Differential Topology

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De Rham theory

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Chapter 3

Morse theory

3.1 Morse functions

Definition 3.1.1. Let M be a manifold. A *Morse function* is a smooth function $f : M \to \mathbb{R}$ such that all critical points are nondegenerate.

Proposition 3.1.2. *Let M be an embedded submanifold of $\mathbb{R}^n$. For almost every point $p \in \mathbb{R}^n$, the function $f : M \to \mathbb{R} : x \mapsto \|x - p\|^2$ is Morse.*

Proof. Suppose that $p \in \mathbb{R}^n$ makes f be not Morse so that it possesses a degenerate critical point. Note that the notation x can denote not only a point variable on M but also the embedding map $M \hookrightarrow \mathbb{R}^n$. Let $N \subset M \times \mathbb{R}^n$ be the normal bundle of the tangent bundle TM and define a map $\varphi : N \to \mathbb{R}^n$ such that $\varphi(x, y) = x + y$. We claim that the point $(x, p - x)$ is contained in N and φ is critical at this point if f is degenerate at x.

The differential of f is

$$df_x(v) = 2(x - p) \cdot dx(v) = 2(x - p) \cdot v,$$

so x is critical point if and only if $x - p$ is proportional to $T_x M$.

Let $\{x^i\}_{i=1}^m$ be orthonormal coordinates for M and let $\{e_j\}_{j=1}^{n-m}$ be an orthonormal frame field of N. Define coordinate functions $\{x^i, y^j\}$ on the manifold N by

$$x^i(x, y) := x^i(x), \quad \text{and} \quad y^j(x, y) := y \cdot e_j(x).$$

Then,

$$\left\{ \frac{\partial x}{\partial x^1}, \cdots, \frac{\partial x}{\partial x^m}, \frac{\partial y}{\partial y^1}, \cdots, \frac{\partial y}{\partial y^{n-m}} \right\}$$

always form an orthonormal basis on $\mathbb{R}^n$ and

Since

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial x}{\partial x^i} + \frac{\partial y}{\partial x^i} \quad \text{and} \quad \frac{\partial \varphi}{\partial y^j} = \frac{\partial y}{\partial y^j},$$

we have

$$\begin{aligned}
&\frac{\partial \varphi}{\partial x^i} \cdot \frac{\partial x}{\partial x^k} = \delta_{ik} - y \cdot \frac{\partial^2 x}{\partial x^i \partial x^k}, &\qquad &\frac{\partial \varphi}{\partial x^i} \cdot \frac{\partial y}{\partial y^l} = -y \cdot \frac{\partial^2 y}{\partial x^i \partial y^l}, \\
&\frac{\partial \varphi}{\partial y^j} \cdot \frac{\partial x}{\partial x^k} = 0, & &\frac{\partial \varphi}{\partial y^j} \cdot \frac{\partial y}{\partial y^l} = \delta_{jl}.
\end{aligned}$$

To represent $d\varphi(\partial_{x^1}, \cdots, \partial_{y^{n-m}})$ with matrix, we can write

$$\begin{pmatrix} \frac{\partial \varphi}{\partial x^i} \\ \frac{\partial \varphi}{\partial y^j} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x^k} & \frac{\partial y}{\partial y^l} \end{pmatrix} = \begin{pmatrix} \mathrm{id} - y \cdot \frac{\partial^2 x}{\partial x^i \partial x^k} & -y \cdot \frac{\partial^2 y}{\partial x^i \partial y^l} \\ 0 & \mathrm{id} \end{pmatrix}.$$

Then,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = 2\left(\mathrm{id} + (x-p) \cdot \frac{\partial^2 x}{\partial x^i \partial x^j}\right)$$

deduces that $d\varphi$ is not surjective at $(x, p-x)$. Therefore, by the Sard theorem, set of such p has measure zero. □

Proposition 3.1.3. *Let M be a manifold. The set of Morse functions is dense in $C^\infty(M)$.*

Proof. Let f be a smooth function on M. Embed M in $\mathbb{R}^{d-1}$ such that $x \mapsto (x_2, \cdots, x_d)$. Then, $x \mapsto (f(x), x_2, \cdots, x_d)$ gives an embedding into R^d. Define a sequence $\{\varepsilon_n\}_n \subset \mathbb{R}^n$ such that $\varepsilon_n \to 0$ and the sequence of functions

$$f_n(x) := \frac{\|x + ne_1 + \varepsilon_n\|^2 - n^2}{2n}$$

is Morse, where $\{e_i\}$ denotes the standard basis of $\mathbb{R}^d$. This can be done by the previous proposition. Then,

$$\begin{aligned} f_n(x) &= \frac{(f(x) + n + \varepsilon_n \cdot e_1)^2 + \cdots + (x_n + \varepsilon_n \cdot e_d)^2 - n^2}{2n} \\ &= f(x) + \frac{\|x + \varepsilon_n\|}{2n} + \varepsilon_n \cdot e_1 \end{aligned}$$

proves that $\|f_n - f\|_{C^k(K)} \to 0$ on every compact $K \subset M$. □

Theorem 3.1.4 (Morse lemma)**.** *Let p be a nondegenerate critical point of a Morse function f on a manifold M. Then, there exists a local chart (U, φ) of p such that*

$$f \circ \varphi^{-1}(x_1, \cdots, x_m) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

for some k. This chart is called Morse chart.

Proof. □

Corollary 3.1.5. *The critical points of a Morse function are isolated. In particular, on a compact manifold are finitely many critical points of a Morse function.*

3.2 Pseudo-gradients

Definition 3.2.1. Let f be a Morse function on a manifold M. A *pseudo-gradient* adapted to f is a vector field X such that

(a) $df(X) < 0$ at all noncritical points,

(b) there is a Morse chart at critical points in which $X = \operatorname{grad} f$, where the metric is induced from the chart.

Proposition 3.2.2. *A pseudo-gradient always exists for any Morse functions.*

Proof. Cover the manifold with charts such that every critical point is contained in a unique chart, which is Morse. For each chart (U, φ), we can define a vector field on U by

$$X := -d\varphi^{-1}(\operatorname{grad}(f \circ \varphi^{-1})),$$

using the standard metric on $\varphi(U)$. Then, we have

$$df(X) = -\langle \operatorname{grad}(f \circ \varphi^{-1}), \operatorname{grad}(f \circ \varphi^{-1}) \rangle \leq 0,$$

where the equality holds only at critical points. With a partition of unity, the vector fields are combined and easily checked to be pseudogradient. □

Definition 3.2.3. Let p be a critical point of a Morse function f on a manifold M. Denote $\varphi^s : M \to M$ by the flow of a pseudo-gradient. A *stable manifold* is defined as

$$W^s(p) := \{ x \in M : \lim_{s \to \infty} \varphi^s(x) = p \},$$

and an *unstable manifold* is defined as

$$W^u(p) := \{ x \in M : \lim_{s \to -\infty} \varphi^s(x) = p \}.$$

Proposition 3.2.4. *The stable manifolds and unstable manifolds are manifolds. Further, they are diffeomorphic open disks. Moreover, the index of p is equal to*

$$\dim W^u(p) = \operatorname{codim} W^s(p)$$

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Chapter 4

Chapter 5

Part III

Topological quantum field theory

Chapter 6

Chern-Simons theory

6.1 (Lie algebra-valued forms)**.** Let $\pi : P \to M$ be a smooth principal G-bundle for a compact Lie group G. The three vector bundles $P \times \mathfrak{g}$, TP, T^*P over P and their section spaces

$$\Gamma(P \times \mathfrak{g}) = \Omega^0(P, \mathfrak{g}), \qquad \Gamma(TP) = \mathfrak{X}(P), \qquad \Gamma(T^*P) = \Omega^1(P)$$

admit smooth right G-actions ad^{-1}, dR, $(dR^*)^{-1}$ respectively. The actions are not equivariant in the sense that they trivially act on the base space P. The right G-action on $\Omega^1(P, \mathfrak{g})$ is given in the identification $\Omega^1(P, \mathfrak{g}) := \Omega^0(P, \mathfrak{g}) \otimes_P \Omega^1(P)$ with the tensor product bundle as $\mathrm{ad}^{-1} \otimes (dR^*)^{-1}$.

$$((dR_h^*)^{-1}(dR_g^*)^{-1}\omega)(X) = \omega(dR_{g^{-1}} dR_{h^{-1}} X) = \omega(d(R_{(gh)^{-1}})X) = ((dR_{gh}^*)^{-1}\omega)(X)$$

notation: if F is a vector space, then

$$\Omega^k(M, F) := \Gamma(M \times F) \otimes_M \Omega^k(M),$$

and if E is a vector bundle over M, then

$$\Omega^k(M, E) := \Gamma(E) \otimes_M \Omega^k(M).$$

trace and determinant.

Consider the universal enveloping algebra $U(\mathfrak{g})$. Then, $\Omega(P, U(\mathfrak{g}))$ is an algebra bundle over P because it is the tensor product of two algebra bundles over P. Concretely, if we denote the multiplication of $\omega_1 \in \Omega^k(P, U(\mathfrak{g}))$ and $\omega_2 \in \Omega^l(P, U(\mathfrak{g}))$ by $\omega_1 \wedge \omega_2 \in \Omega^{k+l}(P, U(\mathfrak{g}))$, then it is computed in the geometric wedge product convention as

$$(\omega_1 \wedge \omega_2)(X_1, \cdots, X_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) \omega_1(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \omega_2(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)})$$

and we also define $[\omega_1, \omega_2] \in \Omega^{k+l}(P, U(\mathfrak{g}))$ such that

$$[\omega_1, \omega_2](X_1, \cdots, X_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) [\omega_1(X_{\sigma(1)}, \cdots, X_{\sigma(k)}), \omega_2(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)})]$$

and

$$d\omega(X_0,X_1,\cdots,X_k) := \frac{1}{k+1}\sum_{0\le i\le k}(-1)^i X_i(\omega(X_0,\cdots,\widehat{X}_i,\cdots,X_k)) \\ + \frac{1}{k+1}\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\cdots,\widehat{X}_i,\cdots,\widehat{X}_j,\cdots,X_k).$$

Let $\omega\in\Omega^1(P,\mathfrak{g})$. We can embed $\Omega(P,\mathfrak{g})\subset\Omega(P,U(\mathfrak{g}))$ to do computations. Then,

$$(\omega\wedge\omega)(X,Y)=\omega(X)\omega(Y)-\omega(Y)\omega(X)=[\omega(X),\omega(Y)]$$

and

$$[\omega,\omega](X,Y)=[\omega(X),\omega(Y)]-[\omega(Y),\omega(X)]=2[\omega(X),\omega(Y)]$$

imply that $\omega\wedge\omega=\frac{1}{2}[\omega,\omega]\in\Omega^2(P,\mathfrak{g})$. We also have

$$d\omega(X,Y)=\frac{1}{2}(X(\omega(Y))-Y(\omega(X)))+\frac{1}{2}\omega([X,Y]).$$

The coefficient conventions are not so important that coefficients are eventually cancelled when we write an equation of forms.

6.2 (Ehresmann connections)**.** Let $\pi:E\to M$ be a smooth fiber bundle. An *Ehresmann connection* on E is a vector subbundle $HE\to E$ of the tangent bundle TE such that $VE\oplus HE=TE$, where $VE\to E$ is defined by the kernel of $TE\to TM$. It is the choice of the splitting section of the exact sequence

$$0\to VE\to TE\to\pi^*TM\to 0.$$

This horizontal subbundle HE gives rise to a surjective linear bundle map $TE\to VE$ to the vertical subbundle, and also a cartesian square

$$\begin{tikzcd} HE \arrow[r, "d\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TM \arrow[d] \\ E \arrow[r, "\pi"] & M, \end{tikzcd}$$

so that we have a Lie algebra bundle map $\mathfrak{X}(M)\to\mathfrak{X}(E)$ with horizontal image, called the *horizontal lift*.

parallel transport

6.3 (Connection forms)**.** Let $\pi:P\to M$ be a smooth principal G-bundle. Keep in mind that the vertical bundle VP is canonically trivialized by the right G-equivariant bundle map $P\times\mathfrak{g}\to VP\subset TP$ constructed by the image of the injective linear bundle map

$$P\times\mathfrak{g}\to T(P\times G)\to TP,\qquad\text{over } P,$$

where the first map is $P\times\mathfrak{g}\to T(P\times G):(u,A)\mapsto((u,e),(0,A))$ and the second map is the differential $T(P\times G)\to TP$ of the principal action $P\times G\to P$. This trivialization induces a right G-equivariant P-linear embedding $\Omega^0(P,\mathfrak{g})\to\mathfrak{X}(P)$, and for a constant section $A\in\Omega^0(P,\mathfrak{g})$ we can assign the *fundamental vector field* $A^\#\in\mathfrak{X}(P)$.

A *connection form* is simply a right G-equivariant left inverse of this fundamental vector field map. More precisely, it is defined as a Lie algebra-valued 1-form $\omega\in\Omega^1(P,\mathfrak{g})$ or a P-linear map $\omega:\mathfrak{X}(P)\to\Omega^0(P,\mathfrak{g})$ which is

(i) right G-invariant in the sense that $\omega(dR_gX)=\mathrm{ad}_g^{-1}(\omega(X))\in\Omega^0(P,\mathfrak{g})$ for all $X\in\mathfrak{X}(P)$,

(ii) vertical in the sense that $\omega(A^{\#}) = A$ for constant sections $A \in \mathfrak{g}$.

A connection form decompose the vector bundle TP into the direct sum of VP and the kernel, which gives rise to the corresponding horizontal subbundle. A connection form ω projects $X \in \mathfrak{X}(P)$ to the vertical one.

(a) We can also define a connection as an right G-invariant Ehresmann connection $HP \to P$.

Proof. (a) Since a right G-equivariant Ehresmann connection gives rise to a linear map $TP \to VP$, so by composition with $VP \cong P \times \mathfrak{g}$, we get the corresponding connection form $TP \to P \times \mathfrak{g}$. □

6.4. Note that G is itself a principal G-bundle over a point. There is a natural connection form $\omega_G \in \Omega^1(G, \mathfrak{g})$ called the *Maurer-Cartan form*, defined such that $\omega_G : \mathfrak{X}(G) \to \Omega^0(G, \mathfrak{g}) : X \mapsto (g \mapsto dL_g^{-1}(X|_g))$. The right G-invariance of the Maurer-Cartan form ω_G is due to

$$\omega_G(dR_h(v)) = dL_{gh}^{-1} dR_h(v) = dL_h^{-1} dR_h dL_g^{-1}(v) = \operatorname{ad}_h^{-1} \omega_G(v), \qquad v \in T_g G,\ h \in G.$$

We can check ω_G is a connection form.

6.5 (Curvature form)**.** Let P be a principal G-bundle over M. The *curvature form* of the connection form $\omega \in \Omega^1(P, \mathfrak{g})$ can be defined either the covariant derivative of ω or the Cartan structural equation $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P, \mathfrak{g})$.

The curvature form is horizontal in the sense that for every vertical vector field $X \in \mathfrak{X}(P)$ we have $\iota_X \Omega = 0 \in \Omega^1(P, \mathfrak{g})$.

6.6 (Exterior covariant derivatives)**.** Let $\pi : P \to M$ be a smooth principal G-bundle. Let F be a faithful representation of G. We have a right G-equivariant trivial vector bundle $P \times F$ over P, where the action is given such that $(p, f)g = (pg, g^{-1}f)$, and the associated vector bundle $E := P \times_G F$ over M.

We say a F-valued form $\omega \in \Omega^k(P, F)$ is *horizontal* if $\iota_X \omega = 0$ for vertical $X \in \mathfrak{X}(P)$. We have

$$\Omega^k(M, E) \cong \Omega_h^k(P, F)^G,$$

where $\Omega_h^k(P, F)^G$ denotes the set of right G-invariant horizontal F-valued k-forms on P, since we have a cartesian square

$$\begin{tikzcd} P \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ P \arrow[r] & M. \end{tikzcd}$$

The exterior derivative $d : \Omega^k(P, F)^G \to \Omega^{k+1}(P, F)^G$ does not preserve horizontality in general. For a connection form $\omega \in \Omega^1(P, \mathfrak{g})^G$, we can define the *exterior covariant derivative* $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ is defined such that we have a commutative diagram

$$\begin{tikzcd} \Omega^k(M, E) \arrow[r, "\nabla"] \arrow[d, equal] & \Omega^{k+1}(M, E) \arrow[d, equal] \\ \Omega_h^k(P, F)^G \arrow[r, "d_\omega"] & \Omega_h^{k+1}(P, F)^G, \end{tikzcd}$$

where d_ω is defined by

$$d_\omega \psi(X_0, \cdots, X_k) := d\psi(HX_0^*, \cdots, HX_k^*),$$

where $HX_i = X_i - \omega(X_i) \in \mathfrak{X}(P)$ are horizontal components of $X_i \in \mathfrak{X}(P)$. The most important case is $k = 0$.

$$\nabla_X s =$$

6.7 (Local expression of principal connections)**.** Let $\pi : P \to M$ be a smooth principal G-bundle, where G is a compact Lie group. Then, we can fix $\{\varphi_\alpha\}$ be a local trivialization of π such that $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is right G-equivariant. Consider a bundle map

$$(\Omega^1(U_\alpha, \mathfrak{g}), \mathrm{ad}^{-1} \otimes \mathrm{id}) \to (\Omega^1(\pi^{-1}(U_\alpha), \mathfrak{g}), \mathrm{ad}^{-1} \otimes (dR^*)^{-1}) : A \mapsto \varphi_\alpha^*(\mathrm{pr}_2^*(\omega_G) + \mathrm{pr}_1^*(A))$$

along the map $\pi^{-1}(U_\alpha) \to U_\alpha$, where $A \in \Omega^1(U_\alpha, \mathfrak{g})$ and the vertical term ω_G denotes the Maurer-Cartan form. In the physical contexts of gauge theory such as the standard model, the 1-form $A \in \Omega^1(U_\alpha, \mathfrak{g})$ is mainly used to describe principal connections.

(a) The above local representation is a right G-equivariant affine bundle map.

(b) The above local representation is injective and the image is exactly the set of connection forms on the trivial principal G-bundle $\pi^{-1}(U_\alpha)$ over U_α.

(c) What is the meaning of the expression $\nabla = d + A$?

(d) Local expression of $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$?

(e) What is the meaning of the expression $dA \wedge A + \frac{2}{3} A \wedge A \wedge A$ and how can we write it in terms of principal connections.

Proof. (a) We interpret $A \in \Omega^1(U_\alpha, \mathfrak{g})$ as $A : \mathfrak{X}(U_\alpha) \to \Omega^0(U_\alpha, \mathfrak{g})$ so that for every $X \in \mathfrak{X}(U_\alpha)$,

(c)
Let $L \to M$ be a trivial line bundle.
Let $s \in \Omega^0(M, L)$, so that $ds : \Omega^1(M, L)$ or

$$ds : \mathfrak{X}(M) \to \Omega^0(M, L) : X \mapsto X^\mu \partial_\mu s.$$

Let $A \in \Omega^1(M, \mathfrak{u}(1))$ or

$$A : \mathfrak{X}(M) \to \Omega^0(M, \mathfrak{u}(1)) : X \mapsto X^\mu A_\mu.$$

We have

$$\nabla_X s = (\nabla s)(X) = ((d + A)s)(X) = X^\mu \partial_\mu s + X^\mu A_\mu s$$

□

$$S := \frac{k}{4\pi} \int_M \mathrm{tr}\left(dA \wedge A + \frac{2}{3} A \wedge A \wedge A \right)$$

In this case, the field equation is $F = 0$.

6.1 Chern-Weil theory

$$(\mathrm{Sym}^n \mathfrak{g}^*)^G \cong (\mathrm{Sym}^n \mathfrak{t}^*)^W \cong H^{2n}(BG, \mathbb{R}).$$

Given a principal G-bundle $P \to X$, this isomorphism induces a ring homomorphism

$$\mathrm{CW} : (\mathrm{Sym}^* \mathfrak{g}^*)^G \to H^{\mathrm{ev}}(X, \mathbb{R}),$$

called the *Chern-Weil homomorphism*. In fact, when X is a smooth manifold, then there is a direct construction of the Chern-Weil homomorphism using connections. Choose any connection ω on P, and let Ω be the curvature.

$$(\mathrm{Sym}^n\,\mathfrak{g}^*)^G \otimes \Omega^{2n}(P, \mathfrak{g}^{\otimes n}) \to \Omega^{2n}(P)$$

6.2 Chern-Simons invariants

6.3 Differential cohomology

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