Harmonic Analysis

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Contents

Part I

Fourier analysis

Chapter 1

Fourier series

1.1 Fourier series in L^p spaces

1.1.

$$\|\hat{f}\|_{\ell^1(\mathbb{Z})} \lesssim \|f\|_{W^{1,1+\varepsilon}(\mathbb{T})}.$$

Inversion theorem is an approximation problem given by $\mathcal{F}^*\mathcal{F} = \lim_{n\to\infty} \mathcal{F}_n^*\mathcal{F}$. The condition $\hat{f} \in \ell^1(\mathbb{Z})$ is a condition just for defining $\mathcal{F}^*\hat{f}$ wihtout using distribution theory, and it does not affect the inversion phenomena. The approximation, in other words, can be seen as an extension method for $\mathcal{F}^* : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ on $c_0(\mathbb{Z})$. Note that $\mathcal{F}_n^*$ on $c_0(\mathbb{Z})$ cannot be bounded directly without distribution theory, but $\mathcal{F}_n^*\mathcal{F}$ on $L^p(\mathbb{T})$ can be bounded well.

1.2 Summability methods

- If $\mathcal{F}_n^*$ is the standard partial sum, then $\mathcal{F}_n^*\mathcal{F}$ is the Dirichlet kernel.
- If $\mathcal{F}_n^*$ is the Cesàro mean, then $\mathcal{F}_n^*\mathcal{F}$ is the Fejér kernel.
- If $\mathcal{F}_r^*$ is the Abel sum, then $\mathcal{F}_r^*\mathcal{F}$ is the Poisson kernel.
- In Fourier transform, we often use the Gauss-Weierstrass kernel.

The injectivity of $\mathcal{F}$ is not an easy problem, which comes from the inversion theorem.

1.2 (Dirichlet kernel)**.** The *Dirichlet kernel* is a function $D_n : \mathbf{T} \to \mathbb{R}$ defined by

$$D_n = \widehat{\mathbf{1}_{|k|\le n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = \mathbf{1}_{|k|\le n}.$$

This is because they are invariant under inverse, in other words, they are even.

(a)

$$D_n(x) = \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.$$

(b) If $f \in \mathrm{Lip}(\mathbf{T})$, then $D_n * f \to f$ pointwisely as $n \to \infty$.

(c)

$$\|D_n\|_{L^1(\mathbf{T})} \gtrsim \log n.$$

Proof.

$$\begin{aligned} D_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\ &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}. \end{aligned}$$

(c) By (2) $\sin x \leq x$ for $x \in [0, \pi/2]$, (3) change of variable,

$$\begin{aligned} \|D_n\|_{L^1(\mathbf{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}\right| dx \\ &\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin\frac{2n+1}{2}x|}{x}\, dx \\ &= \frac{2}{\pi}\int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &= \frac{2}{\pi}\sum_{k=0}^{2n}\int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &\geq \frac{2}{\pi}\sum_{k=0}^{2n}\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi}\, dx \\ &\geq \frac{4}{\pi^2}\sum_{k=0}^{2n}\frac{1}{1+k} \\ &\geq \frac{4}{\pi^2}\log(2n+2). \end{aligned}$$

..? ☐

1.3 (Fejér kernel)**.** The *Fejér kernel* is

(a)

$$K_n(x) = \frac{1}{n+1}\frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}.$$

Proof. Since

$$\begin{aligned} D_n(x) = &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2}, \end{aligned}$$

by telescoping, we get

$$\begin{aligned} \sum_{k=0}^{n} D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}. \end{aligned}$$

$\square$

Two important results from Fejér kernel:

1. If $f(x-)$, $f(x+)$ exist and $S_n f(x)$ converges, then $S_n f(x) \to \frac{1}{2}(f(x-)+f(x+))$.
2. (If $f \in L^1(\mathbf{T})$, then $\sigma_n f \to f$ a.e.)
3. If $f \in L^1(\mathbf{T})$, then $S_n f \to f$ in L^1 and L^2.
4. If f is continuous and $\widehat{f} \in L^1(\mathbb{Z})$, then $S_n f \to f$ uniformly.
5. Since $\sigma_n f$ is a trigonometric polynomial, the set of trigonometric polynomials are dense in $L^1(\mathbf{T})$ and $L^2(\mathbf{T})$.

1.3 Pointwise convergence of Fourier series

BV function: Dini, Jordan's criterion

1.4 (Riemann localization principle).

Exercises

1.5 (Gibbs phenomenon).

1.6 (Du Bois-Reymond function).

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

2.1 (Riemann-Lebesgue lemma).

Lp extension

Gaussian function computation: differential equation method, contour integral method inversion theorem

2.2 (Plancherel theorem).

2.2 Tempered distributions

2.3 (Cauchy principal value). indented contour, imaginary shift, Feynman's trick

Exercises

2.4 (Sampling theorem).

$$\mathcal{F}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(\xi) = \operatorname{sinc}(\xi/2)$$

$\operatorname{sinc} \in L^{1+\varepsilon}(\mathbb{R})$.

2.5 (Poisson summation formula).

2.6 (Uncertainty principle).

Problems

1. Find all $\alpha > 0$ such that

   $$\lim_{x\to\infty} x^{-\alpha} \int_0^x f(y)\,dy = 0$$

   for all $f \in L^3([0,\infty))$.

Chapter 3

Hilbert transform

3.1 Harmonic conjugate

3.2 Kernel representation

3.3 Fourier series in L^p space

Part II

Singular integral operators

Chapter 4

Calderón-Zygmund theory

4.1 Convolution type operators

4.1 (Calderón-Zygmund decomposition of sets). Let $f \in L^1(\mathbb{R}^d)$. Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf := \sup_n E_n|f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|}\int_{Q_x} |f| > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \le \frac{1}{\lambda}\|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\mathbb{R}^d\setminus\Omega)} \le \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|}\int_{Q_x} |f| \le 2^d\lambda.$$

4.2 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|}\int_{Q_x}|f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \le 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof. □

4.3 (L^p boundedness of Calderón-Zygmund operators). Let $T : C_c^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that $Tf(x) = K * f(x)$ for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \operatorname{supp} f$. We say T is called a *Calderón-Zygmund* operator if

(i) T is L^2-bounded: we have

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2},$$

(ii) T satisfies the *Hörmander condition*: we have

$$\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx \lesssim 1$$

for every $y > 0$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2}\|Tg\|^2_{L^2(\Omega)} \leq \frac{4C}{\lambda^2}\|g\|^2_{L^2(\Omega)} \leq \frac{4C}{\lambda^2}\|g\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega^*} |Tb(x)|\,dx \leq \frac{2}{\lambda}\sum_i \int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\operatorname{supp} b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y)b_i(y)\,dy = \int_{Q_i} [K(x-y)-K(x)]b_i(y)\,dy,$$

and

$$\int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d\setminus Q_i^*} |K(x-y)-K(x)|\,dx\,dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.4 (Hölder boundedness of Calderón-Zygmund operators).

4.2 Truncated integrals

Homogeneous kernels

4.3 A_p weights

4.4 Bounded mean oscillation

Exercises

4.5 (Size and cancellation condition). Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$. We say the condition $|K(x)| \lesssim |x|^{-d}$ for $x \neq 0$ as the *size condition*, and say the condition $\int_{r<|x|<R} K(x)\,dx = 0$ for all $0 < r < R < \infty$ as the *cancellation condition*. If K satisfies the size, cancellation, and Hörmander condition, then it is L^2 bounded, hence Calderón-Zygmund.

4.6 (Gradient size condition). Let $|\nabla K(x)| \lesssim |x|^{-d-1}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

4.7 (Riesz potential).

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Almost orthogonality

Carleson measures, paraproducts

6.1 Coltar lemma

6.2 $T(1)$ theorem

Part III

Oscillatory integral operators

Chapter 7

Chapter 8

Restriction

Chapter 9

Part IV

Pseudo-differential operators

Chapter 10

Symbols

10.1 Kohn-Nirenberg calculus

$S^m_{\rho,\delta}$

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \lesssim \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}.$$

Let a be a symbol on $M = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$. Then, the associated ΨDO is

$$T_a\psi(x) := \frac{1}{(2\pi)^d} \iint e^{i\langle x-y,\xi\rangle} a(x,\xi)\psi(y)\,dy\,d\xi.$$

For parameters $0 \leq \lambda \leq 1$ and $h > 0$, let

$$\hat{a}\psi(x) := \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle} a((1-\lambda)x + \lambda y, \xi)\psi(y)\,dy\,d\xi.$$

For example, regardless of h and λ,

$$\hat{\xi}\psi(x) = \frac{h}{i}\psi'(x)$$

and

$$\hat{H}\psi(x) = -h^2\Delta\psi(x) + V(x)\psi(x),$$

where $V : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ and $H : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ such that

$$H(x,\xi) := |\xi|^2 + V(x).$$

$$\frac{d}{dt}a(t) = \{a(t), H\} = X_H a(t)$$

$$\frac{d}{dt}\hat{a}(t) = \frac{d}{dt}e^{\frac{i}{h}t\hat{H}}\hat{a}e^{-\frac{i}{h}t\hat{H}} = -\frac{i}{h}[\hat{a}(t), \hat{H}]$$

Chapter 11

Semiclassical analysis

11.1 Weyl calculus

11.1 (Composition of Weyl quantization).

11.2 Heisenberg group

Chapter 12

Microlocal analysis