0.1. Let (T_n) be a sequence in $B(X,Y)$. If T_n coverges then $\|T_n\|$ is bounded by the uniform boundedness principle.

0.2. We show that there is no projection from ℓ^∞ onto c_0.

(a) Show that a Banach space X is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of X.

0.3 (Bounded below maps in Banach spaces)**.** Let $T : X \to Y$ be a bounded linear map between Banach spaces. Show that the following statements are equivalent:

(a) It is bounded below.

(b) It is injective and has closed range.

(c) It is a isometric isomorphism onto its image.

0.4 (Bounded below maps in Hilbert spaces)**.** Let $T : H \to K$ be a bounded linear operator between Hilbert spaces. Show that the following statements are equivalent:

(a) It is bounded below.

(b) It has a left inverse.

(c) Its adjoint has right inverse.

(d) The product T^*T is invertible.

In particular, a normal operator in $B(H)$ is bounded below if and only if it is invertible.

0.5 (Injectivity and surjectivity of dual map)**.** Let $T : X \to Y$ be a bounded linear operator between Banach spaces and $T^* : Y^* \to X^*$ be its dual.

(a) Show that T^* is injective if and only if T has dense range.

(b) Show that T^* is surjective if and only if T is bounded below.

0.6. For $T \in B(H)$, we have an obvious fact $(\operatorname{im} T)^\perp = \ker T^*$. If T is normal, then the kernel of T and T^* are equal.

(a) Show that if T is surjective bounded operator, then T is invertible.

0.7 (Schur's property of ℓ^1)**.** .

0.8. Let $\varphi : L^\infty([0,1]) \to \ell^\infty(\mathbb{N})$ be an isometric isomorphism. Suppose φ is realised as a sequence of bounded linear functionals on L^∞.

(a) Show that $\varphi^*(\ell^1) \subset L^1$ where ℓ^1 and L^1 are considered as closed linear subspaces of $(\ell^\infty)^*$ and $(L^\infty)^*$ respectively.

(b) Show that φ^* is indeed an isometric isomorphism, and deduce φ cannot be realised as bounded linear functionals on L^∞.

0.9 (Predual correspondence)**.** Let X be a Banach space and Z be a linear subspace of X^*. Define $\varphi : X \to Z^*$ as the restriction of the dual map of inclusion $Z \subset X^*$.

(a) Show that if φ is an isometric isomorphism, then closed ball of X is compact Hausdorff in $\sigma(X, Z)$.

(b) Show that the converse holds by using Goldstine's theorem.

0.10 (Operator monotonicity of square and commitativity)**.** Let $\mathcal{A}$ be a C^*-algebra in which the square function is operator monotone, that is, $0 \le a \le b$ implies $a^2 \le b^2$ for any positive elements a and b in $\mathcal{A}$. We are going to show that $\mathcal{A}$ is necessarily commutative. Let a and b denote arbitrary positive elements of $\mathcal{A}$.

(a) Show that $ab + ba \ge 0$.

(b) Let $ab = c + id$ where c and d are self adjoints. Show that $d^2 \le c^2$.

(c) Suppose $\lambda > 0$ satisfies $\lambda d^2 \le c^2$. Show that $c^2 d^2 + d^2 c^2 - 2\lambda d^4 \ge 0$.

(d) Show that $\lambda (cd + dc)^2 \le (c^2 - d^2)^2$.

(e) Show that $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \le c^2$ and deduce $d = 0$.

(f) Extend the result for general exponent: $\mathcal{A}$ is commitative if $f(x) = x^\beta$ is operator monotone for $\beta > 1$.

0.11 (Compact left multiplications and SOT)**.** Let T_n be a sequence of bounded linear operators on a Hilbert space that converges in SOT. For compact K, $T_n K$ converges in norm, but $K T_n$ generally does not unless T is self-adjoint.

0.12. Let X be a closed subspace of a Banach space Y and

$$i : X \to Y$$

the inclusion. Suppose X and Y have preduals X_* and Y_* respectively. Let

$$j := i^*|_{Y_*} : Y_* \to Z \subset X^*,$$

where $Z := i^*(Y_*)^-$. Then we can show

$$j^* : Z^* \subset X^{**} \to Y$$

coincides with i on $X \cap Z^*$. From the existence of X_* we have $X^{**} \to X$, which is restricted to define a map $k : Z^* \to X$.

$$\begin{tikzcd} & X \arrow[r, "i"] & Y \\ X^{**} \arrow[ur] \arrow[r] & Z^* \arrow[u, "k"] \arrow[ur, "j"] & \end{tikzcd}$$

We can show k is an isomorphism so that we have

$$X_* \cong Y_* / Y_* \cap \ker(i^*).$$

0.13 (Injective *-homomorphism is an isometry)**.**

0.1 Topological measures

0.14. Let X be compact. A positive linear functional ρ on $C(X)$ is bounded with norm $\rho(1)$.

Proof. Since $0 \le \rho(\|f\| \pm f) = \|f\|\rho(1) \pm \rho(f)$, we have $|\rho(f)| \le \rho(1)\|f\|$. □

0.15. Let X be a locally compact Hausdorff space.

(a) The Baire σ-algebra is generated by compact G_δ sets.

(b) If X is second countable, then every Baire set is Borel.

Solution. (b) (A second countable locally compact space is σ-compact.

Since X is σ-compact and Hausdorff, every closed set is a countable union of compact sets, so the Borel σ-algebra on X is generated by compact sets.)

Since locally compact Hausdorff space is regular, the Urysohn metrization implies X is metrizable, and every closed sets in metrizable space is G_δ set. □

0.1.1 The Riesz-Kakutani theorem for positive linear functionals

0.16. Let X be compact. There is a map from the set of finite Baire measures to the set of positive linear functionals on $C(X)$.

Solution. A function in $C(X)$ is Baire measurable and bounded. Thus the integration is well-defined. □

0.17. Let X be compact. There is a map from the set of positive linear functionals on $C(X)$ to the set of finite regular Borel measures.

Solution. i. and ii. and iii. of Theorem 7.2. □

0.18. Let X be compact. Let ρ be a positive linear functional on $C(X)$. Let ν be the regular Borel measure associated to ρ. Then, $\rho(f) = \int f\, d\nu$.

Solution. iv. of Theorem 7.2. □

0.19. Let X be compact. Let ν be a finite regular Borel measure. Let ν' be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\, d\nu$. Then, $\nu = \nu'$ on Borel sets.

Solution. Theorem 7.8. □

The two results above establish the correspondence between positive linear functionals and regular Borel measures. The following is an additional topic: Borel extension of Baire measures.

0.20. Let X be compact. Let μ be a finite Baire measure. Let ν be the regular Borel measure associated to the positive linear functional $f \mapsto \int f\,d\mu$. Then, $\mu = \nu$ on Baire sets.

Solution. Let μ, ν be finite Baire measures. Enough to show if $\int f\,d\mu = \int f\,d\nu$ then $\mu = \nu$ according to the preceding two results.

Enough to show the regularity of Baire measures. $\square$

- A second countable locally compact space is σ-compact.
- A σ-compact locally compact space is paracompact.
- A second countable regular space is paracompact.
- A locally compact Hausdorff space is regular.

semiring σ-finiteness implies the uniqueness

0.2 Problems

0.21. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that f is identically zero if $f'(x) = f(x)^2$ for all x.

0.22. Let $f(x) = x(1+x)^{-1}$ be a function on $\mathbb{R}_{\geq 0}$. Show that a C*-algebra $\mathcal{A}$ is commutative if and only if f is operator subadditive in $\mathcal{A}$.

0.23. Let T be an invertible linear operator on a normed space. Show that $T^{-2} + \|T\|^{-2}$ is injective if it is surjective.

0.24 (Diophantine euqations)**.** (a) Show that there is no integral solution of the equation $x^7 + 7 = y^2$.

(b) Show that if $(x^2 + y^2 + z^2)/(xy + yz + zx)$ is an integer, then it is not divided by 3.

(c) Show that there is no non-trivial integral solution of $x^4 - y^4 = z^2$.