Differential Equations

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Contents

Part I

Linear ordinary differential equations

Chapter 1

Constant coefficient equations

1.1 Characteristic equations

1.2 Complex roots

1.3 Repeated roots

Chapter 2

Variable coefficient equations

2.1 Series solution

2.2 Fuch's theorem

2.3 Orthogonal polynomials

2.4 Sturm-Liouville theory

2.5 The Frobenius method

Fuch's theorem

Chapter 3

Inhomogeneous equations

3.1 Method of undetermined coefficients

3.2 Variation of parameters

3.3 Damped oscillation

3.4 The Laplace transform

discontinuous data gluing

Part II

Nonlinear ordinary differential equations

Chapter 4

Nonlinear ordinary differential equations

4.1 The Picard-Lindelöf theorem

4.2 Integrating factors

Chapter 5

Dynamical systems

5.1 Equillibria

Bifurcations
Stability theory
Hamiltonian systems

5.2 Planar dynamical systems

Examples from ecology, electrical engineerings
Poincaré-Bendixon

Chapter 6

Chaos

Attractors

Part III

Linear partial differential equations

Chapter 7

Laplace's equation

7.1 Harmonic functions

7.1 (Mean value property).

7.2 (Maximum principle).

7.3 (Newtonian potential).

7.4 (Dirichlet problem for half space).

7.5 (Dirichlet problem for open ball).

7.2 Poisson equation

7.6 (Weak derivative).

7.7 (Dirac delta function). Let Ω be an open subset of $\mathbb{R}^d$. The *Dirac delta function* is a linear functional $\delta : C_c^\infty(\Omega) \to \mathbb{R}$ defined by $\delta(\varphi) := \varphi(0)$. We conventionally use the function-like notation $\delta(x)$ to denote $\varphi(0)$ by

$$\int \delta(x)\varphi(x)\,dx.$$

7.8 (Fundamental solution of the Laplace equation). Let $d \geq 2$. The *Fundamental solution of the Laplace equation* is a function $\Phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that solves the boundary value problem

$$\begin{cases} -\Delta\Phi(x) = \delta(x) & \text{in } \mathbb{R}^d, \\ \quad \Phi(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

(a) The funcdamental solution is given by

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d}\frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}.$$

In particular, Φ and $\nabla\Phi$ are locally integrable on $\mathbb{R}^d$ but $\nabla^2\Phi$ is not.

(b) For $u \in C_0^2(\mathbb{R}^d)$,

$$u(x) = -\int \Phi(x-y)\Delta u(y)\,dy.$$

Proof. Note that $\nabla\Phi(y) \cdot \nabla u(x-y)$ is integrable in y. Then,

$$\begin{aligned}
-\int \Phi(y)\Delta u(x-y)\,dy &= -\int \nabla\Phi(y) \cdot \nabla u(x-y)\,dy \\
&= -\lim_{\varepsilon\to\infty} \int_{|y|\ge\varepsilon} \nabla\Phi(y) \cdot \nabla u(x-y)\,dy \\
&= -\lim_{\varepsilon\to\infty} \int_{|y|=\varepsilon} \nabla\Phi(y) u(x-y) \cdot \nu\,dS.
\end{aligned}$$

Since

$$\nabla\Phi(x) = -\frac{1}{\omega_d}\frac{x}{|x|^d}, \qquad \nu = \frac{x}{|x|},$$

we get

$$-\int \Phi(y)\Delta u(x-y)\,dy = \lim_{\varepsilon\to\infty} \frac{1}{\omega_d \varepsilon^{d-1}} \int_{|y|=\varepsilon} u(x-y)\,dS_y = u(x).$$

□

7.9 (Green's function of the Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$ for $d \ge 2$. *Green's function of the Poisson equation* is a function $G : \Omega^2 \setminus \{(x,x) \in \Omega\} \to \mathbb{R}$ that solves the boundary value problem

$$\begin{cases} -\Delta_y G(x,y) = \delta(x-y) & \text{in } y \in \Omega \setminus \{x\}, \\ \quad G(x,y) = 0 & \text{on } y \in \partial\Omega. \end{cases}$$

for each $x \in \Omega$.

Define $\phi : \Omega^2 \to \mathbb{R}$ to be a function that solves the boundary value problem

$$\begin{cases} -\Delta_y \phi(x,y) = 0 & \text{in } y \in \Omega, \\ \quad \phi(x,y) = \Phi(x-y) & \text{on } y \in \partial\Omega. \end{cases}$$

for each $x \in \Omega$. Assume for the domain Ω that there exists a unique ϕ.

(a) Green's function is given by

$$G(x,y) = \Phi(x-y) - \phi(x,y),$$

where Φ is the fundamental solution of the Laplace equation. Physically, $y \mapsto -\phi(x,y)$ has a meaning of the electric potential generated by the induced surface charge of a grounded conductor provided a point charge is at x.

(b) The *Green representation formula* holds: for $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$u(x) = -\int_\Omega G(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} u(y)\nabla_y G(x,y) \cdot \nu\,dS_y.$$

7.10 (Existence and uniqueness of Poisson equation)**.** representation formulas describe the solution assuming

7.3 Helmholtz equation

Chapter 8

Heat equation

8.1 Heat kernel

8.2 Duhamel's principle

8.3 Separation of variables

Chapter 9

Wave equation

9.1 First order partial differential equations

9.2 Initial value problems

d'Alambert
Kirchhoff
odd reflection

9.3 Boundary value problems

Part IV

Nonlinear partial differential equations

Chapter 10

Fluid dynamics

10.1 Burger's equation

10.2 Euler's equation

10.3 Navier-Stokes equation

Chapter 11

Integrable field equations

11.1 Korteweg-de Vries equation

11.2 Boussinesq equation

11.3 Kadomtsev-Petviashvili equation

sine-Gordon equation nonlinear Schřodinger equatoin

Chapter 12

Nonlinear waves and diffusion

12.1 Nonlinear wave equation

12.2 Nonlinear diffusion equation