複素解析学 I 演習 2023 年

問 1 (フックス群としてのモジュラー群)．複素数体 $\mathbb{C}$ の部分集合 A に対して、成分 a, b, c, d が A の元で $ad - bc = 1$ を満たす一次分数変換 $f(z) = (az+b)/(cz+d)$ の集合を $\mathrm{PSL}(2, A)$ と書く．特に $\mathrm{PSL}(2, \mathbb{Z})$ を**モジュラー群**と呼ぶ．上半平面 $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ の部分集合 $D := \{z \in \mathbb{H} : |z| > 1,\ |\mathrm{Re}\, z| < \frac{1}{2}\}$ を定義する．

(1) $\mathrm{PSL}(2, \mathbb{R})$ の元 f は全単射写像 $\mathbb{H} \to \mathbb{H}$ を定義することを示せ．

(2) $\mathrm{PSL}(2, \mathbb{Z})$ は $S(z) := -1/z$ と $T(z) := z + 1$ によって生成されることを示せ．つまり、全ての元が $S^{\pm 1}$ と $T^{\pm 1}$ の有限回の合成として表れることを示せ．

(3) 集合 D は $\mathrm{PSL}(2, \mathbb{Z})$ の**基本領域**であることを示せ．つまり、次の二つが成り立つことを示せ：

  (a) 任意の点 $z \in \mathbb{H}$ に対して $f(z) \in \overline{D}$ を満たす $f \in \mathrm{PSL}(2, \mathbb{Z})$ が少なくとも一つ存在する．

  (b) 任意の点 $z \in \mathbb{H}$ に対して $f(z) \in D$ を満たす $f \in \mathrm{PSL}(2, \mathbb{Z})$ が多くとも一つ存在する．

(4) $\mathrm{PSL}(2, \mathbb{Z})$ は $\mathbb{H}$ に**真性不連続に作用**することを示せ．つまり、任意の点 $z \in \mathbb{H}$ に対して軌道 $\{f(z) : f \in \mathrm{PSL}(2, \mathbb{Z})\}$ が離散集合であることを示せ．

問 2 (カラテオドリ級関数集合の極点)．開単位円板上で定義された正則関数 f が $f(0) = 1$ を満たすとする．もし任意の $|z| < 1$ を満たす複素数 z に対して $\mathrm{Re}\, f(z) > 0$ ならば、f を**カラテオドリ級**の関数という．関数 f が冪級数展開 $f(z) = 1 + 2\sum_{k=1}^{\infty} c_k z^k$ を持つとする．

(1) 正の整数 k と実数 $0 < r < 1$ に対して次の式を示せ：

$$c_k r^k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, f(re^{i\theta}) e^{-ik\theta}\, d\theta.$$

(2) 次の二つの条件が同値であることを示せ：

  (a) 関数 f がカラテオドリ級である．

  (b) 任意の正の整数 n に対して点 $(c_1, \cdots, c_n) \in \mathbb{C}^n$ は $\theta \in [0, 2\pi)$ によって媒介変数表示された曲線 $(e^{-i\theta}, \cdots, e^{-in\theta}) \in \mathbb{C}^n$ の凸包絡の元である．

問 3 (アールフォルス・清水標数)．複素平面上の有理型関数 f を考える．次のように $r \geq 0$ に対する関数 $A(\cdot, f)$ を定義する：

$$A(r, f) := \frac{1}{\pi} \int_{\sqrt{x^2+y^2} \leq r} f^{\#}(x+iy)^2\, dx\, dy, \qquad \text{ただし、} f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2}, \quad z \in \mathbb{C}.$$

関数 $f^{\#}$ を f の**球面導関数**と呼ぶ．

(1) 任意の点 $(x, y) \in \mathbb{R}^2$ に対して、

$$\frac{1}{\pi} f^{\#}(x+iy)^2 = \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)$$

を満たす実平面 $\mathbb{R}^2$ 上の実関数 P と Q を求め、関数 $K(x, y) := 1 + |f(x+iy)|^2$ を用いて表せ．

(2) グリーンの定理と偏角の原理を用いて $r \geq 0$ に対して次の式が成り立つことを示せ：

$$\int_0^r A(t,f)\frac{dt}{t} = \int_0^r n(t,f)\frac{dt}{t} + \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta - \log\sqrt{1+|f(0)|^2}.$$

ただし、$n(r,f)$ は閉円板 $\overline{B(0,r)}$ 内にある重複度を込めて数えた f の極の数である．左辺の関数を f の**アールフォルス・清水標数**と呼ぶ.

(3) 球面導関数 $f^{\#}$ が有界ならば、ある定数 $C>0$ が存在して、全ての $z \in \mathbb{C}$ に対して $|f(z)| \leq Ce^{|z|^2}$ であることを示せ．特に、f は $\mathbb{C}$ 全体上正則である.

問 4 (四分円上のディリクレ問題)**.** 領域 $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0,\ y>0\}$ 上に定義された調和関数 $u \in C^2(\Omega,\mathbb{R})$ が次の境界値条件を満たすとする：各点 $(x_0,y_0) \in \partial\Omega$ に対して

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \begin{cases} 0 & \text{if } y_0>0, \\ 1 & \text{if } y_0=0 \text{ and } 0<x_0<1. \end{cases}$$

(1) 反射原理を用いて u は領域 $\widetilde{\Omega} := \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1,\ x>0\}$ 上の調和関数 $\widetilde{u} \in C^2(\widetilde{\Omega},\mathbb{R})$ に拡張されることを示せ.

(2) 適切な等角変換とポアソン積分を用いて u を求めよ.

Solution of 1. (3) (a) Let $z_0 \in \mathbb{H}$. We may assume $\operatorname{Re} z_0 \in [-\frac{1}{2}, \frac{1}{2})$. For $z \in \mathbb{H}$ satisfying $\operatorname{Re} z \in [-\frac{1}{2}, \frac{1}{2})$, if we define $f_z := T^{-\lfloor \operatorname{Re} Sz + \frac{1}{2} \rfloor} S$, then $\operatorname{Re} f_z(z) \in [-\frac{1}{2}, \frac{1}{2})$. Define a sequence z_n inductively by $z_n := f_{z_{n-1}}(z_{n-1})$ for $n \geq 1$. Then, $\operatorname{Re} z_n \in [-\frac{1}{2}, \frac{1}{2})$ for all n. Since

$$\operatorname{Im} z_n = \frac{\operatorname{Im} z_{n-1}}{(\operatorname{Re} z_{n-1})^2 + (\operatorname{Im} z_{n-1})^2} \geq g(\operatorname{Im} z_{n-1}),$$

where $g(y) := 4y/(1+4y^2)$, since $g^n(y) \uparrow \frac{\sqrt{3}}{2}$ for $0 < y < \frac{\sqrt{3}}{2}$, so there is n such that

$$-\frac{1}{2} \leq \operatorname{Re} z_n < \frac{1}{2}, \qquad \operatorname{Im} z_n > \frac{\sqrt{3}}{4}.$$

If $|z_n| \geq 1$, then we are done, so assume $|z_n| < 1$. Now we have three possibilities: $|z_n - 1| < 1$, $|z_n + 1| < 1$, or $\min\{|z_n - 1|, |z_n + 1|\} \geq 1$. For each case, we can check that $T^{-1}Sz_n$, TSz_n, Sz_n is contained in D, respectively.

(b) Let $w = (az+b)/(cz+d)$. It suffices to show $c = 0$. Suppose $c \neq 0$. Let n be an integer such that $|n - \frac{a}{c}| \leq \frac{1}{2}$. Note that $|z - m| > 1$ and $|w - m| > 1$ for every integer m. Write

$$1 < |w - n| = \left| \frac{az+b}{cz+d} - n \right| \leq \left| \frac{1}{c(cz+d)} \right| + \left| n - \frac{a}{c} \right|.$$

If $|c| \geq 2$, then $|c(cz+d)| \geq 4 \operatorname{Im} z > 2\sqrt{3}$ leads a contradiction. If $|c| = 1$, say $c = 1$, then $|n - a| \leq \frac{1}{2}$ implies $|n - \frac{a}{c}| = 0$ and $|c(cz+d)| = |z + d| > 1$ leads a contradiction. Thus, $c = 0$, and we are done.

(4) Clear from (3). $\square$

Solution of 2. (1) Suppose $k > 0$ first. The Cauchy integral formula writes

$$2c_k k! = [k]fz(0) = \frac{k!}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{k+1}}\,dz = \frac{k!}{2\pi}\int_0^{2\pi}\frac{f(re^{i\theta})}{(re^{i\theta})^k}\,d\theta,$$

and it implies

$$2c_k r^k = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-ik\theta}\,d\theta.$$

Since $f(z)z^k$ is analytic, the Cauchy theorem can be applied to get

$$0 = \frac{1}{2\pi i}\int_{|z|=r} f(z)z^k\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})r^k e^{ik\theta}\,d\theta,$$

and it implies

$$0 = \frac{1}{2\pi}\int_0^{2\pi}\overline{f(re^{i\theta})}e^{-ik\theta}\,d\theta.$$

By combining the above two equations, we obtain the formula. For $k = 0$, applying the Cauchy theorem for f, we have

$$c_0 = f(0) = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z}\,dz = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\,d\theta.$$

Alternatively, we can show the same result using the orthogonal relation of complex exponential functions. An easy computation shows the identity

$$\begin{aligned}
\operatorname{Re} f(re^{i\theta}) &= \frac{1}{2}[f(re^{i\theta}) + \overline{f(re^{i\theta})}] \\
&= \frac{1}{2}\left[\left(1+\sum_{k=1}^{\infty}2c_k(re^{i\theta})^k\right) + \overline{\left(1+\sum_{k=1}^{\infty}2c_k(re^{i\theta})^k\right)}\right] \\
&= \frac{1}{2}\left[\left(1+\sum_{k=1}^{\infty}2c_k r^k e^{ik\theta}\right) + \left(1+\sum_{k=1}^{\infty}2\overline{c_k} r^k e^{-ik\theta}\right)\right] \\
&= \sum_{k=-\infty}^{\infty} c_k r^{|k|}e^{ik\theta}.
\end{aligned}$$

From the uniform convergence of the power series on the compact set $\{z : |z| \le (r+1)/2\}$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\frac{1}{2\pi}\int_0^{2\pi} e^{il\theta}e^{-ik\theta}\,d\theta = \sum_{l=-\infty}^{\infty} c_l r^{|l|}\delta_{kl} = c_k r^{|k|}.$$

(2) (b)$\Rightarrow$(a) Denote by K_n the convex hull of the curve $\theta \mapsto (e^{-i\theta}, \cdots, e^{-in\theta}) \in \mathbb{C}^n$. Suppose first that $(c_1, \cdots, c_n) \in K_n$. For each n, there exists a finite sequence of pairs $(\lambda_{n,j}, \theta_{n,j})_j$ having the following convex combination

$$(c_1, \cdots, c_n) = \sum_j \lambda_{n,j}(e^{-i\theta_{n,j}}, \cdots, e^{-in\theta_{n,j}})$$

with coefficients $\lambda_{n,j} \ge 0$ such that $\sum_j \lambda_{n,j} = 1$. Define

$$f_n(z) := \sum_j \lambda_{n,j}\frac{e^{i\theta_{n,j}} + z}{e^{i\theta_{n,j}} - z},$$

which has positive real part on $|z| < 1$ because $\operatorname{Re}(e^{i\theta_{n,j}} + z)/(e^{i\theta_{n,j}} - z) > 0$ for $|z| < 1$. Then,

$$f_n(z) = \sum_j \lambda_{n,j}(1 + \sum_{k=1}^{\infty} 2e^{-ik\theta_{n,j}}z^k) = 1 + \sum_{k=1}^{n} 2c_k z^k + \sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k$$

implies

$$
\begin{aligned}
|f_n(z)-f(z)| &= \left|\sum_{k=n+1}^{\infty}\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)z^k - \sum_{k=n+1}^{\infty} 2c_k z^k\right| \\
&\le \sum_{k=n+1}^{\infty}\left|\left(\sum_j 2\lambda_{n,j}e^{-ik\theta_{n,j}}\right)-2c_k\right||z|^k \le \sum_{k=n+1}^{\infty} 4|z|^k
\end{aligned}
$$

converges to zero for $|z|<1$. Therefore, f has a non-negative real part on the open unit disk. The non-negativity can be strengthened to positivity by the open mapping theorem so that f belongs to the Carathéodory class.

(a)⇒(b) Conversely, suppose that f is in the Carathéodory class. Let $(\gamma_1,\cdots,\gamma_n)$ be any point on the surface ∂K_n of K_n and S any supporting hyperplane of K_n tangent at $(\gamma_1,\cdots,\gamma_n)$. Let $(u_1,\cdots,u_n)$ be the outward unit normal vector of the supporting hyperplane S. Note that this unit normal vector is uniquely determined for the hyperplane with respect to the induced real inner product structure on the real $2n$-dimensional space $\mathbb{C}^n$ given by

$$
\langle (z_1,\cdots,z_n),(w_1,\cdots,w_n)\rangle = \sum_{k=1}^{n}(\operatorname{Re} z_k \operatorname{Re} w_k + \operatorname{Im} z_k \operatorname{Im} w_k) = \operatorname{Re}\sum_{k=1}^{n} z_k\overline{w}_k.
$$

Then, $\sum_{k=1}^n |u_k|^2 = 1$ and further that the maximum

$$
M := \max_{(x_1,\cdots,x_n)\in K_n} \operatorname{Re}\sum_{k=1}^{n} x_k\overline{u}_k > 0
$$

is attained at $(\gamma_1,\cdots,\gamma_n)$. Our goal is to verify the bound

$$
\operatorname{Re}\sum_{k=1}^{n} c_k\overline{u}_k \le M,
$$

which implies that $(c_1,\cdots,c_n)$ is contained in every half space tangent to K_n so that we finally obtain $(c_1,\cdots,c_n)\in K_n$.

Since for any $\theta\in[0,2\pi)$ the point $(e^{-i\theta},\cdots,e^{-in\theta})$ is in K_n so that

$$
\operatorname{Re}\sum_{k=1}^{n} e^{-ik\theta}\overline{u}_k \le M,
$$

we have for arbitrarily small $\varepsilon>0$ that

$$
\operatorname{Re}\sum_{k=1}^{n} \frac{1}{r^k}e^{-ik\theta}\overline{u}_k \le M+\varepsilon
$$

for any $0<r<1$ sufficiently close to 1, thus we can write

$$
\begin{aligned}
\operatorname{Re}\sum_{k=1}^{n} c_k\overline{u}_k &= \operatorname{Re}\sum_{k=1}^{n}\frac{1}{2\pi r^k}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})e^{-ik\theta}\overline{u}_k\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\operatorname{Re}\sum_{k=1}^{n}\frac{1}{r^k}e^{-ik\theta}\overline{u}_k\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})\,d\theta\cdot(M+\varepsilon) \\
&= M+\varepsilon
\end{aligned}
$$

thanks to the positivity of $\operatorname{Re} f$, and by limiting $r\to 1$ from left we get the desired bound. □

Solution of 3. (1)

$$\frac{du \wedge dv}{\pi(1+u^2+v^2)^2} = d\left(-\frac{v}{2\pi(1+u^2+v^2)}\,du + \frac{u}{2\pi(1+u^2+v^2)}\,dv\right)$$

$$P = -\frac{K_y}{4\pi K}, \qquad Q = \frac{K_x}{4\pi K}.$$

(2)

(3) Since every Taylor coefficient of the log function is real, we have

$$\operatorname{Re}\log f(z) = \frac{1}{2}(\log f(z) + \log \overline{f(z)}) = \log|f(z)|.$$

Take $a \in \mathbb{C}$ and let $r := 2|a|$. By the Schwarz integral formula,

$$\begin{aligned}
\log|f(a)| = \operatorname{Re}\log f(a) &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{re^{i\theta}+a}{re^{i\theta}-a}\operatorname{Re}\log f(re^{i\theta})\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}+a}{re^{i\theta}-a}\right|\log|f(re^{i\theta})|\,d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} 3\log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta \\
&\le \int_0^r A(t,f)\frac{dt}{t} \lesssim \int_0^r t^2\frac{dt}{t} \lesssim |a|^2.
\end{aligned}$$

□

Solution of 4. (1) $(x_0, y_0) \in \partial \widetilde{\Omega}$

$$\lim_{(x,y)\to(x_0,y_0)} \widetilde{u}(x,y) = \begin{cases} 0 & \text{if } y_0 > 0, \\ 2 & \text{if } y_0 < 0. \end{cases}$$

(2) $\widetilde{\Omega}$ is conformally mapped onto the upper half plane by

$$\varphi : z \mapsto \left(\frac{z+i}{iz+1} \right)^2.$$

(3) We can compute

$$|\varphi(x+iy)|^2 = \left(\frac{x^2+(y+1)^2}{x^2+(y-1)^2} \right)^2, \qquad \operatorname{Im} \varphi(x+iy) = \frac{4x(1-x^2-y^2)}{(x^2+(y-1)^2)^2}.$$

For $x^2+y^2>1$ the Poisson kernel gives that

$$\begin{aligned} U(x,y) &= \frac{2}{\pi} \int_{-1}^{1} \frac{y}{(x-t)^2+y^2}\, dt \\ &= \frac{2}{\pi} \left(\tan^{-1} \frac{1-x}{y} + \tan^{-1} \frac{1+x}{y} \right) \\ &= \frac{2}{\pi} \tan^{-1} \frac{2y}{x^2+y^2-1}. \end{aligned}$$

$$u(x,y) = U(\operatorname{Re} \varphi(x+iy), \operatorname{Im} \varphi(x+iy)).$$

Thus we have

$$u(x,y) = \frac{2}{\pi} \tan^{-1} \frac{x(1-x^2-y^2)}{y(1+x^2+y^2)}.$$

□