

Homological Algebra

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1 Day 1: April 6

1. Modules

References: Atsushi Shiho, Yuki Yoshi Kawada

1.1. R -modules

Definition 1.1. Let R be a ring with 1. A (left) R -module is an abelian group M with a map $R \times M \rightarrow M : (a, x) \mapsto ax$ satisfying $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ab)x = a(bx)$, $1x = x$.

Example 1.2. (a) Every abelian group is a \mathbb{Z} -module. The R -module structures on an abelian group M has 1-1 correspondence with the ring homomorphisms $R \rightarrow \text{End}_{\mathbb{Z}}(M)$.

(b) $M = C^\infty(\mathbb{R})$, $R = \mathbb{R}[T]$ a polynomial ring, $R \times M \rightarrow M : (P(T), f(x)) \mapsto P(\frac{d}{dx})f(x)$.

Definition 1.3. A (left) R -submodule of M is a subgroup $N \subset M$ such that $ax \in N$ for $a \in R$, $x \in N$. A (left) R -homomorphism is a group homomorphism $M \rightarrow N$ which preserves the action of R .

Example 1.4. (a) $M = C^\infty(\mathbb{R})$, $R = \mathbb{R}[T]$, then $\varphi : M \rightarrow M : f(x) \mapsto f(x+1)$ is an R -homomorphism.

Definition 1.5. Let $f : M \rightarrow N$ be an R -homomorphism. The kernel of f is $\ker f := \{x \in M : f(x) = 0\} \xrightarrow{i} M$, and the cokernel of f is $N \xrightarrow{p} \text{coker } f := N / \text{im } f$, where the image is $\text{im } f := \{f(x) \in N : x \in M\} \xrightarrow{j} N$.

$$\begin{array}{ccccc} \ker f & \xrightarrow{i} & M & \xrightarrow{f} & N & \xrightarrow{p} & \text{coker } f \\ & & \searrow f & & \nearrow j & & \\ & & \text{im } f & & & & \end{array}$$

On each of them, there is a unique R -module structure such that the each map i, j, p becomes an R -homomorphism respectively.

Theorem 1.6 (Universal property). For the above setting, note that $fi = 0$ and $pf = 0$. If an R -homomorphism $g : M' \rightarrow M$ satisfies $fg = 0$, then there is a unique R -homomorphism $h : M' \rightarrow \ker f$ such that $g = ih$. If an R -homomorphism $g : N \rightarrow N'$ satisfies $gf = 0$, then there is a unique R -homomorphism $h : \text{coker } f \rightarrow N'$ such that $g = hp$.

1.1 Commutative diagrams and exact sequences

Definition 1.7 (Diagram). Among some R -modules suppose we have R -homomorphisms as the following diagram:

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_2 \\ f_3 \downarrow & \searrow f_2 & \downarrow g_1 \\ M_3 & \xrightarrow{g_2} & M_4 \end{array} .$$

Then, if the compositions sharing each source and target coincide, then we say the diagram is commutative. For example, we say the triangle formed by M_2, M_3, M_4 is commutative iff $g_1 = g_2 f_2$.

Definition 1.8 (Sequence). A sequence is a diagram of R -modules placed linearly as

$$\cdots \rightarrow M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \rightarrow \cdots .$$

If $\text{im } f_n = \ker f_{n+1}$ for all n , then we say the sequence is exact.

Example 1.9. (a) $f : M \rightarrow N$ is injective iff $0 \rightarrow M \xrightarrow{f} N$ is exact. $f : M \rightarrow N$ is surjective iff $M \xrightarrow{f} N \rightarrow 0$ is exact.

(b)

$$0 \rightarrow \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \rightarrow 0$$

is exact.

(c)

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

is exact.

(d)

$$0 \rightarrow \mathbb{R} \cos x \oplus \mathbb{R} \sin x \xrightarrow{n} C^\infty(\mathbb{R}) \xrightarrow{\frac{d^2}{dx^2} + 1} C^\infty(\mathbb{R}) \rightarrow 0$$

is exact.

Proposition 1.10 (Five lemma). Suppose each row is exact in the following commutative diagram:

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 & \xrightarrow{f_4} & M_5 \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \downarrow h_5 \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 & \xrightarrow{g_4} & N_5 \end{array}$$

Then,

(a)

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

(b)

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

(c)

$$\begin{array}{cccc} \downarrow & \downarrow \sim & \downarrow \sim & \downarrow \sim \end{array}$$

Proof. (a) We will show $x \in \ker h_3$ is in the image of $f_2 f_1$: $h_3(x) = 0 \implies f_3(x) = 0 \implies x = f_2(y) \implies g_2 h_2(y) = 0 \implies h_2(y) = g_1(z) \implies z = h_1(u) \implies f_1(u) = y$. Then, $x = f_2(y) = f_2 f_1(u) = 0$.

(b) Similar.

(c) Clear. □

Proposition 1.11 (Snake lemma). Suppose the second and the third rows are exact in the following commutative diagram:

$$\begin{array}{ccccccc} & \ker h_1 & & \ker h_2 & & \ker h_3 & \\ & M_1 & & M_2 & & M_3 & 0 \\ 0 & N_1 & & N_2 & & N_3 & \\ & \operatorname{coker} h_1 & & \operatorname{coker} h_2 & & \operatorname{coker} h_3 & \end{array}$$

(a) There is $\delta : \ker h_3 \rightarrow \operatorname{coker} h_1$ such that

$$\ker h_1 \xrightarrow{k_1} \ker h_2 \xrightarrow{k_2} \ker h_3 \xrightarrow{\delta} \operatorname{coker} h_1 \xrightarrow{l_1} \operatorname{coker} h_2 \xrightarrow{l_2} \operatorname{coker} h_3$$

is exact. Here k_1, k_2, l_1, l_2 are induced from f_1, f_2, g_1, g_2 , respectively. The element $\delta(x)$ is determined by u such that $x = f_2(y)$, $z = h_2(y)$, $z = g_1(u)$, and we can check that u does not depend on the choice of y .

(b)

Proof. (a) We have to show the well-definedness of δ , $\ker \subset \operatorname{im}$, and $\operatorname{im} \subset \ker$. Skip. \square

In the general abelian categories, the five lemma and the snake lemma hold but the proofs become more complicated.

1.2 Direct sum, direct product, inductive limit, direct limit

Definition 1.12. Let M_λ be a family of R -modules. The direct product is

$$\prod_{\lambda} M_{\lambda} := \{(x_{\lambda}) : x_{\lambda} \in M_{\lambda}\} \rightarrow M_{\lambda},$$

and the direct sum is the submodule of the direct product such that

$$\bigoplus_{\lambda} M_{\lambda} := \{(x_{\lambda}) : x_{\lambda} = 0 \text{ but finitely many}\} \hookrightarrow \prod_{\lambda} M_{\lambda}$$

Proposition 1.13 (Universal property). (a) For $f_{\mu} : M_{\mu} \rightarrow N$ there is unique $f : \bigoplus_{\lambda} M_{\lambda} \rightarrow N$ such that $f i_{\mu} = f_{\mu}$.

(b) For $g_{\mu} : N \rightarrow M_{\mu}$ there is unique $g : N \rightarrow \prod_{\lambda} M_{\lambda}$ such that $p_{\mu} g = g_{\mu}$.

Remark 1.14. (a) The direct sum and direct product is unique up to isomorphism by the universal property.

(b) For R -homomorphisms $f_{\lambda} : M_{\lambda} \rightarrow N_{\lambda}$ we can induce $\prod_{\lambda} f_{\lambda} : \prod_{\lambda} M_{\lambda} \rightarrow \prod_{\lambda} N_{\lambda}$ and $\bigoplus_{\lambda} f_{\lambda} : \bigoplus_{\lambda} M_{\lambda} \rightarrow \bigoplus_{\lambda} N_{\lambda}$.

(c) In the category of modules, even for infinite indices, direct product and sum commute with the kernel, cokernel, and image. In an abelian category, we may not have infinite direct product/sum.

(d) exactness also preserved under products and sums

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Let $(\Lambda, <)$ be a totally ordered set. By a direct system, we refer the family of R -modules M_λ for each $\lambda \in \Lambda$ and the family of R -homomorphisms $\tau_{\mu\lambda} : M_\lambda \rightarrow M_\mu$ for $\lambda < \mu$ such that $\tau_{\lambda\lambda} = \text{id}_{M_\lambda}$ and $\tau_{\kappa\lambda} = \tau_{\kappa\mu} \tau_{\mu\lambda}$ for $\lambda < \mu < \kappa$.

Example.1.3.3.

- (a) Let $\Lambda = \mathbb{N}$ and $n < m \Leftrightarrow n \mid m$, $M_n = \mathbb{Z}$ and $\tau_{mn}(z) : M_n \rightarrow M_m : z \mapsto (m/n)z$.
- (b) Let M be a R -module, $\{M_\lambda\}$ are finitely generated R -submodules of M , and $\lambda < \mu \Leftrightarrow M_\lambda \subset M_\mu$, with $\tau_{\mu\lambda}$ inclusions.

Definition.

$$\varinjlim M_\lambda = \varinjlim (M_\lambda, \tau_{\mu\lambda}) := \text{coker} \left(\bigoplus_{\substack{(\lambda, \mu) \in \Lambda \\ \lambda < \mu}} M_\lambda \xrightarrow{\Phi} \bigoplus_{\lambda \in \Lambda} M_\lambda \right),$$

where $\Phi((x_{\lambda\mu})) = \sum_{\lambda < \mu} \iota_\mu \tau_{\mu\lambda}(x_{\lambda\mu}) - \iota_\lambda(x_{\lambda\mu})$, and $\iota_\lambda : M_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a componentwise embedding. That is, we want to identify $x \in M_\lambda$ and $\tau_{\mu\lambda}(x) \in M_\mu$ with the map Φ .

Proposition.1.3.4. Let $\tau_\mu : M_\mu \xrightarrow{\iota_\mu} \bigoplus_{\lambda} M_\lambda \twoheadrightarrow \varinjlim M_\lambda$.

- (a) $\tau_\mu = \tau_\kappa \tau_{\kappa\mu}$.
- (b) $M_\mu \xrightarrow{f_\mu} N$ for $\mu \in \Lambda$ are R -homomorphisms, and they satisfy $f_\mu = f_\kappa \tau_{\kappa\mu}$. Then, there is a unique $f : \varinjlim M_\lambda \rightarrow N$ such that $f_\mu = f \tau_\mu$.

For each example in 1.3.3, \mathbb{Q} and M are the direct limits because it satisfies the universal property (1.3.4(b)).

Remark. (1) The direct limit is unique by the universal property up to isomorphism.

(2) If $f_\lambda : M_\lambda \rightarrow M'_\lambda$ are R -homomorphism such that

$$\begin{array}{ccc} M_\lambda & \xrightarrow{f_\lambda} & M'_\lambda \\ \downarrow & & \downarrow \\ M_\mu & \xrightarrow{f_\mu} & M'_\mu \end{array}$$

commutes for all $\lambda < \mu$, then there is a unique f such that

$$\begin{array}{ccccccc} \bigoplus_{\lambda < \mu} M_\lambda & \longrightarrow & \bigoplus_{\lambda} M_\lambda & \longrightarrow & \varinjlim M_\lambda & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \exists! f & & \downarrow \\ \bigoplus_{\lambda < \mu} M'_\lambda & \longrightarrow & \bigoplus_{\lambda} M'_\lambda & \longrightarrow & \varinjlim M'_\lambda & \longrightarrow & 0 \end{array}$$

commutes, and f is denoted by $\varinjlim f_\lambda$. It is by the universal property of cokernel.

Definition.1.3.6. A preordered set Λ is a directed set if $\forall \lambda, \lambda' \in \Lambda$, there is $\mu \in \Lambda$ such that $\lambda, \lambda' < \mu$.

Proposition. If Λ is a directed set, then there is a 1-1 correspondence

$$\left(\prod_{\lambda} M_\lambda \right) / \sim \rightarrow \varinjlim M_\lambda : [x_\lambda] \mapsto \tau_\lambda(x_\lambda),$$

where $x_\lambda \sim y_{\lambda'}$ iff there is $\mu > \lambda, \lambda'$ such that $\tau_{\mu\lambda}(x_\lambda) = \tau_{\mu\lambda'}(y_{\lambda'})$.

Proposition. *If*

$$L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \rightarrow 0$$

is exact, then

$$\operatorname{colim} L_\lambda \rightarrow \operatorname{colim} M_\lambda \rightarrow \operatorname{colim} N_\lambda \rightarrow 0$$

is exact.

Proof. The only non-trivial part is the exactness at $\operatorname{colim} M_\lambda$. We can prove it by diagram chasing. \square

Example. Examples of inverse limit

- (a) projection $\mathbb{Z}/p^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ for $m > n$.
- (b) restriction $C^\infty((-r, r)) \rightarrow C^\infty((-r', r'))$ for $r' > r$.

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Example. Limit preserves injectivity, but not surjectivity: although the diagram

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{Z}_p & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \end{array}$$

commutes, but the induced map $\mathbb{Z} \rightarrow \mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ is not surjective because we have an element $x \in \mathbb{Z}_p$ such that for $\pi_n : \mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ we have $\pi_n(x) \equiv 1 \pmod{p^n}$ for all n .

Lemma (Mittag-Leffler condition). *Let*

$$0 \rightarrow M_n \rightarrow N_n \rightarrow L_n \rightarrow 0$$

be a sequence of exact sequences. Suppose (M_n) satisfies that for each n we have a eventually constant monotonically decreasing sequence

$$M_n \supset \pi_{n,n+1}(M_{n+1}) \supset \pi_{n,n+2}(M_{n+2}) \supset \cdots$$

of submodules of M_n . Then,

$$0 \rightarrow \lim M_n \rightarrow \lim N_n \rightarrow \lim L_n \rightarrow 0.$$

Note that when we consider the sequence of kernels $p^n\mathbb{Z}$ of the maps $\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ in the above example, we can check the sequence does not satisfy the Mittag-Leffler condition.

1.4. Properties of Hom

From now on, we always let R be a commutative ring and M, N be R -modules. Define

$$\text{Hom}_R(M, N) := \{f : M \rightarrow N, R\text{-homomorphism}\}.$$

It is an R -module, which is not the case if R is not commutative. If $\varphi : N_1 \rightarrow N_2$ is an R -homomorphism, then

$$\text{Hom}_R(M, N_1) \rightarrow \text{Hom}_R(M, N_2) : f \mapsto \varphi \circ f$$

is an R -homomorphism. If $\psi : M_1 \rightarrow M_2$ is an R -homomorphism, then

$$\text{Hom}_R(M_2, N) \rightarrow \text{Hom}_R(M_1, N) : f \mapsto f \circ \psi$$

is an R -homomorphism.

Proposition.1.4.1.

(a) *If*

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3$$

is exact, then

$$0 \rightarrow \text{Hom}_R(M, N_1) \rightarrow \text{Hom}_R(M, N_2) \rightarrow \text{Hom}_R(M, N_3)$$

is exact.

(b) *If*

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is exact, then

$$0 \rightarrow \text{Hom}_R(M_3, N) \rightarrow \text{Hom}_R(M_2, N) \rightarrow \text{Hom}_R(M_1, N)$$

is exact.

Proof. (a) If $f_2 \in \text{Hom}_R(M, N_2)$ satisfies $\varphi_2 \circ f_2 = 0$, then by the universal property there exists unique $f_1 : M \rightarrow N_1$ such that the diagram

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \exists! f_1 & \downarrow f_2 & \searrow & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 \xrightarrow{\varphi_2} N_3 \end{array}$$

commutes.

(b) Similar. □

Example. For

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0,$$

The maps

$$0 \cong \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

and

$$\mathbb{Z} \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\circ(\cdot n)} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

are not surjective.

1.5. Projective modules

Definition.1.5.1. An R -module is said to be *projective* if for every surjective $\varphi : N_1 \twoheadrightarrow N_2$ and for every $f : M \rightarrow N_2$, there is a map $\tilde{f} : M \rightarrow N_1$ such that

$$\begin{array}{ccc} & M & \\ \tilde{f} \swarrow & \downarrow f & \\ N_1 & \twoheadrightarrow & N_2 \end{array}$$

commutes, equivalently,

$$\text{Hom}_R(M, N_1) \rightarrow \text{Hom}_R(M, N_2) \rightarrow 0$$

is exact for every exact $N_1 \twoheadrightarrow N_2 \rightarrow 0$.

Proposition.1.5.2. If M is a projective module, then $\text{Hom}_R(M, -)$ is an exact functor.

Proposition.1.5.3. A direct sum of R -modules is projective iff its summands are all projective. In particular, a free R -module is projective.

Corollary.1.5.4. As a corollary, a module M is projective if and only if there is another module N such that $M \oplus N$ is free.

Proof. (\Rightarrow) Take generators of $\{e_\lambda\}_\lambda$ of M . Then, for

$$f : \bigoplus_\lambda R \twoheadrightarrow M : (a_\lambda) \mapsto \sum_\lambda a_\lambda e_\lambda,$$

we have an exact sequence

$$0 \rightarrow \ker f \rightarrow \bigoplus_\lambda R \rightarrow M \rightarrow 0,$$

which is right split by applying the definition of projective modules to extend the codomain of $\text{id}_M : M \rightarrow M$.

(\Leftarrow) Clear from Proposition 1.5.3. □

Corollary.1.5.5. Let R be a PID. Then, since a submodule of a free module is free, so a module is projective if and only if it is free.

1.6. Injective modules

Definition.1.6.1. An R -module is said to be injective if for every injective $\varphi : N_1 \hookrightarrow N_2$ and for every $g : N_1 \rightarrow M$, there is a map $\tilde{g} : N_2 \rightarrow M$ such that

$$\begin{array}{ccc} N_1 & \xhookrightarrow{\varphi} & N_2 \\ \downarrow g & \swarrow \tilde{g} & \\ M & & \end{array}$$

commutes, equivalently,

$$\text{Hom}_R(N_2, M) \rightarrow \text{Hom}_R(N_1, M) \rightarrow 0$$

is exact for every exact $0 \rightarrow N_1 \rightarrow N_2$.

Proposition.1.6.3. An R -module M is injective iff the restriction $\text{Hom}(R, M) \rightarrow \text{Hom}(I, M)$ is surjective for every ideal I of R .

Proof. (\Rightarrow) Clear. (\Leftarrow) Suppose there is $x \in N_2$ such that $N_2 = N_1 + Rx$. Define an ideal I of R such that there is an exact sequence

$$0 \rightarrow I \rightarrow N_1 \oplus R \rightarrow N_2 \rightarrow 0,$$

in which the first map sends b to $(-bx, b)$ and the second map sends (y, a) to $y + ax$. Define $h : I \rightarrow M$ by $h(b) := g(bx)$ and extend it to $\tilde{h} : R \rightarrow M$. Define $\tilde{g} : N_2 \rightarrow M$ by $\tilde{g}(y + ax) := g(y) + \tilde{h}(a)$. We can check it is well-defined from the exactness of the above defining sequence of I . (To be continued..) \square

Corollary.1.6.4. If R is a PID, then an R -module M is injective iff for all $0 \neq a \in R$ the map $M \xrightarrow{a} M$ is surjective.

Proof. Let I be an ideal. If $I = 0$, then clear. If not, we have $I = aR$ for some $0 \neq a \in R$. Then, the restriction $\text{Hom}(R, M) \rightarrow \text{Hom}(I, M)$ is surjective if and only if

$$\begin{array}{ccccccc} M & \xrightarrow{\sim} & \text{Hom}(R, M) & \rightarrow & \text{Hom}(aR, M) & \xrightarrow{\sim} & aM \\ m & \mapsto & (1 \mapsto m) & \mapsto & (a \mapsto am) & \mapsto & am \end{array}$$

is surjective. \square

Example. If $R = \mathbb{Z}$, then \mathbb{Q} and \mathbb{Q}/\mathbb{Z} are injective, and \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ are not injective.

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Proof of 1.6.3. Let \mathcal{S} be the set of all pairs (N, h) such that $N_1 \subset N \subset N_2$ and

$$\begin{array}{ccc} N_1 & \hookrightarrow & N \\ \downarrow & \swarrow & \\ M & & \end{array}$$

commutes, and define a partial order \prec such that $(N, h) \prec (N', h')$ if and only if

$$\begin{array}{ccc} N & \hookrightarrow & N' \\ \downarrow & \swarrow & \\ M & & \end{array}$$

commutes. Since the union of a chain belongs to \mathcal{S} , \mathcal{S} has a maximal element (N_0, h_0) by Zorn's lemma. If $N_0 \subsetneq N_2$, then by taking $x \in N_2 \setminus N_0$, we can show N_0 is not maximal, so $N_0 = N_2$. \square

Proposition.1.6.5. Let M_λ be R -modules, and M be their product. Then, M is injective if and only if every M_λ is injective.

Proof. Apply the definition on the following diagram to show the first row is surjective:

$$\begin{array}{ccc} \text{Hom}_R(N_2, \prod_\lambda M_\lambda) & \longrightarrow & \text{Hom}_R(N_1, \prod_\lambda M_\lambda) \\ \downarrow = & & \downarrow = \\ \prod_\lambda \text{Hom}_R(N_2, M_\lambda) & \longrightarrow & \prod_\lambda \text{Hom}_R(N_1, M_\lambda). \end{array}$$

\square

Proposition.1.6.6. If M is injective \mathbb{Z} -module, then $\text{Hom}_{\mathbb{Z}}(R, M)$ is an injective R -module.

Lemma.1.6.7. Let N be an R -module and M be a \mathbb{Z} -module. Then, $\text{Hom}_{\mathbb{Z}}(R, M)$ is an R -module, and there is a bijection

$$\text{Hom}_R(N, \text{Hom}_{\mathbb{Z}}(R, M)) \cong \text{Hom}_{\mathbb{Z}}(N, M).$$

Proof of Proposition 1.6.6. Apply Lemma 1.6.7 to show the first row is surjective:

$$\begin{array}{ccc} \text{Hom}_R(N_2, \text{Hom}_{\mathbb{Z}}(R, M)) & \longrightarrow & \text{Hom}_R(N_1, \text{Hom}_{\mathbb{Z}}(R, M)) \\ \downarrow = & & \downarrow = \\ \text{Hom}_{\mathbb{Z}}(N_2, M) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(N_1, M). \end{array}$$

\square

Theorem.1.6.8. Every R -module M is embedded in an injective R -module.

Proof. Suppose $R = \mathbb{Z}$. The surjectivity of

$$\bigoplus_{\lambda} \mathbb{Z} \twoheadrightarrow \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$$

implies

$$\text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \hookrightarrow \text{Hom}_{\mathbb{Z}}(\bigoplus_{\lambda} \mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \prod_{\lambda} \mathbb{Q}/\mathbb{Z}.$$

Then, it suffices to prove the canonical map

$$M \rightarrow \text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

is injective. For non-zero $x \in M$, by the injectivity of \mathbb{Q}/\mathbb{Z} , we can extend a \mathbb{Z} -homomorphism $f : \mathbb{Z}x \rightarrow \mathbb{Q}/\mathbb{Z}$ satisfying $f(x) \neq 0$ to a \mathbb{Z} -homomorphism $\tilde{f} : M \rightarrow \mathbb{Q}/\mathbb{Z}$ satisfying $\tilde{f}(x) = f(x) \neq 0$. Therefore, we are done.

Now let R be arbitrary commutative ring. Consider an R -homomorphism

$$\Phi : M \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) : x \mapsto (a \mapsto ax),$$

which is easily checked to be injective by putting $a = 1$. Let M' be an injective \mathbb{Z} -module with an injective \mathbb{Z} -homomorphism $M \rightarrow M'$, and it induces

$$M \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, M').$$

□

1.7. Tensor products

Definition. 1.7.1. Let R be a commutative ring, and M_1, M_2, N be R -modules. Let $\Phi : M_1 \times M_2 \rightarrow N$ be an R -bilinear map. If R is non-commutative, then M_1 and M_2 are set to be right and left R -modules respectively, and Φ is just a \mathbb{Z} -bilinear map but required to satisfy an additional condition $\Phi(-a, -) = \Phi(-, a-)$. Such Φ is called a balanced product.

There is an R -module such that the following universal property holds: for every balanced product $\Phi : M_1 \times M_2 \rightarrow N$, there is a unique R -homomorphism

$$M_1 \times M_2 \xrightarrow{\otimes} M$$

$$N$$

Then, M is called the tensor product of M_1 and M_2 .

Proof. Let \tilde{M} be a free R -module generated by $M_1 \times M_2$. Let \tilde{M}_0 be a R -submodule of \tilde{M} generated by

$$\begin{aligned} (p + p', q) - (p, q) - (p', q), & \quad (p, q + q') - (p, q) - (p, q'), \\ (ap, q) - a(p, q), & \quad (p, aq) - a(p, q). \end{aligned}$$

Let $M := \tilde{M}/\tilde{M}_0$. Then, it satisfies the universal property (Exercise!).

□

Remark 4.1.1.7.2.

- (a) The tensor product is unique.
- (b) $M_1 \otimes M_2$ is an R -module.
- (c) For $f_1 : M_1 \rightarrow M'_1$ and $f_2 : M_2 \rightarrow M'_2$, we have an R -homomorphism $f_1 \otimes f_2 : M_1 \otimes M_2 \rightarrow M'_1 \otimes M'_2$ defined by

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{\otimes} & M_1 \otimes_R M_2 \\ \downarrow & & \downarrow \exists! f_1 \otimes f_2 \\ M'_1 \times M'_2 & \xrightarrow{\otimes} & M'_1 \otimes_R M'_2. \end{array}$$

Proposition 4.2.1.7.3.

- (a) $R \otimes_R M \cong M$.
- (b) $M \otimes_R R \cong M$.
- (c) $(\bigoplus_{\lambda} M_{\lambda}) \otimes_R N \cong \bigoplus_{\lambda} (M_{\lambda} \otimes_R N)$.
- (d) $N \otimes_R (\bigoplus_{\lambda} M_{\lambda}) \cong \bigoplus_{\lambda} (N \otimes_R M_{\lambda})$.

Proof. Use the universal properties for the right-hand sides. □

Proposition 4.3.1.7.4. *Let R be commutative.*

- (a) $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)$.
- (b) $M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1$.

Proof. (a) Use the universal property.
 (b) Omitted. □

Proposition 4.4.1.7.5. *If*

$$M_1 \xrightarrow{f} M_2 \rightarrow M_3 \rightarrow 0$$

is exact, then

$$N \otimes_R M_1 \rightarrow N \otimes_R M_2 \rightarrow N \otimes_R M_3 \rightarrow 0$$

is exact.

Proof. We can construct a unique Ψ by the universal property of $N \otimes M_2$ so that the following diagram commutes.

$$\begin{array}{ccccccc} N \otimes_R M_1 & \longrightarrow & N \otimes_R M_2 & \longrightarrow & \text{coker}(\text{id}_N \otimes f) & \longrightarrow & 0 \\ \otimes \uparrow & & \otimes \uparrow & & \exists! \Psi \uparrow & & \\ N \times M_1 & \longrightarrow & N \times M_2 & \longrightarrow & N \times M_3 & & \end{array}$$

Therefore, we can check $\text{coker}(\text{id}_N \otimes f)$ satisfies the universal property. □

Example. We have

$$M/IM \cong (R \otimes M)/(I \otimes M) \cong (R/I) \otimes M.$$

If $M = R/I$, then

$$I/I^2 \rightarrow R/I \rightarrow (R/I)^{\otimes 2} \rightarrow 0$$

is exact, and the first map is not injective.

Direct limit.

$$(\text{colim}_{\lambda} N_{\lambda}) \otimes_R M \cong \text{colim}_{\lambda} (N_{\lambda} \otimes_R M).$$

Proof.

$$\begin{array}{ccccccc} (\bigoplus_{\lambda < \mu} N_{\lambda}) \otimes_R M & \longrightarrow & (\bigoplus_{\lambda} N_{\lambda}) \otimes_R M & \longrightarrow & \text{coker} & \longrightarrow & 0 \\ \otimes \uparrow & & \otimes \uparrow & & \exists! \Psi \uparrow & & \\ \bigoplus_{\lambda < \mu} (N_{\lambda} \otimes_R M) & \longrightarrow & \bigoplus_{\lambda} (N_{\lambda} \otimes_R M) & \longrightarrow & \text{colim}_{\lambda} (N_{\lambda} \otimes_R M) & & \end{array}$$

□