Algebraic Topology

Ikhan Choi

February 18, 2023

Contents

Part I

Homology

Chapter 1

Axiomatic homology theory

1.1 Singular homology

1.2 Eilenberg-Steenrod axioms

Mayer-Vietoris sequence

Chapter 2

Computation of homology

2.1 Cellular homology

CW complex, equivalence,

2.2 Simplicial homology

geometric realization, equivalence, smith normal form, simplicial approximation,

Chapter 3

Cohomology

cup product Universal coefficient theorem

3.1 Poincaré duality

Part II

Homotopy

Chapter 4

Fundamental groups

4.1

4.1. A *homotopy of paths* is a continuous map $h : I \times I \to X$ such that $h(0,) = x_0$ and

(a) linear homotopy

(b) reparametrization

4.2. The fundamental group is a group composition

4.3 (Van Kampen theorem)**.**

4.2 Covering spaces

path lifting property

Chapter 5

Fibration

5.1 Homotopy lifting property

Locally trivial bundles

pullback bundles: universal property, functoriality, restriction, section prolongation

5.2 Obstruction theory

5.3 Hurewicz theorem

$H_\bullet(\Omega S_n)$ and $H_\bullet(U(n))$ Spin, $\mathrm{Spin}_{\mathbb{C}}$ structure

Chapter 6

Spectral sequences

6.1 Serre spectral sequence

(Lyndon-Hochschild-Serre)

6.2 Adams spectral sequence

Part III

Fiber bundles

Chapter 7

Fiber bundles

7.1 Principal bundles

7.1 (Structure groups)**.** Let G be a topological group and Y be a left G-space. A *G-bundle* or a *fiber bundle with structure group G* with fiber Y is a fiber bundle $p : E \to B$, together with a local trivialization $\varphi = \{\varphi_i : p^{-1}(U_i) \to U_i \times Y\}_i$ such that the transition map are given by

$$\tau_{ij}(b, y) := \varphi_j \varphi_i^{-1}(b, y) = (b, g_{ij}(b)y), \qquad \forall i, j,\ b \in U_i \cap U_j,\ y \in Y,$$

where $g_{ij} : U_i \cap U_j \to G$ are maps.

A *G-bundle map* is a bundle map $(u, f) : (E_1, B_1) \to (E_2, B_2)$ such that

$$\varphi_{2,j} u \varphi_{1,i}^{-1}(b, y) = (f(b), h_{ij}(b)y), \qquad \forall i, j,\ b \in U_{1,i} \cap f^{-1}(U_{2,j}),\ y \in Y,$$

where $h_{ij} : U_{1,i} \cap f^{-1}(U_{2,j}) \to G$ are maps. If $B_1 = B_2 = B$, a G-bundle map over B is a G-bundle map (u, f) such that $f = \mathrm{id}_B$. We denote by $\mathbf{Bun}_Y(B)$ the category of G-bundles over B with fiber Y.

(a) If B is locally compact and Hausdorff, then every fiber bundle with fiber Y has a structure group $\mathrm{Homeo}(Y)$.

(b) A G-bundle map (u, f) is an isomorphism if and only if f is a homeomorphism.

Proof. (a)

(b) $\square$

7.2 (Fiber bundle construction theorem)**.** Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 1-cocyle* on $\mathcal{U}$ with coefficients in G is a set of maps $g = \{g_{ij} : U_i \cap U_j \to G\}_i$ such that the following *cocycle condition* holds:

$$g_{ik}(b) = g_{jk}(b) g_{ij}(b), \qquad \forall i, j, k,\ b \in U_i \cap U_j \cap U_k.$$

The set of Čech 1-cocycles on $\mathcal{U}$ with coeffients in G is denoted by $\check{Z}^1(\mathcal{U}, G)$.

Let $g \in \check{Z}^1(\mathcal{U}, G)$ and Y a left G-space. We will construct a G-bundle with fiber Y that is trivialized over $\mathcal{U}$ in which the transition maps are given by g. Define

$$E := \left(\coprod_i (U_i \times Y) \right) / \sim,$$

where $\sim$ is an equivalence relation generated by

$$(i, b, y) \sim (j, b, g_{ij}(b)y), \qquad \forall i, j,\ b \in U_i \cap U_j,\ y \in Y.$$

Also define $p : E \to B : [i, b, y] \mapsto b$ and $\varphi_i^{-1} : U_i \times Y \to p^{-1}(U_i) : (b, y) \mapsto [i, b, y]$, which are clearly continuous and surjective even without the cocycle condition.

(a) φ_i^{-1} is injective.

(b) φ_i^{-1} is open.

(c) The transition maps from the local trivialization $\varphi = \{\varphi_i\}_i$ coincides with the cocycle g.

Proof. (a) Suppose $\varphi_i^{-1}(b,y) = \varphi_i^{-1}(b',y')$. Since $(i,b,y) \sim (i,b',y')$, we have $b = b'$ and there is a sequence

$$y' = g_{i_{n-1}i_n}(b)g_{i_{n-2}i_{n-1}}(b)\cdots g_{i_0i_1}(b)y,$$

where $i_0 = i_n = i$. By applying the cocycle condition inductively, we obtain $y = y'$, which implies the injectivity of φ_i^{-1}.

(b) The map φ_i^{-1} factors through $\coprod_i (U_i \times Y)$ such that

$$\varphi_i^{-1} : U_i \times Y \to \coprod_i (U_i \times Y) \xrightarrow{\pi} p^{-1}(U_i).$$

Since the inclusion to disjoint union is open, it suffices to show the quotient map $\pi : \coprod_i (U_i \times Y) \to E$ is open. Let $V \subset \coprod_i (U_i \times Y)$ be open. Observe for each pair of i and j that

$$\pi^{-1}\pi(V \cap (U_i \times Y)) \cap (U_j \times Y)$$

is open because it is exactly same as the inverse image of the open set $V \cap (U_i \times Y)$ under the map $(U_i \cap U_j) \times Y \subset U_j \times Y \to U_i \times Y : (b,y) \mapsto (b, g_{ij}(b)y)$. Here we used the cocycle condition of g. Therefore,

$$\pi^{-1}\pi(V) = \bigcup_{ij} \pi^{-1}\pi(V \cap (U_i \times Y)) \cap (U_j \times Y)$$

is open, hence the open π.

(c) Clear by the cocycle condition. □

7.3 (Cohomologous transitions)**.** Let $\mathcal{U} = \{U_i\}_i$ be an open cover of a topological space B, and G be a topological group. A *Čech 0-cochain* on $\mathcal{U}$ with coefficients in G is a set of maps $h = \{h_i : U_i \to G\}_i$. The group of Čech 0-cochains on $\mathcal{U}$ with coefficients in G is denoted by $\check{C}^0(\mathcal{U}, G)$.

The *first Čech cohomology group* of $\mathcal{U}$ with coefficients G is the orbit space of an action on $\check{Z}^1(\mathcal{U}, G)$ by $\check{C}^0(\mathcal{U}, G)$ defined as follows:

$$(hg)_{ij}(b) := h_j(b)g_{ij}(b)h_i(b)^{-1}, \qquad \forall i,j,\ b \in U_i \cap U_j,$$

which is denoted by $\check{H}^1(\mathcal{U}, G)$. We define the *first Čech cohomology group* of B with coefficients in G as the direct limit

$$\check{H}^1(B, G) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G).$$

Let Y be a left G-space, and let $\mathrm{Bun}_Y(B)$ be the set of isomorphism classes of G-bundles over B with fiber Y.

(a) $\mathrm{Bun}_Y(B) \to \check{H}^1(B, G)$ is well-defined.

(b) $\mathrm{Bun}_Y(B) \to \check{H}^1(B, G)$ is surjective.

(c) $\mathrm{Bun}_Y(B) \to \check{H}^1(B, G)$ is injective if Y is faithful.

Proof. (a) Suppose $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be isomorphic G-bundles with fiber Y. Let $u : E_1 \to E_2$ be a G-bundle isomorphism. By considering the refinement, we can find an open cover $\mathcal{U} = \{U_i\}_i$ of B on which E_1 and E_2 are simultaneously locally trivialized.

$$g_1 := \{g_{1,ij} : U_i \cap U_j \to G\}.$$

□

7.4 (Principal bundles)**.** Let G be a topological group, and X be a left *principal homogeneous G-space*, i.e. a free and transitive left G-space such that $G \times X \to X \times X : (g, x) \mapsto (gx, x)$ is a homeomorphism.

A *principal G-bundle* is a G-bundle $p : P \to B$ with fiber X, often together with a fiber-preserving continuous right action $\rho : P \times G \to P$ such that each chart $\varphi_i : p^{-1}(U_i) \to U_i \times X$ induces a principal homogeneous right action on $\{b\} \times X \subset U_i \times X$ which commutes with the left action. The right action ρ is called the *principal right action* or *(global) gauge transformation*. Note that for each $b \in B$ the *fiber* $\{b\} \times X$ has commuting left and right actions, but the *fiber* $p^{-1}(b)$ cannot be given a natural left action from local trivializations.

The category of principal G-bundles over B is denoted by $\mathbf{Prin}_G(B)$, and the morphisms are usually defined as right G-equivariant maps. Then, we may consider the forgetful functor $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$.

(a) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is fully faithful, i.e. a bundle map $u : P_1 \to P_2$ over B is a G-bundle map if and only if it is a right G-equivariant map.

(b) $\mathbf{Prin}_G(B) \to \mathbf{Bun}_X(B)$ is surjective, i.e. every G-bundle with fiber X has a principal right action.

(c) A principal bundle is trivial if it has a global section.

Proof. (a) Let $u : P_1 \to P_2$ be a G-bundle map over B so that there is a map $g_i : U_i \to G$ for each i such that

$$\varphi_i u \varphi_i^{-1}(b, x) = (b, g_i(b)x), \qquad \forall i,\ b \in U_i,\ x \in X.$$

If we write $\rho_s = \rho(-, s)$, then the induced action $\varphi_i \rho_s \varphi_i^{-1}$ on $U_i \times X$ commutes with $\varphi_i u \varphi_i^{-1}$. Now for every $e \in P_1$, we have

$$\begin{aligned}
\rho_s u(e) &= \varphi_i^{-1}(\varphi_i \rho_s \varphi_i^{-1})(\varphi_i u \varphi_i^{-1})\varphi_i(e) \\
&= \varphi_i^{-1}(\varphi_i u \varphi_i^{-1})(\varphi_i \rho_s \varphi_i^{-1})\varphi_i(e) \\
&= u\rho_s(e),
\end{aligned}$$

therefore u is right G-equivariant.

Conversely, let $u : P_1 \to P_2$ be a right G-equivariant map. By fixing $x_0 \in X$ and using the fact that the left action is free and transitive, define $g_i : U_i \to G$ such that

$$(b, g_i(b)x_0) = \varphi_i u \varphi_i^{-1}(b, x_0).$$

The function g_i is continuous since it factors as

$$b \mapsto (b, x_0) \xmapsto{\varphi_i u \varphi_i^{-1}} (b, g_i(b)x_0) \mapsto g_i(b)x_0 \mapsto g_i(b).$$

The last map is continuous since X is a principal homogeneous space. Then, for every $(b, x) \in U_i \times X$, there is a unique $s = s(b, x) \in G$ such that

$$\varphi_i \rho_s \varphi_i^{-1}(b, x_0) = (b, x),$$

so we have

$$\begin{aligned}
\varphi_i u \varphi_i^{-1}(b, x) &= (\varphi_i u \varphi_i^{-1})(\varphi_i \rho_s \varphi_i^{-1})(b, x_0) \\
&= \varphi_i u \rho_s \varphi_i^{-1}(b, x_0) \\
&= \varphi_i \rho_s u \varphi_i^{-1}(b, x_0) \\
&= (\varphi_i \rho_s \varphi_i^{-1})(\varphi_i u \varphi_i^{-1})(b, x_0) \\
&= (\varphi_i \rho_s \varphi_i^{-1}) g_i(b)(b, x_0) \\
&= g_i(b)(\varphi_i \rho_s \varphi_i^{-1})(b, x_0) \\
&= g_i(b)(b, x) \\
&= (b, g_i(b)x).
\end{aligned}$$

Hence, u is a G-bundle map.

(b) Fix $x_0 \in X$ and consider the homeomorphism $G \to X : g \to gx_0$. Define a right action

$$X \times G \to X : (gx_0, s) \mapsto gsx_0.$$

It defines a right principal homogeneous X and commutes with the left action,

Define $\rho : P \times G \to P$ such that

$$\varphi_i \rho_s \varphi_i^{-1}(b, x) = (b, xs).$$

It is well defined, fiber preserving, continuous. also for any b and any chart φ_j containing b, the action on $\{b\} \times X$ defines a principal homogeneous as we have seen. Therefore, ρ is a gauge tranformation.

(c)

□

7.5 (Associated bundles)**.**

$$\mathrm{Prin}_G(B) \xrightarrow{\sim} \mathrm{Bun}_X(B) \xrightarrow{\sim} \check{H}^1(B, G) \hookrightarrow \mathrm{Bun}_F(B)$$

can be given in a more simple way.

7.2 Classifying spaces

Let $\mathrm{Prin}_G(B)$ be the set of isomorphism classes of principal G-bundles. Then, we have a contravariant functor

$$\mathrm{Prin}_G : \mathbf{hTop}_{\mathrm{para}} \to \mathbf{Set}$$

such that there is a natural isomorphism between contravariant functors

$$[-, BG] \to \mathrm{Prin}_G.$$

7.6 (Homotopy properteis)**.** Let $p : E \to B$ be a vector bundle

(a) If $p_1 : E_1 \to B \times [0, \frac{1}{2}]$ and $p_2 : E_2 \to B \times [\frac{1}{2}, 1]$ are trivial, then

(b) If $f, g : B' \to B$ are homotopic, then $f^*\xi \cong g^*\xi$.

7.7 (Finite type)**.**

7.3 Reduction of structure groups

7.4 Vector bundles

subbundles, quotient bundles, bundle maps, constant rank, then ker, im, coker bundles are locally trivial so that they are vector bundles. pullback: vector bundle structure

vector fields(trivial subbundles), parallelizable bundle operations: sum, tensor, dual, hom, exterior reduction and metrics

7.8 (Vector bundles)**.** Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be vector bundles.

(a) A vector bundle map u over B is a vector bundle isomorphism if and only if it is a fiberwise linear isomorphism.

Let $1 \le n \le \infty$. If $f, g : B \to G_k(\mathbb{F}^n)$ such that $f^*(\gamma_{k^n}) \cong g^*(\gamma_{k^n})$, then $jf \simeq jg$, where $j : G_k(\mathbb{F}^n) \to G_k(\mathbb{F}^{2n})$ is the natural inclusion.

7.9. Riemannian and Hermitian metrics

Exercises

group quotient gives a principal G-bundle.

Chapter 8

Characteristic classes

Chapter 9

K-theory

bott periodicity Hopf invariant

Part IV

Stable homotopy theory

equivariant topology chromatic homotopy theory spectral sequences orthogonal spectra abstract homotopy theory Kervaire invariant problem