POSITIVE HAHN-BANACH SEPARATIONS IN OPERATOR ALGEBRAS

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ABSTRACT.

1. INTRODUCTION

- definition and properties of $f_\varepsilon(t) := (1+\varepsilon t)^{-1}t$
- commutant Radon-Nikodym, relation between $\{\omega' \in M_*^+ : \omega' \le \omega\}$ and $\{h \in \pi(M)'^+ : h \le 1\}$, order preserving linear map
- Mazur lemma

Definition 1.1 (Hereditary subsets)**.** Let E be a partially ordered real vector space. We say a subset F of the positive cone E^+ is *hereditary* if $0 \le x \le y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. A $*$-subalgebra B of a $*$-algbera A is called *hereditary* if the positive cone B^+ is a hereditary subset of A^+. We define the *positive polar* of F as the positive part of the real polar

$$F^{r+} := \{x^* \in (E^*)^+ : \sup_{x \in F} x^*(x) \le 1\}.$$

An example that is a non-hereditary closed convex subset of a C*-algebra is $\mathbb{C}1$ in any unital C*-algebra.

Definition 1.2 (Lower dominated sequences)**.** Let E be a partially ordered real vector space. A sequence $x_n \in E$ is called *lower dominated* if there is $x \in E$ such that $x \le x_n$ for all n. If E is the self-adjoint part of the predual of a von Neumann algebra where the Jordan decomposition holds, then we can change the definition such that $x \in -E^+$.

2. POSITIVE HAHN-BANACH SEPARATION THEOREMS

Now we start with the positive Hahn-Banach separation for von Neumann algebras, and will close this section with the same theorem for C*-algebras.

Theorem 2.1 (Positive Hahn-Banach separation for von Neumann algebras)**.** *Let M be a von Neumann algebra.*

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$. In particular, if $x \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega \le 1$ on F.*

(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then $F_* = F_*^{r+r+}$. In particular, if $\omega \in M_*^+ \setminus F_*$, then there is $x \in M^+$ such that $\omega(x) > 1$ and $x \le 1$ on F_*.*

Proof. (1) Since the positive polar is represented as the real polar

$$F^{r+} = F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r,$$

the positive bipolar can be written as $F^{r+r+} = (F - M^+)^{rr+} = (\overline{F - M^+})^+$ by the usual real bipolar theorem, where the closure is for the σ-weak topology. Because $F = (F - M^+)^+ \subset (\overline{F - M^+})^+$, it suffices to prove the opposite inclusion $(\overline{F - M^+})^+ \subset F$.

Let $x \in (\overline{F - M^+})^+$. Take a net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly, and take a net $y_i \in F$ such that $x_i \le y_i$ for each i. Suppose we may assume that the net x_i is bounded. For sufficiently small ε so that the bounded net x_i has the spectra in $[-(2\varepsilon)^{-1}, \infty)$, we have $f_\varepsilon(x_i) \to f_\varepsilon(x)$ σ-strongly, and hence σ-weakly. On the other hand, by the hereditarity and the σ-weak compactness of F, we may asumme that the bounded net $f_\varepsilon(y_i) \in F$ converges σ-weakly to a point of F by taking a subnet. Then, we have $f_\varepsilon(x) \in F - M^+$ by

$$0 \le f_\varepsilon(x) = \lim_i f_\varepsilon(x_i) \le \lim_i f_\varepsilon(y_i) \in F,$$

thus we have $x \in F$ since $f_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. What remains is to prove the existence of a bounded net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly.

Define a convex set

$$G := \left\{ x \in \overline{F - M^+} : \begin{array}{c} \text{there is a sequence } x_m \in F - M^+ \\ \text{such that } -2x \le x_m \uparrow x \ \sigma\text{-weakly} \end{array} \right\} \subset M^{sa},$$

where x_m denotes a sequence. In fact, it has no critical issue on allowing x_m to be uncountably indexed. Since we clearly have $F - M^+ \subset G$ and every non-decreasing net with supremum is bounded and σ-strongly convergent, it suffices to show that G, or equivalently its intersection with the closed unit ball by the Krein-Smŭlian theorem, is σ-strongly closed. Let $x_i \in G$ be a net such that $\sup_i \|x_i\| \le 1$ and $x_i \to x$ σ-strongly. For each i, take a sequence $x_{im} \in F - M^+$ such that $-2x_i \le x_{im} \uparrow x_i$ σ-strongly as $m \to \infty$, and also take $y_{im} \in F$ such that $x_{im} \le y_{im}$. Since $\|x_{im}\| \le 2\|x_i\| \le 2$ is bounded, it implies that there is a bounded net x_j in $F - M^+$ such that $x_j \to x$ σ-strongly, and we can choose arbitrarily small $\varepsilon > 0$ such that $\sigma(x_j) \subset [-(2\varepsilon)^{-1}, \infty)$ for all j. Since $f_\varepsilon(x_j)$ converges to $f_\varepsilon(x)$ σ-strongly and $f_\varepsilon(y_j)$ is a bounded net for each $\varepsilon > 0$ so that we may assume that the net $f_\varepsilon(y_j)$ is σ-weakly covergent by taking a subnet, we have $f_\varepsilon(x) \in F - M^+$ by

$$f_\varepsilon(x) = \lim_j f_\varepsilon(x_j) \le \lim_j f_\varepsilon(y_j) \in F,$$

where the limits are in the σ-weak sense. By taking ε as any decreasingly convergent sequence to zero, we have $x \in G$, hence the closedness of G.

(2) It is enough to prove $(\overline{F_* - M_*^+})^+ \subset F_*$, where the closure is for the weak topology or equivalently in norm by the convexity of $F_* - M_*^+$, so we begin our proof by fixing $\omega \in (\overline{F_* - M_*^+})^+$. For a sequence $\omega_n \in F_* - M_*^+$ such that $\omega_n \to \omega$ in norm of M_*, we can take $\varphi_n \in F_*$ such that $\omega_n \le \varphi_n$ for all n. By modifying ω_n into $\omega_n - (\omega_n - \omega)_+ \in F_* - M_*^+$ and taking a rapidly convergent subsequence, we may assume $\omega_n \le \omega$ and $\|\omega - \omega_n\| \le 2^{-n}$ for all n. If we consider the Gelfand-Naimark-Segal representation $\pi : M \to B(H)$ associated to a positive normal linear functional

$$\widetilde{\omega} := \sum_n (\omega - \omega_n) + \omega + \sum_n 2^{-n} \left(\frac{[\omega_n]}{1 + \|\omega_n\|} + \frac{\varphi_n}{1 + \|\varphi_n\|} \right)$$

on M with the canonical cyclic vector Ω, we can construct commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(M)'$ of $\omega, \omega_n, \varphi_n$ with respect to $\widetilde{\omega}$ respectively. Since $-1 \le h_n \le h$ is bounded, $h_n \to h$ in the weak operator topology of $\pi(M)'$. By the Mazur lemma, we can take a net h_i by convex combinations of h_n such that $h_i \to h$ strongly in $\pi(M)'$, and the corresponding linear functionals ω_i and φ_i satisfy $\omega_i \le \varphi_i$ with $\varphi_i \in F_*$ by the convexity of F_* so that $\omega_i \in F_* - M_*^+$. The net h_i can be taken to be a sequence in fact because $\pi(M)'$ is σ-finite by the existence of the separating vector Ω, but it is not necessary in here. For each i and $0 < \varepsilon < 1$, define

$$h_\varepsilon := f_\varepsilon(h), \quad h_{i,\varepsilon} := f_\varepsilon(h_i), \quad k_{i,\varepsilon} := f_\varepsilon(k_i)$$

in $\pi(M)'$, where the functional calculi are well-defined because $-1 \le h_i$ and $0 \le h, k_i$ for all i, and define k_ε as the σ-weak limit of the bounded net $k_{i,\varepsilon}$, which may be assumed to be σ-weakly convergent. Define $\omega_\varepsilon, \omega_{i,\varepsilon}, \varphi_{i,\varepsilon}, \varphi_\varepsilon$ as the corresponding normal linear functionals on M to $h_\varepsilon, h_{i,\varepsilon}, k_{i,\varepsilon}, k_\varepsilon$. Note that $\varphi_i \in F_*$. The hereditarity of F_* and $0 \le \varphi_{i,\varepsilon} \le \varphi_i$ imply $\varphi_{i,\varepsilon} \in F_*$, and the weak closedness of F_* and the weak convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$ in M_* imply $\varphi_\varepsilon \in F^*$. From $\omega_i \le \varphi_i$, we can deduce $0 \le \omega_\varepsilon \le \varphi_\varepsilon$ by considering the operator monotonicity f_ε and taking the weak limit on i. Thus again, the hereditarity of F_* implies $\omega_\varepsilon \in F^*$, and the weak closedness of F_* and the weak convergence $\omega_\varepsilon \to \omega$ in M_* imply $\omega \in F^*$. $\square$

Now we prepare some lemmas for the positive Hahn-Banach separation theorem for C*-algebras.

Lemma 2.2. *Let A be a C*-algebra, and let F^* be a weakly* closed convex hereditary subset of A^{*+}. If $\omega \in A^{*sa}$ is approximated weakly* by a lower dominated sequence of $F^* - A^{*+}$, then it is approximated in norm by a sequence of $F^* - A^{*+}$.*

Proof. Let $\omega_n \in F^* - A^{*+}$, $\varphi_n \in F^*$, $\widetilde{\omega}_0 \in A^{*+}$ be such that $\omega_n \to \omega$ weakly* in A^* and $-\widetilde{\omega}_0 \le \omega_n \le \varphi_n$ for all n. Consider the Gelfan-Naimark-Segal representation $\pi : A \to B(H)$ of

$$\widetilde{\omega} := \widetilde{\omega}_0 + [\omega] + \sum_n 2^{-n}\left(\frac{[\omega_n]}{1 + \|\omega_n\|} + \frac{\varphi_n}{1 + \|\varphi_n\|}\right)$$

with the canonical cyclic vector $\Omega \in H$. Define the commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(A)'$ of $\omega, \omega_n, \varphi_n$ with respect to $\widetilde{\omega}$.

Consider the range $0 < \varepsilon \le \frac{1}{2}$ for ε. Since $-1 \le h_n, h$ and $0 \le k_n$, the functional calculus $h_{n,\varepsilon} := f_\varepsilon(h_n)$ and $k_{n,\varepsilon} := f_\varepsilon(k_n)$ are well-defined in $\pi(A)'$. The bounded sequences $h_{n,\varepsilon}$ and $k_{n,\varepsilon}$ have weakly convergent subnets in $\pi(A)'$, and denote their limits by h_ε and k_ε respectively. Be cautious that $h'_\varepsilon := f_\varepsilon(h)$ may not be equal to h_ε. By the operator concavity of the function f_ε and the σ-finiteness of $\pi(A)'$, the Mazur lemma retakes sequences $\omega_n \in F^* - A^{*+}$ and $\varphi_n \in F^*$ such that $\omega_n \le \varphi_n$ for all n and $h_{n,\varepsilon} \to h_\varepsilon$ and $k_{n,\varepsilon} \to k_\varepsilon$ strongly. We may assume

$$\|(h_{n,\varepsilon} - h_\varepsilon)\Omega\| < n^{-1}, \qquad \|(h_{n,\varepsilon} - h_\varepsilon)h\Omega\| < n^{-1}$$

for all n uniformly on ε, which will be used later. Note also that we have the identity

$$(1 + \varepsilon h)(h'_\varepsilon - h_{n,\varepsilon})(1 + \varepsilon h) = (h - h_n) + \varepsilon(h - h_n)(1 + \varepsilon h_n)^{-1}(h - h_n).$$

Denote by $\omega_{n,\varepsilon}, \omega_\varepsilon, \omega'_\varepsilon, \varphi_{n,\varepsilon}, \varphi_\varepsilon$ the linear functionals in A^{*sa} corresponded to operators in the commutant $h_{n,\varepsilon}, h_\varepsilon, h'_\varepsilon, k_{n,\varepsilon}, k_\varepsilon \in \pi(A)'$. It follows clearly that $\omega_{n,\varepsilon} \to \omega_\varepsilon$ and $\varphi_{n,\varepsilon} \to \varphi_\varepsilon$ as $n \to \infty$, and $\omega'_\varepsilon \uparrow \omega$ as $\varepsilon \to 0$, weakly in A^*. If we prove $\omega'_\varepsilon - \omega_\varepsilon \to 0$ weakly in A^* as $\varepsilon \to 0$, then since $\omega_{n,\varepsilon} \le \varphi_{n,\varepsilon} \in F^*$ implies $\omega_\varepsilon \le \varphi_\varepsilon \in F^*$, we obtain the weak convergence $\omega_\varepsilon \to \omega$ in A as $\varepsilon \to 0$ with $\omega_\varepsilon \in F^* - A^{*+}$. A desired sequence by applying the Mazur lemma on ω_ε after taking ε to be a decreasing sequence that converges to zero.

Thus, what remains is to prove $\omega'_\varepsilon - \omega_\varepsilon \to 0$ weakly in A^* as $\varepsilon \to 0$. Fix $x \in A^{**}$ with $\|x\| \le 1$. The one-parameter family $(h'_\varepsilon - h_\varepsilon)\pi(x)\Omega$ of vectors is uniformly bounded on $0 < \varepsilon \le \frac{1}{2}$ by the uniform boundedness principle because for each $\eta \in H$, fixing any n, say $n = 1$, we have

$$\begin{aligned}
&|\langle (h'_\varepsilon - h_\varepsilon)\pi(x)\Omega, \eta\rangle| \\
&\quad \le |\langle (h'_\varepsilon - h_{1,\varepsilon})\pi(x)\Omega, \eta\rangle| + |\langle (h_{1,\varepsilon} - h_\varepsilon)\pi(x)\Omega, \eta\rangle| \\
&\quad \le |\langle (1+\varepsilon h)^{-1}(h - h_1)(1+\varepsilon h)^{-1}\pi(x)\Omega, \eta\rangle| \\
&\qquad + \varepsilon|\langle (1+\varepsilon h)^{-1}(h-h_1)(1+\varepsilon h_1)^{-1}(h-h_1)(1+\varepsilon h)^{-1}\pi(x)\Omega, \eta\rangle| \\
&\qquad + \|(h_{1,\varepsilon} - h_\varepsilon)\Omega\|\|\pi(x^*)\eta\| \\
&\quad \le 4\|h - h_1\|\|\Omega\|\|\eta\| + 4\|h - h_1\|^2\|\Omega\|\|\eta\| + \|\eta\|,
\end{aligned}$$

which is uniformly bounded on ε. We further have $(h'_\varepsilon - h_\varepsilon)\pi(x)\Omega \to 0$ weakly in H as $\varepsilon \to 0$, which can be shown as follows. By the boundedness of $(h'_\varepsilon - h_\varepsilon)\pi(x)\Omega$, it is enough to choose $\pi(b)\Omega$ with $b \in A$ satisfying $\|b\| \le 1$ for the test vector. As $\|(h'_\varepsilon - h_\varepsilon)\pi(b)\Omega\|$ is uniformly bounded on ε because $b \in A^{**}$, we can also prove $\|(h'_\varepsilon - h_\varepsilon)h\pi(b)\Omega\|$ is uniformly bounded in the same manner but using $\|(h_{1,\varepsilon} - h_\varepsilon)h\Omega\| < 1$ instead of $\|(h_{1,\varepsilon} - h_\varepsilon)\Omega\| < 1$. Choose their common bound $C > 0$. For an arbitrarily fixed $\delta > 0$, take $a \in A$ such that $\|(\pi(x) - \pi(a))\Omega\| < \delta C^{-1}$ and $\|a\| \le 1$ by the Kaplansky density, and fix n such that $|(\omega - \omega_n)(b^*a)| < \delta$ and $n > \frac{9}{4}\|\Omega\|\delta^{-1}$. Then,

$$\begin{aligned}
&|\langle (h'_\varepsilon - h_\varepsilon)\pi(x)\Omega, \pi(b)\Omega\rangle| \\
&\quad < |\langle (h'_\varepsilon - h_\varepsilon)\pi(a)\Omega, \pi(b)\Omega\rangle| + \delta \\
&\quad < |\langle (h'_\varepsilon - h_\varepsilon)(1+\varepsilon h)\pi(a)\Omega, (1+\varepsilon h)\pi(b)\Omega\rangle| + O(\varepsilon) + \delta \\
&\quad < |\langle (h'_\varepsilon - h_{n,\varepsilon})(1+\varepsilon h)\pi(a)\Omega, (1+\varepsilon h)\pi(b)\Omega\rangle| + \delta + O(\varepsilon) + \delta \\
&\quad \le |(\omega - \omega_n)(b^*a)| + \varepsilon|\langle (1+\varepsilon h_n)^{-1}(h-h_n)\pi(a_\varepsilon)\Omega, (h-h_n)\pi(b)\Omega\rangle| + \delta + O(\varepsilon) + \delta \\
&\quad < \delta + \varepsilon(1-\varepsilon)^{-1}\|(h-h_n)\|^2\|\Omega\|^2 + \delta + O(\varepsilon) + \delta,
\end{aligned}$$

where $O(\varepsilon)$ can be computed as $C(2\varepsilon + \varepsilon^2)\|\Omega\|$, so we have

$$\limsup_{\varepsilon \to 0} |\langle (h'_\varepsilon - h_\varepsilon)\pi(x)\Omega, (1+\varepsilon h)\pi(b)\Omega\rangle| \le 3\delta.$$

Since $\delta > 0$ was taken arbitrarily, we finally have $(h'_\varepsilon - h_\varepsilon)\pi(x)\Omega \to 0$ weakly in H, which implies $\omega'_\varepsilon - \omega_\varepsilon \to 0$ weakly in A^* as $\varepsilon \to 0$. □

The following lemma is a modification of the Krein-Šmulian theorem, and it can be proved in a similar way to the proof of the original theorem.

Lemma 2.3. *Let A be a C^*-algebra, and C_n^* be a non-decreasing sequence of weakly*-closed convex subsets of A^{*sa}, whose union C_∞^* contains A^{*+}. If a norm closed convex subset G^* of A^{*sa} has the property that $G^* \cap C_n^*$ is weakly* closed for each n, then $G^* \cap C_\infty^*$ is relatively weakly* closed in C_∞^*.*

Proof. Fix an element ω_0 of $C_\infty^* \setminus G^*$. It is enough to construct an element a of A^{sa} separating a norm open ball centered at ω_0 from G^*. Since G^* is norm closed, there exists $r > 0$ such that $G^* \cap B(\omega_0, r) = \varnothing$. By replacing G^* to $r^{-1}(G^* - \omega_0)$ and C_n^* to

$r^{-1}(C_n^*-\omega_0)$, we may assume $G^*\cap B(0,1)=\varnothing$, and the claim follows if we prove there is $a\in A^{sa}$ separating $B(0,1)$ and G^*. The condition $A^{*+}\subset C_\infty^*$ becomes $A^{*+}-\omega_0\subset C_\infty^*$. Letting the index n start from one, we may also replace C_n^* to $n(C_n^*\cap B(0,1))$ since its union is still C_∞^*. Note that C_n^* is bounded for each n, and we can easily see that $G^*\cap C_1^*=\varnothing$ and $n^{-1}C_n^*\subset(n+1)^{-1}C_{n+1}^*$.

Note that for any Banach space X, if F is a bounded subset of X, then by endowing with the discrete topology on F, we have a natural bounded linear operator $\ell^1(F)\to X$ by completeness of X, with its dual $X^*\to\ell^\infty(F)$. We will construct a bounded subset F of A^{sa} such that the subset $G^*\cap C_\infty^*$ of A^{*sa} induces a subset of the smaller subspace $c_0(F)$ of $\ell^\infty(F)$ via the restriction map $A^{*sa}\to\ell^\infty(F)$, and also such that it satisfies $G^*\cap C_\infty^*\cap F^\circ=\varnothing$, where $F^\circ:=\{\omega\in A^{*sa}:\sup_{a\in F}|\omega(a)|\le 1\}$ denotes the absolute polar of F. If such a set $F\subset X$ exists, then the image of $G^*\cap C_\infty^*$ in $c_0(F)$ is a convex set disjoint to the closed unit ball of $c_0(F)$ by the condition $G^*\cap C_\infty^*\cap F^\circ=\varnothing$. Therefore, there exists a separating linear functional $l\in\ell^1(F)$ by the Hahn-Banach separation, and it induces a linear functional separating G^* and the unit ball of A^{*sa}. Then, we are done.

Let $F_0:=\{0\}\subset A^{sa}$. As an induction hypothesis on n, suppose for each $0\le k\le n-1$ we already have a finite subset F_k of $(C_k^*)^\circ$ such that

$$G^*\cap C_n^*\cap\left(\bigcup_{k=0}^{n-1}F_k\right)^\circ=\varnothing.$$

If every finite subset F_n of $(C_n^*)^\circ$ satisfies

$$G^*\cap C_{n+1}^*\cap\left(\bigcup_{k=0}^{n-1}F_k\right)^\circ\cap F_n^\circ\neq\varnothing,$$

then since they are weakly* compact, the finite intersection property leads a contradiction because the intersection of all absolute polars F_n° of finite subsets F_n of $(C_n^*)^\circ$ is C_n^*, which is the polar of all union of finite subsets F_n of $(C_n^*)^\circ$ by the bipolar theorem. Thus, we can take a finite subset F_n of $(C_n^*)^\circ$ such that

$$G^*\cap C_{n+1}^*\cap\left(\bigcup_{k=0}^{n}F_k\right)^\circ=\varnothing.$$

Let $F:=\bigcup_{k=0}^\infty F_k$. Then, we have $G^*\cap C_\infty^*\cap F^\circ=\varnothing$, and every element of C_∞^* is restricted to F to define an element of $c_0(F)$ because for each $\omega\in C_n^*$ and $k\ge 0$ we have

$$\omega(F_{n+k})\subset\omega((C_{n+k}^*)^\circ)\subset\frac{n}{n+k}\omega((C_n^*)^\circ)\subset[-\frac{n}{n+k},\frac{n}{n+k}].$$

Finally, for any $\omega\in A^{*sa}$, if we enumerate F as a sequence f_m, then

$$|\omega(f_m)|\le|(\omega_+-\omega_0)(f_m)|+|(\omega_--\omega_0)(f_m)|\to 0,$$

so the uniform boundedness principle concludes that F is bounded. Therefore, the set F satisfies the properties we desired. □

Theorem 2.4 (Positive Hahn-Banach separation for C*-algebras)**.** *Let A be a C*-algebra.*

(1) *If F is a norm closed convex hereditary subset of A^+, then $F=F^{r+r+}$. In particular, if $a\in A^+\setminus F$, then there is $\omega\in A^{*+}$ such that $\omega(a)>1$ and $\omega\le 1$ on F.*

(2) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$. In particular, if $\omega \in A^{*+} \setminus F^*$, then there is $a \in A^+$ such that $\omega(a) > 1$ and $a \le 1$ on F^*.*

Proof. (1) We directly prove the separation without invoking the arguments of positive bipolars. Denote by F^{**} the σ-weak closure of F in the universal von Neumann algebra A^{**}. We first show that F^{**} is hereditary subset of A^{**+}. Suppose $0 \le x \le y$ in A^{**} and $y \in F^{**}$. Then, there is $z \in A^{**}$ such that $x^{\frac{1}{2}} = zy^{\frac{1}{2}}$. Take bounded nets b_i in F and c_i in A such that $b_i \to y$ and $c_i \to z$ σ-strongly* in A^{**} using the Kaplansky density. We may assume the indices of these two nets are same. Since both the multiplication and the involution of a von Neumann algebra on bounded parts are continuous in the σ-strong* topology, and since the square root on a positive bounded interval is a strongly continuous function, we have the σ-strong* limit

$$x = y^{\frac{1}{2}} z^* z y^{\frac{1}{2}} = \lim_i b_i^{\frac{1}{2}} c_i^* c_i b_i^{\frac{1}{2}},$$

so we obtain $x \in F^{**}$ from $b_i^{\frac{1}{2}} c_i^* c_i b_i^{\frac{1}{2}} \in F$. Thus, F^{**} is hereditary in A^{**+}.

Let $a \in A^+ \setminus F$. Observe that we have $a \in A^{**+} \setminus F^{**}$ because if $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, meaning that $a_i \to a$ weakly in A and by the weak closedness of F in A we get a contradiction $a \in F^{**} \cap A = F$. By Theorem 2.1, there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega \le 1$ on $F \subset F^{**}$, so it completes the proof.

(2) As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, so take $\omega \in (\overline{F^* - A^{*+}})^+$, where the closure is for the weak* topology. We first prove it when A is separable, which makes the weak* topology on any bounded part of A^{*sa} metrizable. Consider the following convex set

$$G^* := \left\{ \omega \in \overline{F^* - A^{*+}} : \begin{array}{c} \text{there is a lower dominated sequence } \omega_n \in F^* - A^{*+} \\ \text{such that } \omega_n \to \omega \text{ weakly}^* \text{ in } A^* \end{array} \right\}.$$

We can easily see that $F^* - A^{*+} \subset G^*$, and we claim G^* is the weak* closure. If the claim is true, then we have $G^* = \overline{F^* - A^{*+}}$, and it follows that $\omega \in F^*$ by Lemma 2.2 and Theorem 2.1 (2), so we are done. To prove G^* is weakly* closed, we can take a sequence $\omega_n \in G^*$ such that $\omega_n \to \omega$ weakly* in A^* by the Krein-Šmulian theorem, and we will prove $\omega \in G^*$. Because ω belongs to the relative weak* closure of $G^* \cap C_\infty^*$ in C_∞^*, where

$$C_n^* := \{\omega' \in A^{*sa} : -\sum_{k \le n} \omega_{k-} - \omega_- \le \omega'\}, \qquad C_\infty^* := \bigcup_n C_n^*,$$

so if we prove that G^* is norm closed and $G^* \cap C_n^*$ is weakly* closed for each n, then we obtain $\omega \in G^*$ by Lemma 2.3, and it completes the proof. Since the limit of a norm convergent sequence in G^* can be approximated by a lower dominated sequence in G^* as in the proof of Theorem 2.1 (2), and since every sequence in C_n^* is lower dominated, now it suffices to show $\omega \in G^*$ when it is the weak* limit of a lower dominated sequence $\omega_n \in G^*$. Take $\widetilde{\omega} \in A^{*+}$ such that $-\widetilde{\omega} \le \omega_n$ for all n. Since ω_n is weakly* approximated by a lower dominated sequence in $F^* - A^{*+}$ by definition of G^*, applying Lemma 2.2 for each n, we can find a sequence $\omega_{nm} \in F^* - A^{*+}$ such that $\omega_{nm} \to \omega_n$ in norm of A^* as $m \to \infty$. After modifying ω_{nm} into $\omega_{nm} - (\omega_{nm} - \omega_n)_+ \in F^* - A^{*+}$ to assume $\omega_{nm} \le \omega_n$, if we take a subsequence to have $\|\omega_n - \omega_{nm}\| < 2^{-(n+m)}$, then $\widetilde{\omega}_n := \sum_m (\omega_n - \omega_{nm})$

satisfies $-\widetilde{\omega}_n \le \omega_{nm} - \omega_n$ for all n and m, and $\|\widetilde{\omega}_n\| \le 2^{-n}$. Then, for the diagonal sequence $\omega_{nn} \in F^* - A^{*+}$, we have $\omega_{nn} \to \omega$ weakly* by

$$|(\omega_{nn} - \omega)(a)| \le |(\omega_{nn} - \omega_n)(a)| + |(\omega_n - \omega)(a)| \le 2^{-2n}\|a\| + |(\omega_n - \omega)(a)| \to 0$$

as $n \to \infty$ for each $a \in A$, and it is lower dominated by $-\widetilde{\omega} - \sum_n \widetilde{\omega}_n$, therefore we get the claim $\omega \in G^*$.

Now we consider a general C*-algebra A. For a separable C*-subalgebra B of A, we define a set

$$F_B^* := \{\omega \in B^{*+} : \text{there is } \varphi \in F^* \text{ such that } \omega \le \varphi \text{ on } B^+\}.$$

It is clearly a convex hereditary subset of B^{*+}, and to prove the weak* closedness via the Krein-Šmulian theorem, take a sequence $\omega_{B,n} \in F_B^*$ such that $\omega_{B,n} \to \omega_B$ weakly* in B^*. Let $\varphi_n \in F^*$ be a sequence such that $\omega_{B,n}(b) \le \varphi_n(b)$ on $b \in B^+$, and let $\omega_n \in A^{*+}$ be the extension of $\omega_{B,n}$ for each n. Consider the Gelfand-Naimark-Segal representation $\pi : A \to B(H)$ associated to the positive linear functional

$$\widetilde{\omega} := \sum_n 2^{-n}\left(\frac{\omega_n}{1 + \|\omega_n\|} + \frac{\varphi_n}{1 + \|\varphi_n\|}\right),$$

with the canonical cyclic vector Ω. Let $p \in B(H)$ be the orthogonal projection onto the closed linear subspace $\overline{\pi(B)\Omega} \subset H$. Then, $\omega_n(b) \le \varphi_n(b)$ on $b \in B^+$ implies $ph_np \le pk_np$, so it follows that $ph_{n,\varepsilon}p \le pk_{n,\varepsilon}p$. Taking weakly convergent subnets of $h_{n,\varepsilon}$ and $k_{n,\varepsilon}$, we may define the limits h_ε and k_ε, and $ph_\varepsilon p \le pk_\varepsilon p$ implies that the corresponding functionals have the relations $\omega_\varepsilon(b) \le \varphi_\varepsilon(b)$ for all $b \in B^+$. We clearly have $\varphi_\varepsilon \in F^*$, so $\omega_{B,\varepsilon} = \omega_\varepsilon|_B \in F_B^*(?)$. Since $\omega_{B,\varepsilon} \uparrow \omega_B$ weakly in B^*, we only need to prove the convex set F_B^* is norm closed. Take a sequence $\omega_{B,n} \in F_B^*$ again, but at this time such that $\omega_{B,n} \to \omega_B$ in norm of B^*, together with $\varphi_n \in F^*$ such that $\omega_{B,n}(b) \le \varphi_n(b)$ on $b \in B^+$....

Let $\omega \in (\overline{F^* - A^{*+}})^+$, where the closure is taken in the weak* topology. Our goal is to show $\omega \in F^*$. Take a net $\omega_i \in F^* - A^{*+}$ and $\varphi_i \in F^*$ such that $\omega_i \to \omega$ weakly* in A^* and $\omega_i \le \varphi_i$ for each i. We have $\varphi_i|_B \in F_B^*$ and $\omega_i|_B \in F_B^* - B^{*+}$, with the weak* convergence $\omega_i|_B \to \omega_B$ in B^*, thus we have $\omega|_B \in (\overline{F_B^* - B^{*+}})^+ = F_B^*$ because B is separable. If we consider the non-decreasing net of all separable C*-subalgebras B_j of A, then the restriction $\omega|_{B_j}$ of ω on B_j belongs to the set $F_{B_j}^*$, so there is a net $\varphi_j \in F^*$ such that $\omega(b) \le \varphi_j(b)$ on $b \in B_j^+$ for each j, and by the Hahn-Banach extension, we obtain a net $\omega_j \in F^* - A^{*+}$ such that $\omega_j(b) = \omega(b)$ on $b \in B_j^+$ for all j.

Let A_0^{**} be the set of all elements of A^{**} whose left or right support projection is σ-finite. It is known that A_0^{**} is an algebraic ideal of A^{**}. Let $x \in A_0^{**+}$.

$(\omega_j - \omega)(x^2) \to 0$.

Thus, ω belongs to the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$.

Suppose ω does not belong to the weak closure of $F^* - A^{*+}$. Then, there is $x \in A^{**+}$ such that $\omega(x^2) > 1$ and $x^2 \le 1$ on $F^* - A^{*+}$ by Theorem 2.1 (2). Let $\{p_i\}_{i \in I}$ be a maximal orthogonal family of σ-finite projections of the von Neumann algebra A^{**}.

Consider the bounded linear maps

$$\Gamma : c_0(I) \to A : (\lambda_i)_{i\in I} \mapsto \sum_i \lambda_i x p_i x,$$
$$\Gamma^* : A^* \to \ell^1(I) : \omega' \mapsto (\omega'(xp_ix))_{i\in I},$$
$$\Gamma^{**} : \ell^\infty(I) \to A^{**} : (\lambda_i)_{i\in I} \mapsto \sum_i \lambda_i x p_i x,$$

Γ is a topological embedding so that Γ^* is surjective?

$\Gamma^*(F^*)$ weakly* closed convex hereditary?

$\Gamma^*(F^* - A^{*+}) = \Gamma^*(F^*) - \ell^1(I)^+$?

We can check $\Gamma^*(\omega_j) \to \Gamma^*(\omega)$ weakly* so that $\Gamma^*(\omega) \in \Gamma^*(F^*)$. We can take a net $\omega_k \in F^* - A^{*+}$ such that $\Gamma^*(\omega_k) \to \Gamma^*(\omega)$ weakly, and it follows a contradiction

$$1 \geq \omega_k(x^2) = \langle 1, \Gamma^*(\omega_k)\rangle \to \langle 1, \Gamma^*(\omega)\rangle = \omega(x^2) > 1.$$

Therefore, ω is a positive functional contained in the weak closure of $F^* - A^{*+}$, so $\omega \in F^*$ by Theorem 2.1 (2).

—

If we prove the closability of positive quadratic forms...

If we prove F_B^* is weakly* closed, then there is a net $\varphi_{j,\varepsilon} \in F^*$ such that $\omega(b) \leq \varphi_j(b)$ for $b \in B_j^+$, and hence $\omega(y) \leq \varphi_j(y)$ for $y \in B_j^{**+}$. Assuming $k_{j,\varepsilon} \to k_\varepsilon$ weakly in $\pi(A)'$, we have $\varphi_{j,\varepsilon}(x) \to \varphi_\varepsilon(x)$ for $x \in \mathfrak{M}$.

If $\varphi_{j,\varepsilon}(x) \to \varphi_\varepsilon(x)$ for $x \in \mathfrak{M}$ implies $\varphi_{j,\varepsilon}(x) \to \varphi_\varepsilon(x)$ for all $x \in A^{**}$, then $\varphi_\varepsilon \in F^*$.

If $\mathfrak{M} \cap B_j^{**}$ is σ-weakly dense in B_j^{**}, then $\omega_\varepsilon(y) \leq \varphi_{j,\varepsilon}(y)$ for $y \in B_j^{**+}$. Then, we have $\omega_\varepsilon(x) \leq \varphi_\varepsilon(x)$ for all $x \in A^{**+}$. Then, $\varphi_\varepsilon \in F^*$ implies $\omega_\varepsilon \in F^*$. Then, $\omega \in F^*$.

—

If $\widetilde{\omega}_i$ is a positive norm preserving extension of ω_i, then $\widetilde{\omega}_i \leq \varphi_i$? no.

$\omega_n \to \omega$ in norm and $\omega_n \leq \omega$. Let

Considering $k_{i,\varepsilon} \to k_\varepsilon$ so that $\varphi_\varepsilon \in F^*$. We need weak* convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$.

$\widetilde{\omega}_{i,\varepsilon} \leq \varphi_{i,\varepsilon}$

$\widetilde{\omega}_\varepsilon \leq \varphi_\varepsilon$

$\varphi_i = \omega_i^\sim + (\varphi_i - \omega_i)^\sim$ on B

$\|\omega_i^\sim + (\varphi_i - \omega_i)^\sim\| \leq \|\omega_i^\sim\| + \|(\varphi_i - \omega_i)^\sim\| = \|\omega_i\| + \|\varphi_i|_B - \omega_i\| \leq \|\varphi_i|_B\|$

Here we let ψ be a faithful semi-finite normal weight on A^{**}, and let $\pi : A^{**} \to B(H)$ be the Gelfand-Naimark-Segal representation associated to ψ, together with the left A^{**}-liner map $\Lambda : \mathfrak{N}_\psi \to H$ of dense range such that $\psi(x^*x) = \|\Lambda(x)\|^2$ for all $x \in \mathfrak{N}_\psi$. Note that because the weight ψ is faithful and semi-finite, Λ is injective and σ-weakly densely defined, meaning that $\mathfrak{M}_\psi$ is a hereditary σ-weakly dense $*$-subalgebra of A^{**}. Construct the commutant Radon-Nikodym derivatives h, k_j of ω, φ_j with respect to ψ. Here k_j is a positive self-adjoint operator defined by the Friedrichs extension such that $\operatorname{ran}\Lambda \subset \operatorname{dom} k_j$ for all j. Taking a subnet, we may assume that there is $k_\varepsilon \in \pi(A)'^+$ satisfying $f_\varepsilon(k_j) \to k_\varepsilon$ σ-weakly. Because of the operator concavity of f_ε (more detail), we can take a net $\varphi_l \in F^*$ such that $f_\varepsilon(k_l) \to k_\varepsilon$ σ-strongly, where k_l are again the commutant Radon-Nikodym derivatives of φ_l defined by the Friedrichs extension.

Since is a strongly continuous function, we have $(f_\varepsilon(k_l)-k_\varepsilon)\to 0$ σ-strongly, so if we define $\varphi_{l,\varepsilon}\in F^*-A^{*+}$ and $\varphi_\varepsilon\in A^{*+}$ such that

$$\varphi_{l,\varepsilon}(x^*x):=\langle (f_\varepsilon(k_l)-(f_\varepsilon(k_l)-k_\varepsilon)_+)\Lambda(x),\Lambda(x)\rangle,\quad \varphi_\varepsilon(x^*x):=\langle k_\varepsilon\Lambda(x),\Lambda(x)\rangle$$

for each $x\in\mathfrak{N}_\psi$, then we have $\varphi_{l,\varepsilon}\to\varphi_\varepsilon$ pointwisely on $\mathfrak{M}_\psi$ and $\varphi_{l,\varepsilon}\le\varphi_\varepsilon$ for all l.

How to dominate $\varphi_{l,\varepsilon}$ from below?

By Lemma 2.2, we have $\varphi_{l,\varepsilon}\to\varphi_\varepsilon$ weakly in A^*, so Theorem 2.1 (2) implies that $\varphi_\varepsilon\in(\overline{F^*-A^{*+}}^w)^+=F^*$. If we define $\omega_\varepsilon\in A^{*+}$ and $\varphi_{j,\varepsilon}\in F^*$ by

$$\omega_\varepsilon(x^*x):=\langle f_\varepsilon(h)\Lambda(x),\Lambda(x)\rangle,\quad \varphi_{j,\varepsilon}(x^*x):=\langle f_\varepsilon(k_j)\Lambda(x),\Lambda(x)\rangle$$

for each $x\in\mathfrak{N}_\psi$, then since $\omega\le\varphi_j$ on B_j^+ implies $\omega_\varepsilon\le\varphi_{j,\varepsilon}$ on B_j^+, the weak* limit $\omega_\varepsilon\le\lim_j\varphi_{j,\varepsilon}=\varphi_\varepsilon$ deduces $\omega_\varepsilon\in F^*-A^{*+}$. Since $\omega_\varepsilon\to\omega$ pointwisely on $\mathfrak{M}_\psi$ and $0\le\omega_\varepsilon\le\omega$ for all $0<\varepsilon$, we have $\omega\in F^*$ by Lemma 2.2 and Theorem 2.1 (2).

□

3. Applications to weight theory

Corollary 3.1. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\left\{\begin{array}{c}\text{subadditive normal}\\ \text{weights of } M\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{hereditary closed}\\ \text{convex subsets of } M_*^+\end{array}\right\}$$

$$\varphi\qquad\longmapsto\qquad\{\omega\in M_*^+:\omega\le\varphi\}$$