

Harmonic Analysis

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Part I

Fourier analysis

Chapter 1

Fourier series

1.1 Fourier series in L^p spaces

1.1.

$$\|\hat{f}\|_{\ell^1(\mathbb{Z})} \lesssim \|f\|_{W^{1,1+\varepsilon}(\mathbb{T})}.$$

Inversion theorem is an approximation problem given by $\mathcal{F}^*\mathcal{F} = \lim_{n \rightarrow \infty} \mathcal{F}_n^*\mathcal{F}$. The condition $\hat{f} \in \ell^1(\mathbb{Z})$ is a condition just for defining $\mathcal{F}^*\hat{f}$ without using distribution theory, and it does not affect the inversion phenomena. The approximation, in other words, can be seen as an extension method for $\mathcal{F}^* : \ell^1(\mathbb{Z}) \rightarrow C(\mathbb{T})$ on $c_0(\mathbb{Z})$. Note that \mathcal{F}_n^* on $c_0(\mathbb{Z})$ cannot be bounded directly without distribution theory, but $\mathcal{F}_n^*\mathcal{F}$ on $L^p(\mathbb{T})$ can be bounded well.

- If \mathcal{F}_n^* is the standard partial sum, then $\mathcal{F}_n^*\mathcal{F}$ is the Dirichlet kernel.
- If \mathcal{F}_n^* is the Cesàro mean, then $\mathcal{F}_n^*\mathcal{F}$ is the Fejér kernel.
- If \mathcal{F}_r^* is the Abel sum, then $\mathcal{F}_r^*\mathcal{F}$ is the Poisson kernel.
- In Fourier transform, we often use the Gauss-Weierstrass kernel.

The injectivity of \mathcal{F} is not an easy problem, which comes from the inversion theorem.

1.2 Pointwise convergence of Fourier series

BV function: Dini, Jordan's criterion

1.2 (Riemann localization principle).

Exercises

1.3 (Gibbs phenomenon).

1.4 (Du Bois-Reymond function).

Chapter 2

Fourier transform

2.1 Fourier transform in L^p space

2.1 (Riemann-Lebesgue lemma).

L^p extension

Gaussian function computation: differential equation method, contour integral method inversion theorem

2.2 (Plancherel theorem).

2.2 Tempered distributions

2.3 (Cauchy principal value). indented contour, imaginary shift, Feynman's trick

Exercises

2.4 (Sampling theorem).

$$\mathcal{F}\mathbf{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) = \text{sinc}(\xi/2)$$

$\text{sinc} \in L^{1+\varepsilon}(\mathbb{R})$.

2.5 (Poisson summation formula).

2.6 (Uncertainty principle).

Problems

1. Find all $\alpha > 0$ such that

$$\lim_{x \rightarrow \infty} x^{-\alpha} \int_0^x f(y) dy = 0$$

for all $f \in L^3([0, \infty))$.

Chapter 3

Part II

Singular integral operators

Chapter 4

Caldéron-Zygmund theory

4.1 Hilbert transform

4.2 Calderón-Zygmund operators of convolution type

4.1 (Calderón-Zygmund decomposition of sets). Let $E_n f$ be the conditional expectation with respect to the σ -algebra generated by dyadic cubes with side length 2^{-n} . Let $Mf = \sup_n E_n |f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|} \int_{Q_x} |f| > \lambda.$$

- (a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω .
- (b) We have an weak type estimate $|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1}$.
- (c) $\|f\|_{L^\infty(\mathbb{R}^d \setminus \Omega)} \leq \lambda$.
- (d) For $x \in \Omega$

$$\frac{1}{|Q_x|} \int_{Q_x} |f| \leq 2^d \lambda.$$

4.2 (Calderón-Zygmund decomposition of functions). Let

$$g(x) := \begin{cases} |f(x)| & , x \notin \Omega \\ \frac{1}{|Q_x|} \int_{Q_x} |f| & , x \in \Omega \end{cases}$$

and $b_i := (|f| - g)\chi_{Q_i}$ so that $|f| = g + b$ where $b = \sum_i b_i$.

- (a) $\|g\|_{L^1} = \|f\|_{L^1}$ and $\|g\|_{L^\infty} \lesssim_d \lambda$.
- (b) $\|b\|_{L^1} \leq 2\|f\|_{L^1}$ and $\int b_i = 0$.

Proof.

□

4.3 (Calderón-Zygmund operators of convolution type). Let $T : C_c^\infty(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$ be a *singular integral operator of convolution type* in the sense that there is $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap \mathcal{D}'(\mathbb{R}^d)$ such that

$$Tf(x) = \int K(x-y)f(y)dy$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$, whenever $x \notin \text{supp } f$. We say T is called a *Calderón-Zygmund operator* if

(i) T is L^2 -bounded: we have

$$\|Tf\|_{L^2} \lesssim \|f\|_{L^2},$$

(ii) T satisfies the *Hörmander condition*: we have

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \lesssim 1$$

for every $y > 0$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2 -boundedness implies

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda} \|f\|_{L^1}.$$

(c)

Proof. (a) Using the Chebyshev inequality and the Hölder inequality,

$$|\{x : |Tg(x)| > \frac{\lambda}{2}\}| \leq \frac{4}{\lambda^2} \|Tg\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^2(\Omega)}^2 \leq \frac{4C}{\lambda^2} \|g\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)}.$$

(b) Write

$$|\{x : |Tb(x)| > \frac{\lambda}{2}\} \setminus \Omega^*| \leq \frac{2}{\lambda} \int_{\mathbb{R}^d \setminus \Omega^*} |Tb(x)| dx \leq \frac{2}{\lambda} \sum_i \int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx.$$

Since $x \in \mathbb{R}^d \setminus Q_i^*$ does not belong to $\text{supp } b_i \subset Q_i$ and $\int b_i = 0$, we have

$$Tb_i(x) = \int_{Q_i} K(x-y) b_i(y) dy = \int_{Q_i} [K(x-y) - K(x)] b_i(y) dy,$$

and

$$\int_{\mathbb{R}^d \setminus Q_i^*} |Tb_i(x)| dx = \int_{Q_i} |b_i(y)| \int_{\mathbb{R}^d \setminus Q_i^*} |K(x-y) - K(x)| dx dy \lesssim \|b_i\|_{L^1}.$$

(We need to show it is valid even though b_i is not smooth)

(c)

□

4.3 L^2 -boundedness of truncated integrals

4.4 Calderón-Zygmund operators of non-convolution type

standard kernels

Exercises

4.4 (Gradient size condition). Let $|\nabla K(x)| \lesssim \frac{1}{|x|^{d+1}}$ for $x \neq 0$. Then, convolution with K is a Calderón-Zygmund operator.

Chapter 5

Littlewood-Paley theory

5.1 Littlewood-Paley decomposition

5.2 Multiplier theorems

Chapter 6

Part III

Oscillatory integral operators

Chapter 7

Stationary phase

Chapter 8

Restriction and Kekeya problems

Chapter 9

Dispersive equations

Part IV

Pseudo-differential operators

Chapter 10

10.1

$S_{\rho, \delta}^m$

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \lesssim \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}.$$

Let a be a symbol on $M = \mathbb{R}_x^d \times \mathbb{R}_\xi^d$. Then, the associated Ψ DO is

$$T_a \psi(x) := \frac{1}{(2\pi)^d} \iint e^{i\langle x-y, \xi \rangle} a(x, \xi) \psi(y) dy d\xi.$$

For parameters $0 \leq \lambda \leq 1$ and $h > 0$, let

$$\hat{a}\psi(x) := \frac{1}{(2\pi h)^d} \iint e^{\frac{i}{h}\langle x-y, \xi \rangle} a((1-\lambda)x + \lambda y, \xi) \psi(y) dy d\xi.$$

For example, regardless of h and λ ,

$$\hat{\xi}\psi(x) = \frac{h}{i}\psi'(x)$$

and

$$\hat{H}\psi(x) = -h^2 \Delta \psi(x) + V(x)\psi(x),$$

where $V : \mathbb{R}_x^d \times \mathbb{R}_\xi^d \rightarrow \mathbb{R}$ and $H : \mathbb{R}_x^d \times \mathbb{R}_\xi^d \rightarrow \mathbb{R}$ such that

$$H(x, \xi) := |\xi|^2 + V(x).$$

$$\frac{d}{dt}a(t) = \{a(t), H\} = X_H a(t)$$

$$\frac{d}{dt}\hat{a}(t) = \frac{d}{dt}e^{\frac{i}{h}t\hat{H}}\hat{a}e^{-\frac{i}{h}t\hat{H}} = -\frac{i}{h}[\hat{a}(t), \hat{H}]$$

Chapter 11

Semiclassical analysis

11.1 Quantization

11.1 (Composition of Weyl quantization).

Chapter 12

Microlocal analysis