Contents

1 Unified error analysis

1.1 Approximation of Banach spaces

We follow closely Temam for the abstract error analysis.

Definition 1.1 (Approximation)**.** Let X be a Banach space. An *approximation* of X is an indexed family X_h of finite-dimensional normed spaces, with a *prolongation operator* $P_h \in B(X_h, X)$ and a *restriction operator* $R_h \in B(X, X_h)$ such that $R_h P_h = \mathrm{id}_{X_h}$. The operator $T_h := P_h R_h$ is called the *truncation operator*.

Definition 1.2 (Errors)**.** Let X_h be an approximation of a Banach space X. For $x \in X$ and $x_h \in X_h$, the quantities $\|P_h x_h - x\|$ and $\|x_h - R_h x\|_h$ are called the *strong error* and the *discrete error* of x and x_h. The quantity $\|x - P_h R_h x\|$ is called the *truncation error*.

Definition 1.3 (Stable and convergent approximations)**.** An approximation is said to be *convergent* if the truncation error converges to zero. An approximation is said to be *stable* if the prolongation and restriction operators are uniformly bounded.

Lemma 1.1. *Let X_h be an approximation of a Banach space X. If X_h is stable and convergent, then the convergences in discrete error and strong error are equivalent:*

$$DE \le RC \cdot SE,$$

$$SE \le PC \cdot DE + TE.$$

Example 1.1.

1.2 Discretization scheme

An operator $L : \mathcal{X} \to \mathcal{Y}$ is *well-posed* if there is a continuous linear operator $L^{-1} : Y \to X$ such that $LL^{-1} = \mathrm{id}_Y$, where $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are dense embeddings. Say, consider the spaces $\mathcal{X}$ and $\mathcal{Y}$ as space of distributions.

We will always assume $L : X \to Y$ is a right invertible(i.e. well-posed) linear operator between Banach spaces.

Definition 1.4 (Discretization scheme)**.** A *discretization scheme* or a *scheme* of L is an indexed family $L_h \in L(X_h, Y_h)$ of invertible linear operators, with X_h an approximation of X and a surjective $\Pi_h \in B(Y, Y_h)$.

Note that we never use the prolongation operator for Y_h. The restriction operator Π_h is taken to be routine, but the approximation X_h of X is where we should take subtly. The art of numerical analysis begins with the choice of X_h. The following diagram does not commute, but *approximately* commute.

$$\begin{tikzcd}
X \arrow[r, "L"] \arrow[d, dashed, bend right, "R_h"'] & Y \arrow[d, "\Pi_h"] \\
X_h \arrow[r, "L_h"] \arrow[u, dashed, bend right, "P_h"'] & Y_h
\end{tikzcd}$$

If a scheme is given, we will always write $x_h := L_h^{-1}\Pi_h Lx$ the *approximate solution* for every $x \in X$.

Definition 1.5 (Properties of schemes)**.** Let L_h be a scheme of L. We say the scheme L_h is *stable* if L_h^{-1} is uniformly bounded, *convergent* if the discrete error converges to zero;

$$DE = \|x_h - R_h x\|_h \to 0,$$

and *consistent* if the consistency error converges to zero;

$$CE = \|\Pi_h Lx - L_h R_h x\|_h \to 0.$$

Theorem 1.2 (Lax equivalence)**.** *Let L_h be a scheme of L. If L_h is consistent, then it is stable if and only if it is convergent.*

Proof. One direction is clear from

$$DE = \|x_h - R_h x\|_h \le \|L_h^{-1}\|\|\Pi_h Lx - L_h R_h x\|_h = SC \cdot CE.$$

Conversely, suppose the scheme is convergent. For any $y_h \in Y_h$, since Π_h and L are surjective, there is x such that $y_h = \Pi_h Lx$ so that $L_h^{-1} y_h = x_h$ is convergent since

$$DE = \|x_h - R_h x\|_h \le RC \cdot SE.$$

By the uniform boundedness principle, L_h^{-1} is uniformly bounded. □

1.3 Consistency analysis

1.4 Stability analysis

Von Neumann theory

1.5 Applications

Example 1.2. Consider

$$\begin{cases} u'(x) - u(x) = f(x) & \text{in } x \in (0,1), \\ u(0) = c. \end{cases}$$

Let $X := C^1([0,1])$, $Y := C([0,1]) \times \mathbb{R}$, and $Au(x) := (u'(x) - u(x), u(0))$. Then it is well-posed since there is $E : Y \to X$ defined by

$$E(f,c)(x) := c + \int_0^x e^{-y} f(y)\, dy$$

satisfies

Example 1.3. Consider

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } x \in (0,1)^2, \\ u(x) = 0 & \text{on } x \in \partial(0,1)^2. \end{cases}$$

Let $X =$, $Y =$, Au

Example 1.4. Consider

$$\begin{cases} \partial u(t,x) = \Delta u(t,x) & \text{in } (t,x) \in (0,\infty) \times (0,1), \\ u(0,x) = f(x) & \text{on } x \in [0,1], \\ u(t,0) = 0 & \text{on } t \in [0,\infty), \\ u(t,1) = 0 & \text{on } t \in [0,\infty), \end{cases}$$

Let $X =$, $Y =$, Au

u_j^n, $t = t_0 + nk$, $x = x_0 + jh$

2 Banach space valued integral

2.1 (Simple functions). Let (Ω, μ) be a measure space and X a topological space with the Borel measurable structure. By a *simple function* we mean a measurable function $s : \Omega \to X$ of finite image.

Suppose X satisfies the following property: for each open set $U \subset X$ there is a countable family $\{U_i\}_{i=1}^\infty$ of open sets such that $U = \bigcup_{i=1}^\infty U_i$ and $\overline{U}_i \subset U$. This is the "pointwise approximation condition" for the target space X. In particular, every metrizable space satisfy this condition.

(a) The pointwise limit of a sequence of measurable functions is measurable.

(b) A measurable function is the pointwise limit of a sequence of simple functions, if X is separable and metrizable.

(c) The pointwise limit of a net of simple functions may not be measurable, even for real-valued functions; nets are not appropriate for measure theory.

Proof. (a) Suppose a sequence $(f_n)_n$ of measurable functions converges pointwisely to a function f. For fixed open $U \subset X$ we claim

$$f^{-1}(U) = \bigcup_{i=1}^\infty \liminf_{n\to\infty} f_n^{-1}(U_i).$$

If it is true, then $f^{-1}(U)$ is the countable set operation of measurable sets $f_n^{-1}(U_i)$. Let $\{U_i\}_i$ be the countable family associated to U taken by the "pointwise approximation condition".

($\subset$) If $\omega \in f^{-1}(U)$, then for some i we have $f(\omega) \in U_i$, so $f_n(\omega)$ is eventually in U_i, thus we have $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$.

($\supset$) If $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$ for some i, then $f_n(\omega)$ is eventually in U_i, so $f(\omega) \in \overline{U}_i \subset U$, thus we have $\omega \in f^{-1}(U)$.

(b) Suppose there is a increasing sequence of finite tagged partitions $\mathcal{P}_n \subset \mathcal{B}$ satisfying the following property: for each open-neighborhood pair (x, U) there is n and i such that $P_{n,i} \in \mathcal{P}_n$ and $x \in P_{n,i} \subset U$. We denote the tags by $t_{n,i} \in P_{n,i}$ for each $P_{n,i} \in \mathcal{P}_n$. Define

$$s_n(\omega) := t_{n,i} \quad \text{for} \quad f(\omega) \in P_{n,i}.$$

To show $s_n(\omega) \to f(\omega)$, fix an open $f(\omega) \in U \subset X$. Then, there is n_0 such that there is a sequence $(P_{n,i_n})_{n=n_0}^\infty$ satisfying $P_{n,i_n} \in \mathcal{P}_n$ and $f(\omega) \in P_{n,i_n} \subset U$. Then, for all $n \geq n_0$, we have for $f(\omega) \in P_{n,i_n}$ that $s_n(\omega) = t_{n,i_n} \in P_{n,i_n} \subset U$.

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem.

□

2.2 (Strong measurability). Let (Ω, μ) be a measure space and X a Banach space. A function $f : \Omega \to X$ is said to be *strongly measurable* or *Bochner measurable* if it is a pointwise limit of a sequence of simple functions.

(a) A

2.3 (Pettis measurability theorem). Furthermore, not only measurable, weakly measurables are also approximated by simples if X is separable Banach. We introduce the notion of separably valued functions.

If we associate a complete measure, then the pointwise convergence is relaxed to the almost everywhere convergence....

2.4 (Bochner integral). Let (Ω, μ) be a measure space and X a Banach space.

if there is a net of simple functions $(s_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$\int_\Omega \|f(\omega) - s_\alpha(\omega)\| \, d\mu \to 0$$

for $\alpha \in \mathcal{A}$.

3 Kinetic theory

3.1 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 3.1 (Velocity averaging). *Let L be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. Then,*

$$\| \int u\varphi \, dv\|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|^{1/2}_{L^2_{t,x,v}} \|Lu\|^{1/2}_{L^2_{t,x,v}}$$

for $\varphi \in C_c^\infty(\mathbb{R}_v^n)$, *Proof.* Let $m(t,x) = \int u\varphi\, dv$. By Fourier transform with respect to t and x, we have

$$\hat{u}(\tau,\xi,v) = \frac{1}{i}\frac{\widehat{Lu}(\tau,\xi,v)}{\tau + v\cdot\xi}$$

and

$$\hat{m}(\tau,\xi) = \int \hat{u}(\tau,\xi,v)\varphi(v)\, dv.$$

Fixing τ, ξ, decompose the integral and use Hölder's inequality to get

$$\begin{aligned}|\hat{m}(\tau,\xi)| &\leq \int_{|\tau+v\cdot\xi|<\alpha} |\hat{u}\varphi|\, dv + \int_{|\tau+v\cdot\xi|\geq\alpha} \frac{|\widehat{Lu}\varphi|}{|\tau+v\cdot\xi|}\, dv \\ &\leq \|\hat{u}\|_{L^2_v}^{1/2} \Big(\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\, dv\Big)^{1/2} + \|\widehat{Lu}\|_{L^2_v}^{1/2} \Big(\int_{|\tau+v\cdot\xi|\geq\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|^2}\, dv\Big)^{1/2},\end{aligned}$$

where $\alpha > 0$ is an arbitrary constant that will be determined later. Let

$$I_s(\tau,\xi,\alpha) := \int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\, dv, \qquad I_n(\tau,\xi,\alpha) := \int_{|\tau+v\cdot\xi|\geq\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|}\, dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}, \qquad I_n \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

Define coordinates (v_1, v_2) on $\mathbb{R}_v$ as follows:

$$v_1 := \frac{\tau + v\cdot\xi}{|\xi|} \in \mathbb{R}\,, \qquad v_2 := v - \frac{v\cdot\xi}{|\xi|^2}\xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v_1 - \frac{\tau}{|\xi|})^2 + |v_2|^2 \qquad \text{and} \qquad \int dv = \iint dv_2\, dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| \leq R$. If $\frac{|\tau|-\alpha}{|\xi|} > R$, then the region of integration vanishes so that $I_s = 0$. If $|\tau| \le \alpha + R|\xi|$, then

$$\begin{aligned}
I_s &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \int_{|v_2|^2 \le R^2-(v_1-\frac{\tau}{|\xi|})^2} dv_2\, dv_1 \\
&\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1|\le R} \int_{|v_2|\le R} dv_2\, dv_1 \\
&\lesssim \min\{\frac{2\alpha}{|\xi|}, R\} \cdot R^{n-1} \\
&\simeq \frac{1}{\sqrt{1+(\frac{|\xi|}{\alpha})^2}} \\
&\lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}.
\end{aligned}$$

For the second integral, suppose that φ is supported on $|v| < R$ so that $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < R$. Then,

$$\begin{aligned}
I_n &\lesssim \int_{|v_1|\ge\frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<R} \int_{|v_2|<R} \frac{1}{v_1^2|\xi|^2} dv_2\, dv_1 \\
&\simeq \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-R\} \le v_1 < \frac{|\tau|}{|\xi|}+R} \frac{1}{v_1^2|\xi|^2} dv_1 \\
&\simeq \frac{1}{|\xi|^2}\Big(\frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-R\}} - \frac{1}{\frac{|\tau|}{|\xi|}+R}\Big).
\end{aligned}$$

If $\frac{|\tau|}{|\xi|} - R > \frac{\alpha}{|\xi|}$, then

$$I_n \lesssim \frac{2R}{\tau^2 - (R|\xi|)^2} < \frac{2R}{\alpha(|\tau|+R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

If $|\tau| \le \alpha + R|\xi|$, then

$$I_n \lesssim \frac{1}{|\xi|}\frac{(|\tau|+R|\xi|)-\alpha}{\alpha(|\tau|+R|\xi|)} \le \frac{2R}{\alpha(|\tau|+R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

To sum up, we have

$$|\widehat{m}(\tau,\xi)| \lesssim \frac{1}{(\tau^2+|\xi|^2)^{1/4}}\Big(\sqrt{\alpha}\cdot\|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}}\cdot\|\widehat{Lu}\|_{L_v^2}^{1/2}\Big).$$

Letting $\alpha = \sqrt{\|\widehat{Lu}\|_{L_v^2}/\|\widehat{u}\|_{L_v^2}}$ and squaring,

$$(\tau^2+|\xi|^2)^{1/2}|\widehat{m}(\tau,\xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2}\|\widehat{Lu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on $\mathbb{R}_\tau \times \mathbb{R}^n_\xi$ and Plancheral's theorem gives

$$\|m\|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|^{1/2}_{L^2_{t,x,v}} \|Lu\|^{1/2}_{L^2_{t,x,v}}.$$

$\square$

Corollary 3.2. *Let $\mathcal{F}$ be a family of functions on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ and $L\mathcal{F}$ are bounded in $L^2_{t,x,v}$, then $\int \mathcal{F}\varphi\, dv$ is bounded in $H^{1/2}_{t,x}$.*

Theorem 3.3. *Let $\mathcal{F}$ be a family of functions on $I_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ is weakly relatively compact and $L\mathcal{F}$ is bounded in $L^1_{t,x,v}$, then $\int \mathcal{F}\varphi\, dv$ is relatively compact in $L^1_{t,x}$.*

4 Representation formulas

Theorem 4.1. *Define $\Phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ by*

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log|x| & , \ d = 2, \\ \dfrac{\Gamma(\frac{d}{2}+1)}{d(d-2)\pi^{d/2}} \dfrac{1}{|x|^{d-2}} & , \ d \geq 3. \end{cases}$$

1. *$u = \Phi$ solves*
$$-\Delta u = \delta.$$
2. *$u = \Phi * f$ solves*
$$-\Delta u = f.$$

Proof.

1. Fix $\varphi \in C^\infty_c$. We want to show
$$-\int \Phi \Delta\varphi = \varphi(0).$$
Divide and apply Stokes' theorem twice to get
$$\begin{aligned} \int \Phi\Delta\varphi &= \int_{|x|<\varepsilon} \Phi\Delta\varphi + \int_{|x|\geq\varepsilon} \Phi\Delta\varphi \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi - \int_{|x|\geq\varepsilon} \nabla\Phi\cdot\nabla\varphi + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma. \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi + \int_{|x|\geq\varepsilon} \varphi\Delta\Phi - \int_{|x|=\varepsilon} \varphi\nabla\Phi\cdot d\sigma + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi - \int_{|x|=\varepsilon} \varphi\nabla\Phi\cdot d\sigma + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma. \end{aligned}$$

The first integral is bounded as

$$|\int_{|x|<\varepsilon}\Phi\Delta\varphi|\lesssim_{\varphi}|\int_{|x|<\varepsilon}\Phi|\lesssim\begin{cases}\varepsilon^2|\log\varepsilon| & ,\ d=2,\\ \varepsilon^2 & ,\ d\geq 3.\end{cases}$$

The third integral is bounded as

$$|\int_{|x|=\varepsilon}\Phi\nabla\varphi\cdot d\sigma|\lesssim_{\varphi}|\int_{|x|=\varepsilon}\Phi\, d\sigma|\lesssim\begin{cases}\varepsilon|\log\varepsilon| & ,\ d=2,\\ \varepsilon & ,\ d\geq 3.\end{cases}$$

For the second integral, since

$$\nabla\Phi=-\frac{1}{d\,\alpha(d)}\frac{x}{|x|^d},$$

we have

□

5 Sturm-Liouville theory

5.1 Self-adjointness

Let $I=[a,b]$ and

$$L=-\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right],$$
$$0\leq p(x)\in C^\infty(I),\quad q(x)\in C^\infty(I),\quad 0<w(x)\in C^\infty(I).$$

We expect L to be self-adjoint. In this regard, our interest is ellimination of the difference term

$$\langle f,Lg\rangle-\langle Lf,g\rangle=p(f'g-fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\dfrac{d^2}{dx^2}$	$[a,b]$	Periodic
Helmholtz	$L = -\dfrac{d^2}{dx^2}$	$[a,b]$	Separated Robin
Legendre	$L = -\dfrac{d}{dx}\left((1-x^2)\dfrac{d}{dx}\right)$	$[-1,1]$	None
A. Legendre	$L = -\left[\dfrac{d}{dx}\left((1-x^2)\dfrac{d}{dx}\right) - \dfrac{m^2}{1-x^2}\right]$	$[-1,1]$	Dirichlet
Hermite	$L = -e^{x^2}\left[\dfrac{d}{dx}\left(e^{-x^2}\dfrac{d}{dx}\right)\right]$	$(-\infty,\infty)$	Polynomial growth
Laguerre			

5.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that p does not vanish on the boundary of I that we should cancel $f'g - fg'|_a^b$. View the Sturm-Liouville operator L as a non-densely defined operator on the space $C^\infty(I)$ with inner product $\langle f, g\rangle = \int_I fgw$ with domain

$$V = \{u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0,\ \beta_0 u(b) + \beta_1 u'(b) = 0\},$$

the subspace for the *separated* Robin boundary condition.

Proposition 5.1. *The operator $L : V \to C^\infty(I)$ is self-adjoint when $C^\infty(I)$ has the inner product $\langle f, g\rangle = \int_I fgw$.*

We are interested in the eigenvalue problem of $L : V \to C^\infty(I)$ on V. Fortunately, if we choose a constant $z \in \mathbb{C}\setminus\mathbb{R}$, then $(L-z)^{-1} : C^\infty(I) \to V$ is well-defined.

Proposition 5.2. *If z is not an eigenvalue of L, then $L - z : V \to C^\infty(I)$ is bijective.*

Proof. The injectivity follows from the definition of eigenvalues. We may assume that L is injective by translation $q \mapsto q - \lambda$.

Suppose $f \in C^\infty(I)$. The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$-pu'' - p'u' - qu = fw,$$
$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Let u_a, u_b be the unique solutions of the corresponding homogeneous equation with initial conditions

$$u_a(a) = -\alpha_1, \quad u_a'(a) = \alpha_0, \quad u_b(b) = -\beta_1, \quad u_b'(b) = \beta_0.$$

Then we can define $L^{-1} : C^\infty([0,1]) \to D(L)$ by

$$L^{-1}f(x) := u_a(x) \int_x^b \frac{u_b}{W[u_a, u_b]} \frac{f}{(-p)} w + u_b(x) \int_a^x \frac{u_a}{W[u_a, u_b]} \frac{f}{(-p)} w,$$

where $W[u_a, u_b] := u_a u_b' - u_b u_a'$ denotes the Wronskian. This formula is derived from variation of parameters: we can compute c_a and c_b from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u_a' & u_b' \end{pmatrix} \begin{pmatrix} c_a' \\ c_b' \end{pmatrix} \quad \Longrightarrow \quad L(c_a u_a + c_b u_b) = f.$$

Then, we can check that

$$L^{-1} L u = u$$

for $u \in D(L)$ by computation, which implies L is surjective. □

5.3 Legendre's equation

The Legendre equation is

$$(1 - x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx}\left((1 - x^2)\frac{d}{dx}\right).$$

Since $p(\pm 1) = 0$, the operator $L : C^\infty([-1,1]) \to C^\infty([-1,1])$ is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

l	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2}\log\frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x\log\frac{1+x}{1-x}, \; \cdots \in L^2(-1,1) \setminus C^\infty([-1,1])$$

as eigenvectors of L, then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned}\langle Q_0, Lf\rangle - \langle LQ_0, f\rangle &= p(x)\big(Q_0'(x)f(x) - Q_0(x)f'(x)\big)\Big|_{-1}^{1} \\ &= f(1) - f(-1)\end{aligned}$$

does not vanish in general even for $f \in C^\infty([-1,1])$.

5.4 Bessel's equation

The Bessel equation is

$$x^2u'' + xu' + (k^2x^2 - \nu^2)u = 0, \qquad \text{on } (0,\infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \nu^2\frac{1}{x}\right].$$

6 Peetre's theorem

Lemma 6.1. *Suppose a linear operator $L : C_c^\infty(M) \to C_c^\infty(M)$ satisfies*

$$\operatorname{supp}(Lu) \subset \operatorname{supp}(u) \quad \textit{for} \quad u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m, we can find $u \in C_c^\infty(U \backslash \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C. We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x.

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing \quad \text{for} \quad i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m:

$$\begin{aligned}
\sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\
&\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\
&< \infty.
\end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

□

7 Characteristic curve

Algorithm:

(a) Establish the associated vector field by substituting $u \mapsto y$.

(b) Find the integral curve.

(c) Eliminate the auxiliary variables to get an algebraic equation.

(d) Verify the computed solution is in fact the real solution.

Proposition 7.1. *Suppose that there exists a smooth solution $u : \Omega \to \mathbb{R}_y$ of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, (t,x) \in \Omega \subset \mathbb{R}_{t\geq 0} \times \mathbb{R}_x, \\ \quad u(0,x) = x, at\ x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t,x)$.

Let $\gamma : I \to \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\widetilde{\gamma} : \widetilde{I} \to M$ as the maximal integral curve of the vector field

$$\widetilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\widetilde{\gamma}(0) = \gamma(0)$. Since X and $\widetilde{X}$ coincide on M, the curve $\widetilde{\gamma}$ is also an integral curve of X with $\widetilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\widetilde{I} \subset I$ and $\gamma(\theta) = \widetilde{\gamma}(\theta)$ for all $\theta \in \widetilde{I}$.

Since M is closed in E, the open interval $\widetilde{I} = \gamma^{-1}(M)$ is closed in I, hence $\widetilde{I} = I$ by the connectedness of I. □

Definition 7.1. The projection of the integral curve γ onto Ω is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X. Once we find a necessary condition of the form of algebraic equation, we can demostrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u, we can solve the ODE: let $\gamma(\theta)=(t(\theta),x(\theta),y(\theta))$ be the integral curve of X such that $\gamma(0)=(0,\xi,\xi)$. Then, the system of ODEs

$$
\begin{aligned}
\frac{dt}{d\theta} &= 1, & t(0)&=0,\\
\frac{dx}{d\theta} &= y(\theta)^2, & x(0)&=\xi,\\
\frac{dy}{d\theta} &= 0, & y(0)&=\xi
\end{aligned}
$$

is solved as

$$t(\theta)=\theta, \qquad y(\theta)=\xi, \qquad x(\theta)=\xi^2\theta+\xi.$$

Therefore,

$$u(t,x)=\frac{-1+\sqrt{1+4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega=\{(t,x): tx>-\tfrac{1}{4}\}.$$

7.1 Wave equation

$$
\begin{aligned}
&u_{tt}-c^2u_{xx}=0 \quad \text{for} \quad t,x>0,\\
&u(0,x)=g(x), \qquad u(0,x)=h(x), \qquad u_x(t,0)=\alpha(t).
\end{aligned}
$$

Define $v:=u_t-cu_x$. Then we have

$$
\begin{cases}
v_t+cv_x=0 & t,x>0,\\
v(0,x)=h(x)-cg'(x).
\end{cases}
$$

By method of characteristic,

$$v(t,x)=h(x-ct)-cg'(x-ct).$$

Then, we can solve two system

$$
\begin{cases}
u_t-cu_x=v, & x>ct>0,\\
u(0,x)=g(x),
\end{cases}
$$

and

$$
\begin{cases}
u_t-cu_x=v, & ct>x>0,\\
u_x(t,0)=\alpha(t),
\end{cases}
$$

For the first system, introducing parameter $\xi > 0$,

$$\frac{dt}{d\theta} = 1, \qquad \frac{dx}{d\theta} = -c, \qquad \frac{dy}{d\theta} = -v(t,x),$$
$$t(0) = 0, \qquad x(0) = \xi, \qquad y(0) = g(\xi)$$

is solved as

$$t(\theta) = \theta, \qquad x(\theta) = -c\theta + \xi, \qquad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta')\, d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned} u(t,x) &= g(\xi) - \int_0^\theta v(s, \xi - cs)\, ds \\ &= g(x + ct) \\ &= \frac{3g(x+ct) - g(x-ct)}{2} - \int_0^t h(x + c(t-2s))\, ds \end{aligned}$$

7.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

(a) Suppose $u(0,x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(b) Suppose $u(0,x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(c) Suppose

$$u(0,x) = \begin{cases} 0, & x < 0 \\ times, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and "paste" the solution together.

8 Interchanging limits

8.1 Limit and derivative

Theorem 8.1. *Let f_n be a sequence of absolutely continuous functions such that f_n' converges in L^1 and $f_n(a)$ converges for a point a. Then, the limit and differentiation commutes.*

Proof. Define f such that

$$f(a) = \lim_{n\to\infty} f_n(a) \quad \text{and} \quad f'(x) = \lim_{n\to\infty} f_n'(x).$$

It remains to show $f_n \to f$.

Since

$$f(x) = f(a) + \int_a^x f_n'$$

and

$$f_n(x) = f_n(a) + \int_a^x f_n',$$

we have

$$\lim_{n\to\infty} |f_n(x)-f(x)| \le \lim_{n\to\infty} |f_n(a)-f(a)| + \lim_{n\to\infty} \int |f_n'-f'| = 0. \qquad \square$$

Corollary 8.2. *If $f_n \to f$ in C^1, then $Df_n \to Df$.*

8.2 Limit and integral

We want to find a criterion for This question asks the convergence

$$f_n \to f \quad \text{in} \quad L^1.$$

For a sequence of measurable functions $f_n : (X,\mu) \to \mathbb{R}$, define the maximal function

$$Mf(x) := \sup_n |f_n(x)|.$$

Theorem 8.3 (LDCT)**.** *If $\|Mf\|_{L^1} < \infty$ and $f_n \to f$ a.e., then $f_n \to f$ in L^1.*

continuity application

Theorem 8.4 (Scheffe)**.** *Let $\{f_n\}_n$ be a sequence of nonnegative functions in L^1. Suppose it converges to f pointwisely. Then,*

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1 \quad \Longrightarrow \quad \lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

8.3 Derivative and integral

Define the Newton quotient as

$$D_k f(t,x) := \frac{f(t+k,x)-f(t,x)}{k}$$

for $k \neq 0$. We mainly recognize D_k as an operator that maps $f(0,x)$ to a function of x. Then, we can say that the partial derivative $\partial_t f(0,x)$ is well-defined a.e. x if and only if

$$\lim_{k\to 0} D_k f(0,x) = \partial_t f(0,x) \qquad \text{a.e. } x.$$

We may ask about conditions for the following to hold:

$$\lim_{h\to 0} D_k f(0,x) = \partial_t f(0,x) \qquad \text{in } L^1_x(X).$$

This question naturally arise because it implies the commutability

$$\frac{d}{dt}\int f(t,x)\,dx = \int \frac{\partial}{\partial t} f(t,x)\,dx$$

at $t = 0$. As necessary conditions to formalize the statement, we must basically assume that $f(t,x) \in L_x^1$ for $|t| < \varepsilon$, and $\partial_t f(0,x) \in L_x^1$. Above this, if we give a stronger condition $\operatorname{ess\,sup}_{|t|<\varepsilon} |\partial_t f(t,x)| \in L_x^1$ than $\partial_t f(0,x) \in L_x^1$, then the L_x^1 convergence is obtained.

Theorem 8.5 (Leibniz rule). *Let $f : [0,T] \times X \to \mathbb{R}$ be a curve of integrable functions such that $f(t,x)$ is absolutely continuous in t for a.e. x. If*

$$\int \sup_{0\le t\le T} |f_t(t,x)|\,dx < \infty,$$

then $D_k f(0,x) \to f_t(t,x)$ in L_x^1.

Proof. Our strategy is to apply the Lebesgue dominated convergence theorem. In order to do this, we should control $D_k f(0,x)$ uniformly on k.

The fundamental theorem of calculus for absolute continuous functions implies

$$D_k f(0,x) = \frac{1}{k}\int_0^k \partial_t f(t,x)\,dt,$$

so we have

$$|D_k f(0,x)| \le \frac{1}{k}\int_0^k |\partial_t f(t,x)|\,dt \le \|\partial_t f(x)\|_{L_t^\infty} < \infty$$

and

$$\int |D_k f(0,x)|\,dx < \infty$$

Since the right hand side does not depend on k, the main condition for LDCT is satisfied.

The pointwise convergence (in a.e. sense) is satisfied due to the absolute continuity. By the Lebesgue dominated convergence theorem, we get the desired result. □

Corollary 8.6. *Let*

$$Tf(x) := \int k(x,y) f(y)\,dy.$$

If $|k_x(x,y)|$ is monotone in x and $k_x(x,y)f(y) \in L^1_y$ for all x, then

$$\frac{d}{dx} Tf(x) = \int k_x(x,y) f(y)\, dy.$$

Conditions:

- $\partial_t f \in L^1_{\mathrm{loc},t}$.
- $\partial_t f \in L^1_x$ or $\partial_t f \geq 0$.

Proof: Differentiate

$$\begin{aligned}\int_{t_0}^{t} \int \frac{\partial}{\partial t} f(s,x)\, dx\, ds &= \int \int_{t_0}^{t} \frac{\partial}{\partial t} f(s,x)\, ds\, dx \\ &= \int f(t,x)\, dx - \int f(t_0,x)\, dx.\end{aligned}$$

Let f be a regular distribution, i.e. $f \in L^1_{\mathrm{loc}}$.

(a) $f \in \mathrm{AC}_{\mathrm{loc}}$ iff $f' \in L^1_{\mathrm{loc}}$.

(b) $f \in \mathrm{Lip}$ iff $f' \in L^\infty_{\mathrm{loc}}$.

Here, f' denotes the distributional derivative. For $\mathrm{AC}_{\mathrm{loc}}$ we often say a function is *weakly differentiable*.

9 Existence theorems for ODE

9.1 Picard-Lindelöf theorem

Let $I = [0,T] \subset \mathbb{R}_t$ and $\Omega = \overline{B_r(a)} \subset \mathbb{R}^d_x$. Consider the following initial value problem:

$$x' = f(t,x), \qquad x(0) = a.$$

Theorem 9.1 (Global existence, $\Omega = \mathbb{R}^d$)**.** *If f is $C_t \,\mathrm{Lip}_x$ on $I \times \mathbb{R}^d$, the equation has a unique C^1 global solution on I.*

Proof. Step 1: Construction of an approximation. Define a sequence of functions $\{x_n\}$ as

$$x'_{n+1} = f(t, x_n(t)), \quad x_{n+1}(0) = a; \qquad x_0 \equiv a.$$

These inductive linear equations are classically solved with the explicit formula

$$x_{n+1}(t) = a + \int_0^t f(s, x_n(s))\, dx.$$

The sequence clearly belongs to $C^1(I) \subset C(I)$.

Step 2: Convergence of the approximation. Let

$$\sup_{t\in I} |f(t,x) - f(t,y)| \le K|x-y| \qquad \text{and} \qquad \sup_{t\in I} |f(t,a)| \le M.$$

First we have

$$|x_1(t) - x_0(t)| \le \int_0^t |f(s,a)|\, ds \le Mt.$$

By induction, we have

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\le \int_0^t |f(s, x_n(s)) - f(s, x_{n-1}(s))|\, ds \\ &\le K \int_0^t |x_n(s) - x_{n-1}(s)|\, dx \\ &\le MK^n \int_0^t \frac{s^n}{n!}\, ds \\ &= MK^n \frac{t^{n+1}}{(n+1)!}. \end{aligned}$$

This proves the absolute convergence

$$\sum_{n=0}^{n} \|x_{n+1} - x_n\|_\infty \lesssim e^{KT} - 1,$$

hence x_n uniformly converges in a local time.

$$|x'_{n+1}(t) - x'_n(t)| \le |f(t, x_n(t)) - f(t, x_{n-1}(t))| \le K|x_n(t) - x_{n-1}(t)| \le MK^{n+1} \frac{t^{n+1}}{(n+1)!}.$$

Step 3: Verification of the approximation. Let x^* be the limit of x_n. Then, by limiting

$$x_{n+1}(t) = a + \int_0^t f(s, x_n(s))\, ds,$$

we get

$$x^*(t) = a + \int_0^t f(s, x^*(s))\, ds.$$

Thus, x^* is a solution and it is easy to check x^* is C^1. □

Theorem 9.2 (Local existence)**.** *If f is $C_t \operatorname{Lip}_x$ on $I \times \Omega$, then the equation has a unique C^1 local solution.*

The interval of existence may be arbitrarily chosen such that

$$T \le R \cdot \|f\|^{-1}_{C_{t,x}(I\times\Omega)}.$$

Proof. Define $\varphi : C([0,T], \overline{B(x_0,R)}) \to C([0,T], \overline{B(x_0,R)})$ as:

$$\varphi(x)(t) := x_0 + \int_0^t f(s, x(s))\, ds.$$

It is well-defined since

$$\begin{aligned}|\varphi(x)(t) - x_0| &\le \int_0^t |f(s,x(s))|\, ds \\ &\le TM \le R.\end{aligned}$$

It is a contraction since we have

$$\begin{aligned}|\varphi(x)(t) - \varphi(y)(t)| &\le \int_0^t |f(s,x(s)) - f(s,y(s))|\, ds \\ &\le \int_0^t K|x(s) - y(s)|\, ds \\ &\le TK\|x(s) - y(s)\|\end{aligned}$$

so that

$$\|\varphi(x) - \varphi(y)\| \le TK\|x - y\|$$

□

The above one looses the Lipschitz condition to local condition.

10 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.

- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0,1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every seperable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem -> continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable stict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by $\mathcal{T}_\beta$ coincides with $\mathcal{T}_\alpha$)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- Th integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x}: D' \to D'$ is continuous. They commute (Schwarz theorem holds).
- $D \to S \to L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

11 Ultrafilter

Definition 11.1. An *ultrafilter* is a synonym for maximal filter. If we sat $\mathcal{U}$ is an *ultrafilter on a set A*, then it means $\mathcal{U}$ is a maximal filter as a directed subset of $\mathcal{P}(A)$.

existence of ultrafilter.

Theorem 11.1. *Let $\mathcal{U}$ be an ultrafilter on a set A and X be a compact space. For a function $f: A \to X$, the limit $\mathcal{U}$-$\lim f$ always exists.*

Theorem 11.2. *Let $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$ be a product space of compact spaces X_α. A net $f : \mathcal{D} \to X$ has a convergent subnet.*

1. Use Tychonoff. Compactness and net compactness are equivalent. □

2. It is a proof without Tychonoff. Let $\mathcal{U}$ be a ultrafilter on a set $\mathcal{D}$ contatining all $\uparrow d$. Define a directed set $\mathcal{E} = \{(d,U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$ as $(d,U) \succ (d',U')$ for $U \subset U'$. Let $f : \mathcal{E} \to X$ be a subnet of $f : \mathcal{D} \to X$ defined by $f_{(d,U)} = f_d$.

By the previous theorem, $\mathcal{U}$-$\lim \pi_\alpha f_d \in X_\alpha$ exsits for each α. Define $f \in X$ such that $\pi_\alpha f = \mathcal{U}$-$\lim \pi_\alpha f_d$. Let $G = \prod_\alpha G_\alpha \subset X$ be any open neighborhood of f. Then, $\pi_\alpha f \in G_\alpha$ and we have $G_\alpha = X_\alpha$ except finite. For α, we can take $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$ by definition of convergence with ultrafilter Since $U_\alpha = \mathcal{D}$ except finites, we can take an upper bound $U_0 \in \mathcal{U}$ of $\{U_\alpha\}_\alpha$. Then, by taking any $d_0 \in U_0$, we have $f_{(d,U)} \in G$ for every $(d,U) \succ (d_0,U_0)$. This means $f = \lim_{\mathcal{E}} f_{(d,U)}$, so we can say $\lim_{\mathcal{E}} f_{(d,U)}$ exists. □

12 Selected analysis problems

12.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1,2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \le x - x_0 \le x \le 2^k$, $x - x_0 \in A_k$.

$$|A_k| \ge \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\
&\geq \sum_{k=1}^{N} (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\
&= \sum_{k=1}^{N} \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&> \sum_{k=1}^{N} \frac{2^k}{2^{k+2}} \frac{1}{7k} \\
&= \frac{1}{28} \sum_{k=1}^{N} \frac{1}{k} \\
&\to \infty.
\end{aligned}$$

□

12.2. If $|xf'(x)| \leq M$ and $\frac{1}{x}\int_0^x f(y)\,dy \to L$, then $f(x) \to L$ as $x \to \infty$.

Solution. Since

$$\begin{aligned}
|f(x) - \frac{F(x)-F(a)}{x-a}| &\leq \frac{1}{x-a}\int_a^x |f(x)-f(y)|\,dy \\
&= \frac{1}{x-a}\int_a^x (x-y)|f'(c)|\,dy \\
&\leq \frac{M}{x-a}\int_a^x \frac{x-y}{c}\,dy \\
&\leq M\frac{x-a}{a}
\end{aligned}$$

by the mean value theorem and

$$f(x)-L = \left[f(x) - \frac{F(x)-F(a)}{x-a}\right] + \frac{x}{x-a}\left[\frac{F(x)}{x} - L\right] + \frac{a}{x-a}\left[\frac{F(a)}{a} - L\right],$$

we have for any $\varepsilon > 0$

$$\limsup_{x\to\infty} |f(x)-L| \leq \varepsilon$$

where a is defined by $\frac{x-a}{a} = \frac{\varepsilon}{M}$. □

12.3. Let $f_n : I \to I$ be a sequence of real functions that satisfies $|f_n(x) - f_n(y)| \le |x - y|$ whenever $|x - y| \ge \frac{1}{n}$, where $I = [0, 1]$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap I \to I$ pointwisely.

Step [.1] For $n \ge 4$, we claim

$$|x - y| \le \frac{1}{n} \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{5}{n}. \tag{1}$$

Fix $x \in I$ and take $z \in I$ such that $|x - z| = \frac{2}{n}$ so that

$$|f_n(x) - f_n(z)| \le |x - z| = \frac{2}{n}.$$

If y satisfies $|x - y| \le \frac{1}{n}$, then we have $|y - z| \ge |x - z| - |x - y| \ge \frac{1}{n}$, so we get

$$|f_n(y) - f_n(z)| \le |y - z| \le |y - x| + |x - z| \le \frac{3}{n}.$$

Combining these two inequalities proves what we want.

Step [.2] For $\varepsilon > 0$ and $N := \left\lceil \frac{15}{\varepsilon} \right\rceil$ we claim

$$|x - y| \le \frac{1}{N} \quad \text{and} \quad n > N \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{\varepsilon}{3} \tag{2}$$

when $N \ge 4$. It is allowed for $|x - y|$ to have the following two cases:

$$|x - y| \le \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x - y| \le \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \le \frac{5}{n} < \frac{5}{N} \le \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \le |x - y| \le \frac{1}{N} \le \frac{\varepsilon}{15}.$$

Hence the claim is proved.

Step [.3] We will prove f is uniformly continuous. For $\varepsilon > 0$, take $\delta := \frac{1}{N}$, where $N := \left\lceil \frac{15}{\varepsilon} \right\rceil$. We will show

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

for $x, y \in \mathbb{Q} \cap I$ and $N \geq 4$. Fix rational numbers x and y in I which satisfy $|x-y| < \delta$. Since $f_n(x)$ and $f_n(y)$ converges to $f(x)$ and $f(y)$ respectively, we may take an integer n_x and n_y, such that

$$n > n_x \quad \Longrightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{3}$$

and

$$n > n_y \quad \Longrightarrow \quad |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \tag{4}$$

Choose an integer n such that $n > \max\{n_x, n_y, N\}$. Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since f is continuous on a dense subset $\mathbb{Q} \cap I$, it has a unique continuous extension on the whole I. Let it denoted by the same notation f.

Step [.4] Finally, we are going to show $f_n \to f$ uniformly. For $\varepsilon > 0$, let $N := \left\lceil \frac{15}{\varepsilon} \right\rceil$. The uniform continuity of f allows to have $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{2}{3}\varepsilon. \tag{5}$$

Take a rational $r \in I$, depending on $x \in I$, such that $|x - r| < \min\{\frac{1}{N}, \delta\}$. Then, by (2) and (5), given $n > N \geq 4$, we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any $x \in I$. By limiting $n \to \infty$, we obtain

$$\lim_{n \to \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since ε and x are arbitrary, we can deduce the uniform convergence of f_n as $n \to \infty$. $\square$

12.4. A measurable subset of $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

Solution. Let $E \subset \mathbb{R}$ be measurable with $\mu(E) > 0$. We may assume E is bounded so that we have $E \subset I$ for a closed bounded interval since $\mathbb{R}$ is σ-compact. Let n be a positive integer arbitrarily taken. Then, we can find N such that $\sum_{k=1}^{N} \frac{1}{k} > (n-1)\frac{\mu(I)}{\mu(E)}$.

Assume that every point x in E is contained in at most $n-1$ sets among

$$E,\ \frac{1}{2}E,\ \frac{1}{3}E,\ \cdots,\ \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k\in A} \frac{1}{k}E = \varnothing$$

for any subset $A \subset \{1, \cdots, N\}$ with $|A| \geq n$. Define

$$E_A := \bigcap_{k\in A} \frac{1}{k}E \cap \bigcap_{k'\in A} \left(\frac{1}{k'}E\right)^c$$

for $A \subset \{1, \cdots, N\}$. Then, $\mu(E_A) = 0$ for $|A| \geq n$.

Note that we have

$$\mu(\tfrac{1}{k}E) = \sum_{k\in A} \mu(E_A) = \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) = \sum_{k=1}^{N} \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A) = \sum_{|A|<n} |A|\mu(E_A)$$

by double counting, and since E_A are dijoint, we have

$$\sum_{|A|<n} |A|\mu(E_A) = (n-1) \sum_{0<|A|<n} \mu(E_A) \leq (n-1)\mu(I),$$

hence a contradiction to

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) > (n-1)\mu(I).$$

Therefore, we may find an element x that belongs to $\frac{1}{k}E$ for $k \in A$, where $A \subset \{1, \cdots, N\}$ with $|A| = n$. Then, $ax \in E$ for all $a \in A \subset \mathbb{Z}$. $\square$

13 Physics problem

13.1 Resonance

Let m, b, k, A, ω_d be positive real constants. Consider an underdamped oscillator with sinusoidal diving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0, \; x'(0) = 0.$$

There are some observations:

(a) The underdamping condition means $b^2 - 4mk < 0$ so that the roots of characteristic equation are imaginary.

(b) The positivity of m, b implies the real part of solution that will be denoted by $-\beta = -\frac{b}{2m}$ is negative; it shows exponential decay of solutions.

(c) Introducing the natural frequency $\omega_n = \sqrt{k/m}$, we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega t.$$

(d) The complementary solution is computed as

$$x_c(t) = x_0 \, e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \to \infty} x_c(t) = 0.$$

(e) The condition $\beta > \omega_n$ is equivalent to that the oscillator is underdamped.

(f) Let m, k be fixed. Then, the solution x_c decays most fastly when b satisfied $b^2 = 4mk$, equivalently, $\beta = \omega_n$.

(g) When $\omega_d = \omega_n$ such that the amplitude of particular solution diverges.