Geometry

Ikhan Choi

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Part I

Classical geometry

Chapter 1

Euclidean geometry

1.1 Plane geometry

1.2 Solid geometry

1.3 Axiomatization

Chapter 2

Non-Euclidean geometry

2.1 Absolute geometry

axioms 1 to 4

2.2 Spherical and elliptic geometry

axioms 2 and 4

2.3 Hyperbolic geometry

axiomes 1 to 4

Models of hyperbolic geometry (metric description) Elementary figures Isometries Length, volume, angle

Chapter 3

Non-metric geometry

3.1 Ordered and incidence geometry

axioms 1 and 2

3.2 Affine and projective geometry

axioms 1,2,5

3.3 Conformal and inversive geometry

Part II

Smooth surfaces

Chapter 4

Smooth manifolds

4.1 Local coordinates

4.2 Space curves

4.3 Space surfaces

Reparametrizations

Theorem 4.3.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 4.3.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 4.3.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 4.3.4. *Metric coefficients*

$$\begin{aligned} \langle \alpha_x, \alpha_x \rangle &=: g_{11} & \langle \alpha_x, \alpha_y \rangle &=: g_{12} \\ \langle \alpha_y, \alpha_x \rangle &=: g_{21} & \langle \alpha_y, \alpha_y \rangle &=: g_{22} \end{aligned}$$

Theorem 4.3.5 (Normal coordinates)**.** *...?*

Differentiation of tangent vectors

Definition 4.3.6. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x, y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x, y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 4.3.7 (Gauss formula, Γ_{ij}^k, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$ and $\{L_{ij}\}_{i,j=1}^2$ by the Gauss formula

$$\alpha_{xx} =: \Gamma_{11}^1\alpha_x + \Gamma_{11}^2\alpha_y + L_{11}\,\nu, \qquad \alpha_{xy} =: \Gamma_{12}^1\alpha_x + \Gamma_{12}^2\alpha_y + L_{12}\,\nu,$$
$$\alpha_{yx} =: \Gamma_{21}^1\alpha_x + \Gamma_{21}^2\alpha_y + L_{21}\,\nu, \qquad \alpha_{yy} =: \Gamma_{22}^1\alpha_x + \Gamma_{22}^2\alpha_y + L_{22}\,\nu.$$

The *Christoffel symbols* refer to eight functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^2$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned}
\partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\
&= X^i (\partial_i Y^k)\alpha_k + X^i Y^j \partial_i \alpha_j \\
&= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k\right)\alpha_k + X^i Y^j L_{ij}\,\nu.
\end{aligned}$$

Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu\rangle = \frac{1}{2}\partial_X \langle \nu, \nu\rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 4.3.8. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 4.3.9. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y)\rangle = \langle X, -\partial_Y \nu\rangle = \langle \partial_Y X, \nu\rangle = \langle \partial_X Y, \nu\rangle = \langle \mathcal{S}(X), Y\rangle. \qquad \square$$

Theorem 4.3.10. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i\alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j\alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}_i^j X_i$. Then,

$$g_{ik} X^i \mathcal{S}_j^k Y^j = \langle X, \mathcal{S}(Y)\rangle = \langle \partial_X Y, \nu\rangle = X^i Y^j L_{ij}$$

implies $g_{ik}\,\mathcal{S}_j^k = L_{ij}$. $\square$

Chapter 5

Fundamental forms

5.1 Riemannian metrics

5.2 Gaussian curvatures

Theorema egregium surfaces of constant gaussian curvature

Definition 5.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$\begin{aligned} E &:= \langle \alpha_x, \alpha_x \rangle = g_{11}, \qquad & F &:= \langle \alpha_x, \alpha_y \rangle = g_{12}, \qquad & G &:= \langle \alpha_y, \alpha_y \rangle = g_{22}, \\ L &:= \langle \alpha_{xx}, \nu \rangle = L_{11}, \qquad & M &:= \langle \alpha_{xy}, \nu \rangle = L_{12}, \qquad & N &:= \langle \alpha_{yy}, \nu \rangle = L_{22}. \end{aligned}$$

Corollary 5.2.2. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\begin{aligned} \nu_x &= \frac{FM - GL}{EG - F^2}\alpha_x + \frac{FL - EM}{EG - F^2}\alpha_y, \\ \nu_y &= \frac{FN - GM}{EG - F^2}\alpha_x + \frac{FM - EN}{EG - F^2}\alpha_y. \end{aligned}$$

Theorem 5.2.3.

$$\Gamma_{ij}^l = \frac{1}{2} g^{kl}(g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2}(\log g)_x = \Gamma_{11}^1.$$

$$\nu_x \times \nu_y = K\sqrt{\det g}\ \nu.$$

$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$

$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det \begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

5.1 (Gaussian curvature formula). (a) In general,

$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) For orthogonal coordinates such that $F \equiv 0$,

$$K = -\frac{1}{2\sqrt{\det g}}\left((\frac{1}{\sqrt{\det g}}E_y)_y + (\frac{1}{\sqrt{\det g}}G_x)_x \right).$$

(c) For $f(x,y,z)=0$,

$$K=-\frac{1}{|\nabla f|^4}\begin{vmatrix}0 & \nabla f\\ \nabla f^T & \mathrm{Hess}(f)\end{vmatrix},$$

where ∇f denotes the gradient $\nabla f=(f_x,f_y,f_z)$.

(d) (Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then

$$K=-\tau^2.$$

(e) (Brioschi) E,F,G describes K.

Proof. (a) Clear.

(b) We have $GM=EM$ and

$$\nu_x=-\frac{L}{E}\alpha_x-\frac{M}{G}\alpha_y,\qquad \nu_y=-\frac{M}{E}\alpha_x-\frac{N}{G}\alpha_y.$$

$$\nu_x\times\nu_y=\frac{LN-M^2}{EG}\alpha_x\times\alpha_y$$

After curvature tensors...

□

5.2 (Computation of Gaussian curvatures). (a) (Monge's patch) For $(x,y,f(x,y))$,

$$K=\frac{f_{xx}f_{yy}-f_{xy}^2}{(1+f_x^2+f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t)=(r(t),z(t))$ be a plane curve with $r(t)>0$. If $t\mapsto(r(t),z(t))$ is a unit-speed curve, then

$$K=-\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

Proof. (b) Let

$$\alpha(\theta,t)=(r(t)\cos\theta,r(t)\sin\theta,z(t))$$

be a parametrization of a surface of revolution. Then,

$$\begin{aligned}\alpha_\theta&=(-r(t)\sin\theta,r(t)\cos\theta,0)\\ \alpha_t&=(r'(t)\cos\theta,r'(t)\sin\theta,z'(t))\\ \nu&=\frac{1}{\sqrt{r'(t)^2+z'(t)^2}}(z'(t)\cos\theta,z'(t)\sin\theta,-r'(t)),\end{aligned}$$

and

$$\begin{aligned}\alpha_{\theta\theta}&=(-r(t)\cos\theta,-r(t)\sin\theta,0)\\ \alpha_{\theta t}&=(-r'(t)\sin\theta,-r'(t)\cos\theta,0)\\ \alpha_{tt}&=(r''(t)\cos\theta,r''(t)\sin\theta,z''(t)).\end{aligned}$$

Thus we have

$$E=r(t)^2,\quad F=0,\quad G=r'(t)^2+z'(t)^2,$$

and

$$L=-\frac{r(t)z'(t)}{\sqrt{r'(t)^2+z'(t)^2}},\quad M=0,\quad N=\frac{r''(t)z'(t)-r'(t)z''(t)}{\sqrt{r'(t)^2+z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

□

5.3 (Local isomorphism)**.** Surfaces of the same constant Gaussian curvature are locally isomorphic.

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 6

Part III

Riemann surfaces

Chapter 7

Riemann-Roch theorem

Let X be a compact Riemann surface. Consider a vector space $\mathcal{M}^\times(X) \cup \{0\}$.

$$L(D) := H^0(X, \mathcal{O}(D)) = \{f \in \mathcal{M}^\times(X) : (f) + D \geq 0\} \cup \{0\}.$$

$$\operatorname{Div}(X) = H^0(X, \mathcal{M}^\times/\mathcal{O}^\times) = \Gamma(\mathcal{M}^\times/\mathcal{O}^\times).$$

$$\operatorname{Pic}(X) = H^1(X, \mathcal{O}^\times).$$

First Chern class $H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z})$.

7.1. Let X be a compact Riemann surface. A *Weil divisor* D on X is an element of the free abelian group $\operatorname{Div}(X)$ generated by points of X. By compactness of X, a meromorphic function $f \in \mathcal{M}(X)$ gives rise to a divisor $(f) := \sum_{p \in X} \operatorname{ord}_p(f)p$. Such a divisor is called a *principal divisor*.

Let $D = \sum n_i p_i$ on X be a Weil divisor on X. Each point $P \in X$ has a meromorphic function f on an open neighborhood U of P such that $(f) = D$ on U. It implies that there is a collection $\{f_\alpha\}$ of meromorphic functions f_α defined on U_α, where $\{U_\alpha\}$ is an open cover of X, such that f_α/f_β is a well-defined holomorphic functions on $U_\alpha \cap U_\beta$. The collection $\{f_\alpha\}$ is called a *Cartier divisor*.

A Cartier divisor defines a line bundle.

7.2. Given $\{p_i\}_{i=1}^n$ points and $\{f_i\}_{i=1}^n$ principal parts, there is a meromorphic function f with predescribed principal parts if and only if for every holomorphic 1-form ω we have $\sum_{i=1}^n \operatorname{Res}(f_i\omega, p_i) = 0$.

7.3.

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

The genus can be defined by $g = h^0(X, \Omega^1)$. For algebraic curves, it can be proved as follows: Assuming the Serre duality, we have $\chi(D) = h^0(D) - h^1(D) = l(D) - l(K - D)$ and $\chi(0) = h^0(0) - h^1(0) = 1 - g$. Then, the Riemann-Roch is boiled down to $\chi(D) = \deg(D) + \chi(0)$, which can be shown inductively.

However, we want to prove a compact Riemann surface is projective as an application of the Riemann-Roch theorem, we need to prove the Riemann-Roch theorem without theory of algebraic curves.

(a) If $\deg D < 0$, then $l(D) = 0$.

Proof. (a) Let $f \in L(D) \setminus \{0\}$. Then, $(f) + D \geq 0$ and $\deg(f) = 0$ imply $\deg D \geq 0$, which is a contradiction.

(b) Let $D = 0$. Then, it follows from $l(K) = g$ and $l(0) = 1$.

Let $D > 0$. We may assume $D = \sum_{i=1}^{n} n_i p_i$ with $n_i > 0$. (why?) Let

$$V_i := \Big\{ \sum_{k=-n_i}^{-1} c_k (z - p_i)^k : c_k \in \mathbb{C} \Big\}$$

and $V := \bigoplus_{i=1}^{n} V_i$. (how can we define the principal part of f on Riemann surface?) Then, $\dim V = \deg D$. Define $L(D) \to V$ by principal part at each point p_i.

$\square$

7.4 (Embedding theorem)**.** Let X be a compact Riemann surface. The *complete linear system* of a divisor D on X is

$$|D| := \{(f) + D : f \in \mathcal{O}(X)\}.$$

Then, $|D|$ can be identified with the projective space $(L(D) \setminus \{0\})/\mathbb{C}^\times = \mathbb{CP}^{l(D)-1}$. Let $(f_i)_{i=0}^{l(D)-1}$ be a basis of $L(D)$.

For a linear system Δ of projective dimension $n-1$, we can take (how?) a basis $(f_i)_{i=0}^{n-1}$ such that the following map is well-defined:

$$X \setminus \mathrm{Bl}(\Delta) \to \mathbb{CP}^{n-1} : p \mapsto (f_0 : \cdots : f_{n-1}).$$

Chapter 8

Algebraic curves

8.1

multiplicities, Bezout theorem

8.2

divisors, line bundles euler characteristic (tangent line bundle degree 2-2g, canonical line bundle 2g-2)
$L(D) := \Gamma(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D))$

Jacobian variety (moduli spaces....)

8.1 (Chow theorem)**.** A complex submanifold of a projective space is algebraic.

Chapter 9

Uniformization

The uniformization theorem provides one philosophy to classify compact Riemann surfaces. The universal covering is one of the three: the Riemann sphere, the complex plane, and the open unit disk. Each compact Riemann surface is realized as a quotient of these model space with a properly discontinuous action.

- $g = 0$: Riemann sphere (spherical) $\to$ Riemann sphere itself
- $g = 1$: complex plane (Euclidean) $\to$ elliptic curves
- $g \geq 2$: open unit disk (hyperbolic) $\to$ hyperbolic surfaces, classified by Fuchsian groups(with which properties?)

Part IV

Topological surfaces

Chapter 10

Fundamental groups

10.1 Homotopy

10.1. A *homotopy of paths* is a continuous map $h : I \times I \to X$ such that $h(0,) = x_0$ and

(a) linear homotopy

(b) reparametrization

10.2. The fundamental group is a group composition

10.3 (Van Kampen theorem)**.**

10.2 Covering spaces

10.4 (Path lifting property)**.** Let $p : Y \to X$ be a covering map. For a path $\gamma : [0, 1] \to X$ and a point $y_0 \in Y$ such that $p(y_0) = \gamma(0)$, there is a unique lift $\widetilde{\gamma} : I \to Y$ of γ such that $\widetilde{\gamma}(0) = y_0$.

As a corollary, if γ_0 and γ_1 are end-fixing homotopic and have lifts $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ such that $\widetilde{\gamma}_0(0) = \widetilde{\gamma}_1(0)$, then $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are basepoint-preserving homotopic.

As a corollary, for $p(y_0) = x_0$, the induced map $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective.

Proof. (Uniqueness) The interval $[0, 1]$ can be replaced to any connected set.

(Existence) By the compactness of $[0, 1]$, there is an increasing finite sequence $(t_i)_{i=0}^n$ such that

$$t_0 = 0, \quad t_n = 1, \quad [t_i, t_{i+1}] \subset \gamma^{-1}(U_i), \qquad 0 \le i < n,$$

where U_i is trivializing p.

□

10.5 (Universal covering)**.** connected, locally path connected, semi-locally simply connected

10.6 (Classification of covering spaces)**.** connected, locally path connected, semi-locally simply connected $\pi_1(X, x_0)/p_*(\pi_1(Y, y_0)) \to p^{-1}(x_0)$ is always injective, and bijective if Y is path connected.

Chapter 11

Homology groups

11.1 Singular homology

11.2 Simplicial homology

11.3 Cellular homology

Chapter 12

Classification of surfaces

12.1 Combinatorial surfaces

triangulation orientability euler characteristic genus connected sum