Lebesgue Theory

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Contents

Part I

Measure theory

Chapter 1

Measures and σ-algebras

1.1 Definition of measures

1.2 The Carathéodory extension theorem

1.1 (Outer measures)**.** Let X be a set. An *outer measure* on X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ such that

(i) if $E \subset E'$, then $\mu^*(E) \le \mu^*(E')$, (monotonicity)

(ii) $\mu^*(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu^*(E_i)$. (countable subadditivity)

(a) A function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ is an outer measure if and only if $E \subset \bigcup_{i=1}^\infty E_i$ implies $\mu^*(E) \le \sum_{i=1}^\infty \mu^*(E_i)$.

(b) Let $\mathcal{A} \subset \mathcal{P}(X)$ such that $\varnothing \in \mathcal{A}$. If a function $\rho : \mathcal{A} \to [0,\infty]$ satisfies $\rho(\varnothing) = 0$, then we can associate an outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by defining as

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^\infty \rho(A_i) : E \subset \bigcup_{i=1}^\infty A_i,\ A_i \in \mathcal{A} \right\},$$

where we use the convention $\inf \varnothing = \infty$.

1.2 (Carathéodory measurability)**.** Let μ^* be an outer measure on a set X. A subset $A \subset X$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

e for every subset $E \subset X$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0, \infty]$ is complete.

1.3 (The Carathéodory extension theorem)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ be a semi-ring of sets on a set X and $\rho : \mathcal{A} \to [0, \infty]$ a function with $\rho(\varnothing) = 0$. If the function ρ satisfies

(i) $\rho(A) = \sum_{i=1}^n \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^n \subset \mathcal{A}$, (finite additivity)

(ii) $\rho(A) \le \sum_{i=1}^\infty \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$, ((disjoint) countable subadditivity)

then it is called a *premeasure*. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0, \infty]$ the measure defined from μ^* on Carathéodory measurable subsets. We call μ the *Carathéodory measure* constructed from ρ.

(a) If ρ is finitely additive, then $\mathcal{A} \subset \mathcal{M}$.

(b) If ρ is countably subadditive, then $\mu^*(A) = \rho(A)$ for every $A \in \mathcal{A}$.

(c) If ρ is a premeasure, then μ is an extension of ρ and called *Carathéodory extension* of ρ.

(d) In particular, a premeasure is a priori countably additive in the sense that $\rho(A) = \sum_{i=1}^\infty \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$.

1.4 (Uniqueness of extensions)**.** The Carathéodory extension theorem provides with a uniqueness theorem for measures.

Monotone class lemma: alternative direct proof method without using Carathéodory extension.

Chapter 2

Measures on the real line

Chapter 3

Measurable functions

Part II

Integration

Chapter 4

Lebesgue integration

4.1 Definition of Lebesgue integration

4.2 Convergence theorems

Stein: Egorov $\to$ BCT $\to$ Fatou $\to$ MCT $\to$ L1 is a measure
Stein: BCT + L1 is a measure $\to$ DCT
Folland: MCT $\to$ Fatou $\to$ DCT $\to$ BCT

4.1 (Egorov's theorem)**.** Let Ω be a finite measure space. Let $(f_n : \Omega \to \mathbb{R})_n$ be a sequence of a.e. convergent measurable functions. For $\varepsilon > 0$, there exists a measurable $E_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ and f_n uniformly convergent on E_ε.

Proof. Assume $f_n \to 0$. The set of convergence is

$$\bigcap_{k>0} \bigcup_{n_0>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \},$$

which is a full set. We want to get rid of the dependence on the point x of n_0 in the union $\bigcup_{n_0>0}$. Since

$$\bigcap_{n\geq n_0} \{ x : |f_n(x)| < \frac{1}{k} \}$$

is increasing as $n_0 \to \infty$ to a full set for each $k > 0$, we can find $n_0(k, \varepsilon)$ such that

$$\mu(\bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \}) > \mu(\Omega) - \frac{\varepsilon}{2^k}.$$

Then,

$$\mu(\bigcap_{k>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \}) > \mu(\Omega) - \varepsilon.$$

If we define

$$E_\varepsilon := \bigcap_{k>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \},$$

then for any $k > 0$ and $x \in E_\varepsilon$, and with the $n_0(k,\varepsilon)$ we have chosen, we have

$$n \geq n_0 \quad \Rightarrow \quad |f_n(x)| < \frac{1}{k}.$$

□

4.3 Modes of convergence

Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0, \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > \tfrac{1}{k},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n - f| > \tfrac{1}{k}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n - f| > \tfrac{1}{k}\}.\end{aligned}$$

Since for every k we have

$$\begin{aligned}\limsup_n \{x : |f_n - f| > \tfrac{1}{k}\} &\subset \limsup_{n>k} \{x : |f_n - f| > \tfrac{1}{n}\} \\ &= \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\},\end{aligned}$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\}.$$

4.2. Let (X,μ) be a measure space. Let f_n be a sequence of measurable functions. If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.

Proof. We can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) > \tfrac{1}{2^k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k} - f| > \tfrac{1}{k}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. □

Chapter 5

Product measures

5.1 The Fubini-Tonelli theorem

5.2 The Lebesgue measure on Euclidean spaces

Chapter 6

Lebesgue spaces

6.1 L^p spaces

6.2 L^2 spaces

6.3 The Riesz representation theorem

Part III

Chapter 7

Chapter 8

Bounded linear operators

8.1 (Extension of linear operator)**.** double dual

8.2. Let $T : X \to Y$ be a linear operator. Suppose

$$\|Tx\| \lesssim \|x\|$$

for all $x \in \mathcal{D}$.

(a) If

8.1 Weak L^p spaces

8.2 Interpolation theorems

Chapter 9

Integral operators

9.1 Bounded linear operators

9.2 Regular integral operators

9.3 Convolution type operators

Part IV

Fundamental theorem of calculus

Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i} uv + u\partial_{x_i} v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega \to \mathbb{R}$ such that $f(x, y)$ and $\partial_{x_i} f(x, y)$ are both locally integrable in x and integrable y. Then,

$$\partial_{x_i} \int f(x, y)\, dy = \int \partial_{x_i} f(x, y)\, dy$$

where ∂_{x_i} denotes the weak partial derivative.

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}
(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞
(b) f is AC iff f' is L^1
(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

The Lebesgue differentiation theorem