複素解析学I演習2023年（チョイ）

問 1 (フックス群としてのモジュラー群). 複素数体 $\mathbb{C}$ の部分集合 A に対して、成分 a, b, c, d が A の元で $ad-bc=1$ を満たす一次分数変換 $f(z)=(az+b)/(cz+d)$ の集合を $\mathrm{PSL}(2,A)$ と書く．特に $\mathrm{PSL}(2,\mathbb{Z})$ を**モジュラー群**と呼ぶ．上半平面 $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ の部分集合 $D := \{z \in \mathbb{H} : |z| > 1,\ |\mathrm{Re}\, z| < \frac{1}{2}\}$ を定義する．

(1) $\mathrm{PSL}(2,\mathbb{R})$ の元 f は全単射写像 $\mathbb{H} \to \mathbb{H}$ を定義することを示せ．

(2) $\mathrm{PSL}(2,\mathbb{Z})$ は $S(z) := -1/z$ と $T(z) := z+1$ によって生成されることを示せ．つまり、全ての元が $S^{\pm 1}$ と $T^{\pm 1}$ の有限回の合成として表れることを示せ．

(3) 集合 D は $\mathrm{PSL}(2,\mathbb{Z})$ の**基本領域**であることを示せ．つまり、次の二つが成り立つことを示せ：

- (a) 任意の点 $z \in \mathbb{H}$ に対して $f(z) \in \overline{D}$ を満たす $f \in \mathrm{PSL}(2,\mathbb{Z})$ が少なくとも一つ存在する．
- (b) 任意の点 $z \in \mathbb{H}$ に対して $f(z) \in D$ を満たす $f \in \mathrm{PSL}(2,\mathbb{Z})$ が多くとも一つしか存在しない．

(4) $\mathrm{PSL}(2,\mathbb{Z})$ は $\mathbb{H}$ に**真性不連続に作用**することを示せ．つまり、任意の点 $z \in \mathbb{H}$ に対して軌道 $\{f(z) : f \in \mathrm{PSL}(2,\mathbb{Z})\}$ が離散集合であることを示せ．

問 2 (カラテオドリ級関数集合の極点). 開単位円板上で定義された正則関数 f が $f(0)=1$ を満たすとする．もし任意の $|z|<1$ を満たす複素数 z に対して $\mathrm{Re}\, f(z) > 0$ ならば、f を**カラテオドリ級**の関数という．関数 f が冪級数展開 $f(z) = 1 + 2\sum_{k=1}^{\infty} c_k z^k$ を持つとする．

(1) 正の整数 k と実数 $0<r<1$ に対して次の式を示せ：

$$c_k r^k = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, f(re^{i\theta}) e^{-ik\theta}\, d\theta.$$

(2) 次の二つの条件が同値であることを示せ：

- (a) 関数 f がカラテオドリ級である．
- (b) 任意の正の整数 n に対して点 $(c_1, \cdots, c_n) \in \mathbb{C}^n$ は $\theta \in [0, 2\pi)$ によって媒介変数表示された曲線 $(e^{-i\theta}, \cdots, e^{-in\theta}) \in \mathbb{C}^n$ の凸包絡の元である．

問 3 (アールフォルス・清水標数). 複素平面上の有理型関数 f を考える．次のように $r \geq 0$ に対する関数 $A(\cdot, f)$ を定義する：

$$A(r,f) := \frac{1}{\pi}\int_{\sqrt{x^2+y^2}\leq r} f^{\#}(x+iy)^2\, dx\, dy, \qquad \text{ただし、} f^{\#}(z) := \frac{|f'(z)|}{1+|f(z)|^2}, \quad z \in \mathbb{C}.$$

関数 $f^{\#}$ を f の**球面導関数**と呼ぶ．

(1) 任意の点 $(x,y) \in \mathbb{R}^2$ に対して、

$$\frac{1}{\pi} f^{\#}(x+iy)^2 = \frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)$$

を満たす実平面 $\mathbb{R}^2$ 上の実関数 P と Q を求め、関数 $K(x,y) := 1 + |f(x+iy)|^2$ を用いて表せ．

(2) グリーンの定理と偏角の原理を用いて $r \geq 0$ に対して次の式が成り立つことを示せ：

$$\int_0^r A(t,f)\frac{dt}{t} = \int_0^r n(t,f)\frac{dt}{t} + \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta - \log\sqrt{1+|f(0)|^2}.$$

ただし、$n(r,f)$ は閉円板 $\overline{B(0,r)}$ 内にある重複度を込めて数えた f の極の数である．左辺の関数を f の**アールフォルス・清水標数**と呼ぶ.

(3) 球面導関数 $f^{\#}$ が有界ならば、ある定数 $C>0$ が存在して、全ての $z \in \mathbb{C}$ に対して $|f(z)| \leq Ce^{|z|^2}$ であることを示せ．特に、f は $\mathbb{C}$ 全体上正則である.

問 4 (四分円上のディリクレ問題)．領域 $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0,\ y>0\}$ 上に定義された調和関数 $v \in C^2(\Omega,\mathbb{R})$ が次の境界値条件を満たすとする：各点 $(x_0,y_0) \in \partial\Omega$ に対して

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \begin{cases} 1 & \text{if } y_0 > 0, \\ 0 & \text{if } y_0 = 0 \text{ and } 0 < x_0 < 1. \end{cases}$$

(1) シュワルツの鏡像の原理を用いて v は領域 $\tilde{\Omega} := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0\}$ 上の調和関数 $\tilde{v} \in C^2(\tilde{\Omega},\mathbb{R})$ に拡張されることを示せ.

(2) 適切な等角変換とポアソン積分を用いて v を求めよ.

Solution of 1. (1) Let $f(z) = (az+b)/(cz+d)$ with $a,b,c,d \in \mathbb{R}$ such that $ad - bd = 1$. Since it has the inverse transform $z \mapsto (dz-b)/(-cz+a)$ that is also an element of $\mathrm{PSL}(2,\mathbb{R})$, it is enough to show the well-definedness $f(z) \in \mathbb{H}$ for $z \in \mathbb{H}$. Let $z = x+iy \in \mathbb{H}$ with $y > 0$. Then,

$$\operatorname{Im} f(z) = \operatorname{Im} \frac{ax+b+iay}{cx+d+icy} = \frac{ay(cx+d)-(ax+b)cy}{(cx+d)^2+(cy)^2} = \frac{y}{(cx+d)^2+(cy)^2} > 0,$$

so $f(z) \in \mathbb{H}$.

(2) Let $f(z) = (az+b)/(cz+d)$ with $a,b,c,d \in \mathbb{Z}$ such that $ad - bd = 1$. Consider the following two kinds of moves of f:

- When $|a| < |c|$, we take
$$Sf(z) = \frac{-cz-d}{az+b}.$$
- When $|a| \geq |c| > 0$, with $q,r \in \mathbb{Z}$ such that $a = qc+r$ and $0 \leq r < |c|$, we take
$$T^{-q}f(z) = \frac{rz+b-qd}{cz+d}.$$

By repeating the two moves alternately, we arrive at $c = 0$ in finitely many steps because $|c|$ strictly decreases. Then, since $ad-bc = 1$, we may assume $a = d = 1$ so that $(az+b)/(cz+d) = z+b = T^b(z)$.

(3) (a) Let $z_0 \in \mathbb{H}$. We may assume $\operatorname{Re} z_0 \in [-\frac{1}{2}, \frac{1}{2})$ by taking T^q on z_0 for appropriate $q \in \mathbb{Z}$. Define a sequence $z_n \in \mathbb{H}$ inductively by

$$z_n := T^{-\lfloor \operatorname{Re} S(z_{n-1}) + \frac{1}{2} \rfloor} S(z_{n-1}), \qquad n \geq 1.$$

Then, one can show $\operatorname{Re} z_n \in [-\frac{1}{2}, \frac{1}{2})$ for all n. Since

$$\operatorname{Im} z_n = \operatorname{Im} S(z_{n-1}) = \frac{\operatorname{Im} z_{n-1}}{(\operatorname{Re} z_{n-1})^2 + (\operatorname{Im} z_{n-1})^2} \geq g(\operatorname{Im} z_{n-1}),$$

where $g(y) := 4y/(1+4y^2)$, and since $g^n(y) \uparrow \frac{\sqrt{3}}{2}$ for $0 < y < \frac{\sqrt{3}}{2}$ as $n \to \infty$, there is n such that

$$-\frac{1}{2} \leq \operatorname{Re} z_n < \frac{1}{2}, \qquad \operatorname{Im} z_n > \frac{\sqrt{3}}{4}.$$

If $|z_n| \geq 1$, then we are done, so assume $|z_n| < 1$. Now we have three possibilities: $|z_n - 1| < 1$, $|z_n + 1| < 1$, or $\min\{|z_n - 1|, |z_n + 1|\} \geq 1$. For each case, we can check that $T^{-1}Sz_n$, TSz_n, Sz_n is contained in $\overline{D}$, respectively.

(b) For $z \in D$, let $w = (az+b)/(cz+d) \in D$ with $a,b,c,d \in \mathbb{Z}$ such that $ad - bd = 1$. It suffices to show $c = 0$. Suppose $c \neq 0$. Note that $|z-n| > 1$ and $|w-n| > 1$ for every integer n since $z, w \in D$. Write

$$1 < |w-n| = \left| \frac{az+b}{cz+d} - n \right| \leq \left| \frac{az+b}{cz+d} - \frac{a}{c} \right| + \left| n - \frac{a}{c} \right| = \left| \frac{1}{c(cz+d)} \right| + \left| n - \frac{a}{c} \right|, \qquad n \in \mathbb{Z}.$$

If $|c| \geq 2$, then by taking n such that $|n - (a/c)| \leq \frac{1}{2}$, the estimate $|c(cz+d)| \geq |c|^2 \operatorname{Im} z > 2\sqrt{3}$ leads a contradiction to the above inequality. If $|c| = 1$, then since a/c is an integer, by letting $n = a/c$, we have a contradiction $|c(cz+d)| = |z+cd| > 1$ from the assumption $z \in D$. Thus, $c = 0$, and we are done.

(4) Suppose the orbit $\{f(z) : f \in \mathrm{PSL}(2,\mathbb{Z})\}$ is not discrete. Then, there is $z_0 \in \mathbb{H}$ and a sequence $f_n \in \mathrm{PSL}(2,\mathbb{Z})$ such that $f_n(z) \neq z_0$ for all n and $f_n(z) \to z_0$ as $n \to \infty$. We may assume $z, z_0 \in \overline{D}$ by the part (a) of (3). Consider

$$P := \{I, T, TS, ST^{-1}S = TST, ST^{-1}, S, ST, STS = T^{-1}ST^{-1}, T^{-1}S, T^{-1}\} \subset \mathrm{PSL}(2,\mathbb{Z}).$$

Then, we can check that $\bigcup_{f \in P} f(\overline{D})$ contains an open neighborhood U of $\overline{D}$. For every n that is large enough, from $f_n(\overline{D}) \cap U \neq \emptyset$, it follows that $f_n(D)$ intersects $U \subset \bigcup_{f \in P} f(\overline{D})$, that is, there is $f_0 \in P$ such that $f_n(D) \cap f_0(\overline{D}) \neq \emptyset$, and easily $f_n(D) \cap f_0(D) \neq \emptyset$, because $f(D)$ is open and $f(\overline{D})$ is closed for any $f \in \mathrm{PSL}(2, \mathbb{Z})$. By the part (b) of (3), we can conclude that f_n belongs eventually to P as $n \to \infty$. Since P is a finite set, $f_n(z)$ cannot converge to z_0 unless $f_n(z) = z_0$ for sufficiently large n, therefore the orbit is discrete. $\square$

Remark. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ is called a *Fuchsian group* and a *Kleinian group* respectively. It is known that a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is discrete if and only if it properly discontinuously acts on $\mathbb{H}$. There is a more generalized theorem used for verifying a group generated by several elements of $\mathrm{PSL}(2, \mathbb{R})$ is Fuchsian, the *Poincare polygon theorem*. It states that if there is a polygon in $\mathbb{H}$ satisfying two conditions called a side pairing condition and elliptic cycle condition is realized as a fundamental domain, so the group acts on $\mathbb{H}$ properly discontinuously. $\square$

Solution of 2. (1) Suppose $k > 0$ first. The Cauchy integral formula writes

$$2c_k k! = \frac{\partial^k f}{\partial z^k}(0) = \frac{k!}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{k+1}}\,dz = \frac{k!}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})}{(re^{i\theta})^k}\,d\theta,$$

and it implies

$$2c_k r^k = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}\,d\theta.$$

Since $f(z)z^k$ is analytic, the Cauchy theorem can be applied to get

$$0 = \frac{1}{2\pi i}\int_{|z|=r} f(z)z^k\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})r^k e^{ik\theta}\,d\theta,$$

and it implies

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\overline{f(re^{i\theta})}e^{-ik\theta}\,d\theta.$$

By combining the above two equations, we obtain the formula. For $k = 0$, applying the Cauchy theorem for f, we have

$$c_0 = f(0) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\,d\theta.$$

Alternatively, we can show the same result using the orthogonal relation of complex exponential functions. An easy computation shows the identity

$$\begin{aligned}
\operatorname{Re} f(re^{i\theta}) &= \frac{1}{2}[f(re^{i\theta}) + \overline{f(re^{i\theta})}] \\
&= \frac{1}{2}\left[\left(1 + \sum_{k=1}^{\infty} 2c_k(re^{i\theta})^k\right) + \overline{\left(1 + \sum_{k=1}^{\infty} 2c_k(re^{i\theta})^k\right)}\right] \\
&= \frac{1}{2}\left[\left(1 + \sum_{k=1}^{\infty} 2c_k r^k e^{ik\theta}\right) + \left(1 + \sum_{k=1}^{\infty} 2\overline{c_k} r^k e^{-ik\theta}\right)\right] \\
&= \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta}.
\end{aligned}$$

From the uniform convergence of the power series on the compact set $\{z : |z| \le (r+1)/2\}$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\frac{1}{2\pi}\int_0^{2\pi} e^{il\theta}e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\delta_{kl} = c_k r^{|k|}.$$

(2) (b)⇒(a) Denote by K_n the convex hull of the curve $\theta \mapsto (e^{-i\theta}, \cdots, e^{-in\theta}) \in \mathbb{C}^n$. Suppose first that $(c_1, \cdots, c_n) \in K_n$. For each n, there exists a finite sequence of pairs $(\lambda_{n,j}, \theta_{n,j})_j$ having the following convex combination

$$(c_1, \cdots, c_n) = \sum_j \lambda_{n,j}(e^{-i\theta_{n,j}}, \cdots, e^{-in\theta_{n,j}})$$

with coefficients $\lambda_{n,j} \ge 0$ such that $\sum_j \lambda_{n,j} = 1$. Define

$$f_n(z) := \sum_j \lambda_{n,j}\frac{e^{i\theta_{n,j}} + z}{e^{i\theta_{n,j}} - z},$$

which has positive real part on $|z| < 1$ because $\operatorname{Re}(e^{i\theta_{n,j}} + z)/(e^{i\theta_{n,j}} - z) > 0$ for $|z| < 1$. Then,

$$f_n(z) = \sum_j \lambda_{n,j}(1 + \sum_{k=1}^{\infty} 2e^{-ik\theta_{n,j}}z^k) = 1 + \sum_{k=1}^{n} 2c_k z^k + \sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k$$

implies

$$\begin{aligned}|f_n(z)-f(z)| &= \left|\sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k - \sum_{k=n+1}^{\infty} 2c_k z^k\right| \\ &\le \sum_{k=n+1}^{\infty}\left|\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)-2c_k\right||z|^k \le \sum_{k=n+1}^{\infty} 4|z|^k\end{aligned}$$

converges to zero for $|z|<1$. Therefore, f has a non-negative real part on the open unit disk. The non-negativity can be strengthened to positivity by the open mapping theorem, so f belongs to the Carathéodory class.

(a)$\Rightarrow$(b) Conversely, suppose that f is in the Carathéodory class. Let $(\gamma_1,\cdots,\gamma_n)$ be any point on the surface ∂K_n of K_n and S any supporting hyperplane of K_n tangent at $(\gamma_1,\cdots,\gamma_n)$. Let $(u_1,\cdots,u_n)\in\mathbb{C}^n$ be the outward unit normal vector of the supporting hyperplane S. Note that this outward unit normal vector is uniquely determined for each hyperplane S with respect to the real inner product structure on the $2n$-dimensional real vector space $\mathbb{C}^n$ given by

$$\langle (z_1,\cdots,z_n),(w_1,\cdots,w_n)\rangle = \sum_{k=1}^{n}(\operatorname{Re} z_k \operatorname{Re} w_k + \operatorname{Im} z_k \operatorname{Im} w_k) = \operatorname{Re}\sum_{k=1}^{n} z_k\overline{w}_k.$$

Then, we know that $\sum_{k=1}^{n}|u_k|^2=1$ and the maximum

$$M := \max_{(x_1,\cdots,x_n)\in K_n} \operatorname{Re}\sum_{k=1}^{n} x_k\overline{u}_k > 0$$

is attained at $(\gamma_1,\cdots,\gamma_n)$. Our goal is now to verify the bound

$$\operatorname{Re}\sum_{k=1}^{n} c_k\overline{u}_k \le M$$

from the assumption that f is of Carathéodory class. Once the bound is obtained, then it means that $(c_1,\cdots,c_n)$ is contained in the same side as K_n of arbitrary hyperplanes tangent to K_n, so we finally conclude $(c_1,\cdots,c_n)\in K_n$.

Since for any $\theta\in[0,2\pi)$ the point $(e^{-i\theta},\cdots,e^{-in\theta})$ is in K_n, we have

$$\operatorname{Re}\sum_{k=1}^{n} e^{-ik\theta}\overline{u}_k \le M.$$

For $\varepsilon>0$, we have

$$\operatorname{Re}\sum_{k=1}^{n}\frac{1}{r^k}e^{-ik\theta}\overline{u}_k \le M+\varepsilon$$

for any $0<r<1$ sufficiently close to 1, thus we can write

$$\begin{aligned}\operatorname{Re}\sum_{k=1}^{n} c_k\overline{u}_k &= \operatorname{Re}\sum_{k=1}^{n}\frac{1}{2\pi r^k}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})e^{-ik\theta}\overline{u}_k\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\operatorname{Re}\sum_{k=1}^{n}\frac{1}{r^k}e^{-ik\theta}\overline{u}_k\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\,d\theta\cdot(M+\varepsilon) \\ &= \operatorname{Re} f(0)(M+\varepsilon) = M+\varepsilon\end{aligned}$$

thanks to the part (1) and the positivity of $\operatorname{Re} f$, and by limiting $r\to 1$ from left we get the bound we want. $\square$

Solution of 3. (1) Write $f = u + iv$ for real-valued u and v. Since

$$d\,(P\,dx + Q\,dy) = \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx \wedge dy = \frac{1}{\pi} f^{\#2}\, dx \wedge dy,$$

and since

$$\begin{aligned}
\frac{1}{\pi} f^{\#2}\, dx \wedge dy &= \frac{u_x v_y - u_y v_x}{\pi(1+u^2+v^2)^2}\, dx \wedge dy = \frac{du \wedge dv}{\pi(1+u^2+v^2)^2} \\
&= d\left(-\frac{v}{2\pi(1+u^2+v^2)}\, du + \frac{u}{2\pi(1+u^2+v^2)}\, dv\right) \\
&= d\left(-\frac{v}{2\pi(1+u^2+v^2)}(u_x\, dx + u_y\, dy) + \frac{u}{2\pi(1+u^2+v^2)}(v_x\, dx + v_y\, dy)\right) \\
&= d\left(-\frac{vu_x - uv_x}{2\pi(1+u^2+v^2)}\, dx + \frac{uv_y - vu_y}{2\pi(1+u^2+v^2)}\, dy\right) \\
&= d\left(-\frac{uu_y + vv_y}{2\pi(1+u^2+v^2)}\, dx + \frac{uu_x + vv_x}{2\pi(1+u^2+v^2)}\, dy\right),
\end{aligned}$$

we can check the following satisfy the equation of the problem:

$$P = -\frac{K_y}{4\pi K}, \qquad Q = \frac{K_x}{4\pi K}.$$

(2) Since the equation holds for $r = 0$, it suffices to show the differentiated equation

$$A(r,f) = n(r,f) + \frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log\sqrt{K(r,\theta)}\, d\theta$$

for almost every $r > 0$, where $K(r,\theta) = 1 + |f(re^{i\theta})|^2$. In particular, we will prove this equation for every r such that f does not have a pole a with $|a| = r$. Fix such r and let $\{a_i\}_{i=1}^n$ be poles of f in the region $|z| < r$ with multiplicities m_i for each a_i. Since

$$\begin{aligned}
P\,dx + Q\,dy &= \frac{1}{2\pi}\frac{-K_y\, dx + K_x\, dy}{2K} = \frac{1}{2\pi i}\frac{-iK_y\, dx + K_x\, i dy}{2K} \\
&= \frac{1}{2\pi i}\frac{(K_x - iK_y)(dx + i dy)}{2K} - \frac{1}{2\pi i}\frac{K_x\, dx + K_y\, dy}{2K} \\
&= \frac{1}{2\pi i}\frac{uu_x + vv_x - iuu_y - ivv_y}{1+u^2+v^2}\, dz - \frac{1}{2\pi i}\frac{dK}{2K} \\
&= \frac{1}{2\pi i}\frac{(u_x + iv_x)(u - iv)}{1+u^2+v^2}\, dz - \frac{1}{2\pi i}\frac{d\log K}{2} \\
&= \frac{1}{2\pi i}\frac{f'(z)}{f(z)}\frac{|f(z)|^2}{1+|f(z)|^2}\, dz - \frac{1}{2\pi i}\frac{d\log K}{2},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log\sqrt{K(r,\theta)}\, d\theta &= \frac{r}{2\pi}\int_0^{2\pi} \frac{K_r}{2K}\, d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{uu_r + vv_r}{K}\, d\theta \\
&= \frac{r}{2\pi}\int_0^{2\pi} \frac{u(\cos\theta u_x + \sin\theta u_y) + v(\cos\theta v_x + \sin\theta v_y)}{K}\, d\theta \\
&= \frac{r}{2\pi}\int_0^{2\pi} \frac{\mathrm{Re}[(\cos\theta + i\sin\theta)(u_x + iv_x)(u - iv)]}{K}\, d\theta \\
&= \mathrm{Re}\,\frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{\theta} f'\overline{f}}{1+|f|^2}\, d\theta = \mathrm{Re}\,\frac{1}{2\pi i}\int_{|z|=r} \frac{f'\overline{f}}{1+|f|^2}\, dz \\
&= \mathrm{Re}\int_{|z|=r} (P\,dx + Q\,dy),
\end{aligned}$$

and by the argument principle and $|f(z)| \to \infty$ near the pole $z \to a_i$,

$$\begin{aligned}\int_{|z-a_i|=\varepsilon} (P\,dx + Q\,dy) &= \frac{1}{2\pi i}\int_{|z-a_i|=\varepsilon} \frac{f'(z)}{f(z)} \frac{|f(z)|^2}{1+|f(z)|^2}\,dz \\ &= -m_i - \frac{1}{2\pi i}\int_{|z-a_i|=\varepsilon} \frac{f'(z)}{f(z)} \frac{1}{1+|f(z)|^2}\,dz \to -m_i\end{aligned}$$

as $\varepsilon \to 0$. Then, the Green theorem is applied to have

$$\begin{aligned}A(r,f) &= \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_{|z|\le r,\ \min_i |z-a_i|\ge\varepsilon} f^{\#}(x+iy)^2\,dx\,dy \\ &= \int_{|z|=r} (P\,dx+Q\,dy) - \lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{|z-a_i|=\varepsilon}(P\,dx+Q\,dy) \\ &= \frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log\sqrt{K(r,\theta)}\,d\theta + i\,\mathrm{Im}\int_{|z|=r}(P\,dx+Q\,dy) + \sum_{i=1}^{n} m_i.\end{aligned}$$

Because $\sum_{i=1}^{n} m_i = n(r,f)$ by definition, and seeing the real part, we obtain the desired equation.

(3) Since every Taylor coefficient of the logarithm is real, we have

$$\mathrm{Re}\log f(z) = \frac{1}{2}(\log f(z) + \log\overline{f(z)}) = \log|f(z)|.$$

Take $a \in \mathbb{C}$ and let $r := 2|a|$. By the Schwarz integral formula,

$$\begin{aligned}\log|f(a)| = \mathrm{Re}\log f(a) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{re^{i\theta}+a}{re^{i\theta}-a}\,\mathrm{Re}\log f(re^{i\theta})\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{re^{i\theta}+a}{re^{i\theta}-a}\right| \log|f(re^{i\theta})|\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} 3\log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta \\ &\le 3\int_0^r A(t,f)\frac{dt}{t} \le 3\int_0^r M^2 t^2 \frac{dt}{t} = 6M^2|a|^2,\end{aligned}$$

so $C := e^{6M^2}$ proves the theorem, where M is a bound of the spherical derivative $f^{\#}$. $\square$

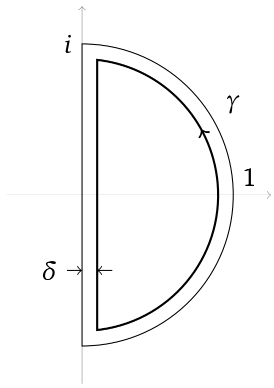
Solution of 4. (1) Identify Ω and $\widetilde{\Omega}$ as subsets of $\mathbb{C}$ by letting $(x, y) = x + iy$. Consider a harmonic conjugate $-u$ of v on Ω such that a function $f(x+iy) := u(x, y) + iv(x, y)$ is holomorphic on Ω. If we define

$$\widetilde{f}(z) := \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0, \\ \overline{f(\overline{z})} & \text{if } \operatorname{Im} z < 0, \end{cases} \qquad z \in \widetilde{\Omega},$$

then $\widetilde{f}$ is holomorphic on $\widetilde{\Omega} \setminus (0, 1)$, and is also continuous on the whole $\widetilde{\Omega}$ because of the boundary condition of v on the real axis. We claim that $\widetilde{f}$ is in fact holomorphic on $\widetilde{\Omega}$. If the claim is true, then $\widetilde{v} := \operatorname{Im} \widetilde{f}$ is the desired extension of v, which satisfies in addition that for $(x_0, y_0) \in \partial\widetilde{\Omega}$ we have

$$\lim_{(x,y)\to(x_0,y_0)} \widetilde{v}(x, y) = \begin{cases} 1 & \text{if } y_0 > 0, \\ -1 & \text{if } y_0 < 0. \end{cases}$$

Let γ be a contour defined for sufficiently small $\delta > 0$ as the following figure:



Denote by $\widetilde{\Omega}_\delta := \{a \in \widetilde{\Omega} : \min_{z_0 \in \partial\widetilde{\Omega}} |z_0 - a| > \delta\}$ the interior of γ. Define a function $\widetilde{g}$ on $\widetilde{\Omega}_\delta$ such that

$$\widetilde{g}(a) := \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z)}{z - a}\, dz, \qquad a \in \widetilde{\Omega}_\delta.$$

Note that the integrand is continuous on the contour γ, and $\widetilde{g}$ is holomorphic on $\widetilde{\Omega}_\delta$ by the Morera theorem, because for every affine triangle σ in the interior of γ we have

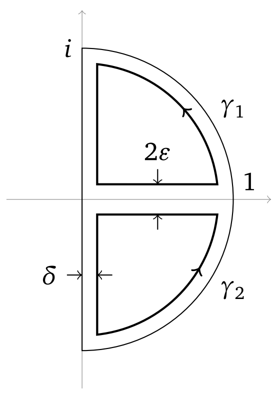
$$\int_\sigma \widetilde{g}(a)\, dz = \int_\sigma \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z)}{z - a}\, dz\, da = \frac{1}{2\pi i} \int_\gamma \left[\int_\sigma \frac{\widetilde{f}(z)}{z - a}\, da \right] dz = 0$$

by the Fubini theorem and the Cauchy theorem for σ.

Moreover, for $a \in \widetilde{\Omega}_\delta \cap \Omega$ we have

$$\widetilde{g}(a) = \lim_{\varepsilon \to 0} \left[\frac{1}{2\pi i} \int_{\gamma_1} \frac{\widetilde{f}(z)}{z - a}\, dz + \frac{1}{2\pi i} \int_{\gamma_1} \frac{\widetilde{f}(z)}{z - a}\, dz \right] = \widetilde{f}(a) + 0 = \widetilde{f}(a),$$

where γ_1 and γ_2 are contours given as the folllowing figure for $\varepsilon > 0$:



The same result holds also for $a \in \widetilde{\Omega}_\delta \setminus \overline{\Omega}$, so we can conclude $\widetilde{g}(a) = \widetilde{f}(a)$ on $a \in \widetilde{\Omega}_\delta \setminus (0,1)$, and by the contintuity of $\widetilde{f}$ and $\widetilde{g}$, we finally have $\widetilde{f} = \widetilde{g}$ so that $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}_\delta$. Since the above arguments make sense for every $\delta > 0$ small enough, the union $\widetilde{\Omega} = \bigcup_{\delta>0} \widetilde{\Omega}_\delta$ implies that the function $\widetilde{f}$ is holomorphic on $\widetilde{\Omega}$.

(2) The domain $\widetilde{\Omega}$ is conformally mapped onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by

$$\varphi : \widetilde{\Omega} \to \mathbb{H} : z \mapsto \left(\frac{z+i}{iz+1} \right)^2 .$$

Note that $\varphi(\Omega) = \{z \in \mathbb{H} : |z| > 1\}$.

We can compute for $(x,y) \in \widetilde{\Omega}$

$$|\varphi(x+iy)|^2 = \left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2} \right)^2, \qquad \operatorname{Im} \varphi(x+iy) = \frac{4x(1-x^2-y^2)}{(x^2+(y-1)^2)^2}.$$

Define a function $V : \mathbb{H} \to \mathbb{R}$ such that $V := \widetilde{v} \circ \varphi^{-1}$. Then, V is a harmonic function satisfying the boundary condition

$$\lim_{(x,y)\to(x_0,0)} V(x,y) = \begin{cases} -1 & \text{if } |x_0| < 1, \\ 1 & \text{if } |x_0| > 1. \end{cases}$$

For $(x,y) \in \varphi(\Omega)$ so that $x^2+y^2 > 1$ the Poisson kernel gives that

$$\begin{aligned} \frac{1-V(x,y)}{2} &= \frac{1}{\pi} \int_{-1}^{1} \frac{y}{(x-t)^2+y^2}\, dt \\ &= \frac{1}{\pi} \left(\tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) \\ &= \frac{1}{\pi} \tan^{-1} \frac{2y}{x^2+y^2-1}, \end{aligned}$$

so

$$V(x,y) = \frac{2}{\pi} \tan^{-1} \frac{x^2+y^2-1}{2y}.$$

Thus we have for $(x,y) \in \Omega$

$$v(x,y) = V(\varphi(x+iy)) = \frac{2}{\pi} \tan^{-1} \frac{y(1+x^2+y^2)}{x(1-x^2-y^2)}. \qquad \square$$