Introduction to Spectral Analysis of Quantum Fields

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105. Let S_n be the symmetrization operator. Show the followings:

(a) $U_\tau S_n = S_n$ for $\tau \in \mathfrak{S}_n$

(b) $S_n^2 = S_n$

(c) $S_n^* = S_n$

Solution. (a)

$$U_\tau S_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} U_\tau U_\sigma = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} U_{\tau\sigma} = S_n.$$

(b)

$$S_n^2 = \frac{1}{(n!)^2}\sum_{\sigma,\tau\in\mathfrak{S}_n} U_{\sigma\tau} = \frac{1}{(n!)^2}\sum_{\sigma,\tau\in\mathfrak{S}_n} U_\sigma = \frac{1}{(n!)^2} n! \sum_{\sigma\in\mathfrak{S}_n} U_\sigma = S_n.$$

(c)

$$S_n^* = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} U_\sigma^* = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} U_{\sigma^{-1}} = S_n.$$

□

106. Let $f_1, f_2, g_1, g_2 \in \mathcal{H}$. Using CCR, compute

$$\langle A^*(f_1)A^*(f_2)\Omega, A^*(g_1)A^*(g_2)\Omega\rangle.$$

Solution. Since $A(f)A^*(g) = A^*(g)A(f) + \langle f, g\rangle$, $A^*(f)\Omega = f$, and $A(f)\Omega = 0$, we have

$$\begin{aligned}
&\langle A^*(f_1)A^*(f_2)\Omega, A^*(g_1)A^*(g_2)\Omega\rangle \\
&\quad = \langle A^*(f_2)\Omega, A(f_1)A^*(g_1)A^*(g_2)\Omega\rangle \\
&\quad = \langle A^*(f_2)\Omega, A^*(g_1)A(f_1)A^*(g_2)\Omega\rangle + \langle f_1, g_1\rangle\langle A^*(f_2)\Omega, A^*(g_2)\Omega\rangle \\
&\quad = \langle A^*(f_2)\Omega, A^*(g_1)A^*(g_2)A(f_1)\Omega\rangle + \langle f_1, g_2\rangle\langle A^*(f_2)\Omega, A^*(g_1)\Omega\rangle + \langle f_1, g_1\rangle\langle A^*(f_2)\Omega, A^*(g_2)\Omega\rangle \\
&\quad = 0 + \langle f_1, g_2\rangle\langle f_2, g_1\rangle + \langle f_1, g_1\rangle\langle f_2, g_2\rangle.
\end{aligned}$$

□

107. Let $f_j, g_j \in \mathcal{H}$. Show that

$$\langle A^*(f_1)\cdots A^*(f_n)\Omega, A^*(g_1)\cdots A^*(g_n)\Omega\rangle = \sum_{\sigma\in\mathfrak{S}_n} \langle f_1, g_{\sigma(1)}\rangle \cdots \langle f_n, g_{\sigma(n)}\rangle.$$

Solution. The case $n = 2$ follows from the problem 106. As the induction hypothesis, suppose the claim is true for $n-1$. Denote by $(1\ k) \in \mathfrak{S}_n$ the transposition which swaps 1 and k, and identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ fixing 1. Then, the coset $(1\ k)\mathfrak{S}_{n-1}$ can be characterized as the set of permutations such that $\sigma(\{2,\cdots,n\}) = \{1,\cdots,n\} \setminus \{k\}$. Then we have

$$\begin{aligned}
&\langle A^*(f_1)\cdots A^*(f_n)\Omega, A^*(g_1)\cdots A^*(g_n)\Omega\rangle \\
&\quad = \langle A^*(f_2)\cdots A^*(f_n)\Omega, A(f_1)A^*(g_1)\cdots A^*(g_n)\Omega\rangle \\
&\quad = \langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_1)A(f_1)A^*(g_2)\cdots A^*(g_n)\Omega\rangle \\
&\qquad\quad + \langle f_1, g_1\rangle\langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_2)\cdots A^*(g_n)\Omega\rangle \\
&\quad = \langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_1)A^*(g_2)A(f_1)A^*(g_3)\cdots A^*(g_n)\Omega\rangle \\
&\qquad\quad + \langle f_1, g_1\rangle\langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_2)\cdots A^*(g_n)\Omega\rangle \\
&\qquad\quad + \langle f_1, g_2\rangle\langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_1)A^*(g_3)\cdots A^*(g_n)\Omega\rangle \\
&\quad = \cdots \\
&\quad = 0 + \sum_{k=1}^{n}\langle f_1, g_k\rangle\langle A^*(f_2)\cdots A^*(f_n)\Omega, A^*(g_1)\cdots A^*(g_{k-1})A^*(g_{k+1})\cdots A^*(g_n)\rangle \\
&\quad = \sum_{k=1}^{n}\sum_{\sigma\in(1\ k)\mathfrak{S}_{n-1}}\langle f_1, g_k\rangle\langle f_2, g_{\sigma(2)}\rangle\cdots\langle f_n, g_{\sigma(n)}\rangle \\
&\quad = \sum_{\sigma\in\mathfrak{S}_n}\langle f_1, g_{\sigma(1)}\rangle\cdots\langle f_n, g_{\sigma(n)}\rangle.
\end{aligned}$$

□

108. Let $f \in \mathcal{H}$. We call

$$\exp f := \sum_{n=0}^{\infty}\frac{A^*(f)^n}{n!}\Omega$$

the *coherent vector*. Let $g \in \mathcal{H}$.

(a) Compute $\langle \exp f, \exp g\rangle$.

(b) Show that $A(g)\exp f = \langle g, f\rangle \exp f$.

Solution. (a) Note that

$$\langle A^*(f_1)\cdots A^*(f_m)\Omega, A^*(g_1)\cdots A^*(g_n)\Omega\rangle = 0$$

for $m \neq n$, then by the problem 107 we have

$$\langle A^*(f)^n\Omega, A^*(g)^n\Omega\rangle = n!\langle f, g\rangle^n,$$

so that

$$\begin{aligned}
\langle \exp f, \exp g\rangle &= \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\langle A^*(f)^n\Omega, A^*(g)^n\Omega\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\langle A^*(f)^n\Omega, A^*(g)^n\Omega\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\langle f, g\rangle^n \\
&= \exp\langle f, g\rangle.
\end{aligned}$$

(b) Since

$$\begin{aligned}
A(g)A^*(f)^n &= A^*(f)A(g)A^*(f)^{n-1} + \langle g, f\rangle A^*(f)^{n-1} \\
&= A^*(f)^2A(g)A^*(f)^{n-2} + 2\langle g, f\rangle A^*(f)^{n-1} \\
&= A^*(f)^nA(g) + n\langle g, f\rangle A^*(f)^{n-1},
\end{aligned}$$

and since $A(g)\Omega = 0$, we have

$$\begin{aligned}
A(g)\exp f &= \sum_{n=0}^{\infty} \frac{1}{n!} A(g)A^*(f)^n\Omega \\
&= 0 + \sum_{n=1}^{\infty} \frac{1}{n!} n\langle g, f\rangle A^*(f)^{n-1}\Omega \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle g, f\rangle A^*(f)^n\Omega \\
&= \langle g, f\rangle \exp f.
\end{aligned}$$

□

109. Let $z \in \mathbb{C}$ and $f, g \in \mathcal{H}$. Show that

$$e^{z\Phi_S(f)}\Omega$$

is an eigenvector of $A(g)$. What is the eigenvalue?

Solution. We first consider an equality

$$A(g)(A(f)+A^*(f))^n = (A(f)+A^*(f))^n A(g) + n\langle g, f\rangle (A(f)+A^*(f))^{n-1},$$

which can be proved by induction

$$\begin{aligned}
A(g)(A(f)+A^*(f))^n &= A(g)(A(f)+A^*(f))^{n-1}(A(f)+A^*(f)) \\
&= [(A(f)+A^*(f))^{n-1}A(g) + (n-1)\langle g, f\rangle (A(f)+A^*(f))^{n-2}](A(f)+A^*(f)) \\
&= (A(f)+A^*(f))^{n-1}A(g)A(f) + (A(f)+A^*(f))^{n-1}A(g)A^*(f) \\
&\quad + (n-1)\langle g, f\rangle (A(f)+A^*(f))^{n-1} \\
&= (A(f)+A^*(f))^{n-1}A(f)A(g) + (A(f)+A^*(f))^{n-1}A^*(f)A(g) \\
&\quad + n\langle g, f\rangle (A(f)+A^*(f))^{n-1} \\
&= (A(f)+A^*(f))^n A(g) + n\langle g, f\rangle (A(f)+A^*(f))^{n-1}.
\end{aligned}$$

Now then we can compute

$$\begin{aligned}
A(g)e^{z\Phi_S(f)}\Omega &= A(g)\sum_{n=0}^{\infty} \frac{(z/\sqrt{2})^n}{n!}(A(f)+A^*(f))^n\Omega \\
&= 0 + \sum_{n=1}^{\infty} \frac{(z/\sqrt{2})^n}{n!} n\langle g, f\rangle (A(f)+A^*(f))^{n-1}\Omega \\
&= \frac{z}{\sqrt{2}}\langle g, f\rangle \sum_{n=0}^{\infty} \frac{(z/\sqrt{2})^n}{n!}(A(f)+A^*(f))^n\Omega \\
&= \frac{z}{\sqrt{2}}\langle g, f\rangle e^{z\Phi_S(f)}\Omega.
\end{aligned}$$

The eigenvalue is $\frac{z}{\sqrt{2}}\langle g, f\rangle$. □

110. Let $z \in \mathbb{C}$ and $f \in \mathcal{H}$. Let

$$F(z) := e^{cz^2} e^{z\Phi_S(f)}\Omega,$$

where $c \in \mathbb{R}$.

(a) Determine c which satisfies $F'(z) = \frac{1}{\sqrt{2}}A^*(f)F(z)$.

(b) Compute $F^{(n)}(0)$.

(c) Rewrite $e^{z\Phi_S(f)}\Omega$ in the coherent vector form, that is, find a constant C and g such that $e^{z\Phi_S(f)}\Omega = C\exp g$.

Solution. (a) For the left-hand side, by interpreting the derivative in the weak limit, we can justify

$$F'(z) = (2cz + \Phi_S(f))F(z).$$

For the right-hand side, since we have similarly to the problem 109 that

$$\begin{aligned}
A^*(f)e^{z\Phi_S(f)}\Omega &= A^*(f)\sum_{n=0}^{\infty}\frac{(z/\sqrt{2})^n}{n!}(A(f)+A^*(f))^n\Omega \\
&= \sum_{n=0}^{\infty}\frac{(z/\sqrt{2})^n}{n!}(A(f)+A^*(f))^n A^*(f)\Omega - \sum_{n=1}^{\infty}\frac{(z/\sqrt{2})^n}{n!}n\langle f,f\rangle(A(f)+A^*(f))^{n-1}\Omega \\
&= e^{z\Phi_S(f)}A^*(f)\Omega - \frac{z}{\sqrt{2}}\langle f,f\rangle e^{z\Phi_S(f)}\Omega \\
&= e^{z\Phi_S(f)}\left(A^*(f) - \frac{z}{\sqrt{2}}\langle f,f\rangle\right)\Omega,
\end{aligned}$$

we obtain

$$\left(2cz + \frac{1}{\sqrt{2}}(A(f)+A^*(f))\right)\Omega = \left(\frac{1}{\sqrt{2}}A^*(f) - \frac{z}{2}\langle f,f\rangle\right)\Omega.$$

Thus we have $c = -\langle f,f\rangle/4$.

(b)

$$F^{(n)}(0) = \left(\frac{1}{\sqrt{2}}A^*(f)\right)^n F(0) = \left(\frac{1}{\sqrt{2}}A^*(f)\right)^n \Omega.$$

(c) Since

$$F(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!}F^{(n)}(0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{\sqrt{2}}A^*(f)\right)^n\Omega = \exp\frac{z}{\sqrt{2}}f,$$

we have $C = e^{-cz^2}$ and $g = \frac{z}{\sqrt{2}}f$. The infinite series in the Taylor expansion is justified in the weak sense. □

111. Let $f, g \in \mathcal{H}$. Show that we have

$$e^{i\Phi_S(f)}e^{i\Phi_S(g)} = ce^{i\Phi_S(f+g)}$$

for some constant c. What is the value of c?

Solution. We can apply the special case of the Baker-Campbell-Hausdorff formula to obtain

$$e^{i\Phi_S(f)}e^{i\Phi_S(g)} = e^{i\Phi_S(f)+i\Phi_S(g)+\frac{1}{2}[i\Phi_S(f),i\Phi_S(g)]} = e^{i\Phi_S(f+g)-\frac{i}{2}\operatorname{Im}\langle f,g\rangle},$$

so we have $c = e^{-\frac{i}{2}\operatorname{Im}\langle f,g\rangle}$. □

112. For a linear subspace $\mathcal{D} \subset \mathcal{H}$, show the following is a $*$-algebra:

$$\mathcal{A} := \mathcal{L}\{e^{i\Phi_S(f)} : f \in \mathcal{D}\}.$$

Solution. We need to show $\mathcal{A}$ is closed under (1) addition, (2) scalar multiplication, (3) multiplication, (4) involution. (1) and (2) are clear and (3) follows from the problem 111. (4) is also clear since $(e^{i\Phi_S(f)})^* = e^{i\Phi_S(-f)}$. □

113. Consider the momentum operator $p = -id/dx$ defined on $L^2(\mathbb{R})$. For $f \in C_0^\infty(\mathbb{R}) \setminus \{0\}$, show the Taylor expansion

$$(e^{iap}f)(x) = \sum_{n=0}^{\infty}\frac{(ia)^n}{n!}(p^nf)(x), \qquad x \in \mathbb{R}$$

does not hold, where $a \in \mathbb{R}$.

Solution. □

114. For an arbitrary self-adjoint operator T on $\mathcal{H}$ and an arbitrary $g \in \mathcal{H}$, show that there is a conjugation J on $\mathcal{H}$ such that

$$JTJ = T, \qquad Jg = g.$$

Solution. By the spectral theorem(VIII.4 in [Reed-Simon I]), we have a finite measure space (M, μ) and a unitary operator $U : \mathcal{H} \to L^2(M, \mu)$ such that $UTU^* = M_\varphi$ for a real-valued function φ on M and $f \in \operatorname{dom} T$ if and only if $M_\varphi U f \in L^2(M, \mu)$, where M_φ denotes the multiplication operator. Consider the polar decomposition $Ug = ru$, where $r(x) \geq 0$ and $|u(x)| = 1$ for a.e. $x \in M$. Define $J : \mathcal{H} \to \mathcal{H}$ such that $Jf := U^* M_{u^2} \overline{Uf}$ for $f \in \mathcal{H}$. Then, J is a conjugation, an antilinear isometric involution. We can check for $f \in \operatorname{dom} T$ that

$$\begin{aligned}
JTJf &= JU^*UTU^*UJf \\
&= JU^*M_\varphi M_{u^2}\overline{Uf} \\
&= JU^*M_{u^2}M_\varphi\overline{Uf} \\
&= JU^*M_{u^2}\overline{M_\varphi Uf} \\
&= JU^*M_{u^2}\overline{UTf} \\
&= JJTf \\
&= Tf
\end{aligned}$$

and

$$Jg = U^*M_{u^2}\overline{Ug} = U^*M_{u^2}\overline{ru} = U^*M_{u^2}ru^{-1} = U^*ru = g.$$ □

115. Prove Lemma 97.

Solution. □