Operator Algebra Seminar Note I

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Contents

Acknowledgement

This note has been written based on the first-year graduate seminar presented at the University of Tokyo in the 2023 Spring semester. Each seminar was delivered for 105 minutes. The main reference for this seminar was Brown-Ozawa, and detailed gaps were filled with the aids of other books such as Takesaki, Murphy, and Paulsen whenever required.

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1.1 Completely positive maps

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* (c.p.) if the inflation $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B}) : [a_{ij}] \mapsto [\varphi(a_{ij})]$ is positive for each $n \geq 1$.

Remark 1.2. For the positivity in matrix algebras, the following equivalent statements are useful.

(a) $[a_{ij}] \in M_n(\mathcal{A})$ is positive.

(b) $[a_{ij}] = [b_{ij}]^*[b_{ij}] = [b_{ji}^*][b_{ij}] = [\sum_k b_{ki}^* b_{kj}]$ for some $[b_{ij}] \in M_n(\mathcal{A})$.

(c) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for a faithful representation $\pi : \mathcal{A} \to B(H)$.

(d) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for every representation $\pi : \mathcal{A} \to B(H)$.

Example 1.3.

(a) A $*$-homomorphism is c.p.

(b) A state is c.p.

(c) A conjugation $B(\widehat{H}) \to B(H) : a \mapsto V^*aV$ is c.p. for every bounded linear $V : H \to \widehat{H}$.

(d) The transpose $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is not c.p.

(e) The convex combination, composition, restriction of c.p. maps is c.p.

Proof. (a) A $*$-homomorphism is positive, and its inflations are all $*$-homomorphisms.

(b) Let $\rho : \mathcal{A} \to \mathbb{C}$ be a state. If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(\mathcal{A})_+$, then we have for $[x_i] \in \ell_2^n$ that

$$\sum_{i,j} \langle \rho(a_{ij})x_j, x_i \rangle_{\mathbb{C}} = \sum_{i,j} \overline{x_i}\rho(a_{ij})x_j = \rho(\sum_{i,j,k} \overline{x_i} b_{ki}^* b_{kj} x_j) = \sum_k \rho((\sum_i b_{ki}x_i)^*(\sum_j b_{kj}x_j)) \geq 0.$$

(c) If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(\widehat{H}))_+$, then we have for $[\xi_i] \in H^n$ that

$$\sum_{i,j} \langle V^* a_{ij} V\xi_j, \xi_i \rangle = \sum_{i,j,k} \langle b_{kj}V\xi_j, b_{ki}V\xi_i \rangle = \sum_k \langle \sum_j b_{kj}V\xi_j, \sum_i b_{ki}V\xi_i \rangle \geq 0.$$

(d) We have a counterexample for $M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The former has an eigenvalues $\{2, 0\}$, and the latter has $\{\pm 1\}$.

(e) Clear. $\square$

Theorem 1.4 (Stinespring dilation)**.** *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \to B(H)$ be a c.p. map. Then, there is a representation $\pi : \mathcal{A} \to B(\widehat{H})$ and a bounded linear operator $V : H \to \widehat{H}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & B(H) \\ B(\widehat{H}) \arrow[ur, "V^* \cdot V"'] & \end{tikzcd}$$

Proof. Define a sesquilinear form on the algebraic tensor product $\mathcal{A} \odot H$ as

$$\langle \sum_j a_j \otimes \xi_j, \sum_i b_i \otimes \eta_i \rangle := \sum_{i,j} \langle \varphi(b_i^* a_j)\xi_j, \eta_i \rangle.$$

It is positive since

$$\sum_{i,j} \langle a_i^* a_j \xi_j, \xi_i \rangle = \sum_{i,j} \langle a_j \xi_j, a_i \xi_i \rangle = \| \sum_i a_i \xi_i \|^2 \geq 0$$

implies

$$\langle \sum_j a_j \otimes \xi_j, \sum_i a_i \otimes \xi_i \rangle = \sum_{i,j} \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle \geq 0.$$

Taking quotient by the left kernel N and completion, we obtain a hilbert space $\widehat{H} := (\mathcal{A} \odot H/N)^-$.

Define $\pi : \mathcal{A} \to B(\widehat{H})$ such that

$$\pi(a)(b \otimes \xi + N) := ab \otimes \xi + N,$$

and define $V : H \to \widehat{H}$ such that

$$V\xi := 1_{\mathcal{A}} \otimes \xi + N.$$

Then for any $\xi, \eta \in H$,

$$\langle V^* \pi(a) V \xi, \eta \rangle = \langle \pi(a)(1_{\mathcal{A}} \otimes \xi + N), 1_{\mathcal{A}} \otimes \xi + N \rangle = \langle a_{\mathcal{A}} \otimes \xi + N, 1_{\mathcal{A}} \otimes \xi + N \rangle = \langle \varphi(a)\xi, \eta \rangle. \quad \square$$

Remark 1.5.

(a) If φ is unital, then V is an isometry since $V^*V = V^*\pi(1)V = \varphi(1) = 1$.

(b) If φ is unital and $H = \mathbb{C}$, then it is just the GNS-construction with the cyclic vector $V1_{\mathbb{C}}$.

(c) If $\varphi : \mathcal{A} \to \mathcal{B}$ is c.p., then by embedding $\mathcal{B}$ into $B(H)$ and applying the Stinespring dilation,

$$\|\varphi(a)\| = \|V^* \pi(a) V\| \leq \|V\| \|a\| \|V\| = \|a\| \|V^* V\| = \|a\| \|\varphi(1)\|$$

implies $\|\varphi\| \leq \|\varphi(1)\|$, hence $\|\varphi\| = \|\varphi(1)\|$.

(d) It has a physical meaning: a unital completely positive map is called quantum channel or quantum operation in quantum information theory. They are interpreted as an evolution in open quantum system, and taking $\widehat{H}$ means introducing a closed ambient system in which unitary evolution occurs.

Theorem 1.6 (Completely positive maps for matrix algebras)**.** *Let $\mathcal{A}$ be a C^*-algebra. Let $e_i \in \ell_2^n$ be standard orthonormal basis and let $e_{ij} = e_i \otimes e_j = |e_i\rangle\langle e_j| \in M_n(\mathbb{C})$ be unit matrix elements.*

(a) *There is a 1-1 correspondence*

$$\mathrm{CP}(M_n(\mathbb{C}), \mathcal{A}) \to M_n(\mathcal{A})_+ : \psi \mapsto [\psi(e_{ij})].$$

(b) *Let $\mathcal{A}$ be unital. There is a 1-1 correspondence*

$$\mathrm{CP}(\mathcal{A}, M_n(\mathbb{C})) \to M_n(\mathcal{A})_+^* : \varphi \mapsto (\widehat{\varphi} : [a_{ij}] \mapsto \sum_{i,j} \langle \varphi(a_{ij}) e_j, e_i \rangle).$$

Proof. (a) Fix $\mathcal{A} \to B(H)$ a faithful representation and just write $\mathcal{A} \subset B(H)$.

Suppose $\psi : M_n(\mathbb{C}) \to \mathcal{A}$ is a c.p. map. Identify $M_n(\mathbb{C}) = B(\ell_2^n)$. Since $[e_{ij}] \in M_n(B(\ell_2^n))_+$ is positive because

$$\sum_{i,j} \langle e_{ij} \xi_j, \xi_i \rangle = \sum_{i,j} \langle e_j, \xi_j \rangle \langle \xi_i, e_i \rangle = |\sum_i \langle e_i, \xi_i \rangle|^2 \geq 0, \qquad \forall [\xi_i] \in (\ell_2^n)^n,$$

it follows that $[\psi(e_{ij})] \in M_n(\mathcal{A})_+$ by the complete positivity of ψ.

Conversely, let $[\psi(e_{ij})] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(H))_+$, For $T = [t_{ij}] \in M_n(\mathbb{C})$ and $\xi, \eta \in H$, write

$$\begin{aligned}\langle \psi(T)\xi, \eta \rangle &= t_{ij} \langle \psi(e_{ij})\xi, \eta \rangle \\ &= t_{ij} \langle b_{kj}\xi, b_{ki}\eta \rangle \\ &= t_{ij}\delta_{kl} \langle b_{lj}\xi, b_{ki}\eta \rangle \\ &= \langle Te_j, e_i \rangle \langle e_l, e_k \rangle \langle b_{lj}\xi, b_{ki}\eta \rangle \\ &= \langle (T \otimes 1 \otimes 1)(e_j \otimes e_l \otimes (b_{lj}\xi)), (e_i \otimes e_k \otimes (b_{ki}\eta)) \rangle.\end{aligned}$$

The summation symols are omitted in each row. Then, if we define

$$V : H \to \ell_2^n \otimes \ell_2^n \otimes H : \xi \mapsto \sum_{i,k} e_i \otimes e_k \otimes (b_{ki}\eta),$$

we have an expression

$$\langle \psi(T)\xi, \eta \rangle = \langle V^*(T \otimes 1 \otimes 1)V\xi, \eta \rangle,$$

which implies that ψ is c.p. because $T \mapsto T \otimes 1_{\ell_2^n} \otimes 1_H$ is a $*$-homomorphism.

(b) Suppose $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ is a c.p. map. Then, $\widehat{\varphi}$ is positive since $[a_{ij}] \in M_n(\mathcal{A})_+$ implies

$$\widehat{\varphi}([a_{ij}]) = \sum_{i,j} \langle \varphi(a_{ij})e_j, e_i \rangle \geq 0.$$

Conversely, let $\widehat{\varphi} \in M_n(\mathcal{A})_+^*$. By the GNS-construction, we have a cyclic representation $\pi : M_n(\mathcal{A}) \to B(H)$ with a cyclic vector $\psi \in H$ such that

$$\widehat{\varphi}([a_{ij}]) = \langle \pi([a_{ij}])\psi, \psi \rangle.$$

For $\xi = \sum_j \xi_j e_j, \eta = \sum_i \eta_i e_i \in \ell_2^n$, write

$$\begin{aligned}\langle \varphi(a)\xi, \eta \rangle &= \sum_{i,j} \langle \varphi(a)\xi_j e_j, \eta_i e_i \rangle = \sum_{i,j} \langle \varphi(\overline{\eta_i} a \xi_j)e_j, e_i \rangle \\ &= \widehat{\varphi}([\overline{\eta_i} a \xi_j]) = \langle \pi([\overline{\eta_i} a \xi_j])\psi, \psi \rangle = \langle \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}]^*[a][\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \psi \rangle \\ &= \langle \pi([a])\pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}])\psi \rangle.\end{aligned}$$

If we define

$$V : \ell_2^n \to H : \xi \mapsto \pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi,$$

then

$$\langle \varphi(a)\xi, \eta \rangle = \langle V^*\pi([a])V\xi, \eta \rangle,$$

so φ is c.p. since $\mathcal{A} \to M_n(\mathcal{A}) : a \mapsto [a]$ is a $*$-homomorphism. □

Theorem 1.7 (Arveson extension). *Let $\mathcal{A} \subset \mathcal{B}$ be C*-algebras such that $1_{\mathcal{B}} \in \mathcal{A}$. Then, every c.p. map $\varphi : \mathcal{A} \to B(H)$ has an norm-preserving c.p. extension $\widetilde{\varphi} : \mathcal{B} \to B(H)$, i.e. $\|\widetilde{\varphi}\| = \|\varphi\|$.*

Proof. Let p_α be the net of projections of finite rank n_α in $B(H)$ with the image V_α, which strongly converges to id_H. Fix α temporarily and let $\varphi_\alpha := p_\alpha \varphi|_{V_\alpha} : \mathcal{A} \to B(V_\alpha)$. Choosing an any orthonormal basis of each V_α, we can rewrite as $\varphi_\alpha : \mathcal{A} \to M_{n_\alpha}(\mathbb{C})$. By the above theorem, we have the associated linear functional $\widehat{\varphi}_\alpha \in M_{n_\alpha}(\mathcal{A})$. Then, the Hahn-Banach extension provides an extension $(\widehat{\varphi}_\alpha)^\sim \in M_{n_\alpha}(\mathcal{B})$, and we can define $\widetilde{\varphi}_\alpha : \mathcal{B} \to M_{n_\alpha}(\mathbb{C})$ as the associated completely positive map. Via the identification $B(V_\alpha) = M_{n_\alpha}(\mathbb{C})$ we used to write $\varphi_\alpha : \mathcal{A} \to M_{n_\alpha}(\mathbb{C})$, we have $\widetilde{\varphi}_\alpha : \mathcal{B} \to B(V_\alpha)$. We can check $\widetilde{\varphi}_\alpha$ actually extends φ_α, i.e. $\widetilde{\varphi}_\alpha(a) = \varphi_\alpha(a)$ for $a \in \mathcal{A}$, by putting $[a\delta_{ik}\delta_{jl}]_{i,j} \in M_{n_\alpha}(\mathcal{A})$ and comparing matrix components for each k, l.

Since $\|\widetilde{\varphi}_\alpha\| = \|\widetilde{\varphi}_\alpha(1)\| = \|\varphi_\alpha(1)\| = \|\varphi_\alpha\| \le \|\varphi\|$, the net $\widetilde{\varphi}_\alpha$ is bounded in $B(\mathcal{B}, B(H))$. The norm-closed unit ball is compact in the point-σ-weak topology $\sigma(B(\mathcal{B}, B(H)), \mathcal{B} \odot L^1(H))$ because it is coarser than the weak* topology $\sigma(B(\mathcal{B}, B(H)), \mathcal{B}\widehat{\otimes}_\pi L^1(H))$. By taking a convergent subnet, we have a limit point $\widetilde{\varphi} : \mathcal{B} \to B(H)$. It is easily seen to be completely positive and extend φ, and satisfies $\|\varphi\| = \|\varphi(1)\| = \|\widetilde{\varphi}(1)\| = \|\widetilde{\varphi}\|$. □

1.2 Enveloping von Neumann algebras

Definition 1.8. For a representation $\pi : \mathcal{A} \to B(H)$ of a C*-algebra $\mathcal{A}$, we define a von Neumann algebra $\mathcal{M}(\pi) := \pi(A)''$ associated to π.

Theorem 1.9 (Sherman-Takeda)**.** *Let $\mathcal{A}$ be a C*-algebra and $\pi_u : \mathcal{A} \to B(H_u)$ the universal representation, the direct sum of all the GNS-representations of states of $\mathcal{A}$. Consider the following three maps*

$$\pi_u : \mathcal{A} \to (\mathcal{M}(\pi_u), \sigma w), \qquad \pi_u^* : \mathcal{M}(\pi_u)_* \to \mathcal{A}^*, \qquad \widetilde{\pi}_u := \pi_u^{**} : \mathcal{A}^{**} \to \mathcal{M}(\pi_u),$$

constructed by adjoints, where $\mathcal{M}(\pi_u)_*$ denotes the set of σ-weakly continuous(= normal) linear functionals on $\mathcal{M}(\pi_u)$.

(a) *π_u^* is isometric.*

(b) *π_u^* is surjective.*

(c) *$\widetilde{\pi}_u$ is an isometric isomorhpism(w.r.t. norms), and is an homeomorphism(w.r.t. weak*-topologies).*

(d) *$\mathcal{A}^{**}$ enjoys a universal property in the sense that for every $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{M}$ to a von Neumann algebra $\mathcal{M}$, there exists a unique normal extension $\widetilde{\varphi} : \mathcal{A}^{**} \to \mathcal{M}$ of φ.*

Proof. (a) It holds for any representation of $\pi : \mathcal{A} \to B(H)$. For each $l \in \mathcal{M}(\pi)_*$, we have

$$\|\pi^*(l)\| = \sup_{\substack{\|a\|\le 1\\ a\in\mathcal{A}}} |l(\pi(a))| = \sup_{\substack{\|b\|\le 1\\ b\in\mathcal{M}(\pi)}} |l(b)| = \|l\|$$

by the Kaplansky density theorem and the σ-weak continuity of l.

(b) Although π_u is an injective $*$-homomorphism and hence is isometric so that its dual $\mathcal{M}(\pi_u)^* \to \mathcal{A}^*$ is surjective by the Hahn-Banach extension, it does not guarantee the σ-weak continuity of the extended linear functional. We claim that every state of $\mathcal{A}$ has a normal extension on $\mathcal{M}(\pi_u)$. If the claim is true, then the Jordan decomposition can be applied to show that every bounded linear functional has a normal extension.

Let ρ be a state of $\mathcal{A}$. If we let ψ be the canonical cyclic vector of the GNS representation $\pi_\rho : \mathcal{A} \to B(H_\rho)$, then the state ρ can be represented as a vector state ω_ψ. Since π_ρ is a subrepresentation of π_u, the unit vector ψ can be seen as an element of H_u, and it defines a normal state of $\mathcal{M}(\pi_u)$.

(c) It is is clear from (a) and (b).

(d) We can define $\widetilde{\varphi}$ as the bitranspose of $\varphi : \mathcal{A} \to (\mathcal{M}, \sigma w)$, and it is a unique extension because $\mathcal{A}$ is σ-weakly dense in $\mathcal{A}^{**}$. □

Remark 1.10. The bidual $\mathcal{A}^{**}$ is frequently viewed as a von Neumann algebra, and we call it the *enveloping von Neumann algebra* of a C*-algebra $\mathcal{A}$. By the universal property, we have a normal $*$-homomorphism $\mathcal{M}(\pi_u) \to \mathcal{M}(\pi)$ that is in fact surjective for every representation π of $\mathcal{A}$, and it fails to be injective even for faithful representations.

Theorem 1.11 (Tomiyama)**.** *Let $\mathcal{B} \subset \mathcal{A}$ be C*-algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a* conditional expectation*, i.e. a contractive idempotent linear map.*

(a) *φ is B-bimodule map.*

(b) φ *is completely positive.*

Proof. Since each conclusion of (a) and (b) still holds for restriction, we may assume $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras by thinking of the bitranspose $\varphi^{**} : \mathcal{A}^{**} \to \mathcal{B}^{**}$.

(a) Since the linear span of projections is σ-weakly dense in a von Neumann algebra, we are enough to show $p\varphi(a) = \varphi(pa)$ and $\varphi(ap) = \varphi(a)p$ for any projection $p \in \mathcal{B}$.

Let $p \in \mathcal{B}$ be a projection and let $a \in \mathcal{A}$. Note that we have

$$p\varphi(a) = pp\varphi(a) = p\varphi(p\varphi(a))$$

and

$$(a - pa)^*(p\varphi(a - pa)) = (p\varphi(a - pa))^*(a - pa) = 0.$$

Then,

$$\begin{aligned}(1+t)^2\|p\varphi(a-pa)\|^2 &= \|p\varphi(a-pa) + tp\varphi(a-pa)\|^2 \\ &= \|p\varphi((a-pa) + tp\varphi(a-pa))\|^2 \\ &\le \|(a-pa) + tp\varphi(a-pa)\|^2 \\ &= \|a-pa\|^2 + t^2\|p\varphi(a-pa)\|^2\end{aligned}$$

implies $p\varphi(a-pa) = 0$ by letting $t \to \infty$. Putting $1_\mathcal{B} - p$ and $1_\mathcal{B}$ instead of p, we obtain $(1_\mathcal{B} - p)\varphi(a - 1_\mathcal{B}a + pa) = 0$ and $\varphi(a - 1_\mathcal{B}a) = 0$, so

$$p\varphi(a) = p\varphi(pa) = \varphi(pa).$$

Similarly, we can show $\varphi(a - ap)p = 0$ and $\varphi(ap)(1-p) = 0$, we are done.

(b) Let $[a_{ij}] \in M_n(\mathcal{A})_+$. Let $\pi : \mathcal{B} \to B(H)$ be a cyclic representation with a cyclic vector ψ. Then, $[\xi_i] \in H^n$ can be replaced to $[\pi(b_i)\psi]$, so we can check the positivity of inflations φ_n as

$$\sum_{i,j}\langle\pi(\varphi(a_{ij}))\pi(b_j)\psi, \pi(b_i)\psi\rangle = \langle\pi(\varphi(\sum_{i,j} b_i^* a_{ij} b_j))\psi, \psi\rangle \ge 0,$$

because it follows $\sum_{i,j} b_i^* a_{ij} b_j \ge 0$ by the positivity of a_{ij} from

$$\langle\pi_\mathcal{A}(\sum_{i,j} b_i^* a_{ij} b_j)\xi, \xi\rangle = \sum_{i,j}\langle\pi_\mathcal{A}(a_{ij})\pi_\mathcal{A}(b_j)\xi, \pi_\mathcal{A}(b_i)\xi\rangle \ge 0,$$

where $\pi_\mathcal{A}$ is any representation of $\mathcal{A}$. □

Theorem 1.12 (Sakai)**.** *Suppose $\mathcal{A}$ is a C^*-algebra which admits a predual F.*

(a) *There is an injective $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{A}^{**}$ with weakly* closed image.*

(b) *π is a topological embedding w.r.t. $\sigma(\mathcal{A}, F)$ and $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$.*

(c) *The predual F is unique in $\mathcal{A}^*$.*

In particular, there is a faithful representation $\mathcal{A} \to B(H)$ whose image is (σ-)weakly closed.

Proof. (a) By taking the adjoint for the inclusion $i : F \hookrightarrow \mathcal{A}^*$, we have a conditional expectation $\varepsilon : \mathcal{A}^{**} \twoheadrightarrow \mathcal{A}$. Its kernel is a $\mathcal{A}$-bimodule, and by the σ-weak density of $\mathcal{A}$ in $\mathcal{A}^{**}$ and the continuity of ε between weak* topologies, so it is in fact a $\mathcal{A}^{**}$-bimodule, which means it is a σ-weakly closed ideal of $\mathcal{A}^{**}$. Thus we have a central projection $z \in \mathcal{A}^{**}$ such that $\ker\varepsilon = (1-z)\mathcal{A}^{**}$.

Define $\pi : \mathcal{A} \to \mathcal{A}^{**}$ such that $\pi(a) := za$. It is clearly a $*$-homomorphism. The injectivity follows from $a = \varepsilon(a) = \varepsilon(za)$ for $a \in \mathcal{A}$. The image is weakly* closed because $\varepsilon(x - \varepsilon(x)) = 0$ implies $z(x - \varepsilon(x)) = 0$ for $x \in \mathcal{A}^{**}$ so that $z\mathcal{A}^{**} = z\mathcal{A}$.

(b) Since $\langle a, f\rangle = \langle \varepsilon(za), f\rangle = \langle za, f\rangle$ for $a \in \mathcal{A}$ and $f \in F$, in which the second equality holds by the definition of ε, it is enough to show $\sigma(z\mathcal{A}, \mathcal{A}^*) = \sigma(z\mathcal{A}, F)$.

For $l \in \mathcal{A}^*$, we claim there exists f such that $\langle za, l\rangle = \langle za, f\rangle$. Define $\widetilde{l} \in \mathcal{A}^*$ such that $\langle x, \widetilde{l}\rangle := \langle zx, l\rangle$ for $x \in \mathcal{A}^{**}$. Then, $\langle zx, l\rangle = \langle z^2x, l\rangle = \langle zx, \widetilde{l}\rangle$ for $x \in \mathcal{A}^{**}$. Suppose $\widetilde{l} \notin F$. Because F is closed in $\mathcal{A}^*$, there is $x \in \mathcal{A}^{**}$ such that $\langle x, \widetilde{l}\rangle \neq 0$ and $\langle x, f\rangle = 0$ for all $f \in F$ by the Hahn-Banach separation. Then, $0 = \langle x, f\rangle = \langle x, i(f)\rangle = \langle \varepsilon(x), f\rangle$ implies $\varepsilon(x) = 0$ so that $zx = 0$, which leads a contradiction $\langle x, \widetilde{l}\rangle = \langle zx, l\rangle = 0$, so we have $\widetilde{l} \in F$.

(c) If closed subspaces F_1 and F_2 of $\mathcal{A}^*$ are preduals of $\mathcal{A}$, then $\sigma(\mathcal{A}, F_1) = \sigma(\mathcal{A}, F_2)$ by the part (b). If $l \in F_1$, which is obviously continuous on $\sigma(\mathcal{A}, F_1)$, and the continuity in $\sigma(\mathcal{A}, F_2)$ implies that l is contained in a linear span of some finitely many elements of F_2, hence $F_1 \subset F_2$. $\square$

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2.1 Nuclear C*-algebras

Proposition 2.1 (Maximal and minimal tensor products)**.** *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$ be C*-algebras and H_1, H_2 be Hilbert spaces.*

(a) *(Continuity of tensor product maps) For any $*$-homomorphisms $\varphi_1 : \mathcal{A}_1 \to \mathcal{B}_1$ and $\varphi_2 : \mathcal{A}_2 \to \mathcal{B}_2$, the $*$-homomorphisms*

$$\varphi_1 \otimes \varphi_2 : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{B}_1 \otimes_{\max} \mathcal{B}_2$$

and

$$\varphi_1 \otimes \varphi_2 : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2$$

are well-defined. The same holds for c.p. maps(see Corollary 3.3 for $\otimes_{\max}$).

(b) *(Universal property) For any $*$-homomorphisms $\varphi_1 : \mathcal{A}_1 \to \mathcal{B}$ and $\varphi_2 : \mathcal{A}_2 \to \mathcal{B}$ whose images are commuting, there exists a unique $*$-homomorhpism*

$$\varphi_1 \times \varphi_2 : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{B}$$

such that $\varphi_1 \times \varphi_2(a_1 \otimes a_2) = \varphi_1(a_1)\varphi_2(a_2)$.

(c) *(Minimal norm is spatial) There is a natural $*$-monomorphism $B(H_1) \otimes_{\min} B(H_2) \hookrightarrow B(H_1 \otimes H_2)$.*

Proof. Omitted. □

Definition 2.2. A C*-algebra $\mathcal{A}$ is called *nuclear* if the canonical surjection $\mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$ is injective for any C*-algebra $\mathcal{B}$.

Example 2.3.

(a) Every finite-dimensional C*-algebra is nuclear.

(b) Every abelian C*-algebra is nuclear.

(c) A non-unital C*-algebra is nuclear if and only if its unitization is nuclear.

(d) A quotient of a nuclear C*-algebra is nuclear.

(e) The inductive limit of nuclear C*-algebras is nuclear.

(f) The tensor product of nuclear C*-algebra is nuclear.

Proof. (a), (b), (e) See Theorem 6.3.9, 6.4.15, and 6.3.10 of [Murphy]. □

2.2 Completely positive approximation property

Definition 2.4. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a c.c.p. map between C*-algebras. We say θ is *factorable* if it factors through a matrix algebra $M_n(\mathbb{C})$. We say θ is *approximable* or *nuclear* if it is a limit of factorable maps in the point-norm topology. When $\mathcal{B}$ is a von Neumann algebra, we say θ is *weakly approximable* or *weakly nuclear* if it is a limit of factorable maps in the point-σ-weak topology.

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, and $\mathcal{M} \subset B(H)$ a von Neumann algebra. Let $\mathcal{F} \subset B(\mathcal{A}, \mathcal{B})$ or $B(\mathcal{A}, \mathcal{M})$ be the set of factorabl maps.*

(a) *$\mathcal{F}$ is convex.*

(b) *In $B(\mathcal{A}, \mathcal{B})$, we have*

$$\overline{\mathcal{F}}^{co} = \overline{\mathcal{F}}^{pt-\|\cdot\|} = \overline{\mathcal{F}}^{pt-w}.$$

(c) *In $B(\mathcal{A},\mathcal{M})$, we have*

$$\overline{\mathbb{R}_{\geq 0}\mathcal{F}}^{pt-\sigma w}\cap B=\overline{\mathcal{F}}^{pt-\sigma w}=\overline{\mathcal{F}}^{pt-wot}=\overline{\mathcal{F}}^{pt-sot},$$

where B denotes the closed unit ball of $B(\mathcal{A},\mathcal{M})$.

Sketch. (a) Let $\mathcal{A}\xrightarrow{\psi_i}M_{n_i}(\mathbb{C})\xrightarrow{\varphi_i}\mathcal{B}$ be c.c.p. maps for $i\in\{0,1\}$. Then, for $t\in[0,1]$ we have a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "(1-t)\psi_0\circ\varphi_0+t\psi_1\circ\varphi_1"] \arrow[d] & & \mathcal{B} \\
\mathcal{A}\oplus\mathcal{A} \arrow[r, "\varphi_0\oplus\varphi_1"'] & M_{n_0}(\mathbb{C})\oplus M_{n_1}(\mathbb{C}) \arrow[r, "((1-t)\psi_0)\oplus(t\psi_1)"'] & \mathcal{B}\oplus\mathcal{B} \arrow[u]
\end{tikzcd}$$

which is commutative, so we are done.

(b), (c) When comparing the strong (operator) topology and the weak (operator) topology, we can use the fact that continuous functionals with respect to both topologies are same and apply the Hahn-Banach separation because $\mathcal{F}$ is convex.

When comparing the weak topology and the σ-weak topology, we can use the compactness of the closed unit ball in the σ-weak topology and use the fact that weak topology is weaker than the σ-weak topology to prove a homeomoprhism.

Finally, for the first equality of (b), see Proposition 3.8.2 in [Brown-Ozawa], which assumes unital maps. □

Theorem 2.6. *Let $\mathcal{A}$ be a C^*-algebra. Then, the identity $\mathcal{A}\to\mathcal{A}$ is approximable if and only if the inclusion $\mathcal{A}\to\mathcal{A}^{**}$ is weakly approximable.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let $E\subset\mathcal{A}$ and $F\subset\mathcal{A}^*$ be any finite subsets and fix $\varepsilon>0$. We may assume E and F are bounded by one and F is positive. We want to construct c.c.p. maps $\mathcal{A}\xrightarrow{\varphi}M_n(\mathbb{C})\xrightarrow{\psi}\mathcal{A}$ such that

$$|l(a-\psi\circ\varphi(a))|<\varepsilon,\qquad a\in E,\ l\in F.$$

By the assumption, we have a net of c.c.p. maps $\mathcal{A}\xrightarrow{\varphi'_\alpha}M_{n_\alpha}(\mathbb{C})\xrightarrow{\psi'_\alpha}\mathcal{A}^{**}$ satisfying

$$|l(a-\psi'_\alpha\circ\varphi'_\alpha(a))|\to 0,\qquad a\in\mathcal{A},\ l\in\mathcal{A}^*.$$

If we choose $0\leq e\leq 1$ in $\mathcal{A}$ such that $l(1-e)<\frac{\varepsilon}{8}$ for all $l\in F$, then

$$\begin{aligned}
l(1-\psi'_\alpha(\mathrm{id}))&=l(1-e)+l(e-\psi'_\alpha\circ\varphi'_\alpha(e))+l(\psi'_\alpha(\varphi'_\alpha(e)-\mathrm{id}))\\
&<\frac{\varepsilon}{8}+l(e-\psi'_\alpha\circ\varphi'_\alpha(e))+0
\end{aligned}$$

implies that we have a c.c.p. map $\mathcal{A}\xrightarrow{\varphi'}M_n(\mathbb{C})\xrightarrow{\psi'}\mathcal{A}^{**}$ in the net such that

$$|l(a-\psi'\circ\varphi'(a))|<\frac{\varepsilon}{4},\qquad a\in E,\ l\in F \tag{1}$$

and

$$l(1-\psi'(\mathrm{id}))<\frac{\varepsilon}{4},\qquad l\in F. \tag{1'}$$

Now we let $\varphi:=\varphi'$ and try to deform ψ' so that $\mathcal{A}$ contains the codomain.

Define $\psi'':M_n(\mathbb{C})\to\mathcal{A}^{**}$ such that

$$\psi''(T):=\frac{1}{n}\mathrm{Tr}(T)(1-\psi'(\mathrm{id}))+\psi'(T).$$

Then, ψ'' is a u.c.p. map and (1') implies

$$|l(\psi' \circ \varphi(a) - \psi'' \circ \varphi(a))| < \frac{\varepsilon}{4}, \qquad a \in E,\ l \in F. \tag{2}$$

Now consider the associated matrix element $[\psi''(e_{ij})] \in M_n(\mathcal{A}^{**})_+$ and the correspondence

$$M_n(\mathcal{A}^*) \xrightarrow{\sim} M_n(\mathcal{A})^* : [l_{ij}] \mapsto ([a_{ij}] \mapsto \sum_{i,j} l_{ij}(a_{ij})).$$

Since $M_n(\mathcal{A})$ is σ-weakly dense in $M_n(\mathcal{A})^{**} = M_n(\mathcal{A}^*)^* = M_n(\mathcal{A}^{**})$, the Kaplansky density theorem implies that the closed ball of $M_n(\mathcal{A})_+$ is σ-weakly dense in the closed ball of $M_n(\mathcal{A}^{**})_+$. For each pair $(a, l) \in E \times F$, if we define $[l_{ij}] \in M_n(\mathcal{A}^*)$ such that $l_{ij} := t_{ij} l$ and $\varphi(a) = [t_{ij}] \in M_n(\mathbb{C})$, then we can take $[a_{ij}] \in M_n(\mathcal{A})$ such that

$$|l(\psi''(\varphi(a)) - \sum_{i,j} t_{ij} a_{ij})| = |\sum_{i,j} l_{ij}(\psi''(e_{ij}) - a_{ij})| = |[l_{ij}]([\psi''(e_{ij}) - a_{ij}])| < \frac{\varepsilon}{4}$$

for all $a \in E$ and $l \in F$. Now we define $\psi''' : M_n(\mathbb{C}) \to \mathcal{A}$ by $\psi'''(e_{ij}) := a_{ij}$ to get

$$|l(\psi'' \circ \varphi(a) - \psi''' \circ \varphi(a))| < \frac{\varepsilon}{4}, \qquad a \in E,\ l \in F \tag{3}$$

and $\psi'' : M_n(\mathbb{C}) \to \mathcal{A}$ is c.p. Note that we may not assume ψ'' is contractive because the correspondence $\mathrm{CP}(M_n(\mathbb{C}), \mathcal{A}) \cong M_n(\mathcal{A})_+$ dose not preserve the norm.

However, when we take ψ''' we can insert an additional condition

$$\|1 - \psi'''(\mathrm{id})\| = \|\psi''(\mathrm{id}) - \psi'''(\mathrm{id})\| < \frac{\varepsilon}{4} \tag{3'}$$

using Mazur's lemma. Define $\psi'''' : M_n(\mathbb{C}) \to \mathcal{A}$ by $\psi'''' := \psi'''/\|\psi'''\|$. Then, (3') implies

$$|l((\psi''' - \psi'''') \circ \varphi(a))| = |l((\|\psi'''\| - 1)\psi''' \circ \varphi(a))| \le \|\|\psi'''\| - 1| < \frac{\varepsilon}{4}, \qquad a \in E,\ l \in F. \tag{4}$$

Combining (1)∼(4), we finally obtain $\psi := \psi''''$ such that

$$|l(a - \psi'''' \circ \varphi(a))| < \frac{\varepsilon}{4} \cdot 4 = \varepsilon, \qquad a \in E,\ l \in F,$$

so we are done. In summary, we have constructed the following maps

$$\begin{array}{rll} \psi' & : M_n(\mathbb{C}) \to \mathcal{A}^{**} & \text{c.c.p.} \\ \psi'' & : M_n(\mathbb{C}) \to \mathcal{A}^{**} & \text{u.c.p.} \\ \psi''' & : M_n(\mathbb{C}) \to \mathcal{A} & \text{c.p.} \\ \psi'''' & : M_n(\mathbb{C}) \to \mathcal{A} & \text{u.c.p.} \end{array}$$

□

2.3 Choi-Effros-Kirchberg characterization

Lemma 2.7 (Bounded Radon-Nikodym theorem). *Let $\mathcal{A}$ be a C^*-algebra. For F a finite dimensional subspace of $\mathcal{A}^*$, there is a cyclic representation $\pi : \mathcal{A} \to B(H)$ with the cyclic vector Ω such that there is a linear map $\pi' : F \to \pi(\mathcal{A})'$ satisfying $l(a) = \langle \pi(a)\pi'(l)\Omega, \Omega\rangle$ for every $l \in F$. The operator $\pi'(l)$ is called the Radon-Nikodym derivative of l with respect to the vector state ω_Ω.*

Proof. Choose a basis $l_1, \cdots, l_n$ of F. With the Jordan decomposition $l_i = l_{i,1} - l_{i,2} + i(l_{i,3} - l_{i,4})$, define a state

$$l_0 := \frac{1}{4n} \sum_{i=1}^{n} \sum_{j=1}^{4} \frac{l_{i,j}}{\|l_{i,j}\|}$$

by averaging, and let $\pi : \mathcal{A} \to B(H)$ be the GNS-representation of l_0 with cyclic vector $\Omega \in H$. Then for each $l \in F$,

$$\sigma(\pi(a)\Omega, \pi(b)\Omega) := l(b^*a)$$

are extended to well-defined bounded sesquilinear forms on H. If we write by $\pi'(l)$ the bounded linear operator associated to the sesquilinear form σ, then for $l \in F$ we have

$$l(b^*a) = \langle \pi'(l)\pi(a)\Omega, \pi(b)\Omega\rangle = \langle \pi(b^*)\pi'(l)\pi(a)\Omega, \Omega\rangle,$$

and by putting $a = 1$ and $b = 1$ respectively, we can conclude $\pi'(l) \in \pi(\mathcal{A})'$ and the desired result. $\square$

Lemma 2.8 (Fell's theorem). *Let $\pi : \mathcal{A} \to B(H)$ be a faithful representation of a C*-algebra. Then, every state of $\mathcal{A}$ is a weak* limit of the convex combination of pullbacks of vector states of $B(H)$.*

Proof. Suppose not and let ω be a counterexample. By the Hahn-Banach separation, there is $a \in \mathcal{A}$ and $r \in \mathbb{R}$ such that

$$\operatorname{Re} \omega_\xi(a) \le r < \operatorname{Re} \omega(a)$$

for all unit vector $\xi \in H$. Defining $h = (a + a^*)/2$, rewrite the above inequality as

$$\omega_\xi(h) \le r < \omega(h).$$

Then, $r - h \ge 0$ by the left inequality, which contradicts to the right inequality. $\square$

Theorem 2.9 (Choi-Effros-Kirchberg). *Let $\mathcal{A}$ be a C*-subalgebra.*

(a) *The identity $\mathcal{A} \to \mathcal{A}$ is approximable.*

(b) *$\mathcal{A}$ is nuclear.*

(c) *$\pi \times i : \mathcal{A} \odot_{\min} \pi(\mathcal{A})' \to B(H)$ is continuous.*

Proof. (a)⇒(b) Recall that every finite-dimensional C*-algebra is nuclear. For any C*-algebra $\mathcal{B}$, we have (by Corollary 3.3) a diagram

$$\begin{tikzcd}
\mathcal{A} \otimes_{\max} \mathcal{B} \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \mathcal{A} \otimes_{\max} \mathcal{B} \\
& M_{n_\alpha}(\mathbb{C}) \otimes_{\max} \mathcal{B} \arrow[ur, dashed] \arrow[dd, dashed, leftrightarrow] & \\
\mathcal{A} \otimes_{\min} \mathcal{B} \arrow[dr, dashed] & & \\
& M_{n_\alpha}(\mathbb{C}) \otimes_{\min} \mathcal{B} &
\end{tikzcd}$$

in which the square at lower left side algebraically commutes for each α and the upper triangle approximately commutes in the point-norm topology because

$$\|a \otimes b - \psi_\alpha \circ \varphi_\alpha(a) \otimes b\|_{\max} = \|a - \psi_\alpha \circ \varphi_\alpha(a)\|\|b\| \to 0$$

for each $a \in \mathcal{A}$, $b \in \mathcal{B}$. Here the dashed arrows mean that they vary as α goes to limit. Hence we have an approximately commuting diagram

$$\begin{tikzcd}
\mathcal{A} \otimes_{\max} \mathcal{B} \arrow[rr] \arrow[dr] & & \mathcal{A} \otimes_{\max} \mathcal{B} \\
& \mathcal{A} \otimes_{\min} \mathcal{B} \arrow[ur, dashed] &
\end{tikzcd}$$

so that we can verify the injectivity of $\mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\max} \mathcal{B}$ (because it is just the identity, so we do not have to take care of the failure of inclusion for maximal tensor products) implies the injectivity of $\mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\min} \mathcal{B}$.

(b)⇒(c) Clear.

(c)⇒(a) Let $E \subset \mathcal{A}$ and $F \subset \mathcal{A}^*$ be finite subsets and fix $\varepsilon > 0$. We want to find c.c.p. maps $\mathcal{A} \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} \mathcal{A}$ such that

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon$$

for $a \in E$ and $l \in F$. To implement the approximation, we would like to regard the inclusion operator as a state of a tensor product C*-algebra via the correspondence

$$B(\mathcal{A}, \mathcal{A}) \subset B(\mathcal{A}, \mathcal{A}^{**}) \xrightarrow{\sim} (\mathcal{A} \otimes_\pi \mathcal{A}^*)^*,$$

which maps the identity map $\mathcal{A} \to \mathcal{A}$ to the linear functional characterized by $a \otimes l \mapsto l(a)$. Since $\mathcal{A}^*$ is not a C*-algebra, we think a "representation" of $\pi' : F \to \pi(\mathcal{A})'$ through the above Radon-Nikodym type result. Let $\pi : \mathcal{A} \to B(H)$ be the cyclic representation obtained from the above Radon-Nikodym theorem and Ω the cyclic vector such that $l(a) = \langle \pi(a)\pi'(l)\Omega, \Omega\rangle$ for $a \in E$ and $l \in F$.

By the assumption, we have a representation

$$\pi \times i : \mathcal{A} \otimes_{\min} \pi(\mathcal{A})' \to B(H).$$

Consdier any faithful representation $\rho : \mathcal{A} \to B(K)$ and the tensor representation

$$\rho \otimes i : \mathcal{A} \otimes_{\min} \pi(\mathcal{A})' \to B(K \otimes H),$$

which is also faithful. By Fell's theorem, the state $\omega_\Omega \circ (\pi \times i)$ on $\mathcal{A} \otimes_{\min} \pi(\mathcal{A})'$ can be approximated by convex combinations of vector states in $B(K \otimes H)$. In particular, by the density of $\pi(\mathcal{A})\Omega$ in H, we have tensors $(\tau_k)_{k=1}^m \subset K \odot \pi(\mathcal{A})\Omega$ such that

$$\left|\omega_\Omega((\pi \times i)(a \otimes \pi'(l))) - \sum_{k=1}^m \lambda_k \omega_{\tau_k}((\rho \otimes i)(a \otimes \pi'(l)))\right| < \varepsilon \tag{†}$$

for all $a \in E$ and $l \in F$, where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda = 1$.

If we write each element $\tau \in K \odot \pi(\mathcal{A})\Omega$ as

$$\tau = \sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega,$$

then

$$\begin{aligned}
\omega_\tau((\rho \otimes i)(a \otimes \pi'(l))) &= \left\langle (\rho(a) \otimes \pi'(l))\Big(\sum_{j=1}^n \eta_j \otimes \pi(b_j)\Omega\Big), \Big(\sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega\Big) \right\rangle \\
&= \sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i\rangle \langle \pi'(l)\pi(b_i^* b_j)\Omega, \Omega\rangle \\
&= l\Big(\sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i\rangle b_i^* b_j\Big).
\end{aligned}$$

If we define c.c.p. maps $\mathcal{A} \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} \mathcal{A}$ for each τ such that

$$\varphi(a) := [\langle \rho(a)\eta_j, \eta_i\rangle], \quad \psi([e_{ij}]) := b_i^* b_j,$$

then we have $\omega_\tau(a \otimes \pi'(l)) = l(\psi \circ \varphi(a))$.

Since $\mu(a \otimes \pi'(l)) = l(a)$ and since the c.c.p. maps that factor through a matrix algebra form a convex set, we have c.c.p. maps $\mathcal{A} \xrightarrow{\varphi} M_n(\mathbb{C}) \xrightarrow{\psi} \mathcal{A}$ such that the inequality (†) is rewritten as

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon,$$

so we are done. □

3 May 26

In the next two subsections, we extend our investigation of famous theorems related to copmletely positive maps: the Stinespring dilation theorem and the Arveson extension theorem.

3.1 Stinespring dilation revisited

Here is a Stinespring dilation theorem for non-unital C*-algebras. We also discuss shortly the minimal Stinespring representation.

Theorem 3.1 (Stinespring dilation). *Let $\mathcal{A}$ be a C*-algebra and $\varphi : \mathcal{A} \to B(H)$ is a c.p. map. Then, there is a representation $\pi : \mathcal{A} \to B(\widehat{H})$ and a bounded linear operator $V : H \to \widehat{H}$ such that $\varphi(a) = V^*\pi(a)V$ for all $a \in \mathcal{A}$. Moreover, we can take π to be* minimal *in the sense that $(\pi(\mathcal{A})VH)^- = \widehat{H}$.*

Proof. In the first day we have defined $\pi : \mathcal{A} \to B(\widehat{H})$ and $V : H \to \widehat{H}$, where $\widehat{H} := (\mathcal{A} \odot H/N)^-$, such that

$$\pi(a)(b \otimes \xi + N) := (ab) \otimes \xi + N, \qquad V\xi := 1_{\mathcal{A}} \otimes \xi + N.$$

Since $\mathcal{A}$ may not have its unit, we want to adjust the definition of V with approximate unit. For an approximate unit e_α of $\mathcal{A}$, let $V_\alpha : H \to \widehat{H}$ be such that

$$V_\alpha \xi := e_\alpha \otimes \xi + N.$$

Recall that if ω is a positive linear functional, then $\lim_\alpha \omega(e_\alpha) = \omega(1)$. For fixed $\xi \in H$, because $\omega_\xi \circ \varphi$ is a positive linear functional and products of approximate units are also approximate units, we have

$$\|(V_\alpha - V_\beta)\xi)\|^2 = \langle \varphi((e_\alpha - e_\beta)^2)\xi, \xi\rangle = \langle \varphi(e_\alpha^2 - e_\alpha e_\beta - e_\beta e_\alpha + e_\beta^2)\xi, \xi\rangle \to 0,$$

so the net $V_\alpha\xi$ is Cauchy. Define $V\xi := \lim_\alpha V_\alpha \xi$. Then,

$$\begin{aligned}
\langle V^*\pi(a)V\xi, \eta\rangle &= \lim_\alpha \langle \pi(a)V_\alpha\xi, V_\alpha\eta\rangle \\
&= \lim_\alpha \langle (ae_\alpha) \otimes \xi + N, e_\alpha \otimes \eta + N\rangle \\
&= \lim_\alpha \langle \varphi(e_\alpha a e_\alpha)\xi, \eta\rangle \\
&= \langle \varphi(a)\xi, \eta\rangle.
\end{aligned}$$

It is easy to check the condition $(\pi(\mathcal{A})VH)^- = \widehat{H}$ holds. $\square$

Remark. For a c.p. map $\varphi : \mathcal{A} \to B(H)$, we can define a *Stinespring dilation* of φ as a triplet (π, K, V), where $\pi : \mathcal{A} \to K$ is a representation and $V : H \to K$ is a bounded linear operator, such that $\varphi(a) = V^*\pi(a)V$. For two Stinespring dilations (π_1, K_1, V_1) and (π_2, K_2, V_2) of φ, we can define morphisms by a bounded linear operator $U : K_2 \to K_1$ such that $\pi_2(a) = U^*\pi_1(a)U$ and $V_2 = UV_1$. The two conditions imply that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\varphi"] \arrow[dr, "\pi_2"] \arrow[ddr, "\pi_1"'] & & B(H) \\
& B(K_2) \arrow[ur, "V_2^*\cdot V_2"] & \\
& B(K_1) \arrow[u, "U^*\cdot U"] \arrow[uur, "V_1^*\cdot V_1"'] &
\end{tikzcd}$$

For arbitrary Stinespring dilation, we may always assume π acts non-degenerately on the subspace VH of K by changing V to $\lim_\alpha \pi(e_\alpha)V$ to make $\operatorname{im} V \subset (\pi(\mathcal{A})VH)^-$. Even we add this in the definition of a Stinespring dilation, the definition of morphisms still left unchanged. Here, the non-denegeracy condition $(\pi(\mathcal{A})VH)^- = \widehat{H}$ for the *minimal* Stinespring dilation is equivalent to that the Stinespring dilation enjoys the universal property, and hence unique up to morhpisms. A nice fact, the isomorhpism U is unitary, which means the minimal Stinespring is unique up to unitary equivalence.

Proposition 3.2 (Multiplicative domain)**.** *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a c.c.p. map between C^*-algebras.*

(a) $|\varphi(a)|^2 \le \varphi(|a|^2)$ *for $a \in \mathcal{A}$.*

(b) *If $|\varphi(a)|^2 = \varphi(|a|^2)$ and $|\varphi(a^*)|^2 = \varphi(|a^*|^2)$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$, respectively.*

(c) *$\{a \in \mathcal{A} : |\varphi(a)|^2 = \varphi(|a|^2)\}$ is a closed under the multiplication. In particular, $\{a \in \mathcal{A} : |\varphi(a)|^2 = \varphi(|a|^2),\ |\varphi(a^*)|^2 = \varphi(|a^*|^2)\}$ is a C^*-subalgebra of $\mathcal{A}$ on which the restriction of φ is a $*$-homomorphism.*

Proof. (a) Let (π, H, V) be the minimal Stinespring dilation of φ. Since φ is contractive, so is V. Then,

$$\varphi(|a|^2) - |\varphi(a)^2| = V^*\pi(a)^*(1_H - VV^*)\pi(a)V \ge 0.$$

(b) For $a \in \mathcal{A}$, the equality $|\varphi(a)|^2 = \varphi(|a|^2)$ holds if and only if $(\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)V = 0$. Then,

$$\varphi(ba) - \varphi(b)\varphi(a) = V^*\pi(b)^*(1_H - VV^*)\pi(a)V = 0.$$

Similar for a^*.

(c) If a and b satisfy the equality, then

$$\begin{aligned}(\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(ab)V &= (\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)\pi(b)V \\ &= (\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)\pi(b)V - [(\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)V]V^*\pi(b)V \\ &= (\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)(\mathrm{id}_H - VV^*)\pi(b)V \\ &= (\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(a)(\mathrm{id}_H - VV^*)^{\frac{1}{2}}[(\mathrm{id}_H - VV^*)^{\frac{1}{2}}\pi(b)V] = 0.\end{aligned}$$ □

Lemma 3.3 (Restriction of product representation)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. and $\Pi : \mathcal{A} \odot \mathcal{B} \to B(H)$ be a $*$-homomorphism. Then, there are $*$-homomorphisms $\pi : \mathcal{A} \to B(H)$ and $\pi' : \mathcal{B} \to B(H)$ with commuting ranges such that $\Pi = \pi \times \pi'$.*

Proof. Let $K := (\Pi(\mathcal{A} \odot \mathcal{B})H)^-$. We first claim that a map $\pi : \mathcal{A} \to B(K)$ defined by

$$\pi(a)(\Pi(\sum_i a_i \otimes b_i)\xi) := \Pi(\sum_i (aa_i) \otimes b_i)\xi$$

for $\xi \in H$ is indeed well-defined and bounded for each $a \in \mathcal{A}$. We can define π' in the same manner.

We may assume $\|a\| \le 1$. It suffices to show the inequality

$$\|\Pi(\sum_i (aa_i) \otimes b_i)\eta\|^2 \le \|\Pi(\sum_i a_i \otimes b_i)\eta\|^2.$$

The left hand side is

$$\|\Pi(\sum_i (aa_i) \otimes b_i)\eta\|^2 = \langle \Pi(\sum_{i,j} (a_i^* a^* a a_j) \otimes (b_i^* b_j))\eta, \eta\rangle$$

and the right hand side is

$$\|\Pi(\sum_i a_i \otimes b_i)\eta\|^2 = \langle \Pi(\sum_{i,j} (a_i^* a_j) \otimes (b_i^* b_j))\eta, \eta\rangle$$

so that there difference is positive because

$$\sum_{i,j} (a_i^*(1 - a^*a)a_j) \otimes (b_i^* b_j) = |\sum_i ((1 - a^*a)^{\frac{1}{2}} a_i) \otimes b_i|^2 \ge 0,$$

where 1 is the unit in the unitization of $\mathcal{A}$ for the continuous functional calculus.

Note that the restrictions π and π' are clearly $*$-homomorphisms, and commuting ranges of π and π' also can be immediately checked. By definition of the restrictions, we have

$$\pi(a)\pi'(b)\eta = \Pi(a \otimes b)\eta$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\eta \in K$. If we extend the domain from K to H by letting $\pi(a)\eta^\perp = \pi'(b)\eta^\perp := 0$ for $\eta^\perp \in H \ominus K$, then since

$$\langle \Pi(a \otimes b)\eta^\perp, \xi\rangle = \langle \eta^\perp, \Pi(a^* \otimes b^*)\rangle = 0$$

for $\xi \in H$ implies $\Pi(a \otimes b)\eta^\perp = 0$, we have

$$\pi(a)\pi'(b)(\eta + \eta^\perp) = \pi(a)\pi'(b)\eta = \Pi(a \otimes b)\eta = \Pi(a \otimes b)(\eta + \eta^\perp),$$

hence $\pi \times \pi' = \Pi$. □

Proposition 3.4 (Maximal tensor products of c.c.p. maps)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be C*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a c.c.p. map. Then, $\varphi \otimes i : \mathcal{A} \otimes_{\max} \mathcal{C} \to \mathcal{B} \otimes_{\max} \mathcal{C}$ is well-defined.*

Proof. First assume $\mathcal{B} \otimes_{\max} \mathcal{C} \subset B(H)$ by taking a faithful representation $b \otimes c \mapsto bc$. By restriction, $\mathcal{B}$ and $\mathcal{C}$ can be seen as commuting C*-subalgebras of $B(H)$. Let $\pi : \mathcal{A} \to B(\widehat{H})$ be the minimal Stinespring representation of the c.c.p. map $\varphi : \mathcal{A} \to \mathcal{B} \subset B(H)$.

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\pi} & B(\widehat{H}) & \overset{\rho}{\dashleftarrow} & \mathcal{C} \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle V^*\cdot V} & & \big\| \\ \mathcal{B} & \hookrightarrow & B(H) & \hookleftarrow & \mathcal{C} \end{array}$$

Now we want to lift the inclusion $\mathcal{C} \subset \mathcal{B}' \subset B(H)$ to obtain a representation $\rho : \mathcal{C} \to \pi(\mathcal{A})' \subset B(\widehat{H})$.

Since $\pi(\mathcal{A})VH$ is dense in $\widehat{H}$, we can try to define ρ by

$$\rho(c)(\sum_i \pi(a_i)V\xi_i) := \sum_i \pi(a_i)Vc\xi_i.$$

It is indeed well-defined from the following inequality:

$$\begin{aligned} \|\sum_i \pi(a_i)Vc\xi_i\|^2 &= \sum_{i,j} \langle c^*\varphi(a_i^*a_j)c\xi_j, \xi_i\rangle \\ &= \langle([c\delta_{ij}]^*[\varphi(a_i^*a_j)][c\delta_{ij}])[\xi_i], [\xi_i]\rangle_{H^n} \\ &= \langle([\varphi(a_i^*a_j)]^{\frac{1}{2}}[c\delta_{ij}]^*[c\delta_{ij}][\varphi(a_i^*a_j)]^{\frac{1}{2}})[\xi_i], [\xi_i]\rangle_{H^n} \\ &\le \|[c\delta_{ij}]\|^2_{M_n(\mathcal{C})}\langle([\varphi(a_i^*a_j)]^{\frac{1}{2}}[\varphi(a_i^*a_j)]^{\frac{1}{2}})[\xi_i], [\xi_i]\rangle_{H^n} \\ &= \|c\|^2\|\sum_i \pi(a_i)V\xi_i\|^2. \end{aligned}$$

Now then we can easily deduce that ρ is linear and preserves the multiplication, and is a $*$-homomorhpism from checking

$$\langle \sum_j \pi(a_j)Vc\xi_j, \sum_i \pi(a_i)V\xi_i\rangle = \langle \sum_j \pi(a_j)V\xi_j, \sum_i \pi(a_i)Vc^*\xi_i\rangle.$$

The commutation with $\pi(\mathcal{A})'$ is clear.

Using the universality, we have a $*$-homomorphism $\pi \times \rho : \mathcal{A} \otimes_{\max} \mathcal{C} \to \mathcal{B} \otimes_{\max} \mathcal{C} \subset B(\widehat{H})$, which satisfies

$$V^*(\pi \times \rho)(a \otimes c)V\xi = V^*\pi(a)\rho(c)V\xi = V^*\pi(a)Vc\xi = \varphi(a)c\xi.$$

Since the product $\varphi(a)c$ has been identified with the simple tensor $\varphi(a) \otimes c \in \mathcal{B} \otimes_{\max} \mathcal{C}$ when we embed $\mathcal{B}$ and $\mathcal{C}$ into $B(H)$, the above equality implies that $V^*(\pi \times \rho)V = \varphi \otimes i$. □

3.2 Arveson extension revisited

Proposition 3.5 (Representation extensioin)**.** *Let $\mathcal{A}$ be a closed ideal of a C^*-algebra $\mathcal{B}$. For a representation $\pi : \mathcal{A} \to B(H)$, there is a representation $\widetilde{\pi} : \mathcal{B} \to B(H)$ which extends π. Also, the extension is unique in the sense $\widetilde{\pi}(b) = \lim_\alpha \pi(e_\alpha b)$ in the strong operator topology, where e_α is an approximate unit of $\mathcal{A}$.*

Proof. Let $K := (\pi(\mathcal{A})H)^-$. Define $\widetilde{\pi} : \mathcal{B} \to B(H)$ such that

$$\widetilde{\pi}(b)(\pi(a)\xi + \eta) := \pi(ba)\xi$$

for $\xi \in H$, $\eta \in H \ominus K$, which is well-defined since

$$\|\pi(ba)\xi\|^2 = \langle \pi(a^*b^*ba)\xi, \xi\rangle \le \|b\|^2 \langle \pi(a^*a)\xi, \xi\rangle = \|b\|^2\|\pi(a)\xi\|^2.$$

It is clearly a $*$-homomorphism and extends π since $\langle \pi(a')\eta, \zeta\rangle = \langle \eta, \pi(a'^*)\zeta\rangle = 0$ for all $\zeta \in H$ implies

$$\widetilde{\pi}(a')(\pi(a)\xi + \eta) = \pi(a'a)\xi = \pi(a')(\pi(a)\xi + \eta).$$

For the uniqueness, we have for each $\pi(a)\xi + \eta \in H$ that

$$\|(\widetilde{\pi}(b) - \pi(e_\alpha b))(\pi(a)\xi + \eta)\|^2 = \|\pi(ba)\xi\|^2 + \|\pi(e_\alpha ba)\xi\|^2 - 2\,\mathrm{Re}\langle \pi(a^*b^*e_\alpha ba)\xi, \xi\rangle \to 0. \qquad \square$$

Theorem 3.6 (Arveson extension)**.** *Let $\mathcal{A} \subset \mathcal{B}$ be C^*-algebras. Let $\varphi : \mathcal{A} \to B(H)$ be a c.p. map and consider the following diagram:*

$$\begin{tikzcd} \mathcal{B} \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ \mathcal{A} \arrow[u] \arrow[r, "\varphi"'] & B(H). \end{tikzcd}$$

(a) *The n.p.c.p.(norm preserving c.p.) extension $\widetilde{\varphi}$ of φ exists if $\mathcal{B}$ is unital and $1_\mathcal{B} \in \mathcal{A}$.*

(b) *The n.p.c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is unital and $\mathcal{B} = \mathcal{A} \oplus \mathbb{C}$.*

(c) *The n.p.c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is non-unital and $\mathcal{B} = \widetilde{\mathcal{A}}$.*

(d) *The n.p.c.p. extension $\widetilde{\varphi}$ of φ always exists.*

Proof. (a) We have proved on the first day.

(b) Define $\widetilde{\varphi}(a + \lambda) = \varphi(a)$. Then, $\widetilde{\varphi} : \mathcal{A} \oplus \mathbb{C} \to B(H)$ is a norm-preserving c.p. extension of φ since

$$\|\widetilde{\varphi}(a + \lambda)\| = \|\varphi(a)\| \le \|\varphi\|\|a\| \le \|\varphi\|\|a + \lambda\|$$

and

$$\|\varphi(a)\| = \|\widetilde{\varphi}(a)\| \le \|\widetilde{\varphi}\|\|a\|.$$

(c) Let $\pi : \mathcal{A} \to B(\widehat{H})$ be the Stinespring representation of φ. Since $\mathcal{A}$ is a closed ideal of $\widetilde{\mathcal{A}}$, we can apply the representation extension for π to get $\widetilde{\pi}$ as follows:

$$\begin{tikzcd} \widetilde{\mathcal{A}} \arrow[dr, "\widetilde{\pi}"] & \\ \mathcal{A} \arrow[u] \arrow[r, "\pi"] \arrow[d, equal] & B(\widehat{H}) \arrow[d, "V^*\cdot V"] \\ \mathcal{A} \arrow[r, "\varphi"] & B(H). \end{tikzcd}$$

Then, the c.p. extension is given by $a \mapsto V^*\widetilde{\pi}(a)V$. To check the norm is preserved, take an approximate unit e_α of $\mathcal{A}$ so that the net $\varphi(e_\alpha^2) = V^*\pi(e_\alpha^2)V$ converges to $V^*\widetilde{\pi}(1)V$ strongly and weakly. For any $\xi \in H$, we have

$$\|\varphi(ae_\alpha)\xi\|^2 = \langle \varphi(ae_\alpha)^*\varphi(ae_\alpha)\xi, \xi\rangle \le \langle \varphi(e_\alpha a^* a e_\alpha)\xi, \xi\rangle \le \|a\|^2\langle \varphi(e_\alpha^2)\xi, \xi\rangle \le \|a\|^2\|\varphi\|^2\|\xi\|^2.$$

Taking $\lim_\alpha$, $\sup_{\|\xi\|=1}$, and $\sup_{\|a\|=1}$, we obtain

$$\|\varphi\|^2 \le \|V^*\widetilde{\pi}(1)V\|^2 \le \|\varphi\|^2.$$

(d) By the part (c), we may assume $\mathcal{B}$ is unital. Let $\widetilde{\mathcal{A}} = \mathcal{A} + 1_{\mathcal{B}}\mathbb{C}$ be the C*-subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and $1_{\mathcal{B}}$. Since $\mathcal{A}$ is an ideal of $\widetilde{\mathcal{A}}$ with codimension at most one, we have three cases:

(i) $\widetilde{\mathcal{A}} = \mathcal{A}$, if $1_{\mathcal{B}} \in \mathcal{A}$,

(ii) $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$, if $\mathcal{A}$ is unital but $1_{\mathcal{B}} \notin \mathcal{A}$

(iii) $\widetilde{\mathcal{A}}$ is the unitization of $\mathcal{A}$, if $\mathcal{A}$ is non-unital.

For the last case reduces to the part (a). By using (b) or (c), we have an extension to $\widetilde{\mathcal{A}}$, and the other two cases also reduce to the part (a). □

We introduce a more controlled version of the Arveson extension, the Trick, named after [Brown-Ozawa]. It is useful when we want to restrict the codomain of an Arveson extension.

Theorem 3.7 (The Trick). *Let $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{C}$ be C*-algebras, and let $\pi : \mathcal{A} \to B(H)$ be a representation.*

(i) If there are C-norm α and β on $\mathcal{A} \odot \mathcal{C}$ and $\mathcal{B} \odot \mathcal{C}$ respectively such that $\mathcal{A} \otimes_\alpha \mathcal{C} \to \mathcal{B} \otimes_\beta \mathcal{C}$ is injective,*

(ii) and if there is a representation $\pi' : \mathcal{C} \to B(H)$ such that $\pi \times \pi' : \mathcal{A} \odot_\alpha \mathcal{C} \to B(H)$ is continuous,

then there is a norm-preserving c.c.p. extension $\varphi : \mathcal{B} \to \pi'(\mathcal{C})'$ of $\pi : \mathcal{A} \to \pi'(\mathcal{C})'$.

Proof. We first claim that we may assume $\mathcal{B}$ and $\mathcal{C}$ are unital. To see this, we must verify the existence of a C*-norm $\widetilde{\beta}$ on $\mathcal{B} \odot \widetilde{\mathcal{C}}$ such that $\mathcal{B} \otimes_\beta \mathcal{C} \to \mathcal{B} \otimes_{\widetilde{\beta}} \widetilde{\mathcal{C}}$ is injective. We can do same thing for $\mathcal{B}$ with symmetry. Take a faithful representation $\rho \times \rho' : \mathcal{B} \otimes_\beta \mathcal{C} \to B(K)$, where $\rho : \mathcal{B} \to B(K)$ and $\rho' : \mathcal{C} \to B(K)$ are restrictions of $\rho \times \rho'$. Then, ρ' can be extended to a faithful representation $\widetilde{\rho}' : \widetilde{\mathcal{C}} \to B(K)$, and it commutes with ρ since the new element $\widetilde{\rho}'(1)$ is the strong limit of elements of $\rho'(\mathcal{C})$. Now consider the product representation $\rho \times \widetilde{\rho}' : \mathcal{B} \odot \widetilde{\mathcal{C}} \to B(K)$. We want to this is faithful. Suppose $(\rho \times \widetilde{\rho}')(\sum_{i=1}^n b_i \otimes c_i) = 0$ and b_i are linearly independent. For approximate units e_α and f_α of $\mathcal{B}$ and $\mathcal{C}$ respectively, we have

$$(\rho \times \rho')(\sum_i b_i e_\alpha \otimes c_i f_\alpha) = (\rho \times \widetilde{\rho}')((\sum_i b_i \otimes c_i)(e_\alpha \otimes f_\alpha)) = 0,$$

which implies $\sum_i b_i e_\alpha \otimes c_i f_\alpha = 0$ for each α. For every sufficiently large α, the finite set $b_i e_\alpha$ should be linearly independent. (Define $W_i := \operatorname{span}(\{b_1, \cdots, b_n\} \setminus \{b_i\})$ and $d_{i,\alpha} := \inf\{\|b_i e_\alpha - b e_\alpha\| : b \in W_i\}$. Then, we can deduce $\min_i d_{i,\alpha} > 0$ for every sufficiently small $\|e_\alpha\|$ from the linear independence of b_i.) Therefore, $c_i f_\alpha = 0$. By $\lim_\alpha$, we can conclude $\rho \times \widetilde{\rho}'$ is faithful so that we have $\mathcal{B} \otimes_\beta \mathcal{C} \to \mathcal{B} \otimes_{\widetilde{\beta}} \widetilde{\mathcal{C}}$ injective with the C*-norm induced from $B(K)$.

Now let $\mathcal{B}$ and $\mathcal{C}$ be unital. Then, by the Arveson extension theorem, there is a c.c.p. map $\Phi : \mathcal{B} \otimes_\beta \mathcal{C} \to B(H)$ which extends $\pi \times \pi'$. Define $\varphi : \mathcal{B} \to B(H)$ by $\varphi(b) := \Phi|_{\mathcal{B}}(b) = \Phi(b \otimes 1)$. Then, φ clearly extends π. Since $\Phi|_{\mathbb{C}1 \otimes \mathcal{C}} = \pi'$ is a $*$-homomorphism, $1 \otimes c$ belongs to the multiplicative domain of Φ, hence we can compute

$$\varphi(b)\pi'(c) = \Phi(b \otimes 1)\Phi(1 \otimes c) = \Phi((b \otimes 1)(1 \otimes c)) = \Phi(1 \otimes c)\Phi(b \otimes 1) = \pi'(c)\varphi(b).$$ □

Corollary 3.8 (Inclusion problem). *Let $\mathcal{A} \subset \mathcal{B}$ be C*-algebras. Then, the following statements are all equivalent:*

(a) *There is a c.c.p. map $\varphi : \mathcal{B} \to \mathcal{A}^{**}$ such that*

$$\begin{tikzcd} \mathcal{B} \arrow[dr, dashed, "\varphi"] & \\ \mathcal{A} \arrow[u] \arrow[r] & \mathcal{A}^{**} \end{tikzcd}$$

is commutative.

(b) *There is a c.c.p. map $\varphi : \mathcal{B} \to \pi(\mathcal{A})''$ such that*

$$\begin{tikzcd} \mathcal{B} \arrow[dr, dashed, "\varphi"] & \\ \mathcal{A} \arrow[u] \arrow[r, "\pi"] & \pi(\mathcal{A})'' \end{tikzcd}$$

is commutative for every representation $\pi : \mathcal{A} \to B(H)$.

(c) *The map $\mathcal{A} \otimes_{\max} C \to \mathcal{B} \otimes_{\max} \mathcal{C}$ is injective for every C*-algebra $\mathcal{C}$.*

Proof. (a)⇒(b) For any representation $\pi : \mathcal{A} \to B(H)$ admits a normal extension $\widetilde{\pi} : \mathcal{A}^{**} \to B(H)$ by the universal property of the enveloping von Neumann algebra. Then, our c.c.p. map is $\widetilde{\pi} \circ \varphi$, where $\varphi : \mathcal{B} \to \mathcal{A}^{**}$ is a c.c.p. map constructed from the assumption (a), because the following diagram commutes:

$$\begin{tikzcd} \mathcal{B} \arrow[dr, dashed, "\varphi"] & \\ \mathcal{A} \arrow[u] \arrow[r] \arrow[d, equal] & \mathcal{A}^{**} \arrow[d, "\widetilde{\pi}"] \\ \mathcal{A} \arrow[r, "\pi"] & \pi(\mathcal{A})''. \end{tikzcd}$$

(b)⇒(a) Clear.

(b)⇒(c) Taking any faithful representation of $\mathcal{A} \otimes_{\max} \mathcal{C}$ and by restriction, we may assume $\mathcal{A}$, $\mathcal{C}$, and $\mathcal{A} \otimes_{\max} \mathcal{C}$ are C*-subalgebras of $B(H)$ such that $\mathcal{A} \subset \mathcal{C}'$. Then, by the assumption (b) we can define a c.c.p. map $\varphi \otimes \mathrm{id}$ such that

$$\begin{tikzcd} \mathcal{B} \otimes_{\max} \mathcal{C} \arrow[dr, dashed, "\varphi\otimes\mathrm{id}"] & \\ \mathcal{A} \otimes_{\max} \mathcal{C} \arrow[u] \arrow[r] & \mathcal{A}'' \otimes_{\max} \mathcal{C} \subset B(H). \end{tikzcd}$$

commutes. Since the horizontal arrow is injective, the vertial arrow is also injective.

(c)⇒(b) This step is corollary of "The Trick". □

3.3 Exact C*-algebras

Let $\mathcal{I}$ be a closed ideal of a C*-algebra $\mathcal{A}$. For a C*-algebra C, do we have an exact sequence

$$0 \to \mathcal{I} \otimes \mathcal{C} \to \mathcal{A} \otimes \mathcal{C} \to (\mathcal{A}/\mathcal{I}) \otimes \mathcal{C} \to 0,$$

where $\otimes = \otimes_{\min}$ or $\otimes_{\max}$? The answer is given as follows:

exact	at left	at middle	at right
$\otimes_{\min}$	True	False	True
$\otimes_{\max}$	True	True	True

injectivity

$$(\mathcal{A}/\mathcal{I}) \otimes_\alpha \xrightarrow{\sim} \frac{\mathcal{A} \otimes_{\max} \mathcal{B}}{\mathcal{I} \otimes_{\max} \mathcal{B}} \to (\mathcal{A}/\mathcal{I}) \otimes_{\max} \mathcal{B}.$$

The surjectivity of $\mathcal{A} \otimes \mathcal{C} \to (\mathcal{A}/\mathcal{I}) \otimes \mathcal{C}$ follows from the fact that the algebraic tensor product is dense and the image of a C*-algebra under a $*$-homomorphism is closed.

3.7.7. passing to subalgebras

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3.7.11. RF Γ is amenable iff $C^*(\Gamma)$ is exact.
Chapter 5: When is $C_r^*(\Gamma)$ exact?
nonamenable residually finite groups (e.g. F_n, $\mathrm{SL}(n, \mathbb{Z})$)

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