

Complex Analysis

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Part I

Holomorphic functions

Chapter 1

Cauchy theory

1.1 Complex differentiability

1.2 Contour integral

Cauchy-Goursat theorem

1.3 Power series

Analyticity, Laurent series,

1.4 Cauchy estimates

1.1. Let $p \in \mathbb{C}[z]$ with $p(z) = \sum_{k=0}^n a_k z^k$.

(a) $|p(z)| \lesssim |z|^n$.

(b) There is $R > 0$ such that $|p(z)| \gtrsim |z|^n$ for $|z| \geq R$.

Proof. If we take $R > 0$ such that

$$\sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2},$$

then $|z| \geq R$ implies

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k \\ &\geq |a_n||z|^n - \sum_{k=0}^{n-1} \frac{|a_k|}{R^{n-k}} |z|^n \\ &\geq \frac{|a_n|}{2} |z|^n. \end{aligned}$$

□

1.2. Let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function on a domain. Then, $\overline{f(z)} = f(\bar{z})$ if and only if $f(z) \in \mathbb{R}$ for $z \in \Omega \cap \mathbb{R}$.

Exercises

- 1.3. If a holomorphic function has positive real parts on the open unit disk then $|f'(0)| < 2 \operatorname{Re} f(0)$.
- 1.4. If at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.
- 1.5. If a holomorphic function on a domain containing the closed unit disk is injective on the unit circle, then so is on the disk.
- 1.6. For a holomorphic function f and every z_0 in the domain, there are $z_1 \neq z_2$ such that $\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(z_0)$.
- 1.7. For two linearly independent entire functions, one cannot dominate the other.
- 1.8. The uniform limit of injective holomorphic function is either constant or injective.
- 1.9. If the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions converges has a limit point, then it compactly converges.
- 1.10. Find all entire functions f satisfying $f(z)^2 = f(z^2)$.
- 1.11. An entire function maps every unbounded sequence to an unbounded sequence is a polynomial.
- 1.12. Let f be a holomorphic function on the open unit disk such that $f(0) = 1$ and $f'(0) > 2$. Then, there is z such that $|z| < 1$ and $f(z)$ is pure imaginary.

Chapter 2

Singularities

2.1 Classification of singularities

Riemann removable singularity theorem, Casorati-Weierstrass theorem, Picard's theorem

2.2 Residue theorem

$$\int_0^{2\pi} \frac{dx}{1+a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \quad -1 < a < 1$$

2.1 (Semicircles). (a)

$$\int_0^\infty \frac{1}{1+x^2} dx =$$

(b)

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(c)

$$\int_0^\infty \frac{\log x}{1+x^2} dx =$$

(d)

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1$$

2.2 (Computation of Fourier transforms).

2.3 (Laplace transforms).

2.4 (Gamma function).

2.3 Zeros and poles

2.5 (Argument principle). (a)

$$\int_\gamma \frac{f'(z)}{f(z)} dz = i \text{winding number}.$$

(b)

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i (\text{number of zeros} - \text{number of poles}).$$

2.6 (Rouché theorem). Let f be a meromorphic function on Ω . Let γ be a curve...

(a) If $h : [0, 1] \times \Omega \rightarrow \mathbb{C}$ is continuous and

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{\gamma} \frac{g'(z)}{g(z)} dz.$$

(b) In particular, if $|g(z)| < |f(z)|$ on $z \in \gamma$, then

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{\gamma} \frac{f'(z) + g'(z)}{f(z) + g(z)} dz.$$

2.7. Fundamental theorem of algebra, proof by the Liouville theorem, and proof by the Rouché theorem.

open mapping theorem

Chapter 3

Polynomial approximation

3.1 Mittag-Leffler theorem

3.2 Weierstrass factorization theorem

3.3 Runge's approximation

Mergelyan

Part II

Geometric function theory

Chapter 4

Conformal mappings

4.1 Riemann sphere

4.2 Open unit disk

4.3 Riemann mapping theorem

Chapter 5

Univalent functions

5.1 Bierbach conjecture

5.2 Harmonic functions

Chapter 6

Maximum principle; Schwarz's lemma, Lindelöf principle,

6.1 Riemann-Hilbert problem

Hilbert transform

6.2 Quasi-conformal mappings

Beltrami equations and Teichmüller theory?

Part III

Riemann surfaces

Chapter 7

Analytic continuation

7.1 Monodromy

7.2 Covering surfaces

7.3 Algebraic functions

7.4 Elliptic curves

Chapter 8

Differential forms

Chapter 9

Uniformization theorem

Part IV

Several complex variables