C*-Algebras

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Contents

Part I

C^*-algebras

Chapter 1

Basic concepts

1.1 Multiplier algebra

1.1 (Multiplier algebra). Let $\mathcal{A}$ be a C*-algebra. A *double centralizer* of $\mathcal{A}$ is a pair (L, R) of bounded linear maps on $\mathcal{A}$ such that $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$. The *multiplier algebra* $M(\mathcal{A})$ of $\mathcal{A}$ is defined to be the set of all double centralizers of $\mathcal{A}$.

1.2 (Essential ideals). (a) Hilbert C*-module description

1.3 (Examples of multiplier algebras). (a) $M(K(H)) \cong B(H)$.

(b) $M(C_0(\Omega)) \cong C_b(\Omega)$.

Proof. (a)

(b) First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega) \cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega) \cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective *-homomorphism $C_b(\Omega) \to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g \in M(C_0(\Omega))^+$. □

1.4 (Strict topology).

1.2 Hereditary C*-subalgebras

1.5 (Hereditary C*-subalgebra and state embedding).

1.3 Tensor products

1.4 State approximation theorems

Chapter 2

Operator systems

Exercises

2.1. Let $\mathcal{B}$ be a hereditary C*-subalgebra of a C*-algebra $\mathcal{A}$. Let $a \in \mathcal{A}^+$. If for any $\varepsilon > 0$ there is $b \in \mathcal{B}^+$ such that $a - \varepsilon \leq b$, then $a \in B^+$.

Proof. To catch the idea, suppose $\mathcal{A}$ is abelian. We want to approximate a by the elements of $\mathcal{B}$ in norm. To do this, for each $\varepsilon > 0$, we want to construct $b' \in \mathcal{B}^+$ such that $a - \varepsilon \leq b' \leq a + \varepsilon$ using b. Taking $b' = \min\{a, b\}$ is impossible in non-abelian case, but we can put $b' = \frac{a}{b+\varepsilon}b$. For a simpler proof, $b' = (\frac{\sqrt{ab}}{\sqrt{b}+\sqrt{\varepsilon}})^2$ is a better choice.

Define

$$b' := \frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}a\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}.$$

Then,

$$\|\sqrt{a} - \sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\|^2 = \|\frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}}a\frac{\sqrt{\varepsilon}}{\sqrt{b}+\sqrt{\varepsilon}}\| \leq \varepsilon$$

implies

$$\lim_{\varepsilon\to 0} b' = \lim_{\varepsilon\to 0}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}}\sqrt{a}\cdot\sqrt{a}\frac{\sqrt{b}}{\sqrt{b}+\sqrt{\varepsilon}} = \sqrt{a}\cdot\sqrt{a} = a.$$

□

Chapter 3

Chapter 4

Part II

Approximation properties

Part III

Constructions

Part IV

Operator K-theory

Chapter 5

Brown-Douglas-Fillmore theory

5.1 (Haagerup property)**.**

Baum-Connes conjecture Non-commutative geometry Elliott theorem

5.1 Approximately finite algebras

Elliott conjecture: amenable simple separable C^*-algerbas are classified by K-theory.