Harmonic Analysis

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Contents

Part I

Fourier analysis

Chapter 1

Fourier series

1.1 Fourier series in L^p spaces

1.1.

$$\|\widehat{f}\|_{\ell^1(\mathbb{Z})} \lesssim \|f\|_{W^{1,1+\varepsilon}(\mathbb{T})}.$$

Inversion theorem is an approximation problem given by $\mathcal{F}^*\mathcal{F} = \lim_{n\to\infty} \mathcal{F}_n^*\mathcal{F}$. The condition $\widehat{f} \in \ell^1(\mathbb{Z})$ is a condition just for defining $\mathcal{F}^*\widehat{f}$ wihtout using distribution theory, and it does not affect the inversion phenomena. The approximation, in other words, can be seen as an extension method for $\mathcal{F}^* : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ on $c_0(\mathbb{Z})$. Note that $\mathcal{F}_n^*$ on $c_0(\mathbb{Z})$ cannot be bounded directly without distribution theory, but $\mathcal{F}_n^*\mathcal{F}$ on $L^p(\mathbb{T})$ can be bounded well.

1.2 Summability methods

- If $\mathcal{F}_n^*$ is the standard partial sum, then $\mathcal{F}_n^*\mathcal{F}$ is the Dirichlet kernel.
- If $\mathcal{F}_n^*$ is the Cesàro mean, then $\mathcal{F}_n^*\mathcal{F}$ is the Fejér kernel.
- If $\mathcal{F}_r^*$ is the Abel sum, then $\mathcal{F}_r^*\mathcal{F}$ is the Poisson kernel.
- In Fourier transform, we often use the Gauss-Weierstrass kernel.

The injectivity of $\mathcal{F}$ is not an easy problem, which comes from the inversion theorem.

1.2 (Dirichlet kernel)**.** The *Dirichlet kernel* is a function $D_n : \mathbf{T} \to \mathbb{R}$ defined by

$$D_n = \widehat{\mathbf{1}_{|k|\le n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = \mathbf{1}_{|k|\le n}.$$

This is because they are invariant under inverse, in other words, they are even.

(a)

$$D_n(x) = \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.$$

(b) If $f \in \mathrm{Lip}(\mathbf{T})$, then $D_n * f \to f$ pointwisely as $n \to \infty$.

(c)

$$\|D_n\|_{L^1(\mathbf{T})} \gtrsim \log n.$$

Proof.

$$\begin{aligned} D_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\ &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}. \end{aligned}$$

(c) By (2) $\sin x \le x$ for $x \in [0, \pi/2]$, (3) change of variable,

$$\begin{aligned} \|D_n\|_{L^1(\mathbf{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\frac{2n+1}{2}x}{\sin\frac{1}{2}x}\right| dx \\ &\ge \frac{2}{\pi}\int_0^{\pi} \frac{|\sin\frac{2n+1}{2}x|}{x}\, dx \\ &= \frac{2}{\pi}\int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &= \frac{2}{\pi}\sum_{k=0}^{2n}\int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &\ge \frac{2}{\pi}\sum_{k=0}^{2n}\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi}\, dx \\ &\ge \frac{4}{\pi^2}\sum_{k=0}^{2n}\frac{1}{1+k} \\ &\ge \frac{4}{\pi^2}\log(2n+2). \end{aligned}$$

..? □

1.3 (Fejér kernel)**.** The *Fejér kernel* is

(a)

$$K_n(x) = \frac{1}{n+1}\frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}.$$

Proof. Since

$$\begin{aligned} D_n(x) = &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2}, \end{aligned}$$

by telescoping, we get

$$\begin{aligned} \sum_{k=0}^{n} D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{\sin^2\frac{n+1}{2}x}{\sin^2\frac{1}{2}x}. \end{aligned}$$

$\square$

Two important results from Fejér kernel:

1. If $f(x-)$, $f(x+)$ exist and $S_n f(x)$ converges, then $S_n f(x) \to \frac{1}{2}(f(x-) + f(x+))$.
2. (If $f \in L^1(\mathbf{T})$, then $\sigma_n f \to f$ a.e.)
3. If $f \in L^1(\mathbf{T})$, then $S_n f \to f$ in L^1 and L^2.
4. If f is continuous and $\widehat{f} \in L^1(\mathbb{Z})$, then $S_n f \to f$ uniformly.
5. Since $\sigma_n f$ is a trigonometric polynomial, the set of trigonometric polynomials are dense in $L^1(\mathbf{T})$ and $L^2(\mathbf{T})$.

1.3 Pointwise convergence of Fourier series

BV function: Dini, Jordan's criterion

1.4 (Riemann localization principle).

Exercises

1.5 (Gibbs phenomenon).

1.6 (Du Bois-Reymond function).

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

2.1 (Riemann-Lebesgue lemma).

Lp extension

Gaussian function computation: differential equation method, contour integral method inversion theorem

2.2 (Plancherel theorem).

2.2 Distributions

2.3 (Cauchy principal value). indented contour, imaginary shift, Feynman's trick

Exercises

2.4 (Sampling theorem).

$$\mathcal{F}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(\xi) = \mathrm{sinc}(\xi/2)$$

$\mathrm{sinc} \in L^{1+\varepsilon}(\mathbb{R})$.

2.5 (Poisson summation formula).

2.6 (Uncertainty principle).

2.7 (Multipole expansion). Let ρ be a compactly supported distribution on $\mathbb{R}^d$. We want to investigate the limit behavior of $\rho(\varepsilon^{-1}x)$ as $\varepsilon \to 0$. More precisely, we want to compute an integer $k \geq d$ such that $\lim_{\varepsilon\to 0+} \varepsilon^{-k}\rho(\varepsilon^{-1}x)$ defines a distribution supported at $\{0\}$, and the coefficients of derivatives of Dirac measures.

We need to introduce quantities called monopole, dipole, quadrapole, octupole, etc.

(a) A distribution supported on $\{0\}$ is a linear combination of the Dirac measure and its derivatives.

(b)

Problems

1. Find all $\alpha > 0$ such that
$$\lim_{x\to\infty} x^{-\alpha} \int_0^x f(y)\,dy = 0$$
for all $f \in L^3([0,\infty))$.

Chapter 3

Hilbert transform

3.1 Harmonic conjugate

3.2 Kernel representation

3.3 Fourier series in L^p space

Part II

Singular integral operators

Chapter 4

Calderón-Zygmund theory

4.1 Convolution type operators

4.1 (Calderón-Zygmund decomposition of sets). Let $f \in L^1(\mathbb{R}^d)$. Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf := \sup_n E_n|f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|}\int_{Q_x} |f| > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \le \frac{1}{\lambda}\|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\mathbb{R}^d\setminus\Omega)} \le \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|}\int_{Q_x} |f| \le 2^d\lambda.$$

4.2 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|}\int_{Q_x} |f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

(a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.

(b) $\|b\|_{L^1} \le 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof. $\square$

4.3 (L^p boundedness of Calderón-Zygmund operators). Let $T : C_c^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that $Tf(x) = K * f(x)$ for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \operatorname{supp} f$. We say T is called a *Calderón-Zygmund* operator if

(i) T is L^2-bounded: we have

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2},$$

(ii) T satisfies the *Hörmander condition*: we have

$$\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx \lesssim 1$$

for every $y>0$.

Let $f=g+b=g+\sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2}\|Tg\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2}\|g\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2}\|g\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda}\int_{\mathbb{R}^d\setminus\Omega^*} |Tb(x)|\,dx \leq \frac{2}{\lambda}\sum_i \int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\operatorname{supp} b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y)b_i(y)\,dy = \int_{Q_i} [K(x-y)-K(x)]b_i(y)\,dy,$$

and

$$\int_{\mathbb{R}^d\setminus Q_i^*} |Tb_i(x)|\,dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d\setminus Q_i^*} |K(x-y)-K(x)|\,dx\,dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.4 (Hölder boundedness of Calderón-Zygmund operators).

4.2 Truncated integrals

Homogeneous kernels

4.3 A_p weights

4.4 Bounded mean oscillation

Exercises

4.5 (Size and cancellation condition). Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$. We say the condition $|K(x)| \lesssim |x|^{-d}$ for $x \neq 0$ as the *size condition*, and say the condition $\int_{r<|x|<R} K(x)\,dx = 0$ for all $0 < r < R < \infty$ as the *cancellation condition*. If K satisfies the size, cancellation, and Hörmander condition, then it is L^2 bounded, hence Calderón-Zygmund.

4.6 (Gradient size condition). Let $|\nabla K(x)| \lesssim |x|^{-d-1}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

4.7 (Riesz potential).

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Almost orthogonality

Carleson measures, paraproducts

6.1 Coltar lemma

6.2 $T(1)$ theorem

Part III

Oscillatory integral operators

Chapter 7

Oscillatory integrals

7.1 (Justification of oscillatory integral)**.** For ϕ, we define a linear functional $O_\phi : A^m_\delta(\mathbb{R}^d) \to \mathbb{C}$ such that

$$O_\phi(a) := \int_{\mathbb{R}^d} e^{i\phi(x)} a(x)\, dx$$

for all $a \in A^m_\delta(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. As a notation, we will use the above integral to denote the value of O_ϕ even for $a \in A^m_\delta(\mathbb{R}^d) \setminus L^1(\mathbb{R}^d)$.

(a) O_ϕ is well-defined and continuous.

(b) The change of variables is justified as follows:

(c) The integral by parts is justified as follows: for $\phi \in$ and $a \in$, we have

$$\int_{\mathbb{R}^d} e^{i\phi(y)} i\partial\phi(y) a(x+y)\, dy = -\int_{\mathbb{R}^d} e^{i\phi(y)} \partial a(x+y)\, dy.$$

(d) The Fubini theorem is justified as follows:

(e) The Fourier inversion is justified as follows:

$$a(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(y)\, dy\, d\xi.$$

Proof. (a) The most difficult part is the construction and the computation of L and its transpose.

(e) Note that the function $(y,\xi) \mapsto a(y)$ belongs to $A^{m'}_\delta(\mathbb{R}^{2d})$ since □

7.2 (Point evaluation of multiplier)**.** We want to show the following point evaluation holds with previously justified oscillatory integral: for each x at which the left-hand side is continuous, we have

$$\Phi(D)a(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\phi(y)} a(x+y)\, dy$$

for all $a \in A^m_\delta$, where $\Phi := \mathcal{F}^* e^{i\phi}$.

7.3 (Stationary phase approximation)**.**

Proof. □

7.4 (Van der Corput lemma)**.**

Dispersive equations and strichartz estimates

Exercises

7.5 (Fresnel phase). We compute L with a specific example

Proof.

$$(1 + xQ^{-1}D)e^{\frac{i}{2}xQx} = \langle x\rangle^2 e^{\frac{i}{2}xQx}.$$

The transpose of $\langle x\rangle^{-2}(1 + xQ^{-1}D)$ is $\langle x\rangle^{-2}(1 + di - 2ix^2 - xD)$ for $Q = I$.

Note that $\langle x\rangle^{-2n}\langle D\rangle^{2n}$ is self-adjoint.

Let Q be a non-degenerate symmetric bilinear form on $\mathbb{R}^d$. Consider a multiplier operator $e^{\frac{i}{2}DQD}$: $\mathcal{S} \to \mathcal{S}$ such that

$$e^{\frac{i}{2}DQD}a(x) := \mathcal{F}^* e^{\frac{i}{2}\xi Q\xi}\mathcal{F}a(x).$$

(a) If $a \in A^m_\delta(\mathbb{R}^d)$, then the pointwise evaluation is given by the oscillatory integral.

$$e^{\frac{i}{2}DQD}a(x) = (2\pi)^{-d}\frac{e^{\frac{i\pi\,\mathrm{sgn}\,Q}{4}}}{|\det Q|^{\frac{1}{2}}}\int_{\mathbb{R}^d} e^{-\frac{i}{2}yQ^{-1}y}a(x+y)\,dy.$$

(b)

$$e^{\frac{i}{2}DQD}a(x) = \sum_{k=0}^{n}\frac{i^k}{2^k k!}(DQD)^k a(x) + r_n(x)$$

□

Chapter 8

Foureir restriction

Kakeya Bochner-Riesz Geometric measure theory

Chapter 9

Part IV

Pseudo-differential operators

Chapter 10

Pseudo-differential calculus

10.1

10.1 (Hörmander symbol classes). Let $m,\rho,\delta\in\mathbb{R}$. The *Hörmander class* $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ of symbols is the set of smooth functions $a\in C^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\lesssim_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}$$

for each $\alpha,\beta\in\mathbb{Z}^d_{\ge0}$.

(a) Fréchet space

10.2 (Asymptotic expansion). Let $\rho,\delta\in\mathbb{R}$ and $(m_k)_{k=0}^\infty\subset\mathbb{R}$ be a sequence with m_0 and $m_k\downarrow-\infty$. Given $a_k\in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d})$, we want to construct $a\in S^{m_0}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a-\sum_{k=0}^n a_k\in S^{m_{n+1}}_{\rho,\delta}(\mathbb{R}^{2d}).\tag{†}$$

The symbol a_0 is called the *principal symbol* of a, or the operator $\mathrm{Op}^t(a)$.

Let $\chi\in C_c^\infty(\mathbb{R}^d_\xi,[0,1])$ be a cutoff function such that

$$\chi(\xi)=\begin{cases}1, & \text{if } |\xi|\le1\\ 0, & \text{if } |\xi|\ge2\end{cases}.$$

(a) If $a\in S^m_{\rho,\delta}$, then $\chi(\varepsilon\xi)a(x,\xi)$ is uniformly bounded in $S^m_{\rho,\delta}$ for $\varepsilon\in(0,1)$ if $\rho\le1$.

(b) There is $a\in S^{m_0}_{\rho,\delta}$ such that (†) if $\rho\le1$.

Proof. (a) On the support of $\xi\mapsto\chi(\varepsilon\xi)$ holds $\langle\xi\rangle<2|\xi|\le4\varepsilon^{-1}$ because $1<\varepsilon^{-1}$, so for each $\alpha,\beta\in\mathbb{Z}^d_{\ge0}$ we have

$$\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta(\chi(\varepsilon\xi)a(x,\xi))| &= |\sum_\tau\binom{\beta}{\tau}\partial_\xi^{\beta-\tau}(\chi(\varepsilon\xi))\partial_x^\alpha\partial_\xi^\tau a(x,\xi)|\\
&= |\sum_\tau\binom{\beta}{\tau}\varepsilon^{|\beta|-|\tau|}\partial_\xi^{\beta-\tau}\chi(\varepsilon\xi)\partial_x^\alpha\partial_\xi^\tau a(x,\xi)|\\
(\because\langle\xi\rangle\le4\varepsilon^{-1})\quad &\le\sum_\tau\binom{\beta}{\tau}(4\langle\xi\rangle^{-1})^{|\beta|-|\tau|}|\partial_\xi^{\beta-\tau}\chi(\varepsilon\xi)||\partial_x^\alpha\partial_\xi^\tau a(x,\xi)|\\
&\lesssim\sum_\tau\binom{\beta}{\tau}\langle\xi\rangle^{-(|\beta|-|\tau|)}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\tau|}\\
(\because\rho\le1)\quad &\le\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|}.
\end{aligned}$$

(b) Because we have $\varepsilon^{-1} \le \langle\xi\rangle$ on the support of $1-\chi(\varepsilon\xi)$, for each k we can take a sequence ε_k small enough such that

$$\max_{\substack{\alpha,\beta\in\mathbb{Z}^d_{\ge 0}\\ |\alpha|+|\beta|\le k}} |\partial_x^\alpha\partial_\xi^\beta((1-\chi(\varepsilon_k\xi))a_k(x,\xi))| \le 2^{-k}\langle\xi\rangle^{m_k+1+\delta|\alpha|-\rho|\beta|}.$$

We may assume $\varepsilon_k \downarrow 0$ so that the following sum is locally finite:

$$a(x,\xi) := \sum_{k=0}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi).$$

If we choose n such that $m_0 \ge m_{n+1}+1$, then in the expansion

$$a(x,\xi) = \sum_{k=0}^{n}(1-\chi(\varepsilon_k\xi))a_k(x,\xi) + \sum_{k=n+1}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi)$$

the first sum clearly belongs to $S^{m_0}_{\rho,\delta}$ and so is the second sum because

$$\begin{aligned}|\partial_x^\alpha\partial_\xi^\beta \sum_{k=n+1}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi)| &\le \sum_{k=n+1}^{\infty} 2^{-k}\langle\xi\rangle^{m_{k+1}+1+\delta|\alpha|-\rho|\beta|}\\ &\le \langle\xi\rangle^{m_{n+1}+1+\delta|\alpha|-\rho|\beta|}\\ &\le \langle\xi\rangle^{m_0+\delta|\alpha|-\rho|\beta|}\end{aligned}$$

for every $\alpha,\beta\in\mathbb{Z}^d_{\ge 0}$. Therefore, $a\in S^{m_0}_{\rho,\delta}$.

Write

$$(a-\sum_{k=0}^{n}a_k)(x,\xi) = \sum_{k=0}^{n}\chi(\varepsilon_k\xi)a_k(x,\xi) + \sum_{k=n+1}^{\infty}(1-\chi(\varepsilon_k\xi))a_k(x,\xi).$$

The first sum belongs to $S^{-\infty}$ because it is compactly supported, and we can also show that the second sum belongs to $S^{m_{n+1}}_{\rho,\delta}$ by decomposing with n' such that $m_{n+1} \ge m'_n + 1$ and by considering the multiplication with a cutoff remains in the same symbol class. □

10.3 (Quantization)**.** The t-quantization of a symbol a is the pseudo-differential operator $\mathrm{Op}^t(a)$ on $\mathcal{S}(\mathbb{R}^d_x)$ defined by

$$\mathrm{Op}^t(a)f(x) := (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a((1-t)x+ty,\xi)f(y)\,dy\,d\xi$$

for all $f\in\mathcal{S}(\mathbb{R}^d)$. Kohn-Nirenberg calculus for $t=0$, Weyl calculus for $t=\frac{1}{2}$.

(a) $\mathrm{Op}^0(a) : \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ is continuous for $a\in\mathcal{S}'(\mathbb{R}^d)$.

(b) $\mathrm{Op}^0(a) : \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ is continuous for $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ if $\delta\le 1$.

Proof. (b) Since $\langle D_y\rangle^2$ is a self-adjoint partial differential operator, for any $n\in\mathbb{Z}_{\ge 0}$ we have

$$\begin{aligned}\mathrm{Op}^0(a)f(x) &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a(x,\xi)f(y)\,dy\,d\xi\\ (\because D_y e^{i(x-y)\xi} = \xi e^{i(x-y)\xi})\quad &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} \langle\xi\rangle^{-2n}\langle D_y\rangle^{2n}e^{i(x-y)\xi}a(x,\xi)f(y)\,dy\,d\xi\\ (\because \text{IBP})\quad &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y)\,dy\,d\xi.\end{aligned}$$

The derivatives of the integrand is integrable with respect to ξ for a sufficiently large n with $m+|\beta|-2n<-d$ because

$$\begin{aligned}
&|\partial_x^\beta(e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))| \\
&\qquad = |\sum_\tau \binom{\beta}{\tau}(i\xi)^{\beta-\tau}e^{i(x-y)\xi}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)\langle D_y\rangle^{2n}f(y)| \\
&\qquad \le \sum_\tau \binom{\beta}{\tau}\langle\xi\rangle^{|\beta|-|\tau|}\langle\xi\rangle^{-2n}|\partial_x^\tau a(x,\xi)||\langle D_y\rangle^{2n}f(y)| \\
(\because a\in S^m_{\rho,\delta})\quad &\qquad \lesssim \sum_\tau \binom{\beta}{\tau}\langle\xi\rangle^{|\beta|-|\tau|}\langle\xi\rangle^{-2n}\langle\xi\rangle^{m+\delta|\tau|}|\langle D_y\rangle^{2n}f(y)| \\
(\because \delta\le 1)\quad &\qquad \lesssim \langle\xi\rangle^{m+|\beta|-2n}|\langle D_y\rangle^{2n}f(y)|,
\end{aligned}$$

so the partial derivative ∂_x commutes with the integral. Since

$$x^\alpha e^{i(x-y)\xi}=(y+D_\xi)^\alpha e^{i(x-y)\xi}=\sum_\sigma\binom{\alpha}{\sigma}y^{\alpha-\sigma}D_\xi^\sigma e^{i(x-y)\xi},$$

we have an expansion

$$\begin{aligned}
x^\alpha\partial_x^\beta\operatorname{Op}^0(a)f(x) &= x^\alpha\partial_x^\beta\int_{\mathbb{R}^{2d}}e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}}x^\alpha\partial_x^\beta(e^{i(x-y)\xi}\langle\xi\rangle^{-2n}a(x,\xi)\langle D_y\rangle^{2n}f(y))\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}}\sum_{\sigma,\tau}\binom{\alpha}{\sigma}\binom{\beta}{\tau}y^{\alpha-\sigma}D_\xi^\sigma e^{i(x-y)\xi}(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)\langle D_y\rangle^{2n}f(y)\,dy\,d\xi \\
&= \int_{\mathbb{R}^{2d}}\sum_{\sigma,\tau}\binom{\alpha}{\sigma}\binom{\beta}{\tau}e^{i(x-y)\xi}(-D_\xi)^\sigma[(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)]y^{\alpha-\sigma}\langle D_y\rangle^{2n}f(y)\,dy\,d\xi.
\end{aligned}$$

Here

$$\sup_{x\in\mathbb{R}^d}|(-D_\xi)^\sigma[(i\xi)^{\beta-\tau}\langle\xi\rangle^{-2n}\partial_x^\tau a(x,\xi)]|$$

is integrable with respect to ξ for sufficiently large n, so with this n we have

$$\sup_{x\in\mathbb{R}^d}|x^\alpha\partial_x^\beta\operatorname{Op}^0(a)f(x)|\lesssim\sum_{\sigma\le\alpha}\sup_{y\in\mathbb{R}^d}|y^{\alpha-\sigma}\langle D_y\rangle^{2n}f(y)|$$

for each $\alpha,\beta\in\mathbb{Z}^d_{\ge 0}$ and all $f\in\mathcal{S}(\mathbb{R}^d)$, which implies $\operatorname{Op}^0(a)f\in\mathcal{S}(\mathbb{R}^d)$. □

10.4 (Change of quantization). Let $m\in\mathbb{R}$, .

(a) $\operatorname{Op}^t(a)=\operatorname{Op}^0(e^{itD_xD_\xi}a)$.

(b) $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ if and only if $e^{itD_xD_\xi}a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$, if $0\le\delta\le\rho\le 1$ and $\delta<1$.

(c) We have the formal adjoint

$$\operatorname{Op}^t(a)^*=\operatorname{Op}^{1-t}(\overline{a}).$$

In particular, we have $\operatorname{Op}^t(a):\mathcal{S}'\to\mathcal{S}'$.

Proof. (a) Note that

$$\begin{aligned}
\mathrm{Op}^t(a)f(x) &= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi}a((1-t)x+ty,\xi)f(y)\,dy\,d\xi \\
(\because \text{Inversion on } \mathbb{R}^{2d})\quad &= (2\pi)^{-3d}\int_{\mathbb{R}^{4d}} e^{i(x-y)\xi}e^{i((1-t)x+ty)x^*+i\xi\xi^*}\widehat{a}(x^*,\xi^*)f(y)\,dx^*\,d\xi^*\,dy\,d\xi \\
&= (2\pi)^{-3d}\int_{\mathbb{R}^{4d}} e^{i(x-y+\xi^*)\xi}\widehat{a}(x^*,\xi^*)e^{i((1-t)x+ty)x^*}f(y)\,dx^*\,d\xi^*\,dy\,d\xi \\
(\because \text{Inversion on } \mathbb{R}^{d})\quad &= -(2\pi)^{-2d}\int_{\mathbb{R}^{2d}} \widehat{a}(x^*,y-x)e^{i((1-t)x+ty)x^*}f(y)\,dx^*\,dy \\
(\because [\xi^*/y-x])\quad &= -(2\pi)^{-2d}\int_{\mathbb{R}^{2d}} e^{i(x+t\xi^*)x^*}\widehat{a}(x^*,\xi^*)f(x+\xi^*)\,dx^*\,d\xi^*.
\end{aligned}$$

(b) We have the oscillatory integral

$$e^{itD_xD_\xi}a(x,\xi) = (2\pi)^{-d}|t|^{-d}\int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta}a(x+y,\xi+\eta)\,dy\,d\eta.$$

Enough to show

$$|\int_{\mathbb{R}^{2d}} e^{-it^{-1}y\eta}a(x+y,\xi+\eta)\,dy\,d\eta| \lesssim \langle\xi\rangle^m.$$

Fix ξ and $\delta\leq\rho$ □

10.5 (Moyal product). Let $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b\in S^l_{\rho,\delta}(\mathbb{R}^{2d})$.

(a) there exists a unique function $a\#^t b\in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^t(x,D)b^t(x,D) = (a\#^t)^t(x,D).$$

(b) It is concretely described by

$$(a\#^t b)(x,\xi) = (2\pi)^{-2}\int_{\mathbb{R}^{4d}} e^{-i(y\eta-z\zeta)}a(x+tz,\xi+\eta)b((1-t)y+x,\xi+\zeta)\,dy\,d\eta\,dz\,d\zeta.$$

(c) If $\delta<\rho$, then

$$a\#^t b(x,\xi) \sim \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{1}{i^k k!}(\partial_y\partial_\eta-\partial_z\partial_\zeta)^k a((1-t)x+tz,\eta)b(tx+(1-t)y,\zeta)\Big|_{\substack{y=z=x\\ \eta=\zeta=\xi}}.$$

10.6 (Parametirx and elliptic operators).

10.2

10.7 (Calderón-Vaillancourt theorem).

Chapter 11

Semiclassical analysis

For parameters $0 \le \lambda \le 1$ and $h > 0$, let

$$\widehat{a}\psi(x) := \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}\langle x-y,\xi\rangle} a((1-\lambda)x + \lambda y, \xi)\psi(y)\, dy\, d\xi.$$

For example, regardless of h and λ,

$$\widehat{\xi}\psi(x) = \frac{h}{i}\psi'(x)$$

and

$$\widehat{H}\psi(x) = -h^2\Delta\psi(x) + V(x)\psi(x),$$

where $V : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ and $H : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ such that

$$H(x,\xi) := |\xi|^2 + V(x).$$

$$\frac{d}{dt}a(t) = \{a(t), H\} = X_H a(t)$$

$$\frac{d}{dt}\widehat{a}(t) = \frac{d}{dt}e^{\frac{i}{h}t\widehat{H}}\widehat{a}e^{-\frac{i}{h}t\widehat{H}} = -\frac{i}{h}[\widehat{a}(t), \widehat{H}]$$

11.1 Heisenberg group

11.2 Phase space transforms

Chapter 12

Microlocal analysis