Analysis VIII/Linear Differential Equations

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On this course

Purpose: We learn basics of pseudodiffernetial operators.

Grading: The grade will be decided by a final report. The report problems will be distributed later in this course.

References:

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- H. Kumano-go, "Pseudo-Differential Operators", MIT Press
- A. Martinez, "An Introduction to Semiclassical and Microlocal Analysis", Springer
- M.A. Shubin, "Pseudodifferntial Operators and Spectral Analysis", Springer
- M. Zworski, "Semiclassical Analysis", Amer. Math. Soc.
- N. Lerner, "Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators", Springer

Chapter 1

Oscillatory Integrals

§ 1.1 Introduction

∘ Notation

In this course we use the notation

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}.$$

We usually let $d \in \mathbb{N}$ be the dimension of the **configuration space**. For any **multi-index** $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we define its **length** and **factorial** as

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = (\alpha_1!) \cdot \cdots \cdot (\alpha_d!),$$

respectively. In addition, for any $\alpha, \beta \in \mathbb{N}_0^d$ we let

$$\alpha \le \beta \quad \overset{\text{def}}{\Longleftrightarrow} \quad \alpha_j \le \beta_j \text{ for all } j = 1, \ldots, d,$$

and define the **binomial coefficient** as

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \quad \text{if } 0 \le \beta \le \alpha, \qquad \binom{\alpha}{\beta} = 0 \quad \text{otherwise,}$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_d - \beta_d)$.

For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \qquad \partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}.$$

Moreover, we introduce the notation

$$D_j = -\mathrm{i}\partial_j, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}.$$

Then, in particular, we have

$$D^\alpha = (-\mathrm{i})^{|\alpha|}\partial^\alpha.$$

Thoughout the course for any $x, \xi \in \mathbb{R}^d$ we write simply

$$x\xi = x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d, \quad x^2 = x \cdot x, \quad |x| = \sqrt{x \cdot x},$$

and we adopt the **Fourier transform** and its inverse defined as extensions from

$$\mathcal{F}u(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} u(x)\,\mathrm{d}x \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d),$$
$$\mathcal{F}^* f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\xi} f(\xi)\,\mathrm{d}\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d),$$

respectively. Note, in particular, for any $u, v \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$

$$(u, v)_{L^2} = (\mathcal{F}u, \mathcal{F}v)_{L^2}, \quad \mathcal{F}^*\xi^\alpha \mathcal{F}u = D^\alpha u,$$

where $(\cdot, \cdot)_{L^2}$ denotes the L^2-**inner product**, being linear and conjugate-linear in the first and second entries, respectively.

Problem. 1. (**Binomial theorem**) Show for any $\alpha \in \mathbb{N}_0^d$ and $x, y \in \mathbb{R}^d$

$$(x+y)^\alpha = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta; \quad \text{In particular,} \quad \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} = 2^{|\alpha|}.$$

2. (**Leibniz rule**) Show for any $\alpha \in \mathbb{N}_0^d$ and $f, g \in C^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha(fg) = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g).$$

◦ **Partial differential operators**

Consider a partial differential operator (PDO) on $\mathbb{R}^d$:

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \qquad a_\alpha \in C^\infty(\mathbb{R}^d).$$

If we let

$$a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

then we can write for any $u \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$

$$Au(x) = a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a(x,\xi) u(y) \, \mathrm{d}y \mathrm{d}\xi.$$

The last integral makes sense even if we replace the polynomial $a(x, \xi)$ in ξ by a **symbol** growing at most polynomially in $\xi \in \mathbb{R}^d$. That is a **pseudodifferential operator** (ΨDO, or PsDO). We are going to develop a pseudodifferential calculus for an appropriate symbol class, and discuss its applications.

Remark. The last integral has to be interpreted as an iterated integral; The integrand is not integrable in (y, ξ). However, we can also justify it as an **oscillatory integral**, as discussed in the following section.

§ 1.2 Oscillatory Integrals

For any $x \in \mathbb{R}^d$ we let

$$\langle x \rangle = (1 + x^2)^{1/2} \in C^\infty(\mathbb{R}^d).$$

Lemma 1.1. 1. For any $x \in \mathbb{R}^d$

$$\frac{1}{\sqrt{2}}(1 + |x|) \le \langle x \rangle \le 1 + |x|.$$

2. For any $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that for any $x \in \mathbb{R}^d$

$$|\partial^\alpha \langle x \rangle| \le C_\alpha \langle x \rangle^{1-|\alpha|}.$$

3. (**Peetre's inequality**) For any $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$\langle x + y \rangle^s \le 2^{|s|} \langle x \rangle^{|s|} \langle y \rangle^s.$$

Proof. 1, 2. We omit the proofs.

3. By the assertion 1 we can estimate

$$\begin{aligned}\langle x+y\rangle &\le 1+|x+y| \le 1+|x|+|y| \\ &\le (1+|x|)(1+|y|) \le 2\langle x\rangle\langle y\rangle.\end{aligned}$$

This implies the assertion for $s \ge 0$. The same estimate also implies

$$\langle y\rangle^{-1} \le 2\langle x\rangle\langle x+y\rangle^{-1}.$$

If we replace x by $-x$, and then y by $x+y$, it follows that

$$\langle x+y\rangle^{-1} \le 2\langle x\rangle\langle y\rangle^{-1},$$

which implies the assertion for $s \le 0$. Hence we are done. $\square$

◦ **Oscillatory Integrals**

For any $m, \delta \in \mathbb{R}$ we define the set of **amplitude functions** as

$$A_\delta^m(\mathbb{R}^d) = \left\{ a \in C^\infty(\mathbb{R}^d);\ \forall \alpha \in \mathbb{N}_0^d \ \sup_{x \in \mathbb{R}^d} \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)| < \infty \right\}.$$

For any $k \in \mathbb{N}_0$ define a **seminorm** $|\cdot|_k$ on $A_\delta^m(\mathbb{R}^d)$ as

$$|a|_k = |a|_{k, A_\delta^m} = \sup \left\{ \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)|;\ |\alpha| \le k,\ x \in \mathbb{R}^d \right\}.$$

Remark. Obviously, $A_\delta^m(\mathbb{R}^d)$ is a **Fréchet space** with respect to the family $\{|\cdot|_k\}_{k \in \mathbb{N}_0}$ of seminorms.

Theorem 1.2. Let Q be a non-degenerate real symmetric matrix of order d, and let $m \in \mathbb{R}$ and $\delta < 1$. Then for any $a \in A^m_\delta(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$ there exists the limit

$$I_Q(a) := \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\, \mathrm{d}x, \tag{$\spadesuit$}$$

and it is independent of choice of $\chi \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exist $k \in \mathbb{N}_0$ and $C > 0$ such that for any $a \in A^m_\delta(\mathbb{R}^d)$

$$|I_Q(a)| \le C|a|_{k, A^m_\delta}.$$

Remark. The last bound implies $I_Q \colon A^m_\delta(\mathbb{R}^d) \to \mathbb{C}$ is continuous.

Proof. Noting that for any $x, y \in \mathbb{R}^d$

$$y\partial\Big(\frac{xQx}{2}\Big) = \frac{1}{2}\sum_{j=1}^{d} y_j(e_jQx + xQe_j) = yQx,$$

we can deduce

$$\mathrm{e}^{\mathrm{i}xQx/2} = {}^tL\mathrm{e}^{\mathrm{i}xQx/2}; \quad {}^tL = \langle x\rangle^{-2}\Big(1 + xQ^{-1}D\Big).$$

Substitute the above identity into the integrand of (♠), and integrate it by parts. Repeat this precedure, and we obtain

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}L^k\Big(\chi(\epsilon x)a(x)\Big)\,\mathrm{d}x$$

for any $k \in \mathbb{N}_0$. Since L is of the form

$$L = c_0 + \sum_{j=1}^{d} c_j\partial_j; \quad c_0 \in A^{-2}_{-1}(\mathbb{R}^d), \quad c_j \in A^{-1}_{-1}(\mathbb{R}^d),$$

there exists $C > 0$ such that for any $\epsilon \in (0,1)$ and $a \in A^m_\delta(\mathbb{R}^d)$

$$\Big| L^k \Big(\chi(\epsilon x) a(x)\Big)\Big| \le C|a|_{k,A^m_\delta} \langle x\rangle^{m-k\,\mathrm{min}\{2,1-\delta\}}. \qquad (\heartsuit)$$

We also note there exists a pointwise limit

$$\lim_{\epsilon\to+0} L^k\Big(\chi(\epsilon x)a(x)\Big) = L^k a(x).$$

Then, if we choose $k \in \mathbb{N}_0$ such that $m - k\,\mathrm{min}\{2, 1-\delta\} < -d$, it follows by the Lebesgue convergence theorem that

$$I_Q(a) = \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} L^k a(x)\,\mathrm{d}x.$$

Certainly the last expression is independent of χ. Combined with ($\heartsuit$), it also implies the asserted bound. We are done. $\square$

Remarks. 1. The limit (♠) from Theorem 1.2 is called an **oscillatory integral**, and is denoted simply by

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\,\mathrm{d}x.$$

The notation is compatible with the case $a \in L^1(\mathbb{R}^d)$.

2. We can also define the oscillatory integral as

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} L^k a(x)\,\mathrm{d}x,$$

where L^k is from the proof of Theorem 1.2. Practically, in order to compute an oscillatory integral we may implement *any* formal integrations by parts until the integrand gets integrable, see also Lemma 1.3.3.

Lemma 1.3. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$.

1. For any $c \in \mathbb{R}^d$
$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \mathrm{e}^{\mathrm{i}cQc/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}yQy/2}\Big(\mathrm{e}^{\mathrm{i}cQy} a(y+c)\Big)\,\mathrm{d}y.$$

2. For any real invertible matrix P of order d
$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py)|\det P|\,\mathrm{d}y.$$

3. For any $\alpha \in \mathbb{N}_0^d$
$$\int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) a(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \partial^\alpha a(x)\,\mathrm{d}x.$$

Proof. 1 and 2. We can prove 1 and 2 very similarly, and here we disucss only 2. Let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then by definition of the oscillatory integral

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\, \mathrm{d}x &= \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) a(x)\, \mathrm{d}x \\ &= \lim_{\epsilon \to +0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} \chi(\epsilon P y) a(Py) |\det P|\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py) |\det P|\, \mathrm{d}y.\end{aligned}$$

This implies the assertion.

3. Similarly to the above, let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \Big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\Big) \chi(\epsilon x) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon\to+0} (-1)^{|\alpha|} \Bigg[\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \chi(\epsilon x) \partial^\alpha a(x)\,\mathrm{d}x \\
&\qquad\qquad + \sum_{|\beta|\ge 1} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \Big(\partial^\beta \chi(\epsilon x)\Big)\Big(\partial^{\alpha-\beta} a(x)\Big)\,\mathrm{d}x \Bigg].
\end{aligned}
$$

For the second integral in the above square brackets we can further implement integrations by parts, e.g., by using L from the proof of Theorem 1.2, and then we can verify that it converges to 0 as $\epsilon \to +0$. Thus we obtain the assertion. $\square$

§ 1.3 Expansion Formula

Definition. Let Q be a non-degenerate real symmetric matrix of order d, and let $u \in \mathcal{S}'(\mathbb{R}^d)$. We define

$$\mathrm{e}^{\mathrm{i}DQD/2}u = \mathcal{F}^* \mathrm{e}^{\mathrm{i}\xi Q\xi/2}\mathcal{F}u \in \mathcal{S}'(\mathbb{R}^d).$$

Theorem 1.4. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$. Then

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}yQ^{-1}y/2} a(x+y)\, \mathrm{d}y.$$

Remark. As for $a \in A^m_\delta(\mathbb{R}^d)$ we can compute pointwise values of $\mathrm{e}^{\mathrm{i}DQD/2}a$ as an oscillatory integral.

Theorem 1.5. There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ and $N \in \mathbb{N}$

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \sum_{k=0}^{N-1} \frac{\mathrm{i}^k}{2^k k!}(DQD)^k a(x) + R_N(a)$$

with

$$\left|R_N(a)\right| \leq \frac{C}{2^N N!} \sum_{|\alpha| \leq d+1} \left\|\partial^{\alpha}(DQD)^N a\right\|_{L^1}.$$

Lemma 1.6. Let Q be a non-degenerate real symmetric matrix of order d. Then

$$\left(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\right)(\xi) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2}}\mathrm{e}^{-\mathrm{i}\xi Q^{-1}\xi/2}.$$

Proof. Step 1. We first let $d = 1$. Since $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is continuous, we can proceed as

$$\begin{aligned}\left(\mathcal{F}\mathrm{e}^{\mathrm{i}Qx^2/2}\right)(\xi) &= \lim_{\epsilon\to+0}\left(\mathcal{F}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)x^2/2}\right)(\xi)\\ &= \lim_{\epsilon\to+0}\left(\epsilon-\mathrm{i}Q\right)^{-1/2}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)^{-1}\xi^2/2}\\ &= \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|Q|^{1/2}}\mathrm{e}^{-\mathrm{i}Q^{-1}\xi^2/2}.\end{aligned}$$

Thus the assertion for $d = 1$ is verified.

Step 2. There exists an invertible real matrix P such that

$$^tPQP = \mathrm{diag}(I_p, -I_q),$$

where I_p, I_q are the identity matrices of order $p, q \in \mathbb{N}_0$ with $p + q = d$, respectively. Changing variables as $x = Py$ and spliting $y = (y', y'') \in \mathbb{R}^p \times \mathbb{R}^q$, we can compute

$$\begin{aligned}
&\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\Big)(P^{-1}\eta) \\
&= \lim_{\epsilon\to+0}\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\mathrm{e}^{-\epsilon x({}^tP^{-1}P^{-1})x}\Big)(P^{-1}\eta) \\
&= \lim_{\epsilon\to+0}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}y\eta}\mathrm{e}^{\mathrm{i}(y'^2-y''^2)/2}\mathrm{e}^{-\epsilon y^2}|\det P|\,\mathrm{d}y \\
&= |\det P|\mathrm{e}^{\mathrm{i}\pi(\mathrm{sgn}\,Q)/4}\mathrm{e}^{-\mathrm{i}(\eta'^2-\eta''^2)/2},
\end{aligned}$$

where in the last equality we use the result from Step 1. Finally let $\eta = P\xi$, and we obtain the assertion. $\square$

Proof of Theorem 1.4. Let $a \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. Then it follows by change of variables, the Plancherel theorem and Lemma 1.6

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}DQD/2}a(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\xi Q\xi/2}\Big(\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}y\xi}a(x+y)\,\mathrm{d}y\Big)\,\mathrm{d}\xi \\
&= \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn}Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}}\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}yQ^{-1}y/2}a(x+y)\,\mathrm{d}y.
\end{aligned}$$

Then, since the right-hand side of the asserted identity is continuous on $A_{\delta}^{m}(\mathbb{R}^d)$ byTheorem 1.2, we obtain the assertion. $\square$

Proof of Theorem 1.5. Recall by Taylor's theorem for any $N \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\mathrm{e}^{\mathrm{i}t} = \sum_{k=0}^{N-1} \frac{(\mathrm{i}t)^k}{k!} + \frac{\mathrm{i}^N}{(N-1)!} \int_0^t \mathrm{e}^{\mathrm{i}s}(t-s)^{N-1}\, \mathrm{d}s,$$

so that we can write

$$\mathrm{e}^{\mathrm{i}\xi Q\xi/2} = \sum_{k=0}^{N-1} \frac{(\mathrm{i}\xi Q\xi)^k}{2^k k!} + r_N(\xi); \quad |r_N(\xi)| \le \frac{|\xi Q\xi|^N}{2^N N!}.$$

Substitute the above expansion into the definition of $\mathrm{e}^{\mathrm{i}DQD/2}a$ and implement the Fourier inversion formula, and then

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \sum_{k=0}^{N-1} \frac{\mathrm{i}^k}{2^k k!}(DQD)^k u(x) + R_N(a)$$

with

$$|R_N(a)| \le \frac{1}{(2\pi)^{d/2} 2^N N!} \int_{\mathbb{R}^d} \Big| \Big(\mathcal{F}(DQD)^N a \Big)(\xi) \Big| \, \mathrm{d}\xi.$$

Finally it suffices to show that for any $v \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$

$$\|\mathcal{F}v\|_{L^1} \le C \sum_{|\alpha| \le d+1} \|\partial^\alpha v\|_{L^1}.$$

However, it is clear since

$$\mathcal{F}v(\xi) = (2\pi)^{-d/2} \langle \xi \rangle^{-2(d+1)} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} (1 + \xi D)^{d+1} v(x) \, \mathrm{d}x.$$

Thus we are done. □

Corollary 1.7 (Stationary phase theorem). There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$, $N \in \mathbb{N}$ and $h > 0$

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/(2h)} a(x)\,\mathrm{d}x$$
$$= \sum_{k=0}^{N-1} \frac{(2\pi)^{d/2} h^{k+d/2} \mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2}(2\mathrm{i})^k k!} \Big((DQ^{-1}D)^k a\Big)(0) + R_N(a,h)$$

with

$$\Big|R_N(a,h)\Big| \le \frac{Ch^{N+d/2}}{|\det Q|^{1/2} 2^N N!} \sum_{|\alpha| \le d+1} \Big\|\partial^\alpha (DQ^{-1}D)^N a\Big\|_{L^1}.$$

Proof. The assertion is clear by Theorems 1.4 and 1.5. $\square$

Remarks. 1. As $h \to +0$, the rapid oscillatory factor $\mathrm{e}^{\mathrm{i}xQx/(2h)}$ cancels contributions from the amplitude a. However, the oscillation is slightly milder at the stationary point $x = 0$ of the phase function. This is why the behavior of a at around $x = 0$ dominates the asymptotics.

2. The **semiclassical parameter** $h > 0$, rooted in the **Planck constant**, plays a fundamental role in the **semiclassical analysis**. However, in this course we do not discuss it.

Problem. Show the following extended version of the "pointwise Fourier inversion formula": For any $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$ and for any $\alpha \in \mathbb{N}_0^d$ and $x' \in \mathbb{R}^d$

$$(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \xi^\alpha a(x) \, \mathrm{d}x \mathrm{d}\xi = (D^\alpha a)(x').$$

Remark. This is an oscillatory integral on $\mathbb{R}^{2d} = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, not on $\mathbb{R}^d$, with a phase function

$$-x\xi = 4^{-1}\Big((x-\xi)^2 - (x+\xi)^2\Big)$$

and an amplitude $\mathrm{e}^{\mathrm{i}x'\xi} \xi^\alpha a(x) \in A^{|\alpha|+\max\{m,0\}}_{\max\{\delta,0\}}(\mathbb{R}^{2d})$.

Solution. By Lemma 1.3 it suffices to prove the assertion for $\alpha = 0$. By definition of oscillatory integrals, take any $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then we can compute

$$\begin{aligned}
&(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon\to+0} (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \chi(\epsilon x)\chi(\epsilon\xi) a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon\to+0} (2\pi\epsilon)^{-d/2} \int_{\mathbb{R}^{d}} (\mathcal{F}\chi)((x-x')/\epsilon)\chi(\epsilon x) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon\to+0} (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} (\mathcal{F}\chi)(\eta)\chi(\epsilon(x'+\epsilon\eta)) a(x'+\epsilon\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} a(x')(\mathcal{F}\chi)(\eta)\,\mathrm{d}\eta \\
&= a(x').
\end{aligned}$$

Hence we are done. $\square$

Chapter 2

Pseudodifferential Calculus

§ 2.1 Pseudodifferential Operators

Definition. Let $m, \rho, \delta \in \mathbb{R}$. We denote by $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ the set of all the functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying that for any $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C > 0$ such that for any $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}.$$

We call $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ the **Kohn–Nirenberg** (or **Hörmander**) **symbol class**, and its element a **symbol of order** m. In addition, we set

$$S^\infty_{\rho,\delta}(\mathbb{R}^{2d}) = \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^{2d}), \quad S^{-\infty}(\mathbb{R}^{2d}) = \bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^{2d}).$$

We often write $S^m(\mathbb{R}^{2d}) = S^m_{1,0}(\mathbb{R}^{2d})$ for short.

Remarks. 1. In order to have an appropriate pseudodifferential calculus available it is typically assumed that

$$0 \leq \delta < \rho \leq 1, \quad \text{or} \quad 1 - \rho \leq \delta < \rho \leq 1.$$

2. Some authors define $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ as the set of all the functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying that for any $\alpha, \beta \in \mathbb{N}_0^d$ and $K \Subset \mathbb{R}^d$ there exists $C > 0$ such that for any $(x, \xi) \in K \times \mathbb{R}^d$

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|}.$$

3. There are many other variations of symbol classes, including semiclassical ones.

4. The symbol class $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ is a Fréchet space with respect to a family of seminorms given by

$$|a|_j = |a|_{j,S^m_{\rho,\delta}} = \sup\Big\{\langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}\Big|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\Big|;\\ |\alpha|+|\beta|\le j,\ (x,\xi)\in\mathbb{R}^{2d}\Big\}.$$

Problem. 1. Show that, if $l \le m$, $\sigma \ge \rho$ and $\epsilon \le \delta$, then

$$S^l_{\sigma,\epsilon}(\mathbb{R}^{2d}) \subset S^m_{\rho,\delta}(\mathbb{R}^{2d}).$$

2. Show that for any $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$, $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ and $\alpha,\beta \in \mathbb{N}_0^d$

$$\partial_x^\alpha\partial_\xi^\beta a \in S^{m+\delta|\alpha|-\rho|\beta|}_{\rho,\delta}(\mathbb{R}^{2d}), \quad ab \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Solution. We omit it. $\square$

Examples. 1. Consider

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha; \quad a_\alpha \in C^\infty(\mathbb{R}^d).$$

If a_α for all $|\alpha| \le m$ satisfy that for any $\beta \in \mathbb{N}_0^d$

$$\sup_{x\in\mathbb{R}^d} |\partial^\beta a_\alpha(x)| < \infty, \tag{$\heartsuit$}$$

then obviously $a \in S^m(\mathbb{R}^{2d})$. Even if a_α dissatisfy ($\heartsuit$), take any $\chi \in C_c^\infty(\mathbb{R}^d)$, and then

$$\chi(x)a(x,\xi) \in S^m(\mathbb{R}^{2d}).$$

We can still discuss local properties of a PDO by letting $\chi(x) = 1$ in a neighborhood of a point of our interest.

2. For any $m \in \mathbb{R}$ we have $\langle \xi \rangle^m \in S^m(\mathbb{R}^{2d})$.

3. Assume $a \in C^\infty(\mathbb{R}^{2d})$ is **positively homogeneous of degree** $m \in \mathbb{R}$ in $|\xi| \geq 1$, i.e., for any $x \in \mathbb{R}^d$, $|\xi| \geq 1$ and $t \geq 1$

$$a(x, t\xi) = t^m a(x, \xi).$$

In addition, assume for simplicity

$$\pi_1(\operatorname{supp} a) \Subset \mathbb{R}^d,$$

where $\pi_1 \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the first projection. Then we have $a \in S^m(\mathbb{R}^{2d})$.

Definition. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$. Define the **pseudodifferential operator** $a(x, D)$ **of order** m as, for any $u \in \mathcal{S}(\mathbb{R}^d)$,

$$a(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a(x, \xi)u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

We denote

$$\Psi^m_{\rho,\delta}(\mathbb{R}^d) = \Big\{a(x, D);\ a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})\Big\},$$

and similarly for $\Psi^\infty_{\rho,\delta}(\mathbb{R}^d)$, $\Psi^{-\infty}(\mathbb{R}^d)$ and $\Psi^m(\mathbb{R}^d)$. In particular, an element of $\Psi^{-\infty}(\mathbb{R}^d)$ is called a **smoothing operator**.

Remarks. 1. Such a systematic procedure to assign operators to symbols is called a **quantization**, as in the quantum mechanics. There are various quantizations.

2. It is also common to use the notation $\mathrm{Op}(a)$ for $a(x, D)$.

3. The **semiclassical pseudodifferential operator** is defined as

$$\mathrm{Op}_h(a) = a(x, hD).$$

Here $h > 0$ is the semiclassical parameter.

4. The operator $\mathrm{e}^{\mathrm{i}DQD/2}$ from the previous chapter may be considered as a pseudodifferential operator, but the associated symbol $\mathrm{e}^{\mathrm{i}\xi Q\xi/2}$ is in a much worse class.

Theorem 2.1. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$. Then $a(x,D)$ is a continuous operator on $\mathcal{S}(\mathbb{R}^d)$.

Proof. For any $N \in \mathbb{N}_0$ we can write

$$a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle\xi\rangle^{-2N} a(x,\xi) \langle D_y\rangle^{2N} u(y)\,\mathrm{d}y\mathrm{d}\xi.$$

Here the integrand is estimated as, for any $\beta \in \mathbb{N}_0^d$,

$$\left|\partial_x^\beta \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle\xi\rangle^{-2N} a(x,\xi) \langle D_y\rangle^{2N} u(y)\right| \le C_\alpha \langle\xi\rangle^{m+|\beta|-2N},$$

and hence we can differentiate $a(x,D)u(x)$ as much as we want by retaking N be larger beforehand. Thus for any $\beta \in \mathbb{N}_0^d$

$$\begin{aligned}\partial^\beta a(x,D)u(x) = (2\pi)^{-d} \sum_{\tau \in \mathbb{N}_0^d} \binom{\beta}{\tau} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi}& \\ \cdot (\mathrm{i}\xi)^{\beta-\tau} \langle\xi\rangle^{-2N} (\partial_x^\tau a)(x,\xi) \langle D_y\rangle^{2N} u(y)\,\mathrm{d}y\mathrm{d}\xi&.\end{aligned}$$

Futhermore, by Lemma 1.3 for any $\alpha \in \mathbb{N}_0^d$

$$x^\alpha \partial^\beta a(x,D)u(x) = (2\pi)^{-d} \sum_{\tau,\sigma\in\mathbb{N}_0^d} \binom{\alpha}{\sigma}\binom{\beta}{\tau} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} y^{\alpha-\sigma}$$
$$\cdot \Big((-D_\xi)^\sigma (\mathrm{i}\xi)^{\beta-\tau} \langle\xi\rangle^{-2N} \partial_x^\tau a\Big)(x,\xi) \langle D_y\rangle^{2N} u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

Therefore for any $k \in \mathbb{N}_0$ by letting N be sufficiently large we can find $C > 0$ and $l \in \mathbb{N}_0$ such that for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$|a(x,D)u|_{k,\mathcal{S}} \leq C|u|_{l,\mathcal{S}}.$$

This implies the assertion. $\square$

§ 2.2 Asymptotic Summation

Theorem 2.2. For each $j \in \mathbb{N}_0$ given $a_j \in S^{m_j}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$m := m_0 > m_1 > m_2 > \cdots > m_j \to -\infty \quad \text{as } j \to \infty,$$

and $\rho, \delta \in \mathbb{R}$. There exists $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that for any $k \in \mathbb{N}_0$

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d}). \tag{$\spadesuit$}$$

Such a is unique up to $S^{-\infty}(\mathbb{R}^{2d})$. Moreover, one can choose $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$\operatorname{supp} a \subset \overline{\left(\bigcup_{j=0}^{\infty} \operatorname{supp} a_j\right)}. \tag{$\heartsuit$}$$

Definition. Under the setting of Theorem 2.2 we write

$$a \sim \sum_{j=0}^{\infty} a_j,$$

and call it the **asymptotic sum** or **asymptotic expansion**. In addition, when $a_0 \not\equiv 0$, we call a_0 the **principal symbol** of a, or of $A := a(x, D)$, and often write it as

$$\sigma(A) = a_0.$$

Note the principal symbol is not unique by definition, and the above identity has to be understood up to lower order errors.

Proof. Step 1. Fix $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi(\xi) = \begin{cases} 0 & \text{for } |\xi| \le 1, \\ 1 & \text{for } |\xi| \ge 2, \end{cases}$$

and we construct $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ of the form

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi)$$

with

$$1 > \epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \to +0.$$

Note the above sum is locally finite, and hence is locally bounded and smooth. Note also, then, ($\heartsuit$) is automatically satisfied.

Step 2. Here we are going to choose

$$1 > \epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \to +0$$

such that for any $j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| + |\beta| \le j$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \le 2^{-j} \langle \xi \rangle^{m_j + 1 + \delta|\alpha| - \rho|\beta|} \tag{$\clubsuit$}$$

For that we note for any $j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C_{j\alpha\beta} > 0$ such that uniformly in $\epsilon \in (0,1)$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon \xi) a_j(x,\xi))\right| \le C_{j\alpha\beta} \langle \xi \rangle^{m_j + \delta|\alpha| - \rho|\beta|}, \tag{$\diamondsuit$}$$

since

$$\epsilon \le 2|\xi|^{-1} \le 4(1 + |\xi|)^{-1} \ \text{ on } \operatorname{supp}\Big(\partial_\xi^\gamma (\chi(\epsilon \xi))\Big) \text{ with } |\gamma| \ge 1.$$

However, since

$$1 \le \epsilon|\xi| \le \epsilon\langle\xi\rangle \quad \text{on} \ \operatorname{supp}\chi(\epsilon\xi),$$

we can further deduce uniformly in $\epsilon \in (0,1)$

$$\left|\partial_x^\alpha \partial_\xi^\beta(\chi(\epsilon\xi)a_j(x,\xi))\right| \le C_{j\alpha\beta}\epsilon\langle\xi\rangle^{m_j+1+\delta|\alpha|-\rho|\beta|}.$$

Now we first choose

$$\epsilon_0 < \min\Big\{1, (C_{000})^{-1}\Big\},$$

and then (♣) is satisfied for $j=0$. Next, suppose we have found $\epsilon_0, \ldots, \epsilon_{j-1}$ as claimed, and then it suffices to choose

$$\epsilon_j < \min\Big\{j^{-1}, \epsilon_{j-1}, 2^{-j}(C_{j\alpha\beta})^{-1};\ |\alpha|+|\beta| \le j\Big\}.$$

Thus by induction we obtain $\epsilon_0, \epsilon_1, \ldots$ as claimed.

Step 3. Here we prove a from Steps 1 and 2 belongs to $S^m_{\rho,\delta}(\mathbb{R}^{2d})$. In fact, for any $\alpha, \beta \in \mathbb{N}_0^d$, if we choose $k \in \mathbb{N}_0$ such that

$$k \geq |\alpha| + |\beta| \quad \text{and} \quad m_k + 1 \leq m,$$

then by ($\diamondsuit$) and ($\clubsuit$)

$$\begin{aligned}
\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| &\leq \sum_{j=0}^{k-1} \left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \\
&\quad + \sum_{j=k}^{\infty} \left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \\
&\leq \sum_{j=0}^{k-1} C_{j\alpha\beta} \langle \xi \rangle^{m_j + \delta|\alpha| - \rho|\beta|} + \sum_{j=k}^{\infty} 2^{-j} \langle \xi \rangle^{m_j + 1 + \delta|\alpha| - \rho|\beta|} \\
&\leq C'_{\alpha\beta} \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|}.
\end{aligned}$$

This implies the claim.

Step 4. Let us verify (♠). For any $k \in \mathbb{N}_0$ decompose

$$a - \sum_{j=0}^{k-1} a_j = \sum_{j=0}^{k-1} \Big(\chi(\epsilon_j \xi) - 1\Big) a_j(x,\xi) + \sum_{j=k}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi).$$

Then the first sum on the right-hand side belongs to $S^{-\infty}(\mathbb{R}^{2d})$ since it vanishes for $|\xi| \geq 2/\epsilon_k$, while the second to $S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d})$ similarly to Step 3. Thus the claim follows.

Step 5. Finally we discuss the uniqueness up to $S^{-\infty}(\mathbb{R}^{2d})$. If both of $a, b \in S^{m}_{\rho,\delta}(\mathbb{R}^{2d})$ satisfy (♠), then for any $k \in \mathbb{N}_0$

$$a - b = \left(a - \sum_{j=0}^{k-1} a_j\right) - \left(b - \sum_{j=0}^{k-1} a_j\right) \in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d}),$$

so that $a - b \in S^{-\infty}(\mathbb{R}^{2d})$. Thus we are done. $\square$

Definition. Let $m \in \mathbb{R}$. $a \in S^m(\mathbb{R}^{2d})$, or $a(x,D) \in \Psi^m(\mathbb{R}^d)$, are **classical** (or **polyhomogeneous**) if a has an expansion

$$a \sim \sum_{j=0}^{\infty} a_j$$

such that, for each $j \in \mathbb{N}_0$, $a_j \in S^{m-j}(\mathbb{R}^{2d})$ is positively homogeneous of degree $m-j$ in $\xi \neq 0$. Although we actually need modifications around $\xi = 0$, we often abuse notation as above. We denote

$$S^m_{\mathrm{cl}}(\mathbb{R}^{2d}) = \Big\{ a \in S^m(\mathbb{R}^{2d});\ a \text{ is classical} \Big\},$$
$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^d) = \Big\{ a(x,D);\ a \in S^m_{\mathrm{cl}}(\mathbb{R}^{2d}) \Big\}.$$

Remark. Under homogeneity the principal symbol is unique.

Examples. 1. Any partial differential operator of order $m \in \mathbb{N}_0$:

$$A = a(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where $a_\alpha \in C^\infty(\mathbb{R}^d)$ has bounded derivatives, is classical. The principal symbol is given by

$$\sigma(A)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

2. For any $m \in \mathbb{R}$ the operator $\langle D \rangle^m \in \Psi^m(\mathbb{R}^{2d})$ is classical. In fact, by the Taylor expansion for any $|\xi| > 1$

$$\begin{aligned} \langle \xi \rangle^m &= |\xi|^m \Big(1 + |\xi|^{-2}\Big)^{m/2} \\ &= \sum_{j=0}^{\infty} \frac{(m/2)(m/2-1)\cdots(m/2-j+1)}{j!} |\xi|^{m-2j}. \end{aligned}$$

Problem (Borel's theorem). Show that, given $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{N}_0^d$, there exists $f \in C^\infty(\mathbb{R}^d)$ such that for any $\alpha \in \mathbb{N}_0^d$

$$(\partial^\alpha f)(0) = c_\alpha.$$

Solution. Step 1. Fix $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| \ge 2, \end{cases}$$

and we construct $f \in C^\infty(\mathbb{R}^d)$ of the form

$$f(x) = \sum_{j=0}^{\infty} \chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha; \quad 1 < R_0 < R_1 < \cdots < R_j \to \infty.$$

Note the above sum is locally finite on $\mathbb{R}^d \setminus \{0\}$, hence locally bounded there. In addition, it is obviously finite at $x = 0$.

Step 2. Here we are going to choose

$$1 < R_0 < R_1 < \cdots < R_j \to \infty$$

such that any $j \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$ with $|\beta| \le j$

$$\left| \partial^\beta \left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right) \right| \le 2^{-j} |x|^{j-1-|\beta|}$$

Note that, thanks to supporting property of $\chi(Rx)$, for any $j \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$ there exists $C_{j\beta} > 0$ such that uniformly in $R \ge 1$

$$\left| \partial^\beta \left(\chi(Rx) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right) \right| \le C_{j\beta} R^{-1} |x|^{j-1-|\beta|}.$$

Then we can discuss similarly to the proof of Theorem 2.2. We omit the details.

Step 3. Now let $\beta \in \mathbb{N}_0^d$, and consider the following series:

$$\sum_{j=0}^{\infty} \partial^\beta\left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha\right) = \sum_{j=0}^{|\beta|} \partial^\beta\left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha\right) + \sum_{j=|\beta|+1}^{\infty} \partial^\beta\left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha\right).$$

The sum is pointwise finite on $\mathbb{R}^d$ similarly to Step 1. Moreover, it is uniformly and absolutely convergent due to the result from Step 2. Since $\beta \in \mathbb{N}_0^d$ is arbitrary, we can conclude $f \in C^\infty(\mathbb{R}^d)$ by induction, and differetiate it under the summation. Thus

$$(\partial^\beta f)(0) = \sum_{j=0}^{\infty} \partial^\beta\left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha\right)\Bigg|_{x=0} = c_\beta.$$

We are done. $\square$