Algebraic Topology

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Contents

I Homology 3

1 Axiomatic homology 4
1.1 Singular homology 4
1.2 Eilenberg-Steenrod axioms 4

2 Homology groups 5
2.1 Cellular homology 5
2.2 Simplicial homology 5

3 Cohomology 6
3.1 Poincaré duality 6

II Homotopy 7

4 Homotopy groups 8

5 Fibration 9
5.1 Homotopy lifting property 9
5.2 Obstruction theory 9
5.3 Hurewicz theorem 9

6 Spectral sequences 10
6.1 Serre spectral sequence 10
6.2 Adams spectral sequence 10

III Fiber bundles 11

7 Fiber bundles 12
7.1 Principal bundles 12
7.2 Classifying spaces 16
7.3 Vector bundles 16

8 Characteristic classes 18

9 K-theory 19

IV Stable homotopy theory 20

10 21

10.1 Generalized homology theory 21

Part I

Homology

Chapter 1

Axiomatic homology

1.1 Singular homology

1.2 Eilenberg-Steenrod axioms

Mayer-Vietoris sequence

Chapter 2

Homology groups

2.1 Cellular homology

CW complex, equivalence,

2.2 Simplicial homology

geometric realization, equivalence, smith normal form, simplicial approximation,

Chapter 3

Cohomology

cup product universal coefficient theorem

3.1 Poincaré duality

Part II

Homotopy

Chapter 4

Homotopy groups

Chapter 5

Fibration

5.1 Homotopy lifting property

Locally trivial bundles

pullback bundles: universal property, functoriality, restriction, section prolongation

5.2 Obstruction theory

5.3 Hurewicz theorem

$H_\bullet(\Omega S_n)$ and $H_\bullet(U(n))$ Spin, $\text{Spin}_\mathbb{C}$ structure

Chapter 6

Spectral sequences

6.1 Serre spectral sequence

(Lyndon-Hochschild-Serre)

6.2 Adams spectral sequence

Part III

Fiber bundles

Chapter 7

Fiber bundles

7.1 Principal bundles

7.1 (Structure groups). Let G be a topological group and F be a left G-space, and $p : E \to B$ be a fiber bundle with fiber F. We say an atlas $\{\varphi_i : p^{-1}(U_i) \to U_i \times F\}_i$ is a *G-atlas* if there is a set $\{g_{ij} : U_i \cap U_j \to G\}_{i,j}$ of maps such that the transition maps are given by

$$\varphi_j \circ \varphi_i^{-1}(b, f) = (b, g_{ij}(b)f), \qquad b \in U_i \cap U_j,\ f \in F.$$

A *G-bundle* with fiber F is a fiber bundle $p : E \to B$ that admits a G-atlas. In this case the group G is called the *structure group* of the fiber bundle. A *G-bundle map* is a bundle map $(\widetilde{u}, u) : (E, B) \to (E', B')$ between G-bundles together with a set $\{h_{ij'} : U_i \cap u^{-1}(U'_{j'}) \to G\}_{i,j'}$ such that

$$\varphi'_{j'} \circ \widetilde{u} \circ \varphi_i^{-1}(b, f) = (u(b), h_{ij'}(b)f), \qquad b \in U_i \cap u^{-1}(U'_{j'}),\ f \in F.$$

If $B = B'$, a G-bundle map over B is a G-bundle map $(\widetilde{u}, u)$ such that $u = \mathrm{id}_B$. We denote by $\mathbf{Bun}_F(B)$ the category of G-bundles over B with fiber F.

(a) If F is a locally compact and locally connected Hausdorff space, then every fiber bundle with fiber F is a $\mathrm{Homeo}(F)$-bundle, where $\mathrm{Homeo}(F)$ is the group of autohomeomorphism group with compact-open topology.

(b) A G-bundle map $(\widetilde{u}, u)$ is an isomorphism if and only if u is a homeomorphism.

(c) A bundle map $(\widetilde{u}, \mathrm{id}_B) : (E, B) \to (E', B)$ is a G-bundle map if and only if there is a set $\{h_i : U_i \to G\}_i$ such that

$$\varphi'_i \circ \widetilde{u} \circ \varphi_i^{-1}(b, f) = (b, h_i(b)f), \qquad b \in U_i,\ f \in F,$$

where $\{U_i\}$ is an open cover over which both E and E' are trivialized.

Proof. (a)

(b) ($\Rightarrow$) Clear.

($\Leftarrow$) The total map $\widetilde{u}$ is continuous bijection because u is a bijection, so it suffices to show $\widetilde{u}^{-1}$ is continuous. Fix $U_i \subset B$ and $U'_{j'} \subset B'$. By substitution of $b' := u(b)$, $f' := h_{ij'}(b)f$, we can write

$$\varphi_i \circ \widetilde{u}^{-1} \circ \varphi'^{-1}_{j'}(b', f') = (u^{-1}(b'), h_{ij'}(u^{-1}(b'))^{-1} f').$$

Since the local trivializations, the inverse operation of G, and the inverse u^{-1} are all continuous, $\widetilde{u}^{-1}$ is also continuous. $\square$

7.2 (Fiber bundle construction theorem). Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 1-cocyle* on $\mathcal{U}$ with coefficients in G is a collection $\{g_{ij} : U_i \cap U_j \to G\}_{i,j}$ of maps such that the following *cocycle condition* holds:

$$g_{ik}(b) = g_{jk}(b)g_{ij}(b), \qquad b \in U_i \cap U_j \cap U_k.$$

The set of Čech 1-cocycles on $\mathcal{U}$ with coeffients in G is denoted by $\check{Z}^1(\mathcal{U}, G)$.

We want to construct a map $\check{Z}^1(\mathcal{U}, G) \to \mathrm{Bun}_F(B)$ for a left G-space F. Let $g \in \check{Z}^1(\mathcal{U}, G)$ and define

$$E := \left(\coprod_i (U_i \times F)\right) / \sim,$$

where $\sim$ is an equivalence relation generated by

$$(b, f, i) \sim (b, g_{ij}(b)f, j), \qquad b \in U_i \cap U_j,\ f \in F.$$

Also define $p : E \to B : [b, f, i] \mapsto b$ and $\varphi_i^{-1} : U_i \times F \to p^{-1}(U_i) : (b, f) \mapsto [b, f, i]$, which are clearly continuous and surjective without the cocycle condition.

(a) φ_i^{-1} is injective.

(b) φ_i^{-1} is open.

(c) The transition maps of the G-atlas $\{\varphi_i\}$ coincides with the cocycle $\{g_{ij}\}$.

Proof. (a) Suppose $\varphi_i^{-1}(b, f) = \varphi_i^{-1}(b', f')$. Since $(b, y, i) \sim (b', y', i)$, we have $b = b'$ and there is a sequence

$$f' = g_{i_{n-1}i_n}(b)g_{i_{n-2}i_{n-1}}(b) \cdots g_{i_0i_1}(b)f,$$

where $i_0 = i_n = i$. By applying the cocycle condition inductively, we obtain $f = f'$, which implies the injectivity of φ_i^{-1}.

(b) The map φ_i^{-1} factors through $\coprod_i (U_i \times F)$ such that

$$\varphi_i^{-1} : U_i \times F \to \coprod_i (U_i \times F) \xrightarrow{\pi} p^{-1}(U_i).$$

Since the canonical inclusion to disjoint union is open, it suffices to show the quotient map $\pi : \coprod_i (U_i \times F) \to E$ is open. Let $V \subset \coprod_i (U_i \times F)$ be open. Observe that

$$\pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open for each pair of i and j because it is exactly same as the inverse image of the open set $V \cap (U_i \times F)$ under the map

$$(U_i \cap U_j) \times F \subset U_j \times F \to U_i \times F : (b, f) \mapsto (b, g_{ij}(b)f).$$

Here we used the cocycle condition of $\{g_{ij}\}$. Therefore,

$$\pi^{-1}\pi(V) = \bigcup_{i,j} \pi^{-1}\pi(V \cap (U_i \times F)) \cap (U_j \times F)$$

is open, hence the open π.

(c) Clear by the cocycle condition. □

7.3 (Cohomologous transitions). Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 0-cochain* on $\mathcal{U}$ with coefficients in G is a collection $\{h_i : U_i \to G\}_i$ of maps. The group of Čech 0-cochains on $\mathcal{U}$ with coefficients in G is denoted by $\check{C}^0(\mathcal{U}, G)$.

The *first Čech cohomology* of $\mathcal{U}$ with coefficients G is the orbit space of an action on $\check{Z}^1(\mathcal{U}, G)$ by $\check{C}^0(\mathcal{U}, G)$ defined as follows:

$$(hg)_{ij}(b) := h_j(b)g_{ij}(b)h_i(b)^{-1}, \qquad b \in U_i \cap U_j,$$

which is denoted by $\check{H}^1(\mathcal{U}, G)$. We define the *first Čech cohomology* of B with coefficients in G as the direct limit of sets

$$\check{H}^1(B, G) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G).$$

Let F be a left G-space, and let $\mathrm{Bun}_F(B)$ be the set of isomorphism classes of G-bundles over B with fiber F.

(a) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G)$ is well-defined.

(b) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G)$ is surjective.

(c) $\mathrm{Bun}_F(B) \to \check{H}^1(B, G/\ker\sigma)$ is injective, where $\sigma : G \to \mathrm{Homeo}(F)$.

Proof. (a) Suppose $p : E_1 \to B$ and $p' : E' \to B$ be isomorphic G-bundles with fiber F. Let $u : E \to E'$ be a G-bundle isomorphism. By considering the refinement, we can find an open cover $\mathcal{U} = \{U_i\}_i$ of B on which E and E' are simultaneously locally trivialized.

$$\{g_{ij} : U_i \cap U_j \to G\}.$$

(b)

(c) □

7.4 (Principal bundles)**.** Let G be a topological group, and X be a left *principal homogeneous G-space*, i.e. a free and transitive left G-space such that the shear map $G \times X \to X \times X : (g, x) \mapsto (gx, x)$ is a homeomorphism.

A *principal G-bundle* is a G-bundle $p : P \to B$ with fiber X, often together with a fiber-preserving continuous right action $\rho : P \times G \to P$ such that each chart $\varphi_i : p^{-1}(U_i) \to U_i \times X$ induces a principal homogeneous right action on $\{b\} \times X \subset U_i \times X$ which commutes with the left action. The right action ρ is called the *principal right action* or *(global) gauge transformation*. Note that for each $b \in B$ the *fiber* $\{b\} \times X$ has commuting left and right actions, but the *fiber* $p^{-1}(b)$ can admit only the principal right action.

The category of principal G-bundles over B is denoted by $\mathbf{Prin}_G(B)$, and the morphisms are usually defined as right G-equivariant maps with respect to the pricipal right actions. Then, we may consider the forgetful functor $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$.

(a) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is fully faithful, i.e. a bundle map $u : P \to P'$ over B is a G-bundle map if and only if it is a right G-equivariant map.

(b) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is surjective, i.e. every G-bundle with fiber X has a principal right action.

(c) A principal bundle is trivial if it has a global section.

Proof. (a) ($\Rightarrow$) Let $u : P \to P'$ be a G-bundle map over B so that there is a set $\{h_i : U_i \to G\}_i$ of maps such that

$$\varphi_i \circ u \circ \varphi_i^{-1}(b, x) = (b, h_i(b)x), \qquad b \in U_i,\ x \in X.$$

If we write $\rho_s : P \to P : e \mapsto \rho(e, s)$ for $s \in G$, then the induced right action $\varphi_i \circ \rho_s \circ \varphi_i^{-1}$ commutes with the left action $\varphi_i \circ u \circ \varphi_i^{-1}$ on $U_i \times X$. Now for every $e \in P_1$, we have

$$\begin{aligned} \rho_s \circ u(e) &= \varphi_i^{-1} \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= \varphi_i^{-1} \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ \varphi_i(e) \\ &= u \circ \rho_s(e), \end{aligned}$$

therefore u is right G-equivariant.

($\Leftarrow$) let $u : P \to P'$ be a right G-equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \to G$ such that

$$(b, g_i(b)x_0) := \varphi_i \circ u \circ \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xmapsto{\varphi_i \circ u \circ \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The continuity of the last map is due to the assumption that the map $(g, x) \mapsto (gx, x)$ is a homeomorphism.

Then, for every $(b, x) \in U_i \times X$ there is a unique $s \in G$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned}
\varphi_i \circ u \circ \varphi_i^{-1}(b, x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b, x_0) \\
&= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b, x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) g_i(b)(b, x_0) \\
&= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b, x_0) \\
&= g_i(b)(b, x) \\
&= (b, g_i(b)x).
\end{aligned}$$

Hence, u is a G-bundle map.

(b) Fix $x_0 \in X$ and consider the homeomorphism $G \to X : g \to gx_0$. Define a right action

$$X \times G \to X : (gx_0, s) \mapsto gx_0 s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X.

Define $\rho : P \times G \to P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b, the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge tranformation.

(c) ($\Rightarrow$) Clear.

($\Leftarrow$) Let $s : B \to E$ be a global section and define

$$\widetilde{u} : B \times X \to E : (b, gx_0) \mapsto s(b)g$$

for any fixed $x_0 \in X$. Then, the continuous map $(\widetilde{f}, \mathrm{id}_B)$ preserves fibers and defines a right G-equivariant isomorphism. $\square$

7.5 (Quotient principal bundles)**.**

7.6 (Reduction of structure groups)**.** Let H be a closed subgroup of G. Then, there is a map $\check{H}^1(B, H) \to \check{H}^1(B, G)$, which is neither in general injective nor surjective. If a G-bundle ξ is contained in the image of $\check{H}^1(B, H)$ through the correspondence $\mathrm{Bun}_F(B) \twoheadrightarrow \check{H}^1(B, G)$, then we may give a H-bundle structure on ξ.

A *reduction* of G to H is a H-structure on a principal G-bundle.

7.2 Classifying spaces

Let $\mathrm{Prin}_G(B)$ be the set of isomorphism classes of principal G-bundles. Then, we have a contravariant functor

$$\mathrm{Prin}_G : \mathbf{Top} \to \mathbf{Set}$$

such that there is a natural transformation between contravariant functors

$$[-, BG] \to \mathrm{Prin}_G,$$

which is an isomorphism on paracompact spaces.

7.7 (Homotopy properteis)**.** Let $p : E \to B$ be a vector bundle

(a) If $p : E \to B \times [0, \frac{1}{2}]$ and $p' : E' \to B \times [\frac{1}{2}, 1]$ are trivial, then

(b) If $f, g : B' \to B$ are homotopic, then $f^*\xi \cong g^*\xi$.

7.8 (Finite type)**.**

7.3 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

7.9 (Vector bundles)**.** Let $p : E \to B$ and $p : E' \to B$ be vector bundles.

(a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \le n \le \infty$. If $f, g : B \to G_k(\mathbb{F}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{F}^n) \to G_k(\mathbb{F}^{2n})$ is the natural inclusion.

7.10. Riemannian and Hermitian metrics

Exercises

7.11. Let G be a topological group, and X be a free right G-space.

(a) If the action is proper and the projection $X \to X/G$ admits local sections, then $X \to X/G$ is a principal G-bundle.

7.12. Suppose $F \to E \to B$ is a principal

(a) If X is contractible, then $X \to$

7.13 (Group quotients)**.** Sufficient conditions for principal bundles. Let G be a Lie group and, X be a free right smooth G-manifold.

(a) If G is compact, then $X \to X/G$ is a principal G-bundle. (Gleason)

(b) The irrational slope provides a counterexample if G is not compact.

(c) Suppose X is a Lie group. If G is a closed subgroup of X, then $X/ \to X/G$ is a principal G-bundle. (Samelson) In particular, if M is a transitive left smooth X-manifold such that G is the isotropy group, then $X \to M$ is a principal G-bundle.

7.14 (Homogeneous spaces). They are all principal bundles.

$$\begin{aligned} \mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n), \\ \mathrm{O}(n-k)\times \mathrm{O}(k) \to \mathrm{O}(n) \to G_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k)\times \mathrm{U}(k) \to \mathrm{U}(n) \to G_k(\mathbb{C}^n), \\ T(n)\cap \mathrm{O}(n) \to \mathrm{O}(n) \to F(\mathbb{R}^n), &\qquad T(n)\cap \mathrm{U}(n) \to \mathrm{U}(n) \to F(\mathbb{C}^n), \\ &\qquad T(n) \to \mathrm{GL}(n,\mathbb{C}) \to F(\mathbb{C}^n), \end{aligned}$$

where $T(n)$ is the group of invertible upper triangular matrices.

$$\mathrm{SO}(n) \to \mathrm{SO}^+(1,n) \to \mathbb{H}^n, \qquad \mathrm{PSO}(2) \to \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2, \qquad ?? \to \mathrm{PSL}(2,\mathbb{C}) \to \mathbb{H}^3,$$

where $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)^+$ is the modular group and $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(1,3)^+$ is the restricted Lorentz group, also called the Möbius group.

7.15 (Hopf fibration). A principal S^1-bundle $S^1 \to S^3 \to S^2$, where we see S^1 as the circle group. The Hopf fibrations are used in describing universal principal bundles off orthogonal or unitary groups. We have principal bundles:

(a) The quaternionic construction gives $S^3 \to S^7 \to S^4$ and the octonianic construction gives $S^7 \to S^{15} \to S^8$. Adams' theorem.

(b) $\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$. In particular, $\mathbb{Z}/2\mathbb{Z} \to S^n \to \mathbb{RP}^n$ for $k = 1$.

(c) $\mathrm{U}(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$. In particular, $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $k = 1$.

Hopf fibration(real, complex, quaternionic, but not octonianic)
In the category of smooth manifolds, if f diffeomorphic, then $\widetilde{f}$ diffeomorphic.

7.16 (Associated bundles).

$$\mathrm{Prin}_G(B) \xrightarrow{\sim} \mathrm{Bun}_X(B) \xrightarrow{\sim} \check{H}^1(B,G) \hookrightarrow \mathrm{Bun}_F(B)$$

can be given in a more simple way.

Chapter 8

Characteristic classes

Chapter 9

K-theory

bott periodicity Hopf invariant

Part IV

Stable homotopy theory

Chapter 10

10.1 Generalized homology theory

A *generalized reduced cohomology theory on pointed CW complexes* is a sequence of functors $\widetilde{E}_q : \mathbf{hCW}_* \to \mathbf{Ab}$ for $q \in \mathbb{Z}$ which is exact and additive, and satisfies the suspension axiom.

10.1. Let X and Y be pointed CW complexes.

(a) Suppose Y is $(n-1)$-connected with non-degenerate base point for some n. Then, $[X, Y] \to [\Sigma X, \Sigma Y]$ is surjective if $\dim X \le 2n-1$, and bijective if $\dim X \le 2n-2$.

10.2. A *spectrum* is a sequence $E := (E_n)_n$ of pointed spaces together with structure maps, either $\sigma_n : \Sigma E_n \to E_{n+1}$ or $\sigma'_n : E_n \to \Omega E_{n+1}$. We have

$$[X, E_n] \xrightarrow{\sigma'_n} [X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}].$$

10.3 (Properties of spectra)**.** A spectrum $E = (E_n)_n$ is called an Ω-*spectrum* if $\sigma'_n : E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence. A *ring spectrum* is a spectrum together with a

(a) E is an Ω-spectrum if and only if $[-, E_n]$ defines a generalized reduced cohomology theory on based CW complexes.

Sphere spectra, Suspension spectra Eilenberg-MacLane spectra(ordinary cohomology theories), K-theory spectra(K-theories), Thom spectra(cobordism theories)

Let E^* be a (generalized) cohomology theory. Then, the computation of $\mathrm{Nat}([-, BO(n)], E^*) \cong E^*(BO(n))$ determines all characteristic classes of real vector bundles.

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem