Foundations of Calculus

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Preface

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Part I

Sequences

Chapter 1

Metric spaces

1.1 Metric spaces

1.1 (Definition of metric spaces)**.** Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, (nondegeneracy)

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$, (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

A pair (X, d) of a set X and a metric on X is called a *metric space*. We often write it simply X.

(a) A normed space X is a metric space with a metric defined by $d(x, y) := \|x - y\|$.

(b) A subset of a metric space is a metric space with a metric given by restriction.

1.2 (System of open balls)**.** A metric is often misunderstood as something that measures a distance between two points and belongs to the study of geoemtry. The main function of a metric is to make a system of small balls, sets of points whose distance from specified center points is less than fixed numbers. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly.

Note that taking either ε or δ in analysis really means taking a ball of the very radius. Investigation of the distribution of open balls centered at a point is now an important problem.

Let X be a metric space. A set of the form

$$\{y \in X : d(x, y) < \varepsilon\}$$

for $x \in X$ and $\varepsilon > 0$ is called an *open ball centered at x with radius ε* and denoted by $B(x, \varepsilon)$ or $B_\varepsilon(x)$.

1.3 (Convergence and continuity in metric spaces)**.** Let $\{x_n\}_n$ be a sequence of points on a metric space (X, d). We say that a point x is a *limit* of the sequence or the sequence *converges to x* if for arbitrarily small ball $B(x, \varepsilon)$, we can find n_0 such that $x_n \in B(x, \varepsilon)$ for all $n > n_0$. If it is satisfied, then we write

$$\lim_{n\to\infty} x_n = x,$$

or simply $x_n \to x$ as $n \to \infty$. We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

A function $f : X \to Y$ between metric spaces is called *continuous at $x \in X$* if for any ball $B(f(x), \varepsilon) \subset Y$, there is a ball $B(x, \delta) \subset X$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. The function f is called *continuous* if it is continuous at every point on X.

(a) A sequence x_n in a metric space X converges to $x \in X$ if and only if $d(x_n, x)$ converges to zero.

(b) Let $f : X \to Y$ be a function between two metric spaces. If there is a constant C such that $d(x, y) \leq C d(f(x), f(y))$ for all x and y in X, then f is continuous. In this case, f is particularly called *Lipschitz continuous* with the *Lipschitz constant* C.

1.4 (Separable metric spaces)**.** separable iff second countable iff lindelof

1.2 Normed spaces

banach space

1.3 Open sets and closed sets

convergence, limit point

1.4 Compact sets

Bolzano-Weierstrass

1.5 Connected sets

Exercises

Problems

Chapter 2

Real sequences

2.1 Monotone sequences

preserving inequalities limsup and liminf monotone convergence

2.2 Extended real numbers

2.1 (Operations in the extended real numbers). We can extend addition (except $\infty + (-\infty)$), subtraction, multiplication (except $\infty \times 0$), division (except dividing by zero).

2.2 (Limits in the extended real numbers).

2.3 Asymptotic analysis

sufficiently large asymptotic expressions growth and decay

Approximate sequences($\varepsilon/3$)

2.3 (Change of limits).

$$|a_n - a| \leq |a_n - b_{mn}| + |b_{mn} - b_m| + |b_m - a|$$

$$\lim_m \sup_n |a_n - b_{mn}| = 0$$

$$\lim_n |b_{mn} - b_m| = 0$$

$$a_n = b_{mn} + c_{mn} \leq b_{mn} + \varepsilon$$

Exercises

2.4.

2.5 (Newton method).

Problems

1. Show that every real sequence $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.

Chapter 3

Series

3.1 Absolute convergence

3.1 (Unconditional convergence).

3.2 Convergence tests

comparison limit comparison cauchy condensation integral....

ratio root

3.2 (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_kB_k - A_{k-1}B_{k-1}$$

$$\sum_{m<k\le n} A_k b_k = A_nB_n - A_mB_m - \sum_{m<k\le n} a_k B_{k-1}.$$

abel test

3.3 (Dirichlet test).

3.4 (Mertens' theorem). If $\sum_{k=0}^\infty a_k$ converges to A absolutely and $\sum_{k=0}^\infty b_k$ converges to B, then their Cauchy product $\sum_{k=0}^\infty c_k$ with $c_k := \sum_{l=0}^k a_l b_{k-l}$ converges to AB.

(a) We have

$$\lim_{m\to\infty} \sup_n \sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l = 0.$$

(b) We have for each m that

$$\lim_{n\to\infty} \sum_{k=1}^m \sum_{l=n-k+1}^n a_k b_l = 0$$

Proof. Let

$$A_n := \sum_{k=0}^n a_k,\ B_n := \sum_{k=0}^n b_k, \quad \text{and} \quad C_n := \sum_{k=0}^n c_k.$$

As $m \to \infty$.

$$\Big|\sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l\Big| \le \sum_{k=m+1}^n |a_k| \Big|\sum_{l=n-k+1}^n b_l\Big| = \sum_{k=m+1}^n |a_k||B_n - B_{n-k}| \lesssim \sum_{k=m+1}^\infty |a_k| \to 0.$$

For fixed m, as $n \to \infty$,

$$\Big|\sum_{k=0}^{m} \sum_{l=n-k+1}^{n} a_k b_l\Big| \le \sum_{k=0}^{m} |a_k| \Big|\sum_{l=n-k+1}^{n} b_l\Big| = \sum_{k=0}^{m} |a_k||B_n - B_{n-k}| \to \sum_{k=0}^{m} |a_k||B - B| = 0.$$

We will prove

$$A_n B_n - C_n = \sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l \to 0$$

as $n \to \infty$. For $\varepsilon > 0$, take m such that

$$|\sup_n \sum_{k=m+1}^{n} \sum_{l=n-k+1}^{n} a_k b_l| < \varepsilon.$$

Then for every n we have

$$|\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l| \le \varepsilon + |\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l|.$$

Taking limits $n \to \infty$ and $\varepsilon \to 0$ in order, we are done. □

Exercises

3.5 (Cesàro mean)**.**

3.6 (Recursive sine sequence)**.** Let $a_{n+1} = \sin a_n$ and $a_n = 1$. We can use $\sin x = x - \frac{x^3}{6} + O(x^5)$.

$$a_n = \sqrt{3} n^{-\frac{1}{2}} - \frac{3\sqrt{3}}{20} n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}).$$

3.7 (Convergence rates of recursive sequences)**.** If $a_{n+1} = a_n - f(a_n)$, $f(0) = 0$, $f(x) > 0$ for $0 < x < \varepsilon$, $f \in C^2$? then

$$f'(a_n) \sim \lim_{x \to 0+} \frac{f'(x)^2}{f''(x) f(x)} \frac{1}{n}.$$

Problems

1. If $a_n \to 0$, then $\frac{1}{n} \sum_{k=1}^{n} a_k \to 0$.
2. If $a_n \ge 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ also diverges.
3. Show that if $a_n \ge 0$ and $\sum a_n < \infty$, then there are sequences $b_n \downarrow 0$ and $\sum c_n < \infty$ such that $a_n = b_n c_n$. (Very special case of the Cohen factorization)

Part II

Functions

Chapter 4

Continuity

4.1 Intermediate and extreme value theorems

left and right limits semicontinuous

4.2 Various continuities

Lipschitz uniform cauchy

Exercises

Problems

1. The set of local minima of a convex real function is connected.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The equation $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.
3. A continuous function that takes on no value more than twice takes on some value exactly once.
4. Let f be a function that has the intermediate value property. If the preimage of every singleton is closed, then f is continuous.

Chapter 5

Differentiation

5.1 Differentiability

5.1 (L'hopital's theorem)**.**

5.2 Monotonicty and convexity

5.3 Taylor expansion

5.2 (Rolle's theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a function that is continuous on $[a,b]$ and differentiable on (a,b).

(a) If $f(a) = f(b) = 0$, then there is $c \in (a,b)$ such that $f'(c) = 0$.

(b) Suppose f is $(n+1)$-times differentiable. If $f(a) = f'(a) = \cdots = f^{(n)}(a) = 0$ and $f(b) = 0$, then there is $c \in (a,b)$ such that $f^{(n+1)}(c) = 0$.

Proof. (a) If $f \equiv 0$, then it is clear. If not, we may assume there is $x \in (a,b)$ such that $f(x) > 0$ by multiplying -1. Since f is continuous, by the extreme value theorem, there is $c \in (a,b)$ such that c attains the maximum of f. Then, $f'(c) = 0$.

(b) By the induction, we have $c_n \in (a,b)$ such that $f^{(n)}(c) = 0$. By applying Rolle's theorem (the part (a)) for $f^{(n)}$, we have $c_{n+1} \in (a, c_n)$ such that $f^{(n+1)}(c_{n+1}) = 0$. □

5.3 (Taylor theorem)**.**

5.4 Smooth functions

Exercises

5.4 (Variations on the mean value theorem)**.** Let f be a differentiable function on the unit closed interval.

(a) If $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

(b) If $f(0) = 0$ there is c such that $cf(c) = (1-c)f'(c)$.

5.5 (Dini derivatives)**.**

5.6 (Darboux theorem)**.**

Problems

1. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} f'(x) = b$, then $a = 0$.
2. Let f be a real C^2 function with $f(0) = 0$ and $f''(0) \neq 0$. Defined a function ξ such that $f(x) = xf'(\xi(x))$ with $|\xi| \leq |x|$, we have $\xi'(0) = 1/2$.
3. Let f be a C^2 function such that $f(0) = f(1) = 0$. We have $\|f\| \leq \frac{1}{8}\|f''\|$.
4. A smooth function such that for each x there is n having the nth derivative vanish is a polynomial.
5. If a real C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.
6. Let a real function f be differentiable. For $a < a' < b < b'$ there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b') - f(a') = f'(c')(b' - a')$.

Chapter 6

Integration

6.1 Riemann integral

tagged partition

6.2 Henstock-Kurzweil intergral

bounded compact support <-> lebesgue

6.3 Improper integral

6.4 Fundamental theorem of calculus for continuous functions

Exercises

Problems

1. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.
2. Find all $a > 0$ and $b > 0$ such that $\int_0^\infty x^{-b} |\tan x|^a \,dx$ converges.

*3. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

Part III

Function spaces

Chapter 7

Continuous functions

7.1 Uniform convergence

Proof. Divide the error

$$|f(x_n) - f(x)| \leq |f(x_n) - f_m(x_n)| + |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|.$$

Using the uniform convergence, we can take m such that $\|f_m - f\| < \varepsilon$, so we have

$$|f(x_n) - f(x)| < \varepsilon + |f_m(x_n) - f_m(x)| + \varepsilon.$$

Then, taking $\limsup_{n\to\infty}$ on the both-hand sides, we get

$$\limsup_{n\to\infty} |f(x_n) - f(x)| \leq \varepsilon + 0 + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ has been arbitrarily taken,

$$\lim_{n\to\infty} |f(x_n) - f(x)| = 0.$$

□

7.2 Arzela-Ascoli theorem

7.3 Stone-Weierstrass theorem

7.1 (Bernstein polynomial)**.** We want to show $\mathbb{R}[x]$ is dense in $C([0,1],\mathbb{R})$. Let $f \in C([0,1],\mathbb{R})$ and define *Berstein polynomials* $B_n(f) \in \mathbb{R}[x]$ for each n such that

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

(a) $B_n(f)$ uniformly converges to f on $[0,1]$.

(b) There is a sequence $p_n \in \mathbb{R}[x]$ with $p_n(0) = 0$ uniformly convergent to $x \mapsto |x|$ on $[-1,1]$.

Proof. (b) Let

$$B_n(x) := \sum_{k=0}^{n} \left|1 - \frac{2k}{n}\right| \binom{n}{k} (1-2x)^k (2x-1)^{n-k}.$$

Since $B_n(x) \to |x|$ uniformly on $[-1,1]$ and $B_n(0) \to 0$, we have $B_n(x) - B_n(0) \to |x|$ uniformly on $[-1,1]$. □

7.2 (Taylor series of square root)**.** We want to show the absolute value is approximated by polynomials in $C([-1,1],\mathbb{R})$ in another way. Let

$$f_n(x) := \sum_{k=0}^{n} a_k (x-1)^k$$

be the partial sum of the Taylor series of the square root function $\sqrt{x}$ at $x = 1$.

(a) By Abel's theorem, f_n uniformly converges to $\sqrt{x}$ on $[0,1]$

(b) There is a sequence $p_n \in \mathbb{R}[x]$ with $p_n(0) = 0$ uniformly convergent to $x \mapsto |x|$ on $[-1,1]$.

7.3 (Proof of Stone-Weierstrass theorem)**.** Let X be a compact Hausdorff space and $S \subset C(X,\mathbb{R})$. We say that S *separates points* if for every distinct x and y in X there is $f \in S$ such that $f(x) \neq f(y)$, and that S *vanishes nowhere* if for every x in X there is $f \in S$ such that $f(x) \neq 0$.

Let $\mathcal{A} = \overline{S\mathbb{R}[S]}$ be the real Banach subalgebra of $C(X,\mathbb{R})$ generated by S.

(a) $\mathcal{A}$ is a lattice.

(b) $\mathcal{A}$ is dense in $C(X,\mathbb{R})$.

Locally compact version, complex version

7.4. Some examples

(a) $z\mathbb{R}[z]$ is dense in $C([1,2],\mathbb{R})$.

(b) $\mathbb{C}[z]$ is dense in $C([0,1],\mathbb{C})$.

(c) $z\mathbb{C}[z,\overline{z}]$ is dense in $C(\mathbb{T},\mathbb{C})$.

Exercises

7.5 (Weierstrass' nowhere differentiable function)**.**

Problems

*1. If a sequence of real functions $f_n : [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$, then it has a uniformly convergent subsequence.

Chapter 8

Differentiable functions

8.1 Differentiable class

completeness

8.2 Hölder spaces

8.3 Analytic functions

Power series uniform convergence and absolute convergence, abel theorem? differentiation convergence of radius, complex domain sum, product, composition, reciprocal? closed under uniform convergence identity theorem

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Integrable functions

9.1

9.1 (Lebesgue criterion of Riemann integrability).

Part IV

Multivariable Calculus

Chapter 10

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10.1 Tangent spaces

10.1 (Vector fields).

10.2 Inverse function theorem

Chapter 11

Differential forms

11.1 Multilinear algebra

11.1 (Tensor product).

11.2 (Wedge product).

11.3 (One-forms).

11.4 (Multiple integral). volume forms, stone weierstrass and fubini

11.2 Vector calculus

11.5 (Exterior derivative).

11.6 (Musical isomorphisms).

11.7 (Inner product of differential forms). ONB

11.8 (Hodge star operator). Identification of 2-forms and vector fields

11.9 (Gradient, curl, and divergence).

11.10 (Potentials).

11.11 (Vector calculus identities).

Exercises

11.12 (Multivariable Taylor's theorem). Symmetric product

11.13 (Vector analysis in two dimension).

11.14 (Geometric algebra).

Chapter 12

Stokes theorems

12.1 Local coordinates

12.1 (Spherical coordinates)**.** Let $U = \mathbb{R}^3 \setminus \{(x,y,z) : x = 0,\ y \geq 0\}$.

$$(x,y,z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

for $(r,\theta,\varphi) \in (0,\infty)\times(0,\pi)\times(0,2\pi)$. Orthonormal bases are

$$\left(\partial_r,\ \frac{1}{r}\partial_\theta,\ \frac{1}{r\sin\theta}\partial_\varphi\right),$$

$$(dr,\ r\,d\theta,\ r\sin\theta\,d\varphi),$$

$$(r^2\sin\theta\,d\theta\wedge d\varphi,\ r\sin\theta\,d\varphi\wedge dr,\ r\,dr\wedge d\theta).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

Proof. Write df in the orthonormal basis

$$\begin{aligned} df &= \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial\theta}\,d\theta + \frac{\partial f}{\partial\varphi}\,d\varphi \\ &= \left(\frac{\partial f}{\partial r}\right)dr + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)r\,d\theta + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right)r\sin\theta\,d\varphi. \end{aligned}$$

After taking the Hodge star operator

$$\begin{aligned} *df &= \left(\frac{\partial f}{\partial r}\right)r^2\sin\theta\,d\theta\wedge d\varphi + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)r\sin\theta\,d\varphi\wedge dr + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right)r\,dr\wedge d\theta \\ &= r^2\sin\theta\frac{\partial f}{\partial r}\,d\theta\wedge d\varphi + \sin\theta\frac{\partial f}{\partial\theta}\,d\varphi\wedge dr + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\,dr\wedge\theta, \end{aligned}$$

the differential is computed as

$$\begin{aligned} d*df &= d\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)d\theta\wedge d\varphi + d\left(\sin\theta\frac{\partial f}{\partial\theta}\right)d\varphi\wedge dr + d\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\right)dr\wedge\theta \\ &= \left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right]dr\wedge d\theta\wedge d\varphi, \end{aligned}$$

so that we have

$$\begin{aligned}\Delta f = *d*df &= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\end{aligned}$$

□

12.2 Integration on curves and surfaces

12.2 (Line integral)**.**

12.3 (Surface integral)**.**

12.3 Stokes theorems

12.4 (Bump functions)**.**

12.5 (Partition of unity)**.**

12.6.