Partial Differential Equations

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Contents

I Sobolev spaces **3**

1 Distribution theory **4**
1.1 Space of test functions 4
1.2 Space of distributions 4
1.3 Well-posedness 4

2 Sobolev inequalities **5**
2.1 Approximations 5
2.2 Extensions and restrictions 5
2.3 Sobolev embeddings 5

3 Generalizations of Sobolev spaces **7**
3.1 Fractional Sobolev spaces 7
3.2 Fourier transform methods 7
3.3 Almost everywhere differentiability 7
3.4 Vector-valued functions 7

II Elliptic equations **8**

4 Potential theory **9**
4.1 Mean value property 9
4.2 Weyl's lemma 9

5 Existence theory **10**
5.1 Variational methods 10
5.2 Lax-Milgram theorem 10
5.3 Fredholm alternative 10
5.4 Perron's method 10
5.5 Eigenvalue problems 10

6 Ellipic regularity **11**
6.1 L^p theory 11
6.2 Schauder theory 13
6.3 De Giorgi-Nash-Moser theory 13
6.4 Viscosity solutions 13

III Evolution equations **14**

7 Parabolic equations **15**
7.1 Galerkin approximation 15
7.2 Semigroup theory 15

8 Hyperbolic equations **16**

9 Local and global existence **17**
9.1 Local existence 17
9.2 Global existence 17
9.3 Weak convergence 17

IV Nonlinear equations **18**

10 **19**

11 Hamilton-Jacobi equations **20**

12 Conservation laws **21**

Part I

Sobolev spaces

Chapter 1

Distribution theory

1.1 Space of test functions

1.1. (a) If a test function φ satisfies $\langle 1, \varphi \rangle = 0$, then there is $v \in \mathbb{R}^d$ and a test function ψ such that $\varphi = v \cdot \nabla \psi$.

(b) If a distribution has zero derivative, then it is a constant.

1.2 (Weak* convergence).

1.2 Space of distributions

1.3 (Rigged Hilbert space).

1.3 Well-posedness

1.4 (Extension of linear operators). Let $T : \mathcal{D} \to \mathcal{D}'$ be a contiuous linear operator. We can always define the adjoint $T^* : \mathcal{D} \subset \mathcal{D}'' \to \mathcal{D}'$. The most reasonable extension of T is $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$. For $f \in (T^*(\mathcal{D}))'$, we can define $\langle T(f), \varphi \rangle := \langle f, T^*\varphi \rangle$ for $\varphi \in \mathcal{D}$.

Suppose $T : (\mathcal{D}, \mathcal{T}) \to (T(\mathcal{D}), \mathcal{S})$ is proved to be continuous. If $(\mathcal{D}, \mathcal{T}) \to (T^*(\mathcal{D}))'$ and $(T(\mathcal{D}), \mathcal{S}) \to \mathcal{D}'$ are embeddings, then the extension of T to the completion of $(\mathcal{D}, \mathcal{T})$ agrees with $T : (T^*(\mathcal{D}))' \to \mathcal{D}'$.

For example, if Φ is locally integrable, then since $(T_\Phi)^* = T_{\widetilde{\Phi}}$ and $\Phi * \varphi \in \mathcal{E} = C^\infty$ for $\varphi \in \mathcal{D}$, the convolution operator $T_\Phi : \mathcal{E}' \to \mathcal{D}'$ can be defined on the space of compactly supported distributions.

If $g * f$ is well-defined, is $f * g$ also well-defined? In other words, if $f \in (T_{\widetilde{g}}(\mathcal{D}))'$ so that $g * f \in \mathcal{D}'$, then $g \in (T_{\widetilde{f}}(\mathcal{D}))'$? Are they same?

$$\langle g, \widetilde{f} * \varphi \rangle =$$

Exercises

1.5. * Describe the range of the operator $T : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ defined by $Tf = \Phi * f$ for $d \geq 3$, where Φ is the fundamental solution of Laplace's equation.

Chapter 2

Sobolev inequalities

2.1 Approximations

2.1 (Completeness of Sobolev norms).

2.2 (Difference quotient).

2.3 (Interior approximation).

2.4 (Myers-Serrin theorem).

2.2 Extensions and restrictions

2.5 (Lipschitz boundary).

2.6 (Extension theorem).

2.7 (Trace theorem).

2.8 (Vanishing at boundary). zero trace, whole domain

2.3 Sobolev embeddings

2.9 (Gagliardo-Nirenberg-Sobolev inequality).

2.10 (Hölder spaces).

2.11 (Morrey inequality).

2.12 (Poincaré inequality). BMO

2.13 (Rellich-Kondrachov theorem). Let Ω be bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary. Let $1 \le p < d$ and $1 \le q < p^*$ where $p^* := \frac{dp}{d-p}$ denotes the Sobolev conjugate. Let $(u_n)_n$ be a bounded sequence in $W^{1,p}(\Omega)$. We may assume it is also bounded in $W^{1,1}(\mathbb{R}^d)$ by the embedding $W^{1,p}(\Omega) \subset W^{1,1}(\Omega)$ and the extension theorem. Let η_ε be a standard mollifier.

(a) There is a subsequence of $(\eta_\varepsilon * u_n)_n$ that is Cauchy in $L^q(\Omega)$ for each $\varepsilon > 0$.

(b) $\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^1(\Omega)} \to 0$ as $\varepsilon \to 0$.

(c) $\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \to 0$ as $\varepsilon \to 0$.

(d) There is a subsequence of $(u_n)_n$ that is Cauchy in $L^q(\Omega)$.

(e) $W^{k,p}(\Omega) \to W^{l,q}(\Omega)$ is a compact embedding if

$$\frac{l}{d} - \frac{1}{q} < \frac{k}{d} - \frac{1}{p}.$$

Proof. (a) The sequence $(\eta_\varepsilon * u_n)_n$ is pointwise bounded from

$$\|\eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim_\varepsilon 1,$$

and equicontinuous from

$$\|\nabla \eta_\varepsilon * u_n\|_{C_0(\mathbb{R}^d)} \le \|\nabla \eta_\varepsilon\|_{C_0(\mathbb{R}^d)} \|u_n\|_{L^1(\mathbb{R}^d)} \lesssim_\varepsilon 1.$$

By the Arzela-Ascoli theorem, since $\overline{\Omega}$ is compact, there is a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $C(\overline{\Omega})$, and hence in $L^q(\Omega)$.

(b) Write

$$\begin{aligned}
\eta_\varepsilon * u_n(x) - u_n(x) &= \frac{1}{\varepsilon^d} \int \eta\left(\frac{x-y}{\varepsilon}\right)(u_n(y) - u_n(x))\, dy \\
&= \int \eta(y)(u_n(x - \varepsilon y) - u_n(x))\, dy \\
&= \int \eta(y) \int_0^1 \frac{d}{dt}(u_n(x - t\varepsilon y))\, dt\, dy \\
&= \int \eta(y) \int_0^1 (-\varepsilon y) \cdot \nabla u_n(x - t\varepsilon y)\, dt\, dy.
\end{aligned}$$

Then, since $|y| \ge 1$ if $\eta(y) > 0$,

$$\|\eta_\varepsilon * u_n - u_n\|_{L^1(\mathbb{R}^d)} \le \varepsilon \int \eta(y) \int_0^1 \int |\nabla u_n(x - t\varepsilon y)|\, dx\, dt\, dy = \varepsilon \|\nabla u_n\|_{L^1(\mathbb{R}^d)}.$$

(c) The interpolation

$$\|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_n - u_n\|_{L^1(\Omega)}^{\theta} \|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)}^{1-\theta}$$

for $q = \frac{\theta}{1} + \frac{1-\theta}{p}$ with $0 < \theta \le 1$ and the Gagliardo-Nireberg-Sobolev inequality

$$\|\eta_\varepsilon * u_n - u_n\|_{L^{p^*}(\Omega)} \lesssim \|\eta_\varepsilon * u_n - u_n\|_{W^{1,p}(\Omega)} \lesssim 1$$

give the L^q version of the part (b),

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} \to 0$$

as $\varepsilon \to 0$.

(d) By the part (c), for any $\delta > 0$, there is $\varepsilon > 0$ such that

$$\sup_n \|\eta_\varepsilon * u_n - u_n\|_{L^q(\Omega)} < \frac{\delta}{2},$$

so for a subsequence $(\eta_\varepsilon * u_{n_k})_k$ that is Cauchy in $L^q(\Omega)$, we have

$$\|u_{n_k} - u_{n_{k'}}\|_{L^q(\Omega)} \le \|\eta_\varepsilon * u_{n_k} - \eta_\varepsilon * u_{n_{k'}}\|_{L^q(\Omega)} + \delta,$$

and by the diagonal argument reducing δ to zero, we can construct the desired subsequence.

(e) $\square$

Chapter 3

Generalizations of Sobolev spaces

3.1 Fractional Sobolev spaces

3.2 Fourier transform methods

3.3 Almost everywhere differentiability

Lipschitz, Rademacher

3.4 Vector-valued functions

3.1 (Pettis measurability theorem)**.** Let (Ω, μ) be a measure space and X a Banach space. Let $f : \Omega \to X$ be a function. We say f is *strongly measurable* or *Bochner measurable* if it is a pointwise limit of a sequence of simple functions.

If μ is complete, then all the pointwise convergence discussed here can be relaxed to the almost everywhere convergence.

(a) If f is strongly measurable, then f is Borel measurable.

(b) If f is Borel measurable, then f is weakly measurable.

(c) If f is weakly measurable and separably valued, then f is strongly measurable.

3.2 (Bochner and Pettis integrals)**.** Let (Ω, μ) be a measure space and X a Banach space.

if there is a net of simple functions $(s_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$\int_\Omega \|f(\omega) - s_\alpha(\omega)\| \, d\mu \to 0$$

for $\alpha \in \mathcal{A}$.

Bochner integrable => Pettis integrable => weakly(scalarly) integrable

Part II

Elliptic equations

Chapter 4

Potential theory

4.1 Mean value property

mean value property maximum principle Harnack inequality
potential estimate Hölder estimate

4.2 Weyl's lemma

Chapter 5

Existence theory

5.1 Variational methods

5.2 Lax-Milgram theorem

5.1 (Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$. Consider the problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{, in } x \in \Omega, \\ \quad u(x) = 0 & \text{, on } x \in \partial\Omega. \end{cases}$$

Define a bilinear form B on $H_0^1(\Omega)$ such that

$$B(u, v) := \int \nabla u(x) \cdot \nabla v(x)\, dx.$$

(a) If $u \in H_0^1(\Omega)$ and $f \in \mathcal{D}'(\Omega)$ satisfy $B(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$, then $-\Delta u = f$.

(b) B is another inner product equivalent to $\langle -, - \rangle_{H_0^1(\Omega)}$.

(c) For $f \in H^{-1}(\Omega)$, there is $u \in H_0^{-1}(\Omega)$ such that $-\Delta u = f$.

5.3 Fredholm alternative

5.4 Perron's method

5.5 Eigenvalue problems

Chapter 6

Ellipic regularity

6.1 L^p theory

6.1 (Interior regularity in H^2). Let Ω be bounded open subset of $\mathbb{R}^d$ and $L : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ a uniformly elliptic operator given by

$$Lu := -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu$$

for $a^{ij} \in C^1(\Omega)$, $b^i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)$.

Fix an open subset $U \Subset \Omega$ and $\zeta \in C_c^\infty(\Omega)$ a cutoff function such that $\zeta = 1$ in U. Let $\varphi := -\partial_k^{-h}(\zeta^2\partial_k^h u)$ for $k = 1, \cdots, d$ and sufficiently small $h > 0$.

(a) We have

$$\|\nabla u\|_{L^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$

(b) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \int a^{ij}\partial_i u\partial_j\varphi + \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$.

(c) We have

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}$$

for all u such that $Lu \in L^2(\Omega)$ and $u \in H^1(\Omega)$.

(d) We have

$$\|u\|_{H^2(U)} \lesssim \|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$$

for all u such that $Lu, u \in L^2(\Omega)$.

Proof. (a) Since $\zeta^2 u \in H^1_0(\Omega)$,

$$\begin{aligned}\int \zeta^2|\nabla u|^2 &\lesssim \int a^{ij}\zeta^2\partial_i u\partial_j u \\ &= \int a^{ij}\,\partial_i u\,\partial_j(\zeta^2 u) - \int a^{ij}\,\partial_i u\,\partial_j(\zeta^2)u \\ &= \int (Lu - b^i\partial_i u - cu)\,\zeta^2 u - \int a^{ij}\,\partial_i u\,2\zeta\partial_j\zeta\,u \\ &\lesssim \int (|Lu\,u| + |u\,\zeta\nabla u| + |u|^2 + |u\,\zeta\nabla u|) \\ &\lesssim \int (|Lu|^2 + |u|^2) + \frac{1}{\varepsilon}\int |u|^2 + \varepsilon\int \zeta^2|\nabla u|^2.\end{aligned}$$

Taking small $\varepsilon > 0$, we are done.

(b) Write

$$\begin{aligned}\int a^{ij}\partial_i u\partial_j\varphi &= -\int a^{ij}\partial_i u\partial_j\partial_k^{-h}(\zeta^2\partial_k^h u) \\ &= \int \partial_k^h(a^{ij}\partial_i u)\,\partial_j(\zeta^2\partial_k^h u) \\ &= \int \partial_k^h a^{ij}\,\partial_i u\,\partial_j(\zeta^2)\,\partial_k^h u + \int \partial_k^h a^{ij}\,\partial_i u\,\zeta^2\,\partial_j\partial_k^h u \\ &\quad + \int a^{ij}\,\partial_k^h\partial_i u\,\partial_j(\zeta^2)\,\partial_k^h u + \int a^{ij}\,\partial_k^h\partial_i u\,\zeta^2\,\partial_j\partial_k^h u.\end{aligned}$$

The last term out of the four terms controls the difference quotient $|\partial_k^h\nabla u|$ as

$$\int a^{ij}\,\partial_k^h\partial_i u\,\zeta^2\,\partial_j\partial_k^h u \gtrsim \int \zeta^2|\partial_k^h\nabla u|^2,$$

and the absolute values of other three terms are estimated up to constant by

$$\begin{aligned}&\int \zeta|\nabla u||\partial_k^h u| + \int \zeta^2|\nabla u||\partial_k^h\nabla u| + \int \zeta|\partial_k^h\nabla u||\partial_k^h u| \\ &\quad\lesssim \left(1+\frac{1}{\varepsilon}\right)\int \zeta^2|\nabla u|^2 + \left(1+\frac{1}{\varepsilon}\right)\int |\partial_h^k u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2 \\ &\quad\lesssim \left(1+\frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2.\end{aligned}$$

Therefore,

$$\int \zeta^2|\partial_k^h\nabla u|^2 \lesssim \int a^{ij}\partial_i u\partial_j\varphi + \left(1+\frac{1}{\varepsilon}\right)\int |\nabla u|^2 + \varepsilon\int \zeta^2|\partial_k^h\nabla u|^2,$$

and taking small $\varepsilon > 0$, we are done.

(c) Note that

$$\int a^{ij}\partial_i u\partial_j\varphi = \int (Lu - b^i\partial_i u - cu)\varphi$$

since $\varphi \in H^1_0(\Omega)$. Because

$$\int (Lu - b^i\partial_i u - cu)\varphi \lesssim \frac{1}{\varepsilon}\int (|Lu|^2 + |\nabla u|^2 + |u|^2) + \varepsilon\int |\varphi|^2$$

and

$$
\begin{aligned}
\int |\varphi|^2 &= \int |\partial_k^{-h}(\zeta^2 \partial_k^h u)|^2 \\
&\lesssim \int |\nabla(\zeta^2 \partial_k^h u)|^2 \\
&\lesssim \int |\partial_k^h u|^2 + \int \zeta^2 |\partial_k^h \nabla u|^2 \\
&\lesssim \int |\nabla u|^2 + \int \zeta^2 |\partial_k^h \nabla u|^2,
\end{aligned}
$$

we obtain

$$
\int (Lu - b^i \partial_i u - cu)\varphi \lesssim \frac{1}{\varepsilon} \int (|Lu|^2 + |u|^2) + \left(\varepsilon + \frac{1}{\varepsilon}\right) \int |\nabla u|^2 + \varepsilon \int \zeta^2 |\partial_k^h \nabla u|^2.
$$

Taking small $\varepsilon > 0$, we are done. □

6.2 Schauder theory

6.3 De Giorgi-Nash-Moser theory

6.4 Viscosity solutions

Part III

Evolution equations

Chapter 7

Parabolic equations

7.1 Galerkin approximation

7.2 Semigroup theory

Chapter 8

Hyperbolic equations

Chapter 9

Local and global existence

9.1 Local existence

contraction mapping

9.2 Global existence

a priori estimates gronwall inequality

9.3 Weak convergence

Part IV

Nonlinear equations

Chapter 10

Chapter 11

Hamilton-Jacobi equations

optimal control viscosity solution

Chapter 12

Conservation laws

shocks NS