# Pseudodifferential Operators

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# 1 Day 1: April 11

Notation

$D_j = (-1)\partial_j$

$$\xi^\alpha \mathcal{F}u = \mathcal{F}D^\alpha u, \quad \xi^\alpha \mathcal{F}^* u = \mathcal{F}^*(-D)^\alpha u, \quad D^\alpha \mathcal{F}u = \mathcal{F}(-x)^\alpha u$$

Let

$$A = \sum a_\alpha(x) D^\alpha, \quad a(x,\xi) = \sum a_\alpha(x)\xi^\alpha.$$

Then,

$$\begin{aligned} Au(x) = \mathcal{F}^* M_{a(x,\xi)} \mathcal{F}u(x) \\ &= (2\pi)^{-1} \int e^{ix\xi} a(x,\xi) \int e^{-iy\xi} u(y)\, dy\, d\xi \\ &= (2\pi)^{-1} \iint e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi. \end{aligned}$$

If $a$ has a polynomial growth in $\xi$, then the integrand $e^{i(x-y)\xi}a(x,\xi)u(y)$ is not integrable in $(y,\xi)$, so we need to justify it as an oscillatory integral.

Japanese bracket, originated by Kitada or Kumano-go (akumade setsu)

$$\langle x+y\rangle^{-2} \le 4\langle x\rangle^2 \langle y\rangle^{-2}$$

$$\langle x^2 + x\rangle \asymp \langle x^2\rangle \asymp \langle x\rangle^2$$

Here we define the amplitude functions as

$$|\partial^\alpha a(x)| \lesssim \langle x\rangle^{m+\delta|\alpha|}, \quad \forall \alpha \in \mathbb{Z}^d_{\ge 0}$$

**Example** (Justification of a qudaratic oscillation)**.** Let $Q$ be a nondegenerate real quadratic form, then for $a \in A^m_\delta(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$,

$$I_Q(a) := \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} e^{i\frac{1}{2}Q(x)} \chi(\varepsilon x) a(x)\, dx$$

exists. The term $e^{i\frac{1}{2}Q(x)}$ oscillates fast where $|x| \gg 1$, the term $\chi(\varepsilon x)$ becomes flatten as $\varepsilon \to 0$. When we do integrate by parts, we integrate the oscillating term, and differentiate the cutoff and amplitude. If we differentiate the amplitude, the integrability is enhanced.

*Proof.* We compute for $Q = I$ as an example. Since

$$De^{i\frac{1}{2}x^2} = xe^{i\frac{1}{2}x^2},$$

we have

$$(1 + x\cdot D)e^{i\frac{1}{2}x^2} = (1+x^2)e^{i\frac{1}{2}x^2}.$$

Define a differential operator $L$ such that

$$^tL := \frac{1}{1+x^2} + \frac{x}{1+x^2}\cdot D = \langle x\rangle^{-2} + \langle x\rangle^{-2} x\cdot D,$$

that is,

$$\begin{aligned} L &= \frac{1}{1+x^2} - D\cdot\frac{x}{1+x^2} - \frac{x}{1+x^2}\cdot D \\ &= \frac{1}{1+x^2} + i\left(\frac{d}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right) - \frac{x}{1+x^2}\cdot D \\ &= (1+(d+2)i)\langle x\rangle^{-2} - 2\langle x\rangle^{-4} - \langle x\rangle^{-2} x\cdot D. \end{aligned}$$

Then ${}^tL$ fixes $e^{i\frac{1}{2}x^2}$, so

$$\int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2}\chi(\varepsilon x)a(x)\,dx = \int_{\mathbb{R}^d} e^{i\frac{1}{2}x^2}L^k[\chi(\varepsilon x)a(x)]\,dx$$

for any $k \geq 0$. Since

$$L = c_0(x) + c_j(x)\partial_j, \qquad c_0 \in A^{-2}_{-1}, \quad c_j \in A^{-1}_{-1},$$

we have

$$|L^k[\chi(\varepsilon x)a(x)]| \lesssim |L^k[a(x)]| \lesssim |a|_k \langle x\rangle^{m-\min\{1-\delta,2\}k}$$

bounded $\varepsilon > 0$.

Then,

$$|L^k[\chi(\varepsilon x)a(x)]| \xrightarrow{\varepsilon\to 0} L^k[a(x)] \quad \text{pointwise.}$$

$\square$

# 2 Day 2: April 18

- Lemma 1.3: Coordinate changes and integration by parts work. Also we can check even if we do coordiante change and differeniation (of oscillating term), we also have oscillatory integral.
- Theorem 1.4: For amplitude functions with $\delta < 1$, an operator *defined* by the multiplier

  $$e^{i\frac{1}{2}Q(D)} : \mathcal{S}' \to \mathcal{S}'$$

  have an explicit expression.
- Theorem 1.5: The above multiplier also can be defined by the Taylor expansion. This kind of theorems may be called a expansion formula (I think).
- Corollary 1.7: We want to have an extension with a parameter. The parameter $h$ is called s semiclassical parameter. As $h \to 0$, the oscillation goes rapid. The name stationary phase is implied by the origin zero is the only critical points of the phase function.
- Lemma 1.6: Here we introduce a sequence of Schwarz functions which converges in $\mathcal{S}'$

  $$e^{-\varepsilon x^2}e^{i\frac{1}{2}Q(x)} \xrightarrow{\varepsilon\to 0} e^{i\frac{1}{2}Q(x)}.$$

  Between the second row and the third row, we have used

  $$\mathcal{F}(e^{i\frac{1}{2}Q(x)}e^{-\varepsilon x({}^tP^{-1}P^{-1})x})(P^{-1}\eta) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P^{-1}\eta}e^{i\frac{1}{2}Q(x)}e^{-\varepsilon x({}^tP^{-1}P^{-1})x}\,dx.$$