

# Three-dimensional Topology

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## Contents

<b>1</b>	<b>Day 1: April 11</b>	<b>2</b>
1.1	Fundamental groups of higher dimensional manifolds . . . . .	2
1.2	Surfaces and their groups . . . . .	3
<b>2</b>	<b>Day 2: April 18</b>	<b>3</b>
2.1	Spherical manifolds . . . . .	4
2.2	Fibered manifolds . . . . .	4
2.3	Heegaard decomposition . . . . .	5
2.4	Dehn surgery . . . . .	5
<b>3</b>	<b>Day 3: April 25</b>	<b>7</b>
3.1	Alexander's theorem . . . . .	7
3.2	Irreducible manifolds . . . . .	7
3.3	Normal surfaces . . . . .	8
<b>4</b>	<b>Day 4: May 9</b>	<b>9</b>
3.4	Prime decomposition theorem . . . . .	9
4.1	Loop theorem . . . . .	10
<b>5</b>	<b>Day 5: May 16</b>	<b>11</b>
<b>6</b>	<b>Day 6: May 23</b>	<b>13</b>
5.1	Essential surfaces . . . . .	13
5.2	Haken hierarchy . . . . .	14

# 1 Day 1: April 11

Plan:

1. Fundamental groups of manifolds
2. Examples and constructions
3. Prime decomposition
4. Loop and sphere theorems
5. Haken manifolds
6. Seifert manifolds
7. JSJ composition
8. Geometrization
9. Virtually special theorems

References:

1. J. Hempel, 3-manifolds
2. W. Jaco, Lectures on three-manifold topology
3. B. Martelli, An introduction to geometric topology
4. Morimoto, An introduction to three-dimensional manifolds (Japanses)

Grading: Submit a report for any three among the exercises given in the lecture (ITC-LMS Kadai).

Cancellation of class: 5/2, 7/11(maybe)

Convention:

- manifold = connected compact orientable smooth manifold
- surface = connected compact orientable smooth 2-dimensional manifold
- tub nbd, isotopy extension, transversality, triangulation,...

## Fundamental group

### 1.1 Fundamental groups of higher dimensional manifolds

**Proposition 1.1.** *Let  $\pi$  be a finitely presented group. Then, for every  $d \geq 4$  there is a  $d$ -manifold  $X$  such that  $\pi_1(X) \cong \pi$ .*

*Proof.* Let  $\pi = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$  be given. If  $Y = (S^1 \times S^{d-1})^{\#n}$ , then  $\pi_1(Y) \cong \langle x_1, \dots, x_n \rangle$  by the van Kampen theorem. Let

$$Z = Y \setminus \left( \bigcup_{i=1}^m \nu(l_i) \right),$$

where  $l_i \subset Y$  is embedded loops representing  $r_i$  and  $\nu$  denotes the open tubular neighborhood. Then,  $\partial Z = \bigcup_{i=1}^m l_i \times S^{d-2}$ . Since  $d \geq 4$ , any loops and disks can be pushed off  $l_1 \cdots, l_n$ , we have an isomorphism  $\pi_1(Z) \rightarrow \pi_1(Y)$ . Then, if we let

$$X = Z \cup_{\partial} \left( \bigcup_{i=1}^m D^2 \times S^{d-2} \right),$$

then  $l_i \times * = \partial(D^2 \times *)$ , we have  $\pi_1(X) \cong \pi_1(Y) / \langle [l_1], \dots, [l_m] \rangle \cong \pi$ . □

## 1.2 Surfaces and their groups

**Theorem 1.2** (Radó, Whitehead). *Every topological surface admits a unique smooth and PL structure.*

**Theorem 1.3.** *Every surface is diffeomorphic to only one of  $\Sigma_{g,b}$ , where  $\Sigma_{g,b} = (T^2)^{\#g} \# (D^2)^{\#b}$ .*

**Corollary 1.4.**  *$S = S^2, T^2, D^2$  are prime, that is,  $S = S_1 \# S_2$  implies  $S_i \approx S^2$  for  $i = 1$  or  $i = 2$ .*

**Remark 1.5.** If the orientability is reduced out, then  $\mathbb{RP}^2$  is prime. Also note that  $T^2 \# \mathbb{RP}^2 \approx (\mathbb{RP}^2)^{\#3}$ .

**Theorem 1.6** (Uniformization). *For every surface  $S \neq D^2$ , its interior admits a complete Riemannian metric of constant curvature*

$$\begin{cases} 1, & \chi(S) > 0 \\ 0, & \chi(S) = 0 \\ -1, & \chi(S) < 0 \end{cases}$$

*with universal covering  $S^2, \mathbb{R}^2, \mathbb{H}^2$ , respectively.*

The hyperbolic plane is  $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$  with the Riemannian metric  $ds^2 = (dx^2 + dy^2)/y^2$ , and  $\text{Isom}^+(\mathbb{H}^2) = \text{PSL}_2(\mathbb{R})$ .

**Proposition 1.7.** *If a surface  $S$  has  $\chi(S) < 0$ , then there is a discrete group  $\Gamma \leq \text{PSL}_2(\mathbb{R})$  such that  $S \approx \mathbb{H}^2/\Gamma$ . In particular,  $\pi_1(S)$  is isomorphic to  $\Gamma$ .*

We have

$$\pi_1(\Sigma_{g,b}) \cong F_{2g+b-1} \quad \text{and} \quad \pi_1(\Sigma_g) \cong \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1], \dots, [a_g, b_g] \rangle.$$

**Proposition 1.8.**  *$\pi_1(\Sigma_g)$  is torsion free.*

**Exercise 1.** Prove Proposition 1.7.

**Theorem 1.9** (Newman).  *$\pi_1(\Sigma_{g,b})$  is linear over  $\mathbb{Z}$ , that is, is isomorphic to a subgroup of  $\text{GL}_n(\mathbb{R})$ . For example, we can check  $F_n \hookrightarrow F_2 \hookrightarrow \text{SL}_2(\mathbb{Z})$  according to the pingpong lemma.*

Over  $\mathbb{R}$ , we may embed  $\pi_1(S) \hookrightarrow \text{PSL}_2(\mathbb{Z}) \cong \text{SO}_{1,2}^+(\mathbb{R}) < \text{GL}_3(\mathbb{R})$  if  $\chi(S) < 0$ ,

**Definition.** A group  $\pi$  is called residually finite (RF) if for every  $1 \neq \gamma \in \pi$  there is a group homomorphism  $\varphi : \pi \rightarrow G$  to a finite group  $G$  such that  $\varphi(\gamma) \neq 1$ . A subgroup  $\pi' < \pi$  is called separable if there is a group homomorphism  $\varphi : \pi \rightarrow G$  to a finite group  $G$  such that  $\varphi(\gamma) \notin \varphi(\pi')$ . In particular,  $\pi$  is residually finite if the trivial subgroup is separable in  $\pi$ . A group  $\pi$  is called locally extended residually finite (LERF) if every finitely generated subgroup of  $\pi$  is separable.

**Theorem 1.10** (Mal'cev). *Every finitely generated linear group over a field is residually finite.*

Over  $\mathbb{Z}$ , if  $1 \neq (a_{ij}) \in \pi < \text{GL}_n(\mathbb{Z})$  given, then for  $m > \max_{i,j} |a_{ij}|$  if we let  $\varphi_m : \pi \hookrightarrow \text{GL}_n(\mathbb{Z}) \rightarrow \text{GL}_n(\mathbb{Z}/m\mathbb{Z})$ , then  $\varphi_m((a_{ij})) \neq 1$ .

**Theorem 1.11** (Scott).  *$\pi_1(\Sigma_{g,b})$  is LERF.*

## 2 Day 2: April 18

### Examples and constructions of 3-manifolds

**Theorem 2.1** (Moise). *Every topological 3-manifold (not necessarily compact, connected, orientable) admits a unique smooth and PL structure.*

## 2.1 Spherical manifolds

Recall

$$\begin{aligned} S^3 &:= \{x \in \mathbb{R}^4 : |x| = 1\} \\ &= \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \\ &= \{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}. \end{aligned}$$

### Lens spaces

Let  $p, q \in \mathbb{Z}$ ,  $p > 0$ ,  $(p, q) = 1$ . Then,  $\mathbb{Z}/p\mathbb{Z} = \langle \zeta = \exp(2\pi\sqrt{-1}/p) \rangle$  acts on  $S^3$  such that  $\zeta \cdot (z, w) = (\zeta z, \zeta^q w)$ . Then, the Lens spaces are defined as

$$L(p, q) := S^3 / (\mathbb{Z}/p\mathbb{Z}) \quad \text{with} \quad \pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}.$$

For example,  $L(1, 1) = S^3$  and  $L(2, 1) = \mathbb{RP}^3$ .

**Theorem 2.2** (Reidemeister).

- (a)  $L(p, q) \simeq L(p, q')$  (homotopy equiv) if and only if there is  $a \in \mathbb{Z}$  such that  $qq' \equiv \pm a^2 \pmod{p}$ .
- (b)  $L(p, q) \approx L(p, q')$  (diffeo) if and only if  $q' \equiv \pm q^{\pm 1} \pmod{p}$ .

For example,  $L(7, 1) \simeq L(7, 2)$  since  $1 \cdot 2 \equiv 3^2 \pmod{7}$ , but  $L(7, 1) \not\approx L(7, 2)$  since  $2 \not\equiv \pm 1 \pmod{7}$ .

*Sketch of ( $\Leftarrow$ ).* Direct construction. (a) With the linking form  $H_1(L) \times H_1(L) \rightarrow \mathbb{Q}/\mathbb{Z}$ . (b) Reidemeister torsion.  $\square$

### General quotients

A spherical manifold is the orbit space  $S^3/\Gamma$ , where  $\Gamma$  is a finite subgroup of  $\text{SO}(4)$  and  $\Gamma \curvearrowright S^3$  freely.

**Example.** With an action  $\langle -1, i, j, k \rangle \curvearrowright S^3$ , we obtain the prism manifold.

**Example.** With an action of the binary icosahedral group  $\Gamma = \mathbb{Z}/2\mathbb{Z} \rtimes A_5$  on  $S^3$ , we obtain the Poincaré sphere. We have  $H_*(S^3/\Gamma) \cong H_*(S^3)$ . If we take  $3/10$  turn instead of  $1/10$ , we have the Seifert-Weber space.

## 2.2 Fibered manifolds

### Twisted bundles

$$N_{g,b} = (\mathbb{RP}^2)^{\#g} \# (D^2)^{\#b}.$$

Let  $D$  be a polygon with oriented sides  $a_1, a'_1, a_2, a'_2, \dots, a_g, a'_g$ .

$$N_g \widetilde{\times} [0, 1] := D \times [0, 1] / \sim,$$

where  $(x, t) \sim (x', 1 - t)$  for  $x \in a_i$ ,  $x' \in a'_i$ ,  $t \in [0, 1]$  with  $[x] = [x'] \in N_g$ , and it is orientable.

$$N_g \widetilde{\times} S^1 := N_g \widetilde{\times} [0, 1] / (x, 0) \sim (x, 1), \quad x \in N_g$$

$$N_{g,b} \widetilde{\times} S^1 := N_g \widetilde{\times} S^1 \setminus \nu(b \text{ fibers}).$$

**Exercise 2.** Show the following:

- (a)  $\mathbb{RP}^2 \widetilde{\times} [0, 1] \approx \mathbb{RP}^3 \setminus (\text{open ball})$ .
- (b)  $\mathbb{RP}^2 \widetilde{\times} S^1 \approx \mathbb{RP}^3 \# \mathbb{RP}^3$ .
- (c)  $N_{1,1} \widetilde{\times} S^1 \approx N_2 \widetilde{\times} [0, 1]$ .

## Mapping tori

The mapping class group is

$$\mathcal{M}_{g,b} := \text{Diff}^+(\Sigma_{g,b}, \partial \Sigma_{g,b}) / \text{isotopy relative to } \partial.$$

**Theorem 2.3** (Dehn, Lickorish).  $\mathcal{M}_{g,b}$  is finitely generated by Dehn twists.

For examples,  $\mathcal{M}_0 = \mathcal{M}_{0,1} = 1$  by the Alexander trick, and  $\mathcal{M}_{0,2} \cong \mathbb{Z}$ ,  $\mathcal{M}_1 = \mathcal{M}_{1,1} \cong \text{SL}_2(\mathbb{Z})$ .

Let  $\varphi \in \mathcal{M}_{g,b}$ . Then, a mapping torus is defined by

$$M_\varphi := \Sigma_{g,b} \times [0, 1] / (\varphi(x), 0) \sim (x, 1).$$

## 2.3 Heegaard decomposition

A manifold with a boundary

$$H_g := D^3 \cup (D^2 \times [0, 1])^{\sqcup g}$$

is called the handle body with genus  $g$ . Then,  $\pi_1(H_g) \cong F_g$  and  $\partial H_g \approx \Sigma_g$ . Let  $\varphi : \partial H_g \rightarrow \partial H_g$  be an orientation-preserving diffeomorphism, i.e. an element of the mapping class group  $\mathcal{M}_g$ . If a 3-manifold  $M$  satisfies

$$M \approx H_g \approx H_g,$$

then the right-hand side is called the Heegaard decomposition (splitting) of  $M$ .

**Proposition 2.4.** Every closed 3-manifold admitting a Heegaard decomposition of genus 0 is diffeomorphic to  $S^3$ .

*Proof.*  $\mathcal{M}_0 = 1$ . □

**Proposition 2.5.** Every closed 3-manifold admitting a Heegaard decomposition of genus 1 is diffeomorphic to  $S^3$ ,  $S^2 \times S^1$ , or  $L(p, q)$ .

**Exercise 3.** Prove the above proposition.

**Theorem 2.6.** Every closed 3-manifold  $M$  admits a Heegaard decomposition along some  $\Sigma_g$ .

*Proof.* Pick a triangulation  $T$  of  $M$ . Then,  $H := \overline{\nu(T^{(1)})}$  is a handlebody. Then,  $H' := M \setminus \text{Int} H$  is also a handlebody. Since  $M$  is orientable, so are  $H$  and  $H'$ , thus we are done. □

There is another proof using Morse theory.

**Corollary 2.7.** The fundamental group of every closed 3-manifold admits a finite presentation of deficiency 0, i.e. the number of generators is equal to the number of relations.

*Proof.* Apply the van Kampen theorem to

$$M = H_g \cup H_g = H_g \cup (D^2 \times [-\varepsilon, \varepsilon])^{\sqcup g} \cup D^3. \quad \square$$

## 2.4 Dehn surgery

Let  $L$  be a link. The link exterior is the set  $E_L = S^3 \setminus \nu(L)$ .

**Proposition 2.8.** Let  $M$  be a 3-manifold, and  $T \subset \partial M$  a torus component. Let  $h : \partial(D^2 \times S^1) \rightarrow T$  be a diffeomorphism. Then,  $M \cup_h (D^2 \times S^1)$  is determined only by  $\pm[h(\partial D^2 \times *)] \in H_1(T)$ .

*Proof.* Write

$$M \cup_h (D^2 \times S^1) = M \cup_h (D^2 \times (-\varepsilon, \varepsilon)) \cup D^3. \quad \square$$

For a knot  $K$ , there are two generators  $\mu$  and  $\lambda$ , called the meridian and the longitude, of  $H_1(\partial E_K)$  such that  $\ker(H_1(\partial E_K) \rightarrow H_1(E_K) = \mathbb{Z}\mu)$  is generated by  $\lambda$ .

**Exercise 4.** Show that  $L(p, q)$  with  $p \neq 0$  and  $S^2 \times S^1$  are obtained by the  $p/q$ -Dehn surgery along the unknot.

**Theorem 2.9** (Lickorish-Wallace). *Every closed 3-manifold can be obtained by an (integral)-Dehn surgery along some link in  $S^3$ .*

*Sketch.* Heegaard decomposition and  $\mathcal{M}_g = \langle \text{Dehn twists} \rangle$ . Each Dehn twist realizes the Dehn surgery steps. □

### 3 Day 3: April 25

#### Prime decomposition

##### 3.1 Alexander's theorem

**Theorem 3.1.** *Every (smooth) embedding  $S^2 \subset \mathbb{R}^3$  bounds some (smooth) embedding  $D^3 \subset \mathbb{R}^3$ .*

*Remark.* The above theorem does not hold in the category of topological spaces. Alexander's horned sphere is one of the counterexamples.

If  $\mathbb{R}^d$  for  $d \geq 5$ , then more complicated result such as h-cobordism theorem must be used to obtain the same conclusion.

*Sketch.* Isotope such a sphere  $S$  so that the coordinate  $z : S \rightarrow \mathbb{R}$  is a Morse function. Assume that for all  $p \neq q \in \text{Crit}(z)$ , then  $z(p) \neq z(q)$ . We use induction on  $(m, n)$ , where  $m$  is the number of saddles and

$$n := \min\{\#\pi_0(S \cap z^{-1}(r)) : r \text{ is a regular value s.t. } z(p) < r < z(q) \text{ for some saddles } p, q\}.$$

Note that  $\#(\text{minima of } z) - m + \#(\text{maxima of } z) = \chi(S) = 2$ .

For the case  $m = 0$  so that there are only one minimum and maximum, then we can construct a ball by applying the Jordan-Schönflies theorem to each level.

For the case  $m = 1$ , then only four types appear: a jelly bean, a red blood cell, and their upside down versions. Apply the Jordan Schönflies again.

For the case  $m \geq 2$ , let  $r$  be a regular value realizing the value of  $n$ . Let  $D$  be union of the closure of the interior of the innermost circles of  $S \cap z^{-1}(r)$ . Replace  $S$  by  $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{-\varepsilon, \varepsilon\})$ . Then, each connected component has lower  $(m, n)$  so that it bounds a ball. Attaching all balls bounded by the components with balls  $D \times [-\varepsilon, \varepsilon]$ ,  $S$  also bounds a ball.  $\square$

##### 3.2 Irreducible manifolds

The connected sum is defined as

$$M \# N := (M \setminus (\text{open ball})) \cup_{\partial} (N \setminus (\text{open ball})).$$

**Proposition 3.2.**

- (a)  $M \# N \approx N \# M$ .
- (b)  $(M_1 \# M_2) \# M_3 \approx M_1 \# (M_2 \# M_3)$ .
- (c)  $M \# S^3 \approx M$ .

We say a manifold  $M$  is *prime* if  $M = N_1 \# N_2$  implies  $N_1 \approx S^3$  or  $N_2 \approx S^3$ . We say a 3-manifold  $M$  is *irreducible* if every embedding  $S^2 \subset M$  bounds some embedding  $D^3 \subset M$ . In other words, in a prime manifold every separating sphere bounds a ball, in an irreducible manifold every sphere bounds a ball.

**Corollary 3.3.** *Every irreducible 3-manifold is prime.*

**Corollary 3.4.** *By Theorem 3.1,  $S^3$  is irreducible.*

**Theorem 3.5.**

- (a)  $S^2 \times S^1$  is prime, but is not irreducible.
- (b) Every closed prime 3-manifold which is not irreducible is diffeomorphic to  $S^2 \times S^1$ .

*Proof.* (a) A sphere  $S^2 \times *$  cannot bound any  $D^3 \subset S^2 \times S^1$  because  $[S^2 \times *] \neq 0 \in H_2(S^2 \times S^1)$ , so  $S^2 \times S^1$  is not irreducible.

Suppose  $S^2 \times S^1 = N_1 \# N_2$ . Since  $\pi_1(N_1) * \pi_1(N_2) \cong \mathbb{Z}$ , one of  $\pi_1(N_1)$  or  $\pi_1(N_2)$  is trivial. Assume  $\pi_1(N_1)$  is trivial and let  $B := N_1 \setminus (\text{open ball})$ . Since  $B$  is also simply connected, it lifts diffeomorphically into the universal cover  $S^2 \times \mathbb{R}$  of  $S^2 \times S^1$ . Because  $S^2 \times \mathbb{R} \approx \mathbb{R}^3 \setminus \{0\}$ , we have an embedding  $B \subset \mathbb{R}^3$ . Because  $\partial B \approx S^2$ , by Theorem 3.1 we have  $B \approx D^3$ , so  $N_1 \approx S^3$ .

(b) If every sphere in  $M$  is separating, then it has to be irreducible since  $M$  is prime, so such  $M$  contains a nonseparating sphere  $S$ . Let  $\gamma$  be an arc connecting the inside and the outside of  $\partial \nu(S)$ . If we let  $M' := \overline{\nu(S)} \cup \overline{\nu(\gamma)}$ , then  $\partial M' \approx S^2 \# (S^1 \times I) \# S^2 \approx S^2$  is a separating sphere. Since  $M \setminus \text{Int} M' \approx D^3$  because  $M'$  is not simply connected and  $M$  is prime,  $M$  is diffeomorphic to  $S^2 \times S^1$ .  $\square$

**Proposition 3.6.** *If a covering space  $\tilde{M}$  of  $M$  is irreducible, then so is  $M$ .*

**Exercise 4.** Prove Proposition 3.6.

*Remark.* The converse of Proposition 3.6 is also known to be true.

### 3.3 Normal surfaces

Fix a 3-manifold  $M$  and its triangulation  $T$ . A (possibly disconnected) subsurface  $S \subset M$  is called a *normal surface* with respect to  $T$  if  $S$  is a union of *normal disks*, defined as seven types of disks in a given tetrahedron: four triangles and three quadrilaterals.

**Proposition 3.7.** *Every (possibly disconnected) subsurface  $S \subset M$  becomes a normal surface with respect to  $T$  by isotopies and the following operations:*

- (i) *Replace  $S$  by  $(S \setminus \partial D \times (-\varepsilon, \varepsilon)) \cup (D \times \{\pm \varepsilon\})$  for a disk  $D$  satisfying  $D \cap (S \cup \partial M) = \partial D$ .*
- (ii) *Remove a components of  $S$  contained in a ball in  $M$ .*

*Proof.* Isotope  $S$  so that  $S \pitchfork T$ . It is sufficient to realize the following:

- (a) For every tetrahedron  $\Delta^{(3)} \subset T^{(3)}$ ,  $S \cap \Delta^{(3)} = \coprod (\text{disks})$ .
- (b) For every disk component  $D$  in (a) and for every edge  $\Delta^{(1)} \in T^{(1)}$ ,  $\#(D \cap \Delta^{(1)}) \leq 1$ .
- (c) For every triangle  $\Delta^{(2)} \subset T^{(2)}$ ,  $S \cap \Delta^{(2)} = \coprod (\text{arcs})$ .

For (a), if there is a non-disk connected component of  $S \cap \Delta^{(3)}$ , perform (i) along innermost loops in  $S \cap \partial \Delta^{(3)}$ . Then, perform (ii) for closed components of  $S$  contained in  $\Delta^{(3)}$ .

For (b), if there is a disk component of  $S \cap \Delta^{(3)}$  which intersects an edge  $\Delta^{(1)}$  more than twice, then for an inner most pair of two points connected by an arc in  $D$ , push the arc to the outside  $\Delta^{(3)}$  with an ambient isotopy.

For (c)

$\square$



## 4 Day 4: May 9

### 3.4 Prime decomposition theorem

**Proposition 3.8.** *Let  $T$  be a triangulation of  $M$  and let  $S \subset M$  be a normal surface with respect to  $T$ . If no component of  $M \setminus \nu(S)$  is a trivial  $[0, 1]$ -bundle, then*

$$\# \pi_0(S) \leq 6 \cdot \# T^{(3)} + \text{rk} H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}).$$

*Proof.* Let  $M_0$  be a component of  $M \setminus \nu(S)$  such that for each tetrahedron  $\Delta^{(3)}$  the intersection  $M_0 \cap \Delta^{(3)}$  is a trivial  $[0, 1]$ -bundle over a normal disk  $D_0(\Delta^{(3)}) \subset \Delta^{(3)}$ . Let  $S' = (S \setminus \sqcup \partial M_0) \sqcup (\sqcup D_0)$ . Then,

$$\# \pi_0(S) = \# \pi_0(S') = \text{rk} H_2(S', \partial S'; \mathbb{Z}/2\mathbb{Z}),$$

and since

$$H_3(M, \partial M \cup S'; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_2(S', \partial S'; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$$

is exact, we have

$$\text{rk} H_2(S', \partial S'; \mathbb{Z}/2\mathbb{Z}) \leq \text{rk} H_3(M, \partial M \cup S'; \mathbb{Z}/2\mathbb{Z}) + \text{rk} H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}).$$

If we let  $M' := (M \setminus \nu(S)) \setminus \sqcup M_0$ , then by the excision theorem,

$$\text{rk} H_3(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = \text{rk} H_3(M', \partial M'; \mathbb{Z}/2\mathbb{Z}) = \pi_0(M').$$

Every component of  $M'$  is not a trivial  $[0, 1]$ -bundle over a normal disk in some tetrahedron in  $T$ . The number of such non-trivial components of the complement  $\Delta^{(3)} \setminus \nu(S)$  is at most 6.  $\square$

**Theorem 3.9** (Prime decomposition theorem; Kneser, Milnor). *Every 3-manifold  $M \neq S^3$  admits a decomposition*

$$M = N_1 \# \cdots \# N_n, \quad N_i : \text{prime} \neq S^3.$$

*Moreover, such a decomposition is unique up to diffeomorphisms and permutations.*

*Proof.* (Existence) If  $M$  contains a non-separating sphere, then there is  $M'$  such that  $M = M' \# (S^2 \times S^1)$  as in the proof of Proposition 3.5. Since we have  $H_1(M) = H_1(M') \oplus \mathbb{Z}$ , the number of such  $S^2 \times S^1$  factors is finite, so by cutting out all of them, we may assume that every sphere in  $M$  is separating.

Let  $S$  be an arbitrary union of spheres in  $M$  such that

$$\text{no components of } M \setminus \nu(S) \text{ is diffeomorphic to } S^3 \setminus (\text{balls}). \quad (\dagger)$$

It suffices to show that  $\pi_0(S)$  is finite. Let  $T$  be a triangulation of  $M$  and  $S' \subset M$  be a normal surface with respect to  $T$  obtained from  $S$  as Proposition 3.7. Suppose that a component  $S_0 \subset S$  is contained in a ball. Then by the Alexander theorem such an innermost component would bound a ball, which contradicts to  $(\dagger)$ , hence the operation (ii) in Proposition 3.7 does not occur when we construct  $S'$ . Each operation (i) splits a component  $S_0 \subset S$  into two spheres  $S_1$  and  $S_2$ , and one of them satisfies  $(\dagger)$ . Therefore, there exists a union of components  $S'' \subset S'$  satisfying the same condition with  $S$  with  $\pi_0(S'') = \pi_0(S)$ , which is bounded by Proposition 3.8.

(Uniqueness) Suppose

$$P_1 \# \cdots \# P_m \# (S^2 \times S^1)^{\#k} = Q_1 \# \cdots \# Q_n \# (S^2 \times S^1)^{\#l},$$

where  $P_i$  and  $Q_j$  are irreducible but not  $S^3$ . It suffices to find a union of spheres  $T \subset M$  such that

$$(P) \quad M \setminus \nu(T) = \bigsqcup_{i=1}^m (P_i \setminus (\text{balls})) \sqcup (\bigsqcup S^3 \setminus (\text{balls})),$$

$$(Q) \quad M \setminus \nu(T) = \bigsqcup_{j=1}^n (Q_j \setminus (\text{balls})) \sqcup (\bigsqcup S^3 \setminus (\text{balls})).$$

Pick unions of spheres  $R, S \subset M$  such that  $R$  satisfies (P),  $S$  satisfies (Q), and  $R \pitchfork S$ . If  $R \cap S = \emptyset$ , then let  $T := R \sqcup S$ . If not, pick components  $R_0 \subset R$  and  $S_0 \subset S$  such that  $R_0 \cap S_0 \neq \emptyset$ . Split  $R_0$  along an innermost disk in  $S_0$  into two spheres  $R_1$  and  $R_2$ , and replace  $R$  by  $R' := (R \setminus R_0) \sqcup R_1 \sqcup R_2$ . Then,  $R'$  satisfies (P) and  $\# \pi_0(R' \cap S) < \# \pi_0(R \cap S)$  so that such operations reduce to the case of  $R \cap S = \emptyset$ .  $\square$

**Exercise 5.** Show that the existence of the prime decomposition theorem follows from the Poincaré conjecture and Grushko's theorem:  $\text{rk } G * H = \text{rk } G + \text{rk } H$ , where the rank is defined to be the minimum number of generators.

**Theorem 3.10** (Kneser conjecture, proved by Stallings). *Let  $M$  be an oriented 3-manifold such that  $\pi_1(M) = G_1 * G_2$ . Then, there are oriented 3-manifolds  $N_1$  and  $N_2$  such that  $M = N_1 \# N_2$  and  $\pi_1(N_i) \cong G_i$  for  $i \in \{1, 2\}$ .*

For this theorem, it suffices to study irreducible 3-manifolds.

## Loop and sphere theorems

### 4.1 Loop theorem

**Theorem 4.1** (Loop theorem). *For a subsurface  $S \subset M$ , there is a continuous embedding  $f : (D^2, \partial D^2) \rightarrow (M, S)$  such that  $f|_{\partial D^2} \neq \text{null homotopic in } S$ . In particular, we may assume  $f$  is smooth.*

*Remark.* The loop theorem holds also for noncompact or nonorientable 3-manifolds.

**Corollary 4.2** (Dehn's lemma). *Every loop in  $\partial M$  null homotopic in  $M$  bounds some embedded disk in  $M$ .*

*Proof.* Apply the loop theorem for the tubular neighborhood of the loop in  $\partial M$ .  $\square$

**Theorem 4.3.** *Let  $K \subset S^3$  be a knot and  $E_K := S^3 \setminus \nu(K)$ . Then,  $K$  is unknot if and only if  $\pi_1(E_K) \cong \mathbb{Z}$ .*

*Proof.*  $(\Rightarrow)$  Clear.

$(\Leftarrow)$  Take a longitude  $\lambda \in \pi(\partial E_K) = H_1(\partial E_K)$  such that

$$\lambda \in \ker(\pi_1(\partial E_K) \rightarrow H_1(E_K)) \subset \iota_*^{-1}([\pi_1(E_K), \pi_1(E_K)]),$$

where  $\iota : \partial E_K \hookrightarrow E_K$ . Then,  $\pi_1(E_K) \cong \mathbb{Z}$  implies  $\lambda = 1$ , and apply Dehn's lemma.  $\square$

**Exercise 6.** Let  $M$  be an irreducible 3-manifold and  $T \subset M$  a torus. Show that the followings are equivalent:

- (i)  $\pi_1(T) \rightarrow \pi_1(M)$  is not injective.
- (ii)  $T$  bounds some embedded solid torus  $D^2 \times S^1$  in  $M$  or  $T$  is contained in some embedded ball in  $M$ .

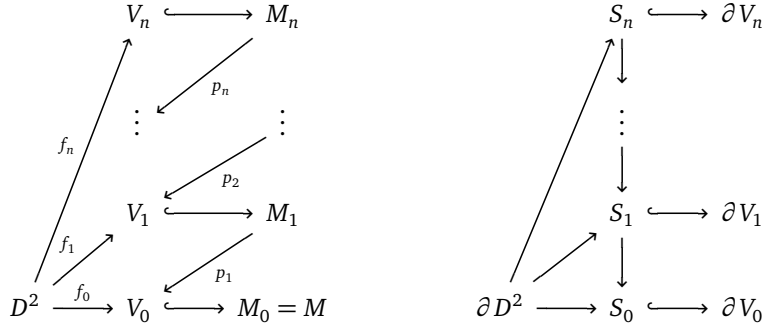
## 5 Day 5: May 16

**Theorem 4.1** (Loop theorem, Papakyriakopoulos). *For a subsurface  $S$  of 3-manifold  $M$ , if there is a continuous map  $f : (D^2, \partial D^2) \rightarrow (M, S)$  such that  $f|_{\partial D^2}$  is not null-homotopic, then there is a smooth embedding  $g : (D^2, \partial D^2) \rightarrow (M, S)$  such that  $g|_{\partial D^2}$  is not null-homotopic.*

*Proof.* Step 1: *Tower construction.* Pick triangulations of  $(M, S)$  and  $D^2$ .

Let  $f_0 : (D^2, \partial D^2) \rightarrow (M, S)$  be a simplicial map such that  $f_0 \simeq f$ . Define  $V_0 \subset M$  to be a 3-submanifold deformation retracting to  $f_0(D^2)$  and  $S_0 \subset \partial V_0$  a component of  $S \cap V_0$  containing  $f_0(\partial D^2)$ .

Given the  $i$ -th level, for any double covering  $p_{i+1} : M_{i+1} \rightarrow V_i$ , let  $f_{i+1} : (D^2, \partial D^2) \rightarrow (M_{i+1}, p_{i+1}^{-1}(S_i))$  be a lift of  $f_i$ . Similarly as  $i+1=0$ , define  $V_{i+1} \subset M_{i+1}$  to be a 3-manifold deformation retracting to  $f_{i+1}(D^2)$  and  $S_{i+1} \subset \partial V_{i+1}$  the component of  $p_{i+1}^{-1}(S_i) \cap V_{i+1}$  containing  $f_{i+1}(\partial D^2)$ .



Since

$$\# \text{ of simplices of } f_0(D^2) < \# \text{ of simplices of } f_1(D^2) < \dots \leq \# \text{ of simplices of } D^2,$$

there is  $n$  such that  $V_n$  has no double cover. If we let  $N_i := \ker(\pi_1(S_i) \rightarrow \pi_1(S))$ , then  $f_{i*}([\partial D^2]) \notin N_i$  implies that  $N_i \neq \pi_1(S_i)$  for each  $i$ .

Step 2: *Embedding on the top.* Consider the following exact sequence:

$$H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(V_n; \mathbb{Z}/2\mathbb{Z}).$$

Since  $V_n$  has no double cover, the Poincaré duality writes

$$H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) \cong H^1(V_n; \mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(\pi_1(V_n), \mathbb{Z}/2\mathbb{Z}) = 0.$$

Also we have

$$H_1(V_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{H}_1(V_n) \otimes \mathbb{Z}/2\mathbb{Z} = 0,$$

which implies  $H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) = 0$  so that  $\partial V_n \cong \coprod S^2$ .

Note that  $S_n \approx S^2 \setminus (\text{disks})$  and  $\pi_1(S_n)$  is generated by components of  $\partial S_n$ . Since  $N_n \neq \pi_1(S_n)$ , there is a component  $l \subset \partial S_n$  such that  $[l] \notin N_n$ . Pick

Step 3: *Descending the tower.* We shall show that given an embedding  $g_i : (D^2, \partial D^2) \rightarrow (V_i, S_i)$  such that  $g_{i*}([\partial D^2]) \notin N_i$ , there is such an embedding  $g_{i-1}$ . Then,  $g_0$  is the desired embedding.

Isotope  $p_i \circ g_i$  so that self-intersections are simple double curves. These are modified by the following operations which give  $g_{i-1}$ .

Case 2 double arcs These are cancelled in two ways and at least one of the results satisfies  $g_{i-1}([\partial D^2]) \notin N_{i-1}$ .  $\square$

Let  $X$  be a path-connected space and  $x_0 \in X$ . Then,

$$\pi_n(X) := \pi_n(X, x_0) = \{f : (S^n, *) \rightarrow (X, x_0)\} / \text{homotopy fixing base points}.$$

For a covering  $p : \tilde{X} \rightarrow X$ , the induced homomorphism  $p_* : \pi_n(\tilde{X}) \rightarrow \pi_n(X)$  is an isomorphism for  $n \geq 2$ . We will use the following two theorems without proofs

**Theorem** (Whitehead theorem). Let  $X, Y$  be CW-complexes. If  $\varphi : X \rightarrow Y$  satisfies  $\varphi_* : \pi_n(X) \rightarrow \pi_n(Y)$  is an isomorphism for each  $n$ , then  $\varphi$  is a homotopy equivalence.

**Theorem** (Hurewicz theorem). For,  $h_n : \pi_n(X) \rightarrow H_n(X) : [f] \mapsto f_*([S_n])$ ,  $h_1$  is the abelianization, and  $h_{n+1}$  is an isomorphism if  $\pi_i(X) = 0$  for  $1 \leq i \leq n$ .

**Theorem 4.4** (Sphere theorem, Papakyriakopoulos). For a 3-manifold  $M$  with  $\pi_2(M) \neq 0$ , there exists an embedding  $f : S^2 \rightarrow M$  such that  $[f] \neq 0$ .

**Corollary 4.5.** If  $M$  is irreducible, then  $\pi_2(M) = 0$ .

**Theorem 4.6.** Let  $M$  be an irreducible 3-manifold with infinite fundamental group.

- (a)  $M$  is aspherical, i.e.  $\pi_n(M) = 0$  for  $n \geq 2$ .
- (b)  $\pi_1(M)$  is torsion-free.

It is well-known that a manifold  $M$  is aspherical if and only if  $M \simeq K(\pi_1(M), 1)$ .

**Lemma 4.7.** Let  $m \geq 2$ . Just use the identity  $H_*(\pi) = H_*(K(\pi, 1))$  as the definition of group homology. Then,

$$H_n(\mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}/m\mathbb{Z} & n : \text{odd}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact,  $K(\mathbb{Z}/m\mathbb{Z}, 1) \simeq S^\infty/(\mathbb{Z}/m\mathbb{Z})$ .

*Proof of Theorem 4.6.* (a) By Corollary 4.5, we have  $\pi_2(M) = 0$ . If we let  $\tilde{M}$  be the universal cover of  $M$ , which is not compact because  $\pi_1(M)$  is infinite, so that we have

$$\pi_3(M) = \pi_3(\tilde{M}) = H_3(\tilde{M}) = 0.$$

Inductively, and since  $\dim M = 3$ , we have for  $n \geq 4$  that

$$\pi_n(M) = \pi_n(\tilde{M}) = H_n(\tilde{M}) = 0.$$

(b) Suppose that there is a non-trivial finite cyclic subgroup  $C \leq \pi_1(M)$ . If we let  $\hat{M}$  be the cover of  $M$  corresponding to  $C$ , then  $\pi_1(\hat{M}) = C$ . It follows from (i) that  $\hat{M} = K(C, 1)$ . By Lemma 4.7, we have  $H_n(\hat{M}) \neq 0$  for odd  $n$ , but  $\dim \hat{M} = 3$  implies that  $H_n(\hat{M}) = 0$  for  $n \geq 4$ .  $\square$

**Theorem 4.8.** The only abelian groups appearing as the fundamental group of closed (orientable) 3-manifolds are  $1, \mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}/n\mathbb{Z}$ .

*Remark.* If we also consider non-orientable 3-manifolds, then  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  also appears as  $\pi_1(\mathbb{RP}^2 \times S^1)$ .

**Exercise 7.** Show that the only abelian groups admitting presentations of deficiency zero are

$$1, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/n\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

(Hint: use Hopf's formula  $H_2(\pi) = R \cap [F, F]/[F, R]$ , where  $\pi = F/R$ .)

*Proof of Theorem 4.8.* By Corollary 2.7 and Exercise 7, it suffices to show  $\pi_1(M) \not\cong \mathbb{Z}^2, \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ . By the prime decomposition theorem, we may assume that  $M$  is irreducible. By the part (b) of Theorem 4.6,  $\pi_1(M) \not\cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ . Now suppose that  $\pi_1(M) \cong \mathbb{Z}^2$ .

Since  $M$  is aspherical by Theorem 4.6,  $M \cong K(\mathbb{Z}^2, 1) = \mathbb{T}^2$ . However,  $\mathbb{Z} = H^3(M) \cong H^3(\mathbb{T}^2) = 0$  leads to a contradiction.  $\square$

## 6 Day 6: May 23

### Haken manifolds

In the following, we only consider *irreducible* 3-manifolds, i.e. every embedded sphere bounds a ball.

#### 5.1 Essential surfaces

When we say subsurfaces  $S$ , we always assume  $S \cap \partial M = \partial S$ .

A subsurface  $S \subset M$  which is not a sphere is called *incompressible* if there is no compressing disk. A *compressing disk* of a subsurface  $S$  is a disk  $D \subset M$  such that  $D \cap S = \partial D$  and  $\partial D$  bounds no disk in  $S$ . The second condition intuitively says that the compression does not generate spheres so that the disk compresses  $S$  non-trivially.

A subsurface  $S \subset M$  which is not a sphere is called *boundary incompressible* if there is no boundary compressing disks. A *boundary compressing disk* of  $S$  is a disk  $D$  such that arcs  $\gamma = D \cap S$  and  $\gamma' = D \cap \partial M$  form the boundary  $\partial D = \gamma \cup \gamma'$  and  $\gamma \cup \gamma''$  does not bound a disk in  $S$  for any arc  $\gamma \subset \partial S$ .

A surface  $S$  is called *boundary parallel* if  $S$  can be isotoped into  $\partial M$ .

A (possibly disconnected) subsurface  $S \subset M$  is called an *essential surface* if every component  $S_0$  of  $S$  is incompressible, boundary incompressible, and not boundary parallel. ( $S_0$  must not be a sphere)

**Proposition 5.1.** *For a subsurface  $S \subset M$ ,  $S$  is incompressible if and only if  $\pi_1(S) \rightarrow \pi_1(M)$  is injective.*

*Proof.* The loop theorem. □

**Proposition 5.2.** *For nonzero  $\alpha \in H_2(M, \partial M)$ , there is an essential surface  $S \subset M$  with  $[S] = \alpha$ .*

Let  $S$  be a possibly disconnected subsurface of  $M$ . We define a complexity  $c(S) := \sum_i (2 - \chi(S_i))^2$ , where  $S_i$  are components of  $S$ .

**Lemma 5.3.** *If  $S'$  is obtained from  $S$  by the surgery along (boundary) compressing disks, then  $c(S') < c(S)$ .*

*Proof.* We may assume  $S$  is connected. Note that every component has  $\chi < 2$  since we are not allowing spheres after surgery. If  $S'$  is connected, then since  $\chi(S') = \chi(S) + 2$ , we are done. If  $S' = S_1 \sqcup S_2$ , then since  $\chi(S_1) + \chi(S_2) = \chi(S') = \chi(S) + 2$ , we are done. □

*Proof of Proposition 5.2.* Pick a (possibly disconnected) subsurface  $S_0 \subset M$  with  $[S_0] = \alpha$ . (Choose  $f : M \rightarrow S^1$  so that  $[f] \in [M, S^1] = H^1(M)$  is equal to the Poincaré dual of  $\alpha$  and let  $S_0 := f^{-1}$ (a regular value)) Perform the surgery  $S_0$  along (boundary) compressing disks until every components of  $S_0$  becomes (boundary) incompressible. By Lemma 5.3, this operation stops, and before and after of the compression are homologous. Removing all spheres and boundary parallel components of  $S_0$ , now we have a desired subsurface  $S$  with  $[S] = [S_0] = \alpha$ . Since  $\alpha \neq 0$ ,  $S \neq \emptyset$ . □

A 3-manifold is a *Haken manifold* if it admits an essential surface in  $M$ .

**Corollary 5.4.** *If  $b_1(M) > 0$ , then  $M$  is Haken.*

*Proof.* Prop 5.2. □

**Corollary 5.5.** *If there is a non-sphere component of  $\partial M$ , then  $M$  is Haken.*

*Proof.* Since  $H_1(\partial M)$  contains an element of infinite order, the exact sequence

$$H^1(M) \cong H_2(M, \partial M) \rightarrow H_1(\partial M) \rightarrow H_1(M)$$

implies  $b_1(M) > 0$ . □

**Theorem 5.6** (Thurston). *For the figure eight knot, the Dehn filling by a rational number  $r = p/q$  is Haken if and only if  $r = 0, \pm 4$ .*

## 5.2 Haken hierarchy

**Proposition 5.7.** *Let  $S \subset M$  be an essential surface. Let  $M' = M \setminus \nu(S)$ . Then, (a)  $M'$  is irreducible and (b) every closed incompressible surface in  $M'$  is incompressible also in  $M$ .*

*Proof.* (a) Let  $S' \subset M'$  be arbitrary sphere. Since  $M$  is irreducible,  $S'$  bounds a ball  $B \subset M$ . Then, we have  $S \subset B$  or  $S \cap B = \emptyset$ . Since  $B$  contains no closed incompressible surface,  $B \cap S = \emptyset$ , so  $B \subset M'$ .

(b) Suppose that there is a closed incompressible surface  $S' \subset M'$  not incompressible in  $M$ . Let  $D \subset M$  be a compressing disk for  $S'$  such that  $D \pitchfork S$  and  $\# \pi_0(D \cap S)$  is minimal. Since  $S$  is essential, an innermost loop in  $S \cap D \subset S$  bounds a disk  $D' \subset S$  (if we take any compressible disk of  $S$  in  $M$  whose boundary is  $S \cap D$ , then the assumption  $S$  is essential implies that the compression gives a sphere; we can take a disk  $D'$ ). Since  $M$  is irreducible,  $D \cup D'$  bounds a ball in  $M$ , so  $D$  can be isotoped such that  $\# \pi_0(D \cap S)$  would reduce, which is a contradiction.  $\square$

**Proposition 5.8.** *If  $\partial M \neq \emptyset$  and  $M$  contains no closed incompressible surface, then  $M$  is a handlebody.*

*Proof.* Cut  $M$  along essential disks into  $\coprod_i M_i$  until every component of  $\partial M_i$  is incompressible. Suppose that there is  $i_0$  such that  $M_{i_0} \not\approx D^3$ . By Proposition 5.7 (a), every component of  $\partial M_{i_0}$  is not a sphere. By Proposition 5.7. (b), the components of  $\partial M_{i_0}$  are closed incompressible surfaces in  $M$  which contradicts the assumption. Therefore, for every  $i$  we have  $M_i \approx D^3$ ,  $M$  is a handlebody.  $\square$

**Theorem 5.9** (Haken hierarchy). *For every Haken manifold  $M$ , there is a sequence of irreducible (possibly disconnected) 3-manifolds  $M = M_0, M_1, \dots, M_n$  such that*

- (a)  $M_{i+1} = M_i \setminus \nu(S_i)$  for some essential surface  $S_i \subset M_i$
- (b)  $M_n \approx \sqcup D^3$ .

The strongest feature of the Haken hierarchy is the possibility of using induction.

**Theorem 5.10** (Waldhausen). *Two closed Haken manifolds with isomorphic fundamental groups are diffeomorphic.*

*Proof.* Induction on the Haken hierarchy.  $\square$

*Remark.* When  $\partial M \neq \emptyset$ , the theorem also holds under a certain additional condition on isomorphisms on  $\pi_1$ .