Fano Threefolds

Ikhan Choi
Lectured by Hiromu Tanaka
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Contents

1 Day 1: April 6

Grade: solve 2∼4 exercises (report)

Throughout this lecture,

- we work over $\mathbb{C}$.
- A projective scheme is a projective scheme over $\mathbb{C}$, i.e. a closed subscheme of $\mathbb{P}^N_{\mathbb{C}}$ for some N.
- A variety is an integral scheme which is separated and of finite type over $\mathbb{C}$.

Definition 1.1. A Fano variety is a smooth projective variety X such that $-K_X$ is ample.

Definition 1.2. Let X be a smooth variety. A canonical divisor K_X is a Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X := \bigwedge^{\dim X} \Omega^1_X \in \mathrm{Pic}(X)$. ($\Omega$ is a locally free sheaf of rank($= \dim X$)) the canonical divisor

Example 1.3. If X is a smooth projective curve, then X is Fano iff $X \equiv \mathbb{P}^1$.

Proof.
1. A divisor D on X is ample iff $\deg D > 0$. ($\deg D = \sum a_i$ for $D = \sum a_i P_i$)
2. $\deg K_X = 2g - 2$, ($g := h^1(X, \mathcal{O}_X) \in \mathbb{Z}_{2n}$)
3. $g = 0$ iff $X = \mathbb{P}^1$.

Moreover, $\mathbb{P}^n$ is Fano. □

Example 1.4. Let $X \subset \mathbb{P}^N$: smooth hypersurface of deg d. For example, we may consider $X = \{x_0^d + \cdots + x_N^d\}$. Then, X is Fano iff $d \leq N$.

Proof. (Sketch) By the adjunction formula,

$$\mathcal{O}_X(K_X) \cong \mathcal{O}_{\mathbb{P}^N}(K_{\mathbb{P}^N} + X)|_X \cong \mathcal{O}_{\mathbb{P}^N}(-N - 1 - d)|_X.$$

Then, $\mathrm{Pic}\,\mathbb{P}^N = \{\mathcal{O}_{\mathbb{P}^N}(m) | m \in \mathbb{Z}\} \cong \mathbb{Z}$ (group isomoprhism). □

Why 3-folds? It is started by Gino Fano (1904∼), and the following theorem gives a motivation:

Theorem 1.5 (Lüroth,1876). *$\mathbb{C} \subset K \subset \mathbb{C}(x)$ be field extensions. Assume the trenscendental degree of K is one. Then, $K \cong \mathbb{C}(y)$.*

The Lüroth problem states that: if $\mathbb{C} \subset K \subset \mathbb{C}(x_1, \cdots, x_n)$ field extensions, assuming the trenscendental degree of K is n, then $K \cong \mathbb{C}(y_1, \cdots, y_n)$?

Theorem 1.6 (Castelnuovo, 1886). *The Lüroth problem is true if $n = 2$.*

The idea of this theorem is to convert Lüroth problem into a geometric version. A field extension $K \subset \mathbb{C}(x)$ corresponds to a dominant rational map $\mathbb{P}^1_{\mathbb{C}} \to X$, and the trenscendental degree one is equivalent to that X is curve. Here we may assume X to be a smooth projective curve. So, the Lüroth theorem can be restated as

Theorem 1.7. *If $\mathbb{P}^1_{\mathbb{C}} \twoheadrightarrow X$ for a smooth projective curve X, then $X \cong \mathbb{P}^1_{\mathbb{C}}$.*

For $n = 2$, we consider the rationality criterion.

Theorem 1.8. *Let X be a smooth projective surface. Then, X is rational iff $H^1(X, \mathcal{O}_X) = H^0(X, 2K_X) = 0$*

Example 1.9. If a surface X is del Pezzo(=Fano surface), then X is rational. It is because if $-K_x$ is ample then $H^0(X, 2K_X) = 0$ ($\because$ if not, then $2K_X$ is linearly equivalent to an effective divisor D, and $2(-K_X)^2 = 2K_X \cdot K_X = D \cdot K_X = \sum a_i C_i \cdot K_x \geq 0$.) Also, by the Kodaira vanishing, we have $H^1(X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X(K_X + (-K_X))) = 0$.

How about $n = 3$? We may consider

- Three-dimensional rationality criterion?
- Fano hypersurface $X \subset \mathbb{P}^4$ are rational?

To settle the second question, Fano studied similar and easier Fano threefolds.

Theorem 1.10. *There is a counterexample to Lüroth's problem. Specifically, if X is the complete intersection of deg 2 hypersurface and deg 3 hypersurface in $\mathbb{P}^5$, X is not rational (1908, Fano), but X is unirational (1912, Enriques).*

Theorem 1.11 (1942, G. Fano)**.** *There is a hypersurface of degree 3 $X \subset \mathbb{P}^4$ which is not rational but unirational.*

Remark 1.12. The proof by Fano is not rigorous, so the second question(rationality of hypersurface) is now considered as results of

- Clemes-Griffiths (deg$=3$)
- Iskovskih-Manin (deg≥ 4)

Classificaiton of Fano 3-folds

Two invariants: Picard number ρ and index r.

Definition 1.13. Let X be a smooth projective variety.

$$\rho = \rho(X) := \dim_{\mathbb{Q}}((\mathrm{Pic}X \otimes_{\mathbb{Z}} \mathbb{Q})/\equiv) \in \mathbb{Z}_{\geq 0}.$$

It is equal to $\dim_{\mathbb{Q}}((\mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q})/\equiv$, where $\mathrm{Div}X$ is the group of Weil divisors so that $\mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q}$ contains the formal linear combinations of prime divisors over $\mathbb{Q}$, and where the quivalence relation is given by $D \equiv D'$ iff $D \cdot C = D' \cdot C$ for every curve on X. From the intersection theory, $D \cdot C = \mathcal{O}_X(D) \cdot C = \deg(\mu^*\mathcal{O}_X(D))$ for $\mu : C^N \to C \hookrightarrow X$ (composition of normal and closed immersion). Then, $D \in \mathrm{Div}X \otimes_{\mathbb{Z}} \mathbb{Q}$ implies that there is $m \in \mathbb{Z}_{\geq 0}$ such that $mD \in \mathrm{Div}X$, then $D \cdot C := \frac{1}{m}((mD) \cdot C)$.

Remark 1.14. Let Xbe a Fano variety. Then, $\mathrm{Pic}X \cong \mathrm{Pic}X/\equiv \cong \mathbb{Z}^{\oplus \rho(X)}$. In particular, $D \sim D'$ implies $D \equiv D'$.

Definition 1.15. Let X be a Fano variety.

$$r = r_X := \text{the largest positive integer that divides } K_X,$$

that is, there is a divisor H such that $-K_X \sim rH$, but for $s > r$ there is no divisor H such that $-K_X \sim sH$.

We shall prove $1 \leq r \leq \dim X + 1$(for $\dim X = 3$, then $r = 1, 2, 3, 4$).

Example 1.16. Let $X = \mathbb{P}^3$. Then, $\mathrm{Pic}X \cong \mathbb{Z}H$, where H is a hyperplane, and $-K_x \equiv\sim 4H$, hence $\rho = 1$ and $r = 4$.

So here is the outline:

1. $r \geq 2$: Iskovskih, Fujita
2. $\rho = r = 1$: Iskovskih, Fujita
3. $\rho \geq 2$: Mori-Mukai

For 1, Δ-genus(Fujita) is used, and for 2 and 3, the cone theorem(minimal model program) is used. When $\dim X = 2$, using MMP, a del Pezzo surface X is reduced to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. When $\dim X = 3$, we have primitive Fano threefolds.

Our plan:

1. Cone theorem(mainly 2-dim)
2. $r \geq 2$
3. $\rho = r = 1$
4. $\rho \geq 2$ (primitive)
5. $\rho \geq 2$ (imprimitive)

Cone theorem

Theorem 1.17 (Cone theorem, Mori, 1982)**.** *Let X be a Fano variety. Then, there is rational curves $l_1, \cdots, l_m$ such that*

$$NE(X) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[l_i] \quad \textit{and} \quad -K_X \cdot l_i \leq \dim X + 1.$$

When $\rho = 3$, $NE(X) \subset N_1(X) \cong \mathbb{R}^{\rho(X)}$ is a triangular pyramid.

Definition 1.18. Let X be a smooth projective variety.

1. $Z_1(X) := \bigoplus_{C:\text{curve on } X} \mathbb{Z}C$,
2. $N_1(X) := (Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}) / \equiv$, where $Z \equiv Z'$ iff $L \cdot Z = L \cdot Z'$ for all $L \in \operatorname{Pic} X$.

It is well-known that

$$N_1(X) \times \left(\frac{\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{R}}{\equiv} \right) \to \mathbb{R}$$

induces a bijection

$$N_1(X) \to \operatorname{Hom}_{\mathbb{R}} \left(\frac{\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{R}}{\equiv}, \mathbb{R} \right),$$

therefore $\dim_{\mathbb{R}} N_1(X) = \rho(X)$.

Definition 1.19. Let X be a smooth projective variety.

1. For $Z \in Z_1(X) \otimes \mathbb{R}$, denote by $[Z] \in N_1(X)$ the numerical equivalence class of Z.
2. For $Z \in Z_1(X) \otimes \mathbb{R}$ is an effective 1-cycle.
3. $NE(X) := \{[Z] \in N_1(X) : Z \text{ effective 1-cycles}\}$

Remark 1.20. $NE(X)$ is a convex cone.

Example 1.21. Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. Let $l_i = \pi_i^{-1}(*)$ for $i = 1, 2$ be any fibers. Then, $NE(X) = \mathbb{R}\geq_0 [l_1] + \mathbb{R}_{\geq 0}[l_2]$. One direction is clear, and for the opposite, pick $[D] = [a_1 C_1 + \cdots + a_r C_r] \in NE(X)$ ($a_i \geq 0$). It is enough to show $C_i \equiv b_1 l_1 + b_2 l_2$ for some $b_1, b_2 \geq 0$. Fix a curve C on X. Note that since $\operatorname{Pic} X = \mathbb{Z}l_1 \oplus \mathbb{Z}l_2$, we have $C \equiv b_1 l_1 + b_2 l_2$, so $0 \leq C \cdot l_i = (b_1 l_1 + b_2 l_2) \cdot l_i = b_i l_1 \cdot l_2 > 0$, we are done.

References for surfaces:

- Beauville: Complex algebraic surfaces (over $\mathbb{C}$),
- Bădescu: Algebraic surfaces

References for cone thm:

- Kollár-Mori: Birational geometry of algebraic varieties
- Debarre: Higher-dimensional algebraic geometry

2 Day 2: April 13

Extremal rays

Definition 2.1. Let X be a Fano variety. A ray R is called an extremal ray (of $NE(X)$ or of X) if $\zeta, \xi \in NE(X)$ and $\zeta + \xi \in R$ imply $\zeta, \xi \in R$.

Theorem 2.2 (Contraction theorem)**.** *Let X be a Fano variety, $R = \mathbb{R}_{\geq 0}[l]$ an extremal ray for a curve l on X. Then, there is a unique morphism $f : X \to Y$ such that*

(i) Y is a projective normal variety,

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Y$,

(iii) For a curve C on X, $f(C)$ is point iff $[C] \in R$.

Note that such f can define the associated extremal ray. Moreover, we have $\rho(X) = \rho(Y) + 1$ and an exact sequence $0 \to \operatorname{Pic} Y \xrightarrow{f^*} \operatorname{Pic} X \xrightarrow{\cdot l} \mathbb{Z}$. The morphism f is called the contraction morphism of R.

Proof. See [Kollár-Mori]. □

Theorem 2.3. *Let X be a del Pezzo surface. Let $R = \mathbb{R}_{\geq 0}[l]$ be an extremal ray for a curve l on X and $f : X \to Y$ be its contraction. Then, one of the following holds:*

(A) l is a (-1)-curve and f is a blow down of l (hence $\dim Y = 2$),

(B) $\dim Y = 1$ (i.e. Y is a smooth projective curve) and $\rho(X) = 2$, and f is a $\mathbb{P}^1$-bundle with fiber l.

(C) $\dim Y = 0$ (i.e. $Y = \operatorname{Spec} \mathbb{C}$) and $\rho(X) = 1$.

Remark 2.4. Let Y be a smooth projective surface and $f : X \to Y$ be the blowup at a point $P \in Y$. Then, $l := f^{-1}(p)$ satisfies $l \cong \mathbb{P}^1$ and $l^2 = -1$; called (-1)-curve. In this case we say f is the blowdown of l.

Remark 2.5. Let X be a del Pezzo surface and $\rho(X) = 1$. Then, it is known that $X \cong \mathbb{P}^2$.

Exercise 2.6. Show the above remark.

Remark 2.7. Let X be a smooth projective rational surface. If there is no (-1)-curve on X, then $X \cong \mathbb{P}^2$ or X is isomorphic to the Hirzeburch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, where $n \in \mathbb{Z}_{\geq 0} \setminus \{1\}$.

Remark 2.8. Let X be a del Pezzo surface and $f : X \to Y$ be a $\mathbb{P}^1$-bundle on a smooth projective curve Y. Then, $Y = \mathbb{P}^1$ and $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \in \{0, 1\}$.

Sketch. Leray spectral sequence gives $H^1(Y, f_*\mathcal{O}_X (= \mathcal{O}_Y)) \hookrightarrow H^1(X, \mathcal{O}_X) = 0$, so $H^1(Y, \mathcal{O}_Y) = 0$ implies $Y = \mathbb{P}^1$.

Also, $\mathbb{P}^1$-bundle, $X \cong \mathbb{P}_{\mathbb{P}^1}(E)$ of rank two, it is well known that $E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ and $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a) \oplus \mathcal{O}(b)) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(b-a))$ for $n := b - a \geq 0$. It is known that for a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ there is a section c such that $c^2 = -n$, then $n \in \{0, 1\}$. □

Lemma 2.9. *Let X be a del Pezzo surface and C a curve on X. Then, $C^2 \geq -1$.*

Proof. Write $(K_X + C) \cdot C = 2h^1(C, \mathcal{O}_C) - 2$. Recall that $(\omega_X \otimes \mathcal{O}_X(C))|_C \cong \omega_C$ holds even if C is a singular curve. Hence, $C^2 \geq -K_X \cdot C - 2 \geq 1 - 2 = -1$. □

Example 2.10. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $l_i = \pi_i^{-1}(*)$ fibers. Then, each projection map π_i corresponds to the extremal rays $\mathbb{R}_{\geq 0}[l_i]$.

Example 2.11. Let $X = \mathbb{P}^2$. Then, $NE(X) = \mathbb{R}_{\geq 0}[l] = \mathbb{R}_{\geq 0}[l'] = \cdots$ since $N_1(X) = \mathbb{R}^{\rho(X)} = \mathbb{R}$.

Example 2.12. Let $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, which is del Pezzo. Then, if f is a blowdown of a section $l \cong \mathbb{P}^1$, then $\rho(Y) = 1$ and $Y \cong \mathbb{P}^2$. Then, we have two extremal rays $[l]$ and $[l']$ which correspond to f and π respectively.

Remark 2.13. Let X be a del Pezzo surface with $\rho(X) \geq 3$. Then,

$$\{\text{extremal rays}\} \longleftrightarrow \{(-1)\text{-curves}\}.$$

Therefore, a del Pezzo surface has a finitely many (-1)-curves.

Example 2.14. Let $f: X \to \mathbb{P}^2$ be a blowup at two points P and Q with $l_P = f^{-1}(P)$ and $l_Q = f^{-1}(Q)$. Lifting a line m passing through P and Q, we obtain m_X the proper transform of m. Then, $\rho(X) = 3$ and $NE(X) = \mathbb{R}_{\geq 0}[l_P] + \mathbb{R}_{\geq 0}[l_Q] + \mathbb{R}_{\geq 0}[m_X]$.

Remark 2.15. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface, for example, $X: x^3 + y^3 + z^3 + w^3 = 0$. It is well-known that X has exactly 27 (-1)-curves so that $NE(X) = \sum_{i=1}^{27} \mathbb{R}_{\geq 0}[l_i]$.

Remark 2.16. Minimal model program for del Pezzo surfaces.

$$\begin{tikzcd}
 & X:\ \text{del Pezzo surface} \arrow[d] & \\
 & \exists(-1)\text{-curve on } X? \arrow[dl, "YES"'] \arrow[dr, "NO"] & \\
f: X \to Y \text{ a blowdown of a } (-1)\text{-curve} \arrow[d] & & \exists \text{ extremal } \mathbb{R}_{\geq 0}[l] \text{ s.t. } l \text{ is not } (-1)\text{-curve} \arrow[d] \\
\text{we can show } Y \text{ is del Pezzo} & & f: X \to Y \text{ the contraction of } \mathbb{R}_{\geq 0}[l] \arrow[dl, "\dim Y=1"'] \arrow[d, "\dim Y=0"] \\
 & X \cong \mathbb{P}^1 \times \mathbb{P}^1 & X \cong \mathbb{P}^2 \ (2.5)
\end{tikzcd}$$

Remark. Let $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ with $n \in \{0, 1\}$.
If $n = 0$, then $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$.
If $n = 1$, then $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, there is a (-1)-curve on X (cf.(2.11))

Outline of (2.3). For an extremal ray $R = \mathbb{R}_{\geq 0}[l]$, (A) for $l^2 < 0$, (B) for $l^2 = 0$, (C) for $l^2 > 0$. □

Proposition 2.17. *Let X be a del Pezzo surface and l be a curve on X with $l^2 < 0$. Then,*

(a) *l is a (-1)-curve,*

(b) *$\mathbb{R}_{\geq 0}[l]$ is an extremal ray,*

(c) *the contraction of R is the blowdown of l.*

In particular, $\dim Y = \dim X = 2$.

Proof. (a) We will show the following statements are equivalent:

(i) l is a (-1)-curve,

(ii) $l \cong \mathbb{P}^1$ and $l^2 = -1$,

(iii) $K_X \cdot l = l^2 = -1$,

(iv) $K_X \cdot l < 0$ and $l^2 < 0$.

Here X is a smooth projective surface and l a curve on it. Note (i) and (ii) are equivalent by definition. The equivalence between (ii) and (iii) is due to $(K_X + l) \cdot l = 2h^1(l, \mathcal{O}_l) - 2 \geq -2$. The equivalence between (iii) and (iv) is clear.

(b) Omitted.

(c) Let $f: X \to Y$ blowdown of l and $P := f(l)$. Recall that f is a contraction of R iff

(i) Y is a projective normal variety,

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Y$,

(iii) for a curve C on X, $f(C)$ is a point iff $[C] \in \mathbb{R}_{\geq 0}[l]$.

It follows (ii) from the following lemma (2.18). For (iii), ($\Rightarrow$) is clear. ($\Leftarrow$) Suppose $[C] \in \mathbb{R}_{\geq 0}[l]$ and $C \neq l$ so that $C \cdot l \geq 0$. Then, $C \equiv al$ for $a \in \mathbb{R}_{\geq 0}$, and $a > 0$ since $C \cdot H = al \cdot H$ for ample H. Now $0 \leq C \cdot l = al \cdot l = a(>0) \cdot l^2(=-1) < 0$, a contradiction. □

Lemma 2.18. *If f is a projective birational morphism of normla varieties, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof. Consider the Stein factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

such that $g_*\mathcal{O}_X = \mathcal{O}_Z$ and h finite. Then,

$$\begin{tikzcd} K(X) & & K(Y) \arrow[ll, "\cong"'] \arrow[dl] \\ & K(Z) \arrow[ul] & \end{tikzcd}$$

implies $Z \xrightarrow{h} Y$ is finite birational morphism, and $A \hookrightarrow B$ is integral extension with $K(A) = K(B)$ where $\operatorname{Spec} A \subset Y$ is affine open and $\operatorname{Spec} B$ is given by the pullback(inverse image of h), hence $A = B$. □

Lemma 2.19. *Let X be a del Pezzo surface and $\mathbb{R}_{\geq 0}[l]$ be an extremal ray for a curve l on X, whose contraction is $f : X \to Y$. Then,*

(A) $l^2 < 0$ *iff* $\dim Y = 2$,

(B) $l^2 = 0$ *iff* $\dim Y = 1$,

(C) $l^2 > 0$ *iff* $\dim Y = 0$.

Proof. Next lecture. □

Proposition 2.20 ((B))**.** *If $l^2 = 0$, then the fiber is isomorphic to $\mathbb{P}^1$.*

Proof. For $P \in Y$, let $F := f^*P = \sum_{i=1}^r a_i C_i$ with $a_i \in \mathbb{Z}_{>0}$ and C_i prime divisors.

Claim 2.21. *Every fiber is irreducible.*

Proof. If it is reducible, then there are $C_1 \neq C_2$ in the fiber, then

$$F \cdot C_1 = (\sum_{i=1}^r a_i C_i) \cdot C_1 = a_1 C_1^2 + (\text{positive}),$$

so $C_1^2 < 0$. Then, $C_i \equiv b_i l$, so $C_1^2 < 0$ implies $l^2 < 0$ and $C_1 \cdot C_2 \geq 0$ implies $l^2 \geq 0$, a contradiction. □

We can show that every fiber F is reduced:

$$(K_X + F) \cdot F = K_X \cdot F + F^2 = K_X \cdot F + 0 < 0,$$

by the adjunction, $F \cong \mathbb{P}^1$. □

3 Day 3: April 20

Nef divisors and big divisors

Our today's goal is to prove Lemma 2.19.

Remark 3.1. Since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, $f : X \to Y$ is surjective so that $\dim Y \in \{0,1,2\}$. If we prove (A) and (C) in the Lemma 2.19, then we are enough.

Proof of Lemma 2.19 (A). ($\Rightarrow$) Proposition 2.17.

($\Leftarrow$) Note that $\dim X = \dim Y$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ imply f is birational. For an ample Cartier divisor A_Y on Y, f^*A_Y is a big divisor(defined later). Then,

$$f^*A_Y \cdot l = \deg(f^*A_Y|_l) = \deg(i^*f^*A_Y) = \deg((f|_l)^*j^*A_Y) = \deg((f|_l)^*\mathcal{O}_{f(l)}) = \deg \mathcal{O}_l = 0,$$

where $i : l \hookrightarrow X$ and $j : f(l) = * \hookrightarrow Y$ such that $fi = jf|_l$.

We can define f^*A_Y to be a big divisor if and only if there is $m \in \mathbb{Z}_{>0}$ such that mf^*A_Y is the sum of an ample divisor A and an effective divisor E. Then, $A \cdot l + E \cdot l = 0$ implies $E \cdot l < 0$, so if we write $E = \sum a_i C_i$, then $l = C_i$ for some i, hence $l^2 < 0$. □

Definition 3.2. Let X be a projective normal variety and D a Cartier divisor. Then, D is called to be big if and only if there are $m \in \mathbb{Z}_{>0}$, an ample Cartier divisor A, and an effective Cartier divisor E such that $mD = A + E$.

Remark 3.3. In the above definition, the equality $mD = A + E$ can be replaced by $\sim$ or $\equiv$.

Remark 3.4. A divisor D is big iff nD is big for all $n \in \mathbb{Z}_{>0}$ iff nD is big for some $n \in \mathbb{Z}_{>0}$.

Proposition 3.5. *Let $f : X \to Y$ be a birational morphism of projective normal varieties. For a Cartier divisor D on Y, f^*D is big iff D is big.*

Proof. Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, by tensoring $\mathcal{O}_Y(mD)$ we get

$$\mathcal{O}_Y(mD) = (f_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(mD) = f_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(mD)) = f_*f^*\mathcal{O}_Y(mD)$$

(the second equality is due to the projection formula), so

$$H^0(Y, \mathcal{O}_Y(mD)) = H^0(Y, f_*f^*\mathcal{O}_Y(mD)) = H^0(X, f^*\mathcal{O}_Y(mD)) = H^0(X, \mathcal{O}_X(mf^*(D))).$$

Therefore, f^*D is big iff D is big by Proposition 3.6. □

Proposition 3.6. *Let X be a projective normal variety and D a Cartier divisor on X. Then D is big iff there is $c \in \mathbb{Q}_{>0}$ such that for all sufficiently large m we have*

$$h^0(X, \mathcal{O}_X(mD)) > c \cdot m^{\dim X}.$$

Proof. ($\Rightarrow$) We may assume $D = A + E$ with A ample and E effective. Then, $H^0(X, mD) = H^0(X, m(A + E)) \hookleftarrow H^0(X, mA)$ by

$$0 \to \mathcal{O}_X(-mE) \to \mathcal{O}_X \to \mathcal{O}_{mE} \to 0.$$

Thus we may assume D is ample.

It is well-known that

$$\chi(X, mD) = \frac{D^{\dim X}}{(\dim X)!} m^{\dim X} + O(m^{\dim X - 1}) \in \mathbb{Z}[m]$$

from the Riemann-Roch, and by the Serr vanishing we have $\chi(X, mD) = h^0(X, mD)$ for large m, and we also have $D^{\dim X} > 0$ by Nakai's criterion.

($\Leftarrow$) Fix A a very ample divisor on X. We may assume by Bertini that A is a normal prime divisor. We have

$$0 \to \mathcal{O}_X(mD - A) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_A \to 0,$$

and $\mathcal{O}_X(mD)|_A \cong \mathcal{O}_A(mD_A)$ for some Cartier divisor D_A on A such that $\mathcal{O}_X(D)|_A \cong \mathcal{O}_A(D_A)$.

Write

$$0 \to H^0(X, mD - A) \to H^0(X, mD) \to H^0(A, mD_A).$$

Here $h_0(X, mD) \geq c \cdot m^{\dim X}$ and $h^0(A, mD_A) \leq b \cdot m^{\dim A}$ by the Exercise 3.7, we have $H^0(X, mD - A) \neq 0$ for some $m > 0$, i.e. $mD - A$ is linearly equivalent to an effective divisor. □

Exercise 3.7. Let Z be a projective normal variety and D a Cartier divisor on Z. Show that there exists $b > 0$ such that $h^0(Z, mD) \leq b \cdot m^{\dim Z}$ for all $m \in \mathbb{Z}_{>0}$. If you want, you may assume that Z is smooth.

Proof of Lemma 2.19 (C). ($\Leftarrow$) Let $\dim Y = 0$ i.e. $Y = \operatorname{Spec} \mathbb{C}$ with $\rho(X) = \rho(Y) + 1 = 1$, which implies that $l \equiv cA$ for some $c \in \mathbb{Q}$ and an ample divisor A on X. Then, we can prove $c > 0$ from $A \cdot l = A \cdot (cA) = cA^2$, hence $l^2 = (cA) \cdot (cA) = c^2 A^2 > 0$.

($\Rightarrow$) Let $l^2 > 0$. Note that if l is a curve on a smooth projective surface X such that $l^2 > 0$, then l is nef because $l \cdot C > 0$ if $l = C$ and $l \cdot C \geq 0$ if $l \neq C$, and furthermore l is big by Proposition 3.9. Fix C a curve on X. We are enough to show $[C] \in \mathbb{R}_{\geq 0}[l]$. Then, $N_1(X) = \bigoplus_C \mathbb{R}_C / \equiv$ is generated by $[l]$, we get $\rho(X) = \dim N_1(X) = 1$ and $\dim Y = 0$.

Let l be a big divisor so that there is a sufficiently large m with a rational map $f : X \dashrightarrow \mathbb{P}^N$ defined by the complete linear system $|ml|$ whose image is a surface. By considering the defining polynomials of $\varphi(C) = \overline{V_+}(f_1, \cdots, f_r)$ such that $\varphi(ml)$ is a hyperplane section, there must be f_i not vanishing on X, so we have f_i with $\overline{V_+}(f_i) \cap \varphi(X) = \varphi(C) + \varphi(E)$, where $E = \varphi^{-1}(\varphi(E))$. Then, since $\overline{V_+}(f_i) \sim \varphi((\deg f_i) ml)$, which implies $(\deg f_i) ml \sim C + E$. Thus, using the definition of extremal rays, we have $[C] \in \mathbb{R}_{\geq 0}[l]$. □

Definition 3.8. Let X be a projective normal variety. A Cartier divisor D is called nef iff $D \cdot C \geq 0$ for all curves C on X.

Proposition 3.9. *Let X be a projective normal variety and D a nef Cartier divisor. Then, D is big iff $D^{\dim X} > 0$.*

Proof. For simplicity, assume $\dim X = 2$.

($\Rightarrow$) Let $mD = A + E$ with $z \in \mathbb{Z}_{>0}$, A ample, E effective. Since $mD \cdot E \geq 0$ and $mD \cdot A = A^2 + E \cdot A > 0$, we have $(mD)^2 = mD \cdot A + mD \cdot E > 0$.

($\Leftarrow$) We may assume X is smooth by taking a resolution of X (the pullback via a rational map of a nef or big divisor is also nef of big respectively). Take H a very ample divisor on X. We also may assume $H - K_X$ is ample by the Serre criterion. Then,

$$0 \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD + H) \to \mathcal{O}_X(mD + H)|_H \to 0$$

and

$$0 \to H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_X(mD + H)) \to H^0(\mathcal{O}_X(mD + H)|_H)$$

are exact. Note that we have

$$h^0(\mathcal{O}_X(mD + H)) = \chi(X, mD + H) = \frac{(mD + H)^2}{2!} + O(m) \geq c \cdot m^2$$

by the Kodaira vanishing

$$H^i(X, mD + H) = H^i(X, K_X + (mD)_{\text{(it is nef)}} + (H - K_X)_{\text{(it is ample)}}) = 0$$

(sum of nef and ample is ample $\because$ Corollary 3.12.) and $h^0(\mathcal{O}_X(mD + H)|_H) \leq b \cdot m^{\dim H} = b \cdot m$. Therefore, $h^0(X, \mathcal{O}(mD)) \geq c' \cdot m^2$ for some c' and sufficiently large m. □

Remark 3.10. Let X be a projective normal variety with a nef divisor D. Then,

(a) $D \cdot {}^{\forall}(\text{curve}) \geq 0$ (by def),

(b) $D \cdot {}^{\forall}(\text{effective 1-cycle}) \geq 0$.

In particular, $NE(X) \subset D^{\geq 0} := \{\zeta \in N_1(X) : D \cdot \zeta \geq 0\} = D^{>0} \cup D^{\perp}$. In fact,

(c) The Kleiman-Mori cone is contained in $D^{\geq 0}$, i.e. $\overline{NE(X)} \subset D^{\geq 0}$.

Theorem 3.11 (Kleiman's ampleness criterion)**.** *Let X be a projective normal variety and D a Cartier divisor. Then, D is ample iff $\overline{NE(X)} \setminus \{0\} \subset D^{>0}$.*

Proof. Omitted. □

Corollary 3.12. *If N is nef and A is ample, then $N + A$ is ample.*

Proof. $\zeta \in \overline{NE(X)} \setminus \{0\}$ implies $(N + A) \cdot \zeta = N \cdot \zeta + A \cdot \zeta > 0$ because $N \cdot \zeta \geq 0$ and $A \cdot \zeta > 0$. □

Remark 3.13. It is useful to use $\mathbb{Q}$-divisors. For $D \in \operatorname{Div} X \otimes_{\mathbb{Z}} \mathbb{Q}$, D is defined to be nef if there is $m \in \mathbb{Z}_{>0}$ such that D is a nef Cartier divisor, and defined to be ample if there is $m \in \mathbb{Z}_{>0}$ such that D is a ample Cartier divisor. Then, a nef divisor can be approximated by $D = \lim_{\varepsilon \to 0+}(D + \varepsilon A)$.

Theorem 3.14 (Nakai-Moishezon)**.** *Let X be a projective normal variety and D a Cartier divisor. Then, D is ample (resp. nef) iff for a subvariety $Y \subset X$ we have $Y \cdot D^{\dim Y} > 0$ (resp. ≥ 0).*

Proof. For amples, well-known. For nefs, it follows from $Y \cdot D^{\dim Y} = \lim_{\varepsilon \to 0+} Y \cdot (D + \varepsilon A)^{\dim Y} \geq 0$. □