Homological Algebra

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1. Show that if $n \geq 2$ is an integer which is not a power of a prime, then there is a projective $\mathbb{Z}/n\mathbb{Z}$-module which is not free.

Solution. □

2. Show that if n is a power of prime, then every projective $\mathbb{Z}/n\mathbb{Z}$-module is free.

Solution. □

3. Let p be a prime and M_i are abelian groups, where $i \in \{1,2,3\}$. Suppose that $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are group homomorphisms satisfying $g \circ f = 0$, and that the homomorphisms $M_i \to M_i : x \mapsto px$ are injective for all i. Consider a sequence

$$0 \to M_1/p^n M_1 \xrightarrow{f_n} M_2/p^n M_2 \xrightarrow{g_n} M_3/p^n M_3 \to 0,$$

where f_n and g_n are homomorphisms naturally induced from f and g. Show that the following statements are equivalent:

(i) The above sequence is exact for an integer $n \geq 1$.

(ii) The above sequence is exact for all integer $n \geq 1$.

Solution. □

4. Let $R := \mathbb{Z}/n\mathbb{Z}$ for an integer $n \geq 2$.

(1) Show that an R-module M is injective if and only if for every $a \in M \setminus \{0\}$ there exist $b \in M$ and $m \mid n$ such that the order of a is n/m and $a = mb$.

(2) Let m and l be divisors of n. Using an injective resolution of $\mathbb{Z}/m\mathbb{Z}$ in the category of R-modules, compute $\mathrm{Ext}^i_R(\mathbb{Z}/l\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

Solution. □

5. Let $R = \mathbb{C}[x,y]$.

(1) Compute $\mathrm{Ext}^i_R(R/(x,y), R)$.

(2) Are $\mathbb{C}(x,y)$ and $\mathbb{C}(x,y)/\mathbb{C}[x,y]$ injective R-modules?

Solution. □

6. For a prime p, is the ideal (p,x) of $\mathbb{Z}[x]$ a flat $\mathbb{Z}[x]$-module?

Solution. □

7. Let A be a commutative ring and B be a A-algebra. Let d be a positive integer and suppose an A-module M satisfies $\mathrm{Tor}^A_n(B,M) = 0$ for $0 < n \leq d$. Show that for any B-module N we have $\mathrm{Ext}^m_B(B \otimes AM, N) \cong \mathrm{Ext}^m_A(M,N)$ for $0 \leq m \leq d$.

Solution. □

8. Let $L_\bullet$ be a chain complex of finitely generated free abelian groups. Here we do not assume L is bounded below. For a prime p and an integer n, define $r_{n,p} := \dim_{\mathbb{F}_p} H_n(L_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p)$. Show that the following are equivalent:

(i) The integer $r_{n,p}$ does not depend on p for all n.

(ii) The homology group $H_n(L_\bullet)$ is free for all n.

Solution. □

9. Define a category $\mathcal{C}$ as follows: an object is a tuple $\mathcal{M} = (M_0, M_1, f_0, f_1)$ of abelian groups M_0, M_1 and homomorphisms $f_i : M_0 \to M_1$ with $i \in \{0,1\}$, and a morphism between $\mathcal{M} = (M_0, M_1, f_0, f_1)$ and $\mathcal{M}' = (M_0', M_1', f_0', f_1')$ is a pair $\varphi = (\varphi_0, \varphi_1)$ of homomorphisms $\varphi_i : M_i \to M_i'$ such that $\varphi_1 \circ f_j = f_j' \circ \varphi_0$ for $i, j \in \{0,1\}$.

(1) Show that $\mathcal{C}$ is abelian.

(2) For an abelian group N, define objects $r_0(N) := (N, 0, 0, 0)$ and $r_1(N) := (N \otimes N, N, \mathrm{pr}_0, \mathrm{pr}_1)$ in $\mathcal{C}$. Show that for any object $\mathcal{M} = (M_0, M_1, f_0, f_1)$ in $\mathcal{C}$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, r_0(N)) \cong \mathrm{Hom}(M_0, N), \qquad \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, r_1(N)) \cong \mathrm{Hom}(M_1, N).$$

(3) Show that $\mathcal{C}$ has enough injective objects.

(4) Define a functor $F : \mathcal{C} \to \mathbf{Ab}$ such that $F(\mathcal{M}) := \{m \in M_0 : f_0(m) = f_1(m)\}$. Show that $R^1F(\mathcal{M}) = \mathrm{coker}(f_0 - f_1)$ and $R^iF = 0$ for $i \geq 2$, where R^iF denotes the right derived functor.

Solution. □

10. Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $C^{\geq 0}(\mathcal{A})$ be an abelian category of cochain complexes $K^\bullet$ such that $K^n = 0$ for $n < 0$.

(1) For an integer $n \geq 0$, find the right adjoint functor of the functor $e_n^* : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto K^n$.

(2) Show that $C^{\geq 0}(\mathcal{A})$ has enough injective objects.

(3) Show that the right derived functor of the left exact functor $H^0 : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto H^0(K^\bullet)$ is given by $H^n : C^{\geq 0}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto H^n(K^\bullet)$ for $n \geq 0$.

Solution. □

11. Give an example of an abelian category in which the direct product exists and the direct product does not preserve right exact sequences.

Solution. □

12. Give an example of an additive category $\mathcal{C}$ with kernels and cokernels in which a morphism $f : A \to B$ such that $\mathrm{coim} f \to \mathrm{im} f$ is not epi exists.

Solution. □