Representation Theory

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Part I

Finite group representations

Chapter 1

Character theory

1.1 Irreducible representations

1.1 (Definition of group representations).

1.2 (Intertwining maps).

1.3 (Subrepresentations). We say *invariant* or *stable*

1.4 (Irreducible representations). indecomposable and irreducible

1.5 (Maschke's theorem). Let G be a finite group and k a field of characteristic coprime to $|G|$. Let (ρ, V) be a finite-dimensional representation of G over k. Let W be an invariant subspace of V.

(a) There is an invariant subspace $W^\perp$ of V that is a complement of W.

(b) Every finite-dimensional representation of G over k is isomorphic to the direct sum of irreducible representations of G over k.

(c) If $k = \mathbb{R}$ or $\mathbb{C}$, then there is a inner product on V such that $W^\perp$ is orthogonal to W. With this innerproduct, $\rho(g)$ is orthogonal (resp. unitary) for all $g \in G$.

1.6 (Schur's lemma). Let G be a finite group and k a field. Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible representations of G over k. Let $\psi \in \hom_G(V_1, V_2)$ be an intertwining map.

(a) If V_1 and V_2 are not isomorphic, then $\psi = 0$.

(b) If V_1 and V_2 are isomorphic, then ψ is a homothety.

1.2 Group algebra

1.7 (Modules and representations). ring <-> group module <-> representation finitely generated <-> finite dimensional

1.8 (Wedderburn's theorem). central idempotents dimension computation

1.9 (Group algebra). regular representation $k[G]$-module and G-representation correspondence

(a) $\mathbb{C}[G]$ is the direct sum of all irreducible representations.

(b) $|G| = \sum_{[V] \in \widehat{G}} (\dim V)^2$.

1.10. The number of irreducible representations and the number of conjugacy classes double counting on $Z(\mathbb{C}[G])$.

1.3 Characters

1.11 (Space of class functions)**.** Ring and inner product structure on the space of class functions.

(a) $\dim \hom_G(V_1, V_2) = \langle \chi_{V_1}, \chi_{V_2} \rangle$.

(b) Irreducible characters form an orthonormal basis of the space of class functions.

1.12 (Characters classify representations)**.** Let G be a finite group and let $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of G over $\mathbb{C}$.

$$\mathrm{Tr} : \mathbf{Rep}(G) \to \{\text{finite sum of irreducible characters}\}$$

surjectivity: trivial injectivity: Suppose two characters are equal. Maschke -> all characters are sum of irreducible characters Schur -> orthogonality, so the coefficients are all equal irreducible-factor-wisely construct an isomoprhism.

1.13 (Character table)**.** computation of matrix elements by character table abelian group, 1dim rep lifting

S^3	e	(12)	(123)
1	1	1	1
ε	1	-1	1
ρ	2	0	-1

the dual inner product: conjugacy check relation to normal subgroups center of rep
algebraic integer dim of irrep divides group order burnside pq theorem

Chapter 2

Classification of representations

2.1 Symmetric groups

young tableux

2.2 Linear groups over finite fields

GL2 and SL2 over finite fields

2.3 Induced representations

induction and restriction of reps (from and to subgroup) frobenius reciprocity, mackey theory

tensoring, complex, real symmetric, exterior

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Representations of Lie algebras

6.1 Representations of $\mathfrak{sl}(2,\mathbb{C})$

6.1 (Pauli matrices)**.** Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) $\{\sigma_1, \sigma_2, \sigma_3\}$ is a basis of complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, and $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of real Lie algebra $\mathfrak{so}(3)$.

(b) For a unit vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$, $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ has eigenvalues ± 1.

6.2 Universal enveloping algebra

PBW theorem, verma module

6.3 Highest weight theorem

Exercises

6.2 (Triplets and quadraplets)**.** Let (π_2, V_2) be the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of degree two. Consider $V_2 \otimes V_2$. Cartan element S_z. $V_2^{\otimes 3}$.

6.3 (Casimir element)**.** Casimir element decomposes a representation into irreducible representations.

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Lie's three theorems

7.2 Fundamental groups of Lie groups

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8.2 Special unitary groups

8.3 Symplectic groups

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8.1 (Lorentz group). $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$

(a) $O(1,3)$ has four components and $SO^+(1,3)$ is the identity component. Orthochronous $O^+(1,3)$, proper $SO(1,3)$.

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