POSITIVE HAHN-BANACH SEPARATIONS IN OPERATOR ALGEBRAS

IKHAN CHOI

ABSTRACT.

1. INTRODUCTION

- definition and properties of $f_\varepsilon(t) := (1+\varepsilon t)^{-1}t$
- commutant Radon-Nikodym, relation between $\{\omega' \in M_*^+ : \omega' \le \omega\}$ and $\{h \in \pi(M)'^+ : h \le 1\}$, order preserving linear map
- Mazur lemma

Definition 1.1 (Hereditary subsets). Let E be a partially ordered real vector space. We say a subset F of the positive cone E^+ is *hereditary* if $0 \le x \le y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. A $*$-subalgebra B of a $*$-algbera A is called *hereditary* if the positive cone B^+ is a hereditary subset of A^+. We define the *positive polar* of F as the positive part of the real polar

$$F^{r+} := \{x^* \in (E^*)^+ : \sup_{x \in F} x^*(x) \le 1\}.$$

An example that is a non-hereditary closed convex subset of a C*-algebra is $\mathbb{C}1$ in any unital C*-algebra.

Definition 1.2 (Lower dominated sequences). Let E be a partially ordered real vector space. A sequence $x_n \in E$ is called *lower dominated* if there is $x \in E$ such that $x \le x_n$ for all n. If E is the self-adjoint part of the predual of a von Neumann algebra where the Jordan decomposition holds, then we can change the definition such that $x \in -E^+$.

2. POSITIVE HAHN-BANACH SEPARATION THEOREMS

We start with the positive Hahn-Banach separation for von Neumann algebras and their preduals, and will close this section with the same theorem for C*-algebras and their duals.

Theorem 2.1 (Positive Hahn-Banach separation for von Neumann algebras). *Let M be a von Neumann algebra.*

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$. In particular, if $x' \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x') > 1$ and $\omega(x) \le 1$ for $x \in F$.*

(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then $F_* = F_*^{r+r+}$. In particular, if $\omega' \in M_*^+ \setminus F_*$, then there is $x \in M^+$ such that $\omega'(x) > 1$ and $\omega(x) \le 1$ for $\omega \in F_*$.*

Proof. (1) Since the positive polar is represented as the real polar

$$F^{r+}=F^r\cap M_*^+=F^r\cap(-M^+)^r=(F\cup -M^+)^r=(F-M^+)^r,$$

the positive bipolar can be written as $F^{r+r+}=(F-M^+)^{rr+}=(\overline{F-M^+})^+$ by the usual real bipolar theorem, where the closure is for the σ-weak topology. Because $F=(F-M^+)^+\subset(\overline{F-M^+})^+$, it suffices to prove the opposite inclusion $(\overline{F-M^+})^+\subset F$.

Let $x\in(\overline{F-M^+})^+$. Take a net $x_i\in F-M^+$ such that $x_i\to x$ σ-strongly, and take a net $y_i\in F$ such that $x_i\le y_i$ for each i. Suppose we may assume that the net x_i is bounded. For sufficiently small ε so that the bounded net x_i has the spectra in $[-(2\varepsilon)^{-1},\infty)$, we have $f_\varepsilon(x_i)\to f_\varepsilon(x)$ σ-strongly, and hence σ-weakly. On the other hand, by the hereditarity and the σ-weak compactness of F, we may asumme that the bounded net $f_\varepsilon(y_i)\in F$ converges σ-weakly to a point of F by taking a subnet. Then, we have $f_\varepsilon(x)\in F-M^+$ by

$$0\le f_\varepsilon(x)=\lim_i f_\varepsilon(x_i)\le\lim_i f_\varepsilon(y_i)\in F,$$

thus we have $x\in F$ since $f_\varepsilon(x)\uparrow x$ as $\varepsilon\to 0$. What remains is to prove the existence of a bounded net $x_i\in F-M^+$ such that $x_i\to x$ σ-strongly.

Define a convex set

$$G:=\left\{x\in\overline{F-M^+}:\begin{array}{c}\text{there is a sequence } x_m\in F-M^+\\ \text{such that } -2x\le x_m\uparrow x\ \sigma\text{-weakly}\end{array}\right\}\subset M^{sa},$$

where x_m denotes a sequence. In fact, it has no critical issue on allowing x_m to be uncountably indexed. Since we clearly have $F-M^+\subset G$ and every non-decreasing net with supremum is bounded and σ-strongly convergent, it suffices to show that G, or equivalently its intersection with the closed unit ball by the Krein-Smǔlian theorem, is σ-strongly closed. Let $x_i\in G$ be a net such that $\sup_i\|x_i\|\le 1$ and $x_i\to x$ σ-strongly. For each i, take a sequence $x_{im}\in F-M^+$ such that $-2x_i\le x_{im}\uparrow x_i$ σ-strongly as $m\to\infty$, and also take $y_{im}\in F$ such that $x_{im}\le y_{im}$. Since $\|x_{im}\|\le 2\|x_i\|\le 2$ is bounded, it implies that there is a bounded net x_j in $F-M^+$ such that $x_j\to x$ σ-strongly, and we can choose arbitrarily small $\varepsilon>0$ such that $\sigma(x_j)\subset[-(2\varepsilon)^{-1},\infty)$ for all j. Since $f_\varepsilon(x_j)$ converges to $f_\varepsilon(x)$ σ-strongly and $f_\varepsilon(y_j)$ is a bounded net for each $\varepsilon>0$ so that we may assume that the net $f_\varepsilon(y_j)$ is σ-weakly covergent by taking a subnet, we have $f_\varepsilon(x)\in F-M^+$ by

$$f_\varepsilon(x)=\lim_j f_\varepsilon(x_j)\le\lim_j f_\varepsilon(y_j)\in F,$$

where the limits are in the σ-weak sense. By taking ε as any decreasingly convergent sequence to zero, we have $x\in G$, hence the closedness of G.

(2) It is enough to prove $(\overline{F_*-M_*^+})^+\subset F_*$, where the closure is for the weak topology or equivalently in norm by the convexity of $F_*-M_*^+$, so we begin our proof by fixing $\omega\in(\overline{F_*-M_*^+})^+$. For a sequence $\omega_n\in F_*-M_*^+$ such that $\omega_n\to\omega$ in norm of M_*, we can take $\varphi_n\in F_*$ such that $\omega_n\le\varphi_n$ for all n. By modifying ω_n into $\omega_n-(\omega_n-\omega)_+\in F_*-M_*^+$ and taking a rapidly convergent subsequence, we may assume $\omega_n\le\omega$ and $\|\omega-\omega_n\|\le 2^{-n}$ for all n. If we consider the Gelfand-Naimark-Segal representation $\pi:M\to B(H)$ associated to a positive normal linear functional

$$\widehat{\omega}:=\sum_n(\omega-\omega_n)+\omega+\sum_n 2^{-n}\left(\frac{[\omega_n]}{1+\|\omega_n\|}+\frac{\varphi_n}{1+\|\varphi_n\|}\right)$$

on M with the canonical cyclic vector Ω, we can construct commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(M)'$ of $\omega, \omega_n, \varphi_n$ with respect to $\widehat{\omega}$ respectively. Since $-1 \leq h_n \leq h$ is bounded, $h_n \to h$ in the weak operator topology of $\pi(M)'$. By the Mazur lemma, we can take a net h_i by convex combinations of h_n such that $h_i \to h$ strongly in $\pi(M)'$, and the corresponding linear functionals ω_i and φ_i satisfy $\omega_i \leq \varphi_i$ with $\varphi_i \in F_*$ by the convexity of F_* so that $\omega_i \in F_* - M_*^+$. The net h_i can be taken to be a sequence in fact because $\pi(M)'$ is σ-finite by the existence of the separating vector Ω, but it is not necessary in here. For each i and $0 < \varepsilon < 1$, define

$$h_\varepsilon := f_\varepsilon(h), \quad h_{i,\varepsilon} := f_\varepsilon(h_i), \quad k_{i,\varepsilon} := f_\varepsilon(k_i)$$

in $\pi(M)'$, where the functional calculi are well-defined because $-1 \leq h_i$ and $0 \leq h, k_i$ for all i, and define k_ε as the σ-weak limit of the bounded net $k_{i,\varepsilon}$, which may be assumed to be σ-weakly convergent. Define $\omega_\varepsilon, \omega_{i,\varepsilon}, \varphi_{i,\varepsilon}, \varphi_\varepsilon$ as the corresponding normal linear functionals on M to $h_\varepsilon, h_{i,\varepsilon}, k_{i,\varepsilon}, k_\varepsilon$. Note that $\varphi_i \in F_*$. The hereditarity of F_* and $0 \leq \varphi_{i,\varepsilon} \leq \varphi_i$ imply $\varphi_{i,\varepsilon} \in F_*$, and the weak closedness of F_* and the weak convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$ in M_* imply $\varphi_\varepsilon \in F^*$. From $\omega_i \leq \varphi_i$, we can deduce $0 \leq \omega_\varepsilon \leq \varphi_\varepsilon$ by considering the operator monotonicity f_ε and taking the weak limit on i. Thus again, the hereditarity of F_* implies $\omega_\varepsilon \in F^*$, and the weak closedness of F_* and the weak convergence $\omega_\varepsilon \to \omega$ in M_* imply $\omega \in F^*$. □

Now we prepare some lemmas for the positive Hahn-Banach separation theorem for C*-algebras.

Lemma 2.2. *Let A be a C*-algebra, and let G^* be a norm closed and downward closed convex subset of A^{*sa}. If an element of A^{*sa} is approximated weakly* by a lower dominated sequence of G^*, then it is approximated in norm by a sequence of G^*.*

Proof. Let $\omega_n \in F^* - A^{*+}$, $\varphi_n \in F^*$, $\widehat{\omega}_0 \in A^{*+}$ be such that $\omega_n|_B \to \omega|_B$ weakly* in B^* and $-\widehat{\omega}_0 \leq \omega_n \leq \varphi_n$ for all n. Consider the Gelfan-Naimark-Segal representation $\pi : A \to B(H)$ of

$$\widehat{\omega} := \widehat{\omega}_0 + [\omega] + \sum_n 2^{-n} \left(\frac{[\omega_n]}{1 + \|\omega_n\|} + \frac{\varphi_n}{1 + \|\varphi_n\|} \right)$$

with the canonical cyclic vector $\Omega \in H$. Let $\theta^* : \pi(A') \to A^* : y \mapsto (a \mapsto \langle y\pi(a)\Omega, \Omega\rangle)$. Define the commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(A)'$ of $\omega, \omega_n, \varphi_n$ with respect to $\widehat{\omega}$. Note that $-1 \leq h \leq 1$, $-1 \leq h_n$, and $0 \leq k_n$, and the functional calculus $h_{n,\varepsilon} := f_\varepsilon(h_n)$ and $k_{n,\varepsilon} := f_\varepsilon(k_n)$ are well-defined in $\pi(A)'$ if $0 < \varepsilon < 1$.

$\theta^*() \subset F^* - A^{*+}$

Let $h_{1n} := h_n$ and $m \geq 2$. Define

$$S_{(m-1)n} := \operatorname{conv}\{h_{(m-1)n,m^{-1}}, h_{(m-1)(n+1),m^{-1}}, \cdots\}, \quad T_{(m-1)n} := \operatorname{conv}\{h_{(m-1)n}, h_{(m-1)(n+1)}, \cdots\}.$$

If we choose any element $h_{m\infty,m^{-1}} \in \bigcap_n \overline{S_{(m-1)n}}^w$ from a non-empty set, then for each n we can take $y_n \in S_{(m-1)n}$ such that

$$\|(y_n - h_{m\infty,m^{-1}})\Omega\| < \frac{2}{3}2^{-n}, \qquad \|(y_n - h_{m\infty,m^{-1}})h\Omega\| < \frac{2}{3}2^{-n},$$

and there is $h_{mn} \in (T_{(m-1)n} - \pi(A)'^+) \cap \pi(A)'_{\geq -1}$ such that $h_{mn,m^{-1}} = y$, so $\theta^*(h_{mn}) \in F^* - A^{*+}$ because the operator concavity of $f_{m^{-1}}$ implies

$$S_{(m-1)n} \subset f_{m^{-1}}((T_{(m-1)n} - \pi(A)'^+) \cap \pi(A)'_{\geq -1})$$

and we have

$$\theta^*(T_{mn} - \pi(A)'^+) \subset F^* - A^{*+}.$$

Note that $h_{mn} \in (T_{(m-1)n} - \pi(A)'^+) \cap \pi(A)'_{\geq -1}$.

like $T_{mn} \subset T_{(m-1)n} - \varepsilon_{mn}\pi(A)'^+$ to get $T_{mn} \subset T_{2n} - (\sum_m \varepsilon_{mn})\pi(A)'^+$? we want $\lim_n \sum_m \varepsilon_{mn} = 0$. summability of $t_m^2/(t_{m-1}^{-1} - t_m^{-1} + t_m)$?

Since

$$\|(h_{(m-1)n,(m-1)^{-1}} - h_{(m-1)n',(m-1)^{-1}})\Omega\| < 2^{-n}, \qquad n' \geq n,$$

if we write

$$h_{mn,m^{-1}} = \sum_{k\geq n} \lambda_k h_{(m-1)k,m^{-1}} = \sum_{k\geq n} \lambda_k f_{-(m(m-1))^{-1}}(h_{(m-1)k,(m-1)^{-1}}) \geq f_{-(m(m-1))^{-1}}(\sum_{k\geq n} \lambda_k h_{(m-1)k,(m-1)^{-1}})$$

and

$$f_{(m-1)^{-1}}(h_{mn}) = f_{(m(m-1))^{-1}}(h_{mn,m}) \geq \sum_{k\geq n} \lambda_k h_{(m-1)k,(m-1)^{-1}}$$

$$h_{nn} \geq f_{(n-1)^{-1}}(\sum_{k\geq n} \lambda_k h_{(n-1)k,(n-1)^{-1}})$$

$$h_{nn} - h = (h_{nn} - h_{nn,m^{-1}}) + (h_{nn,m^{-1}} - h_{,m^{-1}}) - (h_{,m^{-1}} - h)$$

The bounded sequences $h_{n,\varepsilon}$ and $k_{n,\varepsilon}$ have weakly convergent subnets in $\pi(A)'$, and denote their limits by h_ε and k_ε respectively. Be cautious that $h'_\varepsilon := f_\varepsilon(h)$ is not equal to h_ε in general. By the operator concavity of the function f_ε and the σ-finiteness of $\pi(A)'$, the Mazur lemma retakes sequences $\omega_n \in F^* - A^{*+}$ and $\varphi_n \in F^*$ such that $\omega_n \leq \varphi_n$ for all n and $h_{n,\varepsilon} \to h_\varepsilon$ and $k_{n,\varepsilon} \to k_\varepsilon$ strongly. We may assume

$$\|(h_{n,\varepsilon} - h_\varepsilon)\Omega\| < n^{-1}, \qquad \|(h_{n,\varepsilon} - h_\varepsilon)h\Omega\| < n^{-1}$$

for all n uniformly on ε, which will be used later. Note also that we have the identity

$$(1+\varepsilon h)(h'_\varepsilon - h_{n,\varepsilon})(1+\varepsilon h) = (h - h_n) + \varepsilon(h - h_n)(1+\varepsilon h_n)^{-1}(h - h_n).$$

Denote by $\omega_{n,\varepsilon}, \omega_\varepsilon, \omega'_\varepsilon, \varphi_{n,\varepsilon}, \varphi_\varepsilon$ the linear functionals in A^{*sa} corresponded to operators in the commutant $h_{n,\varepsilon}, h_\varepsilon, h'_\varepsilon, k_{n,\varepsilon}, k_\varepsilon \in \pi(A)'$. It follows clearly that $\omega_{n,\varepsilon} \to \omega_\varepsilon$ and $\varphi_{n,\varepsilon} \to \varphi_\varepsilon$ as $n \to \infty$, and $\omega'_\varepsilon \uparrow \omega$ as $\varepsilon \to 0$, all weakly in A^*. If we prove $\omega'_\varepsilon|_B - \omega_\varepsilon|_B \to 0$ weakly in B^* as $\varepsilon \to 0$, then since $\omega_{n,\varepsilon} \leq \varphi_{n,\varepsilon} \in F^*$ implies $\omega_\varepsilon \leq \varphi_\varepsilon \in F^*$, we obtain the weak convergence $\omega_\varepsilon|_B \to \omega|_B$ in B^* as $\varepsilon \to 0$ with $\omega_\varepsilon \in F^* - A^{*+}$. A desired sequence by applying the Mazur lemma on ω_ε within $\varepsilon < (1-\delta)^{-1}\delta$, where $\delta < 1$ is assumed.

Thus, what remains is to prove $\omega'_\varepsilon|_B - \omega_\varepsilon|_B \to 0$ weakly in B^* as $\varepsilon \to 0$. Consider the normal extension $\pi^{**} : A^{**} \to B(H)$ of the representation π. For $x \in A^{**}$ with $\|x\| \le 1$, the one-parameter family $(h'_\varepsilon - h_\varepsilon)\pi^{**}(x)\Omega$ of vectors is uniformly bounded on $0 < \varepsilon \le \frac{1}{2}$ by the uniform boundedness principle, because for each $\xi \in H$, fixing any n, say $n = 1$, we have

$$\begin{aligned}
&|\langle (h'_\varepsilon - h_\varepsilon)\pi^{**}(x)\Omega, \xi\rangle| \\
&\quad \le |\langle (h'_\varepsilon - h_{1,\varepsilon})\pi^{**}(x)\Omega, \xi\rangle| + |\langle (h_{1,\varepsilon} - h_\varepsilon)\pi^{**}(x)\Omega, \xi\rangle| \\
&\quad \le |\langle (1+\varepsilon h)^{-1}(h-h_1)(1+\varepsilon h)^{-1}\pi^{**}(x)\Omega, \xi\rangle| \\
&\qquad + \varepsilon|\langle (1+\varepsilon h)^{-1}(h-h_1)(1+\varepsilon h_1)^{-1}(h-h_1)(1+\varepsilon h)^{-1}\pi^{**}(x)\Omega, \xi\rangle| \\
&\qquad + \|(h_{1,\varepsilon} - h_\varepsilon)\Omega\|\|\pi^{**}(x^*)\xi\| \\
&\quad \le 4\|h-h_1\|\|\Omega\|\|\xi\| + 4\|h-h_1\|^2\|\Omega\|\|\xi\| + \|\xi\|,
\end{aligned}$$

which is uniformly bounded on ε. If we let $p \in B(H)$ the orthogonal projection on the closed linear subspace $\overline{\pi(B)\Omega}$, then for $y \in B^{**}$ with $\|y\| \le 1$, we further have $p(h'_\varepsilon - h_\varepsilon)\pi^{**}(y)\Omega \to 0$ weakly in $B(pH)$ as $\varepsilon \to 0$, which can be shown as follows. By the boundedness of $(h'_\varepsilon - h_\varepsilon)\pi^{**}(y)\Omega$, it is enough to choose $\pi(c)\Omega$ with $c \in B$ satisfying $\|c\| \le 1$ for the test vector. As $\|(h'_\varepsilon - h_\varepsilon)\pi(c)\Omega\|$ is uniformly bounded on ε because $c \in A^{**}$, we can also prove $\|(h'_\varepsilon - h_\varepsilon)h\pi(c)\Omega\|$ is uniformly bounded in the same manner but using $\|(h_{1,\varepsilon} - h_\varepsilon)h\Omega\| < 1$ instead of $\|(h_{1,\varepsilon} - h_\varepsilon)\Omega\| < 1$. Choose their common bound $C > 0$. For an arbitrarily fixed $\delta > 0$, take $b \in B$ such that $\|(\pi^{**}(y) - \pi(b))\Omega\| < \delta C^{-1}$ and $\|b\| \le 1$ by the Kaplansky density, and fix n such that $|(\omega - \omega_n)(c^*b)| < \delta$ and $n > \frac{9}{4}\|\Omega\|\delta^{-1}$. Then,

$$\begin{aligned}
&|\langle (h'_\varepsilon - h_\varepsilon)\pi^{**}(y)\Omega, \pi(c)\Omega\rangle| \\
&\quad < |\langle (h'_\varepsilon - h_\varepsilon)\pi(b)\Omega, \pi(c)\Omega\rangle| + \delta \\
&\quad < |\langle (h'_\varepsilon - h_\varepsilon)(1+\varepsilon h)\pi(b)\Omega, (1+\varepsilon h)\pi(c)\Omega\rangle| + O(\varepsilon) + \delta \\
&\quad < |\langle (h'_\varepsilon - h_{n,\varepsilon})(1+\varepsilon h)\pi(b)\Omega, (1+\varepsilon h)\pi(c)\Omega\rangle| + \delta + O(\varepsilon) + \delta \\
&\quad \le |(\omega - \omega_n)(c^*b)| + \varepsilon|\langle (1+\varepsilon h_n)^{-1}(h-h_n)\pi(b)\Omega, (h-h_n)\pi(c)\Omega\rangle| + \delta + O(\varepsilon) + \delta \\
&\quad < \delta + \varepsilon(1-\varepsilon)^{-1}\|(h-h_n)\|^2\|\Omega\|^2 + \delta + O(\varepsilon) + \delta,
\end{aligned}$$

where the asymptotic notation $O(\varepsilon)$ can be exactly computed as $C(2\varepsilon + \varepsilon^2)\|\Omega\|$, so we have

$$\limsup_{\varepsilon \to 0} |\langle (h'_\varepsilon - h_\varepsilon)\pi^{**}(y)\Omega, \pi(c)\Omega\rangle| \le 3\delta.$$

Since $\delta > 0$ was taken arbitrarily, we finally have $p(h'_\varepsilon - h_\varepsilon)\pi^{**}(y)\Omega \to 0$ weakly in pH, which implies $\omega'_\varepsilon|_B - \omega_\varepsilon|_B \to 0$ weakly in B^* as $\varepsilon \to 0$. □

The following lemma is a modification of the Krein-Šmulian theorem, and it can be proved in a similar way to the proof of the original theorem.

Lemma 2.3. *Let A be a C^*-algebra, and C_n^* be a non-decreasing sequence of weakly*-closed convex subsets of A^{*sa}, whose union C_∞^* contains A^{*+}. If a norm closed convex subset G^* of A^{*sa} has the property that $G^* \cap C_n^*$ is weakly* closed for each n, then $G^* \cap C_\infty^*$ is relatively weakly* closed in C_∞^*.*

Proof. Fix an element ω_0 of $C_\infty^* \setminus G^*$. It is enough to construct an element a of A^{sa} separating a norm open ball centered at ω_0 from G^*. Since G^* is norm closed, there exists $r > 0$ such that $G^* \cap B(\omega_0, r) = \varnothing$. By replacing G^* to $r^{-1}(G^* - \omega_0)$ and C_n^* to $r^{-1}(C_n^* - \omega_0)$, we may assume $G^* \cap B(0,1) = \varnothing$, and the claim follows if we prove there is $a \in A^{sa}$ separating $B(0,1)$ and G^*. The condition $A^{*+} \subset C_\infty^*$ becomes $A^{*+} - \omega_0 \subset C_\infty^*$. Letting the index n start from one, we may also replace C_n^* to $n(C_n^* \cap B(0,1))$ since its union is still C_∞^*. Note that C_n^* is bounded for each n, and we can easily see that $G^* \cap C_1^* = \varnothing$ and $n^{-1}C_n^* \subset (n+1)^{-1}C_{n+1}^*$.

Note that for any Banach space X, if F is a bounded subset of X, then by endowing with the discrete topology on F, we have a natural bounded linear operator $\ell^1(F) \to X$ by completeness of X, with its dual $X^* \to \ell^\infty(F)$. We will construct a bounded subset F of A^{sa} such that the subset $G^* \cap C_\infty^*$ of A^{*sa} induces a subset of the smaller subspace $c_0(F)$ of $\ell^\infty(F)$ via the restriction map $A^{*sa} \to \ell^\infty(F)$, and also such that it satisfies $G^* \cap C_\infty^* \cap F^\circ = \varnothing$, where $F^\circ := \{\omega \in A^{*sa} : \sup_{a\in F} |\omega(a)| \le 1\}$ denotes the absolute polar of F. If such a set $F \subset X$ exists, then the image of $G^* \cap C_\infty^*$ in $c_0(F)$ is a convex set disjoint to the closed unit ball of $c_0(F)$ by the condition $G^* \cap C_\infty^* \cap F^\circ = \varnothing$. Therefore, there exists a separating linear functional $l \in \ell^1(F)$ by the Hahn-Banach separation, and it induces a linear functional separating G^* and the unit ball of A^{*sa}. Then, we are done.

Let $F_0 := \{0\} \subset A^{sa}$. As an induction hypothesis on n, suppose for each $0 \le k \le n-1$ we already have a finite subset F_k of $(C_k^*)^\circ$ such that

$$G^* \cap C_n^* \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ = \varnothing.$$

If every finite subset F_n of $(C_n^*)^\circ$ satisfies

$$G^* \cap C_{n+1}^* \cap \left(\bigcup_{k=0}^{n-1} F_k\right)^\circ \cap F_n^\circ \neq \varnothing,$$

then since they are weakly* compact, the finite intersection property leads a contradiction because the intersection of all absolute polars F_n° of finite subsets F_n of $(C_n^*)^\circ$ is C_n^*, which is the polar of all union of finite subsets F_n of $(C_n^*)^\circ$ by the bipolar theorem. Thus, we can take a finite subset F_n of $(C_n^*)^\circ$ such that

$$G^* \cap C_{n+1}^* \cap \left(\bigcup_{k=0}^{n} F_k\right)^\circ = \varnothing.$$

Let $F := \bigcup_{k=0}^\infty F_k$. Then, we have $G^* \cap C_\infty^* \cap F^\circ = \varnothing$, and every element of C_∞^* is restricted to F to define an element of $c_0(F)$ because for each $\omega \in C_n^*$ and $k \ge 0$ we have

$$\omega(F_{n+k}) \subset \omega((C_{n+k}^*)^\circ) \subset \frac{n}{n+k}\omega((C_n^*)^\circ) \subset [-\frac{n}{n+k}, \frac{n}{n+k}].$$

Finally, for any $\omega \in A^{*sa}$, if we enumerate F as a sequence f_m, then

$$|\omega(f_m)| \le |(\omega_+ - \omega_0)(f_m)| + |(\omega_- - \omega_0)(f_m)| \to 0,$$

so the uniform boundedness principle concludes that F is bounded. Therefore, the set F satisfies the properties we desired. □

Theorem 2.4 (Positive Hahn-Banach separation for C*-algebras). *Let A be a C*-algebra.*

(1) *If F is a norm closed convex hereditary subset of A^+, then $F = F^{r+r+}$. In particular, if $a' \in A^+ \setminus F$, then there is $\omega \in A^{*+}$ such that $\omega(a') > 1$ and $\omega(a) \le 1$ for $a \in F$.*

(2) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$. In particular, if $\omega' \in A^{*+} \setminus F^*$, then there is $a \in A^+$ such that $\omega'(a) > 1$ and $\omega(a) \le 1$ for $\omega \in F^*$.*

Proof. (1) We directly prove the separation without invoking the arguments of positive bipolars. Denote by F^{**} the σ-weak closure of F in the universal von Neumann algebra A^{**}. We first show that F^{**} is hereditary subset of A^{**+}. Suppose $0 \le x \le y$ in A^{**} and $y \in F^{**}$. Then, there is $z \in A^{**}$ such that $x^{\frac12} = zy^{\frac12}$. Take bounded nets b_i in F and c_i in A such that $b_i \to y$ and $c_i \to z$ σ-strongly* in A^{**} using the Kaplansky density. We may assume the indices of these two nets are same. Since both the multiplication and the involution of a von Neumann algebra on bounded parts are continuous in the σ-strong* topology, and since the square root on a positive bounded interval is a strongly continuous function, we have the σ-strong* limit

$$x = y^{\frac12} z^* z y^{\frac12} = \lim_i b_i^{\frac12} c_i^* c_i b_i^{\frac12},$$

so we obtain $x \in F^{**}$ from $b_i^{\frac12} c_i^* c_i b_i^{\frac12} \in F$. Thus, F^{**} is hereditary in A^{**+}.

Let $a \in A^+ \setminus F$. Observe that we have $a \in A^{**+} \setminus F^{**}$ because if $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, meaning that $a_i \to a$ weakly in A and by the weak closedness of F in A we get a contradiction $a \in F^{**} \cap A = F$. By Theorem 2.1, there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega \le 1$ on $F \subset F^{**}$, so it completes the proof.

(2) As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, where the closure is always for the weak* topology through the proof. We first prove it when A is commutative. On a commutative C*-algebra, the rectifier function $\mathbb{R} \to \mathbb{R} : t \mapsto \max\{0, t\}$ is operator monotone. Define

$$G^* := \left\{ \omega \in \overline{F^* - A^{*+}} : \begin{array}{c} \text{there is a bounded net } \omega_j \in F^* - A^{*+} \\ \text{such that } \omega_j \to \omega \text{ weakly}^* \text{ in } A^* \end{array} \right\}.$$

We can easily check $F^* - A^{*+} \subset G^*$ by considering constant sequences. Take a bounded net $\omega_i \in G^*$ in the spirit of the Krein-Šmulian theorem such that $\omega_i \to \omega$ weakly* in A^*. Then, for each i we have a bounded net $\omega_{ij} \in F^* - A^{*+}$ and a net $\varphi_{ij} \in F^*$ such that $\omega_{ij} \le \varphi_{ij}$ for all j and $\omega_{ij} \to \omega_i$ weakly* in A^* by definition of G^*. Since $0 \le \omega_{ij+} \le \varphi_{ij} \in F^*$ implies $\omega_{ij+} \in F^*$ and since it is bounded for each i so that we may assume $\omega_{ij+} \to \omega_i'$ weakly* in A^*, we have $\omega_i \le \omega_i' \in F^*$ by the weak* closedness of F^*. Since $0 \le \omega_{i+} \le \omega_i' \in F^*$ implies $\omega_{i+} \in F^*$ and since it is bounded so that we may assume $\omega_{i+} \to \omega'$ weakly* in A^*, we have $\omega \le \omega' \in F^*$. It implies that $\omega \in F^* - A^{*+} \subset G^*$ and G^* is weakly* closed, so $G^* = \overline{F^* - A^{*+}}$. Therefore, if $\omega \in (\overline{F^* - A^{*+}})^+$, then there is a bounded net $\omega_i \in F^* - A^{*+}$ and a net $\varphi_i \in F^*$ such that $\omega_i \le \varphi_i$ for all i and $\omega_i \to \omega$ weakly* in A^*, so since $0 \le \omega_{i+} \le \varphi_i \in F^*$ implies $\omega_{i+} \in F^*$, and since it is bounded so that we may assume $\omega_{i+} \to \omega'$ weakly* in A^*, we have $\omega \le \omega' \in F^*$, which gives $\omega \in F^*$. This completes the proof of $(\overline{F^* - A^{*+}})^+ \subset F^*$ provided that A is commutative.

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Now we consider a general C*-algebra A. For a separable C*-subalgebra B of A, define

$$F_B^* := \overline{\{\omega_B \in B^{*+} : \text{there is } \omega \in F^* - A^{*+} \text{ such that } \omega|_B = \omega_B\}}^{\|\cdot\|},$$

which is clearly a norm closed and convex, and we can see that it is hereditary in B^{*+} by the positive Hahn-Banach extension. We claim $(\overline{F_B^* - B^{*+}})^+ \subset F_B^*$. As a corollary,

F_B^* becomes weakly* closed, and if A is separable itself, then the proof of the theorem follows by letting $B = A$. Note that the separability of B makes the weak* topology on any bounded part of B^{*sa} metrizable. Consider the following convex set

$$G_B^* := \left\{ \omega_B \in \overline{F_B^* - B^{*+}} : \begin{array}{c} \text{there is a lower dominated sequence } \omega_n \in F^* - A^{*+} \\ \text{such that } \omega_n|_B \to \omega_B \text{ weakly* in } B^* \end{array} \right\}.$$

$$G_B^* := \overline{F_B - B^{*+}}^{\|\cdot\|}$$

We can see that $F_B^* - B^{*+} \subset G_B^*$ by suitably taking the Hahn-Banach extension for the constant sequence of each element of $F^* - A^{*+}$, and it implies $(\overline{F_B^* - B^{*+}})^+ \subset (\overline{G_B^*})^+$. We also have $G_B^{*+} \subset F_B^*$ by Theorem 2.1 (2), so if we prove G_B^* is weakly* closed, then the claim $(\overline{F_B^* - B^{*+}})^+ \subset F_B^*$ follows.

First we prove G_B^* is norm closed in B^*. Let $\omega_{B,n} \in G_B^*$ be a sequence such that $\omega_{B,n} \le \omega_B$ and $\|\omega_{B,n} - \omega_B\| < 2^{-n}$. There is also a lower dominated sequence $\omega_{nm} \in F^* - A^{*+}$ for each n such that $\omega_{nm}|_B \le \omega_{B,n}$ and $\|\omega_{B,n} - \omega_{nm}|_B\| < 2^{-n-m}$ for all m by definition of G_B^* and by Lemma 2.2. Then,

$$-(\omega_B)_- - \sum_n (\omega_B - \omega_{B,n}) - \sum_{n,m} (\omega_{B,n} - \omega_{nm}|_B) \le \omega_{nn}|_B.$$

To prove G_B^* is weakly* closed, we can take a sequence $\omega_{B,n} \in G_B^*$ such that $\omega_{B,n} \to \omega_B$ weakly* in B^* by the Krein-Šmulian theorem and the separability of B. Since G_B^* is norm closed and ω_B belongs to the relative weak* closure of $G_B^* \cap C_\infty^*$ in C_∞^*, where

$$C_n^* := \{\omega_B' \in B^{*sa} : -\sum_{k \le n} (\omega_{B,k})_- - (\omega_B)_- \le \omega_B'\}, \qquad C_\infty^* := \bigcup_n C_n^*,$$

so if we only check $G_B^* \cap C_n^*$ is weakly* closed in B^* for each n, then we obtain $\omega_B \in G_B^*$ by Lemma 2.3, which implies the weak* closedness of G_B^*. Because every sequence in C_n^* is lower dominated, now it suffices to show $\omega_B \in G_B^*$ when it is the weak* limit of a lower dominated sequence $\omega_{B,n} \in G_B^*$. By Lemma 2.2, we may assume $\omega_{B,n} \to \omega_B$ in norm. There is also for each n a sequence $\omega_{nm} \in F^* - A^{*+}$ such that $\omega_{nm}|_B \to \omega_{B,n}$ in norm. Observing there was the same situation in the proof of the norm closedness of G_B^*, we can conclude $\omega_B \in G_B^*$ in the same way, hence G_B^* is weakly* closed.

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So far, we just proved $(\overline{F_B^* - B^{*+}})^+ \subset F_B^*$. Now let $\omega \in (\overline{F^* - A^{*+}})^+$. Take a net $\omega_i \in F^* - A^{*+}$ and $\varphi_i \in F^*$ such that $\omega_i \to \omega$ weakly* in A^* and $\omega_i \le \varphi_i$ for each i. For each separable C*-subalgebra B of A, we have $\varphi_i|_B \in F_B^*$ and $\omega_i|_B \in F_B^* - B^{*+}$ with the weak* convergence $\omega_i|_B \to \omega|_B$ in B^*, thus we have $\omega|_B \in (\overline{F_B^* - B^{*+}})^+ = F_B^*$ because B is separable. If we consider the increasing net of all separable C*-subalgebras $(B_j)_{j\in J}$ of A, then we have $\omega|_{B_j} \in F_{B_j}^*$ so that there is a net $\omega_{(j,\varepsilon)} \in F^* - A^{*+}$ based on the product directed set $\{(j,\varepsilon) : j \in J,\ \varepsilon > 0\}$ such that $\|\omega_{(j,\varepsilon)}|_{B_j} - \omega|_{B_j}\| < \varepsilon$ for each (j,ε).

With this net, as an intermediate step, we prove that ω belongs to the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$, where A_0^{**} denotes the set of all elements of A^{**} whose left or right support projection is σ-finite. Let $x \in A_0^{**+}$ with $\|x\| \le 1$, and let p be the support projection of x. Then, p is σ-finite and we can take a bounded sequence $a_n \in A^+$ such that $\|a_n\| \le 1$ for all n and $a_n p \to x$ σ-strongly in A^{**} by the Kaplansky density theorem and the Mazur lemma, which is sequential because $A^{**}p$ is σ-weakly closed and its bounded part is σ-strongly metrizable. Let p_n be the support projection of a_n for each

n. Since $a_m^{1/k} \to p_m$ and $x^{1/k} \to p$ σ-strongly as $k \to \infty$ for each m, we have $p_m \to p$ σ-strongly as $m \to \infty$. If we choose j_0 such that $a_n \in B_{j_0}$ for all n, then for any $j \succ j_0$, by taking limits $k \to \infty$, $m \to \infty$, and $n \to \infty$ in order on the inequality

$$\begin{aligned}|(\omega_{(j,\varepsilon)} - \omega)(x)| &\le |(\omega_{(j,\varepsilon)} - \omega)(x - a_n p)| + |(\omega_{(j,\varepsilon)} - \omega)(a_n(p - p_m))| \\ &\quad + |(\omega_{(j,\varepsilon)} - \omega)(a_n(p_m - a_m^{1/k}))| + |(\omega_{(j,\varepsilon)} - \omega)(a_n a_m^{1/k})|,\end{aligned}$$

since the final term is uniformly estimated up to ε because $a_n a_m^{1/k} \in B_j$ is uniformly bounded by one, we obtain $\lim_{(j,\varepsilon)}(\omega_{(j,\varepsilon)} - \omega)(x) = 0$. This proves that ω is contained in the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$.

Suppose now $\omega \notin F^*$. Then, there exists $x \in A^{**+}$ such that $\omega(x^2) > 1$ and $\omega'(x^2) \le 1$ for all $\omega' \in F^*$ by Theorem 2.1 (2). Let $\{p_i\}_{i\in I}$ be a maximal orthogonal family of σ-finite projections of the von Neumann algebra A^{**} whose sum is the support projection of x. If we consider order-preserving bounded linear maps $\Gamma : c_0(I) \to A^{**}$ and $\Gamma^* : A^* \to \ell^1(I)$ given by

$$\Gamma((c_i)_{i\in I}) := \sum_i c_i x p_i x, \qquad \Gamma^*(\omega') := (\omega'(x p_i x))_{i\in I},$$

then these maps are in dual, and Γ is extended to the linear map $\Gamma^{**} : \ell^\infty(I) \to A^{**}$ continuous with respect to weak* topologies. Observing that the left and right support projections of an arbitrary element of a von Neumann algebra are Murray-von Neumann equivalent, we can see A_0^{**} is an algebraic ideal of A^{**}, and we have $\Gamma(c_0(I)) \subset A_0^{**}$ due to the fact that each element of $c_0(I)$ has at most countably many non-zero components. Since ω is an element of the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$, we have $\Gamma^*(\omega) \in \overline{\Gamma^*(F^* - A^{*+})}$, where the closure is taken in the weak* topology of $\ell^1(I)$. Since the set $\overline{\Gamma^*(F^* - A^{*+})}$ is contained in the weak* closure of $\overline{\Gamma^*(F^*)} - \ell^1(I)^+$ by $\Gamma^*(F^* - A^{*+}) \subset \Gamma^*(F^*) - \ell^1(I)^+$, whose positive part is $\overline{\Gamma^*(F^*)}$ because $c_0(I)$ is a commutative C*-algebra. Therefore, we have $\Gamma^*(\omega) \in \overline{\Gamma^*(F^*)}$. For any $\delta > 0$, if we choose $c \in c_0(I)^+$ such that $c \le 1$ and $|\langle 1_{\ell^\infty(I)} - c, \Gamma^*(\omega)\rangle| < \delta$ using the Kaplansky density, and choose $\omega' \in F^*$ such that $|\langle c, \Gamma^*(\omega) - \Gamma^*(\omega')\rangle| < \delta$, then we have a contradiction

$$\begin{aligned}1 < \omega(x^2) = \langle 1_{\ell^\infty(I)}, \Gamma^*(\omega)\rangle &\approx_\delta \langle c, \Gamma^*(\omega)\rangle \\ &\approx_\delta \langle c, \Gamma^*(\omega')\rangle \le \langle 1_{\ell^\infty(I)}, \Gamma^*(\omega')\rangle = \omega'(x^2) \le 1,\end{aligned}$$

where the relation symbol $\approx_\delta$ means that the difference converges to zero as $\delta \to 0$, so finally we have $\omega \in F^*$. $\square$

3. Applications to weight theory

The positive Hahn-Banach separation theorem implies a generalization of the Combes theorem on subadditive normal weights.

Corollary 3.1. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\begin{array}{ccc}\left\{\begin{array}{c}\text{subadditive normal}\\ \text{weights of } M\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{hereditary closed}\\ \text{convex subsets of } M_*^+\end{array}\right\} \\ \varphi & \mapsto & \{\omega \in M_*^+ : \omega \le \varphi\}\end{array}$$