Three-dimensional Topology

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University of Tokyo, Spring 2023

April 18, 2023

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Plan:

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References:

1. J. Hempel, 3-manifolds

2. W. Jaco, Lectures on three-manifold topology

3. B. Martelli, An introduction to geometric topology

4. Morimoto, An introduction to three-dimensional manifolds (Japanses)

Grading: Submit a report for any three among the exercises given in the lecture (ITC-LMS Kadai). Cancellation of class: 5/2, 7/11(maybe)
Convention:

- manifold = connected compact orientable smooth manifold

- surface = connected compact orientable smooth 2-dimensional manifold

- tub nbd, isotopy extension, transversality, triangulation,$\cdots$

1. Fundamental group

1.1. Fundamental groups of higher dimensional manifolds

Proposition 1.1. *Let π be a finitely presented group. Then, for every $d \geq 4$ there is a d-manifold X such that $\pi_1(X) \cong \pi$.*

Proof. Let $\pi = \langle x_1, \cdots, x_n \mid r_1, \cdots r_m \rangle$ be given. If $Y = (S^1 \times S^{d-1})^{\#n}$, then $\pi_1(Y) \cong \langle x_1, \cdots x_n \rangle$ by the van Kampen theorem. Let

$$Z = Y \setminus (\coprod_{i=1}^{m} \nu(l_i)),$$

where $l_i \subset Y$ is embedded loops representing r_i and ν denotes the open tubular neighborhood. Then, $\partial Z = \coprod_{i=1}^{m} l_i \times S^{d-2}$. Since $d \geq 4$, any loops and disks can be pushed off $l_1 \cdots, l_n$, we have an isomorphism $\pi_1(Z) \to \pi_1(Y)$. Then, if we let

$$X = Z \cup_\partial (\coprod_{i=1}^{m} D^2 \times S^{d-2}),$$

then $l_i \times * = \partial(D^2 \times *)$, we have $\pi_1(X) \cong \pi_1(Y)/\langle [l_1], \cdots, [l_m] \rangle \cong \pi$. $\qquad \square$

1.2. Surfaces and their groups

Theorem 1.2 (Radó, Whitehead). *Every topological surface admits a unique smooth and PL structure.*

Theorem 1.3. *Every surface is diffeomorphic to only one of $\Sigma_{g,b}$, where $\Sigma_{g,b} = (T^2)^{\#g} \# (D^2)^{\#b}$.*

Corollary 1.4. $S = S^2, T^2, D^2$ *are prime, that is, $S = S_1 \# S_2$ implies $S_i \approx S^2$ for $i = 1$ or $i = 2$.*

Remark 1.5. If the orientability is reduced out, then $\mathbb{RP}^2$ is prime. Also note that $T^2 \# \mathbb{RP}^2 \approx (\mathbb{RP}^2)^{\#3}$.

Theorem 1.6 (Uniformization). *For every surface $S \neq D^2$, its interior admits a complete Riemannian metric of constant curvature*

$$\begin{cases} 1, & \chi(S) > 0 \\ 0, & \chi(S) = 0 \\ -1, & \chi(S) < 0 \end{cases}$$

with universal covering S^2, $\mathbb{R}^2$, $\mathbb{H}^2$, respectively.

The hyperbolic plane is $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, and $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$.

Proposition 1.7. *If a surface S has $\chi(S) < 0$, then there is a discrete group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ such that $S \approx \mathbb{H}^2/\Gamma$. In particular, $\pi_1(S)$ is isomorphic to Γ.*

We have

$$\pi_1(\Sigma_{g,b}) \cong F_{2g+b-1} \quad \text{and} \quad \pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1], \cdots, [a_g, b_g] \rangle.$$

Proposition 1.8. $\pi_1(\Sigma_g)$ *is torsion free.*

Exercise 1. Prove Proposition 1.7.

Theorem 1.9 (Newman). $\pi_1(\Sigma_{g,b})$ *is linear over $\mathbb{Z}$, that is, is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$. For example, we can check $F_n \hookrightarrow F_2 \hookrightarrow \mathrm{SL}_2(\mathbb{Z})$ according to the pingpong lemma.*

Over $\mathbb{R}$, we may embed $\pi_1(S) \hookrightarrow \mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{SO}_{1,2}^+(\mathbb{R}) < \mathrm{GL}_3(\mathbb{R})$ if $\chi(S) < 0$,

Definition. A group π is called residually finite(RF) if for every $1 \neq \gamma \in \pi$ there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \neq 1$. A subgroup $\pi' < \pi$ is called separable if there is a group homomorphism $\varphi : \pi \to G$ to a finite group G such that $\varphi(\gamma) \notin \varphi(\pi')$. In particular, π is residually finite if the trivial subgroup is separable in π. A group π is called locally extended residually finite(LERF) if every finitely generated subgroup of π is separable.

Theorem 1.10 (Mal'cev). *Every finitely generated linear group over a field is residually finite.*

Over $\mathbb{Z}$, if $1 \neq (a_{ij}) \in \pi < \mathrm{GL}_n(\mathbb{Z})$ given, then for $m > \max_{i,j} |a_{ij}|$ if we let $\varphi_m : \pi \hookrightarrow \mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, then $\varphi_m((a_{ij})) \neq 1$.

Theorem 1.11 (Scott). $\pi_1(\Sigma_{g,b})$ *is LERF.*

2 Day 2: April 18

2. Examples and constructions of 3-manifolds

Theorem 2.1 (Moise). *Every topological 3-manifold(not neccesarily compact, connected, orientable) admits a unique smooth and PL structure.*

2.1 Spherical manifolds

Recall

$$S^3 := \{x \in \mathbb{R}^4 : |x| = 1\}$$
$$= \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$$
$$= \{a + bi + cj + dk \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\}.$$

Lens spaces

Let $p, q \in \mathbb{Z}$, $p > 0$, $(p, q) = 1$. Then, $\mathbb{Z}/p\mathbb{Z} = \langle \zeta = \exp(2\pi\sqrt{-1}/p) \rangle$ acts on S^3 such that $\zeta \cdot (z, w) = (\zeta z, \zeta^q w)$. Then, the Lens spaces are defined as

$$L(p, q) := S^3/(\mathbb{Z}/p\mathbb{Z}) \quad \text{with} \quad \pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}.$$

For example, $L(1, 1) = S^3$ and $L(2, 1) = \mathbb{RP}^3$.

Theorem 2.2 (Reidemeister)**.**

(a) $L(p, q) \simeq L(p, q')$ *(homotopy equiv) if and only if there is $a \in \mathbb{Z}$ such that $qq' \equiv \pm a^2$ (mod p).*

(b) $L(p, q) \approx L(p, q')$ *(diffeo) if and only if $q' \equiv \pm q^{\pm 1}$ (mod p).*

For example, $L(7, 1) \simeq L(7, 2)$ since $1 \cdot 2 \equiv 3^2$ (mod 7), but $L(7, 1) \approx L(7, 2)$ since $2 \not\equiv \pm 1$ (mod 7).

Sketch of ($\Leftarrow$). Direct construction. (a) With the linking form $H_1(L) \times H_1(L) \to \mathbb{Q}/\mathbb{Z}$. (b) Reidemeister torsion. $\qquad\square$

General quotients

A spherical manifold is the orbit space S^3/Γ, where Γ is a finite subgroup of $SO(4)$ and $\Gamma \curvearrowright S^3$ freely.

Example. With an action $\langle -1, i, j, k \rangle \curvearrowright S^3$, we obtain the prism manifold.

Example. With an action of the binary icosahedral group $\Gamma = \mathbb{Z}/2\mathbb{Z} \rtimes A_5$ on S^3, we obtain the Poincaré sphere. We have $H_*(S^3/\Gamma) \cong H_*(S^3)$. If we take $3/10$ turn instead of $1/10$, we have the Seifert-Weber space.

2.2 Fibered manifolds

Twisted bundles

$$N_{g,b} = (\mathbb{RP}^2)^{\# g} \# (D^2)^{\# b}.$$

Let D be a polygon with oriented sides $a_1, a_1', a_2, a_2', \cdots, a_g, a_g'$.

$$N_g \widetilde{\times} [0, 1] := D \times [0, 1]/\sim,$$

where $(x, t) \sim (x', 1 - t)$ for $x \in a_i$, $x' \in a_i'$, $t \in [0, 1]$ with $[x] = [x'] \in N_g$, and it is orientable.

$$N_g \widetilde{\times} S^1 := N_g \widetilde{\times} [0, 1]/(x, 0) \sim (x, 1), \ x \in N_g$$

$$N_{g,b} \widetilde{\times} S^1 := N_g \widetilde{\times} S^1 \setminus \nu(b \text{ fibers}).$$

Exercise 2. Show the following:

(a) $\mathbb{RP}^2 \widetilde{[0, 1]} \approx \mathbb{RP}^3 \setminus (\text{open ball})$.

(b) $\mathbb{RP}^2 \widetilde{\times} S^1 \approx \mathbb{RP}^3 \# \mathbb{RP}^3$.

(c) $N_{1,1} \widetilde{\times} S^1 \approx N_2 \widetilde{\times} [0, 1]$.

Mapping tori

The mapping class group is

$$\mathcal{M}_{g,b} := \mathrm{Diff}^+(\Sigma_{g,b}, \partial\Sigma_{g,b})/\text{isotopy relative to } \partial.$$

Theorem 2.3 (Dehn, Lickorish)**.** $\mathcal{M}_{g,b}$ *is finitely generated by Dehn twists.*

For examples, $\mathcal{M}_0 = \mathcal{M}_{0,1} = 1$ by the Alexander trick, and $\mathcal{M}_{0,2} \cong \mathbb{Z}$, $\mathcal{M}_1 = \mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})$. Let $\varphi \in \mathcal{M}_{g,b}$. Then, a mapping torus is defined by

$$M_\varphi := \Sigma_{g,b} \times [0,1]/(\varphi(x), 0) \sim (x, 1).$$

2.3 Heegaard decomposition

A manifold with a boundary

$$H_g := D^3 \cup (D^2 \times [0,1])^{\sqcup g}$$

is called the handle body with genus g. Then, $\pi_1(H_g) \cong F_g$ and $\partial H_g \approx \Sigma_g$. Let $\varphi : \partial H_g \to \partial H_g$ be an orientation-preserving diffeomorphism, i.e. an element of the mapping class group $\mathcal{M}_g$. If a 3-manifold M satisfies

$$M \approx H_g \approx H_g,$$

then the right-hand side is called the Heegaard decomposition(splitting) of M.

Proposition 2.4. *Every closed 3-manifold admitting a Heegaard decomposition of genus 0 is diffeomorphic to S^3.*

Proof. $\mathcal{M}_0 = 1$. $\qquad\square$

Proposition 2.5. *Every closed 3-manifold admitting a Heegaard decomposition of genus 1 is diffeomorphic to S^3, $S^2\times$, or $L(p,q)$.*

Exercise 3. Prove the above proposition.

Theorem 2.6. *Every closed 3-manifold M admits a Heegaard decomposition along some Σ_g.*

Proof. Pick a triangulation T of M. Then, $H := \overline{\nu(T^{(1)})}$ is a handlebody. Then, $H' := M \setminus \mathrm{Int}H$ is also a handlebody. Since M is orientable, so are H and H', thus we are done. $\qquad\square$

There is another proof using Morse theory.

Corollary 2.7. *The fundamental group of every closed 3-manifold admits a finite presentation of deficiency 0, i.e. the number of generators is equal to the number of relations.*

Proof. Apply the van Kampen theorem to

$$M = H_g \cup H_g = H_g \cup (D^2 \times [-\varepsilon, \varepsilon])^{\sqcup g} \cup D^3. \qquad\square$$

2.4 Dehn surgery

Let L be a link. The link exterior is the set $E_L = S^3 \setminus \nu(L)$.

Proposition 2.8. *Let M be a 3-manifold, and $T \subset \partial M$ a torus component. Let $h : \partial(D^2 \times S^1) \to T$ be a diffeomorphism. Then, $M \cup_h (D^2 \times S^1)$ is determined only by $\pm[h(\partial D^2 \times *)] \in H_1(T)$.*

Proof. Write

$$M \cup_h (D^2 \times S^1) = M \cup_h (D^2 \times (-\varepsilon, \varepsilon)) \cup D^3. \qquad\square$$

For a knot K, there are two generators μ and λ, called the meridian and the longitude, of $H_1(\partial E_K)$ such that $\ker(H_1(\partial E_K) \to H_1(E_K) = \mathbb{Z}\mu)$ is generated by λ.

Exercise 4. Show that $L(p,q)$ with $p \neq 0$ and $S^2 \times S^1$ are obtained by the p/q-Dehn surgery along the unknot.

Theorem 2.9 (Lickorish-Wallace). *Every closed 3-manifold can be obtained by an (integral)-Dehn surgery along some link in S^3.*

Sketch. Heegaard decomposition and $\mathcal{M}_g = \langle \text{Dehn twists} \rangle$. Each Dehn twist realizes the Dehn surgery steps. $\qquad\square$