Differential Equations

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Contents

Part I

Linear ordinary differential equations

Chapter 1

Constant coefficient equations

1.1 Characteristic equations

1.2 Complex roots

1.3 Repeated roots

Chapter 2

Variable coefficient equations

2.1 Series solution

2.2 Fuch's theorem

2.3 Orthogonal polynomials

2.4 Sturm-Liouville theory

2.5 The Frobenius method

Fuch's theorem

Chapter 3

Inhomogeneous equations

3.1 Method of undetermined coefficients

3.2 Variation of parameters

3.3 Damped oscillation

3.4 The Laplace transform

discontinuous data gluing

Part II

Nonlinear ordinary differential equations

Chapter 4

Nonlinear ordinary differential equations

4.1 The Picard-Lindelöf theorem

4.2 Integrating factors

Chapter 5

Dynamical systems

5.1 Equillibria

Bifurcations
Stability theory
Hamiltonian systems

5.2 Planar dynamical systems

Examples from ecology, electrical engineerings
Poincaré-Bendixon

Chapter 6

Chaos

Attractors

Part III

Linear partial differential equations

Chapter 7

Laplace's equation

7.1 Harmonic functions

7.1 (Mean value property).

7.2 (Maximum principle).

7.3 (Newtonian potential).

7.4 (Dirichlet problem for half space).

7.5 (Dirichlet problem for open ball).

7.6 (Fundamental solution of the Laplace equation). Consider a boundary problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \mathbb{R}^d_x, \\ \quad u(x) = 0 & \text{on } |x| = \infty. \end{cases}$$

A function

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

defined on $\mathbb{R}^d_x$ for $d \geq 2$ is called *fundamental solution of Laplace's equation*.

(a) Φ and $\nabla\Phi$ are locally integrable on $\mathbb{R}^d_x$ but $\Delta\Phi$ is not.

(b) $\Delta\Phi$ is a tempered distribution on $\mathbb{R}^d_x$.

(c) $-\Delta\Phi(x) = \delta(x)$ in $\mathbb{R}^d_x$.

(d) u solves the boundary problem if and only if it satisfies a representation formula $u = \Phi * f$, if $\Phi * f$ is a well-defined distribution on $\mathbb{R}^d_x$.

Proof. (c) Let $\varphi \in \mathcal{D}(\mathbb{R}^d_x)$. Then, $\nabla\Phi(x)\cdot\nabla\varphi(x) \in L^1(\mathbb{R}^d_x)$ gives

$$\begin{aligned}
-\int \Phi(x)\Delta\varphi(x)\,dx &= -\lim_{\varepsilon\to\infty}\int_{|x|\geq\varepsilon} \nabla\Phi(x)\cdot\nabla\varphi(x)\,dx \\
&= -\lim_{\varepsilon\to\infty}\int_{|x|=\varepsilon} \nabla\Phi(x)\varphi(x)\cdot\nu\,dS + \lim_{\varepsilon\to\infty}\int_{|x|\geq\varepsilon} \Delta\Phi(x)\varphi(x)\,dx.
\end{aligned}$$

Since

$$\nabla\Phi(x) = -\frac{1}{\omega_d}\frac{x}{|x|^d}, \qquad \nu = \frac{x}{|x|},$$

and $\Delta\Phi(x) = 0$ for $x \neq 0$, we get

$$-\int \Phi(x)\Delta\varphi(x)\,dx = \lim_{\varepsilon\to\infty}\frac{1}{\omega_d\varepsilon^{d-1}}\int_{|x|=\varepsilon}\varphi(x)\,dS = \varphi(x).$$

(d) Note that $\Phi = \widetilde{\Phi}$. If u is a solution of the boundary problem, then

$$\langle \Phi * f, \varphi\rangle = \langle f, \Phi * \varphi\rangle = \langle u, -\Delta(\Phi * \varphi)\rangle = \langle u, \Phi * (-\Delta\varphi)\rangle = \langle u, \varphi\rangle.$$

Conversely, if we let $u = \Phi * f$, then

$$\langle u, -\Delta\varphi\rangle = \langle \Phi * f, -\Delta\varphi\rangle = \langle f, \widetilde{\Phi} * (-\Delta\varphi)\rangle = \langle f, \Phi * (-\Delta\varphi)\rangle = \langle f, \varphi\rangle$$

and □

7.7 (Green's function)**.** Let U be a bounded open subset of $\mathbb{R}^d_x$ with C^1 boundary. Consider a boundary value problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } U, \\ \quad u(x) = g(x) & \text{on } \partial U. \end{cases}$$

A *corrector* is a function $\phi(x,y)$ on $U \times U$ defined as the solution of the boundary value problem

$$\begin{cases} -\Delta_y\phi(x,y) = 0 & \text{in } y \in U, \\ \quad \phi(x,y) = \Phi(x-y) & \text{on } y \in \partial U, \end{cases}$$

for each $x \in U$. We assume a well-known fact that the solution ϕ uniquely exists and $\phi \in H^1(U)$, proved later. Then, *Green's function* for U is a function on $U \times U$ defined by

$$G(x,y) := \Phi(x-y) - \phi(x,y).$$

(a) If $g(x) = 0$ on ∂U, then for $x \in U$,

$$u(x) = -\int_U G(x,y)\Delta u(y)\,dy.$$

(b) If $f(x) = 0$ in U, then for $x \in U$,

$$u(x) = \int_{\partial U} u(y)\nabla_y G(x,y) \cdot \nu\, dS(y).$$

(c) u solves the boundary problem if and only if it satisfies a representation formula

$$u(x) = \int_U G(x,y) f(y)\,dy + \int_{\partial U} g(y)\nabla_y G(x,y)\nu \cdot dS(y),$$

if the right-hand side is well defined distribution on $\mathbb{R}^d_x$.

Proof. □

7.2 Green's representation formula

Chapter 8

Heat equation

8.1 Heat kernel

8.2 Duhamel's principle

Chapter 9

Wave equation

9.1 First order partial differential equations

9.2 Initial value problems

d'Alambert
Kirchhoff
odd reflection

9.3 Boundary value problems

Part IV

Nonlinear partial differential equations

Chapter 10

Fluid dynamics

Burger's equation
Euler's equation
Navier-Stokes equation

Chapter 11

Integrable field equations

Korteweg-de Vries equation
Boussinesq equation
Kadomtsev-Petviashvili equation
sine-Gordon equation nonlinear Schr̆odinger equatoin

Chapter 12

Nonlinear waves and diffusion

Nonlinear wave equation
Nonlinear diffusion equation