Smooth Manifolds

Ikhan Choi

July 8, 2023

Contents

Part I

Smooth manifolds

Chapter 1

Smooth structures

1.1 Local coordinates

1.1 (Atlases)**.** Let M be a topological space and consider a fixed positive integer m. An m-dimensional (local) *coordinate system*, or (local) *chart* is a pair (U,φ) consisting of an open set $U \subset M$ and a topological embedding $\varphi : U \to \mathbb{R}^m$. The embedding φ is called a *coordinate map*, and each component of φ with respect to a basis of $\mathbb{R}^m$ is called a *coordinate function*.

An m-dimensional *atlas* on M is an indexed family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ of m-dimensional local charts such that every point is contained in some U_α, that is, $\{U_\alpha\}_\alpha$ is a cover of M. In geography, an atlas means a book of maps of Earth. A term *locally Euclidean space* is sometimes used to refer a topological space M together with an m-dimensional atlas.

(a)

1.2 (Smooth atlases)**.** Let M be a topological space and consider a fixed positive integer m. We say that two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on M are (smoothly) *compatible* if the *transition map*

$$\tau_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth. If every pair of two charts in an atlas is compatible, then we say that the atlas is *smooth*.

Given a smooth atlas $\mathcal{A}$, we can define the *smoothness* of a function $f : M \to \mathbb{R}$ with respect to $\mathcal{A}$ as follows: we say that f is smooth if its *coordinate representation*

$$f \circ \varphi^{-1} : \varphi_\alpha(U) \to \mathbb{R}$$

is smooth for all $(U,\varphi) \in \mathcal{A}$.

Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *compatible* or *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. A *smooth structure* on M is a maximal smooth atlas $\mathcal{A}$; there is no smooth atlas $\mathcal{A}'$ that contains $\mathcal{A}$ properly.

(a) For a given smooth atlas, every transition map is a diffeomorphism.

(b) If two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, then a function $f : M \to \mathbb{R}$ is smooth with respect to $\mathcal{A}_1$ if and only if it is smooth with respect to $\mathcal{A}_2$.

(c) There is a one-to-one correspondence between smooth structures and equivalence classes of smooth atlases. Therefore, we can describe a smooth structure by giving a particular smooth atlas.

1.3 (Manifolds)**.** A *topological manifold* is defined as a second-countable and Hausdorff space together with a maximal atlas, and a *smooth manifold* is defined as a second-countable and Hausdorff space together with a smooth structure. The term *manifold* may refer to any of either a topological or a smooth manifold, which depends on contexts of each reference.

(a) The long line admits a smooth structure, and it is Hausdorff but not second countable.

(b) The line with two origins admits a smooth structure, and it is second countable but not Hausdorff.

1.4 (Partition of unity).

1.5 (Smooth maps and diffeomorphisms). Let scalar functions, scalar fields

1.6 (Embedded manifolds). a *embedded manifold* or a *regular manifold. parametrization*

If $\alpha : U \to \mathbb{R}^n$ is a topological embedding, then we can endow with a unique smooth structure on $\operatorname{im}\alpha$ such that α is smooth.(?)

(a) The image of a regular parameterization is an embedded manifold.

(b) Every open subset of a embedded manifold is a embedded manifold.

(c) Monge patch.

(d) The sphere $S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ is a regular surface.

(e) The set $\{ (x, y) \in \mathbb{R}^2 : y^2 = x^3 + x^2 \}$ is not a regular curve.

(f) The set $\{ (x, y) \in \mathbb{R}^2 : y = |x| \}$ is not a regular curve.

1.2 Tangent spaces

1.7 (Tangent spaces of embedded manifolds). Let M be an m-dimensional embedded manifold in $\mathbb{R}^n$. For a point $p \in M$, take a parameterization α for M at p, and let $x := \alpha^{-1}(p)$ be the coordinates of p. The *tangent space* T_pM of M at p is defined as the image of $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$.

(a) T_pM is a m-dimensional vector subspace of $\mathbb{R}^n$ with a basis $\{\partial_i \alpha(x)\}_{i=1}^m$.

(b) If $v \in T_pM$, then we have a smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(c) If we have a smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, then $v \in T_pM$.

(d) The definition of T_pM is independent on the parameterization α.

1.8 (Tangent spaces as equivalence classes of curves).

1.9 (Tangent spaces as derivations).

the space of derivations on the ring of smooth functions, the dual space of algebraically defined cotangent spaces.

1.3 Differentials

Exercises

1.1 (Polar coordinates). Let $M = \mathbb{R}^2 \setminus \{0\}$. Define a chart (U, φ) by

$$U := \{ (x, y) \in M : x \neq 0 \text{ or } y > 0 \}$$

and $\varphi = (r, \theta) : U \to \mathbb{R}^2$ such that

$$r(x, y) := \sqrt{x^2 + y^2}, \quad \theta(x, y) := \tan^{-1} \frac{y}{x},$$

where $\tan^{-1}(t) := \int_0^t (1 + s^2)^{-1}\, ds$.

(a) The chart (U, φ) is compatible with the standard smooth structure inherited from $\mathbb{R}^2$.

(b) We have

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

1.2 (Spheres). Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface given by

$$\alpha(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2}\right).$$

This map gives a parametrization for the sphere S^2 without the north pole $(0,0,1)$, and is called the *stereographic projection*.

Spherical coordinates

(a) All charts above are compatible.

(b) There exists at least two charts in an atlas on S^n.

(c) For the height function $z : S^2 \to \mathbb{R}$ given by $z(x,y,z) := z$, we have $\partial_x z(x,y) = 4x/(1+x^2+y^2)^2$.

1.3 (Projective spaces). $S^n \to \mathbb{R}P^n$

1.4 (Stiefel and Grassmann varieties). $G_1^{n+1} \cong \mathbb{R}P^n$

1.5 (Parallelizable spheres).

1.6 (Tagent space of matrix groups). Jacobi formula

1.7 (Recovery of compact smooth manifolds). Let M be a compact smooth manifold. C^∞ functor is a fully faithful contravariant functor.

(a) Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

Chapter 2

Tensor fields

2.1 Vector fields

2.1 (Vector fields). Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization with $M = \operatorname{im}\alpha$. A *vector field* is a map $X : M \to \mathbb{R}^n$ such that $X \circ \alpha : U \to \mathbb{R}^n$ is smooth. A *tangent vector field* is a vector field $X : M \to \mathbb{R}^n$ such that $X|_p \in T_pM$. The set of tangent vector fields is often denoted by $\mathfrak{X}(M)$.

2.2. Let $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a parametrization $M = \operatorname{im}\alpha$.

(a) The coordinate representation of a function $f : M \to \mathbb{R}$ is

$$f \circ \alpha : U \to \mathbb{R}.$$

(b) The (external) coordinate representation of a vector field $X : M \to \mathbb{R}^n$ is

$$X \circ \alpha : U \to \mathbb{R}^n.$$

(c) The coordinate representation of a tangent vector field $X : M \to \mathbb{R}^n$ is

$$(X^1 \circ \alpha, \cdots, X^m \circ \alpha) : U \to \mathbb{R}^m$$

where $X = \sum_i X^i \alpha_i$.

2.3. Let α be an m-dimensional parametrization with $M = \operatorname{im}\alpha$. The value of $\partial_i \alpha = \alpha_i : M \to \mathbb{R}^3$ is always a tanget vector at each point $p = \alpha(x)$, and α_i becomes a vector field.

Let s be either a smooth function or vector field on α. Then, we can compute the directional derivative as

$$\partial_i s := \partial_i(s \circ \alpha) = \partial_t(s \circ \gamma)$$

by taking $\gamma(t) = \alpha(x + te_i)$, where e_i is the i-th standard basis vector for $\mathbb{R}^m$.

2.4. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. Let $v = \sum_i v^i \alpha_i|_p \in T_pM$ be a tangent vector at $p = \alpha(x)$. For a function $f : M \to \mathbb{R}$, its partial derivative is defined by

$$\partial_v f(p) := \sum_{i=1}^m v^i \partial_i(f \circ \alpha)(x) \in \mathbb{R}.$$

For a vector field $X : M \to \mathbb{R}^n$, its partial derivative is defined by

$$\partial_v X|_p := \sum_{i=1}^m v^i \partial_i(X \circ \alpha)(x) \in \mathbb{R}^n.$$

This definition is not dependent on parametrization α.

2.5. Let M be the image of a parametrization. Let X be a tangent vector field on M.

(a) If f is a function, then so is $\partial_X f$.

(b) If Y is a vector field, then so is $\partial_X Y$.

(c) If Y is a tangent vector field, then so is $\partial_X Y - \partial_Y X$.

Proof. (a) and (b) are clear. For (c), if we let $X = \sum_i X^i \alpha_i$ and $Y = \sum_j Y^j \alpha_j$ for a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$, then

$$\begin{aligned}\partial_X Y - \partial_Y X &= \partial_X(\textstyle\sum_j Y^j \alpha_j) - \partial_Y(\sum_i X^i \alpha_i)\\ &= \textstyle\sum_j[(\partial_X Y^j)\alpha_j + Y^j \partial_X \alpha_j] - \sum_i[(\partial_Y X^i)\alpha_i + X^i \partial_Y \alpha_i]\\ &= \textstyle\sum_j[(\partial_X Y^j)\alpha_j + Y^j \sum_i X^i \partial_i \alpha_j] - \sum_i[(\partial_Y X^i)\alpha_i + X^i \sum_j Y^j \partial_i \alpha_j]\\ &= \textstyle\sum_j(\partial_X Y^j)\alpha_j - \sum_i(\partial_Y X^i)\alpha_i\\ &= \textstyle\sum_i(\partial_X Y^i - \partial_Y X^i)\alpha_i.\end{aligned}$$

□

2.6. Let M be the image of a parametrization α. For derivatives of functions on M by tangent vectors, we will use

$$\partial_{\alpha_i} f = \partial_i f, \quad \partial_{\alpha_t} f = \partial_t f = f', \quad \partial_{\alpha_x} f = \partial_x f = f_x.$$

For derivatives of vector fields on M by tangent vectors, we will use

$$\partial_{\alpha_i} X = \partial_i X, \quad \partial_{\alpha_t} X = \partial_t X = X', \quad \partial_{\alpha_x} X = \partial_x X = X_x.$$

We will *not* use f_i or X_i for $\partial_i f$ and $\partial_i X$ because it is confusig with coordinate representations, and *not* use the nabula symbol ∇_v in this sense because it will be devoted to another kind of derivatives introduced in Section 4.

2.2 Tensor fields of higher order

tensor bundle tensor fields,

2.3 Differential forms

forms, exterior structures, pullback, interior product

2.4 Lie derivatives

2.7 (Integral curves).

Exercises

2.8 (Orientation).

Chapter 3

Submanifolds

3.1 Constant rank theorem

3.1 (Constant rank theorem)**.** Let M and N be smooth manifolds of dimensions m and n, and $f : M \to N$ a smooth map. Let $p \in M$ and $q \in N$ such that $f(p) = q$. For each pair of local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$, we can introduce functions $a : \varphi(U) \to \mathbb{R}^k$ and $b : \varphi(U) \to \mathbb{R}^{n-k}$ such that the coordinate representation $\widetilde{f} : \varphi(U) \to \psi(V)$ of f is written as

$$\widetilde{f}(x, y) := \psi \circ f \circ \varphi^{-1}(x, y) = (a(x, y), b(x, y))$$

for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{m-k}$ with $(x, y) \in \varphi(U)$. Then, the differential df on U is represented by its Jacobian matrix

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix}.$$

Suppose the differential of f has a locally constant rank k at p.

(a) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and $\partial a/\partial x$ is a $k \times k$ invertible matrix everywhere.

(b) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & 0 \end{pmatrix}.$$

(c) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and

$$D\widetilde{f}|_{(x,y)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) There exists local charts (U, φ) at p and (V, ψ) at q such that $f(U) \subset V$ and $\widetilde{f}(x, y) = (x, 0)$.

Proof. (a) Let (U, φ) and (V, ψ) be local charts at p and q such that $f(U) \subset V$ and the Jacobian matrix $D\widetilde{f}|_{(x,y)}$ is of rank k for every $(x, y) \in \varphi(U)$. For each $(x, y) \in \varphi(U)$, the matrix $D\widetilde{f}|_{(x,y)}$ has an invertible $k \times k$ minor submatrix. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ be permutation matrices that reorder the coordinates in such a way that the invertible $k \times k$ minor submatrix becomes the leading principal minor submatrix.

Define reparametrizations $\varphi' := A \circ \varphi : U \to A(\varphi(U))$ and $\psi' := B \circ \psi : V \to B(\psi(V))$. Then, they are clearly local charts and

$$D(\psi' \circ f \circ \varphi'^{-1}) = D(B \circ \psi \circ f \circ \varphi^{-1} \circ A^{-1}) = B \circ D\widetilde{f} \circ A^{-1}$$

has an invertible leading principal minor submatrix of dimension $k \times k$ at every $(x, y) \in \varphi(U)$.

(b) Let (U, φ) and (V, ψ) be local charts at p and q satisfying the conditions given in the part (a). Consider a map $F : \varphi(U) \to \mathbb{R}^m$ defined by

$$F(x, y) := (a(x, y), y).$$

Then, since

$$DF|_{(x,y)} = \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix}$$

is smooth and invertible everywhere on $\varphi(U)$, there exists an open neighborhood $\varphi(U') \subset \varphi(U)$ of $\varphi(p)$ such that the restriction $F : \varphi(U') \to F(\varphi(U'))$ is a diffeomorphism by the inverse function theorem.

Define a reparamterization $\varphi' := F \circ \varphi : U' \to F(\varphi(U'))$. Then, it is clearly a local chart and

$$\begin{aligned} D(\psi \circ f \circ \varphi'^{-1}) &= D(\psi \circ f \circ \varphi^{-1} \circ F^{-1}) = D\widetilde{f} \circ (DF)^{-1} \\ &= \begin{pmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial a}{\partial x}\right)^{-1} & -\left(\frac{\partial a}{\partial x}\right)^{-1} \frac{\partial a}{\partial y} \\ 0 & \mathrm{id}_{m-k} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ * & 0 \end{pmatrix}. \end{aligned}$$

The last equality holds because the transpose of this matrix has rank k, and the conditions are satisfied with the local charts (U', φ') and (V, ψ).

(c) Let (U, φ) and (V, ψ) be local charts at p and q satisfying the conditions given in the part (b). Then, we have $\widetilde{f}(x, y) = (x, b(x))$ for all $(x, y) \in \varphi(U)$. Consider a map $G : \psi(V) \to \mathbb{R}^n$ defined by

$$G(x, z) := (x, z - b(x)).$$

Then, since

$$DG|_{(x,z)} = \begin{pmatrix} \mathrm{id}_k & 0 \\ -\frac{\partial b}{\partial x} & \mathrm{id}_{n-k} \end{pmatrix}$$

is smooth and invertible everywhere on $\psi(V)$, there exists an open neighborhood $\psi(V') \subset \psi(V)$ of $\psi(q)$ such that the restriction $G : \psi(V') \to G(\psi(V'))$ is a diffeomorphism by the inverse function theorem.

Define a reparamterization $\psi' := G \circ \psi : V' \to G(\psi(V'))$. Then, it is clearly a local chart and

$$\begin{aligned} D(\psi' \circ f \circ \varphi^{-1}) &= D(G \circ \psi \circ f \circ \varphi^{-1}) = DG \circ D\widetilde{f} \\ &= \begin{pmatrix} \mathrm{id}_k & 0 \\ -\frac{\partial b}{\partial x} & \mathrm{id}_{n-k} \end{pmatrix} \begin{pmatrix} \mathrm{id}_k & 0 \\ \frac{\partial b}{\partial x} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, the conditions are satisfied with the local charts (U, φ) and (V', ψ').

(d) Let (U, φ) and (V, ψ) be local charts at p and q satisfying the conditions given in the part (c). Then, by translating constants for these local coordinate systems, we obtain $\widetilde{f}(x, y) = (x, 0)$. □

3.2 (Preimage theorem). Let M and N are smooth manifolds of dimensions m and n. Let $f : M \to N$ be a smooth map. A *critical point* is a point $p \in M$ such that $df|_p$ is not surjective, and a *critical value* is a point $q \in N$ such that $f(p) = q$ for some critical point p. If $q \in N$ is not a critical value, then it is called a *regular value*.

Suppose $q \in N$ is a regular value of f, and $p \in M$ be any points satisfying $f(p) = q$. We will show that $f^{-1}(q)$ is an embedded submanifold of M. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have local charts (U, φ) and (V, ψ) at p and q such that

$$\varphi(p) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad \psi(q) = 0 \in \mathbb{R}^n, \quad \text{and} \quad \widetilde{f}(x, y) = x.$$

(a) $(U \cap f^{-1}(q), \varphi|_{U\cap f^{-1}(q)})$ is an $(m-n)$-dimensional chart at p on $f^{-1}(q)$.

(b) The charts of the form $(U \cap f^{-1}(q), \varphi|_{U\cap f^{-1}(q)})$ defines a smooth atlas.

(c) The inclusion is an embedding.

Proof. (a) Note that every open subset of $U \subset f^{-1}(q)$ is of the form $W \cap f^{-1}(q)$ for an open set $W \subset U$. Since $\varphi(W)$ is open in $\mathbb{R}^m$ for any open $W \subset U$,

$$\begin{aligned}\varphi(W \cap f^{-1}(q)) &= \varphi(W) \cap \varphi(f^{-1}(q)) \\ &= \varphi(W) \cap \widetilde{f}^{-1}(\psi(q)) \\ &= \varphi(W) \cap \widetilde{f}^{-1}(0) \\ &= \varphi(W) \cap (\{0\} \times \mathbb{R}^{m-n})\end{aligned}$$

is open in $\{0\} \times \mathbb{R}^{m-n}$. It means that the restriction of φ on $U \cap f^{-1}(q)$ is an injective open map, so it is a topological embedding into the Euclidean space $\{0\} \times \mathbb{R}^{m-n}$.

□

3.2 Embeddings

3.3 (Immersion is a local embedding)**.** Let $f : M \to N$ be an immersion at $p \in M$. Then, there is a local chart (V, ψ) at $f(p)$ such that

(a) $W = f(M) \cap V$ is an embedded submanifold of V,

(b) there is a retract $V \to W$.

Proof. Since the set of full rank matrices is open, the rank of df is locally contant at p. By the constant rank theorem, we have

$$\varphi(p) = 0 \in \mathbb{R}^m, \quad \psi(f(p)) = (0,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \quad \text{and} \quad \widetilde{f}(x) = (x, 0).$$

Let $W := f(M) \cap V$. Then, the injectivity of φ shows that

$$\psi(W) = \psi(f(U)) = \psi \circ f \circ \varphi^{-1}(\varphi(U)) = \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : x \in \varphi(U)\}$$

is an open subset of $\mathbb{R}^m$, so $(W, \psi|_W)$ is a chart at $f(p)$.

Transition maps are smooth?

The inclusion is a smooth embedding? □

3.4 (Extension of smooth functions)**.** from an embedded manifold.

Let $f : M \to N$ be an injective immersion. There exists unique smooth structure on $f(M)$ such that f and i are smooth.

Let $f : M \to N$ be an embedding. There exists unique smooth structure on $f(M)$ such that i are smooth.

3.3 Distributions

3.5 (Foliation)**.**

Part II

Riemannian manifolds

Chapter 4

Metrics and connections

4.1 Riemannian metric

We say a quantity is *intrinsic* in two different contexts: one is the embedding independency, and the other is the coordinates independency.

Riemannian measure

- Intrinsic: g_{ij}, Γ_{ij}^k, K, $R^l{}_{ijk}$;
- Not intrinsic: ν, L_{ij}, κ_i, H.

Example 4.1.1. Let $\alpha : (-\log 2, \log 2)\times(0, 2\pi) \to \mathbb{R}^3$ and $\beta : (-\frac{3}{4}, \frac{3}{4})\times(0, 2\pi) \to \mathbb{R}^3$ be regular surfaces given by

$$\alpha(x, \theta) = (\cosh x \cos\theta,\ \cosh x \sin\theta,\ x), \qquad \beta(r, z) = (r \cos z,\ r \sin z,\ z).$$

Their Riemannian metrics are

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix}_{(\alpha_x, \alpha_\theta)}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 + r^2 \end{pmatrix}_{(\beta_r, \beta_z)}.$$

Define a map $f : \operatorname{im}\alpha \to \operatorname{im}\beta$ by

$$f : \alpha(x, \theta) \mapsto \beta(\sinh x, \theta) = (r(x, \theta), z(x, \theta)).$$

The Jacobi matrix of f is computed

$$df|_{\alpha(x,\theta)} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}_{(\alpha_x, \alpha_\theta)\to(\beta_r, \beta_z)}.$$

Since f is a diffeomorphism and

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 + r^2 \end{pmatrix} \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix},$$

the map f is an isometry.

4.2 Connections

4.1 (Affine connection)**.** Let M be a smooth manifold An *affine connection* on M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : (X, Y) \mapsto \nabla_X Y$$

such that

(i) $C^\infty(M)$-linear in the first argument X,

(ii) the *Leibniz rule*

$$\nabla_X(fY) = XfY + f\nabla_X Y$$

for $f \in C^\infty(M)$ in the second argument Y is satisfied.

4.2 (Levi-Civita connection)**.** Let M be a Riemannian manifold. A *metric connection* is an affine connection ∇ such that $\nabla g = 0$. A *Levi-Civita connection* is a metric connection ∇ such that $\nabla T = 0$.

(a) ∇ is a metric connection if and only if $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$.

(b) ∇ is a Levi-Civita connection if and only if $\nabla_X Y - \nabla_Y X = [X, Y]$.

(c) There exists a unique Levi-Civita connection on M.

Proof. (Uniqueness) Suppose ∇ is a Levi-Citiva connection on M.

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = \partial_X\langle Y, Z\rangle + \partial_Y\langle X, Z\rangle - \partial_Z\langle X, Y\rangle \\ - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle. \end{aligned}$$

(Existence) □

4.3. Let S be a regular surface embedded in $\mathbb{R}^3$. If we define Christoffel symbols as the Gauss formula, then

$$\mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S) : (X^i\alpha_i, Y^j\alpha_j) \mapsto \left(X^i\partial_i Y^k + X^iY^j\Gamma_{ij}^k\right)\alpha_k$$

defines a Levi-Civita connection.

4.4 (Connection form)**.**

4.5 (Covariant derivative as orthogonal projection)**.** We are going to think about "intrinsic" derivatives for tangent vectors. For coordinate independence, directional derivatives of a tangent vector field should be at least a tangent vector field, which is false for the obvious partial derivatives in the embedded surface setting; for example, T is a tangent vector, but $\mathrm{N} = \kappa\mathrm{T}'$ is not tangent.

Recall that the Gauss formula reads

$$\partial_i\alpha_j = \Gamma_{ij}^k\alpha_k + L_{ij}\nu$$

so that we have

$$\begin{aligned} \partial_X Y &= X^i\partial_i(Y^j\alpha_j) \\ &= X^i(\partial_i Y^k)\alpha_k + X^iY^j\partial_i\alpha_j \\ &= \left(X^i\partial_i Y^k + X^iY^j\Gamma_{ij}^k\right)\alpha_k + X^iY^jL_{ij}\nu. \end{aligned}$$

If we write $\nabla_X Y = \left(X^i\partial_i Y^k + X^iY^j\Gamma_{ij}^k\right)\alpha_k$, then it embodies the orthogonal projection of $\partial_X Y$ onto its tangent space, and we have

$$\partial_X Y = \nabla_X Y + \mathrm{II}(X, Y)\nu.$$

Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $\operatorname{im}\alpha = M$. Let $X = X^i\alpha_i$ and $Y = Y^j\alpha_j$ be tangent vector fields on M. The *covariant derivative* of Y along X is defined as the orthogonal projection of the partial derivative $\partial_X Y$ onto the tangent space:

$$\nabla_X Y := \left(X^i\partial_i Y^k + X^iY^j\Gamma_{ij}^k\right)\alpha_k.$$

(a) Covariant derivatives are intrinsic. In other words, the above definition does not depend on the choice of parametrizations.

Proof. Recall that the Christoffel symbols transform as follows:

$$X^i Y^j \Gamma_{ij}^k = X^a Y^b \left(\Gamma_{ab}^c + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} \right) \frac{\partial x^k}{\partial x^c}.$$

Thus, we have

$$\begin{aligned}
&\left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k \\
&= X^a \frac{\partial}{\partial x^a} \left(Y^c \frac{\partial x^k}{\partial x^c} \right) \alpha_k + X^a Y^b \left(\frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} + \Gamma_{ab}^c \right) \frac{\partial x^k}{\partial x^c} \alpha_k \\
&= X^a \frac{\partial Y^c}{\partial x^a} \alpha_c + X^a Y^b \left(\frac{\partial^2 x^k}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial x^k} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} \right) \alpha_c + X^a X^b \Gamma_{ab}^c \alpha_c \\
&= \left(X^a \partial_a Y^c + X^a Y^b \Gamma_{ab}^c \right) \alpha_c
\end{aligned}$$

since

$$\frac{\partial^2 x^j}{\partial x^a \partial x^b} \frac{\partial x^c}{\partial x^j} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^a} \left(\frac{\partial x^j}{\partial x^b} \frac{\partial x^c}{\partial x^j} \right) = \partial_a \delta_b^c = 0.$$

□

4.3 Geodesics

Geodesic equation Hopf-Rinow theorem Exponential map, Gauss lemma Jacobi fields Cartan-Hadamrd

Chapter 5

Curvature

Chapter 6

Part III

Lie groups

Chapter 7

Lie correspondence

7.1 Exponential map

7.1 (Exponential map).

7.2 (Surjectivity of exponential map).

7.3 (Lie functor).

7.2 Lie's second theorem

7.4 (Derivative of the exponential map). Let G be a Lie group.

(a)
$$\frac{d}{ds}\exp(sX) = \exp(sX)X$$
for $s \in \mathbb{R}$ and $X \in \mathfrak{g}$.

(b)
$$\frac{\partial}{\partial s}$$

7.5 (Baker-Campbell-Hausdorff formula). Let G be a Lie group. Let $X, Y \in \mathfrak{g}$ such that $\exp(X)\exp(Y)$ Define
$$Z(t) := \log(\exp(X)\exp(tY))$$

7.6. (a) The Lie functor
$$\mathrm{Lie} : \mathrm{LieGrp}_{simple} \to \mathrm{LieAlg}_{\mathbb{R}}$$
is fully faithful.

7.3 Lie's third theorem

7.7 (Ado's theorem).

7.8 (Lie's third theorem). Also called the Cartan-Lie theorem.

(a) The Lie functor
$$\mathrm{Lie} : \mathrm{LieGrp}_{simple} \to \mathrm{LieAlg}_{\mathbb{R}}$$
is essentially surjective.

7.4 Fundamental groups of Lie groups

Chapter 8

Compact Lie groups

8.1 Special orthogonal groups

8.2 Special unitary groups

8.3 Symplectic groups

Exercises

8.1 (Lorentz group). $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}^+(1,3)$

(a) $O(1,3)$ has four components and $SO^+(1,3)$ is the identity component. Orthochronous $O^+(1,3)$, proper $SO(1,3)$.

Chapter 9

Representations of Lie groups

9.1 Peter-Weyl theorem

9.2 Spin representations

Clifford algebra