Topological Tensor Products

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May 6, 2023

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Notation

$L(X,Y)$	the set of bounded linear operators from X to Y
$B(X,Y)$	the set of bounded bilinear forms on $X \times Y$
$F(X,Y)$	the set of continuous finite-rank linear operators from X to Y
B_X	closed unit ball of a normed space X
S_X	unit sphere of a normed space X
$X \otimes Y$	algebraic tensor product of X and Y
X^*	continuous dual space
$X^\#$	algebraic dual space

1 Universal properties

1.1 (Algebraic tensor product of vector spaces)**.** Let X and Y be vector spaces. The *algebraic tensor product* is a vector space $X \otimes Y$ with a bilinear map $\otimes : X \times Y \to X \otimes Y$ such that the following universal property: for any vector space Z and any bilinear map $\sigma : X \times Y \to Z$, there exists a unique linear map $\widetilde{\sigma} : X \otimes Y \to Z$ such that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "\sigma"'] & X \otimes Y \arrow[d, dashed, "\widetilde{\sigma}"] \\ & Z \end{tikzcd}$$

is commutative.

(a) The tensor product $X \otimes Y$ always exists.

(b) We have linear maps $L(X,Z) \otimes L(Y,W) \to L(X \otimes Y, Z \otimes W)$ and $B(L(X,Z), L(Y,Z)) \to L(X \otimes Y, Z)$.

(c) Every element $t \in X \otimes Y$ is represented as $t = \sum_{i=1}^n x_i \otimes y_i$ such that $\{x_i\}$ is linearly indpendent. In this case, if $t = 0$ then $y_i = 0$ for all i.

Proof. (a) Let T be the set of formal linear combinations of $X \times Y$, that is, an element of T has the form $\sum_{i=1}^n a_i \cdot (x_i, y_i)$ for $x_i \in X$, $y_i \in Y$, and scalars a_i. Define $T_0 \subset T$ to be a linear space spanned by the elements of the following four types:

$$\begin{gathered} (x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'), \\ (ax, y) - a(x, y), \qquad\qquad (x, ay) - a(x, y). \end{gathered}$$

Then, the quotient space T/T_0 satisfies the universal property with the bilinear map $X \times Y \to T/T_0 : (x, y) \mapsto (x, y) + T_0$. □

1.2 (Algebraic tensor product of involutive algebras)**.**

2 Banach spaces

2.1 (Subcross norms)**.**

2.2 (Injective tensor products)**.** Let X and Y be Banach spaces. Define the *injective norm* ε on $X \otimes Y$ such that

$$\varepsilon\left(\sum_{i=1}^n x_i \otimes y_i\right) := \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{Y^*}}} \left| \sum_{i=1}^n \langle x_i, x^* \rangle \langle y_i, y^* \rangle \right|.$$

We denote by $X \otimes_\varepsilon Y$ the algebraic tensor product with the injective norm, and by $X \widehat{\otimes}_\varepsilon Y$ its completion.

(a) $X \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F((X^*, w^*), (Y, w))$.

(b) $X^* \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F(X, Y)$.

2.3 (Projective tensor products)**.** Let X and Y be Banach spaces. Define the *projective norm* π on $X \otimes Y$ such that

$$\pi(t) := \inf\left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : t = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

We denote by $X \otimes_\pi Y$ the algebraic tensor product with the projective norm, and by $X \widehat{\otimes}_\pi Y$ its completion.

(a) There are natural isometric isomorphisms $(X \otimes_\pi Y)^* \cong B(X, Y) \cong L(X, Y^*)$.

(b)

2.4 (Hilbert space tensor product). Let $\varphi : H \odot K \to L(H^*, K)$. Then, $\lambda(\xi) = \|\varphi(\xi)\|$, $\gamma(\xi) = \operatorname{tr}(|\varphi(\xi)|)$, so $H \otimes_\lambda K \cong K(H^*, K)$ and $H \otimes_\gamma K \cong L^1(H^*, K)$.

2.5 (Nuclear operators).

$$X^* \otimes_\pi Y \to X^* \otimes_\varepsilon Y \xrightarrow{\sim} F(X, Y) \xrightarrow{1} K(X, Y)$$

defines

$$J : X^* \widehat{\otimes}_\pi Y \to K(X, Y).$$

Define $N(X, Y) := \operatorname{im} J$.

2.6 (Grothendieck theorem). Let Y^* be an RNP space. Then, there is an isometric isomorphism $(X \widehat{\otimes}_\varepsilon Y)^* \cong N(X, Y^*)$.

3 Approximation property

3.1 (Approximation property of locally convex spaces).

3.2 (Approximation property of Banach spaces).

3.3 (Approximation property of dual Banach spaces).

3.4 (Mazur's goose). (a) If X has a Schauder basis, then it has the approximation property.

4 Nuclear spaces

5 C*-algebras

von Neumann algebras, Haagerup tensor product