# Noncommutative geometry

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# Part I

# Noncommutative topology

# Chapter 1

# Kasparov category

## 1.1

**1.1** (Pro-C$^*$-algebras)**.** A *pro-C$^*$-algebra* is a complete topological $*$-algebra whose topology is generated by C$^*$-semi-norms. We adopt the convention that a *homomorphism* between pro-C$^*$-algebras means a continuous $*$-homomorphism.

(a) A topological $*$-algebra is a pro-C$^*$-algebra if and only if it is an inverse limit of unital C$^*$-algebras.

*Proof.* (a) Let $A$ be a pro-C$^*$-algebra. The set of continuous C$^*$-seminorms on $A$ is a directed set. Construct an inverse system... Since every C$^*$-algebra is a maximal ideal of a unital C$^*$-algebra of codimension one, we may assume that the objects in this inverse system is unital... Also, elements of $A$ are represented by coherent sequences. $\square$

# Part II

# Spectral triples

## Chapter 2

# Index theory

### 2.1 Dirac operators

### 2.2 Fredholm theory of Mishchenko and Fomenko

## Chapter 3

# Quantum metric spaces

# Chapter 4

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## 4.1 Quantum Hall effect

A wave function is a section $\psi$ of a U(1)-line bundle $\mathcal{L} \to M$, more generally, a section $\psi$ of a Hermitian $G$-vector bundle $\mathcal{V} \to M$. Galilean invariant momentum operator? We fix a connection $\nabla$ on $\mathcal{V}$, which can be forgotten if $M$ is contractible and no magnetic fields are considered. A $\mathrm{U}(N)$-gauge fixing is just a choice of a local field of orthonormal frames on the intersection of two charts of $M$, making $\mathcal{V}$ locally trivially $\mathbb{C}^N$ with the standard basis. After gauge fixing, the covariant derivative is represented locally as $\nabla_j \psi = (\partial_j - iA_j)\psi$, where $A = \sum_j A_j dx^j$ is a $\mathbb{R} = -i\mathfrak{u}(1)$-valued one-form.

The connection $\nabla$ is independent If $M$ has non-trivial holonomy (the Aharonov-Bohm flux), then it comes into play.

On $M = \mathbb{R}^2$ with a connection $\nabla$ such that the curvature form is $\mathcal{F}^\nabla = b \cdot \mathrm{vol}_M$ for some $b \in \mathbb{R} \setminus \{0\}$, then the *Landau operator*, which is a free Hamiltonian defined by the connection Laplacian or the Bochner Laplacian $H := \nabla^* \nabla = -\mathrm{tr}(\nabla^2) \geq 0$, its spectrum is known to be $\sigma(H) = (2\mathbb{N} + 1)|b|$. This discreteness of the spectrum is called the *Landau quantization*.

We consider a spin manifold $M$ with the spinor bundle $\mathcal{S}$, and the sections of $\mathcal{S} \otimes \mathcal{V} \to M$ represent the space of wave functions. curvature of a spin connection the Laplacians

# Chapter 5

# Infinite dimensional manifolds

## 5.1

Loop spaces, Loop groups