

# Category Theory

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# **Part I**

# Chapter 1

## Categories

set theoretical issues morphisms monic

### 1.1 Functors

fully faithful, essentially surjective natural transformations and equivalence 2-category

### 1.2 Categorical constructions

opposite category product category disjoint union category comma category(slice category, morphism category)

### 1.3 Representable functors

Yoneda embedding gives fully faithful functors  $h : \mathcal{C} \rightarrow \mathbf{PSh}(\mathcal{C})$  and  $k : \mathcal{C}^{\text{op}} \rightarrow \mathbf{coPSh}(\mathcal{C})$ . A presheaf  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  is representable if and only if it is essentially contained in the image of the Yoneda embedding.

**1.1 (Yoneda lemma).** Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor from a locally small category  $\mathcal{C}$ . Fix  $c \in \text{Ob}(\mathcal{C})$ . we can define a function

$$\text{Nat}(\text{Hom}(c, -), F) \rightarrow F(c).$$

A *representation* of  $F$  is a pair  $(c, \eta)$  of an object  $c \in \mathcal{C}$  and a natural isomorphism  $\eta : \text{Hom}(c, -) \rightarrow F$ .

A *universal element* of  $F$  is a pair  $(c, x)$  with  $x \in F(c)$  such that for any pair  $(d, y)$  with  $y \in F(d)$  there is a unique morphism  $f \in \text{Hom}(c, d)$  satisfying  $F(f) : x \mapsto y$ .

(a)

*Proof.* (a) Consider the diagram

$$\begin{array}{ccc}
 \text{Hom}(c, c) & \xrightarrow{\eta_c} & F(c) \\
 \downarrow \cong & \text{id}_c \mapsto x := \eta_c(\text{id}_c) & \downarrow F(f) \\
 & \downarrow f \mapsto \eta_d(f) := F(f)(x) & \\
 \text{Hom}(c, d) & \xrightarrow{\eta_d} & F(d)
 \end{array}$$

For a natural transformation  $\eta : \text{Hom}(c, -) \rightarrow F$ , define  $x := \eta_c(\text{id}_c)$  in  $F(c)$ . For  $x \in F(c)$ , conversely, define a  $\eta_d : \text{Hom}(c, d) \rightarrow F(d)$  by  $\eta_d(f) := F(f)(x)$  for  $d \in \text{Ob}(\mathcal{C})$  and  $f \in \text{Hom}(c, d)$ . Then, the collection  $\eta = \{\eta_d : d \in \text{Ob}(\mathcal{C})\}$  provides a natural transformation because for each  $g \in \text{Hom}(d, e)$  we can check the diagram

$$\begin{array}{ccc} \text{Hom}(c, d) & \xrightarrow{\eta_d} & F(d) \\ g \circ - \downarrow & & \downarrow F(g) \\ \text{Hom}(c, e) & \xrightarrow{\eta_e} & F(e) \end{array}$$

commutes from

$$F(g)(\eta_d(f)) = F(g)(F(f)(x)) = F(g \circ f)(x) = \eta_e(g \circ f), \quad f \in \text{Hom}(c, d).$$

The correspondences  $\eta \mapsto x$  and  $x \mapsto \eta$  are inverses of each other, hence the bijection. □

## Chapter 2

# Limits

### 2.1 Construction

products, equalizers, pullbacks

preservation, reflection, creation completeness functoriality

limit-preserving filtered limit-preserving product-preservig mono-preserving

filtered limits

# Chapter 3

## 3.1 Adjunctions

## 3.2 Monads

## 3.3 Kan extensions

## Part II



## Chapter 4

# Chapter 5

## Abelian categories

### 5.1 Regular and exact categories

split, regular, strong effective, normal, strict

A kernel pair of a morphism  $f$  is the pullback of  $(f, f)$ .

A category is called *regular* if every kernel pair admits a coequalizer.

5.1. A regular category is called *exact* if every equivalence relation is given by a kernel pair.

(a)

The category  $\mathbf{Grp}$  is regular but not coregular, since there is a monomorphism which is not regular.

### 5.2 Additive and abelian categories

5.2 (Additive categories). A *pre-additive category* is an  $\mathbf{Ab}$ -enriched category. A *semi-additive category* is one of the followings:

- (i) a pointed  $\mathbf{CMon}$ -enriched category.
- (ii) a category with finite biproducts.

An *additive category* is one of the followings:

- (i) a pointed  $\mathbf{Ab}$ -enriched category.
- (ii)  $\mathbf{Ab}$ -enriched category with finite biproducts.
- (a) additive completion by formally adjoining finite biproducts.
- (b) additive structures on a semi-additive category is unique.

The notion of kernels and cokernels can be defined in a  $\mathbf{Set}_*$ -enriched category. In additive category, we have a natural  $\mathbf{Set}_*$ -enrichment.

5.3 (Pre-abelian categories). A *pre-abelian category* is one of the followings:

- (i) an additive category in which every morphism has the kernel and cokernel.
- (ii) a finitely bicomplete pre-additive category.
- (a)

**5.4** (Semi-abelian categories in the sense of Jenelidze-Márkin-Tholen). A pointed, Baar-exact, proto-modular, with binary coprouducts.

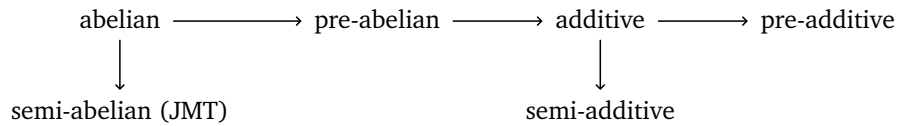
- (a) short five lemma,  $3 \times 3$  lemma, snake lemma, noether isomorphism hold.
- (b) long exact homology sequence
- (c) Every semi-abelian category is exact.
- (d) Every semi-abelian category is finitely bicomplete.
- (e) In general, a semi-abelian category is not pre-additive nor semi-additive.

**5.5** (Abelian categories). An *abelian category* is a  $\mathbf{Ab}$ -enriched category which is finitely bicomplete and satisfies the first isomorphism theorem.

PULLBACK/PUSHOUT, ZERO, EXACT!

- (a) A category is abelian if and only if it is additive and exact.

**5.6** (Freyd-Mitchell embedding).



- Pre-abelian: abelian topological groups, Banach spaces, Fréchet spaces.
- Semi-abelian: groups, non-unital algebras, Lie algebras,  $C^*$ -algebras, compact Hausdorff (profinite) spaces.
- Additive: projective modules

The first isomorphism theorem states that  $\text{coim} \rightarrow \text{im}$  is an isomorphism. The normal subobjects and the first isomorphism theorem is generalized in the context of protomodular categories. The cokernel may not be defined. The category of unital rings is not semi-abelian but protomodular.

- A *protomodular category*
- A *homological category* is a pointed regular protomodular category. (five, nine, snake, long exact sequence, noether isomorphism)
- A *semi-abelian category* is a homological category that is Barr-exact and finite coproducts (free products).

## 5.3

How to manipulate triangles and squares! We use the following definitions: a triangle is a complex of two morphisms, and a square is a commutative square.

A square is just a triangle  $A \rightarrow B \oplus C \rightarrow D$ .

$$\begin{array}{ccccc}
 & & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \\
 0 & \longrightarrow & A' & \longrightarrow & B' \\
 & & & & \\
 & & & & KC \\
 & & & & \downarrow \\
 & & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \\
 0 & \longrightarrow & A' & \longrightarrow & B'
 \end{array}$$

## Chapter 6

# Tensor categories

### 6.1 Monoidal categories

closed, symmetric, cartesian coherence theorem, closure theorem

**6.1** (Monoidal categories). A *monoidal category* is a category  $\mathcal{C}$  equipped with a bifunctor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  such that

- (i) for each triple  $A, B, C \in \mathcal{C}$  there is an isomorphism  $\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$  called the *associator*, satisfying the pentagon identity

$$\begin{array}{ccccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A \otimes B, C, D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A, B, C \otimes D}} & A \otimes (B \otimes (C \otimes D)) \\
 \downarrow \alpha_{A, B, C} \otimes \text{id}_D & & & & \uparrow \text{id}_A \otimes \alpha_{B, C, D} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) & & 
 \end{array}$$

commutes for each  $A, B, C, D \in \mathcal{C}$ .

- (ii) there is a specified object  $I \in \mathcal{C}$  called the *unit object*, and for each  $A \in \mathcal{C}$  there are isomorphisms  $\lambda_A : I \otimes A \rightarrow A$  and  $\rho_A : A \otimes I \rightarrow A$  called the *left unitor* and the *right unitor*, satisfying the triangle identity

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A, I, B}} & A \otimes (I \otimes B) \\
 \searrow \rho_A \otimes \text{id}_B & & \swarrow \text{id}_A \otimes \lambda_B \\
 & A \otimes B & 
 \end{array}$$

commutes for each  $A, B \in \mathcal{C}$ .

We say a monoidal category is *strict* if the associators and unitors are all identity morphisms. A *cartesian* monoidal category is a monoidal category whose monoidal structure  $\otimes$  is given by the categorical product.

**6.2** (Coherence theorem). Let  $\mathcal{C}$  be a monoidal category.

- (a)

$$\begin{array}{ccc}
 (I \otimes A) \otimes B & \xrightarrow{\alpha_{I, A, B}} & I \otimes (A \otimes B) \\
 \searrow \lambda_A \otimes \text{id}_B & & \swarrow \lambda_{A \otimes B} \\
 & A \otimes B & 
 \end{array}$$

- (b)  $\lambda_I = \rho_I$

- (c) The endomorphism monoid  $\text{End}(I)$  is commutative.

(d)  $I$  is unique up to unique isomorphism.

**6.3** (Monoidal functors). coherence maps lax, strong, strict

**6.4** (Enriched categories). Let  $\mathcal{M}$  be a monoidal category. A category  $\mathcal{C}$  is said to be *enriched* over  $\mathcal{M}$  if for each  $A, B \in \mathcal{C}$  there is  $\text{Hom}(A, B) \in \mathcal{M}$  such that

(i) for each  $A, B, C \in \mathcal{C}$  there is a morphism  $\circ_{A,B,C} : \text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ , satisfying

$$\begin{array}{ccc} \text{Hom}(A, B) \otimes \text{Hom}(B, C) \otimes \text{Hom}(C, D) & \xrightarrow{\text{id}_{\text{Hom}(A,B)} \otimes \circ_{B,C,D}} & \text{Hom}(A, B) \otimes \text{Hom}(B, D) \\ \downarrow \circ_{A,B,C} \otimes \text{id}_{\text{Hom}(C,D)} & & \downarrow \circ_{A,B,D} \\ \text{Hom}(A, C) \otimes \text{Hom}(C, D) & \xrightarrow{\circ_{A,C,D}} & \text{Hom}(A, D) \end{array}$$

commutes for each  $A, B, C, D \in \mathcal{C}$ .

(ii) for each  $A \in \mathcal{C}$  there is a morphism  $\text{id}_A : I \rightarrow \text{Hom}(A, A)$ , satisfying

$$\begin{array}{ccccc} I \otimes \text{Hom}(A, B) & & & & \text{Hom}(A, B) \otimes I \\ \downarrow \text{id}_A \otimes \text{id}_{\text{Hom}(A,B)} & \searrow \lambda_{\text{Hom}(A,B)} & & \swarrow \rho_{\text{Hom}(A,B)} & \downarrow \text{id}_{\text{Hom}(A,B)} \otimes \text{id}_B \\ \text{Hom}(A, A) \otimes \text{Hom}(A, B) & \xrightarrow{\circ_{A,A,B}} & \text{Hom}(A, B) & \xleftarrow{\circ_{A,B,B}} & \text{Hom}(A, B) \otimes \text{Hom}(B, B) \end{array}$$

**6.5** (Pointed category). A *pointed category* is a category with a zero object.

(a) A category is  $\text{Set}_*$ -enriched if and only if it admits a zero morphism.

(b) Every pointed category is  $\text{Set}_*$ -enriched.

rigid?

## 6.2 Braided and ribbon categories

## 6.3 Internalization

## 6.4 Tensor and fusion categories