Von Neumann Algebras

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Part I

Chapter 1

Factor classifications

1.1 Factors and traces

Every trace of factor is faithful

1.1. Normal states is a state in which the monotone convergence theorem holds. Precisely, a state ρ is *normal* if a monotone net a_α strongly converges to a then $\rho(a_\alpha) \to \rho(a)$.

1.2

1.2 (Semi-finite traces)**.** Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x \in M^+$ has a net $x_\alpha \in L^1(M,\tau)^+$ such that $x_\alpha \uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x) = \sup\{\, \tau(y) : y \le x,\ y \in L^1(M,\tau)^+ \,\} \quad \text{for} \quad x \in M^+.$$

1.3

Direct integral of factors.

Type I factors. It possess a minimal projection. It is isomorphic to the whole $B(H)$ for some Hilbert space. Therefore, it is classified by the cardinality of H.

Type II factors. No minimal projection, but there are non-zero finite projections so that every projection can be "halved" by two Murray-von Neumann equivalent projections.

In type II_1 factors, the identity is a finite projection Also, Murray and von Neumann showed there is a unique finite tracial state and the set of traces of projections is $[0,1]$. Free probability theory attacks the free groups factors, which are type II_1.

In type II_∞ factors There is a unique semifinite tracial state up to rescaling and the set of traces of projections is $[0,\infty]$.

In type III factors no non-zero finite projections exists. Classified the $\lambda \in [0,1]$ appeared in its Connes spectrum, they are denoted by III_λ. Tomita-Takesaki theory. It is represented as the crossed product of a type II_∞ factor and $\mathbb{R}$.

Amenability, equivalently hyperfiniteness is a very nice condition in von Neumann algebra theory. Group-measure space construction can construct them. There are unique hyperfinite type II_1 and II_∞ factors, and their property is well-known. Fundamental groups of type II factors, discrete group theory, Kazhdan's property (T) are used.

Tensor product facctors such as Araki-Woods factors and Powers factors.

1.4 Hyperfinite factors

weight, trace, state.

finite trace=tracial state.

1.3 (Uniformly hyperfinite algebras)**.** Let $\mathcal{A}$ be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique finite trace.

(b) Every UHF algebra admits a unique finite trace.

(c) Every hyperfinite

1.4 (Classification of UHF algebras)**.**

Chapter 2

Weight theory

Chapter 3

3.1 Connes' bicentralizer problem

Model theory Existentially closed II_1 factors Connes embedding property Gamma
Shlyakhtenko semicircular system
Anantharaman, Popa: An introduction to II_1 factors

Part II

Subfactor theory

The way how quantum systems are decomposed. Quite combinatorial! And has Galois analogy.

3.1 (Jones index theorem)**.** A *subfactor* of a factor M is a factor N containing 1_M.

Tensor categories and topological invariants of 3-folds. Ergodic flows.
standard invariant, Ocneanu's paragroups, Popa's λ-lattices, Jones' planar algebras