E) \le \mu^*(E \cap U) + \mu^*(E \setminus U)$ for any measurable E and open U. For the opposite direction, take $\varepsilon > 0$. Note that we may assume $\mu^*(E) < \infty$. There are open U_i such that $E \subset \bigcup_{i=1}^\infty U_i$ and

$$\mu^*(E) + \frac{\varepsilon}{3} > \sum_{i=1}^\infty \rho(U_i).$$

Take $f_i \in C_c(U_i \cap U,[0,1])$ such that

$$\rho(U_i \cap U) - \frac{1}{3}\cdot\frac{\varepsilon}{2^i} < I(f_i),$$

and take $g_i \in C_c(U_i \setminus \operatorname{supp} f_i,[0,1])$ such that

$$\rho(U_i \setminus \operatorname{supp} f_i) - \frac{1}{3}\cdot\frac{\varepsilon}{2^i} < I(g_i).$$

Then, since $f_i + g_i \in C_c(U_i,[0,1])$, we have

$$\begin{aligned} \rho(U_i) \ge I(f_i+g_i) &> \rho(U_i \cap U) + \rho(U_i \setminus \operatorname{supp} f_i) - \frac{2}{3}\cdot\frac{\varepsilon}{2^i} \\ &\ge \rho(U_i \cap U) + \rho(U_i \setminus U) - \frac{2}{3}\cdot\frac{\varepsilon}{2^i}. \end{aligned}$$

It implies

$$\mu^*(E)+\varepsilon > \sum_{i=1}^\infty \rho(U_i\cap U)+\sum_{i=1}^\infty \rho(U_i\setminus U)) \geq \mu^*(E\cap U)+\mu^*(E\setminus U)$$

because $E\cap U\subset\bigcup_{i=1}^\infty U_i\cap U$ and $E\setminus U\subset\bigcup_{i=1}^\infty U_i\setminus U$.

(c) Note that we have

$$\rho(U)=\sup_{f\in C_c(U,[0,1])} I(f), \qquad \mu(E)=\inf_{\substack{E\subset U\\ U\text{ open}}}\rho(U).$$

We first claim that for $g\in C_c(\Omega,[0,1])$, if K and K' are compact sets such that $g|_K=1$ and $g|_{K'^c}=0$ respectively, then we have

$$\mu(K)\leq I(g)\leq \mu(K').$$

The one inequality directly follows from

$$I(g)\leq \inf_{K'\subset U}\rho(U)=\mu(K').$$

For the other, take sufficiently small $\varepsilon>0$ such that $U:=g^{-1}((1-\varepsilon,1])$ satisfies $K\subset U\subset\operatorname{supp} g$. For any $h\in C_c(U,[0,1])$, the inequality $(1-\varepsilon)h\leq g$ implies $I(h)\leq(1-\varepsilon)^{-1}I(g)$, so

$$\mu(K)\leq\rho(U)\leq I(h)\leq(1-\varepsilon)^{-1}I(g).$$

By limiting $\varepsilon\to0$, we get $\mu(K)\leq I(g)$, the claim proved.

Since $C_c(\Omega)$ is the linear span of $C_c(\Omega,[0,1])$, it is enough to show $I(f)=\int f\,d\mu$ for $f\in C_c(X,[0,1])$. For a fixed positive integer n and for each index $1\leq i\leq n$, let $K_i:=f^{-1}([i/n,1])$ and define

$$f_i(x):=\begin{cases}0 & \text{if } x\in K_{i-1}^c,\\ f(x)-\frac{i-1}{n} & \text{if } x\in K_{i-1}\setminus K_i,\\ \frac{1}{n} & \text{if } x\in K_i,\end{cases}$$

where $K_0:=\operatorname{supp} f$. Note that $nf_i\in C_c(X,[0,1])$ and $f=\sum_{i=1}^n f_i$. For $1\leq i\leq n$ we have $\mu(K_i)<\infty$ because K_i is compact subsets contained in a locally compact Hausdorff space $U:=f^{-1}((0,1])$, but $\mu(K_0)$ is possibly infinite. By the previous claim and the property of integral, we have

$$\frac{\mu(K_i)}{n}\leq I(f_i), \qquad \frac{\mu(K_i)}{n}\leq\int f_i\,d\mu$$

for $1\leq i\leq n$ and

$$I(f_i)\leq\frac{\mu(K_{i-1})}{n}, \qquad \int f_i\,d\mu\leq\frac{\mu(K_{i-1})}{n}$$

for $2\leq i\leq n$. Then, using the above inequalities and $\mu(K_n)\geq0$, we have

$$I(f)\leq I(f_1)+\int f\,d\mu \quad\text{and}\quad \int f\,d\mu\leq\int f_1\,d\mu+I(f).$$

Note that $f_1=\min\{f,1/n\}$ is a sequence of functions indexed by n. By the monotone convergence theorem, $\int f_1\,d\mu\to0$ as $n\to\infty$. We now show $I(f_1)$ converges to zero.

(d) Let μ and ν be finite Borel measures on Ω such that

$$\int g\,d\mu=\int g\,d\nu$$

for all $g \in C(\Omega)$. Let E be any measurable set. Since $\mu + \nu$ is a finite Borel measure, it is regular, and by the Luzin theorem, we have a closed set F and $g \in C(\Omega)$ with $0 \le g \le 1$ such that $\mathbf{1}_E|_F = g|_F$ and $(\mu + \nu)(\Omega \setminus F) < \varepsilon/2$. Then,

$$\begin{aligned} |\mu(E) - \nu(E)| &= |\int \mathbf{1}_E \, d\mu - \int \mathbf{1}_E \, d\nu| \\ &\le \int_{\Omega\setminus F} |\mathbf{1}_E - g| \, d\mu + \int_{\Omega\setminus F} |g - \mathbf{1}_E| \, d\nu \\ &\le 2\mu(\Omega \setminus F) + 2\nu(\Omega \setminus F) < \varepsilon. \end{aligned}$$

By limiting $\varepsilon \to 0$, we have $\mu(E) = \nu(E)$. □

6.7 (Dual of continuous function spaces).

6.4 Hausdorff measures

Exercises

Part III

Linear operators

Chapter 7

Lebesgue spaces

7.1 L^p spaces

7.1 (Hölder inequality).

Proof.

$$\int fg \le C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take C such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \Big(\int |f|^p \Big)^{\frac{1}{p}} \Big(\int |g|^p \Big)^{\frac{1}{q}}.$$

Note that we can show that $1 \le 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \le 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. □

7.2 L^1 spaces

7.2 (Convolution?).

7.3 (Approximate identity?).

7.4 (Continuity of translation?).

7.3 L^2 spaces

7.4 L^∞ spaces

Chapter 8

Bounded linear operators

8.1 Continuity

Schur test

8.2 Density arguments

extension of operators

8.3 Interpolation

weak Lp, marcinkiewicz

Definition 8.3.1. Let f be a measurable function on a measure space (X, μ). The *distribution function* $\lambda_f : [0, \infty) \to [0, \infty)$ is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}).$$

Do not use $\mu(\{x : |f(x)| \geq \alpha\})$. The strict inequality implies the *lower semi-continuity* of λ_f.

(a) For $p > 0$, we have

$$\|f\|_{L^p}^p = p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^p \frac{d\alpha}{\alpha}.$$

Definition 8.3.2.

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0<\alpha<\infty} \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right].$$

Theorem 8.3.1. *For $p \geq 1$ we have $\|f\|_{p,\infty} \leq \|f\|_p$.*

Proof. By the Chebyshev inequality,

$$\sup_{0<\alpha<\infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha)\, d\alpha = \|f\|_{L^p}^p.$$

□

8.1 (Marcinkiewicz interpolation). Let X be a σ-finite measure space and Y be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator $T : L^{p_0}(X) + L^{p_1}(X) \to M(Y)$ has two weak-type estimates

$$\|T\|_{L^{p_0}(X)\to L^{p_0,\infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X)\to L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X)\to L^p(X)} < \infty.$$

Proof. Let $f \in L^p(X)$ and denote $f_h = \chi_{|f|>\alpha} f$ and $f_l = \chi_{|f|\le\alpha} f$. It is easy to show $f_h \in L^{p_0}$ and $f_l \in L^{p_1}$. Then,

$$\begin{aligned}
\|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha)\,\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^p \cdot \mu(|T(f\cdot \mathbf{1}_{|f|>\alpha})| > \alpha)\,\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha)\,\frac{d\alpha}{\alpha} \\
&\le \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0,\infty}}^{p_0}\,\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}} \|Tf_l\|_{L^{p_1,\infty}}^{p_1}\,\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^{p-p_0} \|f_h\|_{p_0}^{p_0}\,\frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{p_1}^{p_1}\,\frac{d\alpha}{\alpha} \\
&\sim \|f\|_p^p.
\end{aligned}$$

by (1) Fubini, (2) Sublinearlity, (3) Chebyshev, (4) Boundedness, (5) Fubini. □

8.2 (Hadamard's three line lemma). Let f be a bounded holomorphic function on the vertical unit stripe $\{z : 0 < \operatorname{Re} z < 1\}$. Then, for $0 < \theta < 1$,

$$\|f\|_{L^\infty(\mathrm{Re}=\theta)} \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta}.$$

Proof. Define

$$g(z) := \frac{f(z)}{\|f\|_{L^\infty(\mathrm{Re}=0)}^{1-z} \|f\|_{L^\infty(\mathrm{Re}=1)}^{z}}, \qquad g_n(z) = g(z) e^{\frac{z^2-1}{n}}.$$

Then we have

1. $g_n \to g$ pointwisely as $n \to \infty$,
2. $g_n(z) \to 0$ uniformly as $\operatorname{Im} z \to \infty$.

The second one is because g is bounded and for $z = x + yi$ we have

$$|g_n(z)| \lesssim |e^{\frac{z^2-1}{n}}| = e^{\mathrm{Re}\,\frac{z^2-1}{n}} = e^{\frac{x^2-y^2-1}{n}} \le e^{\frac{-y^2}{n}}.$$

By (1), it is enough to bound g_n for each n. Truncating the stripe, the outer region is controlled by (2) and the interior region is controlled by the maximum modulus principle. □

8.3 (Riesz-Thorin interpolation). Let X, Y be σ-finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \qquad \frac{1}{q_\theta} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta\to q_\theta} \le \|T\|_{p_0\to q_0}^{1-\theta} \|T\|_{p_1\to q_1}^{\theta}.$$

Proof. Note that

$$\|T\|_{p_\theta \to q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g\rangle|}{\|f\|_{p_\theta}\|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z\rangle = \int \overline{g_z(y)}Tf_z(y)\,dy,$$

where f_z and g_z are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z)+\frac{p_\theta}{p_1}z}\frac{f}{|f|}$$

so that we have $f_\theta = f$ and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for $\mathrm{Re}\, z = x$.

Then,

$$|\langle Tf_z, g_z\rangle| \leq \|T\|_{p_0\to q_0}\|f_z\|_{p_0}\|g_z\|_{q'_0} = \|T\|_{p_0\to q_0}\|f\|_{p_\theta}^{p_\theta/p_0}\|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for $\mathrm{Re}\, z = 0$, and

$$|\langle Tf_z, g_z\rangle| \leq \|T\|_{p_1\to q_1}\|f_z\|_{p_1}\|g_z\|_{q'_1} = \|T\|_{p_1\to q_1}\|f\|_{p_\theta}^{p_\theta/p_1}\|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for $\mathrm{Re}\, z = 1$. By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z\rangle| \leq \|T\|_{p_0\to q_0}^{1-\theta}\|T\|_{p_1\to q_1}^{\theta}\|f\|_{p_\theta}\|g\|_{q'_\theta}$$

for $\mathrm{Re}\, z = \theta$. Putting $z = \theta$ in the last inequality, we get the desired result. $\square$

Chapter 9

Convergence of linear operators

9.1 Translation and multiplication operators

9.2 Convolution type operators

approximation of identity Fejér, Poisson, box?

9.3 Computation of integral transforms

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega_x \times \Omega_y \to \mathbb{R}$ be such that $\partial_{x_i} f$ is well-defined. Suppose f and $\partial_{x_i} f$ are locally integrable in x and integrable y.

Then,

$$\partial_{x_i} \int f(x,y)\,dy = \int \partial_{x_i} f(x,y)\,dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if F is C^2 then the *classical* partial differentiation commute, and if F is not C^2 then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}
(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞
(b) f is AC iff f' is L^1
(c) f is BV iff f' is a finite regular Borel measure

11.1 Absolute continuous measures

11.2 Absolute continuous functions

11.3 Functions of bounded variation

Chapter 12

Lebesgue differentiation theorem

12.1 Hardy-Littlewood maximal function

Let T_m be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m,\ \varepsilon:|\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

12.1 (Hardy-Littlewood maximal function). The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

12.2 (Weak type estimate).

$$\|Mf\|_{1,\infty} \le 3^d \|f\|_{L^1(\Omega)}.$$

(a) Proof by covering lemma.

Proof. (a) By the inner regularity of μ, there is a compact subset K of $\{|Mf| > \lambda\}$ such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every $x \in K$, since $|Mf(x)| > \lambda$, we can choose an open ball B_x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover $\{B_i\}_i$ of K. Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection $\{B_k\}_k$ such that

$$K \subset \bigcup_i Bi \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \le \sum_k 3^d \mu(B_k) \le \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \le \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of B_k's. $\square$

12.3 (Radially bounded approximate identity)**.** If an approximate identity K_n is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every n and x, hence has a weak type estimate.

12.4 (Almost everywhere convergence of operators)**.** Suppose is T_m is a sequence of linear operators such that the maximal function T^*f is dominated by Mf. If $f \in L^1(\Omega)$ and $T_m g \to g$ pointwise for $g \in C(\Omega)$, then $T_m f \to f$ a.e.

Proof. Take $\varepsilon > 0$ and $g \in C(\Omega)$ such that $\|f - g\|_{L^1(\Omega)} < \varepsilon$. Since $T_m g(x) \to g(x)$ pointwise, we have

$$\begin{aligned}
&\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\
&\quad \leq \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \tfrac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \tfrac{\lambda}{2}\}) \\
&\quad \leq \mu(\{x : M(f-g)(x) > \tfrac{\lambda}{2}\}) + \frac{2}{\lambda}\|f - g\|_{L^1(\Omega)} \\
&\quad \lesssim \frac{1}{\lambda}\varepsilon
\end{aligned}$$

for every $\lambda > 0$. Limiting $\varepsilon \to 0$, we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every $\lambda > 0$, hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

Definition 12.1.1.

$$f^*(x) := \lim_{r \to 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\, dy.$$

Theorem 12.1.1 (Lebesgue differentiation)**.** $f^* = 0$ *a.e.*

Proof. Note that $f^* \leq Mf + |f|$ implies

$$\|f^*\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that $g^* = 0$ for $g \in C_c$. Approximate using $f^* = (f - g)^*$. □

Exercises

12.5 (Doubling measure)**.**