Foundations of Calculus

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2.1 Convergence tests

2.1 (Mertens' theorem). If $\sum_{k=0}^{\infty} a_k$ converges to a absolutely and $\sum_{k=0}^{\infty} b_k$ converges to b, then their Cauchy product $\sum_{k=0}^{\infty} c_k$ with $c_k := \sum_{l=0}^{k} a_l b_{k-l}$ converges to ab.

Proof. Let

$$A_n := \sum_{k=0}^{n} a_k,\ B_n := \sum_{k=0}^{n} b_k,\ \text{and } C_n := \sum_{k=0}^{n} c_k$$

so that

$$C_n = \sum_{k=0}^{n}\sum_{l=0}^{k} a_l b_{k-l} = \sum_{l=0}^{n}\sum_{k=l}^{n} a_l b_{k-l} = \sum_{l=0}^{n} a_l \sum_{k=0}^{n-l} b_k = \sum_{l=0}^{n} a_l B_{n-l}.$$

For $\varepsilon > 0$ fix k_0 such that $k \geq k_0$ implies

$$|B_k - B|(\sum_{l=0}^{\infty} |a_l|) < \varepsilon.$$

Then, since we have

$$\begin{aligned}
|C_n - AB| &= |\sum_{k=0}^{n} a_{n-k}(B_k - B) + (A_n - A)B| \\
&\leq \sum_{k=0}^{k_0-1} |a_{n-k}||B_k - B| + \sum_{k=k_0}^{n} |a_{n-k}||B_k - B| + |A_n - A||B| \\
&\leq \max_{n-k_0<k\leq n} |a_k|(\sum_{k=0}^{k_0-1} |B_k - B|) + (\sum_{k=k_0}^{n} |a_{n-k}|) \max_{k_0\leq k\leq n} |B_k - B| + |A_n - A||B|
\end{aligned}$$

and $|a_n|$ and $|A_n - A|$ tend to zero as $n \to \infty$, we get

$$\limsup_{n\to\infty} |C_n - AB| < 0 + \varepsilon + 0.$$

$\square$

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