Algebraic quantum field theory

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Chapter 1

Axiomatic: Osterwalder-Schrader, Wightman, Haag-Kastler

CFT

Statistical physics: Gibbs state by DLR equation, Lieb-Robinson bound, quantum theory

1.1

1.1 (Geometric quantization)**.** On a closed symplectic manifold (X,ω), fix a *pre-quantum line bundle* on X, which is a structured line bundle $(L,\nabla)\to X$ such that the curvature is the symplectic form ω. Define the *pre-quantum operator* $Q: C^\infty(X)\to\operatorname{End}(\Gamma^\infty(L))$ such that

$$Q(f):=-i\nabla_{X_f}+f,$$

where X_f is the Hamiltonian vector field associated to $f\in C^\infty(X)$ with respect to the symplectic structure. polarization and metaplectic correction to construct the Hilbert space.

I think geometric quantizatioin does not work for field theory in general, and the Chern-Simons theory is the one very exceptional.

1.2 (Deformation quantization)**.**

1.3 (Path integral quantization)**.** It generates a FQFT as follows. If we describe the Hilbert space H in FQFT as function spaces to describe the propagator corresponded to the world-volume Σ as a integral operator, we want to compute the kernel in terms of action functional as

$$k(x,y)=\int_{\mathrm{Conf}_\Sigma}[D\varphi]e^{iS(\varphi)}.$$

For almost cases(perhaps), it is done by σ-models, in which we set the configuration as $\mathrm{Conf}_\Sigma:=\mathrm{Map}(\Sigma,X)$ of maps from a world-volume Σ to the target space X. Each function $f\in C^\infty(X)$ defines an operator

1.4. Action functionals

(a) Classical mechanics: Let $\mathrm{Conf}_\Sigma:=C^\infty(\Sigma,X)$, where $\Sigma=[t_i,t_f]$ and $X=T^*\mathbb{R}^3=T\mathbb{R}^3$. For a given background potential $V:X\to\mathbb{R}$, we can define the action $S(\varphi):=\int_\Sigma L(t,\varphi,\dot\varphi)\,dt$, where $L(t,x,v):=\frac{m}{2}\|v\|^2+V(x)$, $m>0$ is the mass parameter.

(b) General relativity: the configuration space is the space of pseudo-metrics, Einstein-Hilbert action

(c) Electromagnetism and Yang-Mills theory: the configuration space is the space of principal bundles with connections, and the action is $S(A):=\frac{1}{2}\langle F,F\rangle$.

(d) Chern-Simons theory: the configuration space is the space of principal bundles with connections on three-dimensional manifolds, and the action is $S(\omega):=\mathrm{CS}_Q(\omega)$, where Q is a symmetric polynomial in Chern-Weil theory.

Except classical mechanics, Σ is the space-time.

In principle, the configuration space is the section space of a fiber bundle, and the Lagrangian is a map from the configuration space to the space of densities, which defines an action functional.

1.2

Consider a Poincaré equivariant topological left(really?) module $H = L^2(H_m)$ over a commutative algebra of the Minkowski space $A = \mathcal{S}(\mathbb{R}^{1,d-1}_x)$. We have a partially defined map $A \to H$ (fourier restriction).

An AQFT on Minkowski space can be seen as a causal conical cyclic Poincaré covariant representation $\phi : H \to B(FH)$ of the left A-module H.

For an ideal I supported on $O \subset \mathbb{R}^{1,d-1}_x$ of A, we can produce a smaller A-module IH and the algebra generated by IH in $B(FH)$ can be considered.

1.5 (Wightman axioms)**.** Let $\mathbb{R}^{1,d-1}_x$ be the Minkowski space and $\mathcal{P}^{\uparrow}_{+}$ the connected component of the Poincaré group. A *Wightman field* is a linear map $\phi : \mathcal{S}(\mathbb{R}^{1,d-1}_x) \to \operatorname{End}(\mathcal{D})$, where $\mathcal{D}$ is an inner product space with completion $\mathcal{H}$, such that

(i) Covariance: there is a representation $U : \mathcal{P}^{\uparrow}_{+} \to U(\mathcal{H})$ such that $\operatorname{Ad} U(\gamma)\phi(f) = \phi(\gamma^* f)$,

(ii) Causality: if the supports of f and g are space-like separated, then $[\phi(f), \phi(g)] = 0$ on D,

(iii) Conicality:

(iv) Cyclicity: there is a Poincaré invaraint cyclic vector Ω in the sense that the span of the set $\{\phi(f_1)\cdots\phi(f_n)\Omega\}$ is dense in $\mathcal{D}$.

1.6 (Free massive bosonic fields)**.** Let $m > 0$, called the mass of a scalar particle, and let $d = 1 + 1$ be a positive integer, called the dimension. Note $\mathcal{P}^{\uparrow}_{+} = \mathbb{R}^{1,1} \rtimes \mathbb{R}$. On the mass shell $H_m := \{p \in \mathbb{R}^{1,1}_p : (p,p) = p_0^2 - p_1^2 = m^2,\ p_0 > 0\}$, the induced metric is Riemannian with the volume form $(p_1^2 + m^2)^{-\frac{1}{2}} dp_1$, so we can define $L^2(H_m)$.

For $f \in \mathcal{S}(\mathbb{R}^{1,1}_x)$, consider the restriction of the Fourier transform $\hat{f} \in L^2(H_m)$.

$$\hat{f}(p) = \int_{\mathbb{R}^{1,1}} e^{i(x,p)} f(x)\, d^2x, \qquad p \in H_m,$$

where d^2x is the Lebesgue measure on $\mathbb{R}^{1,1}$, which is Lorentz invariant. Via the Bosonic Fock space construction $\mathcal{F}^+(L^2(H_m))$, we define a operator-valued distribution

$$\phi : f \mapsto a^\dagger(\hat{f}) + a(\hat{f}),$$

$\phi(f)$ is defined densely on $\mathcal{F}^+(L^2(H_m))$.

(a) ϕ is covariant.

(b) ϕ is local.

(c) ϕ has positive energy.

(d) ϕ admits a vaccum.

(e) ϕ has linear energy bound. In particular, it defines a Araki-Haag-Kastler net.

Proof. (a) Consider a representation $U_m : \mathcal{P}^{\uparrow}_{+} \to U(L^2(H_m))$ on $L^2(H_m)$, defined by

$$(U_m(a,\Lambda)\Psi)(p) := e^{i(a,p)}\Psi(\Lambda^{-1}p), \qquad (a,\Lambda) \in \mathcal{P}^{\uparrow}_{+},\ \Psi \in L^2(H_m).$$

The action $U_m : \mathcal{P}_+^\uparrow \to U(L^2(H_m))$ is extended to $\Gamma(U_m) : \mathcal{P}_+^\uparrow \to U(\mathcal{F}^+(L^2(H_m)))$, called the second quantization. Then, since $\mathcal{F}(a,\Lambda)\mathcal{F}^{-1}$ maps

$$(p \mapsto \int e^{i(x,p)} f(x)\, d^2x) \quad \text{to} \quad (p \mapsto \int e^{i(x,p)} f(\Lambda^{-1}(x-a))\, d^2x = e^{i(a,p)} \int e^{i(x,\Lambda^{-1}p)} f(x)\, d^2x),$$

so it is covariant.

(b) Define the left and right wedges

$$W_L := \{x \in \mathbb{R}^{1,1} : |x_0| \le -x_1\}, \qquad W_R := \{x \in \mathbb{R}^{1,1} : |x_0| \le x_1\}.$$

Suppose f and g are Schwartz functions supported on W_L and W_R respectively. Write

$$[\phi(f), \phi(g)] = [a^\dagger(\widehat{f}), a(\widehat{g})] + [a(\widehat{f}), a^\dagger(\widehat{g})].$$

analytic continuation and residue theorem... If $f(x) \neq 0$ and $g(y) \neq 0$, then x and y are contained in the interior of W_L and W_R, so $(x,y) > 0$.

□

What is the interactions?

Conformal nets, vertex operator algebras, and Segal's picture. Factorization algebra?

1.3

A stochastic process is a family of $*$-homomorphisms $\varphi_t : A \to (N,\tau)$ indexed by t.

A stochastic process $\{\varphi_t\}$ is called *Markov* if there is a unital positive linear semi-group action α on A such that $\varphi_t(a) = \varphi_0(\alpha_t(a))$.

For a unital positive linear semi-group action α on A and an initial condition in A^*, then we can construct a Markov process $\varphi_t : A \to (N,\tau)$. The algbera N does not depend on the initial condition, but the trace τ is determined by the initial condition.