POSITIVE HAHN-BANACH SEPARATION THEOREMS IN OPERATOR ALGEBRAS

IKHAN CHOI

ABSTRACT.

1. INTRODUCTION

In this paper, we prove the following theorem.

Theorem. *Let M be a von Neumann algebra, and let A be a C*-algebra.*

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then for any $x \in M^+ \setminus F$ there exists $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega(x') \le 1$ for all $x' \in F$.*
(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then for any $\omega \in M_*^+ \setminus F_*$ there exists $x \in M^+$ such that $\omega(x) > 1$ and $\omega'(x) \le 1$ for all $\omega' \in F_*$.*
(3) *If F is a norm closed convex hereditary subset of A^+, then for any $a \in A^+ \setminus F$ there exists $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for all $a' \in F$.*
(4) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then for any $\omega \in A^{*+} \setminus F^*$ there exists $a \in A^+$ such that $\omega(a) > 1$ and $\omega'(a) \le 1$ for all $\omega' \in F^*$.*

Here the result (1) was proved by Haagerup in 1975 and it plays a major role in the proof of that σ-weakly lower semi-continuous weight of a von Neumann algebra is given by the pointwise supremum of a set of positive normal linear functionals. However, we give a different proof to motivate the idea of the proof of (4). Haagerup heavily used the σ-strong topology and the strong continuity of continuous bounded functions to prove (1), but such a nice dual topology of the σ-weak topology for von Neumann algebras has no analogy in the dual of C*-algebras. In this background, we give a proof of (1) only using the σ-weak topology, and extend it to prove (4) with the weak* topology.

Contents to write...

- definition and properties of $f_\delta(t) := (1 + \delta t)^{-1} t$
- commutant Radon-Nikodym, completely positive map $\theta : \pi_\psi(M)' \to M_*$ for $\psi \in A^{*+}$ or $\psi \in M_*^+$
- Jordan decomposition and absolute value of linear functionals $[\omega]$
- Mazur lemma

Definition 1.1 (Hereditary subsets)**.** Let E be a partially ordered real vector space. We say a subset F of the positive cone E^+ is *hereditary* if $0 \le x \le y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. When E has a locally convex topology, we define the *positive polar* of a subset F of E as the positive part of the real polar

$$F^{r+} := F^r \cap E^{*+} = \{x^* \in E^{*+} : \sup_{x \in F} x^*(x) \leq 1\}.$$

The following lemma is for the approximation of the functional calculus f_δ in the σ-weak or the weak* topology. Each part will be used in the proof of (1) and (4) respectively.

Lemma 1.2. *Let $\varepsilon, \delta, r > 0$. Suppose $t \in \mathbb{R}$ satisfies $1 + \delta t > 0$.*

(1) *If $|t| \leq r$ and $\delta \leq (\varepsilon/4r^2)^2 \leq (2r)^{\frac{3}{2}}$, then $t \leq f_\delta(t) + (\varepsilon/2)\delta^{\frac{1}{2}}$.*
(2) *If $|t| \leq \delta^{-\frac{1}{6}}$ and $\delta \leq (\varepsilon/8)^6 \leq 2^{-\frac{6}{5}}$, then $t \leq f_\delta(t) + (\varepsilon/4)\delta^{\frac{1}{2}}$.*

Proof. Observe that our inequalities are equivalent, since $1 + \delta t > 0$, to

$$\delta^{\frac{1}{2}}(-t)^2 + \delta(\varepsilon/(2 \text{ or } 4))(-t) - (\varepsilon/(2 \text{ or } 4)) \leq 0.$$

Putting the maximum value of $-t$, the condition for δ can be computed as

$$(\varepsilon/2r)\delta + \delta^{\frac{1}{2}} \leq (\varepsilon/2r^2), \qquad (\varepsilon/4)\delta^{\frac{5}{6}} + \delta^{\frac{1}{6}} \leq (\varepsilon/4),$$

for each case respectively, then we can see $\delta \leq (\varepsilon/4r^2)^2 \leq (2r)^{\frac{3}{2}}$ and $\delta \leq (\varepsilon/8)^6 \leq 2^{-\frac{6}{5}}$ give sufficient conditions. □

2. Positive Hahn-Banach separation theorems

Theorem 2.1. *Let M be a von Neumann algebra, and consider the dual pair (M^{sa}, M_*^{sa}). If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$. In particular, if $x \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega(x') \leq 1$ for $x' \in F$.*

Proof. Since the positive polar is represented as the real polar

$$F^{r+} = F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r,$$

the positive bipolar can be written as $F^{r+r+} = (F - M^+)^{rr+} = (\overline{F - M^+})^+$ by the usual real bipolar theorem, where the closure is for the σ-weak topology. Because $F = (F - M^+)^+ \subset (\overline{F - M^+})^+$, it suffices to prove the opposite inclusion $(\overline{F - M^+})^+ \subset F$.

In the proof of (1), we will always consider the interval $\{m^{-1} : m \in \mathbb{Z}_{>0}\}$ for the domain of δ. Define

$$G := \left\{ x \in M^{sa} : \begin{array}{c} \text{for any } \varepsilon > 0, \text{ there is a sequence } y_\delta \in F \\ \text{indexed on } \delta \leq (1 + \|x\|)^{-1} \text{ such that} \\ \|y_\delta\| \leq \delta^{-1} \text{ and } f_\delta(x) \leq y_\delta + \varepsilon\delta^{\frac{1}{2}} \end{array} \right\}.$$

Note that for $x \in G$ the functional calculus $f_\delta(x)$ is well-defined because $\|x\| < \delta^{-1}$. If $x \in G^+$, then because the sum of closed set and a compact set is closed, we have a decreasing sequence of σ-weakly closed convex hereditary subsets $F_\delta := F + \{x' \in M^+ : x' \leq \delta^{\frac{1}{2}}\}$ of M^+ that satisfies $f_{\delta'}(x) \in F_\delta$ for each $\delta' \leq \delta$, where we let $\varepsilon = 1$. It implies that the σ-weak limit x of $f_\delta(x)$ as $\delta \to \infty$ is contained in the intersection $\bigcap_\delta F_\delta$, so if we write $x = y'_\delta + x'_\delta$ for $y'_\delta \in F$ and $0 \leq x'_\delta \leq \delta^{\frac{1}{2}}$, then since $x'_\delta \to 0$ in norm of M, we have $y'_\delta \to x$ in norm of M, and $x \in F$. This means that $G^+ \subset F$, so it suffices to show $G = \overline{F - M^+}$ to prove $(\overline{F - M^+})^+ \subset F$.

First we can check $F-M^+\subset G$ since if $x\in F-M^+$ with $y\in F$ such that $x\le y$, then $y_\delta:=f_\delta(y)$ satisfies the conditions in the definition of G independently of the value of $\varepsilon>0$, and we also have $G\subset\overline{F-M^+}$ because $f_\delta(x)-\delta^{\frac12}\in F-M^+$ in the definition of G for $\varepsilon=1$ converges to x σ-weakly as $\delta\to0$. It means that we are enough to show that G is σ-weakly closed.

Let $x_i\in G$ be a net such that $x_i\to x$ σ-weakly. By the Krein-Šmulian theorem, we may assume that $\|x_i\|\le r$ for some $r>0$. Assume $\varepsilon\le(2r)^{\frac32}$ and let $\delta_0:=\min\{(\varepsilon/4r^2)^2,(1+r)^{-1}\}$. For $\delta\in(0,\delta_0]$, then since $\delta\le\inf_i(1+\|x_i\|)^{-1}$, we can take sequences $y_{i,\delta}\in F$ following the definition of G such that $f_\delta(x_i)\le y_{i,\delta}+(\varepsilon/2)\delta^{\frac12}$ for all δ and i. Define y_δ by the limit of a σ-weakly convergent subnet of $y_{i,\delta}$. Note that the choice of a subnet depends on δ, but it is not an imporant issue. We clearly have $\|y_\delta\|\le\delta^{-1}$. Since $\|x_i\|\le r$ and $\delta\le(\varepsilon/4r^2)^2\le(2r)^{\frac32}$, we have

$$x_i\le f_\delta(x_i)+(\varepsilon/2)\delta^{\frac12}\le y_{i,\delta}+\varepsilon\delta^{\frac12},$$

so the weak* limit for the subnet gives $f_\delta(x)\le x\le y_\delta+\varepsilon\delta^{\frac12}$. For $\delta\in(\delta_0,(1+\|x\|)^{-1}]$, since $x\le y_{\delta_0}+\varepsilon\delta_0^{\frac12}$, if we define $y_\delta:=f_{\delta-\delta_0}(y_{\delta_0})\in F$, then $\|y_\delta\|\le\delta^{-1}$ and

$$f_\delta(x)\le f_\delta(y_{\delta_0}+\varepsilon\delta_0^{\frac12})\le f_{\delta-\delta_0}(y_{\delta_0})+\varepsilon\delta^{\frac12}=y_\delta+\varepsilon\delta^{\frac12}.$$

Therefore, $x\in G$. □

Theorem 2.2. *Let M be a von Neumann algebra, and consider the dual pair (M_*^{sa},M^{sa}). If F_* is a norm closed convex hereditary subset of M_*^+, then $F_*=F_*^{r+r+}$. In particular, if $\omega\in M_*^+\setminus F_*$, then there is $x\in M^+$ such that $\omega(x)>1$ and $\omega'(x)\le1$ for $\omega'\in F_*$.*

Proof. It is enough to prove $(\overline{F_*-M_*^+})^+\subset F_*$, where the closure is for the weak topology or equivalently in norm by the convexity of $F_*-M_*^+$, so we begin our proof by fixing $\omega\in(\overline{F_*-M_*^+})^+$. Let $\omega_n\in F_*-M_*^+$ be a sequence such that $\omega_n\to\omega$ in norm of M_*, and take $\varphi_n\in F_*$ such that $\omega_n\le\varphi_n$ for all n. By modifying ω_n into $\omega-(\omega-\omega_n)_+=\omega_n-(\omega_n-\omega)_+\in F_*-M_*^+$ and taking a rapidly convergent subsequence, we may assume $\omega_n\le\omega$ and $\|\omega-\omega_n\|\le2^{-n}$ for all n because $\|(\omega_n-\omega)_+\|\le\|\omega_n-\omega\|\to0$. Consider the Gelfand-Naimark-Segal representation $\pi:M\to B(H)$ associated to a positive normal linear functional

$$\psi:=\sum_n(\omega-\omega_n)+\omega+\sum_n2^{-n}\frac{\varphi_n}{1+\|\varphi_n\|}$$

on M and the commutant Radon-Nikodym derivatives h, h_n, and k_n in $\pi(M)'$ with respect to ψ, defined such that

$$\omega=\theta(h),\qquad\omega_n=\theta(h_n),\qquad\varphi_n=\theta(k_n),$$

where $\theta:\pi(M)'\to M_*$ is the commutant Radon-Nikodym map for ψ. Since $-1\le h_n\le h$ is bounded, the weak convergence $\omega_n\to\omega$ implies $h_n\to h$ in the weak operator topology of $\pi(M)'$. By the Mazur lemma, we can take a net h_i in the convex hull of h_n such that $h_i\to h$ strongly in $\pi(M)'$, and the corresponding k_i can be defined such that $\omega_i:=\theta(h_i)$ and $\varphi_i:=\theta(k_i)$ satisfy $\omega_i\le\varphi_i$ with $\varphi_i\in F_*$ by the convexity of F_*. In fact, the net h_i can be taken to be a sequence because $\pi(M)'$ is σ-finite by the existence of the separating vector, but it is not necessary in here. For each i and $0<\delta<1$, define

$$\omega_\delta := \theta(f_\delta(h)), \qquad \omega_{i,\delta} := \theta(f_\delta(h_i)), \qquad \varphi_{i,\delta} := \theta(f_\delta(k_i)),$$

where the functional calculus $f_\delta(h_i)$ is well-defined because $-1 \le h_i$ for all i. Define k_δ as the σ-weak limit of a σ-weakly convergent subnet of $f_\delta(k_i)$, and let $\varphi_\delta := \theta(k_\delta)$. Note that the choice of a subnet depends on δ, but it is not an imporant issue as in the proof of Theorem 2.1. Since $f_\delta(h_i) \to f_\delta(h)$ strongly in $\pi(M)'$ by the strong continuity of f_δ, and since we may assume $f_\delta(k_i) \to k_\delta$ σ-weakly, we have $\omega_{i,\delta} \to \omega_\delta$ and $\varphi_{i,\delta} \to \varphi_\delta$ weakly in M_* for each δ. Then, $0 \le \varphi_{i,\delta} \le \varphi_i$ implies $\varphi_{i,\delta} \in F_*$, and the weak convergence $\varphi_{i,\delta} \to \varphi_\delta$ in M_* implies $\varphi_\delta \in F_*$. On the other hand, $\omega_i \le \varphi_i$ implies $\omega_{i,\delta} \le \varphi_{i,\delta}$ by the operator monotonicity f_δ, and it implies $0 \le \omega_\delta \le \varphi_\delta$ by taking the weak limit on i, so $\omega_\delta \in F_*$. This is a fact that hold independently of the choice of subnet, so the weak convergence $\omega_\delta \to \omega$ in M_* as $\delta \to 0$ implies $\omega \in F_*$, and we can finally get $(\overline{F_* - M_*^+})^+ \subset F_*$. □

Theorem 2.3. *Let A be a C^*-algebra, and consider the dual pair (A^{sa}, A^{*sa}). If F is a norm closed convex hereditary subset of A^+, then $F = F^{r+r+}$. In particular, if $a \in A^+ \setminus F$, then there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for $a' \in F$.*

Proof. We directly prove the separation without invoking the arguments of positive bipolars. Denote by F^{**} the σ-weak closure of F in the universal von Neumann algebra A^{**}. We first show that F^{**} is hereditary subset of A^{**+}. Suppose $0 \le x \le y$ in A^{**} and $y \in F^{**}$. Then, there is $z \in A^{**}$ such that $x^{\frac12} = zy^{\frac12}$. Take bounded nets b_i in F and c_i in A such that $b_i \to y$ and $c_i \to z$ σ-strongly* in A^{**} using the Kaplansky density theorem. We may assume the indices of these two nets are shared by considering the product directed set. Since both the multiplication and the involution of a von Neumann algebra on bounded parts are continuous in the σ-strong* topology, and since the square root on a positive bounded interval is strongly continuous, we have the σ-strong* limit

$$x = y^{\frac12} z^* z y^{\frac12} = \lim_i b_i^{\frac12} c_i^* c_i b_i^{\frac12},$$

so we obtain $x \in F^{**}$ from $b_i^{\frac12} c_i^* c_i b_i^{\frac12} \in F$. Thus, F^{**} is hereditary in A^{**+}.

Let $a \in A^+ \setminus F$. If $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, which means that $a_i \to a$ weakly in A, and by the weak closedness of F in A we get a contradiction $a \in F^{**} \cap A = F$. It implies $a \in A^{**+} \setminus F^{**}$, so by Theorem 2.1, there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for all $a' \in F \subset F^{**}$, and we are done. □

Theorem 2.4. *Let A be a C^*-algebra, and consider the dual pair (A^{*sa}, A^{sa}). If F^* is a weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$. In particular, if $\omega \in A^{*+} \setminus F^*$, then there is $a \in A^+$ such that $\omega(a) > 1$ and $\omega'(a) \le 1$ for $\omega' \in F^*$.*

Proof. As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, where the bar will always mean the weak* closure throughout the whole proof. From now on, we only consider $\delta \in \{2^{-m} : m \in \mathbb{Z}_{>0}\}$. Let

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{for any } \varepsilon > 0, \text{ there are sequences } \psi_\delta \in A^{*+} \text{ and } \varphi_\delta \in F^* \\ \text{indexed on } \delta \le (1+4\|\omega\|)^{-6} \text{ such that} \\ \text{the following five conditions are satisfied:} \\ |\omega(a)| \le \delta^{-\frac{1}{6}} \psi_\delta(a) \text{ for all } a \in A^+,\ \|\psi_\delta\| \le 1,\ \|\varphi_\delta\| \le \delta^{-1}, \\ \omega_\delta \le \varphi_\delta + \varepsilon \delta^{\frac{1}{2}} \psi_\delta, \text{ and } \omega_\delta \to \omega \text{ weakly}^* \text{ in } A^* \text{ as } \delta \to 0 \end{array} \right\},$$

where $\omega_\delta := \theta_\delta(f_\delta(\theta_\delta^{-1}(\omega)))$, and here θ_δ denotes the commutant Radon-Nikodym map associated to ψ_δ. Note that the first condition $|\omega(a)| \le \delta^{-\frac{1}{6}} \psi_\delta(a)$ for all $a \in A^+$ implies ω belongs to the image of θ_δ, and the functional calculus $f_\delta(\theta_\delta^{-1}(\omega))$ in the definition of ω_δ is well-defined since $\|\theta_\delta^{-1}(\omega)\| \le \delta^{-\frac{1}{6}} \le \delta^{-1}$. If $\omega \in G^{*+}$, with sequences $\psi_\delta \in A^{*+}$ and $\varphi_\delta \in F^*$ such that the five conditions hold for $\varepsilon = 1$, and if we let

$$F_\delta^* := F^* + \left\{ \omega' \in A^{*+} : \omega' \le \sum_{\delta' \le \delta} \delta'^{\frac{1}{2}} \psi_{\delta'} \right\}$$

be the non-increasing sequence of weakly* closed convex hereditary subsets of A^{*+} indexed on $\delta \le (1+4\|\omega\|)^{-6}$, then $\omega \ge 0$ implies $\omega_{\delta'} \in F_\delta^*$ for $\delta' \le \delta$, which deduces $\omega \in \bigcap_\delta F_\delta^*$ from the weak* limit $\omega_\delta \to \omega$. If we write $\omega = \varphi'_\delta + \omega'_\delta$ with $\varphi'_\delta \in F^*$ and $0 \le \omega'_\delta \le \sum_{\delta' \le \delta} \delta'^{\frac{1}{2}} \psi_{\delta'}$ for each $\delta \le (1+4\|\omega\|)^{-6}$, then since $\omega'_\delta \to 0$ in norm of A^* as $\delta \to 0$ so that $\varphi'_\delta \in F^*$ converges to ω in norm of A^*, we have $\omega \in F^*$. It means that $G^{*+} \subset F^*$, so we claim $G^* = \overline{F^* - A^{*+}}$ to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$.

Since every element $\omega \in G^*$, letting $\varepsilon = 1$, has a sequence $\omega_\delta - \delta^{\frac{1}{2}} \psi_\delta \in F^* - A^{*+}$ convergent to ω weakly* as $\delta \to 0$, we have $G^* \subset \overline{F^* - A^{*+}}$. For the other direction, suppose first $\omega \in F^* - A^{*+}$ and take any $\varphi \in F^*$ such that $\omega \le \varphi$. Fix $\varepsilon > 0$, and for each $\delta \le (1+4\|\omega\|)^{-6}$ let

$$\psi_\delta := \frac{[\omega]}{1+\|\omega\|} + \frac{\varphi}{(1+\|\omega\|)(1+\|\varphi\|)}, \qquad \varphi_\delta := \theta_\delta(f_\delta(\theta_\delta^{-1}(\varphi))).$$

The first two conditions are easily checked, and if we denote by Ω_δ the canonical cyclic vector of the Gelfand-Naimark-Segal representation of A associated to ψ_δ, then the third condition follows as

$$\|\varphi_\delta\| = \varphi_\delta(1_{A^{**}}) = \langle f_\delta(\theta_\delta^{-1}(\varphi))\Omega_\delta, \Omega_\delta \rangle \le \delta^{-1} \|\Omega_\delta\|^2 = \delta^{-1} \|\psi_\delta\| \le \delta^{-1}.$$

If we let $\omega_\delta := \theta_\delta(f_\delta(\theta_\delta^{-1}(\omega)))$ as in the definition of G^*, then the positivity of θ_δ and the operator monotonicity of f_δ give the fourth condition $\omega_\delta \le \varphi_\delta \le \varphi_\delta + \varepsilon \delta^{\frac{1}{2}} \psi_\delta$, and since ψ_δ is independent of δ so that $f_\delta(\theta_\delta^{-1}(\omega)) \to \theta_\delta^{-1}(\omega)$ strongly as $\delta \to 0$, we have the fifth condition $\omega_\delta \to \omega$ weakly* in A^*. Thus we have $F^* - A^{*+} \subset G^*$, so it is enough to show G^* is weakly* closed to prove the claim.

Let $\omega_i \in G^*$ be a net satisfying $\omega_i \to \omega$ weakly* in A^*, which may be assumed to be bounded by the Krein-Šmulian theorem. Let $\|\omega_i\| \le r$ for some $r > 0$, and in order to show $\omega \in G^*$, we fix $\varepsilon > 0$ and aim to construct an appropriate pair of sequences ψ_δ and φ_δ. Assume $\varepsilon \le 2^{\frac{14}{5}}$ so that $(\varepsilon/8)^6 \le 2^{-\frac{6}{5}}$, and let $\delta_0 := \min\{(\varepsilon/8)^6, (1+4r)^{-6}\}$.

Let $\delta \in (0, \delta_0]$. Since $\delta < \inf_i (1+4\|\omega_i\|)^{-6}$, we can take $\psi_{i,\delta} \in A^{*+}$ and $\varphi_{i,\delta} \in F^*$ for each i and δ following the definition of G^* such that the fourth condition is given by $\omega_{i,\delta} \le \varphi_{i,\delta} + (\varepsilon/4)\delta^{\frac12}\psi_{i,\delta}$, where $\omega_{i,\delta} := \theta_{i,\delta}(f_\delta(\theta_{i,\delta}^{-1}(\omega)))$, and the commutant Radon-Nikodym map $\theta_{i,\delta}$ is for $\psi_{i,\delta}$. Because $\psi_{i,\delta}$ and $\varphi_{i,\delta}$ are bounded nets for each δ, by replacing ω_i to a diagonal subnet of ω_i for iteratively taken nested subnets of ω_i along with the countably many steps of δ, we may assume the nets $\psi_{i,\delta}$ and $\varphi_{i,\delta}$ are weakly* convergent for all δ. See the remark in the below for the detail of this diagonal subnet. We define $\psi_\delta \in A^{*+}$ and $\varphi_\delta \in F^*$ as the weak* limits in A^* of them respectively. Be cautious that we have the weak* convergence $\omega_i \to \omega$ by the initial assumption, but $\omega_{i,\delta}$ may not weakly* converge to $\omega_\delta = \theta_\delta(f_\delta(\theta_\delta^{-1}(\omega)))$. Considering the limits for the three weakly* convergent nets $\omega_i \to \omega$, $\psi_{i,\delta} \to \psi_\delta$, and $\varphi_{i,\delta} \to \varphi_\delta$ in A^* for each δ, we can see that the first three conditions for ω easily follow. Before the check of fourth and fifth conditions, observing that the first conditions for ω_i and ω imply $\|\theta_{i,\delta}^{-1}(\omega_i)\| \le \delta^{-\frac16}$ and $\|\theta_\delta^{-1}(\omega)\| \le \delta^{-\frac16}$, take a note that $\delta \le (\varepsilon/8)^6 \le 2^{-\frac65}$ implies

$$\omega_i \le \omega_{i,\delta} + (\varepsilon/4)\delta^{\frac12}\psi_{i,\delta}, \qquad \omega \le \omega_\delta + (\varepsilon/4)\delta^{\frac12}\psi_\delta.$$

Combining with $\omega_{i,\delta} \le \omega_i$ and $\omega_\delta \le \omega$, we also have

$$|(\omega_\delta - \omega_{i,\delta})(a)| \le |(\omega - \omega_i)(a)| + (\varepsilon/4)\delta^{\frac12}\max\{\psi_{i,\delta}(a), \psi_\delta(a)\}, \qquad a \in A^+.$$

Then, by taking the weak* limit for i on

$$\omega_i \le \omega_{i,\delta} + (\varepsilon/4)\delta^{\frac12}\psi_{i,\delta} \le \varphi_{i,\delta} + (\varepsilon/2)\delta^{\frac12}\psi_{i,\delta},$$

we obtain the fourth condition $\omega_\delta \le \omega \le \varphi_\delta + (\varepsilon/2)\delta^{\frac12}\psi_\delta \le \varphi_\delta + \varepsilon\delta^{\frac12}\psi_\delta$ for ω. On the other hand, if we fix i such that $|(\omega_i - \omega)(a)| < \varepsilon$ which is independent of δ, then

$$\begin{aligned}|(\omega_\delta - \omega)(a)| &\le |(\omega_\delta - \omega_{i,\delta})(a)| + |(\omega_{i,\delta} - \omega_i)(a)| + |(\omega_i - \omega)(a)| \\ &\le |(\omega_{i,\delta} - \omega_i)(a)| + 2|(\omega_i - \omega)(a)| + (\varepsilon/2)\delta^{\frac12}\max\{\psi_{i,\delta}(a), \psi_\delta(a)\} \\ &\le |(\omega_{i,\delta} - \omega_i)(a)| + 2\varepsilon + (\varepsilon/2)\delta^{\frac12}\|a\|,\end{aligned}$$

so taking the limit superior $\delta \to 0$ and the limit $\varepsilon \to 0$ in order on the above estimate, we obtain the weak* convergence $\omega_\delta \to \omega$ as $\delta \to 0$, the fifth condition for ω.

Let $\delta \in (\delta_0, (1+4\|\omega\|)^{-6}]$. Recall that we have $\omega \le \varphi_{\delta_0} + (\varepsilon/2)\delta_0^{\frac12}\psi_{\delta_0}$. Define

$$\psi_\delta := \delta^{\frac16}[\omega] + 4^{-1}\delta_0\varphi_{\delta_0} + 2^{-1}\psi_{\delta_0}, \qquad \varphi_\delta := \theta_\delta(f_{\delta-(\delta_0/4)}(\theta_\delta^{-1}(\varphi_{\delta_0}))).$$

We do not need to check the fifth condition in the range of δ we consider. If we denote $h := \theta_\delta^{-1}(\omega)$, $k_{\delta_0} := \theta_\delta^{-1}(\varphi_{\delta_0})$, and $l_{\delta_0} := \theta_\delta^{-1}(\psi_{\delta_0})$, then since $\|k_{\delta_0}\| \le (\delta_0/4)^{-1}$ and $\|l_{\delta_0}\| \le 2$, we have

$$f_\delta(h) \le f_\delta(k_{\delta_0} + (\varepsilon/2)\delta_0^{\frac12} l_{\delta_0}) \le f_\delta(k_{\delta_0} + \varepsilon\delta_0^{\frac12}) \le f_{\delta-(\delta_0/4)}(k_{\delta_0}) + \varepsilon\delta^{\frac12},$$

and it implies the fourth condition $\omega_\delta \le \varphi_\delta + \varepsilon\delta^{\frac12}\psi_\delta$. The first three conditions are clear. Therefore, $\omega \in G^*$, proving that G^* is weakly* closed, hence the claim $G^* = \overline{F^* - A^{*+}}$ follows. □

Remark 2.5. The ideas of the term *diagonal argument* in analysis are roughly divided into two different situations. Let (X, d) be a metric space. One diagonal argument considers a sequence $x_n \in A$ in a subset $A \subset X$ such that $x_n \to x$ as $n \to \infty$, together with a sequence of sequences $x_{nm} \in B$ in a subset $B \subset A$ such that $x_{nm} \to x_n$ as $m \to \infty$ for each n. If we assume the error estimate $d(x_{nm}, x_n) < m^{-1}$ does not depend on m by taking subsequences, then we can conclude $x_{nn} \to x$ as $n \to \infty$.

The other diagonal argument considers a sequence $x_n \in X$ and a sequence of properties P_m for subsequences of x_n that has an inheritance property in the sense that for each m if a subsequence x_{n_k} of x_n satisfies P_m, then any eventual subsequence $x_{n_{k_j}}$ of x_{n_k} also satisfies P_m. If for each m every subsequence of x_n has a further subsequence satisfying P_m, then we can construct a subsequence of x_n satisfying P_m for all m, by taking a diagonal sequence from an iteratively taken sequence of subsequences.

For convenience, we will call the former by the *diagonal sequence argument*, and the latter by the *diagonal subsequence argument*. The net version of the diagonal sequence argument is famous. For a topological space X, if $I \to X : i \mapsto x_i$ is a net convergent to $x \in X$ and $J_i \to X : j \mapsto x_{ij}$ is a net convergent to $x_i \in X$ for each $i \in I$, then the monotone final function $I \times \prod_{i \in I} J_i \to X : (i, (j_i)_{i \in I}) \mapsto x_{i j_i}$ is a net convergent to x.

What we want in the proof of Theorem 2.4 is a generalized version of the diagonal subsequence argument for nets, which is surprisingly not well-known, and recently there has been an arXiv article. According to this article, for a net $I \to X : i \mapsto x_i$ and a sequence $(P_m)_{m=1}^\infty$ of properties for subnets of x_i satisfying the same inheritance property as above, if we first consider a consecutive sequence of nested subnets

$$\cdots \to I_m \to \cdots \to I_1 \to I \to X$$

such that $I_m \to X$ satisfies $P_{m'}$ for all $m' \leq m$, then we can construct a desired subnet

$$\coprod_m \prod_{m' \leq m} I_{m'} \to I \to X$$

that satisfies P_m for all m. The number of properties P_m cannot be uncountable. The proof seems to be not hard, but I have not read it seriously yet.

Other remarks....

- The restriction of the range $\delta \leq (1 + \|x\|)^{-1}$ is for the well-definedness of the functional calculus $f_\delta(x)$.
- The small perturbation $\varepsilon \delta^{\frac{1}{2}}$ is introduced , The exponent $\frac{1}{2}$ is set because we need $p < 1$ to use $x \leq f_\delta(x) + (\varepsilon/2)\delta^p$ for arbitrarily small $\varepsilon > 0$, provided even though $\|x\|$ is bounded.

—

- For the first condition $|\omega| \leq \delta^{-\frac{1}{6}} \psi_\delta$, the coefficient needs to grow linearly along with the size of ω because we set ψ_δ to be always bounded by one, but we have to remove the explicit contribution of the norm $\|\omega\|$ from the coefficient to make the weak* limits $\omega_i \to \omega$ and $\psi_{i,\delta} \to \psi_\delta$ preserve the inequality for each fixed δ.
- There are four remarks for the bounded range $\delta \leq (1 + 4\|\omega\|)^{-6}$. (1) The necessity of a bound for δ is for the well-definedness of the functional calculus $f_\delta(h)$. (2) The dependence of the bound for δ on the norm $\|\omega\|$ is needed

to fix $\delta > 0$ uniformly on the index i of a bounded net ω_i when we consider limit $\omega_{i,\delta} \to \omega_\delta$ with the Krein-Šmulian. (3) To use $h \le f_\delta(h) + (\varepsilon/4)\delta^{\frac{1}{2}}$ for the growing norm of $\|h\|$ as $\delta \to 0$, it needs to have a sufficiently slow growth rate at least $\|h\| \le \delta^{-\frac{1}{4}}$, but the value $-\frac{1}{6}$ is used because $-\frac{1}{4}$ is not enough to cover the arbitrarily small $\varepsilon > 0$. (4) The number 4 in front of $\|\omega\|$ can be technically any constant greater than 1, and it is introduced to define ψ_δ such that $\|l_{\delta_0}\|$ is uniformly bounded for $\delta \ge \delta_0$.

3. Applications to weight theory

The positive Hahn-Banach separation theorem implies a generalization of the theorems of Combes and Haagerup on normal or lower semi-continuous subadditive weights.

Corollary 3.1. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\left\{\begin{array}{c}\text{normal subadditive}\\ \text{weights of } M\end{array}\right\} \quad\longleftrightarrow\quad \left\{\begin{array}{c}\text{norm closed convex}\\ \text{hereditary subsets of } M_*^+\end{array}\right\}$$
$$\varphi \quad\mapsto\quad \{\omega \in M_*^+ : \omega \le \varphi\}$$

Let φ be a completely additive weight.

$\varphi(\sum_i p_i) = \sum_i \varphi(p_i)$.

We always have $\varphi(\sum_i p_i) \ge \sum_i \varphi(p_i)$.

$\varphi(x) \le 1$ if and only if $\sup_{\omega\in F_*} \omega(x) \le 1$?

$p_i\omega p_i \to \omega$ in norm since $\|\omega - p\omega p\| \le 2\omega(1-p)^{\frac{1}{2}}$, but we do not have $p_i\omega p_i \le \omega$.

$\|\omega - p\omega p\|^2 \le 4\omega(1) - 4\omega(p)$

$4\omega(1) \lesssim 4\omega(p)$

For a normal weight φ, if we find a small closed F_* such that $F_*^{r+} = \{x : \varphi(x) \le 1\}$, then it means that for every ω, $\omega \le \varphi$ implies $\omega \in F_*$...

For $m \in \widehat{M}^+$, $F_* := \{\omega : \omega(m) \le 1\}$ is norm closed. Its positive polar is $\{x : x \le m\}$, and its positive polar is... So $\omega(m) = \sup_{x\le m} \omega(x)$ for any $\omega \in M_*^+$.

$\varphi(m) = ?$