Abstract Harmonic Analysis

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Part I

Locally compact groups

Chapter 1

Locally compact groups

1.1 Haar measures

1.1 (Existence of the Haar measure)**.**

1.2 (Left and right uniformities)**.**

1.3 (Modular functions)**.**

1.4 (Uniformly continuous functions)**.** G acts on $C_u(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$. $g \in C_c(G)$ is two-sided uniformly continuous.

1.2 Convolution algebras

We use the notation $L^p(G)$ for the non-commutative L^p-spaces constructed with the left Haar measure on G, which is a faithful semi-finite normal weight of $L^\infty(G)$. The predual of $L^\infty(G)$ can be identified with $L^1(G)$. The regular representation on $L^2(G)$ is the Gelfand-Naimark-Segal representation associated with the left Haar measure.

1.5 (Convolution algebras of integrable functions)**.** Let G be a locally compact group. Then, $L^1(G)$ is a hermitian Banach $*$-algebra such that

$$(f * g)(x) := (f \otimes g)\Delta(x), \qquad f, g \in L^1(G),\ x \in L^\infty(G).$$

(a) $L^1(G)$ has a two-sided approximate unit in $C_c(G)$.

(b) $\alpha : G \to \mathrm{Aut}(L^1(G))$ is point-norm continuous.

(c) $\lambda : G \to U(L^2(G))$ and $\lambda : L^1(G) \to B(L^2(G))$ are strongly continuous.

(d) Convolution inequalities.

(e) Representation theory equivalence.

Proof. Let U_i be a net of open neighborhoods of the identity e of G. By the Urysohn lemma, there is $e_i \in C_c(U_i)^+$ such that $\|e_i\|_1 = 1$ for each i. We claim that e_i is a two-sided approximate unit for $L^1(G)$. Suppose $g \in C_c(G)$, which is two-sided uniformly continuous. For any $\varepsilon > 0$, choose i_0 such that $\|g - \lambda_s g\| < \varepsilon$ and $\|g - \rho_s g\| < \varepsilon$

for all $s \in U_i$ for $i \succ i_0$. Then, we have

$$\|e_i * g - g\|_1 = \int |e_i * g(t) - g(t)|\,dt \le \iint e_i(s)|g(s^{-1}t) - g(t)|\,ds\,dt$$
$$= \int_{U_i} e_i(s)\|\lambda_s g - g\|_1\,ds < \varepsilon \int e_i(s)\,ds \le \varepsilon,$$

and

$$\|g * e_i - g\|_1 = \int |g * e_i(s) - g(s)|\,ds \le \iint |g(t) - g(s)|e_i(t^{-1}s)\,dt\,ds$$
$$= \iint |g(t) - g(ts)|e_i(s)\,dt\,ds = \int \|g - \rho_s g\|_1 e_i(s)\,ds < \varepsilon \int e_i(s)\,ds \le \varepsilon,$$

and they imply $\lim_i \|e_i * g - g\|_1 = \lim_i \|g * e_i - g\|_1 = 0$. We can approximate $f \in L^1(G)$ with compactly supported continuous functions by the $\varepsilon/3$ argument. □

1.6 (Measure algebras).

1.7 (Group C*-algebras).

1.8 (Group von Neumann algebras). Let G be a locally compact group. Since G is a locally compact Hausdorff space and the left Haar measure is a faithful semi-finite lower semi-continuous weight on the commutative C*-algebra $C_0(G)$, we have a corresponding semi-cyclic representation $m : C_0(G) \to B(L^2(G))$ which is normally extended to a von Neumann algebra $L^\infty(G)$ with $m(L^\infty(G)) = m(C_0(G))''$, and $L^1(G)$ is identified with the predual $L^\infty(G)_*$.

By the left Haar measure, $C_c(G)$ has a natural non-commutative left Hilbert algebra structure

$$(f * g)(s) := \int f(t)g(t^{-1}s)\,dt, \qquad \langle f, g\rangle := \int \overline{g(s)}f(s)\,ds, \qquad f^\sharp(s) := \nabla(s^{-1})\overline{f(s^{-1})},$$

where ∇ is the modular function for G, and it induces the regular representation $\lambda : C_c(G) \to B(L^2(G))$. By the group structure of G, the Hilbert algbera $C_c(G)$ is also a commutative counital multiplier Hopf $*$-algebra

$$(fg)(s) := f(s)g(s), \qquad \Delta f(s,t) = f(st), \qquad f^*(s) := \overline{f(s)}, \qquad \kappa f(s) = f(s^{-1}).$$

We start from this structures.

They satisfy a compatibility condition $\langle fg, h\rangle = \langle f, g^*h\rangle$.

With the integral notation $\lambda(f) = \int \lambda_s f(s)\,ds$, we can write

From now on, we are going to exclude any measure theory and the theory of non-commutative L^p spaces. First, we have the completion $H =: L^2(G)$. Consider two representations

$$\lambda : (C_c(G), *, {}^\sharp) \to B(L^2(G)), \qquad m : (C_c(G), \cdot, {}^*) \to B(L^2(G)).$$

(a) λ is well-defined.

(b) m is well-defined.

Proof. The multiplication representation m is well-defined because for $f \in C_c(G)$ we have $f^*f \in C_c(G) \subset L^2(G)$ so

$$\|m(f)g\|^2 = \langle fg, fg\rangle = \langle f^*fg, g\rangle, \qquad g \in C_c(G).$$

blabla

Note that we have

$$
\begin{aligned}
|\langle\lambda(\xi)\eta,\zeta\rangle|^2 &= |\iint \xi(t)\eta(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\
&\le \iint |\xi(t)||\eta(t^{-1}s)|^2\,ds\,dt \cdot \iint |\xi(t)||\zeta(s)|^2\,ds\,dt \\
&= \|\xi\|_1^2\|\eta\|_2^2\|\zeta\|_2^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\langle\rho(\xi)\eta,\zeta\rangle|^2 &= |\iint \eta(t)\xi(t^{-1}s)\overline{\zeta(s)}\,ds\,dt|^2 \\
&\le \iint |\xi(t^{-1}s)||\eta(t)|^2\,ds\,dt \cdot \iint |\xi(t^{-1}s)||\zeta(s)|^2\,ds\,dt \\
&= \|\xi\|_1\|F\xi\|_1\|\eta\|_2^2\|\zeta\|_2^2
\end{aligned}
$$

imply

$$\|\lambda(\xi)\|_{2\to2} \le \|\xi\|_1, \qquad \|\rho(\xi)\|_{2\to2} \le \sqrt{\|\xi\|_1\|F\xi\|_1}.$$

The equalities do not hold, consider $\|\lambda(\xi)\| = \|\widehat{\xi}\|_\infty$ if $G = \mathbb{R}$.

□

1.9 (Absorption principle)**.** Let G be a locally compact group.

$$w:$$

The *structure operator* of G is an oeprator $w \in U(L^2(G\times G))$ defined such that $w\xi(s,t) := \xi(s,st)$, or $w \in L^\infty(G)\overline{\otimes}W_r^*(G)$ such that $\operatorname{Ad} w(\lambda_s\otimes\lambda_s) := \lambda_s \otimes 1$. If $w(x\otimes x)w^* = x\otimes 1$, then $x = \lambda_s$ for some $s \in G$.

(a) $\lambda\otimes u$ and $\lambda\otimes 1$ are unitarily equivalent. It is called the *Fell absorption principle*.

Proof. The Fell absorption principle states that the composition of equivariant operators

$$
\begin{array}{ccccc}
L^2(G)\otimes H & \xrightarrow{\Delta\otimes 1} & L^2(G)\otimes L^2(G)\otimes H & \xrightarrow{1\otimes ?} & L^2(G)\otimes H \\
\lambda\otimes 1 & \longmapsto & \lambda\otimes\lambda\otimes 1 & \longmapsto & \lambda\otimes u
\end{array}
$$

is unitary.

The structure operator is a special case of the Fell absorption operator

$$
\begin{array}{ccccc}
L^2(G)\otimes L^2(G) & \xrightarrow{\Delta\otimes 1} & L^2(G)\otimes L^2(G)\otimes L^2(G) & \xrightarrow{1\otimes ?} & L^2(G)\otimes L^2(G) \\
\lambda\otimes 1 & \longmapsto & \lambda\otimes\lambda\otimes 1 & \longmapsto & \lambda\otimes\lambda
\end{array}
$$

□

1.3 Fourier and Fourier-Stieltjes algebras

1.10 (Fourier algebras)**.** Let G be a locally compact group. We define the *Fourier algebra* by $A(G) := W_r^*(G)_*$.

(a) $A(G)$ is the set of matrix coefficients of the regular representation $\lambda : G \to U(L^2(G))$, that is, the functions $s \mapsto \langle\lambda(s)\xi,\eta\rangle$ for $\xi,\eta \in L^2(G)$.

(b) $A(G)$ is a dense Banach subalgebra of $C_0(G)$. In particular, $M(G) \to W_r^*(G)$ is a dense embedding.

Proof. □

1.11 (Fourier-Stieltjes algebras). Let G be a locally compact group. We define the *Fourier Stieltjes algebra* by $B(G) := C^*(G)^*$.

(a) $B(G)$ is the linear span of continuous positive definite functions.

(b) On $B(G)_1$, the compact open topology is stronger than the weak* topology.

(c) On $B(G)_1$, the strict topology with respect to $A(G)$ is equivalent to the weak* topology.

Proof. □

dense embeddings among non-commutative algberas and commutative algebras:

$$\begin{array}{ccccccc} L^1(G) & \longrightarrow & C^*(G) & \qquad & A(G) & \longrightarrow & C_0(G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M(G) & \longrightarrow & W_r^*(G). & & B(G) & \longrightarrow & L^\infty(G). \end{array}$$

1.4 Pontryagin duality

1.12 (Locally compact abelian groups). Let G be a locally compact abelian group.

(a) Every irreducible representation of G is one-dimensional, and $\widehat{G}$ is an abelian group.

(b) The compact open topology of $C(G)$ and the weak* topology of $L^\infty(G)$ coincide on $\widehat{G}$, and $\widehat{G}$ is locally compact Hausdorff with this topology.

1.13 (Fourier transforms). Let G be a locally compact abelian group. We introduce the notation $\langle s, p\rangle := p^{-1}(s) \in \mathbb{T}$ for $p \in \widehat{G}$ and $s \in G$. The *Fourier transform* and the *Fourier-Stieltjes transform* of an integrable function $f \in L^1(G)$ and a finite Radon measure $\mu \in M(G)$ are defined by

$$\mathcal{F}f(p) := \int_G \langle s, p\rangle f(s)\, ds, \qquad \mathcal{F}\mu(p) := \int_G \langle s, p\rangle\, d\mu(s) \qquad p \in \widehat{G}.$$

(a) The Fourier transform is restricted to a linear operator $B(G) \cap L^1(G) \to B(\widehat{G}) \cap L^1(\widehat{G})$.

(b) The Fourier transform is uniquely extended to a continuous dense $*$-homomorphism $L^1(G) \to C_0(\widehat{G})$.

(c) The Fourier transform is uniquely extended to a continuous dense $*$-homomorphism $L^1(G) \to B(\widehat{G})$.

(d) The Fourier transform uniquely defines a unitary operator $L^2(G) \to L^2(G)$.

(e) The Fourier-Stietjes transform $M(G) \to L^\infty(G)$ is injective.

Proof. (a) Let $f \in B(G) \cap L^1(G)$.

(b)

(c)

(d) We prove the identity $\|f\|_{L^2(G)} = \|\mathcal{F}f\|_{L^2(\widehat{G})}$ for $f \in B(G) \cap L^1(G)$ and the density of $B(G) \cap L^1(G)$ in $L^2(G)$.

(e) Consider a commutative diagram of Banach $*$-algebras

$$\begin{array}{ccccc} L^1(G) & \xrightarrow{(1)} & C^*(G) & \xrightarrow{(3)} & C_0(\widehat{G}) \\ \downarrow & & & & \downarrow \\ M(G) & \xrightarrow{(2)} & W_r^*(G) & \xrightarrow{(4)} & L^\infty(\widehat{G}) \end{array}$$

The dense injection (1) is by definition of the group C*-algebra. The dense injection (2) is by the dense inclusion $A(G) \to C_0(G)$. The isomorphism (3) is due to the equivalence between representation theories of

G and $C^*(G)$ and the Gelfand duality. The isomorphism (4) is constructed by taking double commutant of $L^1(G)$ in the Plancherel isomorphism $B(L^2(G)) \to B(L^2(\widehat{G}))$. Since the first and third rows are respectively the Fourier transform and Fourier-Stieltjes transform, we are done. $\square$

the decomposition of the regular representation and the Plancherel theorem....

1.14 (Pontryagin duality)**.** Let G be a locally compact abelian group.

(a) The canonical homomorphism $\Phi : G \to \widehat{\widehat{G}}$ defined such that $\Phi(s)(p) = \langle s, p\rangle$ for $s \in G$ and $p \in \widehat{G}$ is a topological isomorphism.

Proof. It suffices to prove that the natural $*$-homomorphisms $C_0(\widehat{\widehat{G}}) \to C_0(G)$ and $M(G) \to M(\widehat{\widehat{G}})$ have dense images. Since the Fourier transform $L^1(G) \to B(\widehat{G})$ is dense, and it factors through $M(G) \to M(\widehat{\widehat{G}})$ with an embedding $M(\widehat{\widehat{G}}) \to B(\widehat{G})$, so $M(G) \to M(\widehat{\widehat{G}})$ is dense. Since the injectivity of the Fourier-Stieltjes transform $M(G) \to L^\infty(\widehat{G})$ implies that its dual $L^1(\widehat{G}) \to C_0(G)$ is dense, and it factors through $C_0(\widehat{\widehat{G}}) \to C_0(G)$ by the Fourier transform, so $C_0(\widehat{\widehat{G}}) \to C_0(G)$ is dense. Therefore, $M(G) \to M(\widehat{\widehat{G}})$ is a $*$-isomorphism. $\square$

Chapter 2

Amenability

Chapter 3

Part II

Representation categories

Chapter 4

Representations of compact groups

4.1 Peter-Weyl theorem

Let G be a compact group. Every representation will assume the strong continuity and the unitarity.

Let π_1 and π_2 be representations, and suppose π_1 is irreducible. If there is a non-zero intertwiner $v \in B(H_1, H_2)$, normalized to have norm one, then $v^*v \in \pi_1(G)' = \mathbb{C}1$ implies that v is an isometry, so π_1 is isomorphic to a subrepresentation of π_2. If π_2 is irreducible, then the existence of non-zero intertwiner is equivalent to that π_1 and π_2 are isomorphic.

Let π_1 and π_2 be representations. Then, any bounded linear operator $w : H_1 \to H_2$ induces an intertwiner $v := \int_G \pi_2(s) w \pi_1(s)^* \, ds : H_1 \to H_2$. For $\xi_1, \eta_1 \in H_1$ and $\xi_2, \eta_2 \in H_2$, if we let $w := \theta_{\xi_1,\xi_2} = \langle \cdot, \xi_1 \rangle \xi_2$, then

$$\begin{aligned} \langle v\eta_1, \eta_2 \rangle &= \int_G \langle \pi_2(s) w \pi_1(s)^* \eta_1, \eta_2 \rangle \, ds \\ &= \int_G \langle \pi_2(s) \langle \pi_1(s)^* \eta_1, \xi_1 \rangle \xi_2, \eta_2 \rangle \, ds \\ &= \int_G \overline{\langle \pi_1(s)\xi_1, \eta_1 \rangle} \langle \pi_2(s)\xi_2, \eta_2 \rangle \, ds. \end{aligned}$$

This implies that matrix coefficients come from non-isomorphic irreducible representations are orthogonal.

For a representation π of G, denote by $A(\pi)$ the linear span of matrix coefficients for π. We prove $\mathcal{O}(G) := \bigcup_\pi A(\pi)$ is dense in $C(G)$, where π runs through all the finite-dimensional irreducible representations of G. Here the irreducibility is redundant because every finite-dimensional representation is decomposed into the direct sum of finite-dimensional irreducible representations.

Note that for the left regular representation $\lambda : G \to U(L^2(G))$ we have $\lambda : L^1(G) \to K(L^2(G))$ and its restriction $\lambda : L^2(G) \to L^2(L^2(G))$ because G is compact. Fix $f \in C(G)$ and let V be an eigenspace of the Hilbert-Schmidt operator $\lambda_f \in L^2(L^2(G))$, which is a finite-dimensional subrepresentation of λ and satisfies $V \subset C(G)$. Let $\{e_i\}$ be an orthonormal basis of V. If $\xi \in V$, then since the contragradient representation λ^* can be defined on V and it is finite-dimensional, we have $\xi \in \mathcal{O}(G)$ by

$$\xi(s) = (\lambda_s^* \xi)(e) = (\sum_i \langle \lambda_s^* \xi, e_i \rangle e_i)(e) = \sum_i e_i(e) \langle \lambda_s^* \xi, e_i \rangle,$$

so $V \in \mathcal{O}(G)$.

For $f \in C(G)$ and $\xi \in L^2(G)$, we can see $\lambda_f \xi$ is uniformly approximated by $\mathcal{O}(G)$ by the spectral truncation of λ_f. Since $C(G) * L^2(G)$ is dense in $C(G)$, the density of $\mathcal{O}(G)$ in $C(G)$ follows.

4.2 Tannaka-Krein duality

4.3 Mackey machine

Example of non-compact Lie groups, Wigner classification

Part III

Topological quantum groups

Chapter 5

Compact quantum groups

5.1 Algebraic compact quantum groups

Multiplier Hopf $*$-algebras

Algebraic quantum groups

idempotent ring assumption

For a monoid, we can associate a bialgebra called the convolution algebra. If the monoid is a group, then the convolution algebra becomes a Hopf algebra.

universal enveloping algebra. q-deformations of the coordinate Hopf algebras $\mathcal{O}(G)$ of a semi-simple complex Lie group, and the universal enveloping algebra $U(\mathfrak{g})$ of a semi-simple complex Lie algebra.

If A is a coalgebra and B is an algebra, then $\mathrm{Hom}_{\mathbb{C}}(A,B)$ becomes an algebra with convolution. If A is a coalgebra, then A^* is an algebra. If A is a bialgebra, then A is a bimodule over A^*.

Duality for finite-dimensional Hopf ($*$-)algebras. dual pairing

5.1 (Algebraic compact quantum groups)**.** Recall that a Hopf algebra A has five linear structure maps the multiplication μ, unit η, comultiplication δ, counit ε, and antipode κ. A *Hopf $*$-algebra* is a Hopf algebra A together with an conjugate-linear involution $* : A \to A$ such that there are commutative diagrams

$$\begin{tikzcd} A\otimes A \arrow[r, "\mu"] \arrow[d, "(*\otimes *)\sigma_A"'] & A \arrow[d, "*"] \\ A\otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "*"'] & A\otimes A \arrow[d, "(*\otimes *)\sigma_A"] \\ A \arrow[r, "\delta"] & A\otimes A \end{tikzcd}$$

where $\sigma_A : A\otimes A \to A\otimes A$ is the swap map. An *algebraic compact quantum group* is defined as a complex Hopf $*$-algebra A together with a unital positive linear functional $h : A \to \mathbb{C}$ satisfying $(h\otimes \mathrm{id})\delta = \eta h = (\mathrm{id}\otimes h)\delta$. It is conventional to use $\mathbb{G}$ to denote a compact quantum group, and we will usually write the underlying Hopf $*$-algebra A as $\mathcal{O}(\mathbb{G})$.

(a) There is a categorical equivalence between commutative compact quantum groups and compact groups.

5.2 Woronowicz compact quantum groups

5.2 (Woronowicz compact quantum groups)**.** From now on, the tensor product of C*-algebras will always be assumed to be the minimal one, if not particularly mentioned. In the sense of Woronowicz, a *compact quantum group* is defined as a unital C*-algebra A together with a coassociative unital $*$-homomorphism $\delta : A \to A\otimes A$ and a state $h : A \to \mathbb{C}$ such that $(1\otimes h)\delta = \eta h = (h\otimes 1)\delta$, where $\eta : \mathbb{C} \to A$ is the unit map. The state h is called the *Haar state*. When we write $\mathbb{G}$ to mean a compact quantum group, then the underlying C*-algebra A is denoted by $C(\mathbb{G})$.

(a) For a C*-algebra A with a coassociative unital $*$-homomorphism $\delta : A \to A \otimes A$, the existence of the Haar state is equivalent to the cancellation property in the sense that the linear spans of the sets $\delta(A)(A \otimes 1)$ and $\delta(A)(1 \otimes A)$ are respectively dense in $A \otimes A$.

$$C_0(G), \quad L^\infty(G), \qquad C^*(G), \quad C_r^*(F), \quad W_r^*(G)$$

$$A(G), B(G)$$

For a compact group G, $C(G)$ has a coalgebra structure induced from $C(G) \subset L^1(G)$.

5.3 (Peter-Weyl theorem)**.** The $*$-subalgebra of matrix coefficients is a Hopf $*$-algebra.

5.4. A *compact algebraic quantum group* is a Hopf $*$-algebra with a positive integral. For a compact quantum group $\mathbb{G}$, the subspace $\mathbb{C}(\mathbb{G})$ spanned by the matrix coefficients of corepresentations is an algebraic quantum group.

5.5. Let $\mathbb{G}$ be a compact quantum group. A *representation* of $\mathbb{G}$ is a corepresentation of $C(\mathbb{G})$.

5.3 Kac algebras

5.6 (Kac algebras)**.** If the Haar state is a trace, then we say the compact quantum group is a *Kac algebra* or is of *Kac type*.

Chapter 6

Locally compact quantum groups

6.1 Locally compact quantum groups

Probably, a Hopf-von Neumann algebra in Enock-Schwartz is just a von Neumann bialgebra in Timmerman, a coinvolutive Hopf-von Neumann algebra in Enock-Schwartz is just a Hopf-von Neumann algebra in Timmerman. Since a locally compact quantum group has counit and antipode as unbounded operators, I do not know if I can say there is a Hopf algebra structure.

6.1 (Locally compact quantum groups)**.** In the sense of Kustermans-Vaes, a *locally compact quantum group* is defined as a von Neumann algebra M together with a coassociative unital normal $*$-homomorphism $\delta : M \to M\overline{\otimes}M$ and faithful semi-finite normal weights φ and ψ such that $(1\otimes\varphi)\delta = \eta\varphi$ on $\mathfrak{M}_\varphi$ and $(\psi\otimes 1)\delta = \eta\psi$ on $\mathfrak{M}_\psi$, where $\eta : \mathbb{C} \to M$ is the unit map. The weight φ and ψ are called the *left* and *right Haar weights* respectively. When we write $\mathbb{G}$ for a locally compact quantum group, the underlying von Neumann algebra is denoted by $L^\infty(\mathbb{G})$.

Recall that $\mathfrak{M}_\varphi$, $\mathfrak{A}_\varphi$, $\mathfrak{N}_\varphi$, $H_\varphi =: L^2(\mathbb{G})$, Λ_φ, Δ_φ, J_φ.

$\mathfrak{N}_\varphi^*\mathfrak{N}_\psi$

6.2 (Fundamental multiplicative unitaries)**.** A *multiplicative unitary* on a Hilbert space H is a unitary operator $W \in B(H\otimes H)$ satisfying the pentagonal identity $W_{12}W_{13}W_{23} = W_{23}W_{12}$ in $B(H\otimes H\otimes H)$, written in the leg numbering notation. It defines a comultiplication $\delta : H \to H\otimes H$ such that $\delta(\xi) := W(\xi\otimes 1)W^*$ for $\xi \in H$.

Let $\mathbb{G}$ be a locally compact quantum group. Then, there is a unique multiplicative unitary W on $L^2(\mathbb{G})$, called the *fundamental multiplicative unitary*, such that

$$W^*(\Lambda_\varphi(x)\otimes\Lambda_\varphi(y)) = (\Lambda_\varphi\otimes\Lambda_\varphi)(\delta(x)(y\otimes 1)), \qquad x, y \in \mathfrak{N}_\varphi.$$

$$\begin{tikzcd}
\mathfrak{N}_\varphi\otimes\mathfrak{N}_\varphi \arrow[r, "\Lambda_\varphi\otimes\Lambda_\varphi"] \arrow[d] & L^2(\mathbb{G})\otimes L^2(\mathbb{G}) \arrow[d, "W^*"] \\
\mathfrak{N}_\varphi\otimes\mathfrak{N}_\varphi \arrow[r, "\Lambda_\varphi\otimes\Lambda_\varphi"] & L^2(\mathbb{G})\otimes L^2(\mathbb{G})
\end{tikzcd}$$

6.3 (Fundamental involutions)**.** Let $\mathbb{G}$ be a locally compact quantum group. Then, there is a closed densely defined conjugate-linear involution $G : \operatorname{dom} G \subset L^2(\mathbb{G}) \to L^2(\mathbb{G})$ such that

$$G\Lambda_\varphi((\psi\otimes\mathrm{id})(\delta(x^*)(y\otimes 1))) = \Lambda_\varphi((\psi\otimes\mathrm{id})(\delta(y^*)(x\otimes 1))), \qquad x, y \in \mathfrak{N}_\varphi^*\mathfrak{N}_\psi.$$

6.4 (Antipode)**.** $\tau_t := \operatorname{Ad}|G|^{-2it}$, $(\sigma_t^\psi\otimes\tau_{-t})\delta = \delta\sigma_t^\psi$, $\delta\tau_t = (\tau_t\otimes\tau_t)\delta$,

For the polar decomposition $G = I|G|$, the *unitary antipode* is defined by $R : \operatorname{dom} R \subset L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) : x \mapsto Ix^*I$. The *antipode* or *coinverse* is $S := R\tau_{-\frac{i}{2}}$

Kac type: trivial scaling group.

6.2 Dual quantum groups

6.3 Crossed products