Homological Algebra

Ikhan Choi
Lectured by Takeshi Tsuji
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Contents

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1. Modules

References: Atsushi Shiho, Yukiyoshi Kawada

1.1. R-modules

Definition 1.1. Let R be a ring with 1. A (left) R-module is an abelian group M with a map $R \times M \to M : (a, x) \mapsto ax$ satisfying $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ab)x = a(bx)$, $1x = x$.

Example 1.2. (a) Every abelian group is a $\mathbb{Z}$-module. The R-module structures on an abelian group M has 1-1 correspondence with the ring homomorphisms $R \to \mathrm{End}_{\mathbb{Z}}(M)$.

(b) $M = C^\infty(\mathbb{R})$, $R = \mathbb{R}[T]$ a polynomial ring, $R \times M \to M : (P(T), f(x)) \mapsto P(\frac{d}{dx})f(x)$.

Definition 1.3. A (left) R-submodule of M is a subgroup $N \subset M$ such that $ax \in N$ for $a \in R$, $x \in N$. A (left) R-homomorphism is a group homomorphism $M \to N$ which preserves the action of R.

Example 1.4. (a) $M = C^\infty(\mathbb{R})$, $R = \mathbb{R}[T]$, then $\varphi : M \to M : f(x) \mapsto f(x+1)$ is an R-homomorphism.

Definition 1.5. Let $f : M \to N$ be an R-homomorphism. The kernel of f is $\ker f := \{x \in M : f(x) = 0\} \xrightarrow{i} M$, and the cokernel of f is $N \xrightarrow{p} \operatorname{coker} f := N/\operatorname{im} f$, where the image is $\operatorname{im} f := \{f(x) \in N : x \in M\} \xrightarrow{j} N$.

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & M \arrow[rr, "f"] \arrow[dr, "f"'] & & N \arrow[r, "p"] & \operatorname{coker} f \\
& & \operatorname{im} f \arrow[ur, "j"'] & &
\end{tikzcd}$$

On each of them, there is a unique R-module structure such that the each map i, j, p becomes an R-homomorphism respectively.

Theorem 1.6 (Universal property)**.** *For the above setting, note that $fi = 0$ and $pf = 0$. If an R-homomorphism $g : M' \to M$ satisfies $fg = 0$, then there is a unique R-homomorphism $h : M' \to \ker f$ such that $g = ih$. If an R-homomorphism $g : N \to N'$ satisfies $gf = 0$, then there is a unique R-homomorphism $h : \operatorname{coker} f \to N'$ such that $g = hp$.*

1.1 Commutative diagrams and exact sequences

Definition 1.7 (Diagram)**.** Among some R-modules suppose we have R-homomorphisms as the following diagram:

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "f_3"'] & M_2 \arrow[d, "g_1"] \arrow[dl, "f_2"'] \\
M_3 \arrow[r, "g_2"] & M_4
\end{tikzcd}.$$

Then, if the compositions sharing each source and target coincide, then we say the diagram is commutative. For example, we say the triangle formed by M_2, M_3, M_4 is commutative iff $g_1 = g_2 f_2$.

Definition 1.8 (Sequence)**.** A sequence is a diagram of R-modules placed linearly as

$$\cdots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \longrightarrow \cdots.$$

If $\operatorname{im} f_n = \ker f_{n+1}$ for all n, then we say the sequence is exact.

Example 1.9. (a) $f : M \to N$ is injective iff $0 \to M \xrightarrow{f} N$ is exact. $f : M \to N$ is surjective iff $M \xrightarrow{f} N \to 0$ is exact.

(b)

$$0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} f \longrightarrow 0$$

is exact.

(c)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is exact.

(d)

$$0 \longrightarrow \mathbb{R}\cos x \oplus \mathbb{R}\sin x \xrightarrow{n} C^\infty(\mathbb{R}) \xrightarrow{\frac{d^2}{dx^2}+1} C^\infty(\mathbb{R}) \longrightarrow 0$$

is exact.

Proposition 1.10 (Five lemma). *Suppose each row is exact in the folloing commutative diagram:*

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & M_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & M_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & M_5 \arrow[d, "h_5"] \\
N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & N_4 \arrow[r, "g_4"] & N_5.
\end{tikzcd}$$

Then,

(a)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, hook] & {} \arrow[d, dashed, hook] & {} \arrow[d, hook] \\
{} & {} & {} & {}
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, dashed, two heads] & {} \arrow[d, two heads] & {} \arrow[d, hook] \\
{} & {} & {} & {}
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
{} \arrow[d, two heads] & {} \arrow[d, "\sim"] & {} \arrow[d, dashed, "\sim"] & {} \arrow[d, "\sim"] & {} \arrow[d, hook] \\
{} & {} & {} & {} & {}
\end{tikzcd}$$

Proof. (a) We will show $x \in \ker h_3$ is in the image of $f_2 f_1$: $h_3(x) = 0 \implies f_3(x) = 0 \implies x = f_2(y) \implies g_2 h_2(y) = 0 \implies h_2(y) = g_1(z) \implies z = h_1(u) \implies f_1(u) = y$. Then, $x = f_2(y) = f_2 f_1 = 0$.

(b) Similar.

(c) Clear. $\square$

Proposition 1.11 (Snake lemma). *Suppose the second and the third rows are exact in the following commutative diagram:*

$$\begin{tikzcd}
 & \ker h_1 & \ker h_2 & \ker h_3 & \\
 & M_1 & M_2 & M_3 & 0 \\
0 & N_1 & N_2 & N_3 & \\
 & \operatorname{coker} h_1 & \operatorname{coker} h_2 & \operatorname{coker} 3 &
\end{tikzcd}$$

(a) *There is* $\delta : \ker h_3 \to \operatorname{coker} h_1$ *such that*

$$\ker h_1 \xrightarrow{k_1} \ker h_2 \xrightarrow{k_2} \ker h_3 \xrightarrow{\delta} \operatorname{coker} h_1 \xrightarrow{l_1} \operatorname{coker} h_2 \xrightarrow{l_2} \operatorname{coker} 3$$

is exact. Here k_1, k_2, l_1, l_2 *are induced from* f_1, f_2, g_1, g_2*, respectively. The element* $\delta(x)$ *is determined by* u *such that* $x = f_2(y)$*,* $z = h_2(y)$*,* $z = g_1(u)$*, and we can check that* u *does not depend on the choice of* y*.*

(b)

Proof. (a) We have to show the well-definedness of δ, $\ker \subset \operatorname{im}$, and $\operatorname{im} \subset \ker$. Skip. □

In the general abelian cateogies, the five lemma and the snake lemma hold but the proofs become more complicated.

1.2 Direct sum, direct product, inductive limit, direct limit

Definition 1.12. Let M_λ be a family of R-modules. The direct product is

$$\prod_\lambda M_\lambda := \{(x_\lambda) : x_\lambda \in M_\lambda\} \twoheadrightarrow M_\lambda,$$

and the direct sum is the submodule of the direct product such that

$$\bigoplus_\lambda M_\lambda := \{(x_\lambda) : x_\lambda = 0 \text{ but finitely many}\} \hookleftarrow M_\lambda$$

Proposition 1.13 (Universal property)**.** (a) *For* $f_\mu : M_\mu \to N$ *there is unique* $f : \bigoplus_\lambda M_\lambda \to N$ *such that* $f i_\mu = f_\mu$*.*

(b) *For* $g_\mu : N \to M_\mu$ *there is unique* $g : N \to \prod_\lambda M_\lambda$ *such that* $p_\mu g = g_\mu$*.*

Remark 1.14. (a) The direct sum and direct product is unique up to isomorphism by the universal property.

(b) For R-homomorphisms $f_\lambda : M_\lambda \to N_\lambda$ we can induce $\prod_\lambda f_\lambda : \prod_\lambda M_\lambda \to \prod_\lambda N_\lambda$ and $\bigoplus_\lambda f_\lambda : \bigoplus_\lambda M_\lambda \to \bigoplus_\lambda N_\lambda$.

(c) In the category of modules, even for infinite indices, direct product and sum commute with the kernel, cokernel, and image. In an abelian category, we may not have infinie direct product/sum.

(d) exactness also preserved under products and sums

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Let $(\Lambda, \prec)$ be a totally ordered set. By a direct system, we refer the family of R-modules M_λ for each $\lambda \in \Lambda$ and the family of R-homomorphisms $\tau_{\mu\lambda} : M_\lambda \to M_\mu$ for $\lambda \prec \mu$ such that $\tau_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$ and $\tau_{\kappa\lambda} = \tau_{\kappa\mu}\tau_{\mu\lambda}$ for $\lambda \prec \mu \prec \kappa$.

Example.1.3.3.

(a) Let $\Lambda = \mathbb{N}$ and $n \prec m \Leftrightarrow n \mid m$, $M_n = \mathbb{Z}$ and $\tau_{mn}(z) : M_n \to M_m : z \mapsto (m/n)z$.

(b) Let M be a R-module, $\{M_\lambda\}$ are finitely generated R-submodules of M, and $\lambda \prec \mu \Leftrightarrow M_\lambda \subset M_\mu$, with $\tau_{\mu\lambda}$ inclusions.

Definition.

$$\varinjlim M_\lambda = \varinjlim(M_\lambda, \tau_{\mu\lambda}) := \operatorname{coker}\Big(\bigoplus_{\substack{(\lambda,\mu)\in\Lambda \\ \lambda\prec\mu}} M_\lambda \xrightarrow{\Phi} \bigoplus_{\lambda\in\Lambda} M_\lambda\Big),$$

where $\Phi((x_{\lambda\mu})) = \sum_{\lambda\prec\mu} \iota_\mu\tau_{\mu\lambda}(x_{\lambda\mu}) - \iota_\lambda(x_{\lambda\mu})$, and $\iota_\lambda : M_\lambda \to \bigoplus_\lambda M_\lambda$ is a componentwise embedding. That is, we want to identify $x \in M_\lambda$ and $\tau_{\mu\lambda}(x) \in M_\mu$ with the map Φ.

Proposition.1.3.4. *Let* $\tau_\mu : M_\mu \xrightarrow{\iota_\mu} \bigoplus_\lambda M_\lambda \twoheadrightarrow \lim_\longrightarrow M_\lambda$.

(a) $\tau_\mu = \tau_\kappa \tau_{\kappa\mu}$.

(b) $M_\mu \xrightarrow{f_\mu} N$ *for* $\mu \in \Lambda$ *are R-homomorphisms, and they satisfy* $f_\mu = f_\kappa\tau_{\kappa\mu}$. *Then, there is a unique* $f : \lim_\longrightarrow M_\lambda \to N$ *such that* $f_\mu = f\tau_\mu$

For each example in 1.3.3, $\mathbb{Q}$ and M are the direct limits because it satisfies the universal property (1.3.4(b))

Remark. (1) The direct limit is unique by the universal property up to isomorphism.

(2) If $f_\lambda : M_\lambda \to M'_\lambda$ are R-homomorphism such that

$$\begin{array}{ccc} M_\lambda & \xrightarrow{f_\lambda} & M\lambda' \\ \downarrow & & \downarrow \\ M_\mu & \xrightarrow{f_\mu} & M'_\mu \end{array}$$

commutes for all $\lambda \prec \mu$, then there is a unique f such that

$$\begin{array}{ccccccc} \bigoplus_{\lambda\prec\mu} M_\lambda & \longrightarrow & \bigoplus_\lambda M_\lambda & \longrightarrow & \lim_\longrightarrow M_\lambda & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \exists! f} & & \downarrow \\ \bigoplus_{\lambda\prec\mu} M'_\lambda & \longrightarrow & \bigoplus_\lambda M'_\lambda & \longrightarrow & \lim_\longrightarrow M'_\lambda & \longrightarrow & 0 \end{array}$$

commutes, and f is denoted by $\lim_\longrightarrow f_\lambda$. It is by the universal property of cokernel.

Definition.1.3.6. A preordered set Λ is a directed set if $\forall \lambda, \lambda' \in \Lambda$, there is $\mu \in \Lambda$ such that $\lambda, \lambda' \prec \mu$.

Proposition. *If* Λ *is a directed set, then there is a 1-1 correspondence*

$$\Big(\coprod_\lambda M_\lambda\Big)/\sim \longrightarrow \varinjlim M_\lambda : [x_\lambda] \mapsto \tau_\lambda(x_\lambda),$$

where $x_\lambda \sim y_{\lambda'}$ *iff there is* $\mu \succ \lambda, \lambda'$ *such that* $\tau_{\mu\lambda}(x_\lambda) = \tau_{\mu\lambda'}(y_{\lambda'})$.

Proposition. *If*

$$L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \longrightarrow 0$$

is exact, then

$$\operatorname{colim} L_\lambda \longrightarrow \operatorname{colim} M_\lambda \longrightarrow \operatorname{colim} N_\lambda \longrightarrow 0$$

is exact.

Proof. The only non-trivial part is the exactness at $\operatorname{colim} M_\lambda$. We can prove it by diagram chasing. $\square$

Example. Examples of inverse limit

(a) projection $\mathbb{Z}/p^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ for $m > n$.

(b) restriction $C^\infty((-r, r)) \to C^\infty((-r', r')$ for $r' > r$.

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Example. Limit preserves injectivity, but not surjectivity: although the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\
\mathbb{Z}_p \arrow[r] & \cdots \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

commutes, but the induced map $\mathbb{Z} \to \mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ is not surjective because we have an element $x \in \mathbb{Z}_p$ such that for $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ we have $\pi_n(x) \equiv 1 \pmod{p^n}$ for all n.

Lemma (Mittag-Leffler condition)**.** *Let*

$$0 \to M_n \to N_n \to L_n \to 0$$

be a sequence of exact sequences. Suppose (M_n) satisfies that for each n we have a eventually constant monotonically decreasing sequence

$$M_n \supset \pi_{n,n+1}(M_{n+1}) \supset \pi_{n,n+2}(M_{n+2}) \supset \cdots$$

of submodules of M_n. Then,

$$0 \to \lim M_n \to \lim N_n \to \lim L_n \to 0.$$

Note that when we consider the seuqence of kernels $p^n\mathbb{Z}$ of the maps $\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ in the above example, we can check the sequence does not satisfy the Mittag-Leffler condition.

1.4. Properties of Hom

From now on, we always let R be a commutative ring and M, N be a R-modules. Define

$$\mathrm{Hom}_R(M,N) := \{f : M \to N, R\text{-homomorphism}\}.$$

It is an R-module, which is not the case if R is not commutative. If $\varphi : N_1 \to N_2$ is an R-homomorphism, then

$$\mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2) : f \mapsto \varphi \circ f$$

is an R-homomorphism. If $\psi : M_1 \to M_2$ is an R-homomorphism, then

$$\mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N) : f \mapsto f \circ \psi$$

is an R-homomorphism.

Proposition.1.4.1.

(a) *If*

$$0 \to N_1 \to N_2 \to N_3$$

is exact, then

$$0 \to \mathrm{Hom}_R(M,N_1) \to \mathrm{Hom}_R(M,N_2) \to \mathrm{Hom}_R(M,N_3)$$

is exact.

(b) *If*

$$M_1 \to M_2 \to M_3 \to 0$$

is exact, then

$$0 \to \mathrm{Hom}_R(M_3,N) \to \mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N)$$

is exact.

Proof. (a) If $f_2 \in \mathrm{Hom}_R(M, N_2)$ satisfies $\varphi_2 \circ f_2 = 0$, then by the universal property there exists unique $f_1 : M \to N_1$ such that the diagram

$$\begin{tikzcd} & & M \arrow[dl, dashed, "\exists! f_1"'] \arrow[d, "f_2"] \arrow[dr] & \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r, "\varphi_2"] & N_3 \end{tikzcd}$$

commutes.

(b) Similar. □

Example. For

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

The maps

$$0 \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

and

$$\mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\circ(\cdot n)} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

are not surjective.

1.5. Projective modules

Definition.1.5.1. An R-module is said to be *projective* if for every surjective $\varphi : N_1 \twoheadrightarrow N_2$ and for every $f : M \to N_2$, there is a map $\widetilde{f} : M \to N_1$ such that

$$\begin{tikzcd} & M \arrow[dl, dashed, "\widetilde{f}"'] \arrow[d, "f"] \\ N_1 \arrow[r, two heads] & N_2 \end{tikzcd}$$

commutes, equivalently,

$$\mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to 0$$

is exact for every exact $N_1 \to N_2 \to 0$.

Proposition.1.5.2. *If M is a projective module, then $\mathrm{Hom}_R(M, -)$ is an exact functor.*

Proposition.1.5.3. *A direct sum of R-modules is projective iff its summands are all projective. In particular, a free R-module is projective.*

Corollary.*1.5.4. As a corollary, a module M is projective if and only if there is another module N such that $M \oplus N$ is free.*

Proof. (⇒) Take generators of $\{e_\lambda\}_\lambda$ of M. Then, for

$$f : \bigoplus_\lambda R \twoheadrightarrow M : (a_\lambda) \mapsto \sum_\lambda a_\lambda e_\lambda,$$

we have an exact sequence

$$0 \to \ker f \to \bigoplus_\lambda R \to M \to 0,$$

which is right split by applying the definition of projective modules to extend the codomain of $\mathrm{id}_M : M \to M$.

(⇐) Clear from Proposition 1.5.3. □

Corollary.*1.5.5. Let R be a PID. Then, since a submodule of a free module is free, so a module is projective if and only if it is free.*

1.6. Injective modules

Definition.1.6.1. An R-module is said to be injective if for every injective $\varphi : N_1 \hookrightarrow N_2$ and for every $g : N_1 \to M$, there is a map $\widetilde{g} : N_2 \to M$ such that

$$\begin{tikzcd} N_1 \arrow[r, hook, "\varphi"] \arrow[d, "g"] & N_2 \arrow[dl, dashed, "\widetilde{g}"] \\ M & \end{tikzcd}$$

commutes, equivalently,

$$\mathrm{Hom}_R(N_2, M) \to \mathrm{Hom}_R(N_1, M) \to 0$$

is exact for every exact $0 \to N_1 \to N_2$.

Proposition.1.6.3. *An R-module M is injective iff the restriction* $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M)$ *is surjective for every ideal I of R.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) Suppose there is $x \in N_2$ such that $N_2 = N_1 + Rx$. Define an ideal I of R such that there is an exact sequence

$$0 \to I \to N_1 \oplus R \to N_2 \to 0,$$

in which the first map sends b to $(-bx, b)$ and the second map sends (y, a) to $y + ax$. Define $h : I \to M$ by $h(b) := g(bx)$ and extend it to $\widetilde{h} : R \to M$. Define $\widetilde{g} : N_2 \to M$ by $\widetilde{g}(y + ax) := g(y) + \widetilde{h}(a)$. We can check it is well-defined from the exactness of the above defining sequence of I. (To be continued..) □

Corollary.1.6.4. *If R is a PID, then an R-module M is injective iff for all $0 \neq a \in R$ the map $M \xrightarrow{\cdot a} M$ is surjective.*

Proof. Let I be an ideal. If $I = 0$, then clear. If not, we have $I = aR$ for some $0 \neq a \in R$. Then, the restriction $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, R)$ is surjective if and only if

$$\begin{array}{ccccccc} M & \xrightarrow{\sim} & \mathrm{Hom}(R, M) & \to & \mathrm{Hom}(aR, M) & \xrightarrow{\sim} & aM \\ m & \mapsto & (1 \mapsto m) & \mapsto & (a \mapsto am) & \mapsto & am \end{array}$$

is surjective. □

Example. If $R = \mathbb{Z}$, then $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, and $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are not injective.

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Proof of 1.6.3. Let $\mathcal{S}$ be the set of all pairs (N,h) such that $N_1 \subset N \subset N_2$ and

$$\begin{tikzcd} N_1 \arrow[r, hook] \arrow[d] & N \arrow[dl] \\ M & \end{tikzcd}$$

commutes, and define a partial order $\prec$ such that $(N,h) \prec (N',h')$ if ando only if

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & N' \arrow[dl] \\ M & \end{tikzcd}$$

commutes. Since the union of a chain belongs to $\mathcal{S}$, $\mathcal{S}$ has a maximal element (N_0,h_0) by Zorn's lemma. If $N_0 \subsetneq N_2$, then by taking $x \in N_2 \setminus N_0$, we can show N_0 is not maximal, so $N_0 = N_2$. $\square$

Proposition.1.6.5. *Let M_λ be R-modules, and M be their product. Then, M is injective if and only if every M_λ is injective.*

Proof. Apply the definition on the following diagram to show the first row is surjective:

$$\begin{tikzcd} \mathrm{Hom}_R(N_2,\prod_\lambda M_\lambda) \arrow[r] \arrow[d, "="] & \mathrm{Hom}_R(N_1,\prod_\lambda M_\lambda) \arrow[d, "="] \\ \prod_\lambda \mathrm{Hom}_R(N_2,M_\lambda) \arrow[r] & \prod_\lambda \mathrm{Hom}_R(N_1,M_\lambda). \end{tikzcd}$$

$\square$

Proposition.1.6.6. *If M is injective Z-module, then $\mathrm{Hom}_\mathbb{Z}(R,M)$ is an injective R-module.*

Lemma.1.6.7. *Let N be an R-module and M be a $\mathbb{Z}$-module. Then, $\mathrm{Hom}_\mathbb{Z}(R,M)$ is an R-module, and there is a bijection*

$$\mathrm{Hom}_R(N,\mathrm{Hom}_\mathbb{Z}(R,M)) \cong \mathrm{Hom}_\mathbb{Z}(N,M).$$

Proof of Proposition 1.6.6. Apply Lemma 1.6.7 to show the first row is surjective:

$$\begin{tikzcd} \mathrm{Hom}_R(N_2,\mathrm{Hom}_\mathbb{Z}(R,M)) \arrow[r] \arrow[d, "="] & \mathrm{Hom}_R(N_1,\mathrm{Hom}_\mathbb{Z}(R,M)) \arrow[d, "="] \\ \mathrm{Hom}_\mathbb{Z}(N_2,M) \arrow[r] & \mathrm{Hom}_\mathbb{Z}(N_1,M). \end{tikzcd}$$

$\square$

Theorem.1.6.8. *Every R-module M is embedded in an injective R-module.*

Proof. Suppose $R = \mathbb{Z}$. The surjectivity of

$$\bigoplus_\lambda \mathbb{Z} \twoheadrightarrow \mathrm{Hom}_\mathbb{Z}(M,\mathbb{Q}/\mathbb{Z})$$

implies

$$\mathrm{Hom}_\mathbb{Z}(\mathrm{Hom}_\mathbb{Z}(M,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_\mathbb{Z}(\bigoplus_\lambda \mathbb{Z},\mathbb{Q}/\mathbb{Z}) = \prod_\lambda \mathbb{Q}/\mathbb{Z}.$$

Then, it suffices to prove the canonical map

$$M \to \mathrm{Hom}_\mathbb{Z}(\mathrm{Hom}_\mathbb{Z}(M,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}))$$

is injective. For non-zero $x \in M$, by the injectivity of $\mathbb{Q}/\mathbb{Z}$, we can extend a $\mathbb{Z}$-homomorphism $f : \mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$ satisfying $f(x) \neq 0$ to a $\mathbb{Z}$-homomorphism $\widetilde{f} : M \to \mathbb{Q}/\mathbb{Z}$ satisfying $\widetilde{f}(x) = f(x) \neq 0$. Threrfore, we are done.

Now let R be arbitrary commutative ring. Consider an R-homomorphism

$$\Phi : M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) : x \mapsto (a \mapsto ax),$$

which is easily checked to be injective by putting $a = 1$. Let M' be an injective $\mathbb{Z}$-module with an injective $\mathbb{Z}$-homomorphism $M \to M'$, and it induces

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M'). \qquad \square$$

1.7. Tensor products

Definition.1.7.1. Let R be a commutative ring, and M_1, M_2, N be R-modules. Let $\Phi : M_1 \times M_2 \to N$ be an R-bilinear map. If R is non-commutative, then M_1 and M_2 are set to be right and left R-modules resepctively, and Φ is just a $\mathbb{Z}$-bilinear map but required to satisfy an additional condition $\Phi(-a, -) = \Phi(-, a-)$. Such Φ is called a balanced product.

There is an R-module such that the following universal property holds: for every balanced product $\Phi : M_1 \to M_2 \to N$, there is a unique R-homomorphism

$$M_1 \times M_2 \xrightarrow{\otimes} M$$

$$N$$

Then, M is called the tensor product of M_1 and M_2.

Proof. Let $\widetilde{M}$ be a free R-module generated by $M_1 \times M_2$. Let $\widetilde{M}_0$ be a R-subodule of $\widetilde{M}$ generated by

$$\begin{aligned} &(p + p', q) - (p, q) - (p', q), \quad &&(p, q + q') - (p, q) - (p, q'), \\ &(ap, q) - a(p, q), &&(p, aq) - a(p, q). \end{aligned}$$

Let $M := \widetilde{M}/\widetilde{M}_0$. Then, it satisfies the universal property(Exercise!). $\square$

Remark 4.1.*1.7.2.*

(a) The tensor product is unique.

(b) $M_1 \otimes M_2$ is an R-module.

(c) For $f_1 : M_1 \to M_1'$ and $f_2 : M_2 \to M_2'$, we have an R-homomorphism $f_1 \otimes f_2 : M_1 \otimes M_2 \to M_1' \otimes M_2'$ defined by

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "\otimes"] \arrow[d] & M_1 \otimes_R M_2 \arrow[d, dashed, "\exists! f_1 \otimes f_2"] \\ M_1' \times M_2' \arrow[r, "\otimes"] & M_1' \otimes_R M_2'. \end{tikzcd}$$

Proposition 4.2.1.7.3.

(a) $R \otimes_R M \cong M$.

(b) $M \otimes_R R \cong M$.

(c) $(\bigoplus_\lambda M_\lambda) \otimes_R N \cong \bigoplus_\lambda (M_\lambda \otimes_R N)$.

(d) $N \otimes_R (\bigoplus_\lambda M_\lambda) \cong \bigoplus_\lambda (N \otimes_R M_\lambda)$.

Proof. Use the universal properties for the right-hand sides. □

Proposition 4.3.1.7.4. *Let R be commutative.*

(a) $(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes_R M_3)$.

(b) $M_1 \otimes_R M_2 \cong M_2 \otimes_R M_1$.

Proof. (a) Use the universal property.
(b) Omitted. □

Proposition 4.4.1.7.5. *If*

$$M_1 \xrightarrow{f} M_2 \to M_3 \to 0$$

is exact, then

$$N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \to 0$$

is exact.

Proof. We can construct a unique Ψ by the universal property of $N \otimes M_2$ so that the following diagram commutes.

$$\begin{tikzcd}
N \otimes_R M_1 \arrow[r] & N \otimes_R M_2 \arrow[r] & \operatorname{coker}(\mathrm{id}_N \otimes f) \arrow[r] & 0 \\
N \times M_1 \arrow[r] \arrow[u, "\otimes"] & N \times M_2 \arrow[r, two heads] \arrow[u, "\otimes"] & N \times M_3 \arrow[u, dashed, "\exists ! \Psi"] &
\end{tikzcd}$$

Therefore, we can check $\operatorname{coker}(\mathrm{id}_N \otimes f)$ satisfies the universal property. □

Example. We have

$$M/IM \cong (R \otimes M)/(I \otimes M) \cong (R/I) \otimes M.$$

If $M = R/I$, then

$$I/I^2 \to R/I \to (R/I)^{\otimes 2} \to 0$$

is exact, and the first map is not injective.

Direct limit.

$$(\operatorname*{colim}_{\lambda} N_\lambda) \otimes_R M \cong \operatorname*{colim}_{\lambda}(N_\lambda \otimes_R M).$$

Proof.

$$\begin{tikzcd}
(\bigoplus_{\lambda<\mu} N_\lambda) \otimes_R M \arrow[r] & (\bigoplus_{\lambda} N_\lambda) \otimes_R M \arrow[r] & \operatorname{coker} \arrow[r] & 0 \\
\bigoplus_{\lambda<\mu}(N_\lambda \otimes_R M) \arrow[r] \arrow[u, "\otimes"] & \bigoplus_{\lambda}(N_\lambda \otimes_R M) \arrow[r, two heads] \arrow[u, "\otimes"] & \operatorname{colim}_\lambda(N_\lambda \otimes_R M) \arrow[u, dashed, "\exists ! \Psi"] &
\end{tikzcd}$$

□

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1.8. Flat modules

Definition (1.8.1)**.** Let R be a commutative ring and M be an R-module. We say M is flat if $\mathrm{id} \otimes f : M \otimes N_1 \to M \otimes N_2$ is injective for every injective $f : N_1 \hookrightarrow N_2$. If R is noncommutative, consider $- \otimes M$ and $M \otimes -$ for left and right modules M respectively.

Example.

(a) A free R-module is flat since tensor product and direct sum satisfy the distribution law.

(b) A direct limit of flat modules is flat. For example, $\mathbb{Q} = \operatorname{colim} \frac{1}{n}\mathbb{Z}$ is flat.

Proposition (1.8.2)**.** *If M is flat, then $M \otimes_R -$ is an exact functor.*

Proposition (1.8.3)**.** *Let M be a left R-module. Then, we can give $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ a right R-module structure by $(fa)(x) = f(ax)$ for $a \in R$ and $x \in M$. For an injective right R-homomorphism $N_1 \hookrightarrow N_2$ between right R-modules, $N_1 \otimes M \to N_2 \otimes M$ is injective if and only if*

$$\mathrm{Hom}_R(N_2, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \xrightarrow{-\circ f} \mathrm{Hom}_R(N_1, \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}))$$

is surjective.

Proof. We first observe that

$$\mathrm{Hom}_{\mathbb{Z}}(N \otimes M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})).$$

Also we have the following from the fact that $\mathbb{Q}/\mathbb{Z}$ is injective: for $\mathbb{Z}$-module homomorphism $f : L_1 \to L_2$, f is injective if and only if $\mathrm{Hom}_{\mathbb{Z}}(L_1, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(L_2, \mathbb{Q}/\mathbb{Z})$ is surjective. □

Remark. If $N \curvearrowleft R \curvearrowleft M \curvearrowleft S$ and $L \curvearrowleft S$, then $\mathrm{Hom}_S(N \otimes_R M, L) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, L))$.

Corollary (1.8.4)**.** *For a left R-module M, M is flat if and only if $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is injective.*

Corollary (1.8.5)**.** *For a right R-module M, M is flat if and only if $I \otimes_R M \to R \otimes_R M = M$ is injective for every right ideal $I \subset R$*

Corollary (1.8.6)**.** *Let R be a PID. Then, M is flat if and only if $M \xrightarrow{\cdot a} M$ is injective for every $a \in R$.*

Proof.

$$M = R \otimes M \cong I \otimes M \hookrightarrow R \otimes M = M.$$

□

2. Complexes

2.1. Definitions

Definition (2.1.1)**.** A chain complex is a pair of a (bilateral) sequence of R-modules C_n and a (bilateral) sequence of R-homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$.

A cochain complex is a pair of a (bilateral) sequence of R-modules C^n and a (bilateral) sequence of R-homomorphisms $d^n : C^n \to C^{n+1}$ such that $d^{n+1} \circ d^n = 0$.

Example (2.1.2)**.** The simplicial homology and the de Rham cohomology.

Remark. It is frequently assumed to be $C_n = 0$ and $C_n = 0$ for negative n.

Definition (2.1.3)**.** Let $C_\bullet$ be a chain complex. Then, $Z_n(C_\bullet) := \ker \partial_n$, $B_n(C_\bullet) := \operatorname{im} \partial_{n+1}$, and $H_n(C_\bullet) := Z_n(C_\bullet)/B_n(C_\bullet)$. For cochain complexes, we can do the same thing.

A chain map between two chain complexes $C_\bullet$ and $C'_\bullet$ is a sequence $f_\bullet = (f_n : C_n \to C'_n)$ such that $\partial'_{n-1} \circ f_n = f_{n-1} \circ \partial_n$. Then, we can check it induces $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$.

A short sequence of chain complexes is said to be exact if the short sequence at each n is exact.

Theorem (2.1.4)**.** *If*

$$0 \to C_\bullet \to C'_\bullet \to C''_\bullet \to 0$$

is exact, then there is a exact sequence

$$\cdots \to H_n(C_\bullet) \to H_n(C'_\bullet) \to H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \to \cdots .$$

Proof.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & Z'_n \arrow[r] \arrow[d] & Z''_n \arrow[d] & \\
0 \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C'_n \arrow[r] \arrow[d] & C''_n \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & C'_{n-1} \arrow[r] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
& C_{n-1}/B_{n-1} \arrow[r] & C'_{n-1}/B'_{n-1} \arrow[r] & C''_{n-1}/B''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

The snake lemma implies the exactness of the first and fourth rows.

$$\begin{tikzcd}
& H_n \arrow[r] \arrow[d] & H'_n \arrow[r] \arrow[d] & H''_n \arrow[d] & \\
& C_n/B_n \arrow[r] \arrow[d] & C'_n/B'_n \arrow[r] \arrow[d] & C''_n/B''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & Z'_{n-1} \arrow[r] \arrow[d] & Z''_{n-1} \arrow[d] & \\
& H_{n-1} \arrow[r] & H'_{n-1} \arrow[r] & H''_{n-1} &
\end{tikzcd}$$

The snake lemma implies the desired boundary map δ_n. $\square$

2.2. Homotopy

Definition (2.2.1)**.** Let $f, g : C \to C'$ be chain maps. A sequence $k = k_\bullet = (k_n : C_n \to C'_{n+1})$ of R-homomorphisms such that $f_n - g_n = k_{n-1} \circ \partial_n + \partial'_{n+1} \circ k_n$ is called a homotopy between f and g.

Proposition (2.2.2)**.** *If $f, g : C_\bullet \to C'_\bullet$ are homotopic, then $H_n(f) = H_n(g)$.*

Example.

(a) Let K be an algebraic extension over $\mathbb{Q}$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[dl] & K[x] \arrow[r, "\frac{d}{dx}"] \arrow[dl, "k^0"'] & K[x] \arrow[r] \arrow[dl, "k^1"'] & 0 \arrow[dl] \\
0 \arrow[r] & K \arrow[r] & K[x] \arrow[r, "\frac{d}{dx}"] & K[x] \arrow[r] & 0
\end{tikzcd}$$

Define

$$k^0(\sum_{n\geq 0} a_n x^n) := a_0, \qquad k^1(\sum_{n\geq 0} a_n x^n) := \sum_{n\geq 0}(n+1)^{-1}a_{n+1}x^{n+1}.$$

Then, k is a homotopy between the zero and the identity, so the cohomology groups are all trivial. (cohomology groups of a exact cochain complex are trivial..?)

(b) Let S be a set and $C^n := \mathrm{Map}(S^{n+1}, M)$ for R-module M.

$$(d^n f)(x_0, \cdots, x_{n+1}) = \sum_{i=0}^{n}(-1)^i f(x_0, \cdots, \check{x}_i, \cdots, x_n).$$

then, id and 0 are homotopic.

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2.3. Double complexes

Definition. A double complex is a family of R-modules $\{C_{p,q}\}$ indexed by $(p,q) \in \mathbb{Z}^2$ together with R-homomorphisms $\partial^I_{p,q} : C_{p,q} \to C_{p-1,q}$ and $\partial^{II}_{p,q} : C_{p,q} \to C_{p,q-1}$ such that

(i) $(C_{\bullet,q}, \partial^I_{\bullet,q})$ and $(C_{p,\bullet}, \partial^{II}_{p,\bullet})$ are chain complexes,

(ii) $\partial^{II} \circ \partial^I + \partial^I \circ \partial^{II} = 0$. (anticommuting squares convention, convenient in defining the total complex)

For a double complex, we can define total complex by

$$T_n := \oplus_{p+q=n} C_{p,q}, \qquad \partial_n : T_n \to T_{n-1} : (a_{p,q})_{p+q=n} \mapsto (\partial^I(a_{p,q})) + (\partial^{II}(a_{p,q})),$$

and it satisfies the axiom of chain complex; $\partial^2 = 0$. The total complex is denoted by $\mathrm{Tot}^{\oplus}(C)$. We can also define with $\times$ instead of $\oplus$ to get $\mathrm{Tot}^{\Pi}(C)$. If $\mathrm{Tot}^{\oplus} = \mathrm{Tot}^{\Pi}$, then we write it as Tot.

Example. Let $C_\bullet$ and $C'_\bullet$ be chain complexes of right and left R-modules(resp.) for a commutative ring R. Then, $D_{p,q} := C_p \otimes_R C_q$ and $\partial^I_{p,q} = \partial_p \otimes \mathrm{id}$, $\partial^{II}_{p,q} = (-1)^p \,\mathrm{id} \otimes \partial_q$ define a double complex, and its total complex is denoted by $C \otimes_R C'$.

Example. Let $C_\bullet$ and $C'^\bullet$ be chain and cochain complexes R-modules for a commutative ring R. Then, $D_{p,q} := \mathrm{Hom}(C_p, C'^q)$ and $d^I_{p,q} = - \circ \partial_{p+1}$, $d^{II}_{p,q} = (-1)^{p+q+1} d^q \circ -$ define a double (cochain) complex, and its total complex is denoted by $\mathrm{Hom}(C, C')$.

Proposition (2.3.1)**.**

(a) *Let $f : C_{\bullet,\bullet} \to C'_{\bullet,\bullet}$; $f_{p,q} : C_{p,q} \to C'_{p,q}$ commutes with ∂^I and ∂^{II}. Suppose there is $N \in \mathbb{Z}$ such that $p < N$ or $q < N$ imply $C_{p,q} = 0$ and $C'_{p,q} = 0$. Suppose also that $H_n(C_{\bullet,q}, \partial^I) \cong H_n(C'_{\bullet,q}, \partial^I)$ for each $n \in \mathbb{Z}$ and $q \in \mathbb{Z}$. Then, $H_n(\mathrm{Tot}(C)) \cong H_n(\mathrm{Tot}(C'))$.*

(b) *Let $f : C^{\bullet,\bullet} \to C'^{\bullet,\bullet}$. Suppose there is $N \in \mathbb{Z}$ such that $p < N$ or $q < N$ imply $C^{p,q} = 0$ and $C'^{p,q} = 0$. If $H^n(C^{\bullet,q}) \cong H^n(C'^{\bullet,q})$ for each $n \in \mathbb{Z}$ and $q \in \mathbb{Z}$, then $H^n(\mathrm{Tot}(C)) \cong H^n(\mathrm{Tot}(C'))$.*

Proof.

$$C^{\leq r}_{p,q} = \begin{cases} 0 & q > r \\ C_{p,q} & q \leq r \end{cases}$$

is a subcomplex of C. Then, we have an exact sequence

$$0 \to C^{\leq r-1} \to C^{\leq r} \to C^{(r)} \to 0$$

of double complexes. Taking Tot, we have

$$\begin{tikzcd}
{} \arrow[r] & H_n(\mathrm{Tot}(C^{\leq r-1})) \arrow[r] \arrow[d] & H_n(\mathrm{Tot}(C^{\leq r})) \arrow[r] \arrow[d] & H_n(\mathrm{Tot}(C^{(r)}_{p,q})) \arrow[r] \arrow[d, "\sim"] & {} \\
{} \arrow[r] & H_n(\mathrm{Tot}(C^{\leq r-1})) \arrow[r] & H_n(\mathrm{Tot}(C^{\leq r})) \arrow[r] & H_n(\mathrm{Tot}(C^{(r)})) \arrow[r] & {}
\end{tikzcd}.$$

Note that $H_n(\mathrm{Tot}(C^{(r)})) = H_{n-r}(C_{\bullet,r})$ gives the isomorphism at the third column. Then, use the five lemma inductively. $\square$

2.4. Ext and Tor

Let C be a chain complex of R-modules and M be an R-module. Then, $C \otimes M$ is a chain complex and $\mathrm{Hom}(C, M)$ is a cochain complex. In this case, we have:

(i) If M is flat, then $H_n(C \otimes_R M) \cong H_n(C) \otimes_R M$.

(ii) If M is injective, then $H_n(\mathrm{Hom}_R(C, M)) \cong \mathrm{Hom}_R(H^n(C), M)$.

We want to measure the failure of this.

Definition (2.4.1). Let M be an R-module.

(a) A *projective resolution* is an exact sequence

$$0 \leftarrow M \xleftarrow{\varepsilon} P_0 \xleftarrow{\partial_1} P_1 \xleftarrow{\partial_2} P_2 \leftarrow \cdots,$$

where P_n is a projective for each n.

(b) A *injective resolution* is an exact sequence

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots,$$

where I^n is a injective for each n.

Proposition (2.4.2). *Every R-module admits a projective resolution and an injective resolution.*

Proof. Every module has a surjection(injection) from(to) a free(injective) module. Then, for the kernel(cokernel) we can do same thing. □

Proposition (2.4.3). *Let $f : M \to M'$ be an R-homomorphism.*

(a) *If $(P_\bullet)$ and $(P'_\bullet)$ are projective resolutions, then there is a chain map $g : P \to P'$. If g and g' are two chain maps between P and P', then g and g' are homotopic.*

(b) *Same for injective resolution.*

Proof. (a) Lift f to get g_0. Restrict to kernel and lift g_0 to get g_1, and so on.
Restrict to kernel and lift $g_0 - g'_0$ to get h_0 □

For an injective resolution I of N, we define $\mathrm{Ext}^n_R(M, N) := H^n(\mathrm{Hom}_R(M, I^\bullet))$.
For a projective resolution P of M, we define $\mathrm{Tor}^R_n(M, N) := H_n(P_\bullet \otimes_R N)$.
(For a flat resolution F of M, we define $\mathrm{Tor}^R_n(M, N) := H_n(F_\bullet \otimes_R N)$).
They do not depend on the choice of resolutions.
For $f : M_1 \to M_2$, we have an induced homomorphism $\mathrm{Ext}^n_R(M_2, N) \to \mathrm{Ext}^n_R(M_1, N)$.
For $f : N_1 \to N_2$, we have an induced homomorphism $\mathrm{Tor}^R_n(M, N_1) \to \mathrm{Tor}^R_n(M, N_2)$.
functoriality.

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Theorem (2.4.4). *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules and N be an R-module. Then, there exist long exact sequences*

(a)

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \\ &\to \mathrm{Ext}^1_R(M_3, N) \to \mathrm{Ext}^1_R(M_2, N) \to \mathrm{Ext}^1_R(M_1, N) \\ &\to \mathrm{Ext}^2_R(M_3, N) \to \mathrm{Ext}^2_R(M_2, N) \to \mathrm{Ext}^2_R(M_1, N) \to \cdots . \end{aligned}$$

(b)

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3) \\ &\to \mathrm{Ext}^1_R(N, M_1) \to \mathrm{Ext}^1_R(N, M_2) \to \mathrm{Ext}^1_R(N, M_3) \\ &\to \mathrm{Ext}^2_R(N, M_1) \to \mathrm{Ext}^2_R(N, M_2) \to \mathrm{Ext}^2_R(N, M_3) \to \cdots . \end{aligned}$$

(c)

$$\begin{array}{ccccccc} \cdots \to \mathrm{Tor}^R_2(M_1, N) & \to & \mathrm{Tor}^R_2(M_2, N) & \to & \mathrm{Tor}^R_2(M_3, N) & \to & \\ \mathrm{Tor}^R_1(M_1, N) & \to & \mathrm{Tor}^R_1(M_2, N) & \to & \mathrm{Tor}^R_1(M_3, N) & \to & \\ M_1 \otimes_R N & \to & M_2 \otimes_R N & \to & M_3 \otimes_R N & \to & 0. \end{array}$$

(d)

$$\begin{array}{ccccccc} \cdots \to \mathrm{Tor}^R_2(N, M_1) & \to & \mathrm{Tor}^R_2(N, M_2) & \to & \mathrm{Tor}^R_2(N, M_3) & \to & \\ \mathrm{Tor}^R_1(N, M_1) & \to & \mathrm{Tor}^R_1(N, M_2) & \to & \mathrm{Tor}^R_1(N, M_3) & \to & \\ N \otimes_R M_1 & \to & N \otimes_R M_2 & \to & N \otimes_R M_3 & \to & 0. \end{array}$$

Proof. (a) For $N \to I^\bullet$ an injective resolution, we have a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(M_3, I^\bullet) \to \mathrm{Hom}_R(M_2, I^\bullet) \to \mathrm{Hom}_R(M_1, I^\bullet) \to 0.$$

(d) For $P_\bullet \to N$ a projective resolution, since a projective module is flat, we have a short exact sequence of chain complexes

$$0 \to P_\bullet \otimes_R M_1 \to P_\bullet \otimes_R M_2 \to P_\bullet \otimes_R M_3 \to 0. \qquad \square$$

Theorem (2.4.5). *We have a natural isomorphism* $\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}_R(P_\bullet, N))$ *for any projective resolution* $P_\bullet \to M$.

Recall that we have to see $\mathrm{Hom}_R(P_\bullet, N)$ as a cochain complex, although the index is placed at lower. The existence of the short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(P_\bullet, M_1) \to \mathrm{Hom}_R(P_\bullet, M_2) \to \mathrm{Hom}_R(P_\bullet, M_3) \to 0$$

implies that $H^n(\mathrm{Hom}_R(P_\bullet, M_i))$ enjoys a long exact sequence.

Let $C^{\bullet,\bullet}$ and $D^{\bullet,\bullet}$ be double complexes bounded below with respect to both directions, i.e. there is N such that if $p \le -N$ or $q \le -N$ then $C^{p,q} = D^{p,q} = 0$, and that $H^p(C^{\bullet,q}) \xrightarrow{\sim} H^p(D^{\bullet,q})$ for all p and q. Then, we have shown that $H^n(\mathrm{Tot}(C^{\bullet,\bullet})) \xrightarrow{\sim} H^n(\mathrm{Tot}(D^{\bullet,\bullet}))$ for every n.

Proposition.

(a) *Let $C^{\bullet,\bullet}$ be a double complex with $C^{p,\bullet} = 0$ for $p < 0$. Let $A^\bullet$ be a cochian complex. Assume $A^\bullet$ and $C^{p,\bullet}$ are bounded below for every p. Suppose $\varepsilon : A^\bullet \to C^{0,\bullet}$ is a cochain map such that*

$$A^q \to C^{0,q} \to C^{1,q} \to \cdots$$

is a resolution for every q. Then, $H^n(A^\bullet) \xrightarrow{\sim} H^n(\mathrm{Tot}(C^{\bullet,\bullet}))$.

(b) *Let $C^{\bullet,\bullet}$ be a double complex with $C^{\bullet,q} = 0$ for $q < 0$. Let $B^\bullet$ be a cochian complex. Assume $B^\bullet$ and $C^{\bullet,q}$ are bounded below for every q. Suppose $\varepsilon : B^\bullet \to C^{\bullet,0}$ is a cochain map such that*

$$B^p \to C^{p,0} \to C^{p,1} \to \cdots$$

is a resolution for every p. Then, $H^n(A^\bullet) \xrightarrow{\sim} H^n(\mathrm{Tot}(C^{\bullet,\bullet}))$.

(c) *Let $C_{\bullet,\bullet}$ be a double complex with $C_{p,\bullet} = 0$ for $p < 0$. Let $A_\bullet$ be a chian complex. Assume $A_\bullet$ and $C_{p,\bullet}$ are bounded below for every p. Suppose $\varepsilon : C_{0,\bullet} \to A_\bullet$ is a chain map such that*

$$\cdots \to C_{1,q} \to C_{0,q} \to A_q$$

is a resolution for every q. Then, $H_n(A_\bullet) \xleftarrow{\sim} H_n(\mathrm{Tot}(C_{\bullet,\bullet}))$.

(d) *Let $C_{\bullet,\bullet}$ be a double complex with $C_{\bullet,q} = 0$ for $q < 0$. Let $B_\bullet$ be a chian complex. Assume $B_\bullet$ and $C_{\bullet,q}$ are bounded below for every q. Suppose $\varepsilon : C_{\bullet,0} \to B_\bullet$ is a chain map such that*

$$\cdots \to C_{p,1} \to C_{p,0} \to B_p$$

is a resolution for every p. Then, $H_n(A_\bullet) \xleftarrow{\sim} H_n(\mathrm{Tot}(C_{\bullet,\bullet}))$.

Proof. Inflate $A^\bullet$ to construct $A^{\bullet,\bullet}$. □

Proof of Theorem 2.4.5. Let $P_\bullet \to M$ and $N \to I^\bullet$ be a projective and an injective resolution respectively. Let $A^q := \mathrm{Hom}_R(M, I^q)$, $B^p := \mathrm{Hom}_R(P_p, N)$, and $C^{p,q} := \mathrm{Hom}_R(P_p, I^q)$. Since $H^n(A^\bullet) = \mathrm{Ext}^n_R(M,N)$, enough to show $H^n(A^\bullet) = H^n(B^\bullet)$. We can define $\varepsilon : A^\bullet \to C^{0,\bullet}$ and $\delta : B^\bullet \to C^{\bullet,0}$. Apply the above proposition to obtain $H^n(A^\bullet) = H^n(\mathrm{Tot}(C^{\bullet,\bullet})) = H^n(B^\bullet)$. □

Proposition (2.4.6)**.** *Let M be an R-module.*

(a) *M is projective $\Leftrightarrow \mathrm{Ext}^n_R(M,N) = 0\ \forall n\ \forall N \Leftrightarrow \mathrm{Ext}^1_R(M,N) = 0\ \forall N$.*
(b) *M is injective $\Leftrightarrow \mathrm{Ext}^n_R(N,M) = 0\ \forall n\ \forall N \Leftrightarrow \mathrm{Ext}^1_R(N,M) = 0\ \forall N$.*
(c) *M is flat $\Leftrightarrow \mathrm{Tor}^R_n(M,N) = 0\ \forall n\ \forall N \Leftrightarrow \mathrm{Tor}^R_1(M,N) = 0\ \forall N$.*

Proof. (a) For (1)⇒(2), use 2.4.5. For (3)⇒(1), use 2.4.4. □

Lemma.

(a) $\mathrm{Ext}^n_R(\bigoplus_\lambda M_\lambda, N) = \prod_\lambda \mathrm{Ext}^n_R(M_\lambda, N)$.
(b) $\mathrm{Ext}^n_R(M, \prod_\lambda N_\lambda) = \prod_\lambda \mathrm{Ext}^n_R(M, N_\lambda)$.
(c) $\mathrm{Tor}^R_n(\bigoplus_\lambda M_\lambda, N) = \bigoplus_\lambda \mathrm{Tor}^R_n(M_\lambda, N)$.
(d) $\mathrm{Tor}^R_n(M, \bigoplus_\lambda N_\lambda) = \bigoplus_\lambda \mathrm{Tor}^R_n(M, N_\lambda)$.

Proposition (2.4.8)**.** *If R is PID, then $\mathrm{Ext}^n_R = \mathrm{Tor}^R_n = 0$ for $n \geq 2$.*

Proof. ($\mathrm{Ext} = 0$) Embed $N \hookrightarrow I_0$ into an injective. Then, $N \to I_0 \to I_1 \to 0$ is an injective resolution because the cokernel I_1 is also injective by the a-times map.

($\mathrm{Tor} = 0$) Consider $P_0 \twoheadrightarrow M$ from a free module. Then, P_0 is torsion-free, P_1 is also torsion-free, so the kernel P_1 is flat. (Or, recall that in PID a module is projective iff free, and that a submodule of a free module is free.) □

Example.

(a) $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$ is a projective resolution of $\mathbb{Z}/m\mathbb{Z}$. Then, taking $\mathrm{Hom}_{\mathbb{Z}}(-, N)$, we have $N \xrightarrow{m} N \to 0$. Thus $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) = H^n(N \xrightarrow{m} N \xrightarrow{0} N)$

(b)