C^*-Algebras

Ikhan Choi

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Contents

Part I

Constructions

Chapter 1

Operator systems and spaces

1.1 Completely positive maps

$|\varphi(a)|^2 \le \|\varphi\|\varphi(|a|^2) \le \|\varphi\|^2\|a\|^2$. If ω is a state, then $|\omega(a)|^2 \le \omega(|a|^2) \le \|a\|^2$.
category of operator systems

1.1 (Choi-Effros characterization).

1.2 (Stinespring dilation).

tensor product of c.p. maps (minimal and maximal)

1.3 (Arveson extension). Trick

1.2 Completely bounded maps

1.3 Subalgebras

1.4 (Hereditary C*-subalgebra). state extension, representation extension(not ideal?)

1.5 (Conditional expectation).

1.6 (Ideals).

1.7 (Enveloping C*-algeberas). Let A be a $*$-algebra. A *C*-norm* is an submultiplicative norm satisfying the C*-identity. Does A have enough $*$-representations?

(a) A complete C*-norm is unique if it exists.
(b) For every C*-norm α on A, there is a $*$-isometry $\pi : A \to B(H)$.
(c) For maximal C*-norm, there is a universal property. The maximal C*-norm can be obtained by running through cyclic representations.

1.4 Tensor products

1.8 (Maximal tensor products). Let A and B be C*-algebras.

(a) A commuting pair of $*$-homomorphisms $\pi : A \to B(H)$ and $\pi' : B \to B(H)$ corresponds to a $*$-homomorphism $\Pi : A \odot B \to B(H)$ via the relation $\Pi(a \otimes b) = \pi(a)\pi'(b)$.

(b) $A \odot B$ admits a $*$-representation and every norms induced from these $*$-representations are uniformly bounded. So, we can define a maximal tensor norm on $A \odot B$.

(c) $a \otimes - : B \to A \odot B$ is bounded for each $a \in A$ with respect to any C*-norm on $A \odot B$. [BO, 3.2.5]

1.9 (Minimal tensor product)**.** spatiality

1.10 (Takesaki theorem)**.**

Tensors with $M_n(\mathbb{C})$, $C_0(X)$.

1.11 (Haagerup tensor product)**.**

Exercises

1.12. Let B be a hereditary C*-subalgebra of a C*-algebra A. Let $a \in A_+$. If for any $\varepsilon > 0$ there is $b \in B_+$ such that $a - \varepsilon \leq b$, then $a \in B_+$.

Proof. To catch the idea, suppose A is abelian. We want to approximate a by the elements of B in norm. To do this, for each $\varepsilon > 0$, we want to construct $b' \in B_+$ such that $a - \varepsilon \leq b' \leq a + \varepsilon$ using b. Taking $b' = \min\{a, b\}$ is impossible in non-abelian case, but we can put $b' = \frac{a}{b+\varepsilon} b$. For a simpler proof, $b' = (\frac{\sqrt{ab}}{\sqrt{b}+\sqrt{\varepsilon}})^2$ is a better choice.

Define
$$b' := \frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}} a \frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}.$$
Then,
$$\|\sqrt{a} - \sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\|^2 = \|\frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}} a \frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}}\| \leq \varepsilon$$
implies
$$\lim_{\varepsilon\to 0} b' = \lim_{\varepsilon\to 0} \frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\sqrt{a} \cdot \sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}} = \sqrt{a} \cdot \sqrt{a} = a.$$
□

Chapter 2

Hilbert C*-modules

2.1

right A convention: to make it commute with the action by adjointable operators.

constructions: direct sum, tensor product, localization

examples: A itself

2.2 Multiplier algebras

2.1 (Double centralizer characterization). Let A be a C*-algebra. A *double centralizer* of A is a pair (L,R) of bounded linear maps on A such that $aL(b) = R(a)b$ for all $a, b \in A$. The *multiplier algebra* $M(A)$ of A is defined to be the set of all double centralizers of A. There is another characterization $M(A) := L_A(A)$, the set of adjointable operators to itself.

2.2 (Cohen factorization theorem).

2.3 (Strict topology). (a) $\|\pi(a - e_\alpha a)\xi\|^2$

2.4 (Essential ideals). (a) Hilbert C*-module description

2.5 (Examples of multiplier algebras). (a) $M(K(H)) \cong B(H)$.

(b) $M(C_0(\Omega)) \cong C_b(\Omega)$.

Proof. (a)

(b) First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega) \cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega) \cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective $*$-homomorphism $C_b(\Omega) \to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g \in M(C_0(\Omega))_+$. □

2.3 C*-correspondences

as a morphism sub and quotient, direct sum, tensor product,

Toeplitz-Cuntz Toeplitz-Pimsner Cuntz-Pimsner Cuntz-Krieger

C*-dynamical systems graph algebras

Coactions and Fell bundles

Induced representations and Morita equivalence KK-theory C*-algebraic quantum groups

Chapter 3

Groups and actions

3.1 Group C* algebras

type I, subhomogeneous

crystallographic discrete heisenberg free groups projectionless of $C^*_r(F_2)$

3.2 Crossed products

3.3 Pimsner algebras

graph algebras

Part II

Properties

Chapter 4

Approximation properties

4.1 Nuclearity and exactness

finite dimensional[BO, 3.3.2], abelian permanence properties completely positive approximation property

$M_n(\mathbb{C})$, $K(H)$, $C_0(X)$.

a separable C*-algebra is nuclear if and only if every factor representation is hyperfinite.

quotients of nuclear local reflexivity

Extension properties weak expectation property relatively weakly injective maximal tensor product inclusion problem

4.2 Quasi-diagonality

Voiculescu theorem

4.1. An operator $x \in B(H)$ is called *quasi-diagonal* if there is a net of projections $p_i \in B(H)$ such that $[p_i, x]$ and $p - \mathrm{id}_H$ converge strongly to zero. A C*-algebra is called *quasi-diagonal* if it admits a faithful representation whose image is quasi-diagonal.

faithful non-degenerate essential representations of a quasi-diagonal C*-algebra are all quasi-diagonal

4.3 AF-embeddability

Chapter 5

Amenability

5.1 Amenable groups

5.2 Amenable actions

crossed products Z_2-grading Connes-Feldman-Weiss Anantharaman-Delaroche Gromov boundaries approximately central structure? dynamical Kirchberg-Phillips

stably finite dynamical Elliott program

Ornstein-Weiss-Rokhlin lemma

5.3 Exact groups

Exact groups

5.4 Other properties

Kazdahn property (T) factorization property Haagerrup property

Kaplansky conjecture

Chapter 6

Simplicity

6.1 Examples from groups

6.2 Elliott program

successful in Kirchberg algebras

https://arxiv.org/pdf/2307.06480.pdf

Elliott classification problem Kirchberg-Phillipes theorem

operator K-theory and its pairing with traces

$\mathcal{Z}$-stability, Rosenberg-Schochet universal coefficient theorem

Connes-Haagerup classification of injective factors

Kirchberg: unital simple separable $\mathcal{Z}$-stable algebra is either purely infinte or stably finite. Haagerup, Blackadar, Handelman: unital simple stably finite algebra has a trace.

Glimm: uniformly hyperfinite algebras Murray-von Neumann: hyperfinite II_1 factors

Part III

Invariants

Chapter 7

Operator K-theory

7.1 Homotopy of C*-algebras

7.1 (Homotopy of $*$-homomorphisms)**.** Let A, B be C*-algebras. Two $*$-homomorphisms in $\mathrm{Mor}(A, B)$ are said to be *homotopic* if they are connected by a path in $\mathrm{Mor}(A, B)$ that is continuous with the point-norm topology.

(a) For pointed compact Hausdorff spaces $(X, x_0), (Y, y_0)$, two pointed maps $\varphi_0, \varphi_1 : X \to Y$ are homotopic if and only if $\varphi_0^*, \varphi_1^* : C_0(Y \setminus \{y_0\}) \to C_0(X \setminus \{x_0\})$ are homotopic.

Proof. (a) Suppose φ_0 and φ_1 are connected by a homotopy φ_t. Fixing $g \in C_0(Y)$ and $t_0 \in I$, we want to show

$$\lim_{t \to t_0} \sup_{x \in X} |g(\varphi_t(x)) - g(\varphi_{t_0}(x))| = 0.$$

Since the function g is uniformly continuous, with respect to an arbitrarily chosen uniformity on Y, so that there is an entourage $E \subset Y \times Y$ such that $(y, y') \in E \circ E$ implies $|g(y) - g(y')| < \varepsilon$. Using compactness we have a finite sequence $(y_i)_{i=1}^n \subset Y$ such that for every y there is y_i satisfying $(y, y') \in E$. Then, $\varphi^{-1}(E[y_i])$ is a finite open cover of $X \times I$, so we have δ such that $|t - t_0| < \delta$ implies for any $x \in X$ the existence of i satisfying $(\varphi_t(x), y_i) \in E$ and $(\varphi_{t_0}(x), y_i) \in E$, which deduces the desired inequality.

Conversely, suppose φ_0^* and φ_1^* are connected by a homotopy φ_t^*. By taking dual, we can induce $\varphi_t : X \to Y$ such that $g(\varphi_t(x)) = (\varphi_t^* g)(x)$ for each $g \in C(Y)$ from φ_t^* via the embedding $X \to M(X)$ by Dirac measures. Let V be an open neighborhood of $\varphi_{t_0}(x_0)$ and take $g \in C(Y)$ such that $g(\varphi_{t_0}(x_0)) = 1$ and $g(y) = 0$ for $y \notin V$. Now we have an open neighborhood U of x_0 such that $x \in U$ implies $|(\varphi_{t_0}^* g)(x) - (\varphi_{t_0}^* g)(x_0)| < \frac{1}{2}$. Also we have $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|\varphi_t^* g - \varphi_{t_0}^* g\| < \frac{1}{2}$. Therefore, $(x, t) \in U \times (t_0 - \delta, t_0 + \delta)$ implies $g(\varphi_t(x)) > 0$, hence $\varphi_t(x) \in V$, which means $X \times I \to Y : (x, t) \mapsto \varphi_t(x)$ is continuous. $\square$

We have $\widetilde{K}^n(X, x_0) = K_n(C_0(X \setminus \{x_0\}))$ for a pointed compact Hausdorff space X. Now then since the inclusion $\{x_0\} \to X$ induces the section so that

$$0 \to K_0(C_0(X \setminus \{x_0\})) \to K_0(C(X)) \to K_0(\{x_0\}) \to 0$$

splits, we have

$$K^0(X) = \widetilde{K}^0(X, x_0) \oplus \mathbb{Z} = K_0(C_0(X \setminus \{x_0\})) \oplus K_0(\{x_0\}) = K_0(C(X))$$

for a compact connected Hausdorff space X. The additivity of K_0 and K^0 removes the connectedness condition.

$$K_0(\mathbb{C}) = \mathbb{Z}, \quad K_0(C_0(\mathbb{R})) = 0, \quad K_1(C_0(\mathbb{R})) = K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$$
$$K^0(*) = \mathbb{Z}, \quad K^0(S^1) = \mathbb{Z}, \quad K^1(S^1) = K^0(S^2) = \mathbb{Z}[x]/(x-1)^2$$

7.2 Brown-Douglas-Fillmore theory

7.2 (Haagerup property).

Baum-Connes conjecture Non-commutative geometry Elliott theorem

7.3 Approximately finite algebras

Elliott conjecture: amenable simple separable C*-algerbas are classified by K-theory. Brattelli diagram

7.4 Fredholm theory of Mishchenko and Fomenko

7.5 Dixmier-Douady theory

7.3 (Banach bundle). A *Banach bundle*, introduced by Fell, is a continuous open surjection $\pi : E \to X$ between topological spaces together with Banach space structure on each fiber $\pi^{-1}(x)$ such that:

(i) the addition $\{(e, e') : \pi(e) = \pi(e')\} \subset E \times E \to E : (e, e') \mapsto e + e'$ is continuous,

(ii) the scalar multiplication $\mathbb{C} \times E \to E : (\lambda, e) \mapsto \lambda e$ is continuous,

(iii) the norm $E \to \mathbb{R}_{\geq 0} : e \mapsto \|e\|$ is continuous,

(iv) the family of subsets $\{e \in B : \pi(e) \in U,\ \|e\| < r\}$ parametrized by open neighborhood $U \subset X$ of x and positive $r \in \mathbb{R}$ forms a neighborhood basis of $0 \in \pi^{-1}(x)$ in E.

The forth condition is equivalent to that if $\|e_i\| \to 0$ and $\pi(e_i) \to x$ then $e_i \to 0_x \in \pi^{-1}(x)$. If the norm satisfies the parallelogram law, then we call the Banach bundle a *Hilbert bundle*.

(a) For a Banach bundle $E \to X$, if X is locally compact Hausdorff and every fiber E_x shares a same finite dimension, then the bundle is locally trivial.

7.4 (Continuous fields of Banach spaces).

7.5 (Banach $C_0(X)$-module). Let $E \to X$ be a Banach bundle.

Fell's condition

A C*-algebra A is called *continuous trace* if the set of all $a \in \mathcal{A}$ such that $\widehat{A} \to \mathbb{R}_{\geq 0} : \pi \mapsto \operatorname{tr}(\pi(a^*a))$ is continuous is dense in A.