Partial Differential Equations

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Part I

Sobolev spaces

Chapter 1

Distribution theory

1.1 Space of test functions

1.1. (a) If a test function φ satisfies $\langle 1, \varphi \rangle = 0$, then there is $v \in \mathbb{R}^d$ and a test function ψ such that $\varphi = v \cdot \nabla \psi$.

(b) If a distribution has zero derivative, then it is a constant.

1.2 (Weak* convergence)**.**

1.2 Space of distributions

1.3 (Rigged Hilbert space)**.**

1.3 Well-posedness

1.4 (Extension of linear operators)**.** Let $T : \mathcal{D} \to \mathcal{D}'$ be a contiuous linear operator. We can always define the adjoint $T^* : \mathcal{D} \subset \mathcal{D}'' \to \mathcal{D}'$. The most reasonable extension of T is $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$. For $f \in (T^*(\mathcal{D}))'$, we can define $\langle T(f), \varphi \rangle := \langle f, T^* \varphi \rangle$ for $\varphi \in \mathcal{D}$.

Suppose $T : (\mathcal{D}, \mathcal{T}) \to (T(\mathcal{D}), \mathcal{S})$ is proved to be continuous. If $(\mathcal{D}, \mathcal{T}) \to (T^*(\mathcal{D}))'$ and $(T(\mathcal{D}), \mathcal{S}) \to \mathcal{D}'$ are embeddings, then the extension of T to the completion of $(\mathcal{D}, \mathcal{T})$ agrees with $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$.

For example, if Φ is locally integrable, then since $(T_\Phi)^* = T_{\widetilde{\Phi}}$ and $\Phi * \varphi \in \mathcal{E} = C^\infty$ for $\varphi \in \mathcal{D}$, the convolution operator $T_\Phi : \mathcal{E}' \to \mathcal{D}'$ can be defined on the space of compactly supported distributions.

If $g * f$ is well-defined, is $f * g$ also well-defined? In other words, if $f \in (T_{\widetilde{g}}(\mathcal{D}))'$ so that $g * f \in \mathcal{D}'$, then $g \in (T_{\widetilde{f}}(\mathcal{D}))'$? Are they same?

$$\langle g, \widetilde{f} * \varphi \rangle =$$

Exercises

Chapter 2

Sobolev inequalities

2.1 Approximations

2.1 (Completeness of Sobolev norms).

2.2 (Difference quotient).

2.3 (Interior approximation).

2.4 (Myers-Serrin theorem).

2.2 Extensions and restrictions

2.5 (Lipschitz boundary).

2.6 (Extension theorem).

2.7 (Trace theorem).

2.8 (Vanishing at boundary). zero trace, whole domain

2.3 Sobolev embeddings

Temporarily we define a *function space* on $\mathbb{R}^d$ as a complete topological vector space X together with embeddings $\mathcal{S}(\mathbb{R}^d) \to X$ and $X \to \mathcal{S}'(\mathbb{R}^d)$. If $\mathcal{S}(\mathbb{R}^d)$ is dense in X, hence so is X in $\mathcal{S}'(\mathbb{R}^d)$, we will say X is *approximable*. We will not take dual spaces for non-approximable spaces, such as $L^\infty(\mathbb{R}^d)$ and $M(\mathbb{R}^d)$.

Let X, Y be function spaces on $\mathbb{R}^d$ such that X is approximable. We claim that if $\|u\|_Y \lesssim \|u\|_X$, then we have embedding $X \subset Y$. Let $u \in X$. Since $\mathcal{S}$ is dense in X, we can take a net $u_\alpha \in \mathcal{S}$ such that $u_\alpha \to u$ in X. Then, u_α is Cauchy in Y by the inequality, we have $v \in Y$ such that $u_\alpha \to v$ in Y. The uniqueness of limits in $\mathcal{S}'$ implies that $u = v$, hence $u \in Y$.

2.9. We introduce the *Sobolev regularity* $\frac{s}{d} - \frac{1}{p}$ for a triple of $s \in \mathbb{R}$, $p \in [1, \infty]$, $d \in \mathbb{Z}_{>0}$, and the *Hölder regularity* $\frac{k+\alpha}{d}$ for a triple $k \in \mathbb{Z}_{\geq 0}$, $\alpha \in [0, 1)$, $d \in \mathbb{Z}_{>0}$.

(a)

$$\|u\|_{W^{k,p}(\mathbb{R}^d)} \lesssim \|u\|_{W^{k',p'}(\mathbb{R}^d)}.$$

(b) If $\frac{k}{d} < \frac{s}{d} - \frac{1}{p}$, then

$$\|\nabla^\alpha u\|_{C_0(\mathbb{R}^d)} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^d)}, \qquad u \in W^{s,p}(\mathbb{R}^d).$$

$\mathcal{S}' = \bigcup_{\alpha,\beta\in\mathbb{Z}^d_{\ge 0}} \langle x\rangle^{-\alpha}\langle\xi\rangle^{-\beta} L^2$.

2.10 (Gagliardo-Nirenberg-Sobolev inequality). If $\frac{1}{d} - \frac{1}{p} = -\frac{1}{p'}$, then

$$\|u\|_{L^{p'}} \lesssim \|\nabla u\|_{L^p}, \qquad u \in C_c^\infty(\mathbb{R}^d).$$

2.11 (Hölder spaces).

2.12 (Morrey inequality).

2.13 (Poincaré inequality). BMO

2.14 (Rellich-Kondrachov theorem). Let Ω be bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary. For $1 \le p < d$, p^* is given by $-\frac{1}{p^*} := \frac{1}{d} - \frac{1}{p}$, called the *Sobolev conjugate*. Let η_ε be a standard mollifier.

(a) The convolution operator $(\eta_\varepsilon * -) : L^1(\Omega) \to C(\overline{\Omega})$ is compact for each $\varepsilon > 0$.

(b) We have

$$\|\eta_\varepsilon * u - u\|_{L^1(\Omega)} \lesssim \varepsilon \|u\|_{W^{1,1}(\Omega)}, \qquad u \in W^{1,1}(\Omega).$$

(c) If $1 \le p < d$ and $1 \le q < p^*$, then there is $\theta > 0$ such that we have

$$\|\eta_\varepsilon * u - u\|_{L^q(\Omega)} \lesssim \varepsilon^\theta \|u\|_{W^{1,p}(\Omega)}, \qquad u \in W^{1,p}(\Omega).$$

(d) If $1 \le p < d$ and $1 \le q < p^*$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

(e) If $\frac{l}{d} - \frac{1}{q} < \frac{k}{d} - \frac{1}{p}$, then the embedding $W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega)$ is a compact.

Proof. (a) The sequence $(\eta_\varepsilon * u_n)_n$ is pointwise bounded from

$$\|\eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim 1, \qquad n \in \mathbb{N},$$

and equicontinuous from

$$\|\nabla \eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\nabla \eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim 1, \qquad n \in \mathbb{N}.$$

By the Arzela-Ascoli theorem, since $\overline{\Omega}$ is compact, there is a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $C(\overline{\Omega})$.

(b) Write

$$\begin{aligned}
\eta_\varepsilon * u_n(x) - u_n(x) &= \int \varepsilon^{-d} \eta(\varepsilon^{-1}(x-y))(u_n(y) - u_n(x))\, dy \\
&= \int \eta(y)(u_n(x - \varepsilon y) - u_n(x))\, dy \\
&= \int \eta(y) \int_0^1 \frac{d}{dt}(u_n(x - t\varepsilon y))\, dt\, dy \\
&= \int \eta(y) \int_0^1 (-\varepsilon y) \cdot \nabla u_n(x - t\varepsilon y)\, dt\, dy.
\end{aligned}$$

Then, since $|y| \ge 1$ if $\eta(y) > 0$,

$$\|\eta_\varepsilon * u_n - u_n\|_{L^1(\mathbb{R}^d)} \le \varepsilon \int \eta(y) \int_0^1 \int |\nabla u_n(x - t\varepsilon y)|\, dx\, dt\, dy = \varepsilon \|\nabla u_n\|_{L^1(\mathbb{R}^d)}.$$

(c) Consider the interpolation

$$\|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_n - u_n\|_{L^1(\Omega)}^\theta \|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)}^{1-\theta}$$

for $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$ with $0 < \theta \le 1$. Since the Gagliardo-Nireberg-Sobolev inequality gives the bound

$$\|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)} \lesssim \|\eta_\varepsilon * u_n - u_n\|_{W^{1,p}(\Omega)} \lesssim 1, \qquad n \in \mathbb{N},\ \varepsilon > 0,$$

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \to 0$$

as $\varepsilon \to 0$.

(d) By the part (c), for any $\delta > 0$, there is $\varepsilon > 0$ such that

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} < \frac{\delta}{2},$$

so for a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $L^q(\Omega)$, we have

$$\|u_{n_k} - u_{n_{k'}}\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_{n_k} - \eta_\varepsilon * u_{n_{k'}}\|_{L^q(\Omega)} + \delta,$$

and by the diagonal argument reducing δ to zero, we can construct the desired subsequence.

(e) $\square$

Chapter 3

Generalizations of Sobolev spaces

3.1 Fractional Sobolev spaces

3.2 Fourier transform methods

3.3 Almost everywhere differentiability

Lipschitz, Rademacher

3.4 Vector-valued functions

3.1 (Pettis measurability theorem)**.** Let (Ω, μ) be a measure space and X a Banach space. Let $f : \Omega \to X$ be a function. We say f is *strongly measurable* or *Bochner measurable* if it is a pointwise limit of a sequence of simple functions.

If μ is complete, then all the pointwise convergence discussed here can be relaxed to the almost everywhere convergence.

(a) If f is strongly measurable, then f is Borel measurable.

(b) If f is Borel measurable, then f is weakly measurable.

(c) If f is weakly measurable and separably valued, then f is strongly measurable.

3.2 (Bochner and Pettis integrals)**.** Let (Ω, μ) be a measure space and X a Banach space. Let $f : \Omega \to X$ be a strongly measurable function. The function f is said to be *Bochner integrable* if there is a net of simple functions $(s_\alpha)_{\alpha \in \mathcal{A}}$ such that

$$\int_\Omega \|f(\omega) - s_\alpha(\omega)\| \, d\mu(\omega) \to 0$$

for $\alpha \in \mathcal{A}$.

(a) f is Bochner integrable if and only if $\int \|f(\omega)\| \, d\mu(\omega) < \infty$.

(b) If f is Bochner integrable, then it is Pettis integrable and the integrals coincides.

Bochner integrable => Pettis integrable => weakly(scalarly) integrable

Part II

Elliptic equations

Chapter 4

Potential theory

4.1 Mean value property

mean value property maximum principle Harnack inequality

potential estimate Hölder estimate

4.2 Weyl's lemma

Exercises

Problems

1. Let $d \geq 3$. Let u be a distribution on $\mathbb{R}^d$ that is harmonic on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity. Then, $u = a_\alpha \partial^\alpha \Phi$.

Chapter 5

Existence theory

5.1 Variational methods

5.2 Lax-Milgram theorem

5.1. Let $L : H \to H$ be a densely defined linear operator. If there is a Hilbert space V containing $\operatorname{dom} L$ and densely embedded in H such that $(u,v) \mapsto \langle Lu, v\rangle_H$ defines a coercive bilinear form on V, then L is admits a surjective closure.

Proof. For $f \in H$, there is $v \in V$ such that $\langle f, \varphi\rangle_H = \langle v, \varphi\rangle_V$ for all $\varphi \in V$. If we let $u := A^{-1}v$, where $A \in B(V)$ is defined such that $\langle L-, -\rangle_H = \langle A-, -\rangle_V$. Then,

$$\langle Lu, \varphi\rangle_H = \langle Au, \varphi\rangle_V = \langle v, \varphi\rangle_V = \langle f, \varphi\rangle_H$$

implies $Lu = f$. $\square$

5.2 (Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the problem

$$\begin{cases} -\Delta u(x) = f(x) & , \text{ in } x \in \Omega, \\ u(x) = 0 & , \text{ on } x \in \partial\Omega. \end{cases}$$

Define a bilinear form B on $H_0^1(\Omega)$ such that

$$B(u,v) := \int \nabla u(x) \cdot \nabla v(x)\, dx.$$

(a) If $u \in H_0^1(\Omega)$ and $f \in \mathcal{D}'(\Omega)$ satisfy $B(u,\varphi) = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, then $-\Delta u = f$.

(b) B is another inner product equivalent to $\langle -, -\rangle_{H_0^1(\Omega)}$.

(c) For $f \in H^{-1}(\Omega)$, there is $u \in H_0^{-1}(\Omega)$ such that $-\Delta u = f$.

5.3 Fredholm alternative

5.4 Perron's method

5.5 Eigenvalue problems

Chapter 6

Ellipic regularity

6.1 L^p theory

6.1 (Interior regularity in H^2). Let Ω be bounded open subset of $\mathbb{R}^d$ and $L : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ a uniformly elliptic operator given by

$$Lu := -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu$$

for $a^{ij} \in C^1(\Omega)$, $b^i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)$.

Fix an open subset $U \Subset \Omega$ and $\zeta \in C_c^\infty(\Omega)$ a cutoff function such that $\zeta = 1$ in U. Let $\varphi := -\partial_k^{-h}(\zeta^2\partial_k^h u)$ for $k = 1, \cdots, d$ and sufficiently small $h > 0$.

(a) We have

$$\|\nabla u\|_{L^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$

(b) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \int a^{ij}\partial_i u\partial_j\varphi + \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$.

(c) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}$$

for all u such that $Lu \in L^2(\Omega)$ and $u \in H^1(\Omega)$.

(d) We have

$$\|u\|_{H^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$.

Proof. (a) Since $\zeta^2 u \in H_0^1(\Omega)$,

$$
\begin{aligned}
\int \zeta^2 |\nabla u|^2 &\lesssim \int a^{ij}\zeta^2 \partial_i u \partial_j u \\
&= \int a^{ij}\ \partial_i u\ \partial_j(\zeta^2 u) - \int a^{ij}\ \partial_i u\ \partial_j(\zeta^2) u \\
&= \int (Lu - b^i\partial_i u - cu)\,\zeta^2 u - \int a^{ij}\ \partial_i u\ 2\zeta\partial_j\zeta\ u \\
&\lesssim \int (|Lu\ u| + |u\ \zeta\nabla u| + |u|^2 + |u\ \zeta\nabla u|) \\
&\lesssim \int (|Lu|^2 + |u|^2) + \frac{1}{\varepsilon}\int |u|^2 + \varepsilon \int \zeta^2|\nabla u|^2.
\end{aligned}
$$

Taking small $\varepsilon > 0$, we are done.

(b) Write

$$
\begin{aligned}
\int a^{ij}\partial_i u \partial_j \varphi &= -\int a^{ij}\partial_i u \partial_j \partial_k^{-h}(\zeta^2 \partial_k^h u) \\
&= \int \partial_k^h(a^{ij}\partial_i u)\ \partial_j(\zeta^2\partial_k^h u) \\
&= \int \partial_k^h a^{ij}\ \partial_i u\ \partial_j(\zeta^2)\ \partial_k^h u + \int \partial_k^h a^{ij}\ \partial_i u\ \zeta^2\ \partial_j\partial_k^h u \\
&\quad + \int a^{ij}\ \partial_k^h\partial_i u\ \partial_j(\zeta^2)\ \partial_k^h u + \int a^{ij}\ \partial_k^h \partial_i u\ \zeta^2\ \partial_j\partial_k^h u.
\end{aligned}
$$

The last term out of the four terms controls the difference quotient $|\partial_k^h \nabla u|$ as

$$
\int a^{ij}\ \partial_k^h\partial_i u\ \zeta^2\ \partial_j\partial_k^h u \gtrsim \int \zeta^2 |\partial_k^h\nabla u|^2,
$$

and the absolute values of other three terms are estimated up to constant by

$$
\begin{aligned}
&\int \zeta|\nabla u||\partial_k^h u| + \int \zeta^2|\nabla u||\partial_k^h\nabla u| + \int \zeta|\partial_k^h\nabla u||\partial_k^h u| \\
&\quad \lesssim \left(1+\frac{1}{\varepsilon}\right)\int \zeta^2|\nabla u|^2 + \left(1+\frac{1}{\varepsilon}\right)\int |\partial_h^k u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2 \\
&\quad \lesssim \left(1+\frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon\int \zeta^2 |\partial_k^h \nabla u|^2.
\end{aligned}
$$

Therefore,

$$
\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \int a^{ij}\partial_i u\partial_j\varphi + \left(1+\frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2,
$$

and taking small $\varepsilon > 0$, we are done.

(c) Note that

$$
\int a^{ij}\partial_i u \partial_j\varphi = \int (Lu - b^i\partial_i u - cu)\varphi
$$

since $\varphi \in H_0^1(\Omega)$. Because

$$
\int (Lu - b^i\partial_i u - cu)\varphi \lesssim \frac{1}{\varepsilon}\int (|Lu|^2 + |\nabla u|^2 + |u|^2) + \varepsilon\int |\varphi|^2
$$

and

$$\begin{aligned}\int |\varphi|^2 &= \int |\partial_k^{-h}(\zeta^2\partial_k^h u)|^2\\ &\lesssim \int |\nabla(\zeta^2\partial_k^h u)|^2\\ &\lesssim \int |\partial_k^h u|^2 + \int \zeta^2|\partial_k^h \nabla u|^2\\ &\lesssim \int |\nabla u|^2 + \int \zeta^2|\partial_k^h \nabla u|^2,\end{aligned}$$

we obtain

$$\int (Lu - b^i\partial_i u - cu)\varphi \lesssim \frac{1}{\varepsilon}\int (|Lu|^2 + |u|^2) + \left(\varepsilon + \frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2.$$

Taking small $\varepsilon > 0$, we are done. □

6.2 Schauder theory

6.3 De Giorgi-Nash-Moser theory

6.4 Viscosity solutions

Part III

Evolution equations

Chapter 7

Parabolic equations

7.1 Galerkin approximation

7.2 Semigroup theory

Chapter 8

Hyperbolic equations

Chapter 9

Local and global existence

9.1 Local existence

contraction mapping

9.2 Global existence

a priori estimates gronwall inequality

9.3 Weak convergence

Part IV

Nonlinear equations

Chapter 10

Chapter 11

Hamilton-Jacobi equations

optimal control viscosity solution

Chapter 12

Conservation laws

shocks NS