Foundations of Calculus

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Part I

Convergence

Chapter 1

Sequences

1.1 Limit of sequences

preserving inequalities limsup and liminf

1.2 Extended real numbers

1.1 (Operations in the extended real numbers)**.** We can extend addition (except $\infty + (-\infty)$), subtraction, multiplication (except $\infty \times 0$), division (except dividing by zero).

1.2 (Limits in the extended real numbers)**.**

1.3 Control of the error

sufficiently large asymptotic expressions

Approximate sequences and change of limits

1.3 (Change of limits)**.**

$$|a_n - a| \le |a_n - b_{mn}| + |b_{mn} - b_m| + |b_m - a|$$

$$\lim_m \sup_n |a_n - b_{mn}| = 0$$

$$\lim_n |b_{mn} - b_m| = 0$$

$$a_n = b_{mn} + c_{mn} \le b_{mn} + \varepsilon$$

1.4 Bounded sequences

monotone convergence Bolzano-Weierstrass

Exercises

1.4.

1.5 (Newton method)**.**

Problems

1. Show that every real sequence $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.

Chapter 2

Series

2.1 Absolute convergence

2.1 (Unconditional convergence).

2.2 Convergence tests

comparison limit comparison cauchy condensation integral....

ratio root

2.2 (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_k B_k - A_{k-1}B_{k-1}$$

$$\sum_{m<k\le n} A_k b_k = A_n B_n - A_m B_m - \sum_{m<k\le n} a_k B_{k-1}.$$

abel test

2.3 (Dirichlet test).

2.4 (Mertens' theorem). If $\sum_{k=0}^{\infty} a_k$ converges to A absolutely and $\sum_{k=0}^{\infty} b_k$ converges to B, then their Cauchy product $\sum_{k=0}^{\infty} c_k$ with $c_k := \sum_{l=0}^{k} a_l b_{k-l}$ converges to AB.

(a) We have

$$\lim_{m\to\infty} \sup_n \sum_{k=m+1}^{n} \sum_{l=n-k+1}^{n} a_k b_l = 0.$$

(b) We have for each m that

$$\lim_{n\to\infty} \sum_{k=1}^{m} \sum_{l=n-k+1}^{n} a_k b_l = 0$$

Proof. Let

$$A_n := \sum_{k=0}^{n} a_k,\ B_n := \sum_{k=0}^{n} b_k, \quad \text{and} \quad C_n := \sum_{k=0}^{n} c_k.$$

As $m \to \infty$.

$$\left|\sum_{k=m+1}^{n} \sum_{l=n-k+1}^{n} a_k b_l\right| \le \sum_{k=m+1}^{n} |a_k| \left|\sum_{l=n-k+1}^{n} b_l\right| = \sum_{k=m+1}^{n} |a_k||B_n - B_{n-k}| \lesssim \sum_{k=m+1}^{\infty} |a_k| \to 0.$$

For fixed m, as $n \to \infty$,

$$\Big|\sum_{k=0}^{m} \sum_{l=n-k+1}^{n} a_k b_l\Big| \le \sum_{k=0}^{m} |a_k| \Big|\sum_{l=n-k+1}^{n} b_l\Big| = \sum_{k=0}^{m} |a_k||B_n - B_{n-k}| \to \sum_{k=0}^{m} |a_k||B - B| = 0.$$

We will prove

$$A_n B_n - C_n = \sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l \to 0$$

as $n \to \infty$. For $\varepsilon > 0$, take m such that

$$|\sup_n \sum_{k=m+1}^{n} \sum_{l=n-k+1}^{n} a_k b_l| < \varepsilon.$$

Then for every n we have

$$|\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l| \le \varepsilon + |\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l|.$$

Taking limits $n \to \infty$ and $\varepsilon \to 0$ in order, we are done. □

Exercises

2.5 (Cesàro mean).

2.6 (Recursive sine sequence). Let $a_{n+1} = \sin a_n$ and $a_n = 1$. We can use $\sin x = x - \frac{x^3}{6} + O(x^5)$.

$$a_n = \sqrt{3} n^{-\frac{1}{2}} - \frac{3\sqrt{3}}{20} n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}).$$

Problems

1. If $a_n \to 0$, then $\frac{1}{n} \sum_{k=1}^{n} a_k \to 0$.
2. If $a_n \ge 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ also diverges.
3. If $a_n \downarrow 0$ and $S_n \le 1 + n a_n$, then $S_n \le 1$.

Chapter 3

Metrics and norms

3.1 Metric spaces

3.1 (Definition of metric spaces)**.** Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(i) $d(x,y) = 0$ if and only if $x = y$, (nondegeneracy)

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$, (symmetry)

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$. (triangle inequality)

A pair (X,d) of a set X and a metric on X is called a *metric space*. We often write it simply X.

(a) A normed space X is a metric space with a metric defined by $d(x,y) := \|x - y\|$.

(b) A subset of a metric space is a metric space with a metric given by restriction.

3.2 (System of open balls)**.** A metric is often misunderstood as something that measures a distance between two points and belongs to the study of geoemtry. The main function of a metric is to make a system of small balls, sets of points whose distance from specified center points is less than fixed numbers. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly.

Note that taking either ε or δ in analysis really means taking a ball of the very radius. Investigation of the distribution of open balls centered at a point is now an important problem.

Let X be a metric space. A set of the form

$$\{y \in X : d(x,y) < \varepsilon\}$$

for $x \in X$ and $\varepsilon > 0$ is called an *open ball centered at x with radius ε* and denoted by $B(x,\varepsilon)$ or $B_\varepsilon(x)$.

3.3 (Convergence and continuity in metric spaces)**.** Let $\{x_n\}_n$ be a sequence of points on a metric space (X,d). We say that a point x is a *limit* of the sequence or the sequence *converges to* x if for arbitrarily small ball $B(x,\varepsilon)$, we can find n_0 such that $x_n \in B(x,\varepsilon)$ for all $n > n_0$. If it is satisfied, then we write

$$\lim_{n\to\infty} x_n = x,$$

or simply $x_n \to x$ as $n \to \infty$. We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

A function $f : X \to Y$ between metric spaces is called *continuous at* $x \in X$ if for any ball $B(f(x),\varepsilon) \subset Y$, there is a ball $B(x,\delta) \subset X$ such that $f(B(x,\delta)) \subset B(f(x),\varepsilon)$. The function f is called *continuous* if it is continuous at every point on X.

(a) A sequence x_n in a metric space X converges to $x \in X$ if and only if $d(x_n, x)$ converges to zero.

(b) Let $f : X \to Y$ be a function between two metric spaces. If there is a constant C such that $d(x, y) \leq Cd(f(x), f(y))$ for all x and y in X, then f is continuous. In this case, f is particularly called *Lipschitz continuous* with the *Lipschitz constant* C.

3.2 Normed spaces

banach space

3.3 Open sets and closed sets

convergence, limit point

3.4 Compact sets

3.5 Connected sets

Exercises

Part II

Real functions

Chapter 4

Continuous functions

4.1 Intermediate and extreme value theorems

4.2 Uniform convergence

Proof. Divide the error

$$|f(x_n) - f(x)| \leq |f(x_n) - f_m(x_n)| + |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|.$$

Using the uniform convergence, we can take m such that $\|f_m - f\| < \varepsilon$, so we have

$$|f(x_n) - f(x)| < \varepsilon + |f_m(x_n) - f_m(x)| + \varepsilon.$$

Then, taking $\limsup_{n\to\infty}$ on the both-hand sides, we get

$$\limsup_{n\to\infty} |f(x_n) - f(x)| \leq \varepsilon + 0 + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ has been arbitrarily taken,

$$\lim_{n\to\infty} |f(x_n) - f(x)| = 0.$$

□

4.3 Arzela-Ascoli theorem

4.4 Stone-Weierstrass theorem

Exercises

Problems

1. The set of local minima of a convex real function is connected.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The equation $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.
3. A continuous function that takes on no value more than twice takes on some value exactly once.
4. Let f be a function that has the intermediate value property. If the preimage of every singleton is closed, then f is continuous.

*5. If a sequence of real functions $f_n\colon [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \le |x - y|$ whenever $|x - y| \ge \frac{1}{n}$, then it has a uniformly convergent subsequence.

Chapter 5

Differentiable functions

5.1 Monotonicty and convexity

5.2 Mean value theorem

Darboux

5.3 Taylor theorem

5.4 Differentiable class

completeness

Exercises

5.1 (Variations on the mean value theorem)**.** Let f be a differentiable function on the unit closed interval.

(a) If $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

(b) If $f(0) = 0$ there is c such that $cf(c) = (1-c)f'(c)$.

5.2 (Convergence rates of recursive sequences)**.** If $a_{n+1} = a_n - f(a_n)$, $f(0) = 0$, $f(x) > 0$ for $0 < x < \varepsilon$, $f \in C^2$? then

$$f'(a_n) \sim \lim_{x\to 0+} \frac{f'(x)^2}{f''(x)f(x)} \frac{1}{n}.$$

Proof. $\square$

Problems

1. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} f'(x) = b$, then $a = 0$.
2. Let f be a real C^2 function with $f(0) = 0$ and $f''(0) \neq 0$. Defined a function ξ such that $f(x) = xf'(\xi(x))$ with $|\xi| \leq |x|$, we have $\xi'(0) = 1/2$.
3. Let f be a C^2 function such that $f(0) = f(1) = 0$. We have $\|f\| \leq \frac{1}{8}\|f''\|$.
4. A smooth function such that for each x there is n having the nth derivative vanish is a polynomial.

5. If a real C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.
6. Let a real function f be differentiable. For $a < a' < b < b'$ there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b') - f(a') = f'(c')(b' - a')$.

Chapter 6

Analytic functions

6.1 Power series

uniform convergence and absolute convergence, abel theorem? differentiation convergence of radius sum, product, composition, reciprocal? closed under uniform convergence

6.2 Complex analytic functions

complex domain (real analytic iff its domain contains real line) convergence of radius, revisited identity theorem

6.3 Special functions

hypergeometric, bessel, gamma, zeta

Exercises

Part III

Integration

Chapter 7

Riemann integral

7.1 Riemann integral

tagged partition

7.2 Henstock-Kurzweil intergral

bounded compact support <-> lebesgue

7.3 Improper integral

7.4 Fundamental theorem of calculus for continuous functions

Exercises

7.1. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

7.2. Find all $a > 0$ and $b > 0$ such that $\int_0^\infty x^{-b}|\tan x|^a\,dx$ converges.

Problems

*1. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

Chapter 8

Integrable functions

8.1

Chapter 9

Part IV

Multivariable Calculus

Chapter 10

Frećhet derivatives

10.1 Tangent spaces

10.1 (Vector fields).

10.2 Inverse function theorem

Chapter 11

Differential forms

11.1 Multilinear algebra

11.1 (Tensor product).

11.2 (Wedge product).

11.3 (One-forms).

11.4 (Multiple integral). volume forms, stone weierstrass and fubini

11.2 Vector calculus

11.5 (Exterior derivative).

11.6 (Musical isomorphisms).

11.7 (Inner product of differential forms). ONB

11.8 (Hodge star operator). Identification of 2-forms and vector fields

11.9 (Gradient, curl, and divergence).

11.10 (Potentials).

11.11 (Vector calculus identities).

Exercises

11.12 (Multivariable Taylor's theorem). Symmetric product

11.13 (Vector analysis in two dimension).

11.14 (Geometric algebra).

Chapter 12

Stokes theorems

12.1 Local coordinates

12.1 (Spherical coordinates)**.** Let $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = 0,\ y \geq 0\}$.

$$(x, y, z) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$$

for $(r, \theta, \varphi) \in (0, \infty) \times (0, \pi) \times (0, 2\pi)$. Orthonormal bases are

$$\left(\partial_r,\ \frac{1}{r}\partial_\theta,\ \frac{1}{r\sin\theta}\partial_\varphi\right),$$

$$(dr,\ r\,d\theta,\ r\sin\theta\,d\varphi),$$

$$(r^2 \sin\theta\,d\theta \wedge d\varphi,\ r\sin\theta\,d\varphi \wedge dr,\ r\,dr \wedge d\theta).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

Proof. Write df in the orthonormal basis

$$\begin{aligned} df &= \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial\theta}\,d\theta + \frac{\partial f}{\partial\varphi}\,d\varphi \\ &= \left(\frac{\partial f}{\partial r}\right) dr + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) r\,d\theta + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right) r\sin\theta\,d\varphi. \end{aligned}$$

After taking the Hodge star operator

$$\begin{aligned} * df &= \left(\frac{\partial f}{\partial r}\right) r^2 \sin\theta\,d\theta \wedge d\varphi + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) r\sin\theta\,d\varphi \wedge dr + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right) r\,dr \wedge d\theta \\ &= r^2 \sin\theta \frac{\partial f}{\partial r}\,d\theta \wedge d\varphi + \sin\theta \frac{\partial f}{\partial\theta}\,d\varphi \wedge dr + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\,dr \wedge \theta, \end{aligned}$$

the differential is computed as

$$\begin{aligned} d * df &= d\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) d\theta \wedge d\varphi + d\left(\sin\theta \frac{\partial f}{\partial\theta}\right) d\varphi \wedge dr + d\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\right) dr \wedge \theta \\ &= \left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right] dr \wedge d\theta \wedge d\varphi, \end{aligned}$$

so that we have

$$\begin{aligned}\Delta f = *d*df &= \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}\end{aligned}$$

□

12.2 Integration on curves and surfaces

12.2 (Line integral).

12.3 (Surface integral).

12.3 Stokes theorems

12.4 (Bump functions).

12.5 (Partition of unity).

12.6.