Algebraic Topology

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Part I

Chapter 1

Convenient categories

1.1 Compactly generated weakly Hausdorff spaces

bicomplete cartesian closed monoidal category. Here, closed means that the right tensoring admits an adjoint called internal hom functor. braided and symmetric...?

A pointed space is a pair of a space and a 0-cell. The smash product is the categorical product in the category of pointed space.

Let Top be the bicomplete cartesian closed monoidal category of CGWH spaces.

1.2 CW complexes

1.3 Simplicial complexes

Chapter 2

Cohomology operations

2.1 Eilenberg-Steenrod axioms

cohomology cup product universal coefficient theorem Poincaré duality

Exercises

characteristic class of projective spaces

Chapter 3

Spectral sequences

3.1 Serre spectral sequence

(Lyndon-Hochschild-Serre)

A *Serre fibration* is a continuous map $Y \to B$ satisfying the homotopy lifting property for CW complexes: for any CW complex A,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\iota_0"'] & Y \arrow[d, "\pi"] \\ A \times I \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & B \end{tikzcd}$$

3.1 (Serre spectral sequence). Let $Y \to B$ be a Serre fibration with fiber space F. Let R be a commutative unital ring.

(a) If B is either simply connected or connected with R has characteristic two, then $\pi_1(B)$ acts trivially on $H_*(F, R)$...?

(b) In this case we have a multiplicative

$$E_2^{p,q} = H^p(B, H^q(F, R)) \to H^{p+q}(Y, R).$$

3.2 Adams spectral sequence

Chapter 4

Fiber bundles

4.1 Principal bundles

4.1 (Locally trivial fiber bundles). A *local trivialization* or an *atlas* of a continuous map $\pi : E \to B$ between topological spaces is a family $\{\varphi_\alpha\}$, indexed by an open cover $\{U_\alpha\}$ of B, of homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$ satisfying $\pi = \mathrm{pr}_1\, \varphi_\alpha$ on $\pi^{-1}(U_\alpha)$. We say π is *locally trivial* if there is an atlas $\{\varphi_\alpha\}$ on π. Note that a locally trivial map does not hold any atlases as its data, but only considers the existence as a property of it.

We define a *fiber bundle* as a locally trivial map $\pi : E \to B$ such that $\pi^{-1}(b)$ is homeomorphic to a topological space F. The spaces F, E, B are called the *fiber space*, *total space*, and *base space* of the fiber bundle π. For a fiber bundle with fiber space F, we may always assume that atlases $\{\varphi_\alpha\}$ have $F_\alpha = F$. A *bundle map* between fiber bundles $\pi : E \to B$ and $\pi' : E' \to B$ is a continuous map $u : E \to E'$ such that $\pi = \pi' u$. The category $\mathrm{Bun}_F(B)$ is defined such that objects are fiber bundles over a base space B with fiber space F and morphisms are bundle maps. Let $\pi : E \to B$ be a fiber bundle with fiber space F.

(a)

(b) π is surjective and open.

4.2 (G-bundles). Let G be a topological group, and F is an effective left G-space. Consider a fiber bundle $\pi : E \to B$ with fiber space F. A *G-atlas* on π is an atlas $\{\varphi_\alpha\}$ with $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$, such that there exists a family $\{g_{\alpha\beta}\}$, which is unique if it exists by effectiveness of F for a given atlas, of continuous maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$\varphi_\alpha \varphi_\beta^{-1}(b, f) = (b, g_{\alpha\beta}(b) f), \qquad b \in U_\alpha \cap U_\beta,\ f \in F.$$

A *G-structure* on π is defined as an equivalence class of G-atlases for π, where two G-atlases for π are defined to be *equivalent* if their union is also a G-atlas for π. A *G-bundle* is a fiber bundle π together with a G-structure. For a G-bundle, there is a unique maximal G-atlas in the G-structure, so a G-structure is sometimes defined as a maximal G-atlas.

A bundle map $u : E \to E'$ between G-bundles $\pi : E \to B$ and $\pi' : E' \to B$ with a common fiber space F is called a *G-bundle map* if there is a family $\{h_\alpha\}$, which is unique if it exists also by effectiveness of F, of continuous maps $h_\alpha : U_\alpha \to G$ such that

$$\varphi'_\alpha u \varphi_\alpha^{-1}(b, f) = (b, h_\alpha(b) f), \qquad b \in U_\alpha,\ f \in F,$$

where $\{\varphi_\alpha\}$ and $\{\varphi'_\alpha\}$ are G-atlases for π and π' indexed by a common open cover $\{U_\alpha\}$. Such an open cover always exists by taking refinement. Note that the definition of G-bundle maps does not depend on the choice of G-atlases in the given G-structure. We will consider G-bundle maps as the morphisms of $\mathrm{Bun}_F(B)$ if F is an effective left G-space.

(a) A G-bundle map u is an isomorphism if and only if u is a homeomorphism. In particular, the category of G-bundles over B with fiber space F is a groupoid.

Proof. □

4.3 (Fiber bundle construction theorem)**.** Let $\{U_\alpha\}$ be an open cover of a space B, and G a topological group. A *Čech 1-cocyle* on $\{U_\alpha\}$ with coefficients in G is a family $\{g_{\alpha\beta}\}$ of maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ satisfying the *cocycle condition*

$$g_{\alpha\gamma}(b) = g_{\alpha\beta}(b)g_{\beta\gamma}(b), \qquad b \in U_\alpha \cap U_\beta \cap U_\gamma.$$

A *Čech 0-cochain* on $\{U_\alpha\}$ with coefficients in G is a family $\{h_\alpha\}$ of maps $h_\alpha : U_\alpha \to G$ of maps without any conditions. The set of Čech 1-cocycles and Čech 0-cochains on $\{U_\alpha\}$ with coefficients in G are denoted by $\check{Z}^1(\{U_\alpha\}, G)$ and $\check{C}^0(\{U_\alpha\}, G)$ repspectively. The *first Čech cohomology* $\check{H}^1(\{U_\alpha\}, G)$ of $\{U_\alpha\}$ with coefficients in G is defined to be the orbit space of an action of $\check{C}^0(\{U_\alpha\}, G)$ on $\check{Z}^1(\{U_\alpha\}, G)$ defined as

$$(\{h_\alpha\}\{g_{\alpha\beta}\})_{\alpha\beta}(b) := h_\alpha(b)g_{\alpha\beta}(b)h_\beta(b)^{-1}, \qquad b \in U_\alpha \cap U_\beta.$$

We define the *first Čech cohomology* of B with coefficients in G as the direct limit of sets

$$\check{H}^1(B, G) := \varinjlim_{\{U_\alpha\}} \check{H}^1(\{U_\alpha\}, G).$$

Let F be an effective left G-space. Let $\mathrm{Bun}_F(B)$ be the category G-bundles over B with fiber space F.

(a) $\mathrm{Bun}_F(B)/\mathrm{iso} \to \check{H}^1(B, G)$ is well-defined and sujective.

(b) $\mathrm{Bun}_F(B)/\mathrm{iso} \to \check{H}^1(B, G)$ is injective.

Proof. (a) Suppose $\pi : E \to B$ and $p\pi' : E' \to B$ are isomorphic G-bundles with fiber space F, and $u : E \to E'$ is a G-bundle isomorphism. Considering the refinement, we fix an open cover $\{U_\alpha\}$ of B on which E and E' are simultaneously locally trivialized.

Now we prove the surjectivity. Let $\{g_{\alpha\beta}\} \in \check{Z}^1(\{U_\alpha\}, G)$. Define

$$E := \left(\bigsqcup_\alpha (U_\alpha \times F)\right) / \sim,$$

where $\sim$ is an equivalence relation generated by

$$(b, g_{\alpha\beta}(b)f)_\alpha \sim (b, f)_\beta, \qquad b \in U_\alpha \cap U_\beta,\ f \in F.$$

Define $\pi : E \to B$ and $\varphi_\alpha^{-1} : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that

$$p([(b, f)_\alpha]) := b, \qquad \varphi_\alpha^{-1}(b, f) := [(b, f)_\alpha], \qquad b \in U_\alpha,\ f \in F.$$

They are clearly continuous and surjective. To show that $\pi : E \to B$ is a fiber bundle, it remains to prove φ_α^{-1} is injective and open.

We first claim that φ_α^{-1} is injective. Assume $\varphi_\alpha^{-1}(b, f) = \varphi_\alpha^{-1}(b', f')$, which means that we have $(b, f)_\alpha \sim (b', f')_\alpha$. Then, we know $b = b'$ and there is a finite sequence $(\alpha_i)_{i=0}^n$ such that $\alpha_0 = \alpha_n = \alpha$ and

$$f = g_{\alpha_0\alpha_1}(b) \cdots g_{\alpha_{n-1}\alpha_n}(b) f'.$$

Applying the cocycle condition inductively, we obtain $f = f'$, which implies the injectivity of φ_α^{-1}.

Next we claim that φ_α^{-1} is open. The map φ_α^{-1} is given by the composition

$$\varphi_\alpha^{-1} : U_\alpha \times F \xrightarrow{\iota} \bigsqcup_\alpha U_\alpha \times F \xrightarrow{\pi} E,$$

where ι and π are the canonical inclusion and the canonical projection. Since ι is clearly open, it suffices to show π is open. Suppose $V \subset \bigsqcup_\alpha U_\alpha \times F$ is open so that we have $V = \bigsqcup_\alpha V_\alpha \times F$ for open subsets $V_\alpha \subset U_\alpha$. Observe that for each β we have $(b', f')_\beta \in \pi^{-1}\pi(V_\alpha \times F)$ if and only if $(b', f')_\beta \sim (b, f)_\alpha$ for some $(b, f) \in V_\alpha \times F$. It is equivalent to that $b = b'$ and

$$f = g_{\alpha_0\alpha_1}(b) \cdots g_{\alpha_{n-1}\alpha_n}(b) f',$$

where $(\alpha_i)_{i=0}^n$ is a finite sequence such that $\alpha_0 = \alpha$ and $\alpha_n = \beta$. Applying the cocycle condition inductively, we can see that it is just $f = g_{\alpha\beta}(b) f'$. Thus, the set $\pi^{-1}\pi(V_\alpha \times F) \cap (U_\beta \times F)$ is exactly the same as the inverse image of the open set $V_\alpha \times F$ under the map

$$(U_\alpha \cap U_\beta) \times F \to U_\alpha \times F : (b, f') \mapsto (b, g_{\alpha\beta}(b) f').$$

It concludes that

$$\pi^{-1}\pi(V) = \bigcup_{\alpha,\beta} \pi^{-1}\pi(V_\alpha \times F) \cap (U_\beta \times F)$$

is open, so $\pi(V)$ is open. Therefore, π is open.

Finally, we can check that the constructed fiber bundles has the transition maps of the G-atlas $\{\varphi_\alpha\}$ coincides with the cocycle $\{g_{\alpha\beta}\}$ up to 0-cochain by the cocycle condition, hence the surjectivity.

(b) □

4.2 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior

reduction and metrics

$$\mathrm{Bun}_{\mathrm{GL}(n,\mathbb{F})}(B) \to \mathrm{Vect}_n^{\mathbb{F}}(B)$$

is an isomorphism-preserving faithful essentially surjective functor, but not full. If we drop the non-invertible morphisms in $\mathrm{Vect}_n(B)$, then the functor becomes an equivalence.

Let $1 \le n \le \infty$. If $f, g : B \to G_k(\mathbb{F}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{F}^n) \to G_k(\mathbb{F}^{2n})$ is the natural inclusion.

4.4 (Vector bundles)**.** A *vector bundle* over a topological field $\mathbb{F}$ is a fiber bundle $\pi : V \to B$ together with continuous maps $+ : V \times_B V \to V$ and $\cdot : \mathbb{F} \times V \to V$ such that each fiber is a finite-dimensional vector space over $\mathbb{F}$. The dimension of a fiber space is uniquely determined, and it is called the *rank* of the vector bundle.

For a vector bundle over $\mathbb{F}$ of rank r, we can fix a vector space $\mathbb{F}^r$ as the fiber space.

(a) A vector bundle over $\mathbb{F}$ of rank r has a unique $\mathrm{GL}(r,\mathbb{F})$-structure whose charts are linear at each fiber.

(b) Whitney sum can be constructed as a pullback, and is a biproduct (in which category?).

(c) tensor product, dual (we have to prove the existence using the fiber bundle construction)

(d) universal properties of sum, tensor, dual.

Proof. Construction of the Whitney sum by the fiber bundle construction theorem: We need to first define a cocycle. Let $V_1 \to B$ and $V_2 \to B$ be vector bundles of rank r_1 and r_2, respectively. Let $\{\varphi_\alpha^1\}$ and $\{\varphi_\alpha^2\}$ be an atlas that locally trivializes both bundles. Then, we have two cocycles $g^1_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r_1, \mathbb{F})$ and $g^2_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r_2, \mathbb{F})$. Define

$$(g^1_{\alpha\beta}, g^2_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{GL}(r_1, \mathbb{F}) \times \mathrm{GL}(r_2, \mathbb{F}) \to \mathrm{GL}(r_1 + r_2, \mathbb{F}).$$

We can easily check that it satisfies the cocycle condition. With the standard left action of $\mathrm{GL}(r_1 + r_2, \mathbb{F})$ on $\mathbb{F}^{r_1} \times \mathbb{F}^{r_2}$, let

$$V_1 \oplus V_2 := \left(\bigsqcup_\alpha (U_\alpha \times \mathbb{F}^{r_1} \times \mathbb{F}^{r_2}) \right) / \sim$$

be the total space defined in the proof of the fiber bundle construction theorem.

Define a function $V_1 \to V_1 \oplus V_2 : v \mapsto [(\varphi_\alpha(v), 0)_\alpha]$, which is well-defined independently of the choice of α, because for any α and β if we let $\varphi_\alpha(v) = (b, f)$ and $\varphi_\beta(v) = (b, f')$ in $(U_\alpha \cap U_\beta) \times \mathbb{F}^r$, then since $f = g^1_{\alpha\beta}(b) f'$ we have

$$(\varphi_\alpha(v), 0)_\alpha = (b, f, 0)_\alpha = (b, g^1_{\alpha\beta}(b) f', 0)_\alpha = (b, (g^1_{\alpha\beta}(b), g^2_{\alpha\beta}(b))(f', 0))_\alpha \sim (b, f', 0)_\beta = (\varphi_\beta(v), 0)_\beta.$$

We need to show this function is continuous, fiber-preserving, fiberwise linear, and satisfies the universal property in the category of vector bundles. □

4.5 (Hermitian bundles). A *Hermitian bundle* is a complex vector bundle $V \to B$ together with $\langle \cdot, \cdot \rangle : V \times_B V \to \mathbb{C}$ such that each fiber is a complex inner product space.

Let $V \to B$ be a Hermitian bundle of rank r. Fix a complex inner product space $\mathbb{C}^r$ as the fiber space.

(a) A Hermitian bundle has a unique $\mathrm{U}(r)$-structure whose charts are unitary at each fiber.

4.6 (Serre-Swan theorem).

Proof. Let B be a compact Hausdorff space. Let $V \to B$ be a vector bundle.

trivialization projection-valued continuous function $B \to M_n(\mathbb{F})$? □

$$\mathrm{Spin}(n) \to \mathrm{SO}(n) \to \mathrm{GL}(n, \mathbb{R})$$

4.3 Principal bundles

4.7 (Principal bundles). A *principal G-bundle* for a topological group G is a fiber bundle $\pi : P \to B$ together with a continuous map $P \times G \to P$ such that each fiber is a right principal homogeneous G-space. Such a right action on P is called the *principal right action*. Having right equivariant maps as morphisms, denote by $\mathrm{Prin}_G(B)$ the category of principal G-bundles over B.

For a principal G-bundle, we may assume the fiber space is the left principal homogeneous G-space G, which is also a right principal homogeneous G-space.

(a) A principal G-bundle has a unique G-structure whose charts are right G-equivariant.

Proof. (a) For any atlas $\{\varphi_\alpha\}$ of the fiber bundle $P \to B$, there is a one-to-one correspondence between families $\{s_\alpha\}$ of local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ and families $\{\psi_\alpha\}$ of fiber-preserving homeomorphisms $\psi_\alpha : U_\alpha \times G \to U_\alpha \times G$ such that $\psi_\alpha \varphi_\alpha$ is right G-equivariant, by the formula $\psi_\alpha \varphi_\alpha(s_\alpha(b) g) = (b, g)$ for

$(b,g)\in U_\alpha\times G$. In other words, if we call the choice of an atlas $\{\varphi_\alpha\}$ together with a family of local sections $\{s_\alpha\}$ a *gauge fixing*, then each gauge fixing gives rise to a right G-equivariant G-atlas. The existence of G-structure follows from the existence of any local section.

For any two gauge fixing $(\{\varphi_\alpha\},\{s_\alpha\})$ and $(\{\varphi'_\beta\},\{s'_\beta\})$, if we define a function $g_{\alpha\beta}:U_\alpha\cap U_\beta\to G$ such that $s_\alpha g_{\alpha\beta}=s_\beta$ on $\pi^{-1}(U_\alpha\cap U_\beta)$, then we can check that

$$\psi_\alpha\varphi_\alpha\varphi'^{-1}_\beta\psi'^{-1}_\beta(b,g)=(b,g_{\alpha\beta}(b)g),\qquad b\in U_\alpha\cap U_\beta,\ g\in G.$$

To complete the proof of the equivalence of $\{\psi_\alpha\varphi_\alpha\}$ and $\{\psi'_\beta\varphi'_\beta\}$, it remains to show that $g_{\alpha\beta}$ is continuous. Obeserve that, writing $U=U_\alpha\cap U_\beta$, the function $g_{\alpha\beta}$ is given by the composition

$$U\xrightarrow{(s_\alpha,s_\beta)}\pi^{-1}(U)\times_U\pi^{-1}(U)\longleftarrow\pi^{-1}(U)\times G\xrightarrow{\mathrm{pr}_2}G\xrightarrow{(\cdot)^{-1}}G,$$

where the inverted morphism at the middle is a continuous bijection defined by the right action at fibers. In the commutative diagram

$$\begin{array}{ccc}\pi^{-1}(U)\times G & \longrightarrow & \pi^{-1}(U)\times_U\pi^{-1}(U)\\ \sim\downarrow & & \downarrow\sim\\ U\times G\times G & \xrightarrow{\sim} & (U\times G)\times_U(U\times G)\end{array}$$

of continuous maps, where the horizontal map in the second row is $(b,g,h)\mapsto((b,g),(b,gh))$, the three maps with tilde are homeomorphisms hence is the other, so the continuity of $g_{\alpha\beta}$ follows. □

4.8 (Associated bundles). Let $\pi:P\to B$ be a principal G-bundle and F be an effective left G-space. Define $\pi:E\to B$ by

$$E:=P\times_G F=P\times F/\sim,\qquad \pi([u,f]):=\pi(u),\qquad u\in P,\ f\in F,$$

where the equivalence relation $\sim$ is generated by

$$(ug,f)\sim(u,gf),\qquad u\in P,\ g\in G,\ f\in F.$$

We need to show the above procedure canonically defines a G-structure for $\pi:E\to B$. The G-bundle $\pi:E\to B$ with fiber space F is called the *associated bundle* to the principal bundle π. We claim that the construction of the associated bundles gives rise to a categorical equivalence $\mathrm{Prin}_G(B)\to\mathrm{Bun}_F(B)$.

(a) $\mathrm{Prin}_G(B)\to\mathrm{Bun}_F(B)$ is a well-defined functor.

(b) $\mathrm{Prin}_G(B)\to\mathrm{Bun}_F(B)$ is fully faithful.

(c) $\mathrm{Prin}_G(B)\to\mathrm{Bun}_F(B)$ is essentially surjective.

Proof. (a) Let $u:P\to P'$ be a right G-equivariant map. By fixing $x_0\in X$ and using the fact that the left action is free and transitive, define $g_\alpha:U_\alpha\to G$ such that

$$(b,g_\alpha(b)x_0):=\varphi_\alpha u\varphi_\alpha^{-1}(b,x_0).$$

The function g_α is continuous since it factors as

$$b\mapsto(b,x_0)\xmapsto{\varphi_\alpha u\varphi_\alpha^{-1}}(b,g_\alpha(b)x_0)\mapsto g_\alpha(b)x_0\mapsto g_\alpha(b).$$

The continuity of the last map is due to the assumption that the map $(g,x)\mapsto(gx,x)$ is a homeomorphism.

Then, for every $(b,x)\in U_i\times X$ there is a unique $s\in G$ such that

$$\varphi_i\circ\rho_s\circ\varphi_i^{-1}(b,x_0)=(b,x),$$

so we have

$$\begin{aligned}
\varphi_i \circ u \circ \varphi_i^{-1}(b,x) &= (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b,x_0) \\
&= \varphi_i \circ u \circ \rho_s \circ \varphi_i^{-1}(b,x_0) \\
&= \varphi_i \circ \rho_s \circ u \circ \varphi_i^{-1}(b,x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1})(b,x_0) \\
&= (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) g_i(b)(b,x_0) \\
&= g_i(b)(\varphi_i \circ \rho_s \circ \varphi_i^{-1})(b,x_0) \\
&= g_i(b)(b,x) \\
&= (b, g_i(b)x).
\end{aligned}$$

Hence, u is a G-bundle map.

(b) Let $u : P \to P'$ be a G-bundle map over B so that there is a set $\{h_i : U_i \to G\}_i$ of maps such that

$$\varphi_i \circ u \circ \varphi_i^{-1}(b,x) = (b, h_i(b)x), \qquad b \in U_i,\ x \in X.$$

If we write $\rho_s : P \to P : e \mapsto \rho(e,s)$ for $s \in G$, then the induced right action $\varphi_i \circ \rho_s \circ \varphi_i^{-1}$ commutes with the left action $\varphi_i \circ u \circ \varphi_i^{-1}$ on $U_i \times X$. Now for every $e \in P_1$, we have

$$\begin{aligned}
\rho_s \circ u(e) &= \varphi_i^{-1} \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(e) \\
&= \varphi_i^{-1} \circ (\varphi_i \circ u \circ \varphi_i^{-1}) \circ (\varphi_i \circ \rho_s \circ \varphi_i^{-1}) \circ \varphi_i(e) \\
&= u \circ \rho_s(e),
\end{aligned}$$

therefore u is right G-equivariant.

(c) Fix $x_0 \in X$ and consider the homeomorphism $G \to X : g \to gx_0$. Define a right action

$$X \times G \to X : (gx_0, s) \mapsto gx_0 s := gsx_0.$$

It defines a right principal homogeneous X that commutes with the left action on X.

Define $\rho : P \times G \to P$ such that

$$\varphi_i \circ \rho_s \circ \varphi_i^{-1}(b,x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b, the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge tranformation.

□

4.9 (Quotient principal bundles).

4.10 (Reduction of structure groups). Let $H \to G$ be a continuous group homomorphism between topological groups. If F is an effective left G-space, then it is an effective left H-space. If $\pi : E \to B$ is a G-bundle with fiber space F, and if $\{\varphi_\alpha\}$ be a G-atlas with $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$,

Then, there is a function $\check{H}^1(B,H) \to \check{H}^1(B,G)$, which is neither in general injective nor surjective. If a G-bundle p is contained in the image of $\check{H}^1(B,H)$ through the correspondence $\mathrm{Bun}_F(B) \twoheadrightarrow \check{H}^1(B,G)$, then we may give a H-bundle structure on p.

A *reduction* of G to H is a H-structure on a principal G-bundle. A reduction is corresponded not to an additional structure itself, but to an equivalence class of additional structures, hence it tells the existence of additional structures.

A *spin structure* on a oriented Euclidean vector bundle $V \to B$ is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)} \to B$ together with a right $\mathrm{Spin}(n)$-equivariant bundle map $P_{\mathrm{Spin}(n)} \to P_{\mathrm{SO}(n)}$. Associated to a spin structure, we can canonically define a spinor bundle $S \to B$.

4.4 Classifying spaces

In this section(?), our goal is to construct BG and prove the natural isomorphism $\mathrm{Prin}_G \to [-, BG]$.

pullback bundles: universal property, functoriality, restriction, section prolongation

4.11 (Pullback bundles)**.** Let $p : E \to B$ be a G-bundle with fiber space F.

(a) If A is paracompact and $f_0, f_1 : A \to B$ are homotopic, then G-bundles $f_0^* p$ and $f_1^* p$ are isomorphic.

Proof. If $p : E \to B \times [0, \frac{1}{2}]$ and $p' : E' \to B \times [\frac{1}{2}, 1]$ are trivial, then

If $f, g : B' \to B$ are homotopic, then $f^*\xi \cong g^*\xi$. □

4.12 (Universal principal bundles)**.** Representability of

$$\mathrm{Prin}_G : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}.$$

Milnor construction.

4.13.

If $EG \to BG$ is universal and BG is paracompact, then BG is unique up to homotopy.

If EG is contractible and BG is paracompact, then $EG \to BG$ is universal.

4.5 Characteristic classes

4.14. Let G be a topological group. Let A be an abelian group and n a positive integer. A *characteristic class* is a cohomology class on the classifying space, i.e. an element of $H^n(BG, A)$. A characteristic class $c \in H^n(BG, A)$ gives rise to a natural transformation $c : \mathrm{Prin}_G \to H^n(-, A)$ via the Brown representation and the Yoneda lemma. Explicitly, for each $X \in \mathrm{Top}$, we have $c : \mathrm{Prin}_G(X) \to H^n(X, A) : E \mapsto f^*c$, where $E \cong f^*(EG)$. Using the Thom isomorphism and Gysin sequences, we can compute the cohomology of classifying spaces, which makes the classification of principal bundles

Suppose we have $BG = K(A, n)$. Take $c \in H^n(BG, A)$ which corresponds to the identity of $\mathrm{End}(A)$ in the isomorphism

$$H^n(BG, A) = H^n(K(A, n), A) \cong \mathrm{Hom}(H_n(K(A, n), \mathbb{Z}), A) = \mathrm{Hom}(\pi_n(K(A, n)), A) = \mathrm{Hom}(A, A) = \mathrm{End}(A),$$

which follows from the Hurewicz theorem. Then, we can show that c defines a natural isomorphism $c : \mathrm{Prin}_G \to H^n(-, A)$. In this case, we can say c is the *characteristic class* for G-bundles.

1. Real line bundles are classified by the first Stiefel-Whitney class $w_1 \in H^1(BG, A) \cong Aw_1$, where

$$G := \mathrm{GL}(1, \mathbb{R}) = \mathbb{Z}/2\mathbb{Z}, \qquad A := \mathbb{Z}/2\mathbb{Z}, \qquad BG = \mathbb{RP}^\infty = K(A, 1).$$

2. Complex line bundles are classified by the first Chern class $c_1 \in H^2(BG, A) \cong Ac_1$, where

$$G := \mathrm{GL}(1, \mathbb{C}) = \mathbb{T}, \qquad A := \mathbb{Z}, \qquad BG = \mathbb{CP}^\infty = K(A, 2).$$

3. Real vector bundles.

$$G := \mathrm{GL}(r, \mathbb{R}), \qquad A := \mathbb{Z}/2\mathbb{Z}, \qquad BG = \mathrm{Gr}_r(\mathbb{R}^\infty).$$

   By Thom and Gysin, we can compute the cohomology ring

$$H^*(BG, A) \cong A[w_1, \cdots, w_r].$$

4.15 (Stiefel-Whitney classes)**.** Let $V \to B$ be a vector bundle of rank r. The

$$H^0(H^0(\mathrm{O}(r))$$

4.16 (Chern classes)**.**

4.17 (Thom isomorphism)**.** Let E be a real vector bundle of rank r over a paracompact arc-connected space B. Let A be a principal ideal domain. Then, there is unique $t(E) \in H^r(E, E_0, A)$ whose image under $H^r(E, E_0, A) \to H^r(E_b, (E_b)_0, A) \cong A$ is the unit for each $b \in B$. With this t, we have an isomorphism

$$\cdot \smile t(E) : H^i(E) \to H^{i+r}(E, E_0), \qquad i \geq 0.$$

The Euler class can be defined from Thom class.

Exercises

4.18. Let G be a topological group, and X be a free right G-space.

(a) If the action is proper and the projection $X \to X/G$ admits local sections, then $X \to X/G$ is a principal G-bundle.

4.19 (Clutching functions)**.**

4.20. Suppose $F \to E \to B$ is a principal

(a) If X is contractible, then $X \to$

4.21 (Group quotients)**.** Sufficient conditions for principal bundles. Let G be a Lie group and M be a free right smooth G-manifold.

(a) If G is compact, then $M \to M/G$ is a principal G-bundle. (Gleason)

(b) The irrational slope provides a counterexample if G is not compact.

(c) Suppose M is a Lie group. If G is a closed subgroup of M, then $M \to M/G$ is a principal G-bundle. (Samelson) In particular, if N is a transitive left smooth M-manifold such that G is the isotropy group, then $M \to N$ is a principal G-bundle.

Proof. (a) We need to check the local triviality from smoothness and the properness from compactness. □

4.22 (Homogeneous spaces)**.** They are all principal bundles.

$$\begin{aligned}
\mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n), \\
\mathrm{O}(n-k) \times \mathrm{O}(k) \to \mathrm{O}(n) \to G_k(\mathbb{R}^n), &\qquad \mathrm{U}(n-k) \times \mathrm{U}(k) \to \mathrm{U}(n) \to G_k(\mathbb{C}^n), \\
T(n) \cap \mathrm{O}(n) \to \mathrm{O}(n) \to F(\mathbb{R}^n), &\qquad T(n) \cap \mathrm{U}(n) \to \mathrm{U}(n) \to F(\mathbb{C}^n), \\
&\qquad T(n) \to \mathrm{GL}(n, \mathbb{C}) \to F(\mathbb{C}^n),
\end{aligned}$$

where $T(n)$ is the group of invertible upper triangular matrices.

$$\mathrm{SO}(n) \to \mathrm{SO}^+(1,n) \to \mathbb{H}^n, \qquad \mathrm{PSO}(2) \to \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{H}^2, \qquad ?? \to \mathrm{PSL}(2,\mathbb{C}) \to \mathbb{H}^3,$$

where $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)^+$ is the modular group and $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SO}(1,3)^+$ is the restricted Lorentz group, also called the Möbius group.

4.23 (Hopf fibration)**.** A principal S^1-bundle $S^1 \to S^3 \to S^2$, where we see S^1 as the circle group. The Hopf fibrations are used in describing universal principal bundles off orthogonal or unitary groups. We have principal bundles:

(a) The quaternionic construction gives $S^3 \to S^7 \to S^4$ and the octonianic construction gives $S^7 \to S^{15} \to S^8$. Adams' theorem.

(b) $\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$. In particular, $\mathbb{Z}/2\mathbb{Z} \to S^n \to \mathbb{RP}^n$ for $k = 1$.

(c) $\mathrm{U}(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$. In particular, $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $k = 1$.

Hopf fibration(real, complex, quaternionic, but not octonianic)
In the category of smooth manifolds, if f diffeomorphic, then $\widetilde{f}$ diffeomorphic.

4.24. (a) $\mathbb{R} \to S^1$ for $\mathbb{Z}$.

(b) $S^\infty \to \mathbb{CP}^\infty$ for $\mathrm{U}(1)$.

(c) $? \to (S^1)^{\vee n}$ for F_n.

(d) $S^\infty \to \mathbb{RP}^\infty$ for $\mathbb{Z}/2\mathbb{Z}$.

(e) $V_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ for $\mathrm{O}(n)$.

Chapter 5

5.1 Obstruction theory

5.2 Hurewicz theorem

$H_\bullet(\Omega S_n)$ and $H_\bullet(U(n))$ Spin, $\mathrm{Spin}_\mathbb{C}$ structure

Chapter 6

Simplicial methods

Part II

Stable homotopy theory

Chapter 7

7.1 Homotopy groups of spheres

Freudenthal suspension theorem. Spanier-Whitehead category, does not contain reduced cohomology theory? Boardman's stable homotopy category, Lima's notion of spectra, Kan's semi-simplicial category, Whitehead's notion of spectra, and finally Adams's construction of stable homotopy category. Bousfield localization is a kind of a category of fractions and Adamas spectral sequence. It leads to the chromatic homotopy.

A commutative monoidal point-set model for the stable homotopy category? Coordinate-free spectra by May, but commutative and associative only up to homotopy. S-modules in EKMM97, symmetric spectra in HSS00, which are shown Quillen equivalent in Sch01. They give closed symmetric monoidal model categories of spectra and model categories of ring spectra. HPS97 axiomatize the stable homotopy theories.

7.1 (Freudenthal suspension theorem). For each r, we have the suspension homomorphism

$$E_n : \pi_{n+r}(S^n) \to \pi_{n+r+1}(S^{n+1}), \qquad n \geq 0,$$

which are isomorphisms if $n > r + 1$. The *stable homotopy groups of spheres* is $\lim_n \pi_{n+r}(S^n)$.

For example, it is known that $\pi_{n+1}(S^n) \cong \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. The computation $\pi_4(S^3)$ is a nice exercise which is done by Serre in his thesis.

Note that $\pi_{n+r}(S^n) = [S^{n+r}, S^n]$. Suspension $\Sigma := S^1 \wedge -$ is a functor, so it defines a function $S : [X, Y] \to [\Sigma X, \Sigma Y]$.

Spanier-Whitehead and Adama category of spectra. Triangulatedness of the homotopy category of a stable model category. a symmetric monoidal smash product and an internal function object.

Part III

Generalized cohomology theories

Chapter 8

8.1. Note that hSpc = hKan = hCW is triangulated.

8.2 (Generalized cohomology theories)**.** A *generalized cohomology theory*, or simply a *cohomology theory*, is defines as a homotopy invariant product-preserving contravariant functor $E^\bullet : \mathrm{Spc}_*^{\mathrm{op}} \to \mathrm{grAb}$, such that

(i) a cofiber sequence is mapped to an exact sequence, (half-exact)

(ii) a natural isomorphism $\sigma : E^{\bullet+1} \circ \Sigma \to E^\bullet$ exists, called the *suspension isomorphism*. (long exact)

A cohomology theory is called *multiplicative* if....

$E^1(\Sigma X) = E^0(X)$.

$K_0(\Sigma A) = K_1(A)$.

Two motivations for spectra:

- representation of cohomology theories
- suspension stabilization

8.3. Let X and Y be pointed CW complexes.

(a) Suppose Y is $(n-1)$-connected with non-degenerate base point for some n. Then, $[X,Y] \to [\Sigma X, \Sigma Y]$ is surjective if $\dim X \le 2n-1$, and bijective if $\dim X \le 2n-2$.

8.4. A *spectrum* is a sequence $E := (E_n)_n$ of pointed spaces together with structure maps, either $\sigma_n : \Sigma E_n \to E_{n+1}$ or $\sigma'_n : E_n \to \Omega E_{n+1}$. We have

$$[X, E_n] \xrightarrow{\sigma'_n} [X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}].$$

8.5 (Properties of spectra)**.** A spectrum $E = (E_n)_n$ is called an *Ω-spectrum* if $\sigma'_n : E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence. A *ring spectrum* is a spectrum together with a

(a) E is an Ω-spectrum if and only if $[-, E_n]$ defines a generalized reduced cohomology theory on based CW complexes.

Sphere spectra, Suspension spectra Eilenberg-MacLane spectra(ordinary cohomology theories), K-theory spectra(K-theories), Thom spectra(cobordism theories)

Let E^* be a (generalized) cohomology theory. Then, the computation of $\mathrm{Nat}([-, BO(n)], E^*) \cong E^*(BO(n))$ determines all characteristic classes of real vector bundles.

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem

Chapter 9

K-theory

9.1 (K-theory of locally compact Hausdorff spaces). compactly supported? one-point compactification? representability?

What relations do we have?

Chapter 10

Cobordisms