Commutative Algebra

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Part I

Affine schemes

Chapter 1

Nullstellensatz

1.1 Radicals

1.2 Affine varieties

1.1 (Weak nullstellensatz).

1.2 (Noether normalization theorem).

$\mathrm{Spec}(\mathbb{C} + (x^2 - 1)\mathbb{C}[x]) = \{0, \mathbb{C}\}$.

Chapter 2

Primary decomposition

2.1 Primary ideals

2.1 (Primary ideals). Let A be a ring. An ideal $\mathfrak{q}$ of A is called *primary* if $A/\mathfrak{q}$ is non-zero and every zero-divisor of $A/\mathfrak{q}$ is nilpotent. Let $\mathfrak{q}$ be a primary ideal of A.

(a) The radical $r(\mathfrak{q})$ is the smallest prime ideal of A containing $\mathfrak{q}$.

2.2 Uniqueness theorems

Noether-Lasker

2.3 Gröbner basis

2.2 (Buchberger algorithm).

Exercises

primary vs prime powers primary vs prime radical

Chapter 3

Localization

3.1

For $f \in A$, $A_f := A[f^{-1}]$.

For $\mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1} A$ is a local ring.

local ring extension of ideals

Since $S^{-1}A$ is a flat A-module for a multiplicative set $S \subset A$, the localization functor $S^{-1} := - \otimes_A S^{-1}A : \mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ is always exact.

$$\operatorname{Spec}(A_{\mathfrak{p}}) \longleftrightarrow \{\mathfrak{q} \in \operatorname{Spec} A : \mathfrak{q} \subset \mathfrak{p}\}.$$

3.2 Valuation

DVR dedekind domains

Part II

Dimension theory

Chapter 4

4.1. Let A be a ring. A strictly increasing finite sequence $(\mathfrak{p}_i)_{i=0}^n$ of prime ideals of A is called a *prime chain* of length n in A. The *height* of a prime ideal $\mathfrak{p}$ of A is the supremum of the length of prime chains containing $\mathfrak{p}$:

$$\operatorname{ht}(\mathfrak{p}) := \sup\{n : \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p},\ \mathfrak{p}_i \in \operatorname{Spec} A\}.$$

The *Krull dimension* or simply the *dimension* of A is the supremum of the heights of prime ideals of A:

$$\dim A := \sup\{\operatorname{ht}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}.$$

4.2 (Krull Hauptidealsatz)**.**

4.3 (Hilbert polynomials)**.**

4.4 (Minimal number of generators of modules)**.** The *embedding dimension* of A is defined $\operatorname{edim} A := \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

For a noetherian local ring A, $\operatorname{edim} A = \dim A$ is the minimal number of generators of $\mathfrak{m}$.

Chapter 5

Homological dimensions

5.1 (First change of rings)**.** Let A be a ring, and let $A' := A/xA$ be the quotient for a regular element $x \in A$. If M' is a non-zero A'-module with $\mathrm{pd}_{A'}(M') < \infty$, then $\mathrm{pd}_A(M') = \mathrm{pd}_{A'}(M') + 1$.

Proof. We introduce the inductive hypothesis on $p' := \mathrm{pd}_{A'}(M')$. Consider a short exact sequence of A'-modules

$$0 \to K' \to F' \to M' \to 0,$$

where F' is a free A'-module.

On the other hand, we get $\mathrm{pd}_{A'}(K') = p' - 1$ because for any A'-module N' we have $\mathrm{Ext}^i_{A'}(K', N') = 0$ for $i \geq p'$ by the long exact sequence of A'-modules

$$0 \cong \mathrm{Ext}^i_{A'}(F', N') \to \mathrm{Ext}^i_{A'}(K', N') \to \mathrm{Ext}^{i+1}_{A'}(M', N') \cong 0, \qquad i \geq p'.$$

Therefore, the inductive hypothesis deduces

$$\mathrm{pd}_A(M') = \mathrm{pd}_A(K') + 1 = \mathrm{pd}_{A'}(K') + 1 = (p' - 1) + 1 = p'.$$

□

$\mathrm{gldim} A = \mathrm{pd}_A(A/\mathfrak{m})$.

5.2. Let A be a noetherian local ring.

Let $d := \dim A$, $g := \mathrm{gldim} A = \mathrm{pd}_A(k)$, and $e := \mathrm{edim} A$.

(a) If A is regular, then A has finite global dimension.

(b) If A has finite global dimension, then A is regular.

Proof. (a)

(b) It suffices to show $e = d$.

We want to find $x \in A$ such that

(i) x is regular in A with $\mathrm{gldim} A' < \infty$,

(ii) $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.

where we denote by $A' := A/xA$ the quotient local ring with the maximal ideal $\mathfrak{m}'$ and the residue field $k' = k$. Since the first condition implies $d = d' + 1$ by the first change of rings and the second condition implies that

$d' = \mathrm{pd}_{A'}(k) < \infty$ implies that we can apply the first change of rings.

We have , $d' = e'$ by the inductive hypothesis, and $e' + 1 = e$ by the fact $\ker(\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$

□