# Homological Algebra

Ikhan Choi

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# Part I

# Chapter 1

# Abelian categories

$$\begin{array}{ccccccc} K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \twoheaddownarrow & & \downarrow & & \\ K' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 0 \end{array}$$

(a) If $A \to A'$ is monic, then $K \to K'$ is monic.

(b) If $B \to B'$ is monic, then $K \to K'$ is epic.

## 1.1 Embedding

A left $R$-module $P$ is projective if and only if the left exact functor $\mathrm{Hom}_R(P, -)$ is exact.

A left $R$-module $I$ is injective if and only if the left exact contravariant functor $\mathrm{Hom}_R(-, I)$ is exact.

**1.1** (Tor functor)**.** Let $R$ be a ring and $M$ be a left $R$-module. We define the *Tor functor* as the left derived functor of the right exact functor $- \otimes_R M : \mathrm{Mod}\text{-}R \to \mathbf{Ab}$

$$\mathrm{Tor}_n^R(N, M) := H_n(P_\bullet \otimes_R M),$$

where $P_\bullet$ is a projective resolution of a right $R$-module $N$.

(a) In fact, the Tor functor may be defined by the left derived functor of the right exact functor $M \otimes_R - : R\text{-}\mathrm{Mod} \to \mathbf{Ab}$ for a right $R$-module $M$.

(b) In fact, only for Tor functors, we may only assume $P_\bullet$ is a flat resolution. (Flat resolution lemma)

**1.2** (Ext functor)**.** Let $R$ be a ring and $M$ be a left $R$-module. We define the *Ext functor* as the right derived functor of left exact functor $\mathrm{Hom}_R(M, -)$

$$\mathrm{Ext}_R^n(M, N) := H^n(M, I^\bullet),$$

where $I^\bullet$ is an injective resolution of $N$.

(a) In fact, the Ext functor may be defined by the right derived functor of the left exact contravariant functor $\mathrm{Hom}(-, M)$.

long exact seuqence

**1.3** (Universal coefficient theorem)**.** Let $R$ be a ring. Let $C_\bullet$ be a chain complex of flat right $R$-modules and $M$ be a left $R$-module.

*Proof.* We first prove the Künneth formula. Note that modules in $Z_\bullet$ and $B_\bullet$ are also flat. We start from that we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0.$$

We have a short exact sequence of chain complexes

$$\mathrm{Tor}_1^R(B_{\bullet-1}, M) \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since modules in $B_{\bullet-1}$ are flat so that $\mathrm{Tor}_1^R(B_{\bullet-1}, M) = 0$, we have a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to C_\bullet \otimes_R M \to B_{\bullet-1} \otimes_R M \to 0.$$

Since $H_n(C_{\bullet-1}) = H_{n-1}(C_\bullet)$ for any chain complex $C$, we have a long exact sequence

$$H_n(B_\bullet \otimes_R M) \to H_n(Z_\bullet \otimes_R M) \to H_n(C_\bullet \otimes_R M) \to H_{n-1}(B_\bullet \otimes_R M) \to H_{n-1}(Z_\bullet \otimes_R M).$$

Since every morphism in $B_\bullet$ and $Z_\bullet$ is zero, we have an exact sequence

$$B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet \otimes_R M) \to B_{n-1} \otimes_R M \xrightarrow{f_{n-1}} Z_{n-1} \otimes_R M.$$

Therefore, we have a short exact sequence

$$0 \to \operatorname{coker} f_n \to H_n(C_\bullet \otimes_R M) \to \ker f_{n-1} \to 0.$$

Since

$$0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$$

is a flat resolution of $H_n(C_\bullet)$, by the flat resolution lemma, we have a long exact sequence

$$\mathrm{Tor}_1^R(Z_n, M) \to \mathrm{Tor}_1^R(H_n(C_\bullet), M) \to B_n \otimes_R M \xrightarrow{f_n} Z_n \otimes_R M \to H_n(C_\bullet) \otimes_R M \to 0.$$

Since $Z_n$ is flat so that $\mathrm{Tor}_1^R(Z_n, M) = 0$, we have

$$\operatorname{coker} f_n = H_n(C_\bullet) \otimes_R M, \quad \ker f_n = \mathrm{Tor}_1^R(H_n(C_\bullet), M).$$

Therefore, we have an exact sequence

$$0 \to H_n(C_\bullet) \otimes_R M \to H_n(C_\bullet \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), M) \to 0.$$

Universal coefficient theorem states that if $R$ is a PID, then the Künneth formula splits non-canonically. $\square$

# Chapter 2

# Cohomology of algberas

## 2.1 Group cohomology

The category of $G$-modules can be identified with the category of $\mathbb{Z}[G]$-modules, which is abelian.

Let $M$ be a $G$-module. The *invariant submodule* of $M$ is denoted by $M^G$. Sending $M$ to $M^G$ yields a functor $\mathrm{Grp} \to \mathrm{Ab}$, which is left exact but not right exact in general. Then we can consider the right derived functor to define cohomology groups. Let us do this concretely.

Let $M$ be a $G$-module. Define $C^n(G,M)$ be the abelian group of all functions $G^n \to M$. The coboundary homomorphism $d : C^n(G,M) \to C^{n+1}(G,M)$ is defined such that

$$d\varphi(g_1,\cdots,g_{n+1}) := g_1\varphi(g_2,\cdots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+2},\cdots,g_{n+1}) + (-1)^{n+1}\varphi(g_1,\cdots,g_n).$$

$$H^0(G,M) = M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M).$$

For $x \in C^0(G,M) = M$, $dx(g) = gx - x$. For $\varphi \in C^1(G,M)$, $d\varphi(g,h) = g\varphi(h) - \varphi(gh) + \varphi(g)$.