Differential Equations

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Ordinary differential equations

Chapter 1

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1.1 (Picard-Lindelöf theorem)**.** Consider the following initial value problem:

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0.$$

Construct an approximate solution $(x_n)_{n=0}^{\infty}$ defined inductively such that $x_0(t) \equiv x_0$ and

$$x'_{n+1}(t) = f(t, x_n(t)), \quad x_{n+1}(0) = x_0.$$

Suppose f satisfies

$$|f(t,x)| \le \frac{R}{T}, \qquad |f(t,x) - f(t,y)| \lesssim |x-y|$$

on the cylinder $[0,T] \times \overline{B(x_0,R)}$.

(a) x_n is in $C^1([0,T], \overline{B(x_0,R)})$.

(b) x_n is Cauchy in $C^1([0,T], \overline{B(x_0,R)})$.

(c) The equation has a unique solution in $C^1([0,T], \overline{B(x_0,R)})$.

Proof. (a) It clearly follows from the explicit formula

$$x_{n+1}(t) = x_0 + \int_0^t f(s, x_n(s))\,ds.$$

(b) Since

$$|x_1(t) - x_0(t)| \le \int_0^t |f(s, x_0)|\,ds \le Mt$$

and

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\le \int_0^t |f(s, x_n(s)) - f(s, x_{n-1}(s))|\,ds \\ &\le K \int_0^t |x_n(s) - x_{n-1}(s)|\,dx \\ &\le MK^n \int_0^t \frac{s^n}{n!}\,ds \\ &= MK^n \frac{t^{n+1}}{(n+1)!}, \end{aligned}$$

we have the convergent series

$$\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|_\infty \le TM \frac{e^{KT} - 1}{KT}.$$

Also,

$$|x'_{n+1}(t) - x'_n(t)| \le |f(t, x_n(t)) - f(t, x_{n-1}(t))| \le K|x_n(t) - x_{n-1}(t)| \le MK^{n+1} \frac{t^{n+1}}{(n+1)!}.$$

(c) Limiting check. □

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$$\int \delta(x)\varphi(x)\,dx.$$

5.8 (Fundamental solution of the Laplace equation). Let $d \geq 2$. The *Fundamental solution of the Laplace equation* is a function $\Phi : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that solves the boundary value problem

$$\begin{cases} -\Delta\Phi(x) = \delta(x) & \text{in } \mathbb{R}^d, \\ \quad \Phi(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

(a) The funcdamental solution is given by

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } d = 2 \\ \frac{1}{(d-2)\omega_d}\frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}.$$

In particular, Φ and $\nabla\Phi$ are locally integrable on $\mathbb{R}^d$ but $\nabla^2\Phi$ is not.

(b) For $u \in C_0^2(\mathbb{R}^d)$,

$$u(x) = -\int \Phi(x-y)\Delta u(y)\,dy.$$

Proof. Note that $\nabla\Phi(y)\cdot\nabla u(x-y)$ is integrable in y. Then,

$$\begin{aligned}-\int \Phi(y)\Delta u(x-y)\,dy &= -\int \nabla\Phi(y)\cdot\nabla u(x-y)\,dy \\ &= -\lim_{\varepsilon\to\infty}\int_{|y|\ge\varepsilon} \nabla\Phi(y)\cdot\nabla u(x-y)\,dy \\ &= -\lim_{\varepsilon\to\infty}\int_{|y|=\varepsilon} \nabla\Phi(y)u(x-y)\cdot\nu\,dS.\end{aligned}$$

Since

$$\nabla\Phi(x) = -\frac{1}{\omega_d}\frac{x}{|x|^d}, \qquad \nu = \frac{x}{|x|},$$

we get

$$-\int \Phi(y)\Delta u(x-y)\,dy = \lim_{\varepsilon\to\infty}\frac{1}{\omega_d\varepsilon^{d-1}}\int_{|y|=\varepsilon} u(x-y)\,dS_y = u(x).$$

□

5.9 (Green's function of the Poisson equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^d$ for $d\ge 2$. *Green's function of the Poisson equation* is a function $G:\Omega^2\setminus\{(x,x)\in\Omega\}\to\mathbb{R}$ that solves the boundary value problem

$$\begin{cases} -\Delta_y G(x,y) = \delta(x-y) & \text{in } y\in\Omega\setminus\{x\}, \\ G(x,y) = 0 & \text{on } y\in\partial\Omega. \end{cases}$$

for each $x\in\Omega$.

Define $\phi:\Omega^2\to\mathbb{R}$ to be a function that solves the boundary value problem

$$\begin{cases} -\Delta_y \phi(x,y) = 0 & \text{in } y\in\Omega, \\ \phi(x,y) = \Phi(x-y) & \text{on } y\in\partial\Omega. \end{cases}$$

for each $x\in\Omega$. Assume for the domain Ω that there exists a unique ϕ.

(a) Green's function is given by

$$G(x,y) = \Phi(x-y) - \phi(x,y),$$

where Φ is the fundamental solution of the Laplace equation. Physically, $y\mapsto -\phi(x,y)$ has a meaning of the electric potential generated by the induced surface charge of a grounded conductor provided a point charge is at x.

(b) The *Green representation formula* holds: for $u\in C^2(\Omega)\cap C(\overline{\Omega})$,

$$u(x) = -\int_\Omega G(x,y)\Delta u(y)\,dy - \int_{\partial\Omega} u(y)\nabla_y G(x,y)\cdot\nu\,dS_y.$$

5.10 (Existence and uniqueness of Poisson equation)**.** representation formulas describe the solution assuming

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