Real Reductive Groups

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We know the finite dimensional representations of complex reductive Lie groups, which has a 1-1 correspondence with finite dimensional(unitary) reps of compact Lie groups via unitarian trick. For example, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$ belong to former, and $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are in the latter.

For the construction and classification of irreducible reps (highest weight theory) of complex reductive Lie groups, we have several methods:

- as quotients of a Verma module,
- as holomorphic sections of line bundles on a flag varieity (Borel-Weil theory).

For infinite dim reps of a real reductive Lie group $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{O}(p,q) = \{g \in M_{p+q}(\mathbb{R}) : {}^t g \mathrm{I}_{p,q} g = \mathrm{I}_{p,q}\}(\mathrm{I}_{p,q} := I_p \oplus (-I_q))$,

- asymptotic behaviors of matrix elements, quotients of principal series representations (Langlands)
- D-modules over flag variety (Beilinson-Bernstein, Brylinski-Kashiwara)
- minimal K-type (Vogar)

Classification of infinite-dimensional unitary reps is still unsolved.

Definition 1.1. A Lie group is informally both a manifold and a group. A C^∞ (complex) manifold is a Hausdorff second countable space that is locally homeomorphic to open sets in $\mathbb{R}^n$ ($\mathbb{C}^n$), such that the transition maps are C^∞ (holomorphic).

A Lie group is a group with a structure of C^∞ manifolds such that maps from the group structures $G \times G \to G : (g, g') \mapsto gg'$ and $G \to G : g \mapsto g^{-1}$ are C^∞. We can do same for complex Lie groups.

Example 1.1 (Lie groups)**.** $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \times)$, $\mathrm{GL}(n,\mathbb{R})$ (C^∞ structure is induced from $\mathbb{R}^{n^2}$ as an open subset), $\mathrm{SL}(n,\mathbb{R})$ (preimage theorem from) are Lie groups.

Example 1.2 (Complex Lie groups)**.** $(\mathbb{C}^n, +)$, $(\mathbb{C}^\times, \times)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$ are complex Lie groups. $U(n)$ is not complex.

Exercise: Check that the above examples.

The definitions of representations differ in references. In this lecture, we follow:

Definition 1.2. Let G be a Lie group, V a fin. dim. vector space over $\mathbb{C}$. A (finite-dimensional) representation is a Lie group homomoprphism $\pi : G \to GL_{\mathbb{C}}(V)$. We can do same for holomorphic representations.

Remark. For a group homomorphism $\pi : G \to \mathrm{GL}(V)$ from a Lie group G, TFAE:

(a) π is C^∞

(b) π is continuous

(c) $G \times V \to V$ is continuous.

Example 1.3. The determinant $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C}) = \mathbb{C}^\times$ and the identity on $\mathrm{GL}(n,\mathbb{C})$ are holomorphic reps of $\mathrm{GL}(n,\mathbb{C})$. Also, $\mu^m : \mathbb{C}^\times \to \mathbb{C}^\times : z \mapsto z^m$ and $\mu^m : \mathrm{U}(1) \to \mathbb{C}^\times$ are holomorphic reps of $\mathbb{C}^\times$ and $\mathrm{U}(1)$.

Definition 1.3. For two reps (π, V), (π', V') of G, we say thery are equivalent if there is a linear isomorphism $i : V \to V'$ such that $\pi(g)i = i\pi'(g)$ for all $g \in G$. For a subspace $W \subset V$, if $\pi(g)(W) \subset W$ for $g \in G$, then we say a representation (π_W, W) is a subrepresentation of (π, V). Irreducible representations are representations having only two subrepresentations. They are "minimal units" of representations.

For reps $(\pi_1, V_1), \cdots, (\pi_n, V_n)$ of G, we define the direct sum as a representation on $V_1 \oplus \cdots \oplus V_n$ with

$$(\pi_1 \oplus \cdots \oplus \pi_n)(g)(v_1, \cdots, v_n) := (\pi_1(g)v_1, \cdots, \pi_n(g)v_n).$$

Proposition 1.1. (a) *If (π, V) is a holomorphic representation of $\mathbb{C}^\times$, then there is $m_1, \cdots, m_n \in \mathbb{Z}$ such that $\pi \sim \mu^{m_1} \oplus \cdots \oplus \mu^{m_n}$.*

(b) *If (π, V) is a holomorphic representation of* $\mathrm{U}(1)$*, then there is $m_1, \cdots, m_n \in \mathbb{Z}$ such that $\pi \sim \mu^{m_1} \oplus \cdots \oplus \mu^{m_n}$.*

Proof. We first show the following lemma: If $(\pi, \mathbb{C}^n)$ is a representation of a Lie group $(\mathbb{R}, +)$, then there is $X \in M_n(\mathbb{C})$ such that $\pi(t) = \exp(tX)$ for $t \in \mathbb{R}$, i.e. it factors through $\mathbb{R} \to M_n(\mathbb{C}) : t \mapsto tX$.

Proof of the lemma: If we take a small open ball U of $M_n(\mathbb{C})$ centered at the origin, then $\exp : U \to \mathrm{GL}(n, \mathbb{C})$ is injective, so we can take t_0 small enough so that $\pi([-t_0, t_0]) \subset \exp(\frac{1}{2}U)$. Let $Y \in U, Z \in \frac{1}{2}U$ such that $\pi(t_0) = \exp(Y)$, $\pi(\frac{t_0}{2}) = \exp(Z)$. Then, $\pi(t_0) = \exp(2Z)$, so $Y = 2Z$. Repeating this, $\pi(\frac{t_0}{2^N}) = \exp(\frac{Y}{2^N})$ for all N. Since $\{\frac{M}{2^N}t_0\}$ is dense in $\mathbb{R}$ and π is continuous, $\pi(at_0) = \exp(aY)\ \forall a \in \mathbb{R}$. Thus we have $X = t_0^{-1}Y$ which satisfies the lemma. (Remark: we only have used the continuity of π, not the smoothness)

Then we back to the proof of the proposition. $\square$