Fiber Bundles

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University of Tokyo, Spring 2023

May 1, 2023

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References: Steenrod, *The topology of fiber bundles*, and Tamaki, *Fiber bundles and homotopy* (Japanese)

1. Introduction

An ultimate goal of topology is to classify topological spaces, up to homeomorphism. If you want to show two spaces are homeomorphic, we should construct a homeomorphism: *Shokuninwaza* (wild knot, Casson handle). If you want to show two spaces are not homeomorphic, then we can investigate topological *properties*, and as their quantitative comparison, we can investigate topological *invariants* Some examples include

- the number of connected componenets,
- the Euler characteristic,
- homology groups,
- homotopy groups,
- the minimal number of open contractible sets to cover the spaces (Lusternik-Schnirelmann category, topological complexity),
- Gelfand-Naimark theorem: $C(X) \cong C(Y)$ implies $X \cong Y$ if they are compact Hausdorff.

We will restrict objects to study. For example, metric spaces, manifolds, CW-complexes. As the assumptions change, invariants may have different appearances. For a manifold X,

$$\chi(X) = \sum_{q=0}^{\infty}(-1)^q \operatorname{rk}_{\mathbb{Z}} H_q(X) = \sum_{q=0}^{\infty}(-1)^q b_q(X).$$

For a CW-complex X,

$$\chi(X) = \sum_{q=0}^{\infty}(-1)^q(\text{the number of } q\text{-cells}).$$

Let M be an connected closed n-dimensional manifold. Some classification results are as follows(up to both homeomorphisms and diffeomorphisms, because $d \leq 2$):

- ($n = 0$) $M \cong *$, and $\chi(*) = 1$.
- ($n = 1$) $M \cong S^1$, and $\chi(S^1) = 0$.
- ($n = 2$)
  - If M is orientable, then $M \cong \Sigma_g$ for $g \geq 0$, and $\chi(\Sigma_g) = 2 - 2g$.
    $\Sigma_0 \cong S^2$, $\Sigma_1 \cong T^2$.
  - If M is not orientable, then $M \cong (\mathbb{RP}^2)^{\#h}$ for $h \geq 1$, and $\chi((\mathbb{RP}^2)^{\#h}) = 2 - h$.
    $\mathbb{RP}^2(\cong \text{Möbius strip} \cup D^2)$, $K = \mathbb{RP}^2 \# \mathbb{RP}^2$

Problem 1. Show $\mathbb{RP}^2 \# T^2 \cong \mathbb{RP}^2 \# K$.

Here are some facts about triangulability:

- Cairns(1935), Whitehead (1940): every C^1-manifold is triangulable (unique as a PL-manifold).
- Rado(1925, $n = 2$), Moise(1952, $n = 3$): for $n \leq 3$, every C^0-manifold is triangulable (unique as a PL-manifold).
- Kirby-Siebermann(1966, $n \geq 5$): for $n \geq 4$, there is a non-triangulable PL-manifold.

- Donaldson, Freedman, Casson: for $n = 4$, there is a non-triangulable manifold as a topological space.
- Manolescu(2013): for $n \geq 5$, there is a non-triangulable manifold as a topological space.

Orientability? For a connected closed surface S, it is orientable iff $H_2(S) \cong \mathbb{Z}$, not orientable iff $H_2(S) \cong 0$. The generator of $H_2(S)$ is called the fundamental class. Orientability asks if the tubular neighborhood of every simple closed curve is homeomorphic to an anulus. It is described by the first Stiefel-Whitney class:

$$w_1(S) \in H^1(S; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H^1(S), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(S), \mathbb{Z}/2\mathbb{Z}).$$

Euler characteristic of manifolds

(0) Odd-dimensional manifolds

Theorem. *For an odd-dimensional closed connected manifold,* $\chi(M^{2n+1}) = 0$.

Proof. If orientable, then $b_0(M) = 1$, $b_3(M) = 1$, $b_1(M) = b_2(M)$ by the Poincaré duality. If not, a double cover is orientable, and $\chi(\widetilde{M}) = 2\chi(M)$. □

(1) Gauss-Bonnet theorem

Theorem (Gauss-Bonnet)**.** *If a smooth manifold M^n embeds into $\mathbb{R}^{n+1}$ (hypersurface), then it is orientable and the Euler characteristic is given by*

$$\chi(M) = \frac{2}{\mathrm{vol}(S^n)} \int_M K \, d\,\mathrm{vol}_M \,.$$

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We have a cohomological interpretation. In the Chern-Weil theory, we have a generalized version of the Gauss-Bonnet theorem for a general compact manifold using the theory of connections. We can interpret $2\,\mathrm{vol}(S^n)^{-1} K \cdot d\,\mathrm{vol}_M$ as a differential form which provides with the Euler characteristic. In the context of the de Rham theorem, we will eventually call the equivalence class of this differential form as the *Euler class*.

(2) Poincaré-Hopf theorem

Let M^n be a orientable connected smooth closed manifold. Let X be a smooth vector field on M such that there are only finitely many zeros $\{p_1, \cdots, p_m\}$. For each p_j, define the index $\mathrm{Ind}(X, p_j)$ as follows: seeing X as a vector field on $\varphi_j(U_j)$ for a chart (U_j, φ_j) not containing zeros of X but p_j and mapping p_j to zero in $\mathbb{R}^n$, we define $\mathrm{Ind}(X, p_j) = \deg f_j$, where $f_j : S_\varepsilon (\approx S^{n-1}) \to S^{n-1} : x \mapsto X_x / \|X_x\|$.

Example. Let $n = 2$. We have indices $1, 1, 1, -1, 0, 2$ for

$$X_1(x, y) = (x, y), \quad X_2(x, y) = (-x, -y), \quad X_3(x, y) = (-y, x),$$

$$X_4(x, y) = (-x, y), \quad X_5(x, y) = \sqrt{x^2 + y^2}(1, 1), \quad X_6(x, y) = (x^2 - y^2, 2xy).$$

Theorem (Poincaré-Hopf)**.**

$$\sum_{j=1}^m \mathrm{Ind}(X, p_j) = \chi(M).$$

We have a cohomological interpretation. Let $c = \sum_{j=1}^{m} \mathrm{Ind}(X, p_j)p_j$ be a singular 0-cycle on M. Then, the Poincaré-Hopf theorem states that we have

$$\begin{array}{ccc} H_0(M) & \xrightarrow{\sim} & \mathbb{Z} \\ p_j & \mapsto & 1 \\ c & \mapsto & \chi(M). \end{array}$$

By the Poincaré duality, we can identify the homology class $[c]$ with a de Rham cohomology class, and the above map is just an integration map.

The cycle c tells us the information of intersections of X and zero section(of the tangent bundle). If TM is trivial, then the zero section does not self-intersection(?) so that $c = 0$. The Euler characteristic measures the twist of a bundle, and the characteristic class generalizes this wakugumi.

2. Fiber bundles

From now we will only consider paracompact Hausdorff spaces. Recall that a space is paracompact iff for every open cover there is a locally finite refinement.

Example. Open sets of $\mathbb{R}^n$, metric spaces, CW-complexes, countable inductive limit of compact spaces are paracompact.

Theorem 2.1. *For any open cover of a paracompact Hausdorff space X, there is a partition of unity subordinate to it.*

Problem 2. Prove the above theorem.

Definition 2.2. Let B be connected(for simplicity). A map $E \to B$ is called a fiber bundle with fiber F, or just a F-bundle, if it is locally trivial: every point $x \in B$ has an open neighborhood U_x such that there is a homeomorphism $\varphi : p^{-1}(U_x) \to U_x \times F$ with $p = \mathrm{pr}_{U_x} \circ \varphi$.

For each $y \in B$ $E_y := p^{-1}(y)$ is homeomorphic to F, and is called the fiber at y. Also, E and B are called the total space and the base space. We somtimes write as $\xi = (F \to E \xrightarrow{p} B)$.

Example.

(a) We say $\mathrm{pr}_1 : B \times F \to B$ is the product or bundle.

(b) $p : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z} : t \mapsto [t]$ is a $\mathbb{Z}$-bundle. In general, a fiber bundle with a discrete fiber is called a covering space.

(c) $p_1 : S^n \to \mathbb{RP}^n = S^n/(x \sim -x)$ is a $\mathbb{Z}/2\mathbb{Z}$-bundle.

(d) $p : S^{2n+1} \to \mathbb{CP}^n = S^{2n+1}/(z \sim uz)$ for $u \in S^1$ is a S^1-bundle. (a generalization of Hopf bundles)

(e) Let M^n be a smooth manifold. Then, the tangent and the contangent bundles are $\mathbb{R}^n$-bundles.

Problem 3. Show that $p : S^{2n+1} \to \mathbb{CP}^n$ is a S^1-bundle by checking concretely its local triviality.

Definition 2.3. If F, E, B are C^r, $p : E \to B$ is C^r, and the local trivialzation is C^r, then we say the fiber bundle is C^r.

Definition 2.4. For $\xi_1 = (F \to E_1 \xrightarrow{p_1} B_1)$, $\xi_2 = (F \to E_2 \xrightarrow{p_2} B_2)$, a bundle map $\Phi = (\widetilde{f}, f) : \xi_1 \to \xi_2$ is a pair of maps $\widetilde{f} : E_1 \to E_2$ and $f : B_1 \to B_2$ such that $f \circ p_1 = p_2 \widetilde{f}$ and the restriction $\widetilde{f} : p_1^{-1}(b) \to p_2^{-1}(f(b))$ is a homeomorphism for every $b \in B$.

If both f and $\widetilde{f}$ are homeomorphisms, then Φ is called a bundle isomorphism. If a bundle is isomorphic to a product bundle, then it is called to be trivial.

Problem 4 For a bundle map Φ, is $\widetilde{f}$ homeomorphic if f is homeomorphic? (If we are doing in the category of smooth manifolds, then the inverse function theorem may be helpful.)

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Transition maps and structure groups

Let $\xi = (F \to E \xrightarrow{p} B)$ be an F-bundle. We have an open cover $\{U_\alpha\}$ such that for each α we have a local trivialization $p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times F$. For $U_\alpha \cap U_\beta \neq \varnothing$, we have a map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F,$$

by which we can define $\widetilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}(b, f) =: (b, \widetilde{g}_{\alpha\beta}(b, f))$. The map $\widetilde{g}_{\alpha\beta}$ is continuous, and we have for each b a homeomorphism

$$g_{\alpha\beta}(b) : F \to F : f \mapsto \widetilde{g}(b, f),$$

that is, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$. If we endow the compact-open topology on $\mathrm{Homeo}(F)$, then $g_{\alpha\beta}$ is continuous.

From definition, $g_{\alpha\beta}(b) \circ g_{\beta\alpha}(b) = \mathrm{id}_F$ for $b \in U_\alpha \cap U_\beta \neq \varnothing$, and $g_{\alpha\beta}(b) \circ g_{\beta\gamma}(b) = g_{\alpha\gamma}(b)$ for $b \in U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ (Note that the second relation implies the first.). The second condition is called the cocycle condition. The maps $\{g_{\alpha\beta}\}$ are called transition maps.

Theorem 2.5. *Let $\{U_\alpha\}$ be an open cover of a connected space B. Suppose we have a collection of continuous maps*

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)\}_{(\alpha,\beta):U_\alpha \cap U_\beta \neq \varnothing}$$

satisfying the cocycle condition.

(♠) Suppose also that F is locally compact, or there exists a topological transformation group G(i.e. G is a topological group such that the group action $G \times F \to F$ is continuous) with

$$\bigcup_{\alpha,\beta} g_{\alpha\beta}(U_\alpha \cap U_\beta) \subset G \subset \mathrm{Homeo}(F).$$

Then, there exists a unique F- bundle $(F \to E \xrightarrow{p} B$ such that it is locally trivializable over $\{U_\alpha\}$ and $\{g_{\alpha\beta}\}$ is the transition maps of the bundle.

The viewpoint of the above theorem is more likely to be the physicist's way of defining manifolds in the sense that they sometimes deifne a manifold as a collection of open subsets of a Euclidean space and transition maps between them.

The condition (♠) gaurantees for the second map in

$$\widetilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times \mathrm{Homeo}(F) \times F \to (U_\alpha \cap U_\beta) \times F$$

$$(b, f) \mapsto (b, g_{\alpha\beta}(b), f) \mapsto (b, g_{\alpha\beta}(f))$$

to be continuous.

Proof. (Sketch) Define

$$\widetilde{E} := \coprod U_\alpha \times F$$

and $E := \widetilde{E}/\sim$, where the equivalence relation $\sim$ is generated by: for each $(b_1, f_1) \in U_\alpha \times F$ and $(b_2, f_2) \in U_\beta \times F$ we have $(b_1, f_1) \sim (b_2, f_2)$ iff $b_1 = b_2$ and $f_1 = g_{\alpha\beta}(b_2)(f_2)$. Let $\pi : \widetilde{E} \to E$ be the canonical projection. Define also

$$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F : [(b, f) \in U_\alpha, F] \mapsto (b, f),$$

which are homeomorphisms by the assumption (♠), satisfying $\mathrm{pr}_1 \circ \varphi_\alpha = p$. □

For the second condition in (♠), G is called a structure group of the F-bundle. From now on, whenever we consider a fiber bundle along with a structure group G, we assume it includes the data of local trivialization.

Remark. We will always think of G for bundle maps between fiber bundles with structure group G. We will frequently consider the maximal transition data and compatible(i.e. satisfying the cocycle condition) local trivializations.

Example.

1. Let $F = V \cong \mathbb{R}^n$ be a real vector space, and $G \in \{\mathrm{GL}(V), \mathrm{SL}(V)\}$ or $G \in \{\mathrm{O}(V), \mathrm{SO}(V)\}$ with a fixed inner product on V. These fiber bundles are called real vector bundles.
2. Let $F = V \cong \mathbb{C}^n$ be a complex vector space, and $G \in \{\mathrm{GL}_{\mathbb{C}}(V)\}$ or $G \in \{\mathrm{U}(V)\}$ with a fixed inner product on V. These fiber bundles are called complex vector bundles.
3. $F = G$ be a Lie group. Then, G-bundle with structure group G is called a principal bundle.
4. Let F be a nice smooth manifold and $G = \mathrm{Diff}^{C^\infty}(F)$ be the group of smooth diffeomorphisms together with the Fréchet topology. Then, we have smooth F-bundles.

Definition 2.6. Let G be a structure group and B be a topological space. If an F-bundle $\xi = (F \to E \to B, G)$ and an F'-bundle $\xi = (F' \to E' \to B, G)$ has the same transition data, then they are called associated bundles.

Example. Let $F = \mathbb{R}^n$ be a real vector space with the standard inner product. Let $G = O(n)$. With $S^{n-1} \subset F$, the sphere bundle inside a real vector bundle is an associated bundle of the original real vector bundle. In particular for $n = 2$ and $G = \mathrm{SO}(2)$, then the circle bundle can be recognized as a principal $SO(2)$-bundle associated to a real plane bundle, and if we see the plane bundle as a complex line bundle, then it corresponds to a pricipal $U(1)$-bundle.

Proposition 2.7. *Let G be a topological group and $\xi = (G \to E \to B, G)$ be a principal G-bundle. Then, there is a natural right action of G on E which is free and the orbit space E/G is homeomorphic to B(transitively act on each fiber).*

Proof. Let $u \in E$ and φ_α a local trivialization containing u such that

$$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G : u \mapsto (p(u), h).$$

We can check the well-definedness of $ug = \varphi_\alpha^{-1}(p(u), hg)$ by

$$\varphi_\beta(ug) = \varphi_\beta \circ \varphi_\alpha^{-1}(p(u), hg) = (p(u), g_{\beta\alpha}(p(u))(hg)) = (p(u), h'g).$$

The right action of G on G is continuous, free, and transitive. The right action of G on E is continuous and free, and $\overline{p} : E/G \to B$ is continuous and bijective. □

Problem 5. Show that $\overline{p}^{-1}$ is also continuous.

Remark. A principal G-bundle may also be defined as follows: a G-bundle such that (1) there is a continuous free right action of G on E which is (2) fiber-preserving and fiberwise transitive, and (3) we can choose *G-equivariant* local trivialization such that $\varphi_\alpha(u) = (p(u), h)$ implies $\varphi_\alpha(ug) = (p(u), hg)$.

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Let G be a topological group. A pricipal G-bundle $(G \to E \to B, G)$ has a continuou free action of G on E.

Remark. For two principal G-bundles, $(\widetilde{f}, f)$ is a bundle map if and only if $\widetilde{f}$ is G-equivariant.

Definition 2.8. Let $\xi = (F \to E \xrightarrow{p} B)$ be a fiber bundle. A continuous map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$ is called a section or a cross section.

An important question asks if there is a section globally defined on the whole B.

Proposition 2.9. *Let $\xi = (G \to E \to B, G)$ be a principal G-bundle. Then, ξ is trivial if and only if it admits a global section.*

Proof. $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Let $s : B \to E$ be a global section. Define

$$\Phi : B \times G \to E : (b, g) \mapsto s(b)g.$$

Then, it is an G-equivariant isomorphism. □

Let X be a right G-space which is free. Then, is X/G a principal G bundle? We have two problems:

(a) Is the inverse image(=orbit) of each point of X/G homeomorphic to G? No, the dynamics $\mathbb{T}^2 \curvearrowleft \mathbb{R}$ with irrational slope.

(b) Does it satisfy the local triviality? No, the translation $\mathbb{R} \leftarrow \mathbb{Q}$.

Proposition 2.10. *Let X be a right G-space which is free. The quotient map $\pi : X \to X/G$ defines a principal G-bundle if and only if $X \curvearrowleft G$ strongly freely(i.e. $X \times X \to G : (x, xg) \mapsto g$ is continuous) and there is a local trivialization for some $y \in X/G$.*

Proof. $(\Rightarrow)$ Clear.

$(\Leftarrow)$

$$\pi^{-1}(U) \to U \times G : s(x)g \mapsto (x, g)$$

is continuous by the strongly free action. It defines local trivializations. □

Theorem 2.11 (Gleason, 1950)**.** *Let M be a smooth manifold and G a compact Lie group which gives a free right smooth action on M. Then, M/G is a smooth manifold such that $M \to M/G$ is a principal G-bundle.*

(Compactness of G implies the properness of the action, and smoothness implies the local triviality)

Corollary 2.12 (Samelson, 1941)**.** *Let H be a compact Lie subgroup of a Lie group G. Then, $G \to G/H$ is a principal H-bundle. In fact, it is sufficient for H to be a closed subgroup of G, even if it is not compact.*

Example.

(a) With an action $S^{2n+1} \curvearrowleft S^1$ such that $(z_0, \cdots, z_n)w = (z_1 w, \cdots, z_n w)$, we have an S^1-bundle

$$S^{2n+1} \to \mathbb{CP}^n : (z_0, \cdots, z_n) \mapsto [z_0 : \cdots : z_n].$$

(b) For $k \leq n$, the Stiefel variety is

$$V_k(\mathbb{R}^n) := \{M \in M_{n,k}(\mathbb{R}) : \mathrm{rk}\, M = k\}.$$

Also define

$$V_k^0(\mathbb{R}^n) := \{M \in V_k(\mathbb{R}^n) : \text{column vectors of } M \text{ are orthonormal}\}$$

and the Grassmannian manifold

$$G_k(\mathbb{R}^n) := \{k\text{-dimensional subspaces of } \mathbb{R}^n\}.$$

With an action $V_k(\mathbb{R}^n) \curvearrowleft \mathrm{GL}(k,\mathbb{R})$ such that $(v_1, \cdots, v_k)X = (v_1X, \cdots, v_kX)$, we have $G_k(\mathbb{R}^n) \cong V_k(\mathbb{R}^n)/\mathrm{GL}(k,\mathbb{R})$ and $G_k(\mathbb{R}^n) \cong V_k^0(\mathbb{R}^n)/\mathrm{O}(k)$. Then, $(\mathrm{O}(k) \to V_k^0(\mathbb{R}^n) \to G_k(\mathbb{R}^n))$ and $(\mathrm{GL}(k,\mathbb{R}) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n))$ are principal bundles.

(c) As a complex version of (b), we have principal bundles $(\mathrm{U}(k) \to V_k^0(\mathbb{C}^n) \to G_k(\mathbb{C}^n))$ and $(\mathrm{GL}(k,\mathbb{C}) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n))$.

Theorem 2.13. *Let M be smooth manifold and suppose we have a transitive smooth left action of a Lie group G on M. Let H be* the *isotropy group. Then, $G/H \to M$ defines a diffeomorphism and $(H \to G \to M)$ is a principal bundle. Such M is called a homogeneous space.*

Example. With an action $\mathrm{SO}(n) \curvearrowright S^{n-1}$, since the isotropy group is isomorphic to $\mathrm{SO}(n-1)$, we have a principal bundle $SO(n-1) \to SO(n) \to S^n$.

We can also see the examples above(Grassmann and Steifel manifolds) as principal bundles on homogeneous spaces with a diffeomorphsim $\mathrm{O}(n-k) \setminus \mathrm{O}(n) \to V_k^0(\mathbb{R}^n) : [A] \mapsto (Ae_1, \cdots, Ae_k)$ and $\mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k) \cong G_k(\mathbb{R}^n)$: principal $\mathrm{O}(k)$-bundle

$$\begin{tikzcd}
\mathrm{O}(k) \arrow[r] & V_k^0(\mathbb{R}^n) \arrow[r, "\sim"] \arrow[d] & \mathrm{O}(n)/\mathrm{O}(n-k) \\
& G_k(\mathbb{R}^n) \arrow[r, "\sim"] & \mathrm{O}(n)/\mathrm{O}(n-k) \times \mathrm{O}(k).
\end{tikzcd}$$

We also have a complex version.