Harmonic Analysis

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Contents

Part I

Fourier analysis

Chapter 1

Fourier series

1.1 Fourier series in L^p spaces

1.1.

$$\|\widehat{f}\|_{\ell^1(\mathbb{Z})} \lesssim \|f\|_{W^{1,1+\varepsilon}(\mathbb{T})}.$$

Inversion theorem is an approximation problem given by $\mathcal{F}^*\mathcal{F} = \lim_{n\to\infty} \mathcal{F}_n^*\mathcal{F}$. The condition $\widehat{f} \in \ell^1(\mathbb{Z})$ is a condition just for defining $\mathcal{F}^*\widehat{f}$ wihtout using distribution theory, and it does not affect the inversion phenomena. The approximation, in other words, can be seen as an extension method for $\mathcal{F}^* : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ on $c_0(\mathbb{Z})$. Note that $\mathcal{F}_n^*$ on $c_0(\mathbb{Z})$ cannot be bounded directly without distribution theory, but $\mathcal{F}_n^*\mathcal{F}$ on $L^p(\mathbb{T})$ can be bounded well.

1.2 Summability methods

- If $\mathcal{F}_n^*$ is the standard partial sum, then $\mathcal{F}_n^*\mathcal{F}$ is the Dirichlet kernel.
- If $\mathcal{F}_n^*$ is the Cesàro mean, then $\mathcal{F}_n^*\mathcal{F}$ is the Fejér kernel.
- If $\mathcal{F}_r^*$ is the Abel sum, then $\mathcal{F}_r^*\mathcal{F}$ is the Poisson kernel.
- In Fourier transform, we often use the Gauss-Weierstrass kernel.

The injectivity of $\mathcal{F}$ is not an easy problem, which comes from the inversion theorem.

1.2 (Dirichlet kernel)**.** The *Dirichlet kernel* is a function $D_n : \mathbf{T} \to \mathbb{R}$ defined by

$$D_n = \widehat{1_{|k|\le n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = 1_{|k|\le n}.$$

This is because they are invariant under inverse, in other words, they are even.

(a)

$$D_n(x) = \frac{\sin \frac{2n+1}{2} x}{\sin \frac{1}{2} x}.$$

(b) If $f \in \mathrm{Lip}(\mathbf{T})$, then $D_n * f \to f$ pointwisely as $n \to \infty$.

(c)

$$\|D_n\|_{L^1(\mathbf{T})} \gtrsim \log n.$$

Proof.

$$\begin{aligned}
D_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\
&= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}.
\end{aligned}$$

(c) By (2) $\sin x \le x$ for $x \in [0, \pi/2]$, (3) change of variable,

$$\begin{aligned}
\|D_n\|_{L^1(\mathbb{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}\right| dx \\
&\ge \frac{2}{\pi}\int_0^{\pi} \frac{|\sin\frac{2n+1}{2}x|}{x}\, dx \\
&= \frac{2}{\pi}\int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\
&= \frac{2}{\pi}\sum_{k=0}^{2n}\int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\
&\ge \frac{2}{\pi}\sum_{k=0}^{2n}\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi}\, dx \\
&\ge \frac{4}{\pi^2}\sum_{k=0}^{2n}\frac{1}{1+k} \\
&\ge \frac{4}{\pi^2}\log(2n+2).
\end{aligned}$$

..? ☐

1.3 (Fejér kernel)**.** The *Fejér kernel* is

(a)

$$K_n(x) = \frac{1}{n+1}\frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}.$$

Proof. Since

$$\begin{aligned}
D_n(x) = &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2},
\end{aligned}$$

by telescoping, we get

$$\begin{aligned}
\sum_{k=0}^{n} D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\
&= \frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}.
\end{aligned}$$

□

Two important results from Fejér kernel:

1. If $f(x-)$, $f(x+)$ exist and $S_n f(x)$ converges, then $S_n f(x) \to \frac{1}{2}(f(x-)+f(x+))$.
2. (If $f \in L^1(\mathbf{T})$, then $\sigma_n f \to f$ a.e.)
3. If $f \in L^1(\mathbf{T})$, then $S_n f \to f$ in L^1 and L^2.
4. If f is continuous and $\widehat{f} \in L^1(\mathbb{Z})$, then $S_n f \to f$ uniformly.
5. Since $\sigma_n f$ is a trigonometric polynomial, the set of trigonometric polynomials are dense in $L^1(\mathbf{T})$ and $L^2(\mathbf{T})$.

1.3 Pointwise convergence of Fourier series

BV function: Dini, Jordan's criterion

1.4 (Riemann localization principle).

Exercises

1.5 (Gibbs phenomenon).

1.6 (Du Bois-Reymond function).

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

There are three conventions for the Fourier transform:

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \begin{cases} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\,dx, \\ (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\,dx, \\ \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\,dx. \end{cases}$$

We will accept the second one.

2.1 (Riemann-Lebesgue lemma)**.** basic estimates

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^d)} \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)}, \qquad f \in L^1(\mathbb{R}^d).$$

$$\|\widehat{f}\|_{L^1(\mathbb{R}^d)} \lesssim \|f\|_{W^{d+1,1}}(\mathbb{R}^d), \qquad f \in W^{d+1,1}(\mathbb{R}^d).$$

Lp extension

2.2 (Fourier inversion)**.** inversion theorem Plancerel

$$\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}, \qquad f \in L^2(\mathbb{R}^d).$$

unitarity

2.3 (Properties)**.** If $D := -i\partial$, then

$$\mathcal{F} D_x \mathcal{F}^* = M_\xi, \qquad D_\xi \mathcal{F} = -\mathcal{F} M_x.$$

$$\mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}} \widehat{f} * \widehat{g}.$$

2.2 Tempered distributions

A routine of Fourier transform computation of $f \in \mathcal{S}'$:

1. Choose a sequence $f_n \in L^1$ such that $f_n \to f$ in $\mathcal{S}'$.
2. Write $\mathcal{F}f_n$ in the integral form.
3. Compute the limit of $\mathcal{F}f_n$ in $\mathcal{S}'$.

Since $g_n \to g$ pointwise implies $g_n \to g$ in $\mathcal{S}'$ if the sequence g_n is dominated by a locally integrable with polynomial growth, we can frequently check the pointwise limit instead of $\mathcal{S}'$.

Methods: approximate identity, indented contour, imaginary shift, Feynman's trick

Examples: $e^{-\frac{1}{2}x^2}$, $e^{\frac{i}{2}x^2}$, $\mathrm{p.v.}\,\frac{1}{x}$, $\mathrm{sgn}(x)$, 1, $\delta(x)$, $\mathrm{sinc}(\frac{x}{2})$, $1_{[-\frac{1}{2},\frac{1}{2}]}$, $\frac{1}{1+x^2}$

Exercises

2.4 (Sampling theorem).

$$\mathcal{F}1_{[-\frac{1}{2},\frac{1}{2}]}(\xi) = \operatorname{sinc}(\xi/2)$$

$\operatorname{sinc} \in L^{1+\varepsilon}(\mathbb{R})$.

2.5 (Gausssian function). Gaussian function computation: differential equation method, contour integral method, imaginary shift

$$\mathcal{F}e^{-x^2}$$

$$\mathcal{F}e^{-\frac{1}{2}xQx} = \frac{e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}}{|\det Q|^{\frac{1}{2}}} e^{-\frac{1}{2}xQ^{-1}x}.$$

$$\mathcal{F}\operatorname{sech}(2\pi)^{\frac{1}{2}}\frac{x}{2} = \operatorname{sech}(2\pi)^{\frac{1}{2}}\frac{x}{2}.$$

2.6.

$$\mathcal{F}1 = (2\pi)^{\frac{1}{2}}\delta$$

$$\mathcal{F}x = (2\pi)^{\frac{1}{2}} i\delta'$$

2.7 (Poisson summation formula).

2.8 (Uncertainty principle).

2.9 (Paley-Wiener theorem). Let f be an integrable compactly supported function. Using the Morera to prove $\hat{f}$ is analytic.

2.10. Let $f \in C^\infty(\mathbb{R})$ and define $f_n(x) := \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(0)x^k$. Suppose $f_n \to f$ pointwise.

(a) f_n never converge to f in the space of tempered distributions. (Hint: use Borel's theorem to construct a Schwarz function)

(b) f_n converges to f in the the space of distributions.

2.11 (Multipole expansion). Let ρ be a compactly supported distribution on $\mathbb{R}^d$. We want to investigate the limit behavior of $\rho(\varepsilon^{-1}x)$ as $\varepsilon \to 0$. More precisely, we want to compute an integer $k \geq d$ such that $\lim_{\varepsilon\to 0+} \varepsilon^{-k}\rho(\varepsilon^{-1}x)$ defines a distribution supported at $\{0\}$, and the coefficients of derivatives of Dirac measures.

We need to introduce quantities called monopole, dipole, quadrapole, octupole, etc.

(a) A distribution supported on $\{0\}$ is a linear combination of the Dirac measure and its derivatives.

(b)

Problems

1. Find all $\alpha > 0$ such that

$$\lim_{x\to\infty} x^{-\alpha} \int_0^x f(y)\,dy = 0$$

for all $f \in L^3([0,\infty))$.

Chapter 3

Part II

Singular integral operators

Chapter 4

Calderón-Zygmund theory

4.1 Convolution type operators

4.1 (Calderón-Zygmund decomposition).

4.2 (Calderón-Zygmund decomposition of sets). Let $f \in L^1(\mathbb{R}^d)$. Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf := \sup_n E_n|f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|}\int_{Q_x} |f| > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \le \frac{1}{\lambda}\|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\mathbb{R}^d\setminus\Omega)} \le \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|}\int_{Q_x} |f| \le 2^d\lambda.$$

4.3 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|}\int_{Q_x}|f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \le 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof. □

4.4 (L^p boundedness of Calderón-Zygmund operators). Let $T : C_c^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that $Tf(x) = K * f(x)$ for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \operatorname{supp} f$. We say T is called a *Calderón-Zygmund operator* if

(i) T is L^2-bounded: we have

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2},$$

(ii) T satisfies the *Hörmander condition*: we have

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \lesssim 1$$

for every $y > 0$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2}\|Tg\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2}\|g\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2}\|g\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega^*} |Tb(x)|\, dx \leq \frac{2}{\lambda}\sum_i \int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\, dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\operatorname{supp} b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y) b_i(y)\, dy = \int_{Q_i} [K(x-y) - K(x)] b_i(y)\, dy,$$

and

$$\int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\, dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d\setminus Q_i^*} |K(x-y) - K(x)|\, dx\, dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.5 (Hölder boundedness of Calderón-Zygmund operators).

4.2 Truncated integrals

Homogeneous kernels

4.3 Hilbert transform

4.6 (Harmonic conjugate).

4.7 (Kernel representation).

4.8 (Fourier series in L^p space).

4.4 A_p weights

4.5 Bounded mean oscillation

Exercises

4.9 (Size and cancellation condition). Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$. We say the condition $|K(x)| \lesssim |x|^{-d}$ for $x \neq 0$ as the *size condition*, and say the condition $\int_{r<|x|<R} K(x)\,dx = 0$ for all $0 < r < R < \infty$ as the *cancellation condition*. If K satisfies the size, cancellation, and Hörmander condition, then it is L^2 bounded, hence Calderón-Zygmund.

4.10 (Gradient size condition). Let $|\nabla K(x)| \lesssim |x|^{-d-1}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

4.11 (Riesz potential).

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Almost orthogonality

Carleson measures, paraproducts

6.1 Coltar lemma

6.2 $T(1)$ theorem

Part III

Oscillatory integral operators

Chapter 7

Oscillatory integrals

7.1 (Justification of oscillatory integral)**.** For a function ϕ with fast growth toward infinity, we want to define a linear functional I_ϕ such that

$$I_\phi(a) := \int_{\mathbb{R}^d} e^{i\phi(x)} a(x)\,dx, \qquad a \in \mathcal{S}(\mathbb{R}^d).$$

A linear functional of the above form is called the *oscillatory integral* with *phase function* ϕ. As a notation, we will use the above integral in the right-hand side to denote the value of I_ϕ even for $a \notin L^1(\mathbb{R}^d)$. Then, we have pointwise justifications for integral calculus.

(a) $I_\phi : A^m_\delta(\mathbb{R}^d) \to \mathbb{C}$ is well-defined and continuous, if ϕ.

(b) The change of variables is justified as follows:

(c) The integral by parts is justified as follows:

$$\int_{\mathbb{R}^d} e^{i\phi(y)} i\partial\phi(y) a(x+y)\,dy = -\int_{\mathbb{R}^d} e^{i\phi(y)} \partial a(x+y)\,dy, \quad x \in \mathbb{R}^d,\ a \in A^m_\delta(\mathbb{R}^d).$$

(d) The Fubini theorem is justified as follows:

(e) The Fourier inversion is justified as follows:

$$a(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(y)\,dy\,d\xi, \quad x \in \mathbb{R}^d,\ a \in A^m_\delta(\mathbb{R}^d).$$

Proof. (a) Note that $A^m_\delta(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is dense in $A^m_\delta(\mathbb{R}^d)$. The most difficult part is the construction and the computation of L and its transpose.

(e) Note that the function $(y,\xi) \mapsto a(y)$ belongs to $A^{m'}_\delta(\mathbb{R}^{2d})$ since □

7.2 (Point evaluation of multiplier)**.** Let $\phi \in$ be a phase function. We want to show the following point evaluation holds with previously justified oscillatory integral:

$$\Phi(D)a(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\phi(y)} a(x+y)\,dy, \qquad x \in \mathbb{R}^d,\ a \in A^m_\delta(\mathbb{R}^d),$$

where $\Phi := \mathcal{F}^* e^{i\phi}$. Which condition for ϕ makes Φ be able to act on $\mathcal{S}'$ by multiplication?

7.3 (Stationary phase approximation)**.**

Proof. □

7.4 (Van der Corput lemma)**.**

Dispersive equations and strichartz estimates

Exercises

7.5 (Fresnel phase)**.** We compute L with a specific example

Proof.

$$(1+xQ^{-1}D)e^{\frac{i}{2}xQx} = \langle x\rangle^2 e^{\frac{i}{2}xQx}.$$

The transpose of $\langle x\rangle^{-2}(1+xQ^{-1}D)$ is $\langle x\rangle^{-2}(1+di-2ix^2-xD)$ for $Q=I$.

Note that $\langle x\rangle^{-2n}\langle D\rangle^{2n}$ is self-adjoint.

Let Q be a non-degenerate symmetric bilinear form on $\mathbb{R}^d$. Consider a multiplier operator $e^{\frac{i}{2}DQD}$: $\mathcal{S}\to\mathcal{S}$ such that

$$e^{\frac{i}{2}DQD}a(x) := \mathcal{F}^* e^{\frac{i}{2}\xi Q\xi}\mathcal{F}a(x).$$

(a) The pointwise evaluation is given by the oscillatory integral.

$$e^{\frac{i}{2}DQD}a(x) = (2\pi)^{-d}\frac{e^{\frac{i\pi}{4}}\,\mathrm{sgn}(Q)}{|\det Q|^{\frac{1}{2}}}\int_{\mathbb{R}^d} e^{-\frac{i}{2}yQ^{-1}y}a(x+y)\,dy, \qquad x\in\mathbb{R}^d,\ a\in A^m_\delta.$$

(b)

$$e^{\frac{i}{2}DQD}a(x) = \sum_{k=0}^{n}\frac{i^k}{2^k k!}(DQD)^k a(x) + r_n(x)$$

$\square$

Chapter 8

Foureir restriction

Kakeya Bochner-Riesz Geometric measure theory

Chapter 9

Part IV

Pseudo-differential operators

Chapter 10

Pseudo-differential calculus

10.1

10.1 (Hörmander symbol classes)**.** Let $m, \rho, \delta \in \mathbb{R}$. The *Hörmander class* $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ of symbols is the set of smooth functions $a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim_{\alpha,\beta} \langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}$$

for each $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$.

(a) Fréchet space

10.2 (Asymptotic expansion)**.** Let $\rho, \delta \in \mathbb{R}$. Let $a_k \in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d})$ for a sequence $(m_k)_{k=0}^\infty \subset \mathbb{R}$ with m_0 and $m_k \downarrow -\infty$. We want to construct $a \in S^{m_0}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a - \sum_{k=0}^{n-1} a_k \in S^{m_n}_{\rho,\delta}(\mathbb{R}^{2d}). \tag{†}$$

The symbol a_0 is called the *principal symbol* of a, or the operator $\mathrm{Op}^t(a)$.

Let $\chi \in C_c^\infty(\mathbb{R}^d_\xi, [0,1])$ be a cutoff function such that

$$\chi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1 \\ 0, & \text{if } |\xi| \geq 2 \end{cases}.$$

(a) If $a \in S^m_{\rho,\delta}$, then $\chi(\varepsilon\xi)a(x,\xi)$ is uniformly bounded in $S^m_{\rho,\delta}$ for $\varepsilon \in (0,1)$ if $\rho \leq 1$.

(b) There is $a \in S^{m_0}_{\rho,\delta}$ such that (†) if $\rho \leq 1$.

Proof. (a) On the support of $\xi \mapsto \chi(\varepsilon\xi)$ holds $\langle\xi\rangle < 2|\xi| \leq 4\varepsilon^{-1}$ because $1 < \varepsilon^{-1}$, so for each $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$ we have

$$\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta(\chi(\varepsilon\xi)a(x,\xi))| &= |\sum_\tau \binom{\beta}{\tau} \partial_\xi^{\beta-\tau}(\chi(\varepsilon\xi))\partial_x^\alpha\partial_\xi^\tau a(x,\xi)| \\
&= |\sum_\tau \binom{\beta}{\tau} \varepsilon^{|\beta|-|\tau|}\partial_\xi^{\beta-\tau}\chi(\varepsilon\xi)\partial_x^\alpha\partial_\xi^\tau a(x,\xi)| \\
(\because \langle\xi\rangle \leq 4\varepsilon^{-1}) \quad &\leq \sum_\tau \binom{\beta}{\tau}(4\langle\xi\rangle^{-1})^{|\beta|-|\tau|}|\partial_\xi^{\beta-\tau}\chi(\varepsilon\xi)||\partial_x^\alpha\partial_\xi^\tau a(x,\xi)| \\
&\lesssim \sum_\tau \binom{\beta}{\tau}\langle\xi\rangle^{-(|\beta|-|\tau|)}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\tau|} \\
(\because \rho \leq 1) \quad &\leq \langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}.
\end{aligned}$$

(b) Because we have $\varepsilon^{-1} \le \langle\xi\rangle$ on the support of $1-\chi(\varepsilon\xi)$, for each k we can take a sequence ε_k small enough such that

$$\max_{\substack{\alpha,\beta\in\mathbb{Z}^d_{\ge 0}\\ |\alpha|+|\beta|\le k}} |\partial_x^\alpha\partial_\xi^\beta((1-\chi(\varepsilon_k\xi))a_k(x,\xi))| \le 2^{-k}\langle\xi\rangle^{m_k+1+\delta|\alpha|-\rho|\beta|}.$$

We may assume $\varepsilon_k \downarrow 0$ so that the following sum is locally finite:

$$a(x,\xi) := \sum_{k=0}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi).$$

If we choose n such that $m_0 \ge m_n + 1$, then in the expansion

$$a(x,\xi) = \sum_{k=0}^{n-1}(1-\chi(\varepsilon_k\xi))a_k(x,\xi) + \sum_{k=n}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi)$$

the first sum clearly belongs to $S^{m_0}_{\rho,\delta}$ and so is the second sum because

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta\sum_{k=n}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi)| &\le \sum_{k=n}^{\infty}2^{-k}\langle\xi\rangle^{m_{k+1}+1+\delta|\alpha|-\rho|\beta|}\\ &\le \langle\xi\rangle^{m_n+1+\delta|\alpha|-\rho|\beta|}\\ &\le \langle\xi\rangle^{m_0+\delta|\alpha|-\rho|\beta|}\end{aligned}$$

for every $\alpha,\beta\in\mathbb{Z}^d_{\ge 0}$. Therefore, $a \in S^{m_0}_{\rho,\delta}$.

Write

$$(a-\sum_{k=0}^{n-1}a_k)(x,\xi) = \sum_{k=0}^{n-1}\chi(\varepsilon_k\xi)a_k(x,\xi) + \sum_{k=n}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi).$$

The first sum belongs to $S^{-\infty}$ because it is compactly supported, and we can also show that the second sum belongs to $S^{m_n}_{\rho,\delta}$ by decomposing with n' such that $m_n \ge m'_n+1$ and by considering the multiplication with a cutoff remains in the same symbol class. $\square$

10.3 (Quantization)**.** For a symbol a defined on $\mathbb{R}^{2d}$ and $t\in[0,1]$, we want to define a pseudo-differential operator $\mathrm{Op}^t(a)$ such that

$$\mathrm{Op}^t(a)f(x) := (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a((1-t)x+ty,\xi)f(y)\,dy\,d\xi, \qquad f\in\mathcal{S}(\mathbb{R}^d).$$

The operator $\mathrm{Op}^t(a)$ is the *t-quantization* of the symbol a. The analysis of t-quantizations is sometimes called the *Kohn-Nirenberg calculus* for $t=0$, the *Weyl calculus* for $t=\frac{1}{2}$.

(a) $\mathrm{Op}^0(a):\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ is well-defined and continuous, if $a\in\mathcal{S}'(\mathbb{R}^2d)$.

(b) $\mathrm{Op}^0(a):\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ is well-defined and continuous, if $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ for $\delta\le 1$.

Proof. (b) Since $\langle D_y\rangle^2$ is a self-adjoint partial differential operator, for any $n\in\mathbb{Z}_{\ge 0}$ we have

$$\begin{aligned}\mathrm{Op}^0(a)f(x) &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a(x,\xi)f(y)\,dy\,d\xi\\ (\because D_ye^{i(x-y)\xi}=\xi e^{i(x-y)\xi})\quad &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}}\langle\xi\rangle^{-2n}\langle D_y\rangle^{2n}e^{i(x-y)\xi}a(x,\xi)f(y)\,dy\,d\xi\\ (\because \text{IBP})\quad &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y)\,dy\,d\xi.\end{aligned}$$

The derivatives of the integrand is integrable with respect to ξ for a sufficiently large n with $m+|\beta|-2n<-d$ because

$$
\begin{aligned}
& |\partial_x^\beta(e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))| \\
& \qquad = |\sum_\tau \binom{\beta}{\tau}(i\xi)^{\beta-\tau}e^{i(x-y)\xi}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)\langle D_y\rangle^{2n}f(y)| \\
& \qquad \leq \sum_\tau \binom{\beta}{\tau}\langle\xi\rangle^{|\beta|-|\tau|}\langle\xi\rangle^{-2n}|\partial_x^\tau a(x,\xi)||\langle D_y\rangle^{2n}f(y)| \\
(\because a\in S^m_{\rho,\delta}) & \qquad \lesssim \sum_\tau \binom{\beta}{\tau}\langle\xi\rangle^{|\beta|-|\tau|}\langle\xi\rangle^{-2n}\langle\xi\rangle^{m+\delta|\tau|}|\langle D_y\rangle^{2n}f(y)| \\
(\because \delta\leq 1) & \qquad \lesssim \langle\xi\rangle^{m+|\beta|-2n}|\langle D_y\rangle^{2n}f(y)|,
\end{aligned}
$$

so the partial derivative ∂_x commutes with the integral. Since

$$
x^\alpha e^{i(x-y)\xi} = (y+D_\xi)^\alpha e^{i(x-y)\xi} = \sum_\sigma \binom{\alpha}{\sigma} y^{\alpha-\sigma}D_\xi^\sigma e^{i(x-y)\xi},
$$

we have an expansion

$$
\begin{aligned}
x^\alpha\partial_x^\beta \operatorname{Op}^0(a)f(x) &= x^\alpha\partial_x^\beta\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}} x^\alpha\partial_x^\beta(e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}}\sum_{\sigma,\tau}\binom{\alpha}{\sigma}\binom{\beta}{\tau}y^{\alpha-\sigma}D_\xi^\sigma e^{i(x-y)\xi}(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)\langle D_y\rangle^{2n}f(y)\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}}\sum_{\sigma,\tau}\binom{\alpha}{\sigma}\binom{\beta}{\tau}e^{i(x-y)\xi}(-D_\xi)^\sigma[(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)]y^{\alpha-\sigma}\langle D_y\rangle^{2n}f(y)\,dy\,d\xi.
\end{aligned}
$$

Here

$$
\sup_{x\in\mathbb{R}^d}|(-D_\xi)^\sigma[(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)]|
$$

is integrable with respect to ξ for sufficiently large n, so with this n we have

$$
\sup_{x\in\mathbb{R}^d}|x^\alpha\partial_x^\beta\operatorname{Op}^0(a)f(x)| \lesssim \sum_{\sigma\leq\alpha}\sup_{y\in\mathbb{R}^d}|y^{\alpha-\sigma}\langle D_y\rangle^{2n}f(y)|
$$

for each $\alpha,\beta\in\mathbb{Z}^d_{\geq 0}$ and all $f\in\mathcal{S}(\mathbb{R}^d)$, which implies $\operatorname{Op}^0(a)f\in\mathcal{S}(\mathbb{R}^d)$. □

10.4 (Change of quantization). Let $m\in\mathbb{R}$, .

(a) $\operatorname{Op}^t(a)=\operatorname{Op}^0(e^{itD_xD_\xi}a)$. In particular, since $M_{e^{itx\xi}}:\mathcal{S}(\mathbb{R}^{2d})\to\mathcal{S}(\mathbb{R}^{2d})$, $\operatorname{Op}^t(a):\mathcal{S}(\mathbb{R}^{2d})\to\mathcal{S}(\mathbb{R}^{2d})$ is well-defined and continuous.

(b) $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ if and only if $e^{itD_xD_\xi}a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$, if $0\leq\delta\leq\rho\leq 1$ and $\delta<1$.

(c) We have the formal adjoint

$$
\operatorname{Op}^t(a)^* = \operatorname{Op}^{1-t}(\overline{a}).
$$

In particular, $\operatorname{Op}^t(a):\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ is well-defined and continuous for $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$.

Proof. (a) Note that

$$\begin{aligned}
\operatorname{Op}^t(a)f(x) &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a((1-t)x+ty,\xi)f(y)\,dy\,d\xi \\
(\because \text{Inversion on } \mathbb{R}^{2d}) \quad &= (2\pi)^{-3d}\int_{\mathbb{R}^{4d}} e^{i(x-y)\xi}e^{i((1-t)x+ty)x^*+i\xi\xi^*}\widehat{a}(x^*,\xi^*)f(y)\,dx^*\,d\xi^*\,dy\,d\xi \\
&= (2\pi)^{-3d}\int_{\mathbb{R}^{4d}} e^{i(x-y+\xi^*)\xi}\widehat{a}(x^*,\xi^*)e^{i((1-t)x+ty)x^*}f(y)\,dx^*\,d\xi^*\,dy\,d\xi \\
(\because \text{Inversion on } \mathbb{R}^{d}) \quad &= -(2\pi)^{-2d}\int_{\mathbb{R}^{2d}} \widehat{a}(x^*,y-x)e^{i((1-t)x+ty)x^*}f(y)\,dx^*\,dy \\
(\because [\xi^*/y-x]) \quad &= -(2\pi)^{-2d}\int_{\mathbb{R}^{2d}} e^{i(x+t\xi^*)x^*}\widehat{a}(x^*,\xi^*)f(x+\xi^*)\,dx^*\,d\xi^*.
\end{aligned}$$

(b) We have the oscillatory integral

$$e^{itD_xD_\xi}a(x,\xi) = (2\pi)^{-d}|t|^{-d}\int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta}a(x+y,\xi+\eta)\,dy\,d\eta.$$

Enough to show

$$\left|\int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta}a(x+y,\xi+\eta)\,dy\,d\eta\right| \lesssim \langle\xi\rangle^m.$$

Fix ξ and $\delta\leq\rho$ □

10.5 (Moyal product)**.** Let $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b\in S^l_{\rho,\delta}(\mathbb{R}^{2d})$.

(a) there exists a unique function $a\#^t b\in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^t(x,D)b^t(x,D) = (a\#^t b)^t(x,D).$$

(b) It is concretely described by

$$(a\#^t b)(x,\xi) = (2\pi)^{-2}\int_{\mathbb{R}^{4d}} e^{-i(y\eta-z\zeta)}a(x+tz,\xi+\eta)b((1-t)y+x,\xi+\zeta)\,dy\,d\eta\,dz\,d\zeta.$$

(c) If $\delta<\rho$, then

$$a\#^t b(x,\xi)\sim\sum_{k\in\mathbb{Z}_{\geq 0}}\frac{1}{i^k k!}(\partial_y\partial_\eta-\partial_z\partial_\zeta)^k a((1-t)x+tz,\eta)b(tx+(1-t)y,\zeta)\Big|_{\substack{y=z=x\\ \eta=\zeta=\xi}}.$$

10.6 (Parametirx and elliptic operators)**.**

10.7 (Calderón-Vaillancourt theorem)**.**

Exercises

Quantization of linera and quadratic exponential symbols.

Chapter 11

Semiclassical analysis

We define for $h > 0$ and $t \in [0,1]$

$$\mathrm{Op}_h^t(a)f(x) := (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{\frac{i}{h}(x-y)\xi} a((1-t)x + ty, \xi) f(y)\, dy\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

$$\mathrm{Op}_h^w(D_x a) = [D_x, \mathrm{Op}_h^w(a)], \qquad \mathrm{Op}_h^w(hD_\xi a) = -[x, \mathrm{Op}_h^w(a)].$$

For example, regardless of $h > 0$ and $t \in [0,1]$,

$$\mathrm{Op}(\xi)\psi(x) = hD\psi(x) = -ih\psi'(x)$$

and

$$\mathrm{Op}(H)\psi(x) = -\frac{h^2}{2m}\Delta\psi(x) + V(x)\psi(x),$$

where

$$H(x,\xi) := \frac{|\xi|^2}{2m} + V(x).$$

In physics, the operator $\mathrm{Op}(H)$ is frequently written as $\widehat{H}$, which will not be used to avoid the confusion regarding the Fourier transform.

$$\frac{d}{dt}a(t) = \{a(t), H\} = X_H a(t)$$

$$\frac{d}{dt}\widehat{a}(t) = \frac{d}{dt}e^{\frac{i}{h}t\widehat{H}}\widehat{a}e^{-\frac{i}{h}t\widehat{H}} = -\frac{i}{h}[\widehat{a}(t), \widehat{H}]$$

Let $J : \mathbb{R}^{2d} \to \mathbb{R}^{2d} : (x,\xi) \mapsto (\xi, -x)$ be a symplectomorphism, the rotation of $\frac{\pi}{2}$ in *clock-wise*. Then, we have

$$\mathcal{F}_h^* \mathrm{Op}_h^w(a)\mathcal{F}_h = \mathrm{Op}_h^w(J^* a).$$

Also,

$$[\mathrm{Op}_h^w(a), \mathrm{Op}_h^w(b)] = \mathrm{Op}_h^w(-ih\{a,b\}) + O(h^2).$$

Since the Weyl quantization has a bound

$$\|\mathrm{Op}_h^w(a)\|_{B(L^2(\mathbb{R}^d))} \lesssim \|a\|_{C_b(\mathbb{R}^{2d})} + O(h^{\frac{1}{2}}), \qquad a \in C_b(\mathbb{R}^{2d}) \cap S^m_{\rho,\delta}(\mathbb{R}^{2d}),$$

for a bounded net $f_h \in L^2(\mathbb{R}^d)$, the positive linear functional $C_0(\mathbb{R}^{2d})$ defined by

$$a \mapsto \langle \mathrm{Op}_h^w(a) f_h, f_h\rangle, \qquad a \in C_0(\mathbb{R}^{2d}) \cap S^m_{\rho,\delta}(\mathbb{R}^{2d})$$

has a limit point in the weak* topology. If a finite Radon measure μ on $\mathbb{R}^{2d}$ is a limit, then μ is called a *semicalssical defect* of the net f_h.

Let p be a symbol such that $|p(x,\xi)| \gtrsim \langle \xi \rangle^k$ for sufficiently large $|\xi|$. This symbol has an interpretation of the Hamiltonian. Suppose the following two conditions are satisfied:

$$\lim_{h\to 0} \| \operatorname{Op}_h^w(p) f_h \|_{L^2(\mathbb{R}^d)} = 0, \qquad \|f_h\|_{L^2(\mathbb{R}^d)} = 1.$$

Then, the support of any semicalssical defect measure μ is contained in $p^{-1}(0)$, called the *characteristic variety* or the *zero energy surface* of the symbol p. We can understand this support restriction as saying that in the semiclassical limit $h \to 0$ all the mass of solution coalesces onto a specific set in phase space. Also we have the flow invariance $\{p, \mu\} = 0$, i.e. $\int_{\mathbb{R}^{2d}} \{p, a\} d\mu = 0$ for all $a \in \mathcal{D}(\mathbb{R}^{2d})$, which means that μ is invariant under the Hamiltonian flow generated by p.

11.1 Heisenberg group

11.2 Phase space transforms

Chapter 12

Microlocal analysis