C^*-Algebras

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Part I

C^*-algebras

Chapter 1

Basic concepts

1.1 Multiplier algebra

1.1 (Multiplier algebra). Let A be a C*-algebra. A *double centralizer* of A is a pair (L,R) of bounded linear maps on A such that $aL(b) = R(a)b$ for all $a, b \in A$. The *multiplier algebra* $M(A)$ of A is defined to be the set of all double centralizers of A.

1.2 (Cohen factorization theorem).

1.3 (Strict topology).

1.4 (Examples of multiplier algebras). (a) $M(K(H)) \cong B(H)$.

(b) $M(C_0(\Omega)) \cong C_b(\Omega)$.

Proof. (a)

(b) First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega) \cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega) \cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective $*$-homomorphism $C_b(\Omega) \to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g \in M(C_0(\Omega))_+$. □

1.5 (Hereditary C*-subalgebra). state extension, representation extension(not ideal?)

1.6 (Essential ideals). (a) Hilbert C*-module description

Exercises

1.7. Let B be a hereditary C*-subalgebra of a C*-algebra A. Let $a \in A_+$. If for any $\varepsilon > 0$ there is $b \in B_+$ such that $a - \varepsilon \leq b$, then $a \in B_+$.

Proof. To catch the idea, suppose A is abelian. We want to approximate a by the elements of B in norm. To do this, for each $\varepsilon > 0$, we want to construct $b' \in B_+$ such that $a - \varepsilon \leq b' \leq a + \varepsilon$ using b. Taking $b' = \min\{a, b\}$ is impossible in non-abelian case, but we can put $b' = \frac{a}{b+\varepsilon} b$. For a simpler proof, $b' = (\frac{\sqrt{ab}}{\sqrt{b}+\sqrt{\varepsilon}})^2$ is a better choice.

Define

$$b' := \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} a \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}}.$$

Then,

$$\|\sqrt{a}-\sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\|^2=\|\frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}}a\frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}}\|\leq\varepsilon$$

implies

$$\lim_{\varepsilon\to 0}b'=\lim_{\varepsilon\to 0}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\sqrt{a}\cdot\sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}=\sqrt{a}\cdot\sqrt{a}=a.$$

□

Chapter 2

Completely positive maps

2.1 Operator systems and spaces

2.2 Dilation theorems

2.3 Extension theorems

Arveson Trick

Chapter 3

Tensor products

3.1 Minimal tensor product

spatiality Takesaki theorem

3.2 Maximal tensor product

universal property restriction theorem c.c.p. tensor product

3.3 Nuclear C*-algebras

finite dimensional, abelian, some constructions

a separable C*-algebra is nuclear if and only if every factor representation is hyperfinite.

Part II

Approximation properties

3.4 Finite dimensional approximation

nuclear and exact C*-algebras

3.5 Voiculescu theorem

3.6 Quasidiagonal C*-algebras

Part III

Constructions

Part IV

Operator K-theory

Chapter 4

4.1 (Homotopy of $*$-homomorphisms)**.** Let A, B be C*-algebras. Two $*$-homomorphisms in $\mathrm{Mor}(A,B)$ are said to be *homotopic* if they are connected by a path in $\mathrm{Mor}(A,B)$ that is continuous with the point-norm topology.

(a) For pointed compact Hausdorff spaces $(X, x_0), (Y, y_0)$, two pointed maps $\varphi_0, \varphi_1 : X \to Y$ are homotopic if and only if $\varphi_0^*, \varphi_1^* : C_0(Y \setminus \{y_0\}) \to C_0(X \setminus \{x_0\})$ are homotopic.

Proof. (a) Suppose φ_0 and φ_1 are connected by a homotopy φ_t. Fixing $g \in C_0(Y)$ and $t_0 \in I$, we want to show

$$\lim_{t \to t_0} \sup_{x \in X} |g(\varphi_t(x)) - g(\varphi_{t_0}(x))| = 0.$$

Since the function g is uniformly continuous, with respect to an arbitrarily chosen uniformity on Y, so that there is an entourage $E \subset Y \times Y$ such that $(y, y') \in E \circ E$ implies $|g(y) - g(y')| < \varepsilon$. Using compactness we have a finite sequence $(y_i)_{i=1}^n \subset Y$ such that for every y there is y_i satisfying $(y, y') \in E$. Then, $\varphi^{-1}(E[y_i])$ is a finite open cover of $X \times I$, so we have δ such that $|t - t_0| < \delta$ implies for any $x \in X$ the existence of i satisfying $(\varphi_t(x), y_i) \in E$ and $(\varphi_{t_0}(x), y_i) \in E$, which deduces the desired inequality.

Conversely, suppose φ_0^* and φ_1^* are connected by a homotopy φ_t^*. By taking dual, we can induce $\varphi_t : X \to Y$ such that $g(\varphi_t(x)) = (\varphi_t^* g)(x)$ for each $g \in C(Y)$ from φ_t^* via the embedding $X \to M(X)$ by Dirac measures. Let V be an open neighborhood of $\varphi_{t_0}(x_0)$ and take $g \in C(Y)$ such that $g(\varphi_{t_0}(x_0)) = 1$ and $g(y) = 0$ for $y \notin V$. Now we have an open neighborhood U of x_0 such that $x \in U$ implies $|(\varphi_{t_0}^* g)(x) - (\varphi_{t_0}^* g)(x_0)| < \frac{1}{2}$. Also we have $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|\varphi_t^* g - \varphi_{t_0}^* g\| < \frac{1}{2}$. Therefore, $(x, t) \in U \times (t_0 - \delta, t_0 + \delta)$ implies $g(\varphi_t(x)) > 0$, hence $\varphi_t(x) \in V$, which means $X \times I \to Y : (x, t) \mapsto \varphi_t(x)$ is continuous. □

We have $\widetilde{K}^n(X, x_0) = K_n(C_0(X \setminus \{x_0\}))$ for a pointed compact Hausdorff space X. Now then since the inclusion $\{x_0\} \to X$ induces the section so that

$$0 \to K_0(C_0(X \setminus \{x_0\})) \to K_0(C(X)) \to K_0(\{x_0\}) \to 0$$

splits, we have

$$K^0(X) = \widetilde{K}^0(X, x_0) \oplus \mathbb{Z} = K_0(C_0(X \setminus \{x_0\})) \oplus K_0(\{x_0\}) = K_0(C(X))$$

for a compact connected Hausdorff space X. The additivity of K_0 and K^0 removes the connectedness condition.

$$K_0(\mathbb{C}) = \mathbb{Z}, \quad K_0(C_0(\mathbb{R})) = 0, \quad K_1(C_0(\mathbb{R})) = K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$$

$$K^0(*) = \mathbb{Z}, \quad K^0(S^1) = \mathbb{Z}, \quad K^1(S^1) = K^0(S^2) = \mathbb{Z}[x]/(x-1)^2$$

Chapter 5

Brown-Douglas-Fillmore theory

5.1 (Haagerup property).

Baum-Connes conjecture Non-commutative geometry Elliott theorem

5.1 Approximately finite algebras

Elliott conjecture: amenable simple separable C^*-algerbas are classified by K-theory.