Homological Algebra

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Contents

Part I

Chapter 1

Abelian categories

$$
\begin{array}{ccccccc}
K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow\!\!\downarrow & & \downarrow & & \\
K' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 0
\end{array}
$$

(a) If $A \to A'$ is monic, then $K \to K'$ is monic.

(b) If $B \to B'$ is monic, then $K \to K'$ is epic.

Chapter 2

A left R-module P is projective if and only if the left exact functor $\mathrm{Hom}_R(P,-)$ is exact.

A left R-module I is injective if and only if the left exact contravariant functor $\mathrm{Hom}_R(-,I)$ is exact.

2.1 (Tor functor)**.** Let R be a ring and M be a left R-module. We define the *Tor functor* as the left derived functor of the right exact functor $-\otimes_R M : \mathrm{Mod}\text{-}R \to \mathbf{Ab}$

$$\mathrm{Tor}_n^R(N,M) := H_n(P_\bullet \otimes_R M),$$

where $P_\bullet$ is a projective resolution of a right R-module N.

(a) In fact, the Tor functor may be defined by the left derived functor of the right exact functor $M \otimes_R - : R\text{-}\mathrm{Mod} \to \mathbf{Ab}$ for a right R-module M.

(b) In fact, only for Tor functors, we may only assume $P_\bullet$ is a flat resolution. (Flat resolution lemma)

2.2 (Ext functor)**.** Let R be a ring and M be a left R-module. We define the *Ext functor* as the right derived functor of left exact functor $\mathrm{Hom}_R(M,-)$

$$\mathrm{Ext}_R^n(M,N) := H^n(M,I^\bullet),$$

where $I^\bullet$ is an injective resolution of N.

(a) In fact, the Ext functor may be defined by the right derived functor of the left exact contravariant functor $\mathrm{Hom}(-,M)$.

long exact seuqence

2.3 (Universal coefficient theorem)**.** Let R be a ring. Let $C_\bullet$ be a chain complex of flat right R-modules and M be a left R-module.

Proof. We first prove the Künneth formula. Note that modules in $Z_\bullet$ and $B_\bullet$ are also flat. We start from that we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0.$$

We have a short exact sequence of chain complexes

$$\mathrm{Tor}_1^R(B_{\bullet-1},M) \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since modules in $B_{\bullet-1}$ are flat so that $\mathrm{Tor}_1^R(B_{\bullet-1},M) = 0$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since $H_n(C_{\bullet-1}) = H_{n-1}(C_\bullet)$ for any chain complex C, we have a long exact sequence

$$H_n(B_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(C_\bullet \otimes_R M) \to H_{n-1}(B_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet \otimes_R M).$$

Since every morphism in $B_\bullet$ and $Z_\bullet$ is zero, we have an exact sequence

$$B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet \otimes_R M) \to B_{n-1} \otimes_R M \xrightarrow{f_{n-1}} Z_{n-1} \otimes_R M.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{coker} f_n \to H_n(C_\bullet \otimes_R M) \to \ker f_{n-1} \to 0.$$

Since

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

is a flat resolution of $H_n(C_\bullet)$, by the flat resolution lemma, we have a long exact sequence

$$\operatorname{Tor}_1^R(Z_n, M) \to \operatorname{Tor}_1^R(H_n(C_\bullet), M) \to B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet) \otimes_R M \to 0.$$

Since Z_n is flat so that $\operatorname{Tor}_1^R(Z_n, M) = 0$, we have

$$\operatorname{coker} f_n = H_n(C_\bullet) \otimes_R M, \quad \ker f_n = \operatorname{Tor}_1^R(H_n(C_\bullet), M).$$

Therefore, we have an exact sequence

$$0 \to H_n(C_\bullet) \otimes_R M \to H_n(C_\bullet \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(C_\bullet), M) \to 0.$$

Universal coefficient theorem states that if R is a PID, then the Künneth formula splits non-canonically.

$\square$