Category Theory

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5.1 Enriched categories

5.1 (Pointed category)**.** A *pointed category* is a category with a zero object.

(a) A category is $\mathbf{Set}_*$-enriched if and only if it admits a zero morphism.

(b) Every pointed category is $\mathbf{Set}_*$-enriched.

Chapter 6

Abelian categories

6.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism f is the pullback of (f, f).

A category is called *regular* if every kernel pair admits a coequalizer.

6.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

The category **Grp** is regular but not coregular, since there is a monomorphism which is not regular.

6.2 Additive and abelian categories

6.2 (Pre-additive categories)**.** A *pre-additive category* is another name of **Ab**-enriched category.

(a) a

6.3 (Semi-additive cateogries)**.** A *semi-additive category* is a category with binary biproducts.

(a) A category is semiadditive if and only if it is pointed **CMon**-enriched.

6.4 (Additive categories)**.** (a) additive completion by formally adjoining finite biproducts.

(b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a $\mathbf{Set}_*$-enriched category.

6.5 (Pre-abelian categories)**.** A *pre-abelian category* is an additive category in which every morphism has the kernel and cokernel. Equivalently, it is a finitely bicomplete pre-additive category.

(a)

6.6 (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen)**.** A pointed, Baar-exact, protomodular, with binary coproudcts.

(a) short five lemma, 3×3 lemma, snake lemma, noehter isomorphism hold.

(b) long exact homology sequence

(c) Every semi-abelian category is exact.

(d) Every semi-abelian category is finitely bicomplete.

(e) In general, a semi-abelian category is not pre-additive nor semi-additive.

6.7 (Abelian categories)**.** We say $\mathcal{C}$ is *abelian* if every morphism has the kernel and cokernel, and every mono and epi is normal.

(a) A category is abelian if and only if it is additive and exact.

6.8 (Freyd-Mitchell embedding)**.**

$$\begin{tikzcd}
\text{abelian} \arrow[r] \arrow[d] & \text{pre-abelian} \arrow[r] & \text{additive} \arrow[r] \arrow[d] & \text{pre-additive} \\
\text{semi-abelian (JMT)} & & \text{semi-additive} &
\end{tikzcd}$$

- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras, C^*-algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

Chapter 7

site, topos $(\infty, 1)$-category