Analysis II

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Contents

Part I

Integration

Chapter 1

Riemann integral

1.1 Riemann integral

We are concerned only with integral on a closed interval, until considering improper integral.

1.1. Let $[a,b] \subset \mathbb{R}$ be a closed interval.

(a) The space of real-valued functions $[a,b] \to \mathbb{R}$ is Dedekind complete.

(b) The space of continuous functions $C([a,b],\mathbb{R})$ is not Dedekind complete.

1.2 (Step functions)**.** A function $f : [a,b] \to \mathbb{R}$ is called a *step function* if it is given by the real linear combination of indicator functions of closed intervals in $[a,b]$.

1.3 (Definition of Riemann integral)**.** Let R be a vector space of some real-valued functions on a closed interval $[a,b]$ containing all step functions. Let R^+ be the subset of all non-negative functions in R. The *integral* can be defined as a map $I : R^+ \to [0,\infty]$ such that

(a) it is additive and homogeneous,

(b) it is normal...

(c) $I(1_{[s,t]}) = t - s$ for all closed intervals $[s,t] \subset [a,b]$.

Such a linear functional is given, then we denote as

$$I(f) = \int_a^b f(x)\,dx, \qquad f \in R.$$

On the space of Riemann integrable functions, the Riemann integral uniquely exists.

(a) The integral $\int_a^b s(x)\,dx := \sum_{i=1}^n c_i(b_i - a_i)$, where $s(x) = \sum_{i=1}^n c_i 1_{[a_i,b_i]}(x)$, is well-defined.

(b) The integral $\int_a^b f(x)\,dx := \lim_{n\to\infty} \int_a^b s_n(x)\,dx$ is well-defined.

Proof. □

simple functions are norm dense in $L^\infty(I)$. step functions are not norm dense in $L^\infty(I)$. step functions are order dense(?) in $L^\infty(I)$.

For a given real function on interval, each (tagged) partition provides a step function. Riemann integral: tagged partition Darboux integral: partition

1.2 Fundamental theorem of calculus

Chapter 2

Lebesgue integral

2.1 Measurability

2.1 (Measurable sets).

2.2 (Measurable functions).

2.3 (Integral of complex-valued functions).

2.2 Improper integral

It is about a infinite measure. For integrable function, it has no problem.

An improper integral must be interpreted as an extension of operators from L^1. There are various way to approximate the improper integral. We need to be able to justify the reason why each specific approximation is reasonable or not.

principal values

Exercises

Problems

1. Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.
2. Find all $a > 0$ and $b > 0$ such that $\int_0^\infty x^{-b}|\tan x|^a\,dx$ converges.
*3. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f(t)\,dt \to L$ then $f(x) \to L$ as $x \to \infty$.
4. Show that for a continuous function $f : [0,1] \to \mathbb{R}$ we have $\int_0^1 x^2 f(x)\,dx = \frac{1}{3} f(c)$ for some $c \in [0,1]$.

Chapter 3

Lebesgue spaces

Proof.

$$\int fg \leq C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take C such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \Big(\int |f|^p \Big)^{\frac{1}{p}} \Big(\int |g|^p \Big)^{\frac{1}{q}}.$$

Note that we can show that $1 \leq 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \leq 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. $\square$

Part II

Multi-variable calculus

Chapter 4

Fréchet derivatives

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4.1 (Vector fields).

4.2 Inverse function theorem

4.3

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5.1 De Rham complex

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5.2 (Wedge product).

5.3 (One-forms).

5.4 (Exterior derivative).

5.2 Riemannian metrics

5.5 (Musical isomorphisms).

5.6 (Inner product of differential forms). ONB

5.7 (Hodge star operator). Identification of 2-forms and vector fields

5.3 Vector calculus

5.8 (Gradient, curl, and divergence).

5.9 (Potentials).

5.10 (Vector calculus identities).

5.4 Integral of differential forms

5.11 (Multiple integral). volume forms, stone weierstrass and fubini

5.12 (C^1 singular chains).

5.13 (Line integrals). A C^1 singular 1-cycle is the formal sum of *contours*, piecewise C^1 closed curves.

5.14 (Surface integrals). A C^1 singular 2-cycles.

Exercises

5.15 (Multivariable Taylor's theorem). Symmetric product

5.16 (Vector analysis in two dimension).

5.17 (Geometric algebra).

Chapter 6

Stokes theorem

6.1

embedded chains instead of manifolds triangulation

6.2 Local coordinates

6.1 (Spherical coordinates)**.** Let $U = \mathbb{R}^3 \setminus \{(x,y,z) : x = 0,\ y \geq 0\}$.

$$(x,y,z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

for $(r,\theta,\varphi) \in (0,\infty)\times(0,\pi)\times(0,2\pi)$. Orthonormal bases are

$$\left\{\partial_r,\ \frac{1}{r}\partial_\theta,\ \frac{1}{r\sin\theta}\partial_\varphi\right\} \subset \mathfrak{X}(U),$$

$$\{dr,\ r\,d\theta,\ r\sin\theta\,d\varphi\} \subset \Omega^1(U),$$

$$\{r^2\sin\theta\,d\theta\wedge d\varphi,\ r\sin\theta\,d\varphi\wedge dr,\ r\,dr\wedge d\theta\} \subset \Omega^2(U).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

Proof. Write df in the orthonormal basis $\{dr,\ r\,d\theta,\ r\sin\theta\,d\varphi\}$ as

$$\begin{aligned} df &= \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial\theta}\,d\theta + \frac{\partial f}{\partial\varphi}\,d\varphi \\ &= \left(\frac{\partial f}{\partial r}\right)dr + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)r\,d\theta + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right)r\sin\theta\,d\varphi. \end{aligned}$$

After taking the Hodge star operator, write in the basis $\{d\theta\wedge d\varphi, d\varphi\wedge dr, dr\wedge d\theta\}$ as

$$\begin{aligned} *df &= \left(\frac{\partial f}{\partial r}\right)r^2\sin\theta\,d\theta\wedge d\varphi + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)r\sin\theta\,d\varphi\wedge dr + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right)r\,dr\wedge d\theta \\ &= r^2\sin\theta\frac{\partial f}{\partial r}\,d\theta\wedge d\varphi + \sin\theta\frac{\partial f}{\partial\theta}\,d\varphi\wedge dr + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\,dr\wedge d\theta. \end{aligned}$$

Then, the differential is computed as

$$
\begin{aligned}
d * df &= d\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) d\theta \wedge d\varphi + d\left(\sin\theta \frac{\partial f}{\partial \theta}\right) d\varphi \wedge dr + d\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \varphi}\right) dr \wedge \theta \\
&= \left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\right] dr \wedge d\theta \wedge d\varphi.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
\Delta f = * d * df &= \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
\end{aligned}
$$

□

6.3 Stokes theorems

6.2 (Bump functions)**.**

6.3 (Partition of unity)**.**

6.4.