

# Functional Analysis

Ikhan Choi

November 23, 2022

# Contents

<b>I</b>	<b>Topological vector spaces</b>	<b>3</b>
<b>1</b>	<b>Locally convex spaces</b>	<b>4</b>
1.1	Vector topologies . . . . .	4
1.2	Seminorms and convex sets . . . . .	4
1.3	Continuous linear functionals . . . . .	4
1.4	Hahn-Banach theorem . . . . .	4
<b>2</b>	<b>Barreled spaces</b>	<b>6</b>
2.1	Uniform boundedness principle . . . . .	6
2.2	Baire category theorem . . . . .	6
2.3	Open mapping theorem . . . . .	7
<b>3</b>	<b>Weak topologies</b>	<b>9</b>
3.1	Dual spaces . . . . .	9
3.2	Weak compactness . . . . .	10
3.3	Weak density . . . . .	10
3.4	Krein-Milman theorem . . . . .	10
3.5	Polar topologies . . . . .	10
<b>II</b>	<b>Banach spaces</b>	<b>12</b>
<b>4</b>	<b>Fréchet, Banach, Hilbert spaces</b>	<b>13</b>
4.1	Banach spaces . . . . .	13
4.2	Hilbert spaces . . . . .	13
<b>5</b>	<b>Bounded linear operators</b>	<b>14</b>
<b>6</b>	<b>Compact operators</b>	<b>15</b>
6.1	Finite-rank operators . . . . .	15
6.2	Fredholm operators . . . . .	15
6.3	Nuclear operators . . . . .	15
<b>III</b>	<b>Spectral theory</b>	<b>17</b>
<b>7</b>	<b>Normal operators</b>	<b>18</b>
7.1	Spectral theorem for compact normal operators . . . . .	18
7.2	Spectral theorem for bounded normal operators . . . . .	18
7.3	Operator topologies . . . . .	18

<b>8</b>	<b>Unbounded operators</b>	<b>20</b>
<b>9</b>	<b>Toeplitz operators</b>	<b>21</b>
<b>IV</b>	<b>Operator algebras</b>	<b>22</b>
<b>10</b>	<b>Banach algebras</b>	<b>23</b>
10.1	Spectra . . . . .	23
10.2	Ideals . . . . .	24
10.3	Gelfand theory . . . . .	25
10.4	Holomorphic functional calculus . . . . .	25
<b>11</b>	<b><math>C^*</math>-algebras</b>	<b>28</b>
11.1	$C^*$ identity . . . . .	28
11.2	Continuous functional calculus . . . . .	29
11.3	Positivity in $C^*$ -algebras . . . . .	30
11.4	Representations of $C^*$ -algebras . . . . .	30
<b>12</b>	<b>Von Neumann algebras</b>	<b>32</b>
12.1	Density theorems . . . . .	32
12.2	Borel functional calculus . . . . .	32
12.3	Factors and traces . . . . .	33

## **Part I**

# **Topological vector spaces**

# Chapter 1

## Locally convex spaces

### 1.1 Vector topologies

1.1 (Canonical uniformity and bornology).

1.2 (Metrizability). Birkhoff-Kakutani

1.3 (Boundedness of linear operators).

### 1.2 Seminorms and convex sets

1.4 (Seminorms).

$$\bigcap_{i=1}^m \{p_i < 1\}$$

Equivalent conditions on the continuity of seminorms

*Proof.*

□

boundedness by seminorms, normability

### 1.3 Continuous linear functionals

1.5. Let  $\bar{x}^* = (x_1^*, \dots, x_n^*) \in X^{*n}$ .  $\bar{x}^* : X \rightarrow \mathbb{F}^n$ . If  $x^* \in X^*$  vanishes on  $\bigcap_{i=1}^n \ker x_i^*$ , then  $x^*$  is a linear combination of  $\{x_i^*\}$ .

### 1.4 Hahn-Banach theorem

1.6 (Hahn-Banach theorem). Let  $X$  be a real vector space. Suppose  $V$  is a linear subspace of  $X$  and  $l : V \rightarrow \mathbb{R}$  is a linear functional dominated by a sublinear functional  $q : X \rightarrow \mathbb{R}$ , that is,  $l(v) \leq q(v)$  for all  $v \in V$ .

- (a) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$ .
- (b) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$  and is dominated by  $q$  if  $\dim X/V = 1$ .
- (c) There is a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  that extends  $l$  and is dominated by  $q$ .

*Proof.* (a) It can be done by the Hamel basis.

(b) Let  $e \in X \setminus V$  so that every vector  $x \in X$  can be uniquely written as  $x = v + te$  with  $v \in V$  and  $t \in \mathbb{R}$ . For  $v_1, v_2 \in V$ ,

$$l(v_1) + l(v_2) = l(v_1 + v_2) \leq q(v_1 + v_2) \leq q(v_1 - e) + q(v_2 + e)$$

implies

$$l(v_1) - q(v_1 - e) \leq -l(v_2) + q(v_2 + e).$$

Define a linear functional  $\tilde{l} : X \rightarrow \mathbb{R}$  such that  $\tilde{l}(v) = l(v)$  and

$$l(v) - q(v - e) \leq \tilde{l}(e) \leq -l(v) + q(v + e)$$

for all  $v \in V$ . Since

$$\tilde{l}(v + te) = l(v) + t\tilde{l}(e) \leq l(v) + t(-l(v) + q(v + e)) = q(v + te)$$

if  $t \geq 0$  and

$$\tilde{l}(v + te) = l(v) + t\tilde{l}(e) \leq l(v) + t(l(-t^{-1}v) - q(-t^{-1}v - e)) = q(v + te)$$

if  $t \leq 0$ , we have  $\tilde{l}(x) \in q(x)$  for all  $x \in X$ .

(c) With  $X$  and  $q$ , Consider a partially ordered set

$$\{(\tilde{V}, \tilde{l}) \mid V \leq \tilde{V} \leq X, \tilde{l} : \tilde{V} \rightarrow \mathbb{R} \text{ is a linear extension of } l \text{ dominated by } q\}$$

such that  $(V_1, l_1) \prec (V_2, l_2)$  if and only if  $V_1 \leq V_2$  and  $l_2|_{V_1} = l_1$ . The nonemptiness and the chain condition is easily satisfied, so it has a maximal element  $(\tilde{V}, \tilde{l})$  by the Zorn lemma. By the part (b), we have  $\tilde{V} = X$ .  $\square$

**1.7 (Complex linear functionals).** Let  $X$  be a complex vector space. Consider a map

$$\begin{array}{ccc} \{\mathbb{C}\text{-linear functionals on } X\} & \rightarrow & \{\mathbb{R}\text{-linear functionals on } X\} \\ l & \mapsto & \operatorname{Re} l. \end{array}$$

Let  $p$  be a seminorm on  $X$  and  $l$  a complex linear functional on  $X$ .

(a) The above map is bijective.

(b)  $|l(x)| \leq p(x)$  if and only if  $|\operatorname{Re} l(x)| \leq p(x)$ .

*Proof.* (b) There is  $\lambda$  such that  $|\lambda| = 1$  and  $l(\lambda x) \geq 0$ . Then,

$$|l(x)| = |\lambda^{-1}l(\lambda x)| = l(\lambda x) = \operatorname{Re} l(\lambda x) \leq p(\lambda x) = p(x).$$

$\square$

**1.8 (Applications of Hahn-Banach theorem).**

## Exercises

**1.9 (Topology of compact convergence).**

## Chapter 2

# Barreled spaces

### 2.1 Uniform boundedness principle

**2.1** (Barreled spaces). Let  $X$  be a topological vector space. A *barrel* is an absorbing, balanced, convex, and closed subset of  $X$ . A *barreled space* is a topological space in which every barrel is a neighborhood of zero.

**2.2** (Uniform boundedness principle). Let  $X$  and  $Y$  be topological vector spaces. Let  $\mathcal{F}$  be a family of continuous linear operator from  $X$  to  $Y$ . Suppose  $\bigcup_{T \in \mathcal{F}} Tx$  is bounded for each  $x \in D$ , where  $D \subset X$ .

- (a) If  $D$  is dense in  $X$ , then  $\bigcap_{T \in \mathcal{F}} T^{-1}\overline{U}$  is absorbing.
- (b) If  $X$  is barreled, then  $\mathcal{F}$  is equicontinuous.

### 2.2 Baire category theorem

**2.3** (Baire spaces). A topological space is called a *Baire space* if the countable intersection of open dense subsets is always dense.

- (a) If a topological vector space is Baire, then it is barreled.
- (b) A Baire space is second category in itself.
- (c) A topological group that is second category in itself is Baire.

**2.4** (Absorbing sets). Let  $X$  be a topological vector space that is Baire. A subset  $U \subset X$  is said to be *absorbing* if for every  $x \in X$  there is a sufficiently large  $t > 0$  such that  $x \in tU$ . Let  $U \subset X$ .

- (a) If  $U$  is closed and absorbing, then  $U$  has non-empty interior.
- (b) If  $U$  is closed and absorbing, then  $U - U$  is a neighborhood of zero.
- (c) If  $U$  is closed, convex, and absorbing, then  $U$  is a neighborhood of zero.

**2.5** (Baire category theorem). The Baire category theorem proves many examples of topological vector space are Baire, in particular barreled.

- (a) A complete metric space is Baire.
- (b) A locally compact Hausdorff space is Baire.

## 2.3 Open mapping theorem

**2.6** (Open mapping theorem). Let  $X$  be a  $F$ -space and  $Y$  a barreled space. Suppose  $T : X \rightarrow Y$  is a continuous and surjective linear operator. Let  $B$  be an open neighborhood of zero in  $X$ .

(a)  $\overline{TB}$  is a neighborhood of zero.

(b)  $TB$  is a neighborhood of zero.

*Proof.* (a) There is an open neighborhood  $U$  of zero such that  $U - U \subset B$ . The set  $\overline{TU}$  is a closed absorbing set because  $T$  is surjective. Since  $Y$  is barreled,  $\overline{TU}$  has a non-empty interior in  $Y$ . Thus,  $\overline{TB} \supset \overline{TU} - \overline{TU}$  is a neighborhood of zero.

(b) Since  $X$  is metrizable, we have a sequence of open neighborhoods  $B_n := \{x : d(x, 0) < 2^{-n}\}$ , where the topology of  $X$  is induced from a metric  $d$ . We claim  $\overline{TB_1} \subset TB_0$ . Take  $y_1 \in \overline{TB_1}$ .

If  $y_n \in \overline{TB_n}$ , then since  $\overline{TB_{n+1}}$  are neighborhoods of zero, we have

$$TB_n \cap (y_n + \overline{TB_{n+1}}) \neq \emptyset.$$

So we can inductively construct sequences  $x_n \in B_n$  and  $y_n \in \overline{TB_n}$  for  $n \geq 2$  such that

$$x_n \in B_n \cap T^{-1}(y_n + \overline{TB_{n+1}})$$

and

$$y_{n+1} := Tx_n - y_n.$$

Then,  $\sum_{n=1}^{\infty} x_n$  converges to  $x \in B_0$ . Therefore,

$$Tx = \sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} (y_{n+1} - y_n) = y_1. \quad \square$$

## Exercises

**2.7.** Let  $(T_n)$  be a sequence in  $B(X, Y)$ . If  $T_n$  converges strongly then  $\|T_n\|$  is bounded by the uniform boundedness principle.

**2.8.** There is a closed absorbing set in  $\ell^2(\mathbb{Z}_{\geq 0})$  that is not a neighborhood of zero;

$$\overline{B}(0, 1) \setminus \bigcup_{i=2}^{\infty} B(i^{-1}e_i, i^{-2})$$

is a counterexample.

**2.9.** There is no metric  $d$  on  $C([0, 1])$  such that  $d(f_n, f) \rightarrow 0$  if and only if  $f_n \rightarrow f$  pointwise as  $n \rightarrow \infty$  for every sequence  $f_n$ . Note that this problem is slightly different to the non-metrizability of the topology of pointwise convergence.

**2.10.** We show that there is no projection from  $\ell^\infty$  onto  $c_0$ .

**2.11** (Schur property).  $\ell^1$

**2.12.** Let  $\varphi : L^\infty([0, 1]) \rightarrow \ell^\infty(\mathbb{N})$  be an isometric isomorphism. Suppose  $\varphi$  is realised as a sequence of bounded linear functionals on  $L^\infty$ .

(a) Show that  $\varphi^*(\ell^1) \subset L^1$  where  $\ell^1$  and  $L^1$  are considered as closed linear subspaces of  $(\ell^\infty)^*$  and  $(L^\infty)^*$  respectively.



- (b) Show that  $\varphi^*$  is indeed an isometric isomorphism, and deduce  $\varphi$  cannot be realised as bounded linear functionals on  $L^\infty$ .

**2.13** (Daugavet property). (a) The real Banach space  $C([0, 1])$  satisfies the Daugavet property.

*Proof.* Let  $T$  be a finite rank operator on  $C([0, 1])$ , and  $e_i$  be a basis of  $\text{im } T$ . Then, for some measures  $\mu_i$ ,

$$Tf(t) = \sum_{i=1}^n \int_0^1 f \, d\mu_i e_i(t).$$

Let  $M := \max \|e_i\|$ .

Take  $f_0$  such that  $\|f_0\| = 1$  and  $\|Tf_0\| > \|T\| - \frac{\varepsilon}{2}$ . Reversing the sign of  $f_0$  if necessary, take an open interval  $\Delta$  such that  $Tf_0(t) \geq \|T\| - \frac{\varepsilon}{2}$  and  $|\mu_i|(\Delta) \leq \frac{\varepsilon}{4nM}$  for all  $i$ . Define  $f_1$  such that  $f_0 = f_1$  on  $\Delta^c$ ,  $f_1(t_0) = 1$  for some  $t_0 \in \Delta$ , and  $\|f_1\| = 1$ . Then,  $\|Tf_1 - Tf_0\| \leq \frac{\varepsilon}{2}$  shows  $Tf_1 \geq \|T\| - \varepsilon$  on  $\Delta$ . Therefore,

$$\|1 + T\| \geq \|f_1 + Tf_1\| \geq f_1(t_0) + Tf_1(t_0) \leq 1 + \|T\| - \varepsilon.$$

□

## Problems

**2.14.** Let  $T$  be an invertible linear operator on a normed space. Then,  $T^{-2} + \|T\|^{-2}$  is injective if it is surjective.

# Chapter 3

## Weak topologies

### 3.1 Dual spaces

3.1 (Bidual).

3.2. Let  $X$  be a locally convex space. The *weak topology* is the topology  $w$  on  $X$  defined by the family of seminorms  $\{x \mapsto |\langle x, \xi \rangle|\}_{\xi \in X^*}$ . The *weak\* topology* is the topology  $w^*$  on  $X^*$  defined by the family of seminorms  $\{\xi \mapsto |\langle x, \xi \rangle|\}_{x \in X}$ . Let  $J : X \rightarrow X^{**}$  be the canonical embedding.

- (a)  $(X, w)$  and  $(X^*, w^*)$  are locally convex.
- (b)  $(X, w)^* = X^*$ .
- (c)  $(X^*, w^*)^* = X$ . Every locally convex space is a dual of a locally convex space.

*Proof.* (a) The Hahn-Banach theorem implies the Hausdorffness.

(c) We will only show  $(X^*, w^*)^* \subset X$ . If  $u \in (X^*, w^*)^*$ , then there are  $x_1, \dots, x_m \in X$  such that

$$|\langle u, \xi \rangle| \leq \sum_{i=1}^m |\langle x_i, \xi \rangle|$$

for all  $\xi \in X^*$ . If we let  $\ker \vec{x} := \bigcap_{i=1}^m \ker x_i$ , then it is a closed subspace of  $X^*$  such that  $\ker \vec{x} \subset \ker u$ , so we have  $u \in \text{span } \vec{x} \subset X$ . □

### 3.3. closure and weak closure of convex subsets

*Proof.* Hahn-Banach □

### 3.4 (Polar).

boundedness, incompleteness

3.5 (Weak convergence by dense set). Let  $X$  be a Banach space,  $D^*$  a subset of  $X^*$ , and  $\overline{D^*}$  the norm closure of  $D^*$ . For example, if  $X$  has a predual  $X_* \subset X^*$  and  $D^*$  is dense in  $X_*$ , then  $\sigma(X, \overline{D^*})$  is the weak\* topology.

- (a) There is a sequence  $x_n \in X$  converges to zero in  $\sigma(X, D^*)$  but not in  $\sigma(X, \overline{D^*})$ .
- (b) A bounded sequence  $x_n \in X$  converges to zero in  $\sigma(X, \overline{D^*})$  if in  $\sigma(X, D^*)$ .

*Proof.* (b) Let  $\xi \in \overline{D^*}$  and choose  $\eta \in D^*$  such that  $\|\xi - \eta\| < \varepsilon$ . Then,

$$|\langle x_n, \xi \rangle| \leq \|x_n\| \|\xi - \eta\| + |\langle x_n, \eta \rangle| \lesssim \varepsilon + |\langle x_n, \eta \rangle| \rightarrow \varepsilon.$$

□

## 3.2 Weak compactness

3.6 (Banach-Alaoglu theorem).

3.7 (Eberlein-Šmulian theorem).

3.8 (James' theorem).

## 3.3 Weak density

Bishop-Phelps theorem

3.9 (Goldstine's theorem). Let  $X$  be a Banach space and  $J : X \rightarrow X^{**}$  the canonical embedding. Our claim is that  $\overline{B}$  is weak\*-dense in  $\overline{B}_{X^{**}}$ . Let  $x_0^{**} \in X^{**}$  with  $\|x_0^{**}\| \leq 1$ , and let

$$\bigcap_{i=1}^m \{x^{**} \in X^{**} : |\langle x^{**} - x_0^{**}, x_i^* \rangle| < \varepsilon\}$$

be an open weak\*-neighborhood of zero in  $X^{**}$  with  $\|x_i^*\| \leq 1$  and  $\varepsilon > 0$ . Let

$$S := \bigcap_{i=1}^m \{x \in X : \langle x, x_i^* \rangle = \langle x_0^{**}, x_i^* \rangle\}.$$

- (a)  $S$  is not empty.
- (b)  $S \cap (1 + \varepsilon)\overline{B}_X$  is not empty for all  $\varepsilon > 0$ .
- (c)  $\overline{B}_X$  is weak\*-dense in  $\overline{B}_{X^{**}}$

*Proof.* (a)

(b) From the part (a), we have  $x \in S$ . Suppose  $S$  does not intersect  $(1 + \varepsilon)\overline{B}_X$ . By the Hahn-Banach theorem, there is  $y^* \in X^*$  such that

$$y^*|_{S-x} = 0, \quad \langle x, y^* \rangle > 1 + \varepsilon, \quad \text{and} \quad \|y^*\| = 1.$$

Since  $S - x = \bigcap_{i=1}^m \ker x_i^*$ , the linear functional  $y^*$  is a linear combination of  $x_1^*, \dots, x_m^*$ , so we have

$$1 + \varepsilon < \langle x, y^* \rangle = \langle x_0^{**}, y^* \rangle \leq \|x_0^{**}\| \|y^*\| \leq 1.$$

(c) Take  $\varepsilon > 0$  such that  $\varepsilon \max_{1 \leq i \leq m} \|x_i^*\| < 1$ . By the part (b), there is  $y \in X$  such that  $\|y\| \leq 1 + \varepsilon$  and  $\langle y, x_i^* \rangle = \langle x_0^{**}, x_i^* \rangle$ . If we let  $x := (1 + \varepsilon)^{-1}y$ , then  $x \in \overline{B}_X$  so that

$$|\langle x - x_0^{**}, x_i^* \rangle| = |\langle x - y, x_i^* \rangle| = |\langle \varepsilon x, x_i^* \rangle| \leq \varepsilon \|x\| \|x_i^*\| < \varepsilon$$

for all  $i$ . □

## 3.4 Krein-Milman theorem

Choquet theory

## 3.5 Polar topologies

Mackey-Arens

## Exercises

**3.10** (James' space). not reflexive but isometrically isomorphic to bidual

**3.11** (Predual correspondence). Let  $X$  be a Banach space. Let

$$\{(Y, \varphi) \mid \varphi : X \rightarrow Y^* \text{ is an isometric isomorphism}\}$$

and

$$\{Z \leq X^* \mid \overline{B_X} \text{ is compact Hausdorff in } (X, \sigma(X, Z))\}.$$

$$(Y, \varphi) \mapsto \text{im } \varphi^*|_{J(Y)}$$

- (a) The map is well-defined.
- (b) The map is surjective. (by Goldstein)
- (c) The map is injective up to isomorphism for  $Y$ .

**3.12.** Let  $X$  be a closed subspace of a Banach space  $Y$  and

$$i : X \rightarrow Y$$

the inclusion. Suppose  $X$  and  $Y$  have preduals  $X_*$  and  $Y_*$  respectively. Let

$$j := i^*|_{Y_*} : Y_* \rightarrow Z \subset X^*,$$

where  $Z := i^*(Y_*)^\perp$ . Then we can show

$$j^* : Z^* \subset X^{**} \rightarrow Y$$

coincides with  $i$  on  $X \cap Z^*$ . From the existence of  $X_*$  we have  $X^{**} \rightarrow X$ , which is restricted to define a map  $k : Z^* \rightarrow X$ .

$$\begin{array}{ccccc} & & X & \xrightarrow{i} & Y \\ & \nearrow & \uparrow k & \nearrow j & \\ X^{**} & \longrightarrow & Z^* & & \end{array}$$

We can show  $k$  is an isomorphism so that we have

$$X_* \cong Y_*/Y_* \cap \ker(i^*).$$

**3.13** (Mazur's lemma).

## **Part II**

# **Banach spaces**

## Chapter 4

# Fréchet, Banach, Hilbert spaces

### 4.1 Banach spaces

dual is Banach. Basis problem, Mazur' duck.

### 4.2 Hilbert spaces

Projections. Reducing subspaces. Hilbert space classification by cardinal. Riesz representation theorem.

4.1. (a) A Banach space  $X$  is isometrically isomorphic to a Hilbert space if there is a bounded linear projection on every closed subspace of  $X$ .

4.2 (Riesz representation theorem). Let  $H$  be a Hilbert space over a field  $\mathbb{F}$ , which is either  $\mathbb{R}$  or  $\mathbb{C}$ .

We use the bilinear form  $\langle -, - \rangle : X \times X^* \rightarrow \mathbb{F}$  of canonical duality. *Dirac* notation  $\langle - | - \rangle$  for the inner product of a complex Hilbert spaces such that  $\langle x, y \rangle = \langle y | x \rangle$ . The Riesz representation theorem states that a continuous linear functional on a Hilbert space is represented by the inner product with a vector.

(a) For each  $x^* \in H^*$ , there is a unique  $x \in H$  such that  $\langle y, x^* \rangle = \langle y, x \rangle$  for every  $y \in H$ .

(b)  $H \rightarrow H^* : x \mapsto \langle -, x \rangle$  is a natural linear and anti-linear isomorphism if  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{C}$ , respectively.

### Exercises

## Chapter 5

# Bounded linear operators

**5.1** (Bounded belowness in Banach spaces). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ . The following statements are equivalent:

- (a)  $T$  is bounded below.
- (b)  $T$  is injective and has closed range.
- (c)  $T$  is a topological isomorphism onto its image.

**5.2** (Bounded belowness in Hilbert spaces). Let  $T \in B(H, K)$  for Hilbert spaces  $H$  and  $K$ . The following statements are equivalent:

- (a)  $T$  is bounded below.
- (b)  $T$  is left invertible.
- (c)  $T^*$  is right invertible.
- (d)  $T^*T$  is invertible.

**5.3** (Injectivity and surjectivity of adjoint). Let  $T \in B(X, Y)$  for Banach spaces  $X$  and  $Y$ .

- (a)  $T^*$  is injective if and only if  $T$  has dense range.
- (b)  $T^*$  is surjective if and only if  $T$  is bounded below.

**5.4** (Normal operators). For  $T \in B(H)$ , we have an obvious fact  $(\text{im } T)^\perp = \ker T^*$ . Suppose  $T$  is normal.

- (a)  $\ker T = \ker T^*$ .
- (b)  $T$  is bounded below if and only if  $T$  is invertible.
- (c) If  $T$  is surjective, then  $T$  is invertible.

**5.5** (Invariant and Reducing subspaces). Let  $K$  be a closed subspace of  $H$ .

- (a)  $K$  is reducing for  $T$  if and only if  $K$  is invariant for  $T$  and  $T^*$ .
- (b)  $K$  is reducing for  $T$  if and only if  $TP = PT$ , where  $P$  is the orthogonal projection on  $K$ .

## Chapter 6

# Compact operators

$K(X, Y)$  is closed in  $B(X, Y)$ .  $K(X)$  is an ideal of  $B(X)$ . adjoint is  $K(X, Y) \rightarrow K(Y^*, X^*)$ . integral operators are compact. riesz operator, quasi-nilpotent operator.

### 6.1 Finite-rank operators

### 6.2 Fredholm operators

**6.1.** A bounded linear operator  $T : X \rightarrow Y$  between Banach spaces is called a *Fredholm* operator if its kernel is finite dimensional and its range is finite codimensional.

(a) A Fredholm operator  $T$  has closed range.

*Proof.* (a) Let  $C$  be a finite dimensional subspace of  $Y$  such that  $\text{im } T \oplus C = Y$ . Let  $\tilde{T} : X / \ker T \rightarrow Y$  be the induced operator of  $T$ . Define  $S : (X / \ker T) \oplus C \rightarrow Y$  such that  $S(x + \ker T, c) := \tilde{T}(x + \ker T) + c$ . Then,  $S$  is a topological isomorphism between Banach spaces by the open mapping theorem, so  $S(X / \ker T \oplus \{0\}) = \text{im } \tilde{T} = \text{im } T$  is closed.  $\square$

**6.2** (Atkinson's theorem). An operator  $T \in B(X, Y)$  is Fredholm if and only if there is  $S \in B(Y, X)$  such that  $TS - I$  and  $ST - I$  is finite rank.

**6.3** (Fredholm index). locally constant, in particular, continuous. composition makes the addition of indices.

### 6.3 Nuclear operators

tensor products

## Exercises

**6.4** (Completely continuous operators). On reflexive spaces, completely continuous operators are same with compact operators.

**6.5** (Dunford-Pettis property). A Banach space  $X$  is said to have the *Dunford-Pettis property* if all weakly compact operators  $T : X \rightarrow Y$  to any Banach space  $Y$  is completely continuous.

(a)  $X$  has the Dunford-Pettis property if and only if for every sequences  $x_n \in X$  and  $x_n^* \in X^*$  that converge to  $x$  and  $x^*$  weakly we have  $x_n^*(x_n) \rightarrow x^*(x)$ .



- (b)  $C(\Omega)$  for a compact Hausdorff space  $\Omega$  has the Dunford-Pettis property.
- (c)  $L^1(\Omega)$  for a probability space  $\Omega$  has the Dunford-Pettis property.
- (d) Infinite dimensional reflexive Banach space does not have the Dunford-Pettis property.

## Problems

1. If  $T \in B(L^2([0, 1]))$  is a compact operator, then for any  $\varepsilon > 0$  there is a constant  $C_\varepsilon > 0$  such that

$$\|Tf\|_{L^2} \leq \varepsilon \|f\|_{L^2} + C_\varepsilon \|f\|_{L^1}.$$

*Proof.* 1. Suppose there is  $\varepsilon > 0$  such that we have sequence  $f_n \in L^2$  satisfying  $\|f_n\|_2 = 1$  and

$$\|Tf_n\|_2 > \varepsilon + n\|f_n\|_1.$$

By the compactness of  $T$ , there is a subsequence  $Tf_{n_k}$  converges to  $g \neq 0$  in  $L^2$ . Then,  $\|f_{n_k}\|_1 \rightarrow 0$  implies  $f_{n_k} \rightarrow 0$  weakly in  $L^2$ , hence also for  $Tf_{n_k}$ . It means  $g = 0$ , which contradicts to the assumption.  $\square$

## **Part III**

# **Spectral theory**

## Chapter 7

# Normal operators

### 7.1 Spectral theorem for compact normal operators

There is an orthonormal basis  $E \subset H$  such that

$$T = \sum_{e \in E} \lambda_e |e\rangle \langle e|.$$

### 7.2 Spectral theorem for bounded normal operators

**7.1 (Spectral measure).** Let  $(\Omega, \mathcal{M})$  be a measurable space and  $H$  a Hilbert space. A *projection valued measure* on  $\Omega$  for  $H$  is a map  $E : \mathcal{M} \rightarrow B(H)$  such that

- (i)  $E(A)$  is an orthogonal projection with  $E(\emptyset) = 0$ ,
- (ii) the set function  $E_{\xi, \eta} : \mathcal{M} \rightarrow \mathbb{C} : A \mapsto \langle E(A)\xi, \eta \rangle$  is a complex measure on  $\Omega$  for each  $\xi, \eta \in H$ .

Let  $\Omega$  be a locally compact Hausdorff space. A *spectral measure* is a projection valued measure  $E$  on the Borel measurable space  $\Omega$  such that  $E_{\xi, \eta}$  is regular.

- (a) The condition (ii) is equivalent to the countable additivity:  $E(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} E(A_i)$  in the strong operator topology of  $B(H)$  for  $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$ .
- (b)  $E(A \cap B) = E(A)E(B)$  for  $A, B \in \mathcal{M}$ .

**7.2.** Let  $T \in B(H)$  be a normal operator. Then, there exists a spectral measure  $E$  on  $\sigma(T)$  for  $H$  such that

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

This spectral measure  $E$  is also called the *resolution of the identity*.

### 7.3 Operator topologies

**7.3 (Compact left multiplications and SOT).** Let  $T_n$  be a sequence of bounded linear operators on a Hilbert space that converges in SOT. For compact  $K$ ,  $T_n K$  converges in norm, but  $K T_n$  generally does not unless  $T$  is self-adjoint.

**7.4.** Let  $f$  be a linear functional on  $B(H)$  for a Hilbert space  $H$ . Then, TFAE:

- (a)  $f$  is WOT-continuous,

(b)  $f$  is SOT-continuous,

(c)  $f(T) = \sum_{i=1}^n \langle Tx_i, y_i \rangle$  for some  $x_i, y_i$ .

*Proof.* (2)  $\Rightarrow$  (3) is the only nontrivial implication. By the definition of SOT, there exists  $v \in \mathcal{H}^n$  such that

$$|f(T)| \leq \|T^{\oplus n} v\|.$$

The functional  $f : \mathcal{A} \rightarrow \mathbb{C}$  factors through  $\mathcal{H}^n$  such that

$$\mathcal{A} \rightarrow v\mathcal{H}^n \rightarrow \mathbb{C}.$$

□

## Chapter 8

# Unbounded operators

Kato-Rellich theorem

## **Chapter 9**

# **Toeplitz operators**

**Part IV**

**Operator algebras**

# Chapter 10

## Banach algebras

### 10.1 Spectra

**10.1** (Banach algebras).

**10.2** (Inverses in Banach algebras). Let  $\mathcal{A}$  be a unital Banach algebra.

- (a) If  $\|a\| < 1$ , then  $1 - a$  is invertible. So  $\mathcal{A}^\times$  is open.
- (b)  $\mathcal{A}^\times \rightarrow \mathcal{A}^\times : a \mapsto a^{-1}$  is differentiable.

**10.3** (Spectrum and resolvent). Let  $a$  be an element of a unital Banach algebra  $\mathcal{A}$ . The *spectrum* of  $a$  in  $\mathcal{A}$  is defined to be the set

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } \mathcal{A}\},$$

and the *resolvent* of  $a$  in  $\mathcal{A}$  is defined to be its complement  $\rho_{\mathcal{A}}(a) := \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ . We can now define the *resolvent map*  $R : \rho_{\mathcal{A}}(a) \rightarrow \mathcal{A}$  such that

$$R(\lambda) = R(\lambda; a) := (\lambda - a)^{-1}.$$

If  $\mathcal{A}$  is clear in its context, we omit it to just write  $\sigma(a)$  and  $\rho(a)$ .

- (a)  $\sigma(a)$  is compact.
- (b)  $\sigma(a)$  is non-empty.
- (c) If  $\mathcal{A}$  is a division ring, then  $\mathcal{A} \cong \mathbb{C}$ . This result is called the *Gelfand-Mazur theorem*.

*Proof.* (b) Suppose the spectrum  $\sigma(a) = \emptyset$  so that  $(\lambda - a)^{-1}$  exists for every  $\lambda \in \mathbb{C}$ . Note that  $a \neq 0$ . Since the resolvent map  $R : \mathbb{C} \rightarrow \mathcal{A}$  is continuous and we have for  $|\lambda| > 2\|a\|$  that

$$\|(\lambda - a)^{-1}\| = \|\lambda^{-1}(1 - \lambda^{-1}a)^{-1}\| = \left\| \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1}a)^k \right\| < (2\|a\|)^{-1} \sum_{k=0}^{\infty} 2^{-k} = \|a\|^{-1},$$

the function  $R$  is bounded. Also, for every  $l \in \mathcal{A}^*$  we have that the function  $\mathbb{C} \rightarrow \mathbb{C} : \lambda \mapsto \langle R(\lambda), l \rangle$  is holomorphic since  $a \mapsto a^{-1}$  is differentiable. Therefore, by the Liouville theorem, the bounded entire function  $\lambda \mapsto \langle R(\lambda), l \rangle$  is constant for all  $l \in \mathcal{A}^*$ . Because  $\mathcal{A}^*$  separates points of  $\mathcal{A}$ , the function  $R$  is constant, which implies  $a \in \mathbb{C}$  and contradicts to  $\sigma(a) = \emptyset$ .

(c)

□

**10.4** (Spectral radius). Let  $a$  be an element of a unital Banach algebra  $\mathcal{A}$ . The *spectral radius* of  $a$  in  $\mathcal{A}$  is defined to be

$$r(a) := \sup_{\lambda \in \sigma(a)} |\lambda|.$$



- (a)  $r(a) \leq \inf_{n \geq 1} \|a^n\|^{\frac{1}{n}}$  for all  $a \in \mathcal{A}$ .
- (b)  $\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$  for all  $a \in \mathcal{A}$ .

*Proof.* (a) Since  $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1}$  exists if  $|\lambda| > \|a\|$ , we have  $r(a) \leq \|a\|$  for all  $a \in \mathcal{A}$ . For every  $\lambda \in \sigma(a)$  and every integer  $n \geq 1$  we have

$$|\lambda|^n = |\lambda^n| \leq r(a^n) \leq \|a^n\|,$$

and it proves  $r(a) \leq \inf_{n \geq 1} \|a^n\|^{\frac{1}{n}}$ .

(b) On the domain  $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$ , on which  $R(\lambda)$  is well-defined, we have a holomorphic function  $\lambda \mapsto \langle R(\lambda), l \rangle$  for each  $l \in \mathcal{A}^*$ . By comparing to the same function but on a smaller domain  $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$ , we can determine the coefficients of the Laurent series of  $\langle R(\lambda), l \rangle$  at infinity as

$$\langle R(\lambda), l \rangle = \left\langle \lambda^{-1} \sum_{k=0}^{\infty} (\lambda^{-1}a)^k, l \right\rangle = \sum_{k=0}^{\infty} \langle a^k, l \rangle \lambda^{-k-1}$$

for each  $l \in \mathcal{A}^*$ .

It implies for each  $\lambda \in \mathbb{C}$  with  $|\lambda| > r(a)$  that the sequence  $(a^k \lambda^{-k-1})_k$  in  $\mathcal{A}$  is weakly bounded, hence is normly bounded by the uniform boundedness principle. Let  $\|a^n\| \leq C_\lambda |\lambda^{n+1}|$  for all  $n \geq 1$ . Then,

$$\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} C_\lambda^{\frac{1}{n}} |\lambda^{n+1}|^{\frac{1}{n}} = |\lambda|$$

for all  $\lambda$  with  $|\lambda| > r(a)$ , so we are done.  $\square$

**10.5** (Spectrum in closed subalgebras). For fixed element, smaller the ambient algebra, less “holes” in the spectrum. Let  $\mathcal{B} \subset \mathcal{A}$  be a closed subalgebra containing  $1_{\mathcal{A}}$ . Note that  $\mathcal{B}$  may be unital even for  $1_{\mathcal{A}} \notin \mathcal{B}$ .

- (a)  $\mathcal{B}^\times$  is clopen in  $\mathcal{A}^\times \cap \mathcal{B}$ .

## 10.2 Ideals

**10.6** (Ideals). (a) If  $I$  is a left ideal, then  $\mathcal{A}/I$  is a left  $\mathcal{A}$ -module.

**10.7** (Modular left ideals). A left ideal  $I$  is called *modular* if there is  $e \in \mathcal{A}$  such that  $a - ae \in I$  for all  $a \in \mathcal{A}$ . The element  $e$  is called a *right modular unit* for  $I$ .

- (a)  $I$  is modular if and only if  $\mathcal{A}/I$  is unital(?).
- (b) A proper modular left ideal is contained in a maximal left ideal.
- (c)  $I$  is a maximal modular left ideal if and only if  $I$  is a modular maximal left ideal.
- (d) There is a non-modular maximal ideal in the disk algebra.

**10.8** (Closed ideals). (a) closure of proper left ideal is proper left.

- (b) maximal modular left ideal is closed.

**10.9** (Unitization). Let  $\mathcal{A}$  be an algebra. Recall that we always assume algebras are associative. Consider an embedding  $\mathcal{A} \rightarrow B(\mathcal{A}) : a \mapsto L_a$ , where  $L_a(b) = ab$ . Define

$$\tilde{\mathcal{A}} := \{ L_a + \lambda \text{id}_{B(\mathcal{A})} : a \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Note that this construction is available even for unital  $\mathcal{A}$ .

- (a) If  $\mathcal{A}$  is normed, then  $\tilde{\mathcal{A}}$  is a normed algebra such that there is an isometric embedding  $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ .

- (b) If  $\mathcal{A}$  is Banach, then  $\tilde{\mathcal{A}}$  is a Banach algebra.
- (c)  $\mathcal{A} \oplus \mathbb{C}$  is topologically isomorphic to  $\tilde{\mathcal{A}}$  as normed spaces.

*Proof.* (a) The space of bounded operators  $B(\mathcal{A})$  is a normed algebra. Then,  $\tilde{\mathcal{A}}$  is a normed  $*$ -algebra with induced norm

$$\|L_a + \lambda \text{id}_{B(\mathcal{A})}\| = \sup_{b \in \mathcal{A}} \frac{\|ab + \lambda b\|}{\|b\|}$$

Then,  $\mathcal{A}$  is a normed  $*$ -subalgebra of  $\tilde{\mathcal{A}}$  because the norm and involution of  $\mathcal{A}$  agree with  $\tilde{\mathcal{A}}$ .

(b) Suppose  $(x_n, \lambda_n)$  is Cauchy in  $\tilde{\mathcal{A}}$ . Since  $\mathcal{A}$  is complete so that it is closed in  $\tilde{\mathcal{A}}$ , we can induce a norm on the quotient  $\tilde{\mathcal{A}}/\mathcal{A}$  so that the canonical projection is (uniformly) continuous so that  $\lambda_n$  is Cauchy. Also, the inequality  $\|x\| \leq \|(x, \lambda)\| + |\lambda|$  shows that  $x_n$  is Cauchy in  $\mathcal{A}$ .

Since a finite dimensional normed space is always Banach and  $\mathcal{A}$  is Banach,  $\lambda_n$  and  $x_n$  converge. Finally, the inequality  $\|(x, \lambda)\| \leq \|x\| + |\lambda|$  implies that  $(x_n, \lambda_n)$  converges.

- (c) Check the topology on  $\mathcal{A} \oplus \mathbb{C}$  in detail...

□

unitization, homomorphisms, category(direct sum, product, etc.)

$B(\mathbb{C}^n)$  is simple, but  $B(X)$  is not simple.

## 10.3 Gelfand theory

Banach algebra of single generator semisimplicity and symmetricity

**10.10** (Spectrum of a Banach algebra). Let  $\mathcal{A}$  be a commutative Banach algebra. A *character* of  $\mathcal{A}$  is a non-zero algebra homomorphism  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ . Denote by  $\sigma(\mathcal{A})$  the set of all characters of  $\mathcal{A}$ . We will show that all characters are bounded. Then, endow with the weak\* topology on  $\sigma(\mathcal{A})$  from the inclusion  $\sigma(\mathcal{A}) \subset \mathcal{A}^*$ . We call this space as the *spectrum* of  $\mathcal{A}$ . Let  $\varphi \in \sigma(\mathcal{A})$ .

- (a)  $\|\varphi\| = 1$ .
- (b) If  $\mathcal{A}$  is unital, then  $\sigma(\mathcal{A})$  is compact and Hausdorff.
- (c) Even if  $\mathcal{A}$  is non-unital,  $\sigma(\mathcal{A})$  is locally compact and Hausdorff.

**10.11** (Gelfand-Naimark representation). Let  $\mathcal{A}$  be a commutative Banach algebra.

$$\Gamma : \mathcal{A} \rightarrow C_0(\sigma(\mathcal{A})).$$

- (a)  $\Gamma(\mathcal{A})$  separates points.
- (b)  $\Gamma$  has closed range if
- (c)  $\Gamma$  is injective if
- (d)  $\Gamma$  is isometric if  $\|r(a)\| = \|a\|$  for all  $a \in \mathcal{A}$ .

## 10.4 Holomorphic functional calculus

**10.12.** Let  $a$  be an element of a unital Banach algebra  $\mathcal{A}$ . Let  $f$  be a holomorphic function on a neighborhood  $U$  of  $\sigma(a)$ . Let  $C$  be a positively oriented smooth simple closed curve in  $U$  enclosing  $\sigma(a)$ . Define  $f(a) \in \mathcal{A}^{**}$  as the Dunford integral

$$\langle f(a), l \rangle := \int_C f(\lambda) \langle R(\lambda), l \rangle d\lambda$$

for all  $l \in \mathcal{A}^*$ .

Let  $\text{Hol}(\sigma(a))$  be the space of all holomorphic functions on a neighborhood of  $\sigma(a)$  endowed with the topology of compact convergence. Note that  $\text{Hol}(\sigma(a))$  is not Banach. We define the *holomorphic functional calculus* by

$$\text{Hol}(\sigma(a)) \rightarrow \mathcal{A} : f \mapsto f(a).$$

It is also called the Riesz or the Riesz-Dunford functional calculus.

- (a)  $f(a) \in \mathcal{A}$ , i.e.  $f(a)$  is given by the Pettis integral.
  - (b)  $f(a)$  is independent of the choice of  $C$ .
  - (c) The functional calculus is an injective algebra homomorphism.
  - (d) The functional calculus is continuous.
  - (e) power series, 1 to 1,  $\lambda$  to  $a$ .
- spectral mapping

## Exercises

**10.13** (Basic properties of spectrum). Let  $\mathcal{A}$  be a unital algebra.

- (a)  $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ .
- (b) If  $\sigma(a)$  is non-empty, then  $\sigma(p(a)) = p(\sigma(a))$ .

*Proof.* (a) Intuitively, the inverse of  $1-ab$  is  $c = 1+ab+abab+\dots$ . Then,  $1+bca = 1+ba+baba+\dots$  is the inverse of  $1-ba$ . □

$$C_b(\Omega) \ell^\infty(S) L^\infty(\Omega) B_b(\Omega) A(\mathbb{D}) B(X)$$

**10.14.** In  $C(\mathbb{R})$ , the modular ideals correspond to compact sets.

**10.15** (Disk algebra). (a) Every continuous homomorphism is an evaluation.

**10.16** (Polynomial convexity). (See Conway)

**10.17** (Inclusion relation on spectra). (a)  $\sigma(a+b) \subset \sigma(a) + \sigma(b)$  and  $\sigma(ab) \subset \sigma(a)\sigma(b)$  for unital cases.

- (b)  $\sigma(a^{-1}) = \sigma(a)^{-1}$  for unital cases.
- (c)  $r(a)^n = r(a^n)$ .

**10.18** (Spectral radius function). (a) upper semi-continuous

**10.19** (Vector-valued complex function theory). Let  $\Omega$  be an open subset of  $\mathbb{C}$  and  $X$  a Banach space. For a vector-valued function  $f : \Omega \rightarrow X$ , we say  $f$  is *differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{-1} (f(\lambda) - f(\lambda_0))$$

exists in  $X$  for every  $\lambda \in \Omega$ , and *weakly differentiable* if the limit

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{-1} \langle f(\lambda) - f(\lambda_0), x^* \rangle$$

exists in  $\mathbb{C}$  for each  $x^* \in X^*$  and every  $\lambda \in \Omega$ . Then, the followings are all equivalent.

- (a)  $f$  is differentiable.
- (b)  $f$  is weakly differentiable.

(c) For each  $\lambda_0 \in \Omega$ , there is a sequence  $(x_k)_{k=0}^{\infty}$  such that we have the power series expansion

$$f(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k x_k,$$

where the series on the right hand side converges absolutely and uniformly on any closed ball in  $\Omega$  centered at  $\lambda_0$ .

**10.20** (Exponential of an operator).

# Chapter 11

## C\*-algebras

### 11.1 C\* identity

11.1 (Involutive Banach algebras). Banach \*-algebra:  $\|a^*\| = \|a\|$ .

11.2 (C\* identity). A normed \*-algebra  $\mathcal{A}$  is called a C\*-algebra if

- (a)  $\mathcal{A}$  is Banach,
- (b)  $\mathcal{A}$  satisfies the C\*-identity:  $\|x^*x\| = \|x\|^2$ .

11.3 (Unitization of C\*-algebras).

$$(L_a + \lambda \text{id}_{B(\mathcal{A})})^* = L_{a^*} + \bar{\lambda} \text{id}_{B(\mathcal{A})}.$$

*Proof.* The C\*-identity easily follows from the following inequality:

$$\begin{aligned} \|(x, \lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\ &= \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\| \\ &= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y)\| \\ &\leq \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y\| \\ &= \|(x, \lambda)^*(x, \lambda)\|. \end{aligned}$$

□

11.4 (Spectra of normal elements). Let  $\mathcal{A}$  be a C\*-algebra, and  $\tilde{\mathcal{A}}$  be its unitization. We say an element  $a \in \tilde{\mathcal{A}}$  is *unitary* if  $a^*a = aa^* = e$ , and say an element  $a \in \mathcal{A}$  is *self-adjoint* if  $a^* = a$ .

- (a) If  $a \in \tilde{\mathcal{A}}$  is unitary, then  $\sigma(a) \subset \mathbb{T}$ .
- (b) If  $a \in \mathcal{A}$  is self-adjoint, then  $\sigma(a) \subset \mathbb{R}$ .
- (c) The converses of the parts (a) and (b) are not generally true.

*Proof.* (a)

(b) We may assume  $\mathcal{A}$  is unital. By the holomorphic functional calculus, we have

$$e^{ia} = \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \in \mathcal{A},$$

and the inverse of  $e^{ia}$  is  $e^{-ia}$ . Since the involution  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  is continuous, we can check  $e^{ia}$  is unitary by

$$(e^{ia})^* = \sum_{n=1}^{\infty} \frac{(-ia)^n}{n!} = e^{-ia}.$$

For every  $\varphi \in \sigma(\mathcal{A})$ , then by the part (a) the equality

$$e^{-\operatorname{Im} \varphi(a)} = |e^{i\varphi(a)}| = |\varphi(e^{ia})| = 1$$

proves  $\varphi(a) \in \mathbb{R}$ , hence  $\sigma(a) \subset \mathbb{R}$ .

(c) Let  $\mathcal{A} = M_2(\mathbb{C})$  and  $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Then,  $\sigma(a) = \{1\}$  but  $a$  is neither unitary nor self-adjoint. We will show in the next section that the converses hold if we assume  $a$  is normal.  $\square$

**11.5** ( $*$ -homomorphisms). (a) determined by self-adjoint elements

(b) norm-decreasing

(c)

## 11.2 Continuous functional calculus

**11.6** (Gelfand-Naimark representation for  $C^*$ -algebras). For a commutative unital  $C^*$ -algebra  $\mathcal{A}$ , consider the Gelfand transform  $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ .

(a)  $\Gamma$  is a  $*$ -homomorphism.

(b)  $\Gamma$  is an isometry.

(c)  $\Gamma$  is a  $*$ -isomorphism.

*Proof.* (a)

(b) Note that we have

$$\|\Gamma a\| = \sup_{\varphi \in \sigma(\mathcal{A})} |\Gamma a(\varphi)| = \sup_{\varphi \in \sigma(\mathcal{A})} |\varphi(a)| = r(a)$$

for all  $a \in \mathcal{A}$ . If we assume  $a$  is self-adjoint, then since  $\|a\|^2 = \|a^*a\| = \|a^2\|$ , the spectral radius coincides with the norm by the Beurling formula for spectral radius in Banach algebras:

$$\|\Gamma a\| = r(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Hence we have for all  $a \in \mathcal{A}$  that

$$\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|(\Gamma a)^* \Gamma a\| = \|\Gamma a\|^2.$$

(c) By the part (a) and (b), the image  $\Gamma(\mathcal{A})$  is a closed unital  $*$ -subalgebra of  $C(\sigma(\mathcal{A}))$ , and it separates points by definition. Then,  $\Gamma(\mathcal{A})$  is dense in  $C(\sigma(\mathcal{A}))$  by the Stone-Weierstrass theorem, which implies  $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A}))$ .  $\square$

**11.7** (Finitely generated  $C^*$ -algebras). joint spectrum.

**11.8** (Continuous functional calculus). Let  $\mathcal{A}$  be a  $C^*$ -algebra, and  $a \in \mathcal{A}$  a normal element. Then, we have an isometric  $*$ -homomorphism

$$C(\sigma(a)) \rightarrow \mathcal{A}$$

defined by the inverse of the Gelfand transform, which we call the *continuous functional calculus*.

(a)  $\operatorname{id} \mapsto a$ .

(b)  $(f + g)(a) = f(a) + g(a)$  and  $(fg)(a)$ .

(c)  $(f \circ g)(a) = f(g(a))$ .

### 11.3 Positivity in $C^*$ -algebras

11.9 (Positive elements). (a) If  $a, b \geq 0$ , then  $a + b \geq 0$ .

(b) If  $a^*a \leq 0$ , then  $a^*a = 0$ .

(c)  $a^*a \geq 0$  for all  $a \in \mathcal{A}$ .

11.10 (Absolute value of an operator).

11.11 (Operator monotone functions). (a) inverse

(b) conjugation

11.12 (Injective  $*$ -homomorphism).

11.13 (Approximate identity). separable?

### 11.4 Representations of $C^*$ -algebras

11.14 (Representation of  $C^*$ -algebras). A *representation* of a  $C^*$ -algebra is a  $*$ -homomorphism  $\pi : \mathcal{A} \rightarrow B(H)$  for a Hilbert space  $H$ .

11.15 (Non-degenerate representation). Let  $\pi : \mathcal{A} \rightarrow B(H)$  be a representation of a  $C^*$ -algebra  $\mathcal{A}$ . We say  $\pi$  is *non-degenerate* if  $\pi(\mathcal{A})H$  is dense in  $H$ .

(a)  $\pi$  is non-degenerate.

(b) For each  $\xi \in H$  there is  $a \in \mathcal{A}$  such that  $\pi(a)\xi \neq 0$ .

(c)  $\pi(e_\alpha) \rightarrow \text{id}_H$  strongly for every approximate identity  $e_\alpha$  of  $\mathcal{A}$ .

11.16 (Cyclic representation). Let  $\pi : \mathcal{A} \rightarrow B(H)$  be a representation of a  $C^*$ -algebra  $\mathcal{A}$ .

(a)

11.17 (Irreducible representation). Let  $\pi : \mathcal{A} \rightarrow B(H)$  be a representation of a  $C^*$ -algebra  $\mathcal{A}$ . We say  $\pi$  is *irreducible* if there is no proper closed subspace  $K \subset H$  such that  $\pi(a)K \subset K$ .

(a)  $\pi$  is irreducible.

(b)  $\pi(\mathcal{A})' = \mathbb{C} \text{id}_H$ .

(c)  $\pi(\mathcal{A})$  is strongly dense in  $B(H)$ .

(d) Every non-zero vector is cyclic.

11.18 (Gelfand-Naimark-Segal representation). Let  $\mathcal{A}$  be a  $C^*$ -algebra, and  $\rho$  be a state on  $\mathcal{A}$ . The *left kernel* of  $\rho$  is defined to be

$$L_\rho := \{a \in \mathcal{A} : \rho(a^*a) = 0\}.$$

(a)  $L_\rho$  is a left ideal of  $\mathcal{A}$ .

(b)  $\langle a + L, b + L \rangle := \rho(b^*a)$  is an inner product on  $\mathcal{A}/L_\rho$ .

(c) There is a unique representation  $\pi_\rho : \mathcal{A} \rightarrow B(H_\rho)$  such that  $\pi_\rho(a)(b + L) := ab + L$  for  $a, b \in \mathcal{A}$ .

(d)  $\pi_\rho : \mathcal{A} \rightarrow B(H_\rho)$  is a cyclic representation.

11.19 (Kadison transitivity theorem).

11.20 (Left ideals).

11.21 (Primitive ideals).

11.22 (Hull-kernel topology).

## Exercises

**11.23.** Let  $\mathcal{B}$  be a hereditary  $C^*$ -subalgebra of a  $C^*$ -algebra  $\mathcal{A}$ . Let  $a \in \mathcal{A}^+$ . If for any  $\varepsilon > 0$  there is  $b \in \mathcal{B}^+$  such that  $a - \varepsilon \leq b$ , then  $a \in \mathcal{B}^+$ .

*Proof.* To catch the idea, suppose  $\mathcal{A}$  is abelian. We want to approximate  $a$  by the elements of  $\mathcal{B}$  in norm. To do this, for each  $\varepsilon > 0$ , we want to construct  $b' \in \mathcal{B}^+$  such that  $a - \varepsilon \leq b' \leq a + \varepsilon$  using  $b$ . Taking  $b' = \min\{a, b\}$  is impossible in non-abelian case, but we can put  $b' = \frac{a}{b+\varepsilon}b$ . For a simpler proof,  $b' = (\frac{\sqrt{ab}}{\sqrt{b}+\sqrt{\varepsilon}})^2$  is a better choice.

Define

$$b' := \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} a \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}}.$$

Then,

$$\|\sqrt{a} - \sqrt{a} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}}\|^2 = \|\frac{\sqrt{\varepsilon}}{\sqrt{b} + \sqrt{\varepsilon}} a \frac{\sqrt{\varepsilon}}{\sqrt{b} + \sqrt{\varepsilon}}\| \leq \varepsilon$$

implies

$$\lim_{\varepsilon \rightarrow 0} b' = \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} \sqrt{a} \cdot \sqrt{a} \frac{\sqrt{b}}{\sqrt{b} + \sqrt{\varepsilon}} = \sqrt{a} \cdot \sqrt{a} = a.$$

□

**11.24** (Operator monotone square). Let  $\mathcal{A}$  be a  $C^*$ -algebra in which the square function is operator monotone, that is,  $0 \leq a \leq b$  implies  $a^2 \leq b^2$  for any positive elements  $a$  and  $b$  in  $\mathcal{A}$ . We are going to show that  $\mathcal{A}$  is necessarily commutative. Let  $a$  and  $b$  denote arbitrary positive elements of  $\mathcal{A}$ .

- (a) Show that  $ab + ba \geq 0$ .
- (b) Let  $ab = c + id$  where  $c$  and  $d$  are self adjoints. Show that  $d^2 \leq c^2$ .
- (c) Suppose  $\lambda > 0$  satisfies  $\lambda d^2 \leq c^2$ . Show that  $c^2 d^2 + d^2 c^2 - 2\lambda d^4 \geq 0$ .
- (d) Show that  $\lambda(cd + dc)^2 \leq (c^2 - d^2)^2$ .
- (e) Show that  $\sqrt{\lambda^2 + 2\lambda - 1} \cdot d^2 \leq c^2$  and deduce  $d = 0$ .
- (f) Extend the result for general exponent:  $\mathcal{A}$  is commutative if  $f(x) = x^\beta$  is operator monotone for  $\beta > 1$ .

**11.25** (States on unitization). Let  $\mathcal{A}$  and  $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$  be a  $C^*$ -algebra and its unitization respectively. Let  $\tilde{\rho} = \rho \oplus \lambda$  be a bounded linear functional on  $\tilde{\mathcal{A}}$ , where  $\rho \in \mathcal{A}^*$  and  $\lambda \in \mathbb{C}^* = \mathbb{C}$ .

- (a)  $\tilde{\rho}$  is positive if and only if  $\lambda \geq 0$  and  $0 \leq \rho \leq \lambda$ .
- (b)  $\tilde{\rho}$  is a state if and only if  $\lambda = 1$  and  $\rho$  is positive with  $\|\rho\| \leq 1$ .
- (c)  $\tilde{\rho}$  is a pure state if and only if  $\lambda = 1$  and  $\rho$  is either a pure state or zero.

**11.26** (Representations of  $C_0(\Omega)$ ). Let  $\mathcal{A} = C_0(\Omega)$  and  $\mu$  be a state on  $\mathcal{A}$ , a regular Borel probability measure on  $\Omega$ .

- (a) The left kernel of  $\mu$  is  $L_\mu = \{f \in \mathcal{A} : f|_{\text{supp } \mu} = 0\}$ .
- (b) The quotient is  $\mathcal{A}/L_\mu \cong C(\text{supp } \mu)$  so that  $H_\mu = L^2(\text{supp } \mu, \mu)$ .
- (c) The canonical cyclic vector is the unity function.

**11.27** (Representations of  $K(H)$ ).

## Problems

- \*1. A  $C^*$ -algebra is commutative if and only if a function  $f(x) = x(1+x)^{-1}$  is operator subadditive.



# Chapter 12

## Von Neumann algebras

### 12.1 Density theorems

**12.1** (Von Neumann algebras). A  $C^*$ -algebra  $\mathcal{A}$  is called a *von Neumann algebra* if there is a isometric  $*$ -homomorphism  $\mathcal{A} \rightarrow B(H)$  for a Hilbert space  $H$  whose image is closed in the weak operator topology.

**12.2** (Vigier theorem). Increasing bounded net is convergent in strong operator topology. The boundedness is important because we have to construct a bounded sesquilinear form using the monotone convergence in  $\mathbb{R}$ .

**12.3** (Bicommutant theorem). Let  $\mathcal{A}$  be a non-degenerate  $C^*$ -subalgebra of  $B(H)$ .

- (a)  $\mathcal{A}'$  and  $\mathcal{A}''$  are weakly closed.
- (b) For  $a \in \mathcal{A}''$  and  $\xi \in H$ , there is a sequence  $a_n \in \mathcal{A}$  such that  $a_n(\xi) \rightarrow a(\xi)$ .
- (c) For  $a \in \mathcal{A}''$  and  $\xi_1, \dots, \xi_m \in H$ , there is a sequence  $a_n \in \mathcal{A}$  such that  $a_n(\xi_i) \rightarrow a(\xi_i)$  for all  $i$ .
- (d)  $\mathcal{A}$  is von Neumann algebra if and only if  $\mathcal{A} = \mathcal{A}''$ .

*Proof.* (b) Let  $K := \overline{\mathcal{A}\xi}$  be the cyclic subspace of  $\xi$  in  $H$  and  $p$  its orthogonal projection. We claim  $a\xi \in K$ . For every  $b \in \mathcal{A}$ , we have  $bK \subset K$  because the multiplication by  $b$  is continuous on  $H$ , and  $b^*K \subset K$  because  $\mathcal{A}$  is self-adjoint. It means that  $K$  reduces all  $b \in \mathcal{A}$ , and then  $bp = pb$  implies  $ap = pa$ , so  $K$  also reduces  $a$ . Therefore,  $aK \subset K$  proves  $a\xi = \lim_{\alpha} e_{\alpha} a\xi \in K$ , where  $e_{\alpha}$  is an approximate identity of  $\mathcal{A}$ .

(e) Since  $\overline{\mathcal{A}}^{\text{wot}}$  is closed convex,  $\overline{\mathcal{A}}^{\text{sot}} = \overline{\mathcal{A}}^{\text{wot}}$ . Also,  $\mathcal{A}''$  is weakly closed,  $\overline{\mathcal{A}}^{\text{wot}} \subset \mathcal{A}''$ . □

**12.4** (Kaplansky density theorem).

### 12.2 Borel functional calculus

resolution of identity normal operator theories: multiplicity, invariant subspaces  $L^{\infty}$  representation

**12.5** (Borel functional calculus). Let  $\mathcal{A}$  be a von Neumann algebra.

$$B^{\infty}(\sigma(a)) \rightarrow \mathcal{A}.$$

- (a) The Borel functional calculus is in general not injective.
- (b) If we endow the topology of pointwise convergence on  $B^{\infty}(\sigma(a))$  and the strong operator topology on  $\mathcal{A}$ , then the Borel functional calculus is continuous.
- (c) not isometric, even if it is injective.

(d) Every von Neumann algebra is the closed span of projections.

**12.6.** (b) By the bounded convergence theorem.

(d) This is because  $\sigma(a) \subset \mathbb{C}$  is compact so that it is separable and metrizable; every bounded measurable function is a pointwise limit of simple functions.

## 12.3 Factors and traces

Every trace of factor is faithful

**12.7.** Normal states is a state in which the monotone convergence theorem holds. Precisely, a state  $\rho$  is *normal* if a monotone net  $a_\alpha$  strongly converges to  $a$  then  $\rho(a_\alpha) \rightarrow \rho(a)$ .