Harmonic Analysis

Ikhan Choi

November 10, 2021

Contents

Part I

Fourier analysis

Chapter 1

Fourier series

Chapter 2

Fourier integrals

2.1 (Fourier transform of regular Borel measures)**.** Let $\mathcal{M}(\mathbb{R}^d)$ be the space of regular Borel complex measures on $\mathbb{R}^d$.

(a) If $f \in \mathcal{M}(\mathbb{R}^d)$, then $\widehat{f} \in C_b(\mathbb{R}^d)$.

(b) If $f \in \mathcal{M}(\mathbb{R}^d)$ and $\widehat{f} \in L^1(\mathbb{R}^d)$, then $f \in L^1(\mathbb{R}^d)$.

(c) Fourier inversion holds for $\mathcal{M}(\mathbb{R}^d)$ in the sense that

Part II

Singular integral operators

Chapter 3

Caldéron-Zygmund theory

Let f be a nonnegative and sufficiently nice function on $\mathbb{R}^d$, and fix $\lambda > 0$.

3.1 (Calderón-Zygmund decomposition of sets). Let $E_n f$ be the conditional expectation with repect to the σ-algebra generated by dyadic cubes with side length 2^{-n}. Let $Mf = \sup_n |E_n f|$ be the maximal function, and let $\Omega := \{x : Mf(x) > \lambda\}$ for fixed $\lambda > 0$. For $x \in \Omega$ let Q_x be the maximal dyadic cube such that $x \in Q_x$ and

$$\frac{1}{|Q_x|} \int_{Q_x} f > \lambda.$$

(a) $\{Q_x : x \in \Omega\}$ is a countable partition of Ω.

(b) We have an weak type estimate $|\Omega| \leq \frac{1}{\lambda} \|f\|_{L^1}$.

(c) $\|f\|_{L^\infty(\Omega^c)} \leq \lambda$.

(d) For $x \in \Omega$

$$\frac{1}{|Q_x|} \int_{Q_x} f \leq 2^d \lambda.$$

3.2 (Calderón-Zygmund decomposition of a function). Let

$$g(x) := \begin{cases} f(x) & , x \notin \Omega \\ \frac{1}{|Q_x|} \int_{Q_x} f & , x \in \Omega \end{cases}$$

and $b_i := (f - g)\chi_{Q_i}$.

(a) $f = g + b$ where $b = \sum_i b_i$.

(b) $\|g\|_{L^1} = \|f\|_{L^1}$.

(c) $\|g\|_{L^\infty} \lesssim_d \lambda$.

(d) $\int_{Q_i} b_i = 0$.

3.3 (Calderón-Zygmund theory for convolution type)**.** Let T be a singular integral operator of convolution type in the sense that there is a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ such that

$$Tf(x) = \int K(x-y)f(y)\,dy$$

whenever $x \notin \operatorname{supp} f$. Suppose the following two conditions are satisfied.

(i) The L^2-boundedness: $\|Tf\|_{L^2} \lesssim \|f\|_{L^2}$.

(ii) The Hörmander condition: $\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \lesssim 1$.

Let $f = g + b = g + \sum_i b_i$ be the Calderón-Zygmund decomposition, and let $\Omega^* := \bigcup_i Q_i^*$ where Q_i^* is the cube with the same center as Q_i and whose sides are $2\sqrt{d}$ times longer.

(a) The L^2-boundedness implies

$$|\{x : |Tg(x)| > \tfrac{\lambda}{2}\}| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(b) The Hörmander condition implies

$$|\{x : |Tb(x)| > \tfrac{\lambda}{2}\} \setminus \Omega^*| \lesssim_d \frac{1}{\lambda}\|f\|_{L^1}.$$

(c) T is weak $(1,1)$.

Chapter 4

Littlewood-Paley theory

Chapter 5

Multiplier theorems

Part III

Pseudo-differential operators

Part IV

Oscillatory integral operators