

Measure Theory

Ikhan Choi

August 21, 2022

Contents

| | | |
|------------|--|-----------|
| I | Measures | 3 |
| 1 | Measure spaces | 4 |
| 1.1 | Measurable spaces | 4 |
| 1.2 | Measure spaces | 4 |
| 1.3 | Carathéodory extension | 4 |
| 2 | Measures on the real line | 8 |
| 3 | Measurable functions | 9 |
| 3.1 | Simple functions | 9 |
| 3.2 | Almost everywhere convergence | 9 |
| II | Lebesgue integral | 12 |
| 4 | Convergence theorems | 13 |
| 4.1 | Definition of Lebesgue integral | 13 |
| 4.2 | Convergence theorems | 13 |
| 4.3 | Radon-Nikodym theorem | 13 |
| 5 | Product measures | 14 |
| 5.1 | Fubini-Tonelli theorem | 14 |
| 5.2 | Lebesgue measure on Euclidean spaces | 14 |
| 6 | Measures on metric spaces | 15 |
| 6.1 | Borel measures | 15 |
| 6.2 | Riesz-Markov-Kakutani representation theorem | 15 |
| 6.3 | Hausdorff measures | 15 |
| III | Linear operators | 16 |
| 7 | Lebesgue spaces | 17 |
| 7.1 | L^p spaces | 17 |
| 7.2 | L^1 spaces | 17 |
| 7.3 | L^2 spaces | 17 |
| 7.4 | L^∞ spaces | 17 |

| | |
|--|---------------|
| 8 Bounded linear operators | 18 |
| 8.1 Continuity | 18 |
| 8.2 Density arguments | 18 |
| 8.3 Interpolation | 18 |
| 9 Convergence of linear operators | 19 |
| 9.1 Translation and multiplication operators | 19 |
| 9.2 Convolution type operators | 19 |
| 9.3 Computation of integral transforms | 19 |
| IV Fundamental theorem of calculus | 20 |
| 10 Weak derivatives | 21 |
| 11 Absolutely continuity | 22 |
| 12 Lebesgue differentiation theorem | 23 |

Part I

Measures

Chapter 1

Measure spaces

1.1 Measurable spaces

1.1 (Measurable spaces).

1.2 Measure spaces

1.2 (Definition of measures). Let (Ω, \mathcal{M}) be a measurable space. A *measure* on \mathcal{M} is a set function $\mu : \mathcal{M} \rightarrow [0, \infty] : \emptyset \mapsto 0$ that is *countably additive*: we have

$$\mu\left(\bigsqcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for $(E_i)_{i=1}^{\infty} \subset \mathcal{M}$. Here the squared cup notation reads the disjoint union.

1.3 (Continuity of measures).

1.4 (Pushforward measures).

1.5 (Complete measures).

1.3 Carathéodory extension

1.6 (Outer measures). Let Ω be a set. An *outer measure* on Ω is a set function $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty] : \emptyset \mapsto 0$ such that

(i) μ^* is *monotone*: we have

$$S_1 \subset S_2 \Rightarrow \mu^*(S_1) \leq \mu^*(S_2)$$

for $S_1, S_2 \in \mathcal{P}(\Omega)$,

(ii) μ^* is *countably subadditive*: we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

Comparing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

- (a) A set function $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty] : \emptyset \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \Rightarrow \mu^*(S) \leq \sum_{i=1}^{\infty} \mu^*(S_i)$$

for $S \in \mathcal{P}(\Omega)$ and $(S_i)_{i=1}^{\infty} \subset \mathcal{P}(\Omega)$.

- (b) For $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ by defining as

$$\mu^*(S) := \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i, B_i \in \mathcal{A} \right\},$$

where we use the convention $\inf \emptyset = \infty$.

Proof.

□

1.7 (Carathéodory measurability). Let μ^* be an outer measure on a set Ω . We want to construct a measure by restriction of μ^* on a properly defined σ -algebra. A subset $E \subset \Omega$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(\Omega)$. Let \mathcal{M} be the collection of all Carathéodory measurable subsets relative to μ^* .

- (a) \mathcal{M} is an algebra and μ^* is finitely additive on \mathcal{M} .
- (b) \mathcal{M} is a σ -algebra and μ^* is countably additive on \mathcal{M} .
- (c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \rightarrow [0, \infty]$ is complete.

Proof.

□

1.8 (Carathéodory extension theorem). The Carathéodory extension is a construction method for a measure extending a given set function ρ . The idea is to restrict the outer measure μ^* associated to ρ in order to obtain a measure μ . We want to find a sufficient condition for μ to be a measure on a σ -algebra containing \mathcal{A} .

For $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$ be a set function. Let $\mu^* : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be the associated outer measure of ρ , and $\mu : \mathcal{M} \rightarrow [0, \infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

- (a) We have $\mu^*|_{\mathcal{A}} = \rho$ if ρ satisfies the monotone countable subadditivity:

$$A \subset \bigcup_{i=1}^{\infty} B_i \Rightarrow \rho(A) \leq \sum_{i=1}^{\infty} \rho(B_i)$$

for $A \in \mathcal{A}$ and $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$.

- (b) We have $\mathcal{A} \subset \mathcal{M}$ if ρ satisfies the following property: for every $B, A \in \mathcal{A}$, and for any $\varepsilon > 0$, there are $\{C_j\}_{j=1}^{\infty}$ and $\{D_j\}_{j=1}^{\infty} \subset \mathcal{A}$ such that

$$B \cap A \subset \bigcup_{j=1}^{\infty} C_j \quad \text{and} \quad B \setminus A \subset \bigcup_{j=1}^{\infty} D_j,$$

and

$$\rho(B) + \varepsilon > \sum_{j=1}^{\infty} \rho(C_j) + \sum_{j=1}^{\infty} \rho(D_j).$$

Proof. (a) Clearly $\mu^*(A) \leq \rho(A)$ for $A \in \mathcal{A}$. We may assume $\mu^*(A) < \infty$. For arbitrary $\varepsilon > 0$ there is $\{B_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(A) + \varepsilon > \sum_{i=1}^\infty \rho(B_i) \geq \rho(A).$$

Limiting $\varepsilon \rightarrow 0$, we get $\mu^*(A) \geq \rho(A)$.

(b) Let $S \in \mathcal{P}(\Omega)$ and $A \in \mathcal{A}$. It is enough to check the inequality $\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \setminus A)$ for S with $\mu^*(S) < \infty$, so we may assume there is a countable family $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. Then, we have $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$ satisfying

$$\mu^*(S) + \varepsilon > \sum_{i=1}^\infty \left(\rho(B_i) + \frac{\varepsilon}{2^{i+1}} \right) > \sum_{i,j=1}^\infty \rho(C_{i,j}) + \sum_{i,j=1}^\infty \rho(D_{i,j}) \geq \mu^*(S \cap A) + \mu^*(S \setminus A).$$

Therefore, A is Carathéodory measurable relative to μ^* . \square

1.9 (Uniqueness of extension of measures). The existence of the Carathéodory extension provides a uniqueness theorem for the extension of measures. The important property here is σ -finiteness: for $\emptyset \in \mathcal{A} \subset \mathcal{P}(\Omega)$, let $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$ be a set function. Then, we say ρ is σ -finite if there is a countable cover $(B_i)_{i=1}^\infty \subset \mathcal{A}$ of Ω such that $\rho(B_i) < \infty$ for each i .

Let μ^* be the outer measure associated to ρ . Let \mathcal{M} be a σ -algebra such that the restriction $\mu^*|_{\mathcal{M}} : \mathcal{M} \rightarrow [0, \infty]$ is a measure, and $\mu : \mathcal{M} \rightarrow [0, \infty]$ be any measure. Suppose further that $\mu^*(A) = \rho(A) = \mu(A)$ for all $A \in \mathcal{A}$. Let $E \in \mathcal{M}$.

- (a) $\mu(E) \leq \mu^*(E)$.
- (b) If $E_1, E_2 \in \mathcal{M}$ satisfy $\mu(E_1) = \mu^*(E_1)$ and $\mu(E_2) = \mu^*(E_2)$, then $\mu(E_1 \cup E_2) = \mu^*(E_1 \cup E_2)$.
- (c) $\mu(E) = \mu^*(E)$ if $\mu^*(E) < \infty$.
- (d) If ρ is σ -finite, then $\mu(E) = \mu^*(E)$ for $\mu^*(E) = \infty$.

Proof. (a) If $\mu^*(E) = \infty$, then $\mu(E) \leq \mu^*(E)$ trivially. Suppose $\mu^*(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Also, we have

$$\mu(E) \leq \mu\left(\bigcup_{i=1}^\infty B_i\right) \leq \sum_{i=1}^\infty \mu(B_i) = \sum_{i=1}^\infty \rho(B_i)$$

whenever $E \subset \bigcup_{i=1}^\infty B_i$, so $\mu(E) \leq \mu^*(E)$.

(b) In the light of the inclusion-exclusion principle,

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2) - \mu^*(E_1 \cap E_2) \leq \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2) = \mu(E_1 \cup E_2)$$

proves the identity we want.

(c) Because $\mu^*(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^*(E) + \varepsilon > \sum_{i=1}^\infty \rho(B_i).$$

Applying the part (b) inductively, we have for every n that

$$\mu\left(\bigcup_{i=1}^n B_i\right) = \mu^*\left(\bigcup_{i=1}^n B_i\right),$$

and by limiting $n \rightarrow \infty$ the continuity from below gives

$$\mu\left(\bigcup_{i=1}^\infty B_i\right) = \mu^*\left(\bigcup_{i=1}^\infty B_i\right).$$

Then, we have

$$\mu^*(E) \leq \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) + \mu(E)$$

and

$$\mu\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) \leq \mu^*\left(\bigcup_{i=1}^{\infty} B_i \setminus E\right) = \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) - \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(B_i) - \mu^*(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu^*(E) < \varepsilon,$$

we get $\mu^*(E) < \mu(E) + \varepsilon$ and $\mu^*(E) \leq \mu(E)$ by limiting $\varepsilon \rightarrow 0$.

(d) Let $(B_i)_{i=1}^{\infty} \subset \mathcal{A}$ be such that $\rho(B_i) < \infty$ and $\Omega = \bigcup_{i=1}^{\infty} B_i$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \geq 2$. Then, $(E_i)_{i=1}^{\infty}$ is a pairwise disjoint cover of Ω with

$$\mu^*(E \cap E_i) \leq \mu^*(E_i) \leq \mu^*(B_i) = \rho(B_i) < \infty$$

for each i , so we have by the part (c) that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \sum_{i=1}^{\infty} \mu^*(E \cap E_i) = \mu^*(E). \quad \square$$

Exercises

1.10 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let \mathcal{A} be a collection of subsets of a set Ω such that $\emptyset \in \mathcal{A}$. We say \mathcal{A} is a semi-ring if it is closed under finite intersection, and the complement is a finite union of elements of \mathcal{A} . We say \mathcal{A} is a semi-algebra

Let \mathcal{A} be a semi-ring of sets over Ω . Suppose a set function $\rho : \mathcal{A} \rightarrow [0, \infty] : \emptyset \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \rho(A_i)$$

for $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

1.11 (Monotone class lemma). A collection $\mathcal{C} \subset \mathcal{P}(\Omega)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let H be a vector space closed under bounded monotone convergence. If $\text{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^\infty(\sigma(\mathcal{A})) \subset H$.

Chapter 2

Measures on the real line

2.1 (Distribution functions).

2.2 (Helly selection theorem).

2.3 (Non-Lebesgue measurable set).

Exercises

2.4 (Steinhaus theorem). Let $E \subset \mathbb{R}$ be Lebesgue measurable with $\lambda(E) > 0$.

- (a) For any $\alpha < 1$, there is an interval $I = [a, b]$ such that $\lambda(E \cap I)/\lambda(I) > \alpha$.
- (b) $E - E$ contains an open interval containing zero.

Proof. (a)

□

Problems

- *1. Every Lebesgue measurable set in \mathbb{R} of positive measure contains an arbitrarily long arithmetic progression.

Chapter 3

Measurable functions

3.1 Simple functions

3.1 (Measurability of pointwise limits). Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

Proof. Let $f(x) = \lim_{n \rightarrow \infty} s_n(x)$.

□

3.2 Almost everywhere convergence

3.2 (Almost everywhere convergence). Let (Ω, μ) be a measure space and let $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence f_n is defined as the set

$$\{x \in \Omega : \lim_{n \rightarrow \infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say f_n converges to f *almost everywhere* with respect to μ if the set of divergence is a null set in μ . We simply write

$$f_n \rightarrow f \text{ a.e.}$$

if f_n converges to f almost everywhere, and we frequently omit the measure μ if it has no confusion.

(a) If μ is complete and, if $f_n \rightarrow f$ a.e., then f is measurable.

3.3 (Tail events). Let (Ω, μ) be a measure space and let $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon > 0} \bigcap_{n > 0} \bigcup_{i \geq n} T_i^\varepsilon,$$

where

$$T_n^\varepsilon := \{x : |f_n(x) - f(x)| \geq \varepsilon\},$$

which is called the *tail event*. The term is originated from probability theory.

(a) $f_n \rightarrow f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n \rightarrow \infty} T_n^\varepsilon) = 0.$$

3.4 (Borel-Cantelli lemma).

3.5 (Convergence in measure). Let (Ω, μ) be a measure space and let $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions. We say f_n converges to a measurable function $f : \Omega \rightarrow \overline{\mathbb{R}}$ in measure if for each $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) = \lim_{n \rightarrow \infty} \mu(T_n^\varepsilon) = 0.$$

- (a) If $f_n \rightarrow f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k} \rightarrow f$ a.e.
- (b) If every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}} \rightarrow f$ a.e., then $f_n \rightarrow f$ in measure.

Proof. (a) Since $\mu(T_n^{1/k}) \rightarrow 0$ for each k as $n \rightarrow \infty$, there is n_k such that

$$\mu(T_{n_k}^{1/k}) < \frac{1}{2^k}.$$

We claim that $f_{n_k} \rightarrow f$ a.e. Since

$$\sum_{k=1}^{\infty} \mu(T_{n_k}^{1/k}) < \infty,$$

by the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k \rightarrow \infty} T_{n_k}^{1/k}) = 0.$$

For each $\varepsilon > 0$,

$$\limsup_{k \rightarrow \infty} T_{n_k}^\varepsilon = \bigcap_{k > \varepsilon^{-1}} \bigcup_{j \geq k} T_{n_j}^\varepsilon \subset \bigcap_{k > \varepsilon^{-1}} \bigcup_{j \geq k} T_{n_j}^{1/k} = \limsup_{k \rightarrow \infty} T_{n_k}^{1/k}$$

implies $f_{n_k} \rightarrow f$ a.e.

(b)

□

3.6 (Egorov theorem). Egorov's theorem informally states that an almost everywhere convergent functional sequence is “almost” uniformly convergent. Through this famous theorem, we introduce a convenient “ $\varepsilon/2^m$ ” argument, occasionally used throughout measure theory to construct a measurable set having a special property.

Let (Ω, μ) be a measure space and let $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions. Our idea is to consider a family of sequences of increasing measurable subsets which converge to full sets. Let

$$E_n^m := \bigcap_{i \geq n} \{x : |f_i(x) - f(x)| < \frac{1}{m}\}.$$

Note that $\Omega \setminus E_n^m = \bigcup_{i \geq n} T_i^{1/m}$.

- (a) Suppose $\mu(\Omega \setminus E_n^m) \rightarrow 0$ as $n \rightarrow \infty$ for each m . Then, for every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and for each m there is n satisfying $K \subset E_n^m$.
- (b) Let $\mu(\Omega) < \infty$. Then, $f_n \rightarrow f$ a.e. if and only if $\mu(\Omega \setminus E_n^m) \rightarrow 0$ as $n \rightarrow \infty$ for each m .
- (c) Let $\mu(\Omega) < \infty$. If $f_n \rightarrow f$ a.e., then for every $\varepsilon > 0$ there is a measurable $K \subset \Omega$ such that $\mu(\Omega \setminus K) < \varepsilon$ and $f_n \rightarrow f$ uniformly on K .

Proof. (a) For each m , we can find n_m such that

$$\mu(\Omega \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(\Omega \setminus K) = \mu\left(\bigcup_{m=1}^{\infty} (\Omega \setminus E_{n_m}^m)\right) \leq \sum_{m=1}^{\infty} \mu(\Omega \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(b) The set of divergence of the sequence f_n is given by

$$\bigcup_{m>0} \bigcap_{n>0} \bigcup_{i \geq n} \{x : |f_i(x) - f(x)| \geq \frac{1}{m}\} = \bigcup_{m>0} \bigcap_{n>0} (\Omega \setminus E_n^m).$$

Then, the convergence $f_n \rightarrow f$ a.e. means that for every fixed m the intersection

$$\bigcap_{n>0} (\Omega \setminus E_n^m) = \limsup_n T_n^m$$

is a null set. Since $\mu(\Omega) < \infty$ and we have $\Omega \setminus E_n^m \supset \Omega \setminus E_{n+1}^m$ clearly by definition, we are done by the continuity from above.

(c) Fix $m > 0$. Since $n \geq n_m$ implies $K \subset E_{n_m}^m \subset E_n^m$, we have

$$n \geq n_m \quad \Rightarrow \quad \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \quad \square$$

Exercises

3.7 (Cauchy's functional equation). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x + y) = f(x) + f(y)$, satisfied for all $x, y \in \mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x) = ax$ for all $x \in \mathbb{R}$, where $a := f(1)$.

- (a) $f(x) = ax$ for all $x \in \mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.
- (b) If f is continuous at a point, then f is linear.
- (c) If f is Lebesgue measurable, then f is linear.

Part II

Lebesgue integral

Chapter 4

Convergence theorems

4.1 Definition of Lebesgue integral

4.2 Convergence theorems

4.1 (Monotone convergence theorem).

4.3 Radon-Nikodym theorem

An integrable function as a measure σ -finite measures

Chapter 5

Product measures

5.1 Fubini-Tonelli theorem

5.2 Lebesgue measure on Euclidean spaces

Chapter 6

Measures on metric spaces

6.1 Borel measures

6.2 Riesz-Markov-Kakutani representation theorem

locally compact

6.3 Hausdorff measures

Part III

Linear operators

Chapter 7

Lebesgue spaces

7.1 L^p spaces

Proof.

$$\int f g \leq C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take C such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |g|^q \right)^{\frac{1}{q}}.$$

Note that we can show that $1 \leq 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \leq 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. \square

7.2 L^1 spaces

7.3 L^2 spaces

7.4 L^∞ spaces

Chapter 8

Bounded linear operators

8.1 Continuity

Schur test

8.2 Density arguments

extension of operators

8.3 Interpolation

weak L_p , marcinkiewicz

Chapter 9

Convergence of linear operators

9.1 Translation and multiplication operators

9.2 Convolution type operators

approximation of identity

9.3 Computation of integral transforms

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W_{\text{loc}}^{1,1}$.

10.1 (Product rule for weakly differentiable functions). We want to show that if u , v , and uv are weakly differentiable with respect to x_i , then $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}u v + u \partial_{x_i}v$.

10.2 (Interchange of differentiation and integration). Let $f : \Omega_x \times \Omega_y \rightarrow \mathbb{R}$ be such that $\partial_{x_i}f$ is well-defined. Suppose f and $\partial_{x_i}f$ are locally integrable in x and integrable in y .

Then,

$$\partial_{x_i} \int f(x, y) dy = \int \partial_{x_i} f(x, y) dy.$$

Chapter 11

Absolutely continuity

- (a) f is Lip_{loc} iff f' is L_{loc}^{∞}
- (b) f is AC_{loc} iff f' is L_{loc}^1
- (a) f is Lip iff f' is L^{∞}
- (b) f is AC iff f' is L^1
- (c) f is BV iff f' is a finite regular Borel measure

Chapter 12

Lebesgue differentiation theorem