Complex Analysis

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Contents

Part I

One complex variable

Chapter 1

Holomorphic functions

1.1 Cauchy theory

1.1 (Holomorphic functions)**.** A *domain* in $\mathbb{C}$ means a non-empty connected open subset of the complex plane $\mathbb{C}$. A complex valued function f defined on a domain $\Omega \subset \mathbb{C}$ is called *holomorphic* if it is C^1 and complex differentiable, that is, the following limit exists for every $a \in \Omega$:

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

The set of all holomorphic functions on Ω is denoted by $\mathrm{Hol}(\Omega)$ or $\mathcal{O}(\Omega)$. Cauchy-Riemann equation can be interpreted as several ways: the matrix representation of df corresponds to a complex number via $x + iy \mapsto \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)$, the closedness of the 1-form $f(z)\,dz$.

Let $f \in C^1(\Omega, \mathbb{C})$ on a domain $\Omega \subset \mathbb{C}$. Write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$.

(a) f is holomorphic if and only if it satisfies the Cauchy-Riemann equation in Ω.

Proof. We may assume $a = 0 \in \Omega$. Since f is C^1, we have the Taylor expansion

$$f(z) - f(0) = u_x(0)x + u_y(0)y + i(v_x(0)x + v_y(0)y) + o(|z|), \qquad z \to 0.$$

($\Rightarrow$) Let $y = 0$ so that $z = x$. Then,

$$f(z) - f(0) = u_x(0)x + iv_x(0)x + o(|x|), \qquad x \to 0$$

implies $f'(0) = u_x(0) + iv_x(0)$. If we let $x = 0$ so that $z = iy$, then

$$f(z) - f(0) = u_y(0)(y) + iv_y(0)y + o(|y|), \qquad y \to 0$$

implies $f'(0) = -iu_y(0) + v_y(0)$.

($\Leftarrow$) If the Cauchy-Riemann equation implies

$$f(z) - f(0) = u_x(0)z + iv_x(0)z + o(|z|), \qquad z \to 0.$$

$\square$

1.2 (Contour integral)**.** We mean by a *contour* on a domain $\Omega \subset \mathbb{C}$ is a formal sum $\gamma = \sum_{i=1}^{n} \gamma_i$ with $n \geq 1$ of C^1 paths $\gamma_i : [a_i, b_i] \to \Omega$ such that $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$ for all $1 \leq i < n$ and $\gamma_n(b_n) = \gamma_1(a_1)$, which we call the components of γ. In other words, a contour can just be regarded as a piecewise C^1 closed curve. A formal sum of contours on Ω whose components are all defined on the unit interval is called a C^1 *singular 1-cycle* on Ω.

The *contour integral* of $f \in \mathrm{Hol}(\Omega)$ along a contour $\gamma = \sum_{i=1}^n \gamma_i$ is defined by

$$\int_\gamma f(z)\,dz := \sum_{i=1}^n \int_{a_i}^{b_i} \gamma_i^*(f(z)\,dz) = \sum_{i=1}^n \int_{a_i}^{b_i} f(\gamma_i(t))\gamma_i'(t)\,dt.$$

(a) The contour integral does not depend on the choice of Ω containing γ, and on the reparametrization of γ.

(b) If we denote by $|z| = 1$ the contour $\gamma(\theta) := e^{i\theta}$ with $\theta \in [0, 2\pi]$, then for $n \in \mathbb{Z}$ we have

$$\int_{|z|=1} z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise .} \end{cases}$$

1.3 (Cauchy theorem)**.** We mean by a *triangle* in a domain $\Omega \subset \mathbb{C}$ a map $\sigma : \Delta \to \Omega$ that has a C^1 extension on a neighborhood of Δ, where

$$\Delta := \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq 1\}.$$

The *boundary* of a triangle σ is a contour defined as $\partial\sigma = \gamma_1 + \gamma_2 + \gamma_3$, where

$$\gamma_1(t) := \sigma(t, 0), \quad \gamma_2(t) := \sigma(1-t, t), \quad \gamma_3(t) := \sigma(0, 1-t), \qquad t \in [0, 1],$$

and a formal sum of the boundary of triangles is called a C^1 *singular 1-boundary* on Ω.

(a) A contour on Ω whose components are defined on the unit interval is null-homotopic if and only if it is the sum of the boundary of some triangles in Ω.

(b) Ω is contractible if and only if Ω is simply connected.

(c) If Ω is simply connected, then for a contour γ and a holomorphic function f on Ω,

$$\int_\gamma f(z)\,dz = 0.$$

Proof. (a) C^1 approximation...

(c) Since f is holomorphic, the 1-form $f(z)\,dz$ is closed. The Stokes theorem writes

$$\int_{\partial\sigma} f(z)\,dz = \int_\sigma d(f(z)\,dz) = 0$$

for arbitrary triangle $\sigma : \Delta \to \Omega$. □

1.4 (Cauchy integral formula)**.** Let f be a holomorphic function on a simply connected domain $\Omega \subset \mathbb{C}$.

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz.$$

Remind the proof of the mean value property for harmonic functions. The proof essentially have a shrinking process using the homotopy and uses the boundedness of the difference quotient. Higher order version: we can prove before the analyticity by interchange of diff and int.

Proof. □

1.5 (Cauchy estimates)**.** (a) If an entire function f satisfies $|f(z)| \lesssim 1 + |z|^n$, then f is a polynomial of degree at most n. In particular, the *Liouville theorem* follows; a bounded entire function is constant.

1.2 Power series

1.6 (Analyticity of holomorphic functions).

$$\sup_{z\in K}\left|\frac{f^{(k)}(z)}{k!}\right|^{\frac{1}{k+1}}<\infty.$$

(a) A real function on $I\subset\mathbb{R}$ is analytic if and only if it has an analytic extension on an open neighborhood Ω of I in $\mathbb{C}$.

1.7 (Identity theorem).

identity theorem for harmonic: on an open set, but not on the real line, e.g. 0 and y

1.8 (Open mapping theorem).

inverse function if $n=1$ open mapping if $n\geq 1$ Maximum principle Schwarz lemma and description of automorphisms of the disk

1.9 (Morera theorem). The C^1 condition in the definition of holomorphic functions is necessary to apply the Stokes theorem when we prove the Cauchy theorem. However, the C^1 condition can be dropped and the pointwise complex differentiability is sufficient to check a function is holomorphic. Let $f\in C(\Omega,\mathbb{C})$ on a domain $\Omega\subset\mathbb{C}$.

(a) If for every point $a\in\Omega$ there is an open neighborhood U of a in Ω in which every affine triangle $\sigma:\Delta\to U$ satisfies $\int_{\partial\sigma}f(z)\,dz=0$, then f is holomorphic. (Morera)

(b) If f is complex differentiable everywhere on Ω, then it is holomorphic. (Goursat)

Proof. (a) Let $U=\{z\in\Omega:|z-a|<\varepsilon\}$ for sufficiently small ε in which every triangle $\sigma:\Delta\to U$ is integrated out by f. If we define

$$F(z):=\int_0^z f(\zeta)\,d\zeta,\qquad z\in U,$$

then by the triangle condition, we have

$$F(z+h)-F(z)=\int_z^{z+h}f(\zeta)\,d\zeta.$$

We can show $F'(z)=f(z)$ by the continuity of f, so F is holomorphic on U. Therefore f is holomorphic because it also has the power series representation as well as F.

(b) We prove $\int_{\partial\sigma}f(z)\,dz=0$ for all affine triangle $\sigma:\Delta\to\Omega$. Suppose not. Then, there is a triangle $\sigma:\Delta\to\Omega$ such that $\int_\sigma f(z)\,dz\neq 0$. By subdivision, we have $\partial\sigma\simeq\sum_{i=1}^4\partial\sigma_i$ with $\operatorname{diam}\sigma_i\leq\frac{1}{2}\operatorname{diam}\sigma$, so there is i such that

$$|\int_{\partial\sigma_i}f(z)\,dz|\geq\frac{1}{4}|\int_\sigma f(z)\,dz|.$$

Then, we have a sequence of affine triangles σ_n such that

$$|\int_{\partial\sigma_n}f(z)\,dz|\geq\frac{1}{4^n}|\int_\sigma f(z)\,dz|.$$

Take $a\in\Omega$ the limit point of the subdivision. By the assumption, there is $\delta>0$ such that

$$|z-a|<\delta\quad\Rightarrow\quad\left|\frac{f(z)-f(a)}{z-a}-f'(a)\right|<\varepsilon,$$

so we see that

$$|\int_{\partial\sigma_n}f(z)\,dz|=|\int_{\partial\sigma_n}(f(z)-f(a)-f'(a)(z-a))\,dz|\leq\varepsilon\sup_{z\in\partial\sigma_n}|z-a|\cdot\operatorname{length}(\partial\sigma_n)\lesssim\frac{\varepsilon}{4^n}.$$

The limit $\varepsilon\to 0$ leads to a contradiction. $\square$

1.3 Harmonic functions on two dimensions

Harmonic conjugate

1.10 (Mean value property).

$$\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})(re^{i\theta})^{-k}\,d\theta = \begin{cases} 0 & \text{if } k < 0 \\ \dfrac{f^{(k)}(0)}{k!} & \text{if } k \geq 0 \end{cases}$$

for r such that f is defined on $\overline{B}_r$.

1.11 (Schwarz integral formula). Let f be a holomorphic function on the open unit disk $\mathbb{D}$. If h is another holomorphic function, then

$$f(a) = \frac{1}{2\pi}\int_{|z|=r} f(z)\left(\frac{z}{z-a} + zh(z)\right)\frac{dz}{iz}$$

for $0 < r < 1$. Schwarz integral formula

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta}+a}{re^{i\theta}-a}\operatorname{Re} f(re^{i\theta})\,d\theta + i\operatorname{Im} f(0).$$

(a) Find the holomorhpic h_a on an open neighborhood of $\mathbb{D}$ in terms of a such that $|z| = 1$ implies $\frac{z}{z-a} + zh_a(z)$ is real.

(b) Poisson kernel.

Proof.

$$h_a(z) =$$

□

Maximum principle; Lindelöf principle,

1.4 Polynomial approximatioin

Mittag-Leffler theorem

1.12 (Compact convergence of holomorphic functions). (a) injectivity preservation: Hurwitz theorem

Principal part For a meromorphic function f, we say a polynomial p without constant term is a *principal part* of f at z_0 if we have a partial fraction decomposition

$$f(z) = p\left(\frac{1}{z-z_0}\right) + h(z),$$

where $h(z)$ is holomorphic at z_0. It is unique. pre-assigned principal parts

Weierstrass factorization theorem Infinite product

Runge's approximation Mergelyan

Exercises

1.13 (Wirtinger derivatives).

1.14 (Branch of logarithm and nth root). on simply connected domain

1.15 (Log$r$ on $\mathbb{C} \setminus \{0\}$). harmonic function wihtout harmonic conjugate?

1.16 (Fundamental theorem of algebra). Let $p \in \mathbb{C}[z]$ be a polynomial of degree n such that

$$p(z) = \sum_{k=0}^{n} c_k z^k, \quad {}_n \neq 0.$$

(a) $|p(z)| \lesssim |z|^n$.

(b) There is $R > 0$ such that $|p(z)| \gtrsim |z|^n$ for $|z| \geq R$.

Proof. (b) We want to justify that the leading term $a_n z^n$ is dominant in the series $\sum_{k=0}^{n} c_k z^k$ when $|z|$ is sufficiently large. Let $\varepsilon > 0$. Since $p(z) - c_n z^n$ is of degree at most $n-1$, we can take $R > 0$ such that for $|z| \geq R$ we can control the relative error as

$$\left| \frac{p(z) - c_n z^n}{c_n z^n} \right| < \varepsilon,$$

which implies

$$|p(z)| \geq (1-\varepsilon)|c_n||z^n|.$$

$\square$

Problems

1. If a holomorphic function has positive real parts on the open unit disk then $|f'(0)| \leq 2\operatorname{Re} f(0)$.
2. If at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.
3. If a holomorphic function on a domain containing the closed unit disk is injective on the unit circle, then so is on the disk.
4. For a holomorphic function f and every z_0 in the domain, there are $z_1 \neq z_2$ such that $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$.
5. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain. Then, $\overline{f(z)} = f(\overline{z})$ if and only if $f(z) \in \mathbb{R}$ for $z \in \Omega \cap \mathbb{R}$.
6. For two linearly independent entire functions, one cannot dominate the other.
7. The uniform limit of injective holomorphic function is either constant or injective.
8. If the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions converges has a limit point, then it compactly converges.
9. Find all entire functions f satisfying $f(z)^2 = f(z^2)$.
10. An entire function maps every unbounded sequence to an unbounded sequence is a polynomial.
11. If a holomorphic function satisfies $\operatorname{Re} f(z) \leq 1 + |z|^2$, then f is a polynomial at most degree two.
12. If $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is a holomorphic function defined on the open unit disk satisfying $\sum_{k=2}^{\infty} k|c_k| \leq |c_1| \neq 0$, then f is injective. (Grunsky coefficients)

Chapter 2

Analytic continuation

2.1 Riemann surfaces

Three perspectives: We can see $\mathbb{P}^1$ as the moduli space of lines, $U_0 \cup U_1$, and $\mathbb{C} \cup \{\infty\}$.

Runge: $\mathbb{C}[z]$ is dense in $\mathcal{O}(\Omega)$ if Ω is simply connected.

Mergelyan: $\mathbb{C}[z]$ is dense in $\mathcal{A}(\overline{\Omega}) := \mathcal{O}(\Omega) \cap C(\overline{\Omega})$.

transformation rule? gluing rule?

2.1 (Riemann sphere)**.**

- analytic continuation by functional equation
- analytic continuation by contour integral

2.2 (Analytic continuation by contour integral)**.** For a not necessarily closed contour γ on Ω,

$$h(a) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz, \qquad a \in \Omega \setminus \operatorname{im}\gamma$$

is a holomorphic function on $\Omega \setminus \operatorname{im}\gamma$. For this, you can use either the power series or the Morera with Fubini.

If f is holomorphic on the complement of a zero-length set(can we describe it with rectifiability?) in Ω, then it is holomorphic. (Painlevé)

2.3 (Branch cut)**.** We can represent f with any coordinate system(usually polar coordinates).

Define $f : \{re^{i\theta} : r > 0, -\pi < \theta < \pi\} \to \mathbb{C}$ such that

$$f(re^{i\theta}) := \log r + i\theta.$$

Then, $e^{f(z)} = z$. Define $f : \{x + iy : y \neq 0 \text{ or } -1 < x < 1\} \to \mathbb{C}$ such that

$$f(z) := \frac{1}{\sqrt{r_+ r_-}} e^{i\frac{\theta_+ + \theta_-}{2}},$$

where $z - 1 = r_+ e^{i\theta_+}$ and $z + 1 = r_- e^{i\theta_-}$. Then, $f(z)$ is a branch of $1/\sqrt{z^2-1}$.

Monodromy Covering surfaces Algebraic functions Elliptic functions Uniformization

Chapter 3

Zeros and poles

3.1 Isolated singularities

3.1 (Isolated singularities). removable singularity, pole, essential singularity

3.2 (Laurent series expansion).

3.3 (Casorati-Weierstrass theorem).

3.4 (Picard's theorems).

3.2 Residue theorem

3.5 (Residue theorem).

3.6 (Unit circle substitution).

$$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \quad -1 < a < 1$$

3.7 (Semicircular contour). We want to justify the following definite integral:

$$\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{2e}.$$

This can be viewed as a special value of the characteristic function of the *Cauchy distribution* in probability theory. Define $f : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}$ and the *semicircular contour* $\gamma = \gamma_1 + \gamma_2$ for $R > 0$ as follows:

$$f(z) := \frac{e^{iz}}{z^2+1}, \qquad \begin{cases} \gamma_1(x) := x & \text{for } x \in [-R, R], \\ \gamma_2(\theta) := Re^{i\theta} & \text{for } \theta \in [0, \pi]. \end{cases}$$

(a) We have

$$\sup_{R>0} \int_{\gamma_2} |e^{iz}|\,|dz| \le 1.$$

This is called the *Jordan lemma*.

(b)

$$\lim_{R\to\infty} \int_{\gamma_i} f(z)\,dz = \begin{cases} 2\int_0^\infty \frac{\cos x}{x^2+1}\,dx & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

(c)

$$\lim_{R\to\infty}\int_\gamma f(z)\,dz = \frac{\pi}{e}.$$

Proof. (a) Let $M_R = \max_{z\in\gamma_2}|h(z)|$. Since $\sin\theta \ge \frac{2}{\pi}\theta$ for $0\le\theta\le\frac{\pi}{2}$, we have

$$\begin{aligned}\left|\int_{\gamma_2} e^{iz}h(z)\,dz\right| &= \left|\int_0^\pi e^{iRe^{i\theta}}h(Re^{i\theta})\,iRe^{i\theta}\,d\theta\right|\\ &\le M_R R\int_0^\pi e^{-R\sin\theta}\,d\theta\\ &= 2M_R R\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta\\ &\le 2M_R R\int_0^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\theta}\,d\theta\\ &= \pi M_R(1-e^{-R}).\end{aligned}$$

So we are done because $\lim_{R\to\infty} M_R = 0$.

(b) For $i=1$, we have

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx = 2\int_0^\infty f(x)\,dx$$

by the definition of improper integrals. For $i=2$, it clearly follows from the part (a).

(c) Note that for sufficiently large R, the function f has only one pole at $z=i$ in the interior of C, which is simple; define $g:\operatorname{int}\gamma\to\mathbb{C}$ such that

$$f(z) =: \frac{g(z)}{(z-i)} = \frac{g(i)}{z-i} + \frac{g(z)-g(i)}{z-i}.$$

Then, by the residue theorem, we obtain

$$\int_\gamma f(z)\,dz = 2\pi i\operatorname{Res}(f,i) = \frac{\pi}{e}$$

for sufficiently large R such that $R>1$. □

3.8 (Indented contour). Indented contour is often used to compute the principal value of integrals. Here we want to justify the *Dirichlet integral* as an example:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Define $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ and the *indented contour* for $r,R>0$ as follows:

$$f(z) = \frac{e^{iz}}{z},\qquad \begin{cases}\gamma_1: x\mapsto x, & x\in[r,R],\\ \gamma_2:\theta\mapsto Re^{i\theta}, & \theta\in[0,\pi],\\ \gamma_3: x\mapsto x, & x\in[-R,-r],\\ \gamma_4:\theta\mapsto re^{\pi-\theta}, & \theta\in[0,\pi].\end{cases}$$

The indented contour is effective when f has a simple pole at zero.

(a)

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_C f(z)\,dz = \begin{cases}0 & \text{if } \gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4\\ 2i\int_0^\infty \frac{\sin x}{x}\,dx & \text{if } \gamma=\gamma_1+\gamma_3\\ 0 & \text{if } \gamma=\gamma_2\\ -\pi i & \text{if } \gamma=\gamma_4.\end{cases}$$

Proof. It follows from the Jordan lemma.

For $\gamma = \gamma_4$, since we have a partial fraction decomposition

$$f(z) = \frac{1}{z} + h(z), \qquad h(z) := \frac{e^{iz} - 1}{z},$$

where h has a removable singularity at zero,

$$\int_{\gamma_4} f(z)\,dz = \int_{\gamma_4} \frac{dz}{z} + \int_{\gamma_4} h(z)\,dz \to -\pi i + 0$$

as $r \to \infty$. □

3.9 (Sector contour). We want to justify the *Fresnel integral*:

$$\int_0^\infty \cos x^2\,dx = \sqrt{\frac{\pi}{8}}.$$

Sector contour is also used to compute the Fourier transform of Gaussian function, which also contains a nonlinear polynomial in a exponential term. Define $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ and the *circular sector contour* for $R > 0$ as follows:

$$f(z) = e^{iz^2}, \qquad \begin{cases} \gamma_1 : x \mapsto x, & x \in [0, R], \\ \gamma_2 : \theta \mapsto Re^{i\theta}, & \theta \in [0, \frac{\pi}{4}], \\ \gamma_3 : x \mapsto (R - x)e^{\frac{\pi}{4}i}, & x \in [0, R]. \end{cases}$$

(a)

Proof. (b)

□

3.10 (Rectangular contour). A rectangular contour is used for the Fourier transform of functions periodic along imaginary direction.

$$\int_0^\infty \frac{\sin x}{e^x - 1}\,dx, \qquad \int_0^\infty \frac{\cos x}{\cosh x}\,dx$$

3.11 (Keyhole contour). the *keyhole contour* or the *Hankel contour*

$$\int_0^\infty \frac{x^{a-1}}{1+x} = \frac{\pi}{\sin \pi a} \quad (0 < a < 1), \qquad \int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}}$$

$\log z$ trick

$$\int_0^\infty \frac{dx}{1 + x^3}$$

3.3 Argument principle

3.12 (Argument principle).

(a) We have a partial fraction decomposition

$$\frac{f'(z)}{f(z)} = \frac{\operatorname{ord}_a(f)}{z - a} + h(z),$$

where h is holomorphic at a.

(b)

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} g(z)\,dz = \sum_a \operatorname{ord}_a(f) g(a).$$

(c) Winding number

Proof.

$$\frac{f'(z)}{f(z)} = \frac{\mathrm{ord}_a(f)}{z-a} + \frac{g'(z)}{g(z)},$$

where $g(z) := f(z)/(z-a)^{\mathrm{ord}_a(f)}$ is holomorphic at a. □

3.13 (Rouché theorem). Let f be a meromorphic function on Ω.

(a) If $h : [0,1] \times \Omega \to \mathbb{C}$ is continuous, then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

In particular, if $|g(z)| < |f(z)|$ on $z \in \gamma$, then

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz.$$

3.4 Nevanlinna theory

3.14 (Poisson-Jensen formula).

3.15 (Nevanlinna functions). Let f be a meromorphic function on a neighborhood of the closed disk $\overline{B(0,r)} \subset \mathbb{C}$ and let $a \in \mathbb{CP}^1$. We count the number of poles in the region $|z| \le r$, counting multiplicity, with the following function

$$n(r,a,f) := \sum_{|z|\le r} (\mathrm{ord}_z(f-a))^+, \qquad n(r,f) := n(r,\infty,f).$$

Note that $n(r,a,f) = n(r,(f-a)^{-1})$ and $n(0,f^{-1}) - n(0,f) = \mathrm{ord}_0 f$. The *Nevanlinna proximity function* is

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta.$$

The *Nevanlinna counting function* is

$$N(r,f) := \int_0^r (n(t,f) - n(0,f))\,\frac{dt}{t} + n(0,f)\log r.$$

The *Nevanlinna characteristic function* is

$$T(r,f) := m(r,f) + N(r,f).$$

3.16 (First fundamental theorem). Jensen formula

3.17 (Second fundamental theorem).

3.18 (Ahlfors-Shimizu formulation). Let f be a meromorphic function on $\mathbb{C}$. Consider the following uniform probability measure on the Riemann sphere

$$d\rho(w) := \frac{du\,dv}{\pi(1+|w|^2)^2}, \qquad w = u+iv.$$

Define

$$A(r,f) := \frac{1}{\pi}\int_{|z|\le r} f^\#(z)^2\,dx\,dy = \int_{|z|\le r} f^* d\rho, \qquad f^\#(z) := \frac{|f'(z)|}{1+|f(z)|^2}.$$

The latter function $f^\#$ is called the *spherical derivative* of f. The *Ahlfors-Shimizu characteristic function* and *proximity function* are defined by

$$T_0(r,f) := \int_0^r A(t,f)\frac{dt}{t}, \qquad m_0(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{1+|f(re^{i\theta})|^2}\,d\theta.$$

(a) $\int \log|f-w|\,d\rho(w) = \log\sqrt{1+|f|^2}$

(b) We have
$$A(r,f) = \int n(r,a,f)\,d\rho(a) = n(r,f) + r\frac{d}{dr}m_0(r,f).$$

(c) We have $T_0(r,f) = T(r,f) + O(1)$ as $r \to \infty$.

Proof. (b)

Let F be the image of the set $\{z : |z| = r\} \cup \{z : f'(z) = 0\}$ under f. Since F and $f^{-1}(F)$ are of measure zero, so we may assume $f : U \to f(U)$ is locally biholomorphic, where $U := \{z : |z| \le r\} \setminus f^{-1}(F)$. So we may define the degree of f, which is locally constant and coincides with $n(r,a,f)$. So the first equality follows from
$$\int_{|z|\le r} f^* d\rho = \int_U f^* d\rho = \int n(r,a,f)\,d\rho(a).$$

By the argument principle,
$$n(t,a,f) - n(t,f) = \frac{1}{2\pi i}\int_{|z|=t} \frac{f'(z)}{f(z)-a}\,dz,$$

and by
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{f(z)-re^{i\theta}}\,d\theta = \frac{1}{2\pi i}\int_{|w|=r} \frac{1}{f(z)}\left(\frac{1}{f(z)-w} + \frac{1}{w}\right)dw = \begin{cases} 1/f(z) & \text{if } r < |f(z)|, \\ 0 & \text{if } r > |f(z)| \end{cases}$$

for fixed $f(z) \in \mathbb{C}$ and $r > 0$, we have
$$\begin{aligned} \int n(t,a,f)\,d\rho(a) - n(t,f) &= \frac{1}{2\pi i}\int_{|z|=t}\int \frac{f'(z)}{f(z)-a}\,d\rho(a)\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=t}\int_0^\infty\int_0^{2\pi} \frac{f'(z)}{f(z)-re^{i\theta}}\frac{r}{\pi(1+r^2)^2}\,d\theta\,dr\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=t}\int_0^{|f(z)|} \frac{2\pi f'(z)}{f(z)}\frac{r}{\pi(1+r^2)^2}\,dr\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=t} \frac{f'(z)\overline{f(z)}}{1+|f(z)|^2}\,dz. \end{aligned}$$

Also,
$$\begin{aligned} t\frac{d}{dt}m_0(t,f) &= \frac{t}{2\pi}\int_0^{2\pi} \frac{d\log\sqrt{1+|f(te^{i\theta})|^2}}{dt}\,d\theta \\ &= \frac{t}{2\pi}\int_0^{2\pi} \frac{1}{2}\frac{f'(te^{i\theta})\overline{f(te^{i\theta})} + f(te^{i\theta})\overline{f'(te^{i\theta})}}{1+|f(te^{i\theta})|^2}e^i\theta\,d\theta \\ &= \frac{1}{2\pi i}\int_{|z|=t} \frac{f'(z)\overline{f(z)}}{1+|f(z)|^2}\,dz. \end{aligned}$$

(c) Two solutions: one is $T_0(r,f) = N(r,f) + m_0(r,f) - m_0(0,f)$. Another is using $T_0(r,f) = \int N(r,a,f)\,d\rho(a)$ and the first fundamental theorem. □

Applications of second fundamental theorem? Borel directions and deficient values?

Exercises

3.19 (The second proof of the fundamental theorem of algebra). by Rouché.

3.20 (Laplace transforms).

3.21 (Gamma function). Hankel representation

3.22 (Abel-Plana formula).

Sokhotski-Plemelj theorem, Kramers-Konig relations, Titchmarsh theorem for Hilbert transform, Phragmén-Lindelöf principle, Carlson's theorem

Problems

1. We have $\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta} = \sqrt{2}\pi$.
2. Find the number of roots of $z^6 + z + 1 = 0$ in $\{x + iy \in \mathbb{C} : x > 0, y > 0\}$.
3. Find the number of roots of $z - e^{-z} = 2$ in the right half plane.
4. If f is an entire function such that $|f(z)| \le e^{|z|^\lambda}$, then $|\{z \in B(0,R) : f(z) = 0\}| \lesssim R^\lambda$.
5. There is no holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that $|f(z)| \to \infty$ for all sequences $z_n \in \mathbb{D}$ with $|z_n| \to 1$.
6. If f is a bounded holomorphic function defined on $\mathbb{C} \setminus E$, where $E \subset [0,1]$ is the Cantor set, then f is constant.
7. Suppose a sequence of nowhere vanishing holomorphic functions f_n on a domain Ω converges to a non-constant function f uniformly on compact sets. Then, f is also nowhere vanishing. (Hurwitz)

Part II

Geometric function theory

Chapter 4

4.1 Conformal mappings

4.1 (Conformality of holomorphic maps). $f' \neq 0$ and f' satisfies the Cauchy-Riemann

4.2 (Möbius transform). generators, fixed points

4.3 (Blaschke factors).

4.4 (Normal family). locally bounded, then compact (Montel)

4.5 (Schwarz lemma).

4.6 (Riemann mapping theorem). Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\Omega \neq \mathbb{C}$.

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is injective and holomorphic, and } f(z_0) = 0\}$$

(a) There exists an injective holomorphic function $f : \Omega \to \mathbb{D}$.

(b) If $0 \in \Omega_1 \subsetneq \mathbb{D}$, then there is a conformal mapping $h : \Omega_1 \to \Omega_2$ such that $h(0) = 0$ and $|h'(0)| > 1$, where $0 \in \Omega_2 \subset \mathbb{D}$.

(c) The supremum of $|f'(0)|$ is attained in $\mathcal{F}$.

(d) There exists a conformal mapping $f : \Omega \to \mathbb{D}$.

Exercises

4.7 (Special solution of Laplace' equation).

4.8 (Normal family for meromorphic functions).

Problems

1. Find a conformal mapping that maps the open unit disk onto $A := \{ z \in \mathbb{C} : \max\{|z|, |z-1|\} < 1 \}$.

Chapter 5

Univalent functions

5.1 Bierbach conjecture

5.2 Riemann-Hilbert problem

Hilbert transform almost everywhere convergence, Hardy-Littlewood maximal function

5.3 Quasi-conformal mappings

Beltrami equations and Teichmüler theory?

5.4 Exercises

5.1 (Carathéodory class). Let f be a holomorphic function on the open unit disk $\mathbb{D}$ such that $\operatorname{Re} f(z) > 0$ for $z \in \mathbb{D}$ and $f(0) = 1$. Show that $|f'(0)| \geq 2$.

Chapter 6

Part III

Several complex variables

Chapter 7

Complex analytic sheaves

7.1 Analytic spaces

7.1 (Complex model spaces)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and A is a subset of X that is the support of a coherent sheaf on X. Then, we can show (really? I hope so.) that there exists a natural sheaf $\mathcal{O}_A$ of rings on X such that A is the support of $\mathcal{O}_A$ and there is locally an exact sequence of sheaves of rings on X

$$\mathcal{O}_U^q \to \mathcal{O}_U \to \mathcal{O}_A|_U \to 0.$$

The kernel of $\mathcal{O}_X \to \mathcal{O}_A$ is called the *ideal sheaf* or the *relation sheaf* of A and denoted by $\mathcal{I}_{(X,A)}$.

Note that we have a canonical ringed space $(\Omega, \mathcal{O}_\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$. A *complex model space* is a subset A of some Ω that is the support of a coherent sheaf on Ω.

Proof. □

7.2 (Complex analytic spaces)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. An *analytic atlas* on X is the family $\{\varphi_i\}$ of maps $\varphi_i : U_i \to \mathbb{C}^{n_i}$ which map open $U_i \subset X$ homeomorphically onto open $\varphi(U_i) \subset \mathbb{C}^{n_i}$, such that φ_i and $\tau_{ij} = \varphi_j \varphi_i^{-1}$ induce the sheaf isomorphisms $\mathcal{O}_{U_i} \to \mathcal{O}_{\varphi(U_i)}$ and $\mathcal{O}_{\varphi_i(U_i \cap U_j)} \to \mathcal{O}_{\varphi_j(U_i \cap U_j)}$. A *complex analytic space* or briefly a *complex space* is a ringed space $(X, \mathcal{O}_X)$ together with an analytic atlas $\{\varphi_i\}$, satisfying an additional condition that X is Hausdorff. We do not have to assume the second countability of X because the partition of unity does not play a role in complex analysis. An *analytic set* in a complex space X is a subset A that is the support of a coherent sheaf on X.

7.2 Oka coherence theorems

$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = 0$ $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C}(z)$ $\mathbb{C}[z] = \mathcal{O}_{\mathbb{P}^1}(\mathbb{C}) \cap \mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1)$ $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PSL}(2, \mathbb{C})$ $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}^1) = \mathbb{C}(z) \cup \{\infty\}$.

Four coherence theorems:

1.
2.
3.
4.

7.3. smooth->normal(integrally closed)->irreducible(integral domain)->reduced(no nilpotents)

7.4 (Reduced points)**.** Rücker nullstellensatz, every section is realized as a family of functions, sheaf map f_* is uniquely lifted

7.5 (Weierstrass preparation theorem). Consider $\mathcal{O}'_0 \subset \mathcal{O}'_0[z_n] \subset \mathcal{O}_0$. Consider B_ρ, where $\rho = (\rho', \rho_n) \in \mathbb{R}^n_{>0}$. Note $\mathcal{O}_0 = \bigcup_\rho B_\rho$.

A Weierstrass polynomial is a monic polynomial in $\mathcal{O}'_0[z_n]$ such that $\frac{d^k w}{dz_n^k}(0,0) = 0$ for all k. We use the convention that the degree and order are with respect to z_n by letting $z' = 0$.

(a) If $f, g \in \mathcal{O}_0$, then there are unique $q \in \mathcal{O}_0$ and $r \in \mathcal{O}'_0[z_n]$ such that $\deg r < \operatorname{ord} g$ and $f = qg + r$.

(b) If $g \in \mathcal{O}_0$, then there is a unique Weierstrass polynomial $w \in \mathcal{O}'_0[z_n]$ and $u \in \mathcal{O}_0^\times$ such that $\deg w = \operatorname{ord} g$ and $g = uw$.

7.6 (Weierstrass isomorphism theorem).

7.7 (First Oka coherence theorem). Let $\mathcal{O} := \mathcal{O}_{\mathbb{C}^n}$ and assume $\mathcal{O}' := \mathcal{O}_{\mathbb{C}^{n-1}}$ is coherent. We prove that $\mathcal{O}$ is coherent at the origin.

(a) Let $0 \neq f_0 \in \mathcal{O}_0$. Then, there is an open neighborhood $U \subset \mathbb{C}^n$ of the origin such that $f \in \mathcal{O}(U)$ and $\mathcal{O}_U \to \mathcal{O}_U / f\mathcal{O}_U$ is a split epi over $\mathcal{O}_U$.

(b) Let $0 \neq f_0 \in \mathcal{O}_0$. Then, there is an open neighborhood $U \subset \mathbb{C}^n$ of the origin such that $f \in \mathcal{O}(U)$ and $\mathcal{O}_U / f\mathcal{O}_U$ is coherent over $\mathcal{O}_U$.

Proof. (a) Note that $\mathcal{O}$ is locally irreducible and has Hausdorff étale.

(b) We may assume $f(0) = 0$, i.e. there is no constant term in the power series f_0, because otherwise it is clear from $\mathcal{O}_U = f\mathcal{O}_U$ for some U. We may assume $f_0(0, z_n) \neq 0$, i.e. there is a monomial of z_n in the power series f_0 by coordinate transform. So, by the above assumptions, we have $\operatorname{ord} f_0 > 0$. By the Weierstrass preparation theorem, there is a Weierstrass polynomial $w_0 \in \mathcal{O}'_0[z_n]$ such that $\deg w = \operatorname{ord} f$ and $f_0\mathcal{O}_0 = w_0\mathcal{O}_0$.

Choose open $U' \subset \mathbb{C}^{n-1}$ such that w_0 has a representative $w \in \mathcal{O}'(U')[z_n]$. Then, we use the Weierstrass isomorphism theorem and the extension principle.

□

7.8 (Local rings on complex analytic spaces).

(a) $\mathcal{O}_{X,x}$ is isomorphic to a quotient of $\mathcal{O}_{\mathbb{C}^n,0}$.

(b) $\mathcal{O}_{X,x}$ is local, noetherian, and henselian.

(c) $\mathcal{O}_{\mathbb{C}^n,0}$ is factorial.

7.3 Levi problem

7.9 (Domains of holomorphy). A domain $\Omega \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there is no domain $\widetilde{\Omega} \subset \mathbb{C}^n$ such that Ω is a proper subset of $\widetilde{\Omega}$ and $\mathcal{O}(\widetilde{\Omega}) \to \mathcal{O}(\Omega)$ is surjective.

(a) For a compact $K \subset \Omega$ such that $\Omega \setminus K$ is connected, $\mathcal{O}(\Omega) \to \mathcal{O}(\Omega \setminus K)$ is surjective. (Hartog extension theorem)

(b) The union of increasing sequence of domains of holomorphy is a domain of holomorphy (Behnke-Stein theorem)

7.10 (Holomorphically convex domains). Let $\Omega \subset \mathbb{C}^n$ be a domain. For compact $K \subset \Omega$, the *holomorphically convex hull* in Ω is the set

$$\widehat{K}_\Omega := \{z \in \Omega : |f(z)| \leq \|f\|_{C(K)} \text{ for } f \in \mathcal{O}(\Omega)\}.$$

We say the domain Ω is *holomorphically convex* if for every compact $K \subset \Omega$ the holomorphically convex hull $\widehat{K}_\Omega$ is compact.

(a) A polydisc, a convex domain is holomorphically convex.

(b) Ω is holomorphically convex if and only if it is a domain of holomorphy if and only if $d(K, \partial\Omega) = d(\widehat{K}_\Omega, \partial\Omega)$ for every compact $K \subset \Omega$(Cartan-Thullen theorem)

7.11 (Plurisubharmonic functions)**.** Let X be a complex analytic space. An upper semi-continuous function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *plurisubharmonic* if for every holomorphic $\varphi : \mathbb{D} \subset \mathbb{C} \to X$ the composition $f \circ \varphi$ is subharmonic.

(a) If Ω is a domain of holomoprhy, then $-\log d$ is plurisubharmonic.

7.12 (Pseudo-convex domains)**.**

7.13 (Levi problem)**.**

Oka lemma?

$$\begin{tikzcd}
& \text{domain of holomorphy} \arrow[dd, dashed, leftrightarrow, "\substack{\text{in } \mathbb{C}^n \\ \text{Cartan-Thullen}}"] \arrow[dr] & & \\
\text{Stein} \arrow[ur] \arrow[dr] & & \text{continuous exhaustive plurisubharmonic} \arrow[r] & \text{pseudo-convex} \\
& \text{holomorphically convex} \arrow[ur] & &
\end{tikzcd}$$

7.4 Cartan theorem

Cartan's theorem B: if $\mathcal{F}$ is a coherent sheaf on a Stein manifold X, then $H^p(X, \mathcal{F}) = 0$ for $p \geq 1$.

Cousin problems in terms of sheaf cohomologies:

1. Characterize the image of $H^0(X, \mathcal{M}) \to H^0(X, \mathcal{M}/\mathcal{O})$.
   It is a generalization of the Mittag-Leffler theorem for prescribed poles. Consider an exact sequence
   $$H^0(X, \mathcal{M}) \to H^0(X, \mathcal{M}/\mathcal{O}) \to H^1(X, \mathcal{O}) = 0.$$
   Then, the first Cousin problem is solved when X is a Stein manifold.
2. Characterize the image of $H^0(X, \mathcal{M}^\times) \to H^0(X, \mathcal{M}^\times/\mathcal{O}^\times)$.
   It is a generalization of the Weierstrass theorem for prescribed zeros. Consider an exact sequence
   $$H^0(X, \mathcal{M}^\times) \to H^0(X, \mathcal{M}^\times/\mathcal{O}^\times) \to H^1(X, \mathcal{O}^\times).$$
   The sheaf $\mathcal{M}^\times/\mathcal{O}^\times$ is the sheaf of Cariter divisors, and line bundles are classified by $H^1(X, \mathcal{O}^\times)$. Considering the exponential exact sequence, we also have an exact sequence
   $$0 = H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) = 0.$$
   Then, the second Cousin problem is solved when X is a Stein manifold such that $H^2(X, \mathbb{Z}) = 0$. We can compute this by the first Chern class.