Fano Threefolds

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Contents

1 Day 1: April 6

Grade: solve 2∼4 exercises (report)

Throughout this lecture,

- we work over $\mathbb{C}$.
- A projective scheme is a projective scheme over $\mathbb{C}$, i.e. a closed subscheme of $\mathbb{P}^N_{\mathbb{C}}$ for some N.
- A variety is an integral scheme which is separated and of finite type over $\mathbb{C}$.

Definition 1.1. A Fano variety is a smooth projective variety X such that $-K_X$ is ample.

Definition 1.2. Let X be a smooth variety. A canonical divisor K_X is a Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X := \bigwedge^{\dim X} \Omega^1_X \in \mathrm{Pic}(X)$. ($\Omega$ is a locally free sheaf of rank($= \dim X$)) the canonical divisor

Example 1.3. If X is a smooth projective curve, then X is Fano iff $X \equiv \mathbb{P}^1$.

Proof. 1. A divisor D on X is ample iff $\deg D > 0$. ($\deg D = \sum a_i$ for $D = \sum a_i P_i$)

2. $\deg K_X = 2g - 2$, ($g := h^1(X, \mathcal{O}_X) \in \mathbb{Z}_{2n}$)

3. $g = 0$ iff $X = \mathbb{P}^1$.

Moreover, $\mathbb{P}^n$ is Fano. □

Example 1.4. Let $X \subset \mathbb{P}^N$: smooth hypersurface of deg d. For example, we may consider $X = \{x_0^d + \cdots + x_N^d\}$. Then, X is Fano iff $d \leq N$.

Proof. (Sketch) By the adjunction formula,

$$\mathcal{O}_X(K_X) \cong \mathcal{O}_{\mathbb{P}^N}(K_{\mathbb{P}^N} + X)|_X \cong \mathcal{O}_{\mathbb{P}^N}(-N - 1 - d)|_X.$$

Then, $\mathrm{Pic}\,\mathbb{P}^N = \{\mathcal{O}_{\mathbb{P}^N}(m) | m \in \mathbb{Z}\} \cong \mathbb{Z}$ (group isomoprhism). □

Why 3-folds? It is started by Gino Fano (1904∼), and the following theorem gives a motivation:

Theorem 1.5 (Lüroth,1876)**.** $\mathbb{C} \subset K \subset \mathbb{C}(x)$ *be field extensions. Assume the trenscendental degree of* K *is one. Then,* $K \cong \mathbb{C}(y)$*.*

The Lüroth problem states that: if $\mathbb{C} \subset K \subset \mathbb{C}(x_1, \cdots, x_n)$ field extensions, assuming the trenscendental degree of K is n, then $K \cong \mathbb{C}(y_1, \cdots, y_n)$?

Theorem 1.6 (Castelnuovo, 1886)**.** *The Lüroth problem is true if* $n = 2$*.*

The idea of this theorem is to convert Lüroth problem into a geometric version. A field extension $K \subset \mathbb{C}(x)$ corresponds to a dominant rational map $\mathbb{P}^1_{\mathbb{C}} \to X$, and the trenscendental degree one is equivalent to that X is curve. Here we may assume X to be a smooth projective curve. So, the Lüroth theorem can be restated as

Theorem 1.7. *If* $\mathbb{P}^1_{\mathbb{C}} \twoheadrightarrow X$ *for a smooth projective curve* X*, then* $X \cong \mathbb{P}^1_{\mathbb{C}}$*.*

For $n = 2$, we consider the rationality criterion.

Theorem 1.8. *Let* X *be a smooth projective surface. Then,* X *is rational iff* $H^1(X, \mathcal{O}_X) = H^0(X, 2K_X) = 0$

Example 1.9. If a surface X is del Pezzo(=Fano surface), then X is rational. It is because if $-K_x$ is ample then $H^0(X, 2K_X) = 0$ ($\because$ if not, then $2K_X$ is linearly equivalent to an effective divisor D, and $2(-K_X)^2 = 2K_X \cdot K_X = D \cdot K_X = \sum a_i C_i \cdot K_x \geq 0$.) Also, by the Kodaira vanishing, we have $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X(K_X + (-K_X))) = 0$.

How about $n = 3$? We may consider

- Three-dimensional rationality criterion?
- Fano hypersurface $X \subset \mathbb{P}^4$ are rational?

To settle the second question, Fano studied similar and easier Fano threefolds.

Theorem 1.10. *There is a counterexample to Lüroth's problem. Specifically, if X is the complete intersection of deg 2 hypersurface and deg 3 hypersurface in $\mathbb{P}^5$, X is not rational (1908, Fano), but X is unirational (1912, Enriques).*

Theorem 1.11 (1942, G. Fano)**.** *There is a hypersurface of degree 3 $X \subset \mathbb{P}^4$ which is not rational but unirational.*

Remark 1.12. The proof by Fano is not rigorous, so the second question(rationality of hypersurface) is now considered as results of

- Clemes-Griffiths (deg$=3$)
- Iskovskih-Manin (deg≥ 4)

Classificaiton of Fano 3-folds

Two invariants: Picard number ρ and index r.

Definition 1.13. Let X be a smooth projective variety.

$$\rho = \rho(X) := \dim_{\mathbb{Q}}((\mathrm{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q})/\equiv) \in \mathbb{Z}_{\geq 0}.$$

It is equal to $\dim_{\mathbb{Q}}((\mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q})/\equiv$, where $\mathrm{Div}X$ is the group of Weil divisors so that $\mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q}$ contains the formal linear combinations of prime divisors over $\mathbb{Q}$, and where the quivalence relation is given by $D \equiv D'$ iff $D \cdot C = D' \cdot C$ for every curve on X. From the intersection theory, $D \cdot C = \mathcal{O}_X(D) \cdot C = \deg(\mu^* \mathcal{O}_X(D))$ for $\mu : C^N \to C \hookrightarrow X$ (composition of normal and closed immersion). Then, $D \in \mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q}$ implies that there is $m \in \mathbb{Z}_{\geq 0}$ such that $mD \in \mathrm{Div}X$, then $D \cdot C := \frac{1}{m}((mD) \cdot C)$.

Remark 1.14. Let Xbe a Fano variety. Then, $\mathrm{Pic}X \cong \mathrm{Pic}X/\equiv \cong \mathbb{Z}^{\oplus \rho(X)}$. In particular, $D \sim D'$ implies $D \equiv D'$.

Definition 1.15. Let X be a Fano variety.

$$r = r_X := \text{the largest positive integer that divides } K_X,$$

that is, there is a divisor H such that $-K_X \sim rH$, but for $s > r$ there is no divisor H such that $-K_X \sim sH$.

We shall prove $1 \leq r \leq \dim X + 1$(for $\dim X = 3$, then $r = 1, 2, 3, 4$).

Example 1.16. Let $X = \mathbb{P}^3$. Then, $\mathrm{Pic}X \cong \mathbb{Z}H$, where H is a hyperplane, and $-K_x \equiv\sim 4H$, hence $\rho = 1$ and $r = 4$.

So here is the outline:

1. $r \geq 2$: Iskovskih, Fujita
2. $\rho = r = 1$: Iskovskih, Fujita
3. $\rho \geq 2$: Mori-Mukai

For 1, Δ-genus(Fujita) is used, and for 2 and 3, the cone theorem(minimal model program) is used. When $\dim X = 2$, using MMP, a del Pezzo surface X is reduced to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. When $\dim X = 3$, we have primitive Fano threefolds.

Our plan:

1. Cone theorem(mainly 2-dim)
2. $r \geq 2$
3. $\rho = r = 1$
4. $\rho \geq 2$ (primitive)
5. $\rho \geq 2$ (imprimitive)

Cone theorem

Theorem 1.17 (Cone theorem, Mori, 1982)**.** *Let X be a Fano variety. Then, there is rational curves $l_1, \cdots, l_m$ such that*

$$NE(X) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[l_i] \quad \textit{and} \quad -K_X \cdot l_i \leq \dim X + 1.$$

When $\rho = 3$, $NE(X) \subset N_1(X) \cong \mathbb{R}^{\rho(X)}$ is a triangular pyramid.

Definition 1.18. Let X be a smooth projective variety.

1. $Z_1(X) := \bigoplus_{C:\text{curve on } X} \mathbb{Z}C$,
2. $N_1(X) := (Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R})/\equiv$, where $Z \equiv Z'$ iff $L \cdot Z = L \cdot Z'$ for all $L \in \operatorname{Pic} X$.

It is well-known that

$$N_1(X) \times \left(\frac{\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{R}}{\equiv}\right) \to \mathbb{R}$$

induces a bijection

$$N_1(X) \to \operatorname{Hom}_{\mathbb{R}}\left(\frac{\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{R}}{\equiv}, \mathbb{R}\right),$$

therefore $\dim_{\mathbb{R}} N_1(X) = \rho(X)$.

Definition 1.19. Let X be a smooth projective variety.

1. For $Z \in Z_1(X) \otimes \mathbb{R}$, denote by $[Z] \in N_1(X)$ the numerical equivalence class of Z.
2. For $Z \in Z_1(X) \otimes \mathbb{R}$ is an effective 1-cycle.
3. $NE(X) := \{[Z] \in N_1(X) : Z \text{ effective 1-cycles}\}$

Remark 1.20. $NE(X)$ is a convex cone.

Example 1.21. Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $l_i = \pi_i^{-1}(*)$ for $i = 1, 2$ be any fibers. Then, $NE(X) = \mathbb{R}_{\geq 0}[l_1] + \mathbb{R}_{\geq 0}[l_2]$. One direction is clear, and for the opposite, pick $[D] = [a_1 C_1 + \cdots + a_r C_r] \in NE(X)$ ($a_i \geq 0$). It is enough to show $C_i \equiv b_1 l_1 + b_2 l_2$ for some $b_1, b_2 \geq 0$. Fix a curve C on X. Note that since $\operatorname{Pic} X = \mathbb{Z} l_1 \oplus \mathbb{Z} l_2$, we have $C \equiv b_1 l_1 + b_2 l_2$, so $0 \leq C \cdot l_i = (b_1 l_1 + b_2 l_2) \cdot l_i = b_i l_1 \cdot l_2 > 0$, we are done.

References for surfaces:

- Beauville: Complex algebraic surfaces (over $\mathbb{C}$),
- Bădescu: Algebraic surfaces

References for cone thm:

- Kollár-Mori: Birational geometry of algebraic varieties
- Debarre: Higher-dimensional algebraic geometry

2 Day 2: April 13

Extremal rays

Definition 2.1. Let X be a Fano variety. A ray R is called an extremal ray (of $NE(X)$ or of X) if $\zeta, \xi \in NE(X)$ and $\zeta + \xi \in R$ imply $\zeta, \xi \in R$.

Theorem 2.2 (Contraction theorem). *Let X be a Fano variety, $R = \mathbb{R}_{\geq 0}[l]$ an extremal ray for a curve l on X. Then, there is a unique morphism $f : X \to Y$ such that*

(i) Y is a projective normal variety,

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Y$,

(iii) For a curve C on X, $f(C)$ is point iff $[C] \in R$.

Note that such f can define the associated extremal ray. Moreover, we have $\rho(X) = \rho(Y) + 1$ and an exact sequence $0 \to \operatorname{Pic} Y \xrightarrow{f^*} \operatorname{Pic} X \xrightarrow{\cdot l} \mathbb{Z}$. The morphism f is called the contraction morphism of R.

Proof. See [Kollár-Mori]. $\square$

Theorem 2.3. *Let X be a del Pezzo surface. Let $R = \mathbb{R}_{\geq 0}[l]$ be an extremal ray for a curve l on X and $f : X \to Y$ be its contraction. Then, one of the following holds:*

(A) l is a (-1)-curve and f is a blow down of l (hence $\dim Y = 2$),

(B) $\dim Y = 1$ (i.e. Y is a smooth projective curve) and $\rho(X) = 2$, and f is a $\mathbb{P}^1$-bundle with fiber l.

(C) $\dim Y = 0$ (i.e. $Y = \operatorname{Spec} \mathbb{C}$) and $\rho(X) = 1$.

Remark 2.4. Let Y be a smooth projective surface and $f : X \to Y$ be the blowup at a point $P \in Y$. Then, $l := f^{-1}(p)$ satisfies $l \cong \mathbb{P}^1$ and $l^2 = -1$; called (-1)-curve. In this case we say f is the blowdown of l.

Remark 2.5. Let X be a del Pezzo surface and $\rho(X) = 1$. Then, it is known that $X \cong \mathbb{P}^2$.

Exercise 2.6. Show the above remark.

Remark 2.7. Let X be a smooth projective rational surface. If there is no (-1)-curve on X, then $X \cong \mathbb{P}^2$ or X is isomorphic to the Hirzeburch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, where $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$.

Remark 2.8. Let X be a del Pezzo surface and $f : X \to Y$ be a $\mathbb{P}^1$-bundle on a smooth projective curve Y. Then, $Y = \mathbb{P}^1$ and $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \in \{0, 1\}$.

Sketch. Leray spectral sequence gives $H^1(Y, f_*\mathcal{O}_X(= \mathcal{O}_Y)) \hookrightarrow H^1(X, \mathcal{O}_X) = 0$, so $H^1(Y, \mathcal{O}_Y) = 0$ implies $Y = \mathbb{P}^1$.

Also, $\mathbb{P}^1$-bundle, $X \cong \mathbb{P}_{\mathbb{P}^1}(E)$ of rank two, it is well known that $E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ and $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(b-a))$ for $n := b - a \geq 0$. It is known that for a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ there is a section c such that $c^2 = -n$, then $n \in \{0, 1\}$. $\square$

Lemma 2.9. *Let X be a del Pezzo surface and C a curve on X. Then, $C^2 \geq -1$.*

Proof. Write $(K_X + C) \cdot C = 2h^1(C, \mathcal{O}_C) - 2$. Recall that $(\omega_X \otimes \mathcal{O}_X(C))|_C \cong \omega_C$ holds even if C is a singular curve. Hence, $C^2 \geq -K_X \cdot C - 2 \geq 1 - 2 = -1$. $\square$

Example 2.10. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $l_i = \pi_i^{-1}(*)$ fibers. Then, each projection map π_i corresponds to the extremal rays $\mathbb{R}_{\geq 0}[l_i]$.

Example 2.11. Let $X = \mathbb{P}^2$. Then, $NE(X) = \mathbb{R}_{\geq 0}[l] = \mathbb{R}_{\geq 0}[l'] = \cdots$ since $N_1(X) = \mathbb{R}^{\rho(X)} = \mathbb{R}$.

Example 2.12. Let $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, which is del Pezzo. Then, if f is a blowdown of a section $l \cong \mathbb{P}^1$, then $\rho(Y) = 1$ and $Y \cong \mathbb{P}^2$. Then, we have two extremal rays $[l]$ and $[l']$ which correspond to f and π respectively.

Remark 2.13. Let X be a del Pezzo surface with $\rho(X) \geq 3$. Then,

$$\{\text{extremal rays}\} \longleftrightarrow \{(-1)\text{-curves}\}.$$

Therefore, a del Pezzo surface has a finitely many (-1)-curves.

Example 2.14. Let $f : X \to \mathbb{P}^2$ be a blowup at two points P and Q with $l_P = f^{-1}(P)$ and $l_Q = f^{-1}(Q)$. Lifting a line m passing through P and Q, we obtain m_X the proper transform of m. Then, $\rho(X) = 3$ and $NE(X) = \mathbb{R}_{\geq 0}[l_P] + \mathbb{R}_{\geq 0}[l_Q] + \mathbb{R}_{\geq 0}[m_X]$.

Remark 2.15. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface, for example, $X : x^3 + y^3 + z^3 + w^3 = 0$. It is well-known that X has exactly 27 (-1)-curves so that $NE(X) = \sum_{i=1}^{27} \mathbb{R}_{\geq 0}[l_i]$.

Remark 2.16. Minimal model program for del Pezzo surfaces.

$$\begin{tikzcd}
 & X:\ \text{del Pezzo surface} \arrow[d] & & \\
 & \exists (-1)\text{-curve on } X? \arrow[dl, "YES"'] \arrow[drr, "NO"] & & \\
f : X \to Y \text{ a blowdown of a } (-1)\text{-curve} \arrow[d] & & & \exists \text{ extremal } \mathbb{R}_{\geq 0}[l] \text{ s.t. } l \text{ is not } (-1)\text{-curve} \arrow[d] \\
\text{we can show } Y \text{ is del Pezzo} & & & f : X \to Y \text{ the contraction of } \mathbb{R}_{\geq 0}[l] \arrow[dl, "\dim Y=1"'] \arrow[d, "\dim Y=0"] \\
 & & X \cong \mathbb{P}^1 \times \mathbb{P}^1 & X \cong \mathbb{P}^2 \ (2.5)
\end{tikzcd}$$

Remark. Let $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ with $n \in \{0, 1\}$.
If $n = 0$, then $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$.
If $n = 1$, then $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, there is a (-1)-curve on X (cf.(2.11))

Outline of (2.3). For an extremal ray $R = \mathbb{R}_{\geq 0}[l]$, (A) for $l^2 < 0$, (B) for $l^2 = 0$, (C) for $l^2 > 0$. □

Proposition 2.17. *Let X be a del Pezzo surface and l be a curve on X with $l^2 < 0$. Then,*

(a) *l is a (-1)-curve,*

(b) *$\mathbb{R}_{\geq 0}[l]$ is an extremal ray,*

(c) *the contraction of R is the blowdown of l.*

In particular, $\dim Y = \dim X = 2$.

Proof. (a) We will show the following statements are equivalent:

(i) l is a (-1)-curve,

(ii) $l \cong \mathbb{P}^1$ and $l^2 = -1$,

(iii) $K_X \cdot l = l^2 = -1$,

(iv) $K_X \cdot l < 0$ and $l^2 < 0$.

Here X is a smooth projective surface and l a curve on it. Note (i) and (ii) are equivalent by definition. The equivalence between (ii) and (iii) is due to $(K_X + l) \cdot l = 2h^1(l, \mathcal{O}_l) - 2 \geq -2$. The equivalence between (iii) and (iv) is clear.

(b) Omitted.

(c) Let $f : X \to Y$ blowdown of l and $P := f(l)$. Recall that f is a contraction of R iff

(i) Y is a projective normal variety,

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Y$,

(iii) for a curve C on X, $f(C)$ is a point iff $[C] \in \mathbb{R}_{\geq 0}[l]$.

It follows (ii) from the following lemma (2.18). For (iii), ($\Rightarrow$) is clear. ($\Leftarrow$) Suppose $[C] \in \mathbb{R}_{\geq 0}[l]$ and $C \neq l$ so that $C \cdot l \geq 0$. Then, $C \equiv al$ for $a \in \mathbb{R}_{\geq 0}$, and $a > 0$ since $C \cdot H = al \cdot H$ for ample H. Now $0 \leq C \cdot l = al \cdot l = a(>0) \cdot l^2(=-1) < 0$, a contradiction. $\square$

Lemma 2.18. *If f is a projective birational morphism of normla varieties, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof. Consider the Stein factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

such that $g_*\mathcal{O}_X = \mathcal{O}_Z$ and h finite. Then,

$$\begin{tikzcd} K(X) & & K(Y) \arrow[ll, "\cong"'] \arrow[dl] \\ & K(Z) \arrow[ul] & \end{tikzcd}$$

implies $Z \xrightarrow{h} Y$ is finite birational morphism, and $A \hookrightarrow B$ is integral extension with $K(A) = K(B)$ where $\operatorname{Spec} A \subset Y$ is affine open and $\operatorname{Spec} B$ is given by the pullback(inverse image of h), hence $A = B$. $\square$

Lemma 2.19. *Let X be a del Pezzo surface and $\mathbb{R}_{\geq 0}[l]$ be an extremal ray for a curve l on X, whose contraction is $f : X \to Y$. Then,*

(A) $l^2 < 0$ *iff* $\dim Y = 2$,

(B) $l^2 = 0$ *iff* $\dim Y = 1$,

(C) $l^2 > 0$ *iff* $\dim Y = 0$.

Proof. Next lecture. $\square$

Proposition 2.20 ((B))**.** *If $l^2 = 0$, then the fiber is isomorphic to $\mathbb{P}^1$.*

Proof. For $P \in Y$, let $F := f^*P = \sum_{i=1}^r a_i C_i$ with $a_i \in \mathbb{Z}_{>0}$ and C_i prime divisors.

Claim 2.21. *Every fiber is irreducible.*

Proof. If it is reducible, then there are $C_1 \neq C_2$ in the fiber, then

$$F \cdot C_1 = (\sum_{i=1}^r a_i C_i) \cdot C_1 = a_1 C_1^2 + (\text{positive}),$$

so $C_1^2 < 0$. Then, $C_i \equiv b_i l$, so $C_1^2 < 0$ implies $l^2 < 0$ and $C_1 \cdot C_2 \geq 0$ implies $l^2 \geq 0$, a contradiction. $\square$

We can show that every fiber F is reduced:

$$(K_X + F) \cdot F = K_X \cdot F + F^2 = K_X \cdot F + 0 < 0,$$

by the adjunction, $F \cong \mathbb{P}^1$. $\square$

3 Day 3: April 20

Nef divisors and big divisors

Our today's goal is to prove Lemma 2.19.

Remark 3.1. Since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, $f : X \to Y$ is surjective so that $\dim Y \in \{0,1,2\}$. If we prove (A) and (C) in the Lemma 2.19, then we are enough.

Proof of Lemma 2.19 (A). ($\Rightarrow$) Proposition 2.17.

($\Leftarrow$) Note that $\dim X = \dim Y$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ imply f is birational. For an ample Cartier divisor A_Y on Y, f^*A_Y is a big divisor(defined later). Then,

$$f^*A_Y \cdot l = \deg(f^*A_Y|_l) = \deg(i^*f^*A_Y) = \deg((f|_l)^*j^*A_Y) = \deg((f|_l)^*\mathcal{O}_{f(l)}) = \deg \mathcal{O}_l = 0,$$

where $i : l \hookrightarrow X$ and $j : f(l) = * \hookrightarrow Y$ such that $fi = jf|_l$.

We can define f^*A_Y to be a big divisor if and only if there is $m \in \mathbb{Z}_{>0}$ such that mf^*A_Y is the sum of an ample divisor A and an effective divisor E. Then, $A\cdot l + E \cdot l = 0$ implies $E \cdot l < 0$, so if we write $E = \sum a_i C_i$, then $l = C_i$ for some i, hence $l^2 < 0$. □

Definition 3.2. Let X be a projective normal variety and D a Cartier divisor. Then, D is called to be big if and only if there are $m \in \mathbb{Z}_{>0}$, an ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.

Remark 3.3. In the above definition, the equality $mD = A + E$ can be replaced by $\sim$ or $\equiv$.

Remark 3.4. A divisor D is big iff nD is big for all $n \in \mathbb{Z}_{>0}$ iff nD is big for some $n \in \mathbb{Z}_{>0}$.

Proposition 3.5. *Let $f : X \to Y$ be a birational morphism of projective normal varieties. For a Cartier divisor D on Y, f^*D is big iff D is big.*

Proof. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, by tensoring $\mathcal{O}_Y(mD)$ we get

$$\mathcal{O}_Y(mD) = (f_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(mD) = f_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(mD)) = f_*f^*\mathcal{O}_Y(mD)$$

(the second equality is due to the projection formula), so

$$H^0(Y, \mathcal{O}_Y(mD)) = H^0(Y, f_*f^*\mathcal{O}_Y(mD)) = H^0(X, f^*\mathcal{O}_Y(mD)) = H^0(X, \mathcal{O}_X(mf^*(D))).$$

Therefore, f^*D is big iff D is big by Proposition 3.6. □

Proposition 3.6. *Let X be a projective normal variety and D a Cartier divisor on X. Then D is big iff there is $c \in \mathbb{Q}_{>0}$ such that for all sufficiently large m we have*

$$h^0(X, \mathcal{O}_X(mD)) > c \cdot m^{\dim X}.$$

Proof. ($\Rightarrow$) We may assume $D = A + E$ with A ample and E effective. Then, $H^0(X, mD) = H^0(X, m(A + E)) \hookleftarrow H^0(X, mA)$ by

$$0 \to \mathcal{O}_X(-mE) \to \mathcal{O}_X \to \mathcal{O}_{mE} \to 0.$$

Thus $h^0(X, mA) \le h^0(X, mD)$ implies that we may assume D is ample.

It is well-known that

$$\chi(X, mD) = \frac{D^{\dim X}}{(\dim X)!} m^{\dim X} + O(m^{\dim X - 1}) \in \mathbb{Z}[m]$$

from the Riemann-Roch, and by the Serre vanishing we have $\chi(X, mD) = h^0(X, mD)$ for large m, and we also have $D^{\dim X} > 0$ by Nakai's criterion.

($\Leftarrow$) Fix A a very ample divisor on X. We may assume by Bertini that A is a normal prime divisor. We have

$$0 \to \mathcal{O}_X(mD-A) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_A \to 0,$$

and $\mathcal{O}_X(mD)|_A \cong \mathcal{O}_A(mD_A)$ for some Cartier divisor D_A on A such that $\mathcal{O}_X(D)|_A \cong \mathcal{O}_A(D_A)$.

Write

$$0 \to H^0(X, mD-A) \to H^0(X, mD) \to H^0(A, mD_A).$$

Here $h_0(X, mD) \geq c \cdot m^{\dim X}$ and $h^0(A, mD_A) \leq b \cdot m^{\dim A}$ by the Exercise 3.7, we have $H^0(X, mD-A) \neq 0$ for some $m > 0$, i.e. $mD - A$ is linearly equivalent to an effective divisor. □

Exercise 3.7. Let Z be a projective normal variety and D a Cartier divisor on Z. Show that there exists $b > 0$ such that $h^0(Z, mD) \leq b \cdot m^{\dim Z}$ for all $m \in \mathbb{Z}_{>0}$. If you want, you may assume that Z is smooth.

Proof of Lemma 2.19 (C). ($\Leftarrow$) Let $\dim Y = 0$ i.e. $Y = \operatorname{Spec} \mathbb{C}$ with $\rho(X) = \rho(Y) + 1 = 1$, which implies that $l \equiv cA$ for some $c \in \mathbb{Q}$ and an ample divisor A on X because every projective variety has an ample divisor. Then, we can prove $c > 0$ from $A \cdot l = A \cdot (cA) = cA^2$, hence $l^2 = (cA) \cdot (cA) = c^2 A^2 > 0$.

($\Rightarrow$) Let $l^2 > 0$. Note that if l is a curve on a smooth projective surface X such that $l^2 > 0$, then l is nef because $l \cdot C > 0$ if $l = C$ and $l \cdot C \geq 0$ if $l \neq C$, and furthermore l is big by Proposition 3.9. Fix C a curve on X. We are enough to show $[C] \in \mathbb{R}_{\geq 0}[l]$. Then, $N_1(X) = \bigoplus_C \mathbb{R}_C / \equiv$ is generated by $[l]$, we get $\rho(X) = \dim N_1(X) = 1$ and $\dim Y = 0$.

Let l be a big divisor so that there is a sufficiently large m with a rational map $f : X \dashrightarrow \mathbb{P}^N$ defined by the complete linear system $|ml|$ whose image is a surface. By considering the defining polynomials of $\varphi(C) = \overline{V_+}(f_1, \cdots, f_r)$ such that $\varphi(ml)$ is a hyperplane section, there must be f_i not vanishing on X, so we have f_i with $\overline{V_+}(f_i) \cap \varphi(X) = \varphi(C) + \varphi(E)$, where $E = \varphi^{-1}(\varphi(E))$. Then, since $\overline{V_+}(f_i) \sim \varphi((\deg f_i) ml)$, which implies $(\deg f_i) ml \sim C + E$. Thus, using the definition of extremal rays, we have $[C] \in \mathbb{R}_{\geq 0}[l]$. □

Definition 3.8. Let X be a projective normal variety. A Cartier divisor D is called nef iff $D \cdot C \geq 0$ for all curves C on X.

Proposition 3.9. *Let X be a projective normal variety and D a nef Cartier divisor. Then, D is big iff $D^{\dim X} > 0$.*

Proof. For simplicity, assume $\dim X = 2$.

($\Rightarrow$) Let $mD = A + E$ with $z \in \mathbb{Z}_{>0}$, A ample, E effective. Since $mD \cdot E \geq 0$ from that D is nef and $mD \cdot A = A^2 + E \cdot A > 0$ from that A is ample, we have $(mD)^{\dim X} = (mD)^2 = mD \cdot A + mD \cdot E > 0$.

($\Leftarrow$) We may assume X is smooth by taking a resolution of X (the pullback via a rational map of a nef or big divisor is also nef of big respectively). Take H a very ample divisor on X. We also may assume $H - K_X$ is ample by the Serre criterion. Then,

$$0 \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD+H) \to \mathcal{O}_X(mD+H)|_H \to 0$$

and

$$0 \to H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_X(mD+H)) \to H^0(\mathcal{O}_X(mD+H)|_H)$$

are exact. Note that we have

$$h^0(\mathcal{O}_X(mD+H)) = \chi(X, mD+H) = \frac{(mD+H)^2}{2!} + O(m) \geq c \cdot m^2$$

by the Kodaira vanishing

$$H^i(X, mD+H) = H^i(X, K_X + (mD)_{(\text{it is nef})} + (H - K_X)_{(\text{it is ample})}) = 0$$

(sum of nef and ample is ample $\because$ Corollary 3.12.) and $h^0(\mathcal{O}_X(mD+H)|_H) \leq b \cdot m^{\dim H} = b \cdot m$. Therefore, $h^0(X, \mathcal{O}(mD)) \geq c' \cdot m^2$ for some c' and sufficiently large m. □

Remark 3.10. Let X be a projective normal variety with a nef divisor D. Then,

(a) $D \cdot {}^{\forall}(\text{curve}) \geq 0$ (by def),

(b) $D \cdot {}^{\forall}(\text{effective 1-cycle}) \geq 0$.

In particular, $NE(X) \subset D^{\geq 0} := \{\zeta \in N_1(X) : D \cdot \zeta \geq 0\} = D^{>0} \cup D^{\perp}$. In fact,

(c) The Kleiman-Mori cone is contained in $D^{\geq 0}$, i.e. $\overline{NE(X)} \subset D^{\geq 0}$.

Theorem 3.11 (Kleiman's ampleness criterion)**.** *Let X be a projective normal variety and D a Cartier divisor. Then, D is ample iff $\overline{NE(X)} \setminus \{0\} \subset D^{>0}$.*

Proof. Omitted. □

Corollary 3.12. *If N is nef and A is ample, then $N + A$ is ample.*

Proof. $\zeta \in \overline{NE(X)} \setminus \{0\}$ implies $(N + A) \cdot \zeta = N \cdot \zeta + A \cdot \zeta > 0$ because $N \cdot \zeta \geq 0$ and $A \cdot \zeta > 0$. □

Remark 3.13. It is useful to use $\mathbb{Q}$-divisors. For $D \in \operatorname{Div} X \otimes_{\mathbb{Z}} \mathbb{Q}$, D is defined to be nef if there is $m \in \mathbb{Z}_{>0}$ such that D is a nef Cartier divisor, and defined to be ample if there is $m \in \mathbb{Z}_{>0}$ such that D is a ample Cartier divisor. Then, a nef divisor can be approximated by $D = \lim_{\varepsilon \to 0+}(D + \varepsilon A)$.

Theorem 3.14 (Nakai-Moishezon)**.** *Let X be a projective normal variety and D a Cartier divisor. Then, D is ample (resp. nef) iff for a subvariety $Y \subset X$ we have $Y \cdot D^{\dim Y} > 0$ (resp. ≥ 0).*

Proof. For amples, well-known. For nefs, it follows from $Y \cdot D^{\dim Y} = \lim_{\varepsilon \to 0+} Y \cdot (D + \varepsilon A)^{\dim Y} \geq 0$. □

4 Day 4: April 27

We study Δ-genus to classify Fano 3-folds with index $r \geq 2$.

Definition 4.1. Let X be a Fano 3-fold. The index $r = r_X \in \mathbb{Z}_{>0}$ is defined such that there is a divisor H with $-K_X \sim rH$ but no divisors H satisfy $-K_X \sim sH$ for $s \in \mathbb{Z}_{r>0}$.

Lemma 4.2. $1 \leq r \leq 4$.

Proof. Cone theorem implies $NE(X) = \sum_{i=1}^{m} \mathbb{R}_{\geq}[l_i]$ with $0 < -K_X \cdot l_i \leq \dim X + 1 = 4$. Then, since $r \leq -K_X \cdot l_i$, we are done. □

Today's goal: $r = 4$ implies $X \cong \mathbb{P}^3$, and $r = 3$ implies $X \cong$ (quadratic) $\subset \mathbb{P}^4$. Here is our outline:

- If $r = 4$, then
- $\Delta(X, H) = 0$ with $-K_X \sim 4H$, then
- $|H|$ is very ample with $H^3 = 1$, then
- $X \cong \mathbb{P}^3$.

We can do $r = 3$ similar.

Δ-genus (1): definition and examples

Definition 4.3. A pair (X, D) is called a polarized variety if X is a projective variety and D is an ample divisor(or invertible sheaf) on X.

Definition 4.4. Let (X, D) be a polarized variety. Then,

$$\Delta(X, D) := \dim X + D^{\dim X} - h^0(X, D).$$

Example 4.5.

(i) Let $n \in \mathbb{Z}_{>0}$. Then,

$$\begin{aligned}\Delta(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) &= \dim \mathbb{P}^1 + \deg \mathcal{O}_{\mathbb{P}^1}(n) - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \\ &= 1 + n - (n+1) = 0.\end{aligned}$$

(ii) Let X be an elliptic curve and D an ample divisor on X. Then, by the Riemann-Roch

$$h^0(X, D) - h^1(X, D) = \chi(X, D) = \deg D + 1 - g = \deg D$$

and the Serre duality $h^1(X, D) = h^0(X, -D) = 0$, we have

$$\Delta(X, D) = 1 + \deg D - \deg D = 1.$$

Example 4.6. Let X be a del Pezzo surface. Then, $\Delta(X, -K_X) = 1$.

Proof. By the Riemann-Roch

$$\chi(X, D) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(-K_X) \cdot (-K_X - K_X)$$

and the Kodaira vanishing

$$\chi(X, -K_X) = \chi(X, K_X + (-2K_X)) = h^0(X, -K_X), \qquad \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X),$$

we have $h^0(X, -K_X) = K_X^2 + 1$. Therefore,

$$\Delta(X, -K_X) = \dim X + (-K_X)^2 - h^0(X, -K_X) = 1.$$

□

Proposition 4.7. *Let X be a Fano 3-fold. Pick a divisor H such that $-K_X \sim rH$.*

(a) *If $r = 4$, then $\Delta(X,H) = 0$ and $H^3 = 1$.*

(b) *If $r = 3$, then $\Delta(X,H) = 0$ and $H^3 = 2$.*

Proposition 4.8 (Riemann-Roch for 3-folds)**.** *Let X be a smooth projective 3-fold and D a divisor. Then,*

(a)
$$\chi(X,D) = \frac{1}{12}D\cdot(D-K_X)\cdot(2D-K_X) + \frac{1}{12}D\cdot c_2(X) + \chi(X,\mathcal{O}_X).$$

(b)
$$-K_X\cdot c_2(X) = 24\chi(X,\mathcal{O}_X).$$

Proof. Omitted. □

Corollary 4.9. *Let X be Fano 3-fold and H an ample divisor such that $H \equiv -qK_X$ with $q \in \mathbb{Q}_{>0}$. Then,*

$$h^0(X,H) = \chi(X,H) = \frac{1}{12}q(q+1)(2q+1)(-K_X)^3 + 2q + 1.$$

As a comment for $\mathbb{Q}_{>0}$, in most cases we have $q^{\pm 1} \in \mathbb{Z}_{>0}$. For example, $H \equiv -\frac{1}{r}K_X$ iff $rH \equiv -K_X$.

Proof. By Propositioin 4.8 and the Kodaira vanishing

$$\chi(X,H) = h^0(X,H), \quad \chi(X,\mathcal{O}_X) = 1,$$

we can complete the proof by simple computation. □

Theorem 4.10. *Let (X,D) be a polarized variety. Then, $\Delta(X,D) > \dim Bs|D|$, where $\dim\emptyset := -1$. In particular, $\Delta(X,D) \geq 0$.*

Proof. We will do if time permits. □

Proof of Proposition 4.7. We only show (a). Note that

$$h^0(X,H) =^{(4.9)} \frac{1}{12}q(q+1)(2q+1)(-K_X)^3 + 2q + 1 = h^0(X,H) = \frac{5}{2}H^3 + \frac{3}{2}$$

since $q = \frac{1}{4}$ and $(-K_X)^3 = (4H)^3 = 64H^3$. Then, Theorem 4.10 and $H^3 \geq 1$ imply

$$0 \geq \Delta(X,H) = \dim X + H^3 - h^0(X,H) = \frac{3}{2}(1-H^3) \leq 0.$$

Therefore, $H^3 = 1$ and $\Delta(X,H) = 0$. □

Remark 4.11. If $r = 4$ and $-K_X \sim 4H$, then $h^0(X,H) = 4$. If $|H|$ is very ample, then $X \hookrightarrow \mathbb{P}^{4-1} = \mathbb{P}^3$, hence $X \cong \mathbb{P}^3$. Thus we are enough to show the complete linear system $|H|$ is very ample.

Theorem 4.12. *Let (X,D) be a polarized variety with $\Delta(X,D) = 0$. Then,*

(a) *N_1 property holds: the section ring $\bigoplus_{m=0}^{\infty} H^0(X,mD)$ of D is generated by $H^0(X,D)$ as a $\mathbb{C}$-algebra.*

(b) *$|D|$ is very ample.*

Exercise 4.13. Show that under the N_1 property, if D is ample, then $|D|$ is very ample.

Proposition 4.14. *Let (X,L) be a polarized variety with invertible sheaf L. Let Y be an integral closed subscheme in $|L|$. For example, if X is normal with $L \cong \mathcal{O}_X(D)$, then $D \sim Y$, and it is a prime divisor. Then,*

(a) $L^{\dim X} = (L|_Y)^{\dim X - 1}$.

(b) $0 \le \Delta(X,L) - \Delta(Y,L|_Y) \le h^1(X,\mathcal{O}_X)$.

(c) $H^0(X,L) \to H^0(Y,L|_Y)$ *is surjective iff* $\Delta(X,L) = \Delta(Y,L|_Y)$.

(d) *Assume the condition in the part (c). Then, if* $L|_Y$ *satisfies* N_1 *property, then so does* L.

Proof of Proposition 4.12 assuming Proposition 4.14. For simplicity, we assume X is smooth. The complete linear system $|D|$ is base point free by $\Delta(X,D) = 0$ and Theorem 4.10 ($\dim Bs|D| < \Delta(X,D)$). Let $Y \in |D|$ be a general member. By Bertini, Y is smooth and connected(D is ample), hence Y is a smooth prime divisor. Applying Proposition 4.14, we have $0 \le \Delta(Y,D|_Y) \le \Delta(X,D) \le 0$. By Proposition 4.14 (d), D satisfies N_1 property from applying the induction hypothesis. □

Remark 4.15. We can check that for a projective curve X we have TFAE:

(i) $X \cong \mathbb{P}^1$,

(ii) $\Delta(X,D) = 0$ for every ample D,

(iii) $\Delta(X,D) = 0$ for an ample D.

Proof of Proposition 4.14. Write $n := \dim X$.

(a) $L^n = L^{n-1} \cdot Y = (L|_Y)^{n-1}$

(b) $\Delta(X,L) = n + L^n - h^0(X,L)$ and $\Delta(Y,L|_Y) = (n-1) + (L|_Y)^{n-1} - h^0(Y,L|_Y)$ imply

$$\Delta(X,L) - \Delta(Y,L|_Y) = 1 + h^0(Y,L|_Y) - h^0(X,L).$$

By taking $- \otimes L$ on

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

we have exact sequences

$$0 \to \mathcal{O}_X \to L \to L|_Y \to 0$$

and

$$0 \to H^0(X,\mathcal{O}_X) \to H^0(X,L) \to H^0(Y,L|_Y) \xrightarrow{\delta} H^1(X,\mathcal{O}_X).$$

Then,

$$h^1(X,\mathcal{O}_X) \ge \dim \operatorname{im} \delta = h^0(Y,L|_Y) - h^0(X,L) + h^0(X,\mathcal{O}_X) = \Delta(X,L) - \Delta(Y,L|_Y)$$

and $\dim \operatorname{im} \delta \ge 0$ implies the desired result.

(c) We have $\delta = 0$ if and only if $\Delta(X,L) = \Delta(Y,L|_Y)$, which is also equivalent to that $H^0(X,L) \to H^0(Y,L|_Y)$ is surjective.

(d) Note that we have a surjection $H^0(X,L) \to H^0(Y,L|_Y)$. Suppose $L|_Y$ satisfies N_1 property. If $\zeta \in H^0(Y,mL|_Y)$, then $\zeta = \sum c\xi_1 \cdots \xi_m$ for $c \in \mathbb{C}$ and $\xi_i \in H^0(Y,L|_Y)$, so we can show the map $H^0(X,mL) \to H^0(Y,mL|_Y)$ is surjective.

It is enough to show $\mu_X : H^0(X,mL) \otimes_{\mathbb{C}} H^0(X,L) \to H^0(X,(m+1)L)$ is surjective.

$$\begin{tikzcd}
 & & H^0(X,mL) \otimes_{\mathbb{C}} H^0(X,L) \arrow[r, two heads] \arrow[d, "\mu_X"] & H^0(Y,mL|_Y) \otimes_{\mathbb{C}} H^0(Y,L|_Y) \arrow[d, two heads, "\mu_Y"] \\
0 \arrow[r] & H^0(X,mL) \arrow[r, "-\otimes s_Y"] & H^0(X,(m+1)L) \arrow[r, two heads, "\pi_{m+1}"] & H^0(Y,(m+1)L|_Y)
\end{tikzcd}$$

For $\zeta \in H^0(X,(m+1)L)$, we have $\zeta_Y := \pi_{m+1}(\zeta) \in H^0(Y,(m+1)L|_Y)$ and there is $\sum c\xi_Y \otimes \eta_Y \in H^0(Y,mL|_Y) \otimes_{\mathbb{C}} H^0(Y,L|_Y)$ and back to obtain $\sum c\xi_X \otimes \eta_X \in H^0(X,mL) \otimes_{\mathbb{C}} H^0(X,L)$ with surjectivity. If we define $\widetilde{\zeta} := \zeta - \mu_X(\sum c\xi_X \otimes \eta_X)$, then $\pi_{m+1}(\widetilde{\zeta}) = \zeta_Y - \zeta_Y = 0$ so that there is $\widetilde{\widetilde{\zeta}} \in H^0(X,mL)$ such that $\widetilde{\zeta} = \widetilde{\widetilde{\zeta}} \otimes s_Y = \mu_X(\widetilde{\widetilde{\zeta}} \otimes s_Y)$, where $V(s_Y) = Y$ (check the exact sequence in the part (b)). Then, $\zeta = \widetilde{\zeta} + \mu_X(\sim)$ belongs to the image of μ_X. □

We now prove Theorem 4.10.

Definition 4.16. Let X be a projective variety and L an ample invertible sheaf. Let $V \subset H^0(X,L)$ be a $\mathbb{C}$-linear subspace. Let $\Delta(X,L,V) := \dim X + L^{\min X} - \dim_{\mathbb{C}} V$. (Note $\Delta(X,L) = \Delta(X,L,H^0(X,L))$)

Theorem 4.17. $\Delta(X,L,V) > \dim Bs|V|$, *where* $|V|$ *is the linear system corresponding to* V.

Proof. We may assume that X is normal and $V = H^0(X,L)$. the normalization of the resolution of the inderminacies of $\varphi_{|L|}$..

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{|L|}"] & \mathbb{P}^N_{\mathbb{C}} & Z := \psi(Y) \arrow[l, hook', "subsp"] \\ Y \arrow[u, "\mu"] \arrow[ur, "\psi"] \arrow[urr, "\psi"'] & & \end{tikzcd}$$

One of the following holds:

(i) $\dim Bs|L| = n$, where $n = \dim X$,

(ii) $\dim Z = 1$,

(iii) $\dim Z \geq 2$ and $\dim Bs|L| = n-1$,

(iv) $\dim Z \geq 2$ and $\dim Bs|L| \leq n-2$,

For the case (i), since $\dim Bs|L| = n$ iff $H^0(X,L) = 0$, we have

$$\Delta(X,L) = n + L^n - h^0(X,L) > n = \dim Bs|L|.$$

For the case (ii), we have $\Delta(X,L) = n + L^n - h^0(X,L)$. Then, $\mu^*L = M + F$ is decomposed into a base point free movable part M and a fixed part F by $L \mapsto \mu^*L$ and $L_Z := \mathcal{O}_{\mathbb{P}^N}(1)|_Z \mapsto M$. Then, with normal X and μ birational we have

$$H^0(X,L) \cong H^0(Y,\mu^*L) \cong H^0(Y,M).$$

Also $H^0(Y,M) \cong H^0(Z,L_Z)$ since the injectivity follows from $\psi_*\mathcal{O}_Y \hookleftarrow \mathcal{O}_Z$ and the surjectivity is due to the fact that the composition $H^0(Y,M) \leftarrow H^0(Z,L_Z) \leftarrow H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ is bijective. Now

$$0 \leq \Delta(Z,L_Z) = 1 + \deg L_Z - h^0(Z,L_Z)$$

and

$$(\mu^*L)^{n-1} \cdot (\psi^*L_Z) = (\deg L_Z) \cdot (\mu^*L)^{n-1} \cdot (\text{a general fiber of } \psi) \geq \deg L_Z$$

because μ^*L is nef and big. Then,

$$L^n = (\mu^*L)^n = (\mu^*L)^{n-1} \cdot (M+F) \geq \deg L_Z + (\mu^*L)^{n-1} \cdot F,$$

and

$$\begin{aligned} \Delta(X,L) &= n + L^n - h^0(X,L) \\ &\geq n + \deg L_Z + (\mu^*L)^{n-1} \cdot F - h^0(Z,L_Z) \\ &= n + \Delta(Z,L_Z) - 1 + (\mu^*L)^{n-1} \cdot F \\ &\geq n - 1 + (\mu^*L)^{n-1} \cdot F \\ &\geq n-1. \end{aligned}$$

If $\dim Bs|L| \leq n-2$, then we are done. If $\dim Bs|L| = n-1$, then $(\mu^*L)^{n-1} \cdot F > 0$ because $\mu(F)$ has dimension $n-1$, so $\Delta(X,L) > n-1$.

For the case (iii) and (iv), see [Fujita]. □

- T. Fujita, Classification $\cdots$ of polarized varieties (Book)
- T. Fujita, On the structure $\cdots$ with Δ-genus zero (Many papers by Fujita)

5 Day 5: May 11

Δ-genus (2): the case $\Delta = 0$

Theorem 5.1. *Let (X, L) be a polarized variety with $\Delta(X, L) = 0$ and $n := \dim X$.*

(a) *If X is smooth, then one of the following holds:*

(A) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$, i.e. there is an isomorphism $\theta : X \to \mathbb{P}^n$ such that $L \cong \theta^*\mathcal{O}(1)$.

(B) $(X, L) \cong (Q^n, \mathcal{O}(1))$, wherer $Q^n \subset \mathbb{P}^{n+1}$ is a quadric hypersurface.

(C) $(X, L) \cong (\mathbb{P}_{\mathbb{P}^1}(E), \mathcal{O}(1))$, where E is a locally free sheaf on $\mathbb{P}^1$ of rank n and $\mathbb{P}_{\mathbb{P}^1}(E)$ is the $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$.

(D) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}(2))$.

(b) *If X is not smooth, then (X, L) is a cone of the part (a). See Remark 5.3.*

Importance of $\Delta = 0$: Hyperelliptic?

Remark 5.2. $\Delta(X, L) = 0$ implies $|L|$ is very ample, hence $\varphi_{|L|} : X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is a closed immersion with $N := h^0(X, L) - 1$. For example,

(A) $\varphi_{|L|} = \mathrm{id}$.

(B) $\varphi_{|L|} : X \hookrightarrow \mathbb{P}^{n+1}$.

(D) $\varphi_{|\mathcal{O}(2)|} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 : [x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : yz : zx]$ (Veronese embedding).

Remark 5.3. For the case (B), via linear transformations we may assume

$$X = \{x_0^2 + \cdots + x_N^2 = 0\} = \mathrm{Proj}\frac{\mathbb{C}[x_0, \cdots, x_N]}{(x_0^2 + \cdots + x_N^2)} \subset \mathbb{P}^N_{\mathbb{C}}.$$

Then,

$$\mathrm{Proj}\frac{\mathbb{C}[x_0, \cdots, x_N, y]}{(x_0^2 + \cdots + x_N^2)} \subset \mathbb{P}^{N+1}_{\mathbb{C}}$$

is a(the) cone of $X \subset \mathbb{P}^N_{\mathbb{C}}$. More generally,

$$\mathrm{Proj}\frac{\mathbb{C}[x_0, \cdots, x_N, y_1, \cdots, y_r]}{(x_0^2 + \cdots + x_N^2)} \subset \mathbb{P}^{N+r}_{\mathbb{C}}$$

is a (generalized) cone of $X \subset \mathbb{P}^N_{\mathbb{C}}$.

Definition 5.4. Let

$$X = \mathrm{Proj}\frac{\mathbb{C}[x_0, \cdots, x_N]}{(f_1, \cdots, f_s)} \subset \mathbb{P}^N_{\mathbb{C}}.$$

Then,

$$\mathrm{Proj}\frac{\mathbb{C}[x_0, \cdots, x_N, y_1, \cdots, y_r]}{(f_1, \cdots, f_s)} \subset \mathbb{P}^{N+r}_{\mathbb{C}}$$

is called a cone of $X \subset \mathbb{P}^N_{\mathbb{C}}$.

Example 5.5 ((A)+(D)). Let $n, r \in \mathbb{Z}_{>0}$. Then,

$$\Delta(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n}(r)) = \dim \mathbb{P}^n + (\mathcal{O}_{\mathbb{P}^n}(r))^n - h^0(\mathbb{P}^n, \mathcal{O}(r)) = n + r^n - \binom{n+r}{n}.$$

For (A), we can check $r = 1$ implies $\Delta(\mathbb{P}^n, \mathcal{O}(1)) = 0$.
For (D), we can check $n = 2$ implies $\Delta(\mathbb{P}^2, \mathcal{O}(r)) = \frac{(r-1)(r-2)}{2}$, hence $\Delta = 0$ if and only if $r \in \{1, 2\}$.

Example 5.6 ((B)). Let $X \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. Let $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_X$. Then, $\Delta(X, \mathcal{O}_X(1)) = \dim X + \mathcal{O}_X(1)^n - h^0(X, \mathcal{O}_X(1))$. Since

$$\mathcal{O}_X(1)^n = (\mathcal{O}_{\mathbb{P}^{n+1}}(1)|_X)^n = \mathcal{O}_{\mathbb{P}^{n+1}}(1)^n \cdot X = \mathcal{O}_{\mathbb{P}^{n+1}}(1)^n \cdot \mathcal{O}_{\mathbb{P}^{n+1}}(2) = 2\mathcal{O}_{\mathbb{P}^{n+1}}(1)^{n+1} = 2$$

and the standard usage of the projection formula and exact sequences implies that

$$0 = H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(-1)) \to H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \to H^0(\mathcal{O}_X(1)) \to H^1(\mathcal{O}_{\mathbb{P}^{n+1}}(-1)) = 0$$

and $h^0(X, \mathcal{O}_X(1)) = h^0(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) = n+2$, so we have $\Delta(X, \mathcal{O}_X(1)) = 0$.

Theorem 5.7 ((A)). *Let (X, L) be a smooth polarized variety and $n := \dim X$. Then, $(X, L) \cong (\mathbb{P}^n, \mathcal{O}(1))$ if and only if $\Delta(X, L) = 0$ and $L^n = 1$.*

Proof. ($\Rightarrow$) By 5.2. ($\Leftarrow$) Since $\Delta(X, L) = n + L^n - h^0(X, L)$, we have $h^0(X, L) = n+1$. Then, we have a closed immersion $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^{h^0(X,L)-1} = \mathbb{P}_{\mathbb{C}}^n$ so that $X \cong \mathbb{P}^n$. □

Similarly we can prove:

Theorem 5.8 ((B)). *Let (X, L) be a smooth polarized variety and $n := \dim X$. Then, $(X, L) \cong (Q, \mathcal{O}_Q(1))$ if and only if $\Delta(X, L) = 0$ and $L^n = 2$.*

Now we are interested in the remaining case: $\Delta(X, L) = 0$ and $L^n \geq 3$.

Remark 5.9 ((C)). Let E be a vector bundle(i.e. locally free sheaf) on $\mathbb{P}^1$ of rank $n \in \mathbb{Z}_{>0}$. It is well known that $E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, $a_i \in \mathbb{Z}$. Let $X := \mathbb{P}_{\mathbb{P}^1}(E) = \mathbb{P}(E)$ and let $\pi : X \to \mathbb{P}^1$ be the bundle projection.

Assume $a_i > 0$ for all i. We will see later that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is very ample.

Remark 5.10. Let (X, L) as in Theorem 5.1.(a)(C). Then, $(X, L) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)), \mathcal{O}(1))$ with $a_i > 0$. Our goal is to verify $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample and $\Delta(\mathbb{P}(E), \mathcal{O}(1)) = 0$.

If $n = 1$, clearly $(\mathbb{P}(E), \mathcal{O}(1)) \cong (\mathbb{P}^1, \mathcal{O}(a))$. If $n = 2$, then fiber is $\cong \mathbb{P}^1$ and $\mathbb{P}(E) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a) \oplus \mathcal{O}(b))$ for $a, b > 0$. If $n = 2$, then fiber is $\cong \mathbb{P}^1$ and $\mathbb{P}(E) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a) \oplus \mathcal{O}(b))$ for $a, b > 0$.

Remark 5.11 (F, D_i, Γ_i). Let $X := \mathbb{P}_{\mathbb{P}^1}(E)$ and $E = \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}^1}(a_j)$. Fix $1 \leq i \leq n$.

(a) For every $p \in \mathbb{P}^1$, $F := \pi^*(p)$ the fiber at p is an effective divisor on X.

(b) $E \xrightarrow{\text{proj}} \mathcal{O}_{\mathbb{P}^1}(a_i)$ is surjective, we also have a surjection $\mathrm{Sym} E \twoheadrightarrow \mathrm{Sym}\mathcal{O}_{\mathbb{P}^1}(a_i)$ between symmetric algebras, so it induces a closed immersion $\gamma_i : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_i)) \hookrightarrow \mathbb{P}(E)$ and they are bundles on $\mathbb{P}^1$. Let $\Gamma_i := \gamma_i(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_i)))$, a section of π.

(c) If we consider projections $E \to \bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}^1}(a_j)$ for each i, then there is a closed immersion

$$\mathbb{P}(\bigoplus_{j \neq i} \mathcal{O}_{\mathbb{P}^1}(a_j)) \to \mathbb{P}(E)$$

from a $\mathbb{P}^{n-2}$-bundle to a $\mathbb{P}^{n-1}$-bundle. Let D_i be this smooth prime divisor on $X = \mathbb{P}(E)$.

Remark 5.12. $D_i \cap \Gamma_i = \varnothing$ since $(F \cap D_i) \cap (F \cap \Gamma_i) = \varnothing$ for each fiber F. For example, $n = 3$, Γ_i is the intersection of D_j and D_k when we restrict them to the fiber F, where $|\{i, j, k\}| = 3$.

Proposition 5.13. $\mathcal{O}_{\mathbb{P}(E)}(1) \sim D_i + a_i F$.

Proof. Let π be the bundle projection. Since $F \cong \mathbb{P}^{n-1}$, $\mathcal{O}_{\mathbb{P}(E)}(1)|_F \cong \mathcal{O}_F(1) := \mathcal{O}_{\mathbb{P}^{n-1}(1)}$ and $D_i|_F \sim \mathcal{O}_F(1)$, thus $\mathcal{O}_{\mathbb{P}(E)}(1) - D_i \equiv_\pi 0$, i.e.

$$(\mathcal{O}_{\mathbb{P}(E)}(1) - D_i) \cdot (\text{curve contracted by } \pi) = 0.$$

There exists $r \in \mathbb{Z}$ such that $\mathcal{O}_{\mathbb{P}(E)}(1) - D_i \sim rF$. Then,

$$0 = D_i \cdot \Gamma_i = (\mathcal{O}_{\mathbb{P}(E)}(1) - rF) \cdot \Gamma_i = \mathcal{O}_{\mathbb{P}(E)}(1) \cdot \Gamma_i - r$$

because for the inclusion $j : \Gamma_i \to X$ we have

$$\mathcal{O}_X(F)|_{\Gamma_i} = \mathcal{O}_X(\pi^* P)|_{\Gamma_i} \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(P)|_{\Gamma_i} \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(1)|_{\Gamma_i} = j^* \pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$$

and it implies $F \cdot \Gamma_i = \deg(F|_{\Gamma_i}) = 1$, so $r = a_i$. □

Proposition 5.14.

(a) $|\mathcal{O}_{\mathbb{P}(E)}(1)|$ *is base point free.*

(b) $\mathcal{O}_{\mathbb{P}(E)}(1)$ *is ample;* $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *is a polarized variety.*

Proof. (a) With different fibers F and F' we have $\mathcal{O}_{\mathbb{P}(E)}(1) \sim D_i + a_i F \sim D_i + a_i F'$. Then,

$$Bs|\mathcal{O}_{\mathbb{P}(E)}(1)| \subset \bigcap_{i=1}^{n} \operatorname{supp}(D_i + a_i F) \cap \bigcap_{i=1}^{n} \operatorname{supp}(D_i + a_i F') = \bigcap_{i=1}^{n} \operatorname{supp} D_i = \varnothing.$$

(b) Let C be a curve on $X = \mathbb{P}(E)$. By the part (a) it is enough to show $\mathcal{O}_{\mathbb{P}(E)}(1) \cdot C > 0$. We have two cases: $\pi(C) = *$ or not.

If $\pi(C)$ is a point p, then $C \subset F = \pi^*(p)$ implies $\mathcal{O}_{\mathbb{P}(E)} \cdot C = (D_i + a_i F) \cdot C \geq D_i \cdot C$ because F is nef, and $D_i \cdot C = (D_i|_F) \cdot C > 0$.(Nakai criterion)

If $\pi(C)$ is not a point, then there is $D_i \not\supset C$. Then, $\mathcal{O}_{\mathbb{P}(E)}(1) \cdot C = (D_i + a_i F) \cdot C \geq a_i F \cdot C \geq a_i > 0$. Here we used $F \cdot C = \deg(\pi^* \mathcal{O}_{\mathbb{P}^1}(p)|_C) = \deg(j^* \pi^* \mathcal{O}_{\mathbb{P}^1}(p)) > 0$. □

Proposition 5.16. $\Delta(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) = 0$.

Proof. Write

$$H^0(\mathbb{P}(E), \mathcal{O}(1)) \cong H^0(\mathbb{P}^1, E) \cong \bigoplus_{j=1}^{n} H^0(\mathbb{P}^1, \mathcal{O}(a_j)).$$

By $a_j > 0$, $h^0(\mathbb{P}(E), \mathcal{O}(1)) = \sum_{j=1}^{n} (a_j + 1)$.

Also, $D_1 \cdot \ldots \cdot D_n = 0$ and $D_2 \cdot \ldots \cdot D_n \cdot F) = (D_2|_F) \cdot \ldots \cdot (D_n|_F) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-1} = 1$ imply

$$\mathcal{O}_{\mathbb{P}(E)}(1)^n = (D_1 + a_1 F) \cdot \ldots \cdot (D_n + a_n F) = 0 + \sum_{j=1}^{n} a_j + (-) \cdot F^2 = \sum_{j=1}^{n} a_j.$$

So we are done. □

6 Day 6: May 18

Fano threefolds with $r = 2$

Here is a key argument: There exists $H' \sim H$ such that H' is a smooth projective divisor. Then, $K_{H'} = (K_X + H')|_{H'} \sim (-2H + H)|_{H'} = -H|_{H'}$, so H' is a del Pezzo surface.

The following theorem is our goal of today.

Theorem 6.1. *Let X be a Fano 3-fold with $r = 2$ so that $-K_X \sim 2H$.*

(a) *If $H^3 \geq 2$, then $|H|$ is base point free.*

(b) *If $H^3 \geq 3$, then $|H|$ has the N_1 property, hence ample.*

Remark 6.2. If $H^3 \in \{1, 2\}$, then Theorem 6.1 is not needed for our classification, so we will only consider $H^3 \geq 3$ from now on.

First, we prove Theorem 6.1 (b) by applying the following theorem for (X, H):

Theorem 6.3. *Let (X, L) be a polarized variety such that* $\dim Bs|L| \leq 0$ *and* $L^{\dim X} \geq 2\Delta(X, L) - 1$. *Then, (X, L) has a ladder (see Definition 6.5).*

Proposition 6.4. *Let X be a Fano 3-fold with $r = 2$, $-K_X \sim 2H$. Then,*

(a) $\Delta(X, H) = 1$,

(b) $\dim Bs|H| \leq 0$,

(c) (X, H) *has a ladder.*

Proof. For (a), by the Riemann-Roch (4.9) with $q = \frac{1}{2}$, we have

$$\Delta(X, H) = \dim X + H^3 - h^0(X, H) = 3 + H^3 - (H^3 + 2) = 1.$$

Then, (b) follows from $\dim Bs|L| < \Delta(X, H) = 1$, and (c) follows from $H^3 \geq 1 = 2\Delta(X, H) - 1$. □

Definition 6.5. Let (X, L) be a polarized variety. An integral scheme Y is a *rung* of (X, L) if $Y \in |L|$, i.e. there is $0 \neq s \in H^0(X, L)$ such that $Y = \{s = 0\} \subset X$. In particular, $\dim Y = \dim X - 1$. When X is normal, a rung Y is just a prime divisor Y such that $L \sim Y$.

A sequence $X = X_0 \supset X_1 \supset \cdots \supset X_{n-1}$ with $n := \dim X$ is a *ladder* of (X, L) if X_i is a rung of $(X_{i-1}, L|_{X_{i-1}})$ for $1 \leq i \leq n-1$. We say a ladder is *regular* if $\Delta(X, L) = \Delta(X_1, L|_{X_1}) = \Delta(X_2, L|_{X_2}) = \cdots$.

Remark 6.6. If $\Delta(X, L) = 0$, then (X, L) has a regular ladder.

Remark 6.7. If $\Delta(X, L) = \Delta(Y, L|Y)$ for a rung Y of (X, L), and if $L|_Y$ has the N_1 property, then L has the N_1 property. Since N_1 property can be checked for one-dimensional X_{n-1}, the existence of a regular ladder implies the N_1 property of L.

Proposition 6.8. *Let X be a Fano 3-fold with $r = 2$, $-K_X \sim 2H$. Then, (X, H) has a regular ladder.*

Proof. By Proposition 6.4, we have a ladder $X = X_0 \supset X_1 \supset X_2$. Let $C := X_2$. Since $X_1 \in |H|$, we may assume $X_1 = H$, which is a prime divisor.

By Propositioin 4.14, we have

$$0 \leq \Delta(X, H) - \Delta(H, H|_H) \leq h^1(X, \mathcal{O}_X) = 0$$

and

$$\leq \Delta(H, H|_H) - \Delta(C, H|_C) \leq h^1(H, \mathcal{O}_H) = 0,$$

so $\Delta(X, H) = \Delta(H, H|_H) = \Delta(C, H|_C)$; the ladder is regular. Here when we compute $h^1(H, \mathcal{O}_H) = 0$, we have used the exact sequence for $0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$ with the Kodaira vanishing theorem. □

Proposition 6.9. *Let X be a Fano 3-fold with $r = 2$, $-K_X \sim 2H$. Let $X \supset H \supset C$ be a regular ladder. Then, C is a projective Gorenstein curve with $h^1(C, \mathcal{O}_C) = 1$.*

Proof. Note that H and C are effective divisors of X and H respectively, hence C is Gorenstein. In general, if V is a Gorenstein variety and W is an effective Cartier divisor, then W is a Gorenstein scheme and $(\omega_V \otimes \mathcal{O}_V(W))|_W \cong \omega_W$. We remark that a vareity is Gorenstein if and only if it is Cohen-Macauley with invertible ω_X. When V and W are normal, then $(K_V + W)|_W \sim K_X$.

Since

$$\omega_H \cong (\omega_X \otimes \mathcal{O}_X(H))|_H \cong \mathcal{O}_X(K_X + H)|_H \cong \mathcal{O}(-H)|_H,$$

we have

$$\omega_C \cong (\omega_H \otimes \mathcal{O}_H(C))|_H = (\mathcal{O}_X(-H)|_H \otimes \mathcal{O}_X(H)|_H)|_C \cong (\mathcal{O}_X|_H)|_C = \mathcal{O}_C,$$

therefore the Serre duality implies $h^1(C, \mathcal{O}_C) = h^0(C, \omega_C) = h^0(C, \mathcal{O}_C) = 1$. □

Proposition 6.10. *Let C be a projective Gorenstein curve with arithmetic $h^1(\mathcal{O}_C) = 1$, i.e. $\omega_C \cong \mathcal{O}_C$. If a Cartier divisor D has dimension $\dim D \geq 3$, then D has the N_1 property.*

We will prove (6.11)⇒(6.13)⇒(6.10).

Proposition 6.11. *Let C be as Proposition 6.10, and D a Cartier divisor.*

(a) *If $\deg D \geq 1$, then $H^1(C, D) = 0$ and $h^0(X, D) = \deg D$.*

(b) *If $\deg D = 1$, then $Bs|D| = P$, where P is a smooth point of C.*

(c) *If $\deg D \geq 2$, then $|D|$ is base point free.*

Proof. (a) Directly follows by the Serre duality and the Riemann-Roch.

(b) By (a), we may assume D is effective. By $\deg D = 1$, D is a smooth point P. Since $h^1(C, D) = 1$, we have $Bs|D| = P$.

(c) Fix a smooth point $Q \in X$. Then, $D - (d-1)Q$ has degree 1, and is linearly equivalent to P by the part (b). Then, $D \sim (d-1)Q + P$, and $Bs|D| \subset \{P, Q\}$.

Let R be any smooth point. Then, we have

$$H^0(C, \mathcal{O}_C(D)) \to H^0(R, \mathcal{O}_C(D)|_R) \to H^1(C, \mathcal{O}_C(D-R)) = 0$$

by (a), $\deg(D-R) \geq 2-1 > 0$, so $Bs|D|$ does not contain smooth points. We are done. □

Exercise 6.12. Let C be a projective Gorenstein curve and D an effective Cartier divisor. If $\deg D = 1$, then $\operatorname{supp} D \subset$ the sum of locus of C.

Proposition 6.13. *Let C be as before, and D a Cartier divisor.*

(a) *If $\deg D \geq 2$, then $\bigoplus_{m=0}^{\infty} H^0(C, mD)$ is generated by $H^0(C, D) \oplus H^0(C, 2D)$ as a $\mathbb{C}$-algebra.*

(b) *If $\deg D \geq 3$, then $\bigoplus_{m=0}^{\infty} H^0(C, mD)$ is generated by $H^0(C, D)$ as a $\mathbb{C}$-algebra; it enjoys the N_1 property.*

Proof. We only show (a).

It is enough to show

$$H^0(C, D) \otimes_{\mathbb{C}} H^0(C, (r+2)D) \to H^0(C, (r+3)D)$$

is surjective for all $r \geq 0$. This follows from the Castelnuovo-Mumford regularity (6.14 + 6.15). $H^1(C, \mathcal{O}_C(D) \otimes \mathcal{O}_C(D)^{1-1}) = 0$, $\mathcal{O}_C(D)$ is 1-regular with respect to $\mathcal{O}_C(D)$, globally defined by (6.11). □

Definition 6.14. Let X be a projective scheme over $\mathbb{C}$, A a globally generated ample invertible sheaf, F a coherent sheaf. For $m \in \mathbb{Z}$, we say F is *m-regular* with respect to A if $H^i(X, F \otimes A^{m-i}) = 0$ for each $i > 0$.

Theorem 6.15. *Notation as in Definition 6.14. Then,*

$$H^0(X,A) \otimes H^0(X, F \otimes A^{m+r}) \to H^0(X, F \otimes A^{m+r+1})$$

is surjective for $r \in \mathbb{Z}_{\geq 0}$.

Proof. See [FGA explained, §5] or [Lazarsfeld, Positivity]. □

Remark 6.16. Let C be a projective Gorenstein curve with $h^1(C, \mathcal{O}_C) = 1$. Then, $C \cong$ (cubic curve) $\subset \mathbb{P}^2_{\mathbb{C}}$. It is because $|3P|$ is very ample by Proposition 6.13.

Proposition 6.17. *Let (X, L) be a polarized variety. Assume X is Cohen-Macauley, $\dim Bs|L| \leq 0$, and $\dim \operatorname{im} \varphi_{|L|} = \dim X$, where $\operatorname{im} \varphi := (\varphi_{|L|}(X \setminus Bs|L|))^-$. Then, X has a ladder.*

Proof. Induction on $\dim X$. For $D \in |L|$ a general member, it suffices to show D is integral and $(D, L|_D)$ satisfies the three assumptions in the statement of this proposition.

For reducedness of D, $R_0 + S_1$, S_1 Cohen-Macauley, $D \cap X_{sm}$ by Bertini. For irreducibility of D, Bertini for irreducibles [Jounalou 83].

For $(D, L|_D)$, the integrality is done. The second assumption $\dim Bs|L|_D| \leq 0$ follows from

$$Bs|L| \cap D = \bigcap_{s \in H^0(X,L)} \operatorname{supp} s \cap D \supset \bigcap_{t \in H^0(D, L|_D)} \operatorname{supp} t = Bs|L|_D|.$$

The third assumption is due to $\dim \varphi_{|L|_D|}(D) \geq \dim \varphi_{|L|}(D) \geq \dim X - 1$, where the second inequality can be proved as follows: if we let X' be the normalization of the resolution of the indeter of $\varphi_{|L|}$, then we have by the Stein factorization that

$$\begin{tikzcd} X' \arrow[r, "\psi \text{ birat}"] \arrow[d, "\mu"] \arrow[dr, "\varphi'"] & Z \arrow[d, "\text{fin}"] \\ X \arrow[r, dashed] & \operatorname{im} \varphi_{|D|} \subset \mathbb{P}^N_{\mathbb{C}}. \end{tikzcd}$$

Suppose $\dim \varphi'(\mu_*^{-1} D) = \dim \varphi_{|L|}(D) < \dim D$. Then, $\mu_*^{-1} D \subset Ex\psi$, $D = \mu(\mu_*^{-1} D) \subset \mu(Ex\psi)$. D general member, infinitely many choices, and only finitely many prime divisors. contradiction. □

Proposition 6.19. *Let (X, L) be a polarized variety and $n := \dim X$. Assume*

(a) *X is smooth,*

(b) *$\dim Bs|L| \leq 0$,*

(c) *$\dim \operatorname{im} \varphi_{|L|} < \dim X$,*

(d) *$L^n \geq 2\Delta(X, L) - 1$.*

Then, every general member of $|L|$ is smooth.

If jikangire: see [Fujita, Book], [Fujita, total deficiency I].

Proof. For simplicity, assume $Bs|L| = P$. If $D \in |L|$ is a general member, then the Bertini implies $D \setminus P$ is smooth. Suppose D is singular at P.

For the blowup $\alpha : X' \to X$, we have a decomposition $\alpha^* L = L' + mE$ $(m \geq 2)$ into a movable part and a fixed part. The normalization $\beta : X'' \to X'$ at the indetersal of $|L'|$, with $\beta * L' = L'' + F$.

$$\begin{tikzcd}
X'' \arrow[d, "\beta"] \arrow[ddr, "\varphi_{|L''|}"] & \\
X' \arrow[d, "\alpha"] \arrow[dr, "\varphi_{|L'|}"] & \\
X \arrow[r, "\varphi_{|L|}"] & Y \subset \mathbb{P}^N_{\mathbb{C}}
\end{tikzcd}$$

where $Y = \varphi_{|L|}(X) = \varphi_{|L'|}(X') = \varphi_{|L''|}(X'')$. Fix H such that $\mathcal{O}_Y(H) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_Y$ so that $L'' = \varphi''^* H$. We can check $\dim Y = n - 1$. Then, since

$$h^0(X, L) = h^0(X', L') = h^0(X'', L'') = h^0(Y, H),$$

we get

$$\Delta(X, L) = n + L^n - h^0(Y, H).$$

Now we have

$$0 \leq \Delta(Y, H) = n - 1 + H^{n-1} - h^0(Y, H) = \Delta(X, L) - L^n - 1 + H^{n-1}.$$

We can show $L^n \geq 2H^{n-1}$ so that we have $L^n \geq 2(L^n + 1 - \Delta(X, L))$, which leads to a contradiction $2\Delta(X, L) - 2 \geq L^n$. $\square$

7 Day 7: May 25

Fano threefolds with $r = 2$: II

Notation 7.1. Today, we will always use the following: X is a Fano 3-fold with $r = 2$, H is a smooth prime divisor such that $-K_X \sim 2H$. In particular, H is a del Pezzo surface.

Outline:

1. $1 \leq H^3 \leq 9$.
2. Case study(e.g. $H^3 = 3$ implies $X = (\deg = 3) \subset \mathbb{P}^4$).

Proposition 7.1.

(a) $H|_H \sim -K_H$.

(b) $1 \leq H^3 \leq 9$.

Proof. (a) $K_H = (K_X + H)|_H \sim (-2H + H)|_H = -H|_H$.

(b) $H^3 = (H|_H)^2 = (-K_H)^2 = K_H^2$. It is well-known that $1 \leq K_H^2 \leq 9$ for a del Pezzro surface H. (If there is a (-1)-curve, then $H \cong \mathbb{P}^2$ or $H \cong \mathbb{P}^1 \times \mathbb{P}^1$. If there is no (-1)-curve, then $K_H^2 < K_{H'}^2$, where $H \to H'$ is a contraction of the (-1)-curve.) □

Theorem 7.2. *We denote by (d) a hypersurface of degree d, denote by $P(a,b,c,\cdots)$ the weighted projective space, and denote by $\cap$ the complete intersection. Then, the followings hold.*

(1) If $H^3 = 1$, then $(6) \subset \mathbb{P}(1,1,1,2,3)$.

(2) If $H^3 = 2$, then $(4) \subset \mathbb{P}(1,1,1,1,2)$.

(3) If $H^3 = 3$, then $(3) \subset \mathbb{P}^4$.

(4) If $H^3 = 4$, then $(2) \cap (2) \subset \mathbb{P}^5$.

(5) If $H^3 = 5$, then $\mathrm{Gr}(2,5) \cap (1) \cap (1) \cap (1) \subset \mathbb{P}^6$. (we have $\mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^6$ by Plücker)

(6) If $H^3 = 6$, then $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

(7) If $H^3 = 7$, then the blowup of $\mathbb{P}^3$ at a point.

Remark. If $H^3 \geq 3$, then $|H|$ is very ample with $\varphi_{|H|} : X \hookrightarrow \mathbb{P}^{H^3+1}$.

Remark 7.3. These are actually Fano 3-folds with $r = 2$. For (3),(4),(6),(7), we can check with the adjunction formula and the Lefschetz hyperplane section theorem.

For (5), the Grassmannian $Y = \mathrm{Gr}(r,n) := \{r\text{-dimensional subspaces of } \mathbb{C}^n\}$ has dimension $r(n-r)$. If $-K_Y \sim nH_Y$ and $\operatorname{Pic} Y \cong \mathbb{Z}H_Y$, then the Plücker embedding is given by $\varphi_{|H_Y|} : Y \hookrightarrow \mathrm{Gr}(1,N) = \mathbb{P}^{N-1} : W \mapsto \bigcap^r W$. By the adjunction formula, if $X := Y \cap (1) \cap (1) \cap (1)$, then $-K_X \sim 2(H_Y|_X)$. Also,

$$H_Y^{\dim Y} = \frac{(2n-4)!}{(n-1)!(n-2)!} = \frac{6!}{4!3!} = 5$$

for $r = 2$ and $n = 5$. See [Eisenbud-Harris 3264].

For (1) and (2), let $Y := \mathbb{P}(1,1,1,1,2)$, for example. Then, its singularity is a single point and it is a normal projective toric variety. Since $\mathrm{Cl} Y = \mathbb{Z}D_0$ and $\operatorname{Pic} Y = \mathbb{Z}(2D_0)$, $-K_Y \sim (1+1+1+1+2)|_{D_0} = 6D_0$ and $X \sim 4D_0$. Since $D_0|_X$ is Cartier by avoiding singularity, $-K_X \sim 2(D_0|_X)$.

Case (4).(Similarly for (3)) The Riemann-Roch gives

$$h^0(X, mH) = \frac{m(m+1)(m+2)}{6} H^3 + m + 1 = \frac{2}{3} m(m+1)(m+2) + m + 1.$$

Then,

$$h^0(X,H)=6,\quad h^0(X,2H)=19,\quad h^0(\mathbb{P}^5,\mathcal{O}(1))=\binom{6}{1}=6,\quad h^0(\mathbb{P}^5,\mathcal{O}(2))=\binom{7}{2}=21.$$

Note

$$X=\mathrm{Proj}\frac{\mathbb{C}[x_0,\cdots,x_5]}{I_X}\hookrightarrow \mathrm{Proj}\mathbb{C}[x_0,\cdots,x_5]=\mathbb{P}^5.$$

With an exact sequence

$$0\to I_X\to\mathcal{O}_{\mathbb{P}^5}\to\mathcal{O}_X\to 0$$

and

$$0\to H^0(\mathbb{P}^5,I_X\otimes\mathcal{O}(2))\to H^0(\mathbb{P}^5,\mathcal{O}(2))\to H^0(X,\mathcal{O}(2)|_X(=2H))\to 0,$$

we have $h^0(\mathbb{P}^5,I_X\otimes\mathcal{O}(2))=21-19=2$ and two quadrics $Q_1,Q_2\subset\mathbb{P}^5$ with $X\subset Q_i$. We also have $Q_1\neq Q_2$ and Q_i are integral (If not, $Q_i=H+H'$ and $X\subset Q_i$ implies $X\subset H$ or $X\subset H'$ by irreducibility, which is absurd to $H^0(\mathbb{P}^5,\mathcal{O}(1))\xrightarrow{\cong}H^0(X,H)$). Then, $X\subset Q_1\cap Q_2$, and they have same degree 4, so $X=Q_1\cap Q_2$. The divisors X and $Q_2|_{Q_1}$(effective Cartier divisor on Q_1)) are effective Weil divisors on Q_1, so $X\leq Q_2|_{Q_1}$ and $X\mathcal{O}(1)^3=Q_2|_{Q_1}\cdot\mathcal{O}(1)^3=Q_2\cdot Q_1\cdot\mathcal{O}(1)^3=4$. Thus $X=Q_2|_{Q_1}$ as Weil divisors on Q_1, and $Q_2|_{Q_1}=Q_1\cap Q_2$ is an integral scheme, so we have a closed immersion $X\to Q_1\cap Q_2$, then by the same dimension we have $X=Q_1\cap Q_2$.

Case (2).(Similarly for (1)) Then we have

(i) $h^0(X,H)=4$ with $H^0(X,H)=\bigoplus_{i=0}^3\mathbb{C}x_i$.

(ii) $h^0(X,2H)=11$ with $H^0(X,H)=(\bigoplus_{i=0}^3\mathbb{C}x_i^2)\oplus(\bigoplus_{0\leq i<j\leq 3}\mathbb{C}x_ix_j)\oplus\mathbb{C}y$.

(iii) $h^0(X,3H)=24$ with

(iv) $h^0(X,4H)=45$.

For (ii) the linear independence of 10 elements are non-trivial. See Proposition 7.5. Note that $X\cong\mathrm{Proj}(\bigoplus_{d=0}^\infty H^0(X,dH))$ and $\bigoplus_{d=0}^\infty H^0(X,dH)$ is generated by $\bigoplus_{d=1}^2 H^0(X,dH)$ by the same argument as in day 6. For degree four, we have three cases Y^2, $Y^1\times(\deg 2$ using $X_0\sim X_3)$, and $Y^0\times(\deg 4$ using $X_0\sim X_3)$. Then, $1+\binom{5}{2}+\binom{7}{3}=46$, i.e. there is a homogeneous polynomial of degree 4 $f(x_0,x_1,x_2,x_3,y)$. We can check $X=\{f=0\}$.

Exercise 7.4. Let X be a smooth projective variety and L be an invertible sheaf. Then, $S^2H^0(X,L)\to H^0(X,L^{\otimes 2})$ is injective, where S^2 means the symmetric product.

So far we have classified $1\leq H^3\leq 4$ (for $H^3=1$ or 3, we can do similarly as $H^3=2$ and 4). We have $H^3\notin\{8,9\}$, and with a more argument we can show $1\leq H^3\leq 5$ if and only if $\rho(X)=1$.

Proposition 7.5. $H^3\neq 9$.

Sketch. If $H^3=9$, then $H\cong\mathbb{P}^2$. We have a torsion-free cokernel for $\mathrm{Pic}X\hookrightarrow\mathrm{Pic}H:\mathcal{O}_X(H)\mapsto\mathcal{O}_X(H)|_H\cong\omega_H^{-1}\cong\mathcal{O}_{\mathbb{P}^2}(3)$ by the Leftschetz hyperplane theorem or some others. Then, there is H' such that $H\sim 3H'$ and $-K_X\sim 2H\sim 6H'$. □

Case (5). In what follows, we consider $H^3=5$ and want to prove $X\cong\mathrm{Gr}(2,5)\cap(1)\cap(1)\cap(1)$. Here is a rough idea:

(A) Let $X\xleftarrow{\sigma}Y$ be a blowup, let $Y\xrightarrow{\psi}Z\subset\mathbb{P}^4$ blowdown. Suppose ψ is a blowup along $B\subset Z$, with $B\cong\mathbb{P}^1$ and $\deg B=3$, a smooth cubic rational curve.

(B) We can recover X from Z a smooth quadric and B a cubic $\mathbb{P}^1$. We can show X does not depend on the choice of (Z,B), so the Fano threefold with $r=2$ and $H^3=5$ is unique(we already have an example).

For (A), we have four steps.

(A1) There is a curve $\Gamma \subset X$ such that $H \cdot \Gamma = 1$, $\Gamma \cong \mathbb{P}^1$, and $N_{\Gamma/X} \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma$, since $\Gamma \subset X \subset \mathbb{P}^6$ induces $\mathcal{O}(1) \mapsto H \mapsto$ a line in Γ. Take the blowup $\sigma : Y \to X$ and let $H_Y := \sigma_*^{-1}H$ (str transform).

(A2) $|H_Y|$ is a base point free and $h^0(Y,H_Y)=5$. Let $Z := \varphi_{|H|}(Y)$.

(A3) $H_Y^3 = 2$, $\psi : Y \to Z$ is birational, and $Z \subset \mathbb{P}^4$ is a quadric hypersurface.

(A4) Also, Z is smooth, and ψ is a blowup along a smooth cubic curve.(Proof omitted)

We omit the proof for (A1) and (A4).

For (A2), let Λ be the linear system consisting of the hyperplane sections $H \subset X$ such that $\Gamma \subset H$. Then, $Bs\Lambda = \Gamma$ scheme-theoretically, σ is the resolution of indet of φ_Λ so that $\sigma^*H = \sigma_*^{-1}H + E = H_Y + E$ with base point free $|H_Y|$. Moreover, $H^0(Y,H_Y) \cong V_\Lambda \subset H^0(X,H)$ with codimension 2, hence $h^0(Y,H_Y)=5$.

For (A3), we omit for $H^3=2$. Note that $2 = H_Y^3 = (\deg\psi) \times H_Z^3 = (\deg\psi) \times (\deg Z)$, where $H_Z := \mathcal{O}_{\mathbb{P}^4}(1)|_Z$. If $\deg Z = 1$, then $H^0(Y,H_Y) \hookleftarrow H^0(Z,H_Y) \leftarrow H_0(\mathbb{P}^4,\mathcal{O}(1))$ is an isomorphism, so we have a contradiction. Therefore, $\deg Z = 1$ and $\deg\psi = 1$, we are done. For (B), let $Z, Z' \subset \mathbb{P}^4$ be

smooth quadric hypersurfaces and $B \subset Z$, $B' \subset Z'$ be smooth cubis rational curves. We want to show that there is $\sigma : \mathbb{P}^4 \xrightarrow{\cong} \mathbb{P}^4$ such that $\sigma(Z) = Z'$ and $\sigma(B) = B'$.

(B1) Let

$$V := \bigcap_{\substack{H:\text{ hyperplane}\\ B\subset H}} = (\text{the smallest linear sub in } \mathbb{P}^4 \text{ containing } B) \cong \mathbb{P}^3$$

$B, B' \subset Z \cap V \cong \mathbb{P}^1 \times \mathbb{P}^1$ smooth quadric surface...

(B2) we can show $\tau(B) = B'$.