Algebraic Topology

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A characteristic class is a natural transformation $\mathrm{Prin}_G = [-, BG] \to H^n(-, A)$ for some n. They are always can be given by the pullback of classes in $H^n(BG, A)$ by the Yoneda lemma.

1. $G = \mathrm{GL}(1, \mathbb{R})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Grp}$. $BG = G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$.

$$(\mathrm{Prin}_G, \otimes) \cong (H^1(-, \mathbb{Z}/2\mathbb{Z}), +).$$

2. $G = \mathrm{GL}(1, \mathbb{C})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Grp}$. $BG = G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

$$(\mathrm{Prin}_G, \otimes) \cong (H^2(-, \mathbb{Z}), +).$$

3. $G = \mathrm{GL}(n, \mathbb{R})$. $\mathrm{Prin}_G : \mathbf{Para}^{\mathrm{op}} \to \mathbf{Set}$. $BG = G_n(\mathbb{R}^\infty)$. By Thom and Gysin,

$$H^*(BG, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, \cdots, w_n].$$

Since there is a special class in $H^n(K(A, n))$ so that the inducing map provides an isomorphism $[X, K(A, n)] = H^n(X, A)$, we have $H^n(B\mathrm{GL}(n, \mathbb{C})) \to H^n(X, A)$.

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A *generalized reduced cohomology theory on pointed CW complexes* is a sequence of functors $\widetilde{E}_q : \mathbf{hCW}_* \to \mathbf{Ab}$ for $q \in \mathbb{Z}$ which is exact and additive, and satisfies the suspension axiom.

9.1. Let X and Y be pointed CW complexes.

(a) Suppose Y is $(n-1)$-connected with non-degenerate base point for some n. Then, $[X, Y] \to [\Sigma X, \Sigma Y]$ is surjective if $\dim X \le 2n-1$, and bijective if $\dim X \le 2n-2$.

9.2. A *spectrum* is a sequence $E := (E_n)_n$ of pointed spaces together with structure maps, either $\sigma_n : \Sigma E_n \to E_{n+1}$ or $\sigma'_n : E_n \to \Omega E_{n+1}$. We have

$$[X, E_n] \xrightarrow{\sigma'_n} [X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}].$$

9.3 (Properties of spectra)**.** A spectrum $E = (E_n)_n$ is called an Ω*-spectrum* if $\sigma'_n : E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence. A *ring spectrum* is a spectrum together with a

(a) E is an Ω-spectrum if and only if $[-, E_n]$ defines a generalized reduced cohomology theory on based CW complexes.

Sphere spectra, Suspension spectra Eilenberg-MacLane spectra(ordinary cohomology theories), K-theory spectra(K-theories), Thom spectra(cobordism theories)

Let E^* be a (generalized) cohomology theory. Then, the computation of $\mathrm{Nat}([-, BO(n)], E^*) \cong E^*(BO(n))$ determines all characteristic classes of real vector bundles.

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem