C*-Algebras

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Contents

Part I

Constructions

Chapter 1

Completely positive maps

1.1 Operator systems and spaces

1.1 (Choi-Effros characterization)**.**

1.2 (Von Neumann inequality)**.**

The set $M_n(A)^+$ is linearly spanned by elements of the form $[a_i^*a_j] \in M_n(A)$ for $[a_i] \in A^n$. A linear map $\varphi : A \to B$ is completely positive if

$$\varphi(a_i^*a_j)$$

1.3 (n-positive maps)**.** Let S be an operator space. Let A and B be C*-algebras.

(a) (Cauchy-Schwarz inequality) If $\varphi : A \to B$ is a 2-positive map, then $\lim_\alpha \|\varphi(e_\alpha)\| = \|\varphi\|$ for any approximate unit (e_α) of A, and

$$\varphi(a)^*\varphi(a) \le \|\varphi\|\varphi(a^*a), \qquad a \in A.$$

(b) (Multiplicative domain) Let $\varphi : A \to B$ be a 4-positive map with $\|\varphi\| = 1$. If $a \in A$ satisfies $\varphi(a)^*\varphi(a) = \varphi(a^*a)$, then $\varphi(b)\varphi(a) = \varphi(ba)$ for all $b \in A$. In particular, if $\varphi : B \to C$ is an extension of a $*$-homomorphism $\pi : A \to C$, then $\varphi(ab) = \pi(a)\varphi(b)$ and $\varphi(ba) = \varphi(b)\pi(a)$ for $a \in A$ and $b \in B$.

Proof. (a) It suffices to show

$$\varphi(a)^*\varphi(a) \le \lim_\alpha \|\varphi(e_\alpha)\|\varphi(a^*a), \qquad a \in A,$$

since

$$\frac{\|\varphi(a)\|^2}{\|a\|^2} \le \lim_\alpha \|\varphi(e_\alpha)\| \frac{\|\varphi(a^*a)\|}{\|a^*a\|}$$

implies $\|\varphi\|^2 \le \lim_\alpha \|\varphi(e_\alpha)\|\|\varphi\|$. Suppose B acts on a Hilbert space H non-degenerately and faithfully. Since φ is 2-positive, we have

$$\begin{pmatrix} \varphi(e_\alpha^2) & \varphi(e_\alpha a) \\ \varphi(a^*e_\alpha) & \varphi(a^*a) \end{pmatrix} = \varphi^{(2)}\left(\begin{pmatrix} e_\alpha^2 & e_\alpha a \\ a^*e_\alpha & a^*a \end{pmatrix}\right) = \varphi^{(2)}\left(\begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix}\right) \ge 0,$$

and it is equivalent to

$$\langle \varphi(e_\alpha^2)\xi, \xi\rangle + 2\,\mathrm{Re}\langle \varphi(e_\alpha a)\eta, \xi\rangle + \langle \varphi(a^*a)\eta, \eta\rangle \ge 0, \qquad \xi, \eta \in H, \quad a \in A.$$

We put $\xi := -(\|\varphi(e_\alpha)\| + \varepsilon)^{-1}\varphi(e_\alpha a)\eta$ for $\varepsilon > 0$ to get

$$\varphi(e_\alpha a)^*\varphi(e_\alpha a) \le \varphi(e_\alpha a)^*[2 - (\|\varphi(e_\alpha)\| + \varepsilon)^{-1}\varphi(e_\alpha^2)]\varphi(e_\alpha a) \le (\|\varphi(e_\alpha)\| + \varepsilon)\varphi(a^*a)$$

We have the desired inequality by taking limits for α and ε.

(b) Since the second inflation $\varphi^{(2)}$ is 2-positive, we may write the Cauchy-Schwarz inequality

$$\varphi^{(2)}\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right)^* \varphi^{(2)}\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \le \varphi^{(2)}\left(\begin{pmatrix} a^*a & a^*b \\ b^*a & b^*b \end{pmatrix}\right),$$

so

$$\begin{pmatrix} 0 & \varphi(a^*b) - \varphi(a^*)\varphi(b) \\ \varphi(b^*a) - \varphi(b^*)\varphi(a) & \varphi(b^*b) - \varphi(b^*)\varphi(b) \end{pmatrix} \ge 0,$$

which implies $\varphi(b^*a) - \varphi(b^*)\varphi(a) = 0$ for any $b \in A$.

Note that $\|\pi\| = 1$ if π is not trivial. Using the above argument for a and a^*, we are done. □

1.4 (Russo-Dye theorem)**.** If $C(X) \to B$ is positive, then it is c.p.

1.5 (Completely positive maps for matrix algebras)**.** Let A be a C*-algebra.

(a) Choi matrix

(b) There is a one-to-one correspondence

$$\mathrm{CP}(M_n(\mathbb{C}), A) \to M_n(A)_+ : \varphi \mapsto [\varphi(e_{ij})].$$

(c) Let A be unital. There is a one-to-one correspondence

$$\mathrm{CP}(A, M_n(\mathbb{C})) \to M_n(A)^*_+ : \varphi \mapsto (s_\varphi : [a_{ij}] \mapsto \sum_{i,j} \langle \varphi(a_{ij})e_j, e_i \rangle).$$

(d) The above correspondences are (maybe?) isometric if we endow the complete norm on CP.

Proof. (b)

□

1.2 Dilations and Extensions

A linear map $\varphi : A \to B(H)$ is completely positive if and only if

$$\sum_{i,j} \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle \ge 0, \qquad (a_i) \in A^n, \ (\xi_i) \in H^n.$$

1.6 (Stinespring dilation)**.** Let A be a C*-algebra and $\varphi : A \to B(H)$ be a c.p. map. A *Stinespring dilation* of φ is a pair (π, V) of a representation $\pi : A \to B(K)$ and a bounded linear operator $V : H \to K$ such that $\varphi(a) = V^*\pi(a)V$ for $a \in A$.

$$\begin{tikzcd} B(K) \arrow[dr, "V^* \cdot V"] & \\ A \arrow[u, "\pi"] \arrow[r, "\varphi"'] & B(H) \end{tikzcd}$$

(a) φ has a Stinespring dilation (π, V) such that $\overline{\pi(A)VH} = K$.

(b) For a non-degenerate Stinespring dilation (π, V) of φ, the operator V is an isometry if and only if $\sup_\alpha \varphi(e_\alpha) = 1$.

Proof. (a) As we have done in the construction of the GNS representation, define a sesquilinear form on the algebraic tensor product $A \odot H$ such that

$$\langle a \otimes \xi, b \otimes \eta \rangle := \langle \varphi(b^* a)\xi, \eta \rangle, \qquad a \otimes \xi, b \otimes \eta \in A \odot H.$$

It is positive semi-definite since the complete positivity of φ implies

$$\langle \sum_j a_j \otimes \xi_j, \sum_i a_i \otimes \xi_i \rangle = \sum_{i,j} \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle \geq 0, \qquad a_i \otimes \xi_i \in A \odot H.$$

Then, we obtain a Hilbert space $K := \overline{A \odot H / N}$, where $N := \{\eta \in A \odot H : \langle \eta, \eta \rangle = 0\}$. The above construction of a Hilbert space is sometimes called the separation and completion.

Define $\pi : A \to B(K)$ such that

$$\pi(a)(b \otimes \eta + N) := ab \otimes \eta + N, \qquad a \in A, \quad b \otimes \eta + N \in K,$$

and $V : H \to K$ such that

$$\langle V\xi, b \otimes \eta + N \rangle := \langle \varphi(b^*)\xi, \eta \rangle, \qquad \xi \in H, \quad b \otimes \eta + N \in K.$$

The operator V is well-defined by the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle \varphi(b^*)\xi, \eta \rangle|^2 = |\langle \xi, \varphi(b)\eta \rangle|^2 &\leq \|\xi\|^2 \langle \varphi(b^*)\varphi(b)\eta, \eta \rangle \\ &\leq \|\xi\|^2 \|\varphi\| \langle \varphi(b^* b)\eta, \eta \rangle = \|\xi\|^2 \|\varphi\| \|b \otimes \eta + N\|^2. \end{aligned}$$

Then, we can check $\pi(a)V\xi = a \otimes \xi + N$ for $a \in A$ and $\xi \in H$ from

$$\begin{aligned} \langle \pi(a)V\xi, b \otimes \eta + N \rangle = \langle V\xi, a^* b \otimes \eta + N \rangle &= \langle \varphi(b^* a)\xi, \eta \rangle \\ &= \langle a \otimes \xi + N, b \otimes \eta + N \rangle, \qquad b \otimes \eta + N \in K, \end{aligned}$$

so it follows that $V^* \pi(a) V = \varphi(a)$ for $a \in A$ from

$$\langle V^* \pi(a) V\xi, \eta \rangle = \langle V\xi, a^* \otimes \eta + N \rangle = \langle \varphi(a)\xi, \eta \rangle, \qquad \xi, \eta \in H,$$

and the condition $\overline{\pi(A)VH} = K$.

$\square$

1.7 (Voiculescu theorem)**.** Let A be a unital C*-algebra. Let $\pi : A \to B(K)$ be a faithful non-degenerate representation and $\varphi : A \to B(H)$ be a u.c.p. map. Suppose further that $\varphi|_{\pi^{-1}(K(K))} = 0$.

When do we need the faithfulness of π? When do we need the unitality of φ? When do we need the separability of A?

(a) φ is weakly* approximated by vector states, if H is one-dimensional. (Glimm)

(b) φ is approximated by isometry conjugations in $B(A, B(H))$, if H is finite-dimensional. (?)

(c) φ is approximated by isometry conjugations in $\varphi + B(A, K(H))$, if H, K are separable.

Proof. (a) Hahn-Banach separation and Weyl-von Neumann theorem.

(b) correspondence for c.p. maps to matrix algebras.

(c) quasi-central approximate unit and block diagonal c.p. maps.

$\square$

1.8 (Arveson extension)**.** Let $A \subset B$ be C*-algebras. Let $\varphi : A \to B(H)$ be a c.p. map and consider the following diagram:

$$\begin{tikzcd} B \arrow[dr, dashed, "\tilde{\varphi}"] & \\ A \arrow[u] \arrow[r, "\varphi"'] & B(H). \end{tikzcd}$$

(a) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if B is unital and $1_B \in A$.

(b) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is unital and $B = A \oplus \mathbb{C}$.

(c) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is non-unital and $B = \widetilde{\mathcal{A}}$.

(d) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ always exists.

1.9 (Representation extension)**.** Let I be a closed ideal of a C*-algebra B. For a representation $\pi : I \to B(H)$, there is a representation $\widetilde{\pi} : B \to B(H)$ which extends π. If π is non-degenerate, the extension is unique and $\pi(e_\alpha b) \to \widetilde{\pi}(b)$ and $\pi(be_\alpha) \to \widetilde{\pi}(b)$ strongly for $b \in B$, where e_α is an approximate unit of I.

Proof. We may assume π is non-degenerate by replacing H to $\overline{\pi(I)H}$. Define $\widetilde{\pi} : B \to B(H)$ such that

$$\widetilde{\pi}(b)(\pi(a)\xi) := \pi(ba)\xi, \qquad a \in I,\ \xi \in H.$$

The well-definedness is from

$$\|\pi(ba)\xi\|^2 = \langle \pi(a^*b^*ba)\xi, \xi\rangle \le \|b\|^2 \langle \pi(a^*a)\xi, \xi\rangle = \|b\|^2 \|\pi(a)\xi\|^2.$$

It is clearly a $*$-homomorphism and extends π.

For the uniqueness, if π is non-degenerate and $\widetilde{\pi}$ is a $*$-homomorphism which extends π, then

$$\widetilde{\pi}(b)(\pi(a)\xi) = \widetilde{\pi}(b)\widetilde{\pi}(a)\xi = \widetilde{\pi}(ba)\xi = \pi(ba)\xi,$$

which is unique by the density of $\pi(I)H$ in H. □

extension of representations for ideals
unique extension of c.p. maps for hereditary subalgebras.

1.3 Completely bounded maps

1.4 Tensor products

1.10 (Maximal tensor products)**.** Let A and B be C*-algebras.

(a) (restrictions) A commuting pair of $*$-homomorphisms $\pi : A \to B(H)$ and $\pi' : B \to B(H)$ corresponds to a $*$-homomorphism $\Pi : A \odot B \to B(H)$ via the relation $\Pi(a \otimes b) = \pi(a)\pi'(b)$.

(b) $A \odot B$ admits a $*$-representation and every norms induced from these $*$-representations are uniformly bounded. So, we can define a maximal tensor norm on $A \odot B$.

(c) $a \otimes - : B \to A \odot B$ is a bounded linear map for each $a \in A$ with respect to any C*-norm on $A \odot B$. [BO, 3.2.5]

1.11 (Minimal tensor product)**.** spatiality

1.12 (Takesaki theorem)**.**

Tensors with $M_n(\mathbb{C})$, $C_0(X)$.

1.13 (Haagerup tensor product)**.**

Trick

Exercises

1.14. Let A be a hereditary C*-subalgebra of a C*-algebra B and let $b \in B_+$. If for any $\varepsilon > 0$ there is $a \in A_+$ such that $b - a \leq \varepsilon$, then $b \in A$.

Proof. For $a \in A_+$ satisfying $b \leq a + \varepsilon \leq (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^2$, define

$$a_\varepsilon := a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} \in A.$$

Then,

$$\|b^{\frac{1}{2}} - b^{\frac{1}{2}} a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\|^2 = \varepsilon \|(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\| \leq \varepsilon.$$

Thus $a_\varepsilon \to b$ in norm as $\varepsilon \to 0$. □

Chapter 2

Hilbert modules

2.1 Hilbert modules

2.1 (Banach modules). Let A be a Banach algebra. A *Banach A-module* is a Banach space $\mathcal{E}$ which is a A-module such that the action is bounded.

(a) (Cohen factorization theorem) If A has a left approximate unit, then $A\mathcal{E}$ is closed in $\mathcal{E}$.

Proof. Suppose ξ belongs to the closure of $A\mathcal{E}$ and take $\varepsilon > 0$. We will construct a decreasing sequence a_n in the unitization $\widetilde{A}$ such that $a_n^{-1}\xi$ and a_n are both Cauchy. In order to do this, we first need to check $a_n^{-1} \in \widetilde{A} \setminus A$ can act on $\mathcal{E}$, which is easy anyway.

Let $a_0 = 1$ and suppose we have defined $a_n \geq 2^{-n}$. Take $b \in A$ and η such that $\|\xi - b\eta\| < \varepsilon 2^{-(2n+1)}$. Take $e \in A$ such that $\|a_n^{-1}b - ea_n^{-1}b\|\|\eta\| < \varepsilon 2^{-(n+1)}$. Now inductively define

$$a_{n+1} := a_n - 2^{-(n+1)}(1-e) \in \widetilde{A}$$

so that $a_{n+1} \geq 2^{-(n+1)}$ is invertible.

Then, we can check a_n is Cauchy whose limit point belongs to A, and $a_n^{-1}\xi$ is Cauchy because by the identity

$$a_{n+1}^{-1} - a_n^{-1} = a_{n+1}^{-1}(a_n - a_{n+1})a_n^{-1} = a_{n+1}^{-1}2^{-(n+1)}(1-e)a_n^{-1}$$

we get

$$\begin{aligned}\|a_{n+1}^{-1}\xi - a_n^{-1}\xi\| &\leq \|a_{n+1}^{-1} - a_n^{-1}\|\|\xi - b\eta\| + \|(a_{n+1}^{-1} - a_n^{-1})b\|\|\eta\| \\ &\leq 2^n \cdot \varepsilon 2^{-(2n+1)} + \varepsilon 2^{-(n+1)} = \varepsilon 2^{-n}.\end{aligned}$$

□

2.2 (Finsler modules). Let A be a C*-algebra.

2.3 (Hilbert modules). Let A be a C*-algebra. A *Hilbert A-module* is a complex linear space $\mathcal{E}$ which is a right A-module together with a

(i) a ring homomorphism $A^{op} \to \mathrm{End}_{\mathbb{C}}(\mathcal{E})$,

(ii) an A-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to A$ which is A-linear in second argument,

which is complete with respect to the norm $\|\xi\| := \|\langle\xi,\xi\rangle\|^{\frac{1}{2}}$.

(a) Examples: A itself, $\ell^2(\mathbb{N},A)$, etc.

(b) direct sum, tensor product, localization

2.4 (Multiplier algebra)**.** Let A be a C*-algebra. A *double centralizer* of A is a pair (L,R) of bounded linear maps on A such that $aL(b)=R(a)b$ for all $a,b\in A$. The *multiplier algebra* $M(A)$ of A is defined to be the set of all double centralizers of A. There is another characterization $M(A):=L_A(A)$, the set of adjointable operators to itself.

double centralizers vs essential ideals, strict topology,

(a) $\|\pi(a-e_\alpha a)\xi\|^2$

(b) If a_α are unitary, the convergences in the strict topology and the weak topology(how to define this?) coincide.

(c) If a_α are increasing, the convergences in the strict topology and the weak topology(how to define this?) coincide.

(d) $M(K(H))\cong B(H)$.

(e) $M(C_0(\Omega))\cong C_b(\Omega)$.

Proof. First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega)\cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega)\cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective $*$-homomorphism $C_b(\Omega)\to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g\in M(C_0(\Omega))_+$. □

2.2 Categorical aspects

2.5 (Morphisms)**.** Let A and B be C*-algebras. A *morphism* from A to B is defined to be a non-degenerate $*$-homomorphism $A\to M(B)$.

(a) The inclusion of a proper closed ideal is never a morphism. A surjective $*$-homomorphism $A\to B$ is always a morphism.

Every morphism $A\to M(B)$ induces the following?:

$$\begin{tikzcd} PS(B) \arrow[r, two heads] \arrow[d] & \widehat{B} \arrow[r, two heads] \arrow[d] & \mathrm{Prim}(B) \arrow[d] \\ PS(A) \arrow[r, two heads] & \widehat{A} \arrow[r, two heads] & \mathrm{Prim}(A). \end{tikzcd}$$

For a short exact sequence

$$0\longrightarrow I\longrightarrow A\longrightarrow B\longrightarrow 0\ ,$$

we have

$$\begin{tikzcd} PS(I) \arrow[r, hook] \arrow[d, two heads] & PS(A) \arrow[d, two heads] & PS(B) \arrow[l, hook'] \arrow[d, two heads] \\ \widehat{I} \arrow[r, hook] \arrow[d, two heads] & \widehat{A} \arrow[d, two heads] & \widehat{B} \arrow[l, hook'] \arrow[d, two heads] \\ \mathrm{Prim}(I) \arrow[r, hook, "\text{open}"'] & \mathrm{Prim}(A) & \mathrm{Prim}(B) \arrow[l, hook', "\text{closed}"] \end{tikzcd}$$

We have to understand C*-algebras in the context of homotopy theory, so the pointed topological spaces must be considered. An open set U of a locally compact Hausdorff space X should be recognized as the quotient space $(X,x)/(A,x)$, where $x\notin U=A^c$, hence the ideal $A(U)$ corresponds and the restriction $A(X)\to A(U)$ does not make sense. In other words, $A(U)$ **is not an analogue of** $\mathcal{O}_X(U)$**, but of** $\mathcal{I}_{(X,X\setminus U)}$. It is fortunate that the kernel of the restriction, an ideal, can be recognized as the function algebra of the complement, which is not the case in algebraic geometry...? (Can define the quotient X/A for an analytic subset of a complex space X?)

Then, how can we understand the sheaf theoretic restriction on an open set in operator algebras? How about Banach or Fréchet algebras? Can we consider a “rigid” Zariksi topology on the spectrum? (Closed sets in C*-context are too flaccid)

2.6 (Modular maximal left ideals)**.**

2.7 (Primitive ideals)**.** hull kernel topology

$$PS(A) \cong \{(\pi,\psi)\}/\sim_u, \qquad \widehat{A} \cong \{\pi\}/\sim_u .$$

A	$PS(A)$	$\widehat{A}$	$\mathrm{Prim}(A)$
$C(X)$	X	X	X
$K(H)$	PH	$*$	$*$
$\widetilde{K}(H)$	?	?	$\{0,K(H)\}$
$B(H)$			

(a) $\mathrm{Prim}(A)$ is locally compact T_0 space.

(b) Two maps $PS(A) \to \widehat{A} \to \mathrm{Prim}(A)$ are continuous surjective open maps

(c) If A is type I, then $\widehat{A} \to \mathrm{Prim}(A)$ is an homeomorphism.

2.8 (Dauns-Hoffman theorem)**.**

2.3 Pimsner algebras

2.9 (C*-correspondences)**.** Let A be a C*-algebra. A *C*-correspondence* over A is a right Hilbert A-module $\mathcal{E}$ together with a $*$-homomorphism $\varphi : A \to B(\mathcal{E})$, called the *left action*. We say $\mathcal{E}$ is *faithful* or *non-degenerate* if φ is faithful or non-degenerate, respectively.

(a) If $\varphi : A \to M(B)$ is a unital completely positive map, then we can construct a natural A-B-correspondence $\mathcal{E}$ by mimicking the GNS construction on $A \odot B$.

(b) If $\varphi : A \to M(B)$ is a non-degenerate $*$-homomorphism, $\varphi \in \mathrm{Mor}(A,B)$ in other words, then we can associate a canonical A-B-correspondence B such that the left action is realized with φ. More precisely, $\iota : \mathcal{E} \to B : a \otimes b \mapsto \varphi(a)b$ provides a well-defined linear isomorphism (surjectivity follows from the density of $\varphi(A)B$ in B and the Cohen factorization theorem) and the two actions on $\mathcal{E}$ is described by $\iota(a\xi b) = \varphi(a)\iota(\xi)b$.

2.10. Let $\mathcal{E}$ be a C*-correspondence over a C*-algebra A. Let B be a C*-algebra and see it as a trivial C*-correspondence over B. A *representation* of $\mathcal{E}$ on B is a pair (π,τ) of a $*$-homomorphism $\pi : A \to B$ and a linear map $\tau : \mathcal{E} \to B$ such that

$$\pi(\langle\xi,\eta\rangle) = \tau(\xi)^*\tau(\eta), \qquad \tau(\varphi(a)\xi) = \pi(a)\tau(\xi).$$

We define the *Katsura ideal*

$$J(\mathcal{E}) := \varphi^{-1}(K(\mathcal{E})) \cap \varphi^{-1}(0)^{\perp}.$$

A *covariant representation* is a representation of $\mathcal{E}$ such that

$$\psi(\varphi(a)) = \pi(a), \qquad a \in J(\mathcal{E}).$$

(a) Let $(A,\mathbb{Z},\alpha)$ be a C*-dynamical system and consider the canonical C*-correspondence A over A with the left action $\varphi := \alpha_1 \in \mathrm{Aut}(A) \subset \mathrm{Mor}(A)$. This correspondence is full, faithful, and non-degenerate. Note that also we have $J(A) = \varphi^{-1}(A) \cap A = A$. If (π,τ) is an any representation of this C*-correspondence A on B, then

How can we decribe representations of C*-correspondence A with left action $\varphi \in \mathrm{Aut}(A)$ in terms of covariant representations of the C*-dynamical system $(A, \mathbb{Z}, \alpha)$ with $\alpha_n = \varphi^n$?

as a morphism sub and quotient, direct sum, tensor product,

Toeplitz-Cuntz Toeplitz-Pimsner Cuntz-Pimsner Cuntz-Krieger

Subproduct systems

2.4 Morita equivalence

Induced representations?

Chapter 3

Constructions

3.1 Categorical constructions

direct limits, tensor products, free products

3.2 Crossed products

3.1 (Group algebras)**.**

type I, subhomogeneous
crystallographic discrete heisenberg free groups projectionless of $C_r^*(F_2)$

3.2 (Enveloping C*-algeberas)**.** Let A be a $*$-algebra. A *C*-norm* is an submultiplicative norm satisfying the C*-identity. Does A have enough $*$-representations?

(a) A complete C*-norm is unique if it exists.

(b) For every C*-norm α on A, there is a $*$-isometry $\pi : A \to B(H)$.

(c) For maximal C*-norm, there is a universal property. The maximal C*-norm can be obtained by running through cyclic representations.

3.3 (C*-dynamical system)**.** Let G be a locally compact group. A *C*-dynamical system* or a *G-C*-algebra* is a C*-algebra A together with a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ that is continuous in the point-norm topology. We will often write a triple (A, G, α) instead of A to refer a C*-dynamical system.

(a) There is an equivalence between categories of locally compact transformation groups and C*-dynamical system on abelian C*-algebras.

On $U(H)$, the strict topology and the strong operator topology are equal. Therefore, we have three topologies to consider: strong, weak, and σ-weak.

3.4 (Covariant representation)**.** Let G be a locally compact group.
A *covariant representation* of a C*-dynamical system (A, G, α) is a G-equivariant $*$-homomorphism $\pi : (A, G, \alpha) \to (B(H), G, \beta)$ for a C*-dynamical system $(B(H), G, \beta)$, where a Hilbert space H.

(a) There exists a unitary representation $u : G \to B(H)$ such that $\pi(\alpha_s a) = u_s \pi(a) u_s^*$.

(b) (Integrated form) There is a one-to-one correspondence between covariant representations of (A, G, α) and $*$-representations of $L^1(G, A)$. (non-degenerate)

Note that we have a homeomorphism $\mathrm{Aut}(K(H)) \cong PU(H)$ between the point-norm topology and the strong operator topology.

$\mathbb{Z}$-action, Homeo-action, left multiplication of subgroup induced representation regular representation $(C_0(G), G, \lambda) \to (B(L^2(G)), G, \lambda)$.

commutative case

3.3 Graph algebras

3.4 Groupoid algebras

Part II

Properties

Chapter 4

Approximation properties

4.1 Nuclearity and exactness

finite dimensional[BO, 3.3.2], abelian, AF permanence properties

4.1 (Completely positive approximation property)**.** Let A be a C*-algebra.

(a) If A has the CPAP, then A is nuclear.

(b) If A is nuclear, then A has the CPAP.

Proof. (b)

Let $E \subset A$ and $F \subset A^*$ be finite subsets and fix $\varepsilon > 0$. We want to find completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon$$

for $a \in E$ and $l \in F$. To implement the approximation, we would like to regard a bounded linear operator on A as a state of a tensor product of C*-algebras, which maps $\theta \in B(A)$ to the linear functional characterized by $a \otimes l \mapsto l(\theta(a))$. However, since A^* is not a C*-algebra, we embed A^* locally in $B(H)$ through the Radon-Nikodym type result. Let $\pi : A \to B(H)$ be the cyclic representation obtained from a positive linear functional that dominates F and Ω the cyclic vector such that there is a linear map $\pi' : F \to \pi(A)'$ satisfying

$$l(a) = \omega_\Omega(\pi(a)\pi'(l)) = \langle \pi(a)\pi'(l)\Omega, \Omega\rangle$$

for $a \in E$ and $l \in F$. Now the duality of A and F is embodied in the tensor product representation

$$\pi \times i : A \otimes_{\max} \pi(A)' \to B(H)$$

together with a cyclic vector $\Omega \in H$. Here the nuclearity is used to write $A \otimes_{\max} \pi(A)' = A \otimes_{\min} \pi(A)'$.

If we take any faithful representation $\rho : A \to B(K)$, then we obtain a fathful representation

$$\rho \otimes i : A \otimes_{\min} \pi(A)' \to B(K \otimes H).$$

By the Hahn-Banach separation, the state $(\pi \times i)^*\omega_\Omega$ on $A \otimes_{\min} \pi(A)'$ can be approximated weakly* by convex combinations of vector states in $B(K \otimes H)$. In particular, by the density of $\pi(A)\Omega$ in H, we have algebraic tensors $(t_k)_{k=1}^m \subset K \odot \pi(A)\Omega$ such that

$$\left|\omega_\Omega((\pi \times i)(a \otimes \pi'(l))) - \sum_{k=1}^m \lambda_k \omega_{t_k}((\rho \otimes i)(a \otimes \pi'(l)))\right| < \varepsilon \qquad (\dagger)$$

for all $a \in E$ and $l \in F$, where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda = 1$.

If we write each element $t \in K \odot \pi(A)\Omega$ as

$$t = \sum_{i=1}^{n} \eta_i \otimes \pi(b_i)\Omega, \qquad \eta_i \in K,\ b_i \in A,$$

then

$$\begin{aligned}\omega_t((\rho \otimes i)(a \otimes \pi'(l))) &= \left\langle (\rho(a) \otimes \pi'(l))\Big(\sum_{j=1}^{n} \eta_j \otimes \pi(b_j)\Omega\Big), \Big(\sum_{i=1}^{n} \eta_i \otimes \pi(b_i)\Omega\Big) \right\rangle \\ &= \sum_{i,j=1}^{n} \langle \rho(a)\eta_j, \eta_i \rangle \langle \pi'(l)\pi(b_i^* b_j)\Omega, \Omega \rangle \\ &= l\Big(\sum_{i,j=1}^{n} \langle \rho(a)\eta_j, \eta_i \rangle b_i^* b_j\Big).\end{aligned}$$

If we define completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ for each τ such that

$$\varphi(a) := [\langle \rho(a)\eta_j, \eta_i \rangle], \quad \psi([e_{ij}]) := b_i^* b_j,$$

then we have $\omega_t(a \otimes \pi'(l)) = l(\psi \circ \varphi(a))$.

Since $\mu(a \otimes \pi'(l)) = l(a)$ and since the completely positive contractions which factor through a matrix algebra form a convex set, we have completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that the inequality (†) is rewritten as

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon,$$

so we are done. □

quotients of nuclear local reflexivity

a separable C*-algebra is nuclear if and only if every factor representation is hyperfinite.

Extension properties weak expectation property relatively weakly injective maximal tensor product inclusion problem

excision: Akemann-Anderson-Pedersen

4.2 Quasi-diagonality

4.2 (Weyl-von Neumann theorem)**.** A self-adjoint bounded operator is quasi-diagonal.

4.3 (Glimm lemma)**.** If a state ω of $B(H)$ vanishes on $K(H)$, then it is a weak* limit of vector states.

4.4 (Voiculescu theorem)**.**

4.5 (Quasi-diagonal algebras)**.** An operator $a \in B(H)$ is called *quasi-diagonal* if there is a net of projections $p_i \in B(H)$ such that $[p_i, a] \to 0$ in norm and $p_i \uparrow \mathrm{id}_H$ strongly. A C*-algebra is called *quasi-diagonal* if it admits a faithful representation whose image is quasi-diagonal.

faithful non-degenerate essential representations of a quasi-diagonal C*-algebra are all quasi-diagonal

locally quasi-diagonal

4.3 AF-embeddability

Chapter 5

Amenability

5.1 Amenable groups

5.2 Amenable actions

crossed products Z_2-grading Connes-Feldman-Weiss Anantharaman-Delaroche Gromov boundaries approximately central structure? dynamical Kirchberg-Phillips

stably finite dynamical Elliott program

Ornstein-Weiss-Rokhlin lemma

5.3 Exact groups

Exact groups

5.4 Other properties

Kazdahn property (T) factorization property Haagerrup property

Kaplansky conjecture

Chapter 6

Simplicity

Furstenburg boundary

Part III

Invariants

Chapter 7

Operator K-theory

7.1 Homotopy of C*-algebras

7.1 (Homotopy of $*$-homomorphisms)**.** Let A, B be C*-algebras. Two $*$-homomorphisms in $\mathrm{Mor}(A, B)$ are said to be *homotopic* if they are connected by a path in $\mathrm{Mor}(A, B)$ that is continuous with the point-norm topology.

(a) For pointed compact Hausdorff spaces $(X, x_0), (Y, y_0)$, two pointed maps $\varphi_0, \varphi_1 : X \to Y$ are homotopic if and only if $\varphi_0^*, \varphi_1^* : C_0(Y \setminus \{y_0\}) \to C_0(X \setminus \{x_0\})$ are homotopic.

Proof. (a) Suppose φ_0 and φ_1 are connected by a homotopy φ_t. Fixing $g \in C_0(Y)$ and $t_0 \in I$, we want to show

$$\lim_{t \to t_0} \sup_{x \in X} |g(\varphi_t(x)) - g(\varphi_{t_0}(x))| = 0.$$

Since the function g is uniformly continuous, with respect to an arbitrarily chosen uniformity on Y, so that there is an entourage $E \subset Y \times Y$ such that $(y, y') \in E \circ E$ implies $|g(y) - g(y')| < \varepsilon$. Using compactness we have a finite sequence $(y_i)_{i=1}^n \subset Y$ such that for every y there is y_i satisfying $(y, y') \in E$. Then, $\varphi^{-1}(E[y_i])$ is a finite open cover of $X \times I$, so we have δ such that $|t - t_0| < \delta$ implies for any $x \in X$ the existence of i satisfying $(\varphi_t(x), y_i) \in E$ and $(\varphi_{t_0}(x), y_i) \in E$, which deduces the desired inequality.

Conversely, suppose φ_0^* and φ_1^* are connected by a homotopy φ_t^*. By taking dual, we can induce $\varphi_t : X \to Y$ such that $g(\varphi_t(x)) = (\varphi_t^* g)(x)$ for each $g \in C(Y)$ from φ_t^* via the embedding $X \to M(X)$ by Dirac measures. Let V be an open neighborhood of $\varphi_{t_0}(x_0)$ and take $g \in C(Y)$ such that $g(\varphi_{t_0}(x_0)) = 1$ and $g(y) = 0$ for $y \notin V$. Now we have an open neighborhood U of x_0 such that $x \in U$ implies $|(\varphi_{t_0}^* g)(x) - (\varphi_{t_0}^* g)(x_0)| < \frac{1}{2}$. Also we have $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|\varphi_t^* g - \varphi_{t_0}^* g\| < \frac{1}{2}$. Therefore, $(x, t) \in U \times (t_0 - \delta, t_0 + \delta)$ implies $g(\varphi_t(x)) > 0$, hence $\varphi_t(x) \in V$, which means $X \times I \to Y : (x, t) \mapsto \varphi_t(x)$ is continuous. □

We have $\widetilde{K}^n(X, x_0) = K_n(C_0(X \setminus \{x_0\}))$ for a pointed compact Hausdorff space X. Now then since the inclusion $\{x_0\} \to X$ induces the section so that

$$0 \to K_0(C_0(X \setminus \{x_0\})) \to K_0(C(X)) \to K_0(\{x_0\}) \to 0$$

splits, we have

$$K^0(X) = \widetilde{K}^0(X, x_0) \oplus \mathbb{Z} = K_0(C_0(X \setminus \{x_0\})) \oplus K_0(\{x_0\}) = K_0(C(X))$$

for a compact connected Hausdorff space X. The additivity of K_0 and K^0 removes the connectedness condition.

$$K_0(\mathbb{C}) = \mathbb{Z}, \quad K_0(C_0(\mathbb{R})) = 0, \quad K_1(C_0(\mathbb{R})) = K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$$
$$K^0(*) = \mathbb{Z}, \quad K^0(S^1) = \mathbb{Z}, \quad K^1(S^1) = K^0(S^2) = \mathbb{Z}[x]/(x-1)^2$$

7.2 K_0 and K_1 groups

local Banach algebras

homotopy invariance relative, reduced theory partially ordered abelian group

unitary Bott periodicity six-term exact sequence

7.3 Equivariant K-theory

7.2 (Pimsner-Voiculescu exact sequence).

Connes' Thom isomorphism

7.4 Cuntz semigroup

nuclear dimension

Chapter 8

KK-theory

8.1 Cuntz pairs

8.2 Kasparov modules

Chapter 9

Part IV

Classification

Chapter 10

Simple nuclear algebras

10.1 AF-algebras

Glimm's classification of UHF algebras Bratteli diagram Elliott's intertwining argument
Separable AF-algebras are classified by pointed ordered K_0.

10.2 Kirchberg-Phillips theorem

10.3 Classifiability

Jiang-Su stability Universal coefficient theorem
Toms-Winter conjecture strongly self-absorbing nuclear dimension
successful in Kirchberg algebras
https://arxiv.org/pdf/2307.06480.pdf
Elliott classification problem Kirchberg-Phillipes theorem
operator K-theory and its pairing with traces
$\mathcal{Z}$-stability, Rosenberg-Schochet universal coefficient theorem
Connes-Haagerup classification of injective factors

Kirchberg: unital simple separable $\mathcal{Z}$-stable algebra is either purely infinte or stably finite. Haagerup, Blackadar, Handelman: unital simple stably finite algebra has a trace.

Glimm: uniformly hyperfinite algebras Murray-von Neumann: hyperfinite II_1 factors

10.4 Inclusions

Chapter 11

Continuous fields

11.1 Fell bundles

11.1 (Banach bundles)**.** A *Banach bundle*, introduced by Fell, is a continuous open surjection $\pi : E \to X$ between topological spaces together with Banach space structure on each fiber $\pi^{-1}(x)$ such that:

(i) the addition $\{(e, e') : \pi(e) = \pi(e')\} \subset E \times E \to E : (e, e') \mapsto e + e'$ is continuous,

(ii) the scalar multiplication $\mathbb{C} \times E \to E : (\lambda, e) \mapsto \lambda e$ is continuous,

(iii) the norm $E \to \mathbb{R}_{\ge 0} : e \mapsto \|e\|$ is continuous,

(iv) the family of subsets

$$\{e \in B : \pi(e) \in U,\ \|e\| < r\}_{U \in N(x), r > 0}$$

forms a neighborhood basis of $0 \in \pi^{-1}(x)$ in E.

The forth condition is equivalent to that if $\|e_i\| \to 0$ and $\pi(e_i) \to x$ then $e_i \to 0_x \in \pi^{-1}(x)$.

(a) For a Banach bundle $E \to X$, if X is locally compact Hausdorff and every fiber E_x shares a same finite dimension, then the bundle is locally trivial.

11.2 (Continuous fields of Banach spaces)**.**

11.3 (Hilbert bundles)**.** A *Hilbert bundle* is a Banach bundle whose norm function satisfies the parallelogram law.

(a) On a compact X, there is an equivalence between the category of Hilbert $C(X)$-modules and the category of Hilbert bundles over X.

(b) On a compact X, there is an equivalence between the category of algebraically finitely generated Hilbert $C(X)$-modules and the category of classical locally trivial finite-rank complex vector bundle over X. It is due to that finitely generatedness implies the projectivity and the Serre-Swan theorem.

11.2 Dixmier-Douady theory

Fell's condition

A C*-algebra A is called *continuous trace* if the set of all $a \in \mathcal{A}$ such that $\widehat{A} \to \mathbb{R}_{\ge 0} : \pi \mapsto \mathrm{tr}(\pi(a^* a))$ is continuous is dense in A.

Dadarlat-Pennig theory

Coactions and Fell bundles