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Analytic Methods in Open String Field Theory

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We review the basics of recent developments of analytic methods in open string field theory. We in particular explain Schnabl's analytic solution for tachyon condensation in detail, assuming only the basic knowledge on conformal field theory.

Subject Index: 128

Contents

§1. Introduction

One of the most important problems in theoretical physics today is to construct a consistent quantum theory including gravity. String theory is expected to provide an important clue to this problem, but its current formulation is not yet complete. First, it is defined only perturbatively with respect to a coupling constant, and we do not

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expect that the perturbative expansion has a finite radius of convergence. Second, only on-shell scattering amplitudes can be calculated and off-shell quantities such as correlation functions are not well defined in string theory. Third, since string theory is independently defined for each consistent background, it is not clear whether there is a universal set of degrees of freedom describing different backgrounds. We could say that we have not yet identified fundamental degrees of freedom of string theory.

One possible approach to these problems is to construct a field theory of strings, which is called *string field theory*. While it is a natural approach in view of the success of quantum field theory, it is possible, and perhaps likely, that string theory will turn out to be formulated nonperturbatively in a completely different way. For example, there has been a long history of developments in formulating string theory using matrix models. We also learned from the AdS/CFT correspondence that string theory on a class of backgrounds can be defined in terms of ordinary field theory without gravity. The quantum formulation of string field theory by path integral does not seem to be the best candidate for the final formulation of string theory.

On the other hand, string field theory can play a role complementary to other approaches in our quest for a consistent formulation of string theory. While string theory as a perturbation theory can only explore an infinitesimal neighborhood of each consistent background, we can in principle discuss the relationship of various backgrounds using string field theory. String field theory can also be thought of as a universal, effective theory when elementary excitations are string-like. To eliminate ghosts from such string-like excitations in unphysical directions, any covariant description in terms of a spacetime field theory would require gauge invariance. Then the gauge invariance seems to determine the interacting theory uniquely. This has been a guiding principle in constructing covariant string field theory. Since consistent backgrounds of string theory are described by conformal field theories in two dimensions, the classical equation of motion of string field theory determined by the spacetime gauge invariance should therefore reproduce this requirement of conformal invariance in the world-sheet perspective. We hope that deeper understanding of the relation between the world-sheet conformal invariance and the spacetime gauge invariance will help us reveal aspects of the non-perturbative theory behind the perturbative string theory.

Recently there have been impressive developments in the research of open string field theory since the construction of an analytic solution by Schnabl.[1] This review is intended to provide the gateway to this exciting area of research without assuming much background.[*] In fact, it is not even necessary to have detailed knowledge on string theory to appreciate the recent analytic development in open string field theory. On the other hand, the basic knowledge of conformal field theory (CFT) is required, but it is sufficient to be familiar with the contents explained in Chapter 2 of the textbook by Polchinski.[3] This review will be understood without much difficulty after learning the chapter, and we mostly follow the conventions of the textbook.

[*] For an exhaustive list of references, see the review by Fuchs and Kroyter.[2]

§2. The basics of open string field theory

2.1. *Construction of the free theory*

Quantum field theory is defined by an action such as

$$S = \int d^4x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi + \dots \Big] . \tag{2·1}$$

If the coupling is weak, we can use perturbation theory. Perturbation theory for on-shell scattering amplitudes takes a form of an expansion in terms of Feynman diagrams. String theory is perturbation theory for on-shell scattering amplitudes with particles in Feynman diagrams replaced by strings. It provides a consistent perturbation theory including gravity. However, we generally do not know where this perturbation theory comes from. As we mentioned in the introduction, string field theory is one natural approach, and it is formulated such that the spectrum of string theory is reproduced from an action with gauge invariance.

Let us consider open bosonic string theory on a flat spacetime in 26 dimensions with Neumann boundary conditions. When the open string is in the ground state of the internal oscillators, it behaves like a tachyonic scalar particle of mass m given by

$$m^2 = -\frac{1}{\alpha'} , \tag{2·2}$$

where the constant α' is related to the string tension T by

$$T = \frac{1}{2\pi\alpha'} . \tag{2·3}$$

The open string in the ground state is thus described by a tachyonic scalar field $T(k)$, where we write it in momentum space. When the open string is in the first-excited states, it behaves like a massless particle described by a massless vector field $A_\mu(k)$. When the open string is in a higher-excited state, it behaves like a massive particle with mass m given by

$$m^2 = \frac{1}{\alpha'} , \frac{2}{\alpha'} , \frac{3}{\alpha'} , \dots . \tag{2·4}$$

Therefore, one way to think about the degrees of freedom of string field theory is to consider a set of an infinite number of spacetime fields $\{ T(k) , A_\mu(k) , \dots \}$, and the action will be a functional of these fields: $S[T(k) , A_\mu(k) , \dots]$.

Let us try to construct such an action for the free theory. First construct an action for $T(k)$ and $A_\mu(k)$. It should be

$$S = -\int \frac{d^{26}k}{(2\pi)^{26}} \Big[\frac{1}{2} T(-k) \Big(k^2 - \frac{1}{\alpha'} \Big) T(k) + \frac{1}{2} A_\mu(-k) \, (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \, A_\nu(k) \Big] . \tag{2·5}$$

Or this action can also be written as

$$S = -\int \frac{d^{26}k}{(2\pi)^{26}} \Big[\frac{1}{2} T(-k) \Big(k^2 - \frac{1}{\alpha'} \Big) T(k) + \frac{1}{2} A_\mu(-k) \, k^2 A^\mu(k) + iB(-k) \, k^\mu A_\mu(k) + \frac{1}{2} B(-k) \, B(k) \Big] \tag{2·6}$$

by introducing an auxiliary field $B(k)$. The equations of motion derived from the second form of the action are

$$\Big(k^2 - \frac{1}{\alpha'} \Big) T(k) = 0 \,, \quad k^2 A_\mu(k) - i k_\mu B(k) = 0 \,, \quad B(k) + i k^\mu A_\mu(k) = 0 \,. \qquad (2\cdot7)$$

If we eliminate $B(k)$ using the last equation, we recover the first form of the action. The two actions are therefore equivalent. The second form of the action is invariant under the gauge transformations given by

$$\delta_\Lambda A_\mu(k) = i k_\mu \Lambda(k) \,, \qquad \delta_\Lambda B(k) = k^2 \Lambda(k) \,. \qquad (2\cdot8)$$

We could in principle proceed to the massive fields and construct the free action of those fields, but it would be formidably complicated. Actually, the action in the second form (2·6), the equations of motion (2·7), and the gauge transformations (2·8) can be written compactly by introducing an object called *string field*. Just as the $SU(2)$ gauge fields $A^a_\mu(x)$ with $a = 1, 2, 3$ can be incorporated into a single 2×2 matrix field $A_\mu(x)$ using the Pauli matrices σ^a as

$$A_\mu(x) = \frac{1}{2} \sum_{a=1}^{3} A^a_\mu(x) \, \sigma^a \,, \qquad (2\cdot9)$$

the component fields $\{ \, T(k) \, , \, A_\mu(k) \, , \, \ldots \}$ can be incorporated into a string field Ψ, which is a state in a boundary CFT in two dimensions. For the open string on a flat spacetime in 26 dimensions with Neumann boundary conditions, the Fock space of the boundary CFT is constructed by the bosonic operators α^μ_n and the fermionic operators b_n and c_n satisfying

$$[\, \alpha^\mu_n, \alpha^\nu_m \,] = n \, \eta^{\mu\nu} \delta_{n+m,0} \,, \qquad \{ \, b_n, c_m \, \} = \delta_{n+m,0} \qquad (2\cdot10)$$

with the state $| \, 0; k \, \rangle$ defined by

$$\begin{aligned} \alpha^\mu_n | \, 0; k \, \rangle &= 0 \quad \text{for} \quad n > 0 \,, \\ b_n | \, 0; k \, \rangle &= 0 \quad \text{for} \quad n > -2 \,, \\ c_n | \, 0; k \, \rangle &= 0 \quad \text{for} \quad n > 1 \,, \\ p_\mu | \, 0; k \, \rangle &= k_\mu | \, 0; k \, \rangle \,. \end{aligned} \qquad (2\cdot11)$$

The component fields $T(k)$, $A_\mu(k)$, and $B(k)$ are incorporated into Ψ as follows:

$$\Psi = \int \frac{d^{26}k}{(2\pi)^{26}} \Big[\, \frac{1}{\sqrt{\alpha'}} \, T(k) \, c_1 | \, 0; k \, \rangle + \frac{1}{\sqrt{\alpha'}} \, A_\mu(k) \, \alpha^\mu_{-1} c_1 | \, 0; k \, \rangle + \frac{i}{\sqrt{2}} \, B(k) \, c_0 | \, 0; k \, \rangle \, \Big] \,. \qquad (2\cdot12)$$

The equations of motion can be written compactly as

$$Q_B \, \Psi = 0 \,, \qquad (2\cdot13)$$

where the *BRST operator* Q_B is given by

$$Q_B = \sum_{n=-\infty}^{\infty} c_n L^{(m)}_{-n} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (m-n) \, {}^\circ_\circ \, c_m c_n b_{-m-n} \, {}^\circ_\circ - c_0 \,. \qquad (2\cdot14)$$

We denote the creation-annihilation normal ordering of $\mathcal{O}$ by $\,^{\circ}_{\circ}\,\mathcal{O}\,^{\circ}_{\circ}$. The creation-annihilation normal ordering for the bc oscillators is defined by

$$^{\circ}_{\circ}\, c_n b_{-n} \,^{\circ}_{\circ} = \begin{cases} c_n b_{-n} & \text{for} \quad n \le 0\,, \\ -\, b_{-n} c_n & \text{for} \quad n > 0\,, \end{cases} \tag{2·15}$$

and the Virasoro operators in the matter sector $L_n^{(m)}$ are given by

$$L_n^{(m)} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \,^{\circ}_{\circ}\, \alpha_m^\mu \alpha_{n-m}^\nu \,^{\circ}_{\circ}\, \eta_{\mu\nu} \quad \text{with} \quad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu\,. \tag{2·16}$$

The gauge transformations can also be written compactly as

$$\delta_\Lambda \Psi = Q_B \Lambda \tag{2·17}$$

with

$$\Lambda = \frac{i}{\sqrt{2}\,\alpha'} \int \frac{d^{26}k}{(2\pi)^{26}}\, \Lambda(k)\, |\, 0; k \,\rangle\,. \tag{2·18}$$

The action can be written as

$$S = -\frac{1}{2} \langle\, \Psi, Q_B \Psi \,\rangle \tag{2·19}$$

using the *BPZ inner product*. We will give a general definition of the BPZ inner product based on the CFT description later, but for the current particular case the BPZ inner product $\langle\, A, B \,\rangle$ can be defined as follows. First, we associate a bra $\langle\, A\,|$ with each state $|\, A\,\rangle$. Its rule is defined recursively. For the state $|\, 0; k\,\rangle$, the corresponding bra is $\langle\, 0; k\,|$:

$$|\, 0; k \,\rangle \to \langle\, 0; k \,|\,. \tag{2·20}$$

Note that this is different from the usual Hermitian conjugation where $\langle\, 0; -k\,|$ is associated with $|\, 0; k\,\rangle$. Then for the state $\mathcal{O}\,|\, A\,\rangle$, the corresponding bra is associated as follows:

$$\mathcal{O}\,|\, A \,\rangle \to \begin{cases} \langle\, A\,|\, \mathcal{O}^\star & \text{when } \mathcal{O} \text{ is Grassmann even}\,, \\ (-1)^A \langle\, A\,|\, \mathcal{O}^\star & \text{when } \mathcal{O} \text{ is Grassmann odd}\,, \end{cases} \tag{2·21}$$

where the BPZ conjugation $\mathcal{O}^\star$ of the operator $\mathcal{O}$ is defined by

$$(\alpha_n^\mu)^\star = (-1)^{n+1} \alpha_{-n}^\mu\,, \quad (b_n)^\star = (-1)^n b_{-n}\,, \quad (c_n)^\star = (-1)^{n+1} c_{-n}\,. \tag{2·22}$$

Here and in what follows a string field in the exponent of -1 denotes its Grassmann property: it is 0 mod 2 for a Grassmann-even string field and it is 1 mod 2 for a Grassmann-odd string field. It follows from this definition that

$$(\mathcal{O}_1 \mathcal{O}_2)^\star = \begin{cases} -\, \mathcal{O}_2^\star\, \mathcal{O}_1^\star & \text{when both } \mathcal{O}_1 \text{ and } \mathcal{O}_2 \text{ are Grassmann odd}\,, \\ \mathcal{O}_2^\star\, \mathcal{O}_1^\star & \text{when at least one of } \mathcal{O}_1 \text{ and } \mathcal{O}_2 \text{ is Grassmann even}\,. \end{cases} \tag{2·23}$$

The bra $\langle A |$ for an arbitrary state $| A \rangle$ has now been defined. Then the contraction $\langle A | B \rangle$ can be reduced to $\langle 0; k | c_{-1} c_0 c_1 | 0; k' \rangle$ using the commutation and anticommutation relations in (2·10) as well as the definition (2·11) of $| 0; k \rangle$. We define

$$\langle 0; k | c_{-1} c_0 c_1 | 0; k' \rangle = (2\pi)^{26} \delta^{(26)}(k + k') \,. \tag{2·24}$$

Now that the contraction $\langle A | B \rangle$ for an arbitrary pair A and B is determined, the BPZ inner product $\langle A, B \rangle$ is defined by

$$\langle A, B \rangle = \langle A | B \rangle \,. \tag{2·25}$$

It follows from the definition that

$$\langle \mathcal{O} A, B \rangle = \begin{cases} \langle A, \mathcal{O}^\star B \rangle & \text{when } \mathcal{O} \text{ is Grassmann even} \,, \\ (-1)^A \langle A, \mathcal{O}^\star B \rangle & \text{when } \mathcal{O} \text{ is Grassmann odd} \,. \end{cases} \tag{2·26}$$

Important properties of the BRST operator and the BPZ inner product are

$$\langle A, B \rangle = (-1)^{AB} \langle B, A \rangle \,, \quad \langle Q_B A, B \rangle = -(-1)^A \langle A, Q_B B \rangle \,, \quad Q_B^2 = 0 \,. \tag{2·27}$$

The first property is not manifest in the definition we just presented, but we will later give a more general definition of the BPZ inner product and prove this property. The second property can also be stated as $Q_B^\star = -Q_B$. Namely, the BRST operator is BPZ odd. The nilpotency of the BRST operator, $Q_B^2 = 0$, can be confirmed using the commutation and anticommutation relations in (2·10), but we will give a definition of the BRST operator for a general boundary CFT and the properties $Q_B^\star = -Q_B$ and $Q_B^2 = 0$ hold for any boundary CFT with central charge $c = 26$.

In this description using the string field, it is straightforward to generalize the construction of the free action to the massive fields. We generalize the string field Ψ in (2·12) by incorporating all the states of ghost number 1, where the ghost number is defined by the number of c oscillators minus the number of b oscillators. We then introduce spacetime fields as coefficients in front of these states. The action (2·19) constructed from the resulting string field is guaranteed to be gauge invariant because of the properties (2·27).

At the end of this subsection, let us briefly discuss gauge fixing. One possible gauge-fixing condition of the action (2·6) is to set $B(k) = 0$. In terms of the string field Ψ in (2·12), this condition can be stated as $b_0 \Psi = 0$. In fact, this condition works as a gauge-fixing condition for the whole string field and is called the *Siegel gauge* condition.

2.2. *The interacting theory*

We have completed the construction of the free action for string field theory. Let us next consider the interacting theory. An important point of string perturbation theory is that there are no Lorentz-invariant interaction points. This means that the form of the interactions is uniquely determined for a given free theory. This is crucially different from ordinary field theories where many interacting theories are possible for a given free theory. In the context of string field theory, we look for an action for the interacting theory which is invariant under nonlinearly extended

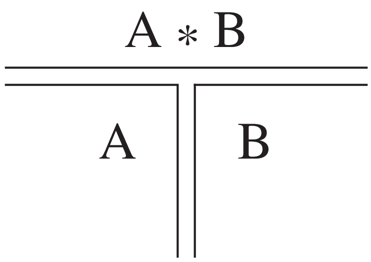

Fig. 1. Witten's star product.

gauge transformations. Just as the gauge transformation of the free $U(N)$ Yang-Mills theory $\delta A_\mu = \partial_\mu \Lambda$ is nonlinearly extended in the interacting theory as $\delta A_\mu = \partial_\mu \Lambda + i(\Lambda A_\mu - A_\mu \Lambda)$ and the Yang-Mills action is invariant under this extended transformation, we look for a nonlinear extension of the gauge transformation $\delta_\Lambda \Psi = Q_B \Lambda$ of the free theory and an action which is invariant under the nonlinear gauge transformation. Such an action was constructed by Witten.[4)] Witten introduced a product $A * B$ of string fields A and B called Witten's *star product* and constructed the following action:

$$S = -\frac{1}{\alpha'^3 g_T^2} \left[\frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle \right], \tag{2·28}$$

where g_T is the open string coupling constant.[*)] The string field is a state in a boundary CFT and is defined on a segment. This is not manifest in the representation using the oscillators, but we will later make it manifest in a different representation. The star product $A * B$ is then defined by taking the BPZ inner product for the right half of the string field A and the left half of the string field B. The resulting state defined on the combined segment for the left half of A and the right half of B is $A * B$. See Fig. 1. Again from our definition of the BPZ inner product using the oscillators it is not clear how to take the BPZ inner product only for a part of the states, but we will explain it later using a different representation. For the moment, it is sufficient to have a rough idea depicted in Fig. 1.

The star product is noncommutative,

$$A * B \neq B * A\,, \tag{2·29}$$

and there is a useful analogy with the matrix multiplication. Let us formally associate the left half of the string field with the left index of the matrix and the right half of the string field with the right index of the matrix:

$$A \longleftrightarrow A_{ij}\,. \tag{2·30}$$

[*)] With our definitions of the BPZ inner product and the star product, The open string coupling constant g_T here is related to the coupling constant g_o and g'_o in Ref. 3) as

$$g_T = g'_o \sqrt{\frac{2}{\alpha'}} = \frac{g_o}{\alpha'}\,.$$

Then the star product $A * B$ corresponds to the matrix multiplication,

$$A * B \longleftrightarrow (AB)_{ij} = A_{ik} B_{kj} \,, \tag{2·31}$$

and the BPZ inner product $\langle\, A, B \,\rangle$ corresponds to the trace of AB:

$$\langle\, A, B \,\rangle \longleftrightarrow \mathrm{tr}\, AB = A_{ij} B_{ji} \,. \tag{2·32}$$

We will give a precise definition of the star product later, but the gauge invariance of the action follows only from the properties (2·27) and the relations

$$\begin{aligned} &\langle\, A, B * C \,\rangle = \langle\, A * B, C \,\rangle \,, \\ &Q_B (A * B) = Q_B A * B + (-1)^A A * Q_B B \,. \end{aligned} \tag{2·33}$$

The first relation corresponds to the associativity of the matrix multiplication,

$$\mathrm{tr}\, A(BC) = \mathrm{tr}\,(AB)C = A_{ij} B_{jk} C_{ki} \,, \tag{2·34}$$

and the second relation is the derivation property of the BRST operator with respect to the star product, which corresponds to the derivation property of the ordinary derivative with respect to the matrix multiplication:

$$\frac{d}{dx} \Big(A_{ij}(x) B_{jk}(x) \Big) = \frac{dA_{ij}(x)}{dx}\, B_{jk}(x) + A_{ij}(x)\, \frac{dB_{jk}(x)}{dx} \,. \tag{2·35}$$

The equation of motion derived from the action is

$$Q_B \Psi + \Psi * \Psi = 0 \,. \tag{2·36}$$

The nonlinearly extended gauge transformation is

$$\delta_\Lambda \Psi = Q_B \Lambda + \Psi * \Lambda - \Lambda * \Psi \,, \tag{2·37}$$

and the action is invariant under this gauge transformation:

$$\delta_\Lambda S = 0 \,. \tag{2·38}$$

The invariance can be shown only using the following relations:

$$\begin{aligned} &\langle\, A, B \,\rangle = (-1)^{AB} \langle\, B, A \,\rangle \,, \quad \langle\, Q_B A, B \,\rangle = -(-1)^A \,\langle\, A, Q_B B \,\rangle \,, \quad Q_B^2 = 0 \,, \\ &\langle\, A, B * C \,\rangle = \langle\, A * B, C \,\rangle \,, \quad Q_B (A * B) = Q_B A * B + (-1)^A A * Q_B B \,. \end{aligned} \tag{2·39}$$

The proof is algebraically the same as that for the Chern-Simons theory in three dimensions, and the action is often called the Chern-Simons-like action.

We will define the BRST operator Q_B, the BPZ inner product, and the star product for general boundary CFT, and the relations (2·39) hold for any boundary CFT with the central charge $c = 26$ in the matter sector. We can therefore construct a gauge-invariant action of open string field theory for any given boundary CFT with $c = 26$ in the matter sector.

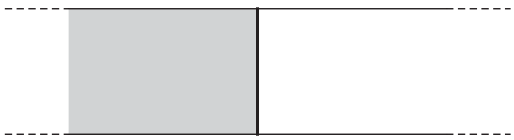

Fig. 2. The strip coordinates.

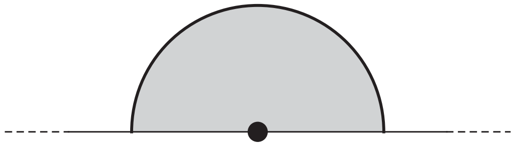

Fig. 3. The upper half-plane. The shaded region of Fig. 2 in the strip coordinate w is mapped to the shaded region of this figure in the coordinate z by the conformal transformation $z = e^w$.

2.3. *CFT description*

The string field of open string field theory is a state in a boundary CFT in two dimensions. In the strip coordinates (τ, σ) with the range $0 \leq \sigma \leq \pi$ and $-\infty < \tau < \infty$, the state is defined on a segment with the same value of τ such as $\tau = 0$. See Fig. 2. CFT is invariant under conformal transformations:

$$z = x + iy \to z' = x' + iy' = f(z) \,. \qquad (2\cdot40)$$

The complex strip coordinate $w = \tau + i\sigma$ can be mapped to the coordinate z of the upper half-plane (UHP) by the conformal transformation given by

$$z = e^w \,. \qquad (2\cdot41)$$

The coordinate z of the upper half-plane is useful because we can use the *state-operator correspondence* to represent states in the Fock space. The segment with $\tau = 0$ in the strip coordinates is mapped to the semi-circle at $|z| = 1$ in the upper half-plane so that the state is defined on this semi-circle. Since $z \to 0$ as $\tau \to -\infty$, specifying an initial condition at $\tau = -\infty$ in the strip coordinates can be translated into an operator insertion at $z = 0$ in the upper half-plane. We denote the operator at the origin of the coordinate z corresponding to the state $|\varphi\rangle$ in the Fock space by $\varphi(0)$:

$$|\varphi\rangle \longleftrightarrow \varphi(0) \,. \qquad (2\cdot42)$$

Namely, the state $|\varphi\rangle$ can be represented as the state we obtain at the semi-circle $|z| = 1$ in the upper half-plane by path integral over the region $|z| < 1$ in the upper half-plane with an insertion of $\varphi(0)$. See Fig. 3.

The matter sector of the boundary CFT depends on the background we chose to formulate string field theory. For the open bosonic string in 26-dimensional flat spacetime, the matter sector consists of 26 scalar fields $X^\mu(z, \bar{z})$ with the equations of motion given by

$$\partial\bar{\partial}X^\mu(z, \bar{z}) = 0 \,, \qquad (2\cdot43)$$

where

$$\partial = \frac{\partial}{\partial z}\,, \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}}\,. \tag{2·44}$$

The ghost sector of the boundary CFT consists of the b ghost and the c ghost whose equations of motion are

$$\bar{\partial} b(z) = 0\,, \quad \partial \tilde{b}(\bar{z}) = 0\,, \quad \bar{\partial} c(z) = 0\,, \quad \partial \tilde{c}(\bar{z}) = 0\,. \tag{2·45}$$

On the upper half-plane of z, the boundary conditions imposed on the real axis are

$$\partial X^\mu(z) = \bar{\partial} X^\mu(\bar{z})\,, \quad b(z) = \tilde{b}(\bar{z})\,, \quad c(z) = \tilde{c}(\bar{z}) \quad \text{at} \quad \Im z = 0\,. \tag{2·46}$$

The solutions to the equations of motion satisfying the boundary conditions can be expanded as follows:

$$\begin{aligned} X^\mu(z,\bar{z}) &= x^\mu - i\alpha' p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\alpha_n^\mu}{n}\Big(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\Big)\,, \\ b(z) &= \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+2}}\,, \qquad \tilde{b}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{b_n}{\bar{z}^{n+2}}\,, \\ c(z) &= \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n-1}}\,, \qquad \tilde{c}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{c_n}{\bar{z}^{n-1}}\,. \end{aligned} \tag{2·47}$$

We use the doubling trick and define $\partial X^\mu(z)$, $b(z)$, and $c(z)$ in the region $\Im z < 0$ by

$$\partial X^\mu(z) \equiv \bar{\partial} X^\mu(\bar{z}')\,, \quad b(z) \equiv \tilde{b}(\bar{z}')\,, \quad c(z) \equiv \tilde{c}(\bar{z}') \quad \text{for} \quad \Im z < 0\,, \quad z' = \bar{z}\,. \tag{2·48}$$

If we write $z = x + iy$ and $z' = x' + iy'$, we can translate $\Im z < 0$ and $z' = \bar{z}$ into $y < 0$ and $x' = x$, $y' = -y$. Therefore, for example, $\partial X^\mu(x,y)$ at $(x,y) = (2,-3)$ is given by $\bar{\partial} X^\mu(x',y')$ at $(x',y') = (2,3)$. In terms of the complex coordinates, it is slightly confusing as $\partial X^\mu(z)$ at $z = x + iy = 2 - 3i$ is given by $\bar{\partial} X^\mu(\bar{z}')$ at $\bar{z}' = x' - iy' = 2 - 3i$. Using the doubling trick, the modes α_n^μ, b_n, and c_n are given by

$$\begin{aligned} \alpha_n^\mu &= i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i}\, z^n\, \partial X^\mu(z)\,, \qquad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu\,, \\ b_n &= \oint \frac{dz}{2\pi i}\, z^{n+1}\, b(z)\,, \quad c_n = \oint \frac{dz}{2\pi i}\, z^{n-2}\, c(z)\,, \end{aligned} \tag{2·49}$$

where the contour of these integrals runs along the unit circle counterclockwise.

In CFT, operator products can be expanded in a complete set of local operators:

$$\mathcal{O}_i(z)\, \mathcal{O}_j(w) = \sum_k c_{ij}^k(z-w)\, \mathcal{O}_k(w)\,. \tag{2·50}$$

The expansions of the operator products $\partial X^\mu(z)\,\partial X^\nu(w)$ and $b(z)\,c(w)$ are given by

$$\begin{aligned}
\partial X^\mu(z)\,\partial X^\nu(w) &= -\frac{\alpha'}{2}\,\eta^{\mu\nu}\frac{1}{(z-w)^2} + :\partial X^\mu(z)\,\partial X^\nu(w): \\
&= -\frac{\alpha'}{2}\,\eta^{\mu\nu}\frac{1}{(z-w)^2} + \sum_{k=0}^{\infty}\frac{1}{k!}\,(z-w)^k :\partial^{k+1}X^\mu\,\partial X^\nu:(w)\,, \\
b(z)\,c(w) &= \frac{1}{z-w} + :b(z)\,c(w): \;= \frac{1}{z-w} + \sum_{k=0}^{\infty}\frac{1}{k!}\,(z-w)^k :(\partial^k b)c:(w)\,,
\end{aligned} \tag{2·51}$$

where conformal normal ordering of $\mathcal{O}$ is denoted by $:\mathcal{O}:$. The singular part of the operator product expansion (OPE) is often written as

$$\partial X^\mu(z)\,\partial X^\nu(w) \sim -\frac{\alpha'}{2}\,\eta^{\mu\nu}\frac{1}{(z-w)^2}\,, \qquad b(z)\,c(w) \sim \frac{1}{z-w}\,. \tag{2·52}$$

Let us use operator product expansions to identify the state $|\mathbf{1}\rangle$ corresponding to the unit operator $\mathbf{1}$ in the state-operator correspondence. The state $|\mathbf{1}\rangle$ is defined at the semi-circle $|z|=1$ in the upper half-plane by path integral over the region $|z|<1$ in the upper half-plane without any insertion at the origin. Now consider the states $\alpha_n^\mu\,|\mathbf{1}\rangle$, $b_n\,|\mathbf{1}\rangle$, and $c_n\,|\mathbf{1}\rangle$. Since no operators are inserted at the origin, we find

$$\begin{aligned}
\oint\frac{dz}{2\pi i}\,z^n\,\partial X^\mu(z) &= 0 \quad \text{for} \quad n\geq 0\,, \\
\oint\frac{dz}{2\pi i}\,z^{n+1}\,b(z) &= 0 \quad \text{for} \quad n\geq -1\,, \\
\oint\frac{dz}{2\pi i}\,z^{n-2}\,c(z) &= 0 \quad \text{for} \quad n\geq 2\,.
\end{aligned} \tag{2·53}$$

These are translated into

$$\begin{aligned}
\alpha_n^\mu\,|\mathbf{1}\rangle &= 0 \quad \text{for} \quad n\geq 0\,, \\
b_n^\mu\,|\mathbf{1}\rangle &= 0 \quad \text{for} \quad n\geq -1\,, \\
c_n^\mu\,|\mathbf{1}\rangle &= 0 \quad \text{for} \quad n\geq 2
\end{aligned} \tag{2·54}$$

so that we identify $|\mathbf{1}\rangle$ with $|\,0;0\,\rangle \equiv |0\rangle$:

$$|\mathbf{1}\rangle = |\,0;0\,\rangle \equiv |0\rangle \longleftrightarrow \mathbf{1}\,. \tag{2·55}$$

We use this identification to fix the normalization of the state-operator correspondence. Let us next consider the operator corresponding to the state $c_1|0\rangle$. Since $c_1|0\rangle = c_1|\mathbf{1}\rangle$ and

$$\oint\frac{dz}{2\pi i}\,\frac{1}{z}\,c(z) = c(0)\,, \tag{2·56}$$

we find that $c(0)$ is the corresponding operator:

$$c_1|0\rangle \longleftrightarrow c(0)\,. \tag{2·57}$$

As an exercise, let us calculate $b_{-1}c_1|0\rangle$ using the state-operator correspondence. Since the state $c_1|0\rangle$ corresponds to the operator $c(0)$, the operator corresponding to $b_{-1}c_1|0\rangle$ can be calculated as follows:

$$\oint \frac{dz}{2\pi i}\, b(z)\, c(0) = \oint \frac{dz}{2\pi i}\, \frac{1}{z} = 1\,, \tag{2·58}$$

where we used the OPE

$$b(z)\, c(0) \sim \frac{1}{z}\,. \tag{2·59}$$

Therefore, we find that

$$b_{-1}c_1|0\rangle = |0\rangle\,, \tag{2·60}$$

which reproduces the result using the anticommutation relation:

$$b_{-1}c_1|0\rangle = \{b_{-1}, c_1\}|0\rangle = |0\rangle\,. \tag{2·61}$$

The action of CFT is invariant under appropriate conformal transformations of operators. Consider a conformal transformation from the coordinate z to the coordinate z' given by $z' = f(z)$. We denote the operator in the coordinate z' transformed from an operator $\varphi(z)$ in the coordinate z by $f \circ \varphi(z)$:

$$\varphi(z) \to f \circ \varphi(z)\,. \tag{2·62}$$

Using this notation, the conformal invariance can be stated as

$$\langle\, \varphi_1(z_1)\ \varphi_2(z_2)\ \ldots\ \varphi_n(z_n)\,\rangle_\Sigma = \langle\, f \circ \varphi_1(z_1)\ f \circ \varphi_2(z_2)\ \ldots\ f \circ \varphi_n(z_n)\,\rangle_{f\circ\Sigma}\,, \tag{2·63}$$

where we denoted by $f \circ \Sigma$ the surface mapped from the surface Σ under the conformal transformation $f(z)$. A primary field of weight h is an operator which transforms as follows:

$$\mathcal{O}(z) \to f \circ \mathcal{O}(z) = \Bigl(\frac{df(z)}{dz}\Bigr)^h\, \mathcal{O}(f(z))\,. \tag{2·64}$$

The operators $\partial X^\mu(z)$, $b(z)$, and $c(z)$ are primary fields. Their weights are 1, 2, and -1, respectively:

$$\begin{aligned} \partial X^\mu(z) \to f \circ \partial X^\mu(z) &= \frac{df(z)}{dz}\, \partial X^\mu(f(z))\,, \\ b(z) \to f \circ b(z) &= \Bigl(\frac{df(z)}{dz}\Bigr)^2\, b(f(z))\,, \\ c(z) \to f \circ c(z) &= \Bigl(\frac{df(z)}{dz}\Bigr)^{-1}\, c(f(z))\,. \end{aligned} \tag{2·65}$$

The operator $\partial c(z)$ is not a primary field. Its transformation can be derived from that of $c(z)$ by taking a derivative with respect to z:

$$\partial c(z) \to f \circ \partial c(z) = \partial c(f(z)) - \frac{f''(z)}{f'(z)^2}\, c(f(z))\,, \tag{2·66}$$

where

$$f'(z) = \frac{df(z)}{dz}\,, \qquad f''(z) = \frac{d^2 f(z)}{dz^2}\,. \tag{2·67}$$

Conformal transformations are generated by the energy-momentum tensor:

$$T(z) \equiv T_{zz}(z)\,, \qquad \widetilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})\,, \qquad T_{z\bar{z}} = 0\,. \tag{2·68}$$

The conservation of the energy-momentum tensor can be stated as

$$\bar{\partial} T(z) = 0\,, \qquad \partial \widetilde{T}(\bar{z}) = 0\,, \tag{2·69}$$

and the energy-momentum tensor satisfies the following boundary condition on the upper half-plane:

$$T(z) = \widetilde{T}(\bar{z}) \quad \text{at} \quad \Im z = 0\,. \tag{2·70}$$

The energy-momentum tensor can be expanded as

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}\,, \qquad \widetilde{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{L_n}{\bar{z}^{n+2}}\,, \tag{2·71}$$

and the modes L_n are called the Virasoro generators. We use the doubling trick for the energy-momentum tensor as well:

$$T(z) \equiv \widetilde{T}(\bar{z}') \quad \text{for} \quad \Im z < 0\,, \quad z' = \bar{z}\,. \tag{2·72}$$

Then the Virasoro generators are given by

$$L_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z)\,, \tag{2·73}$$

where the contour of the integral runs along the unit circle counterclockwise. For the open bosonic string in 26-dimensional flat spacetime, the matter part of the energy-momentum tensor $T^{(m)}$ is given by

$$T^{(m)}(z) = -\frac{1}{\alpha'} :\partial X^{\mu}\, \partial X_{\mu}: (z)\,, \tag{2·74}$$

and the ghost part of the energy-momentum tensor $T^{(bc)}$ is

$$T^{(bc)}(z) = \;:(\partial b) c:(z) - 2\,\partial(:bc:)(z)\,. \tag{2·75}$$

The OPE of the energy-momentum tensor and a primary field $\mathcal{O}$ of weight h is

$$T(z)\,\mathcal{O}(w) \sim \frac{h}{(z-w)^2}\,\mathcal{O}(w) + \frac{1}{z-w}\,\partial\mathcal{O}(w)\,, \tag{2·76}$$

and the OPE of the energy-momentum tensor with itself takes the form

$$T(z)\,T(w) \sim \frac{c}{2}\,\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w)\,, \tag{2·77}$$

where c is a constant called the central charge. The central charge of the scalar field X^{μ} is 1 for each direction, and the central charge of the bc ghost CFT is -26. The total central charge must vanish for a consistent string background. In this case, the total energy-momentum tensor is a primary field of weight 2. For a 26-dimensional spacetime, the total central charge indeed vanishes.

The BRST operator Q_B is an integral of the BRST current $j_B(z)$:

$$Q_B = \oint \frac{dz}{2\pi i}\, j_B(z) \tag{2·78}$$

with

$$j_B(z) = cT^{(m)}(z) + :bc\partial c:(z) + \frac{3}{2}\,\partial^2 c(z)\,. \tag{2·79}$$

We can show using various OPE's that

$$Q_B^2 = 0 \tag{2·80}$$

when the central charge of the matter sector is 26. We can also confirm that the BRST current is a primary field of weight 1 by calculating the OPE of the BRST current with the energy-momentum tensor. Under the conformal transformation $z' = f(z)$, the BRST operator, which is the zero mode of $j_B(z)$, is therefore invariant:

$$\oint \frac{dz}{2\pi i}\, j_B(z) \to \oint \frac{dz'}{2\pi i}\, j_B(z')\,, \tag{2·81}$$

where the contour should be mapped according to $z' = f(z)$. This is an important property of an integral of a primary field of weight 1. Another important property of the BRST operator is that it transforms the b ghost to the energy-momentum tensor:

$$Q_B \cdot b(w) \equiv \oint \frac{dz}{2\pi i}\, j_B(z)\, b(w) = T(w)\,, \tag{2·82}$$

where the contour of the integral encircles the point w counterclockwise. The energy-momentum tensor is invariant under the BRST transformation:

$$Q_B \cdot T(w) = 0\,, \tag{2·83}$$

which is consistent with the nilpotency: $Q_B^2 = 0$. These relations can be translated into the following anticommutation and commutation relations:

$$\{Q_B, b_n\} = L_n\,, \qquad [\,Q_B, L_n\,] = 0\,. \tag{2·84}$$

2.4. *BPZ inner product*

We are now ready to provide the general definition of the BPZ inner product $\langle\, \varphi_1, \varphi_2 \,\rangle$ of the states φ_1 and φ_2 in the Fock space. We prepare the states φ_1 and φ_2 in the upper half-plane using the state-operator correspondence. The BPZ inner product $\langle\, \varphi_1, \varphi_2 \,\rangle$ is defined by the following two-point function:

$$\langle\, \varphi_1, \varphi_2 \,\rangle = \langle\, I \circ \varphi_1(0)\, \varphi_2(0) \,\rangle_{\mathrm{UHP}}\,, \tag{2·85}$$

where the conformal transformation $I(\xi)$ is given by

$$I(\xi) = -\frac{1}{\xi}\,. \tag{2·86}$$

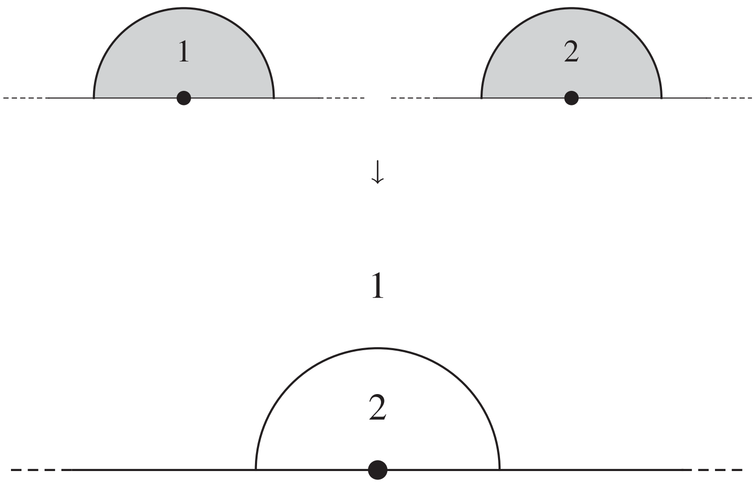

Fig. 4. The definition (2·85) of the BPZ inner product. The half unit disk 1 is mapped to the outer region by the conformal transformation $I(\xi)$, and it is combined with the other half unit disk 2 to form an upper half-plane.

See Fig. 4. In this representation, however, the origin $\xi = 0$ is mapped to the point at infinity by $I(\xi)$ so that the insertion point of the operator $I \circ \varphi_1(0)$ is outside the coordinate patch. We can make a conformal transformation such that both operators are inserted at finite points. For example, we can perform the conformal transformation $h(z)$ given by

$$h(z) = -\frac{z+1}{z-1}\,, \tag{2·87}$$

which maps the upper half-plane to itself. Then the BPZ inner product is expressed as follows:

$$\langle\, \varphi_1, \varphi_2 \,\rangle = \langle\, h \circ I \circ \varphi_1(0)\;\, h \circ \varphi_2(0) \,\rangle_{\rm UHP}\,. \tag{2·88}$$

Since

$$h \circ I(\xi) = h(I(\xi)) = \frac{\xi - 1}{\xi + 1}\,, \tag{2·89}$$

the operator $h \circ I \circ \varphi_1(0)$ is inserted at -1, while $h \circ \varphi_2(0)$ is inserted at 1. See Fig. 5.

Let us now prove the important property of the BPZ inner product

$$\langle\, \varphi_2, \varphi_1 \,\rangle = (-1)^{\varphi_1 \varphi_2} \langle\, \varphi_1, \varphi_2 \,\rangle\,. \tag{2·90}$$

Since $I(I(\xi)) = \xi$, we have

$$\begin{aligned} \langle\, \varphi_2, \varphi_1 \,\rangle &= \langle\, I \circ \varphi_2(0)\;\, \varphi_1(0) \,\rangle_{\rm UHP} = \langle\, I \circ I \circ \varphi_2(0)\;\, I \circ \varphi_1(0) \,\rangle_{\rm UHP} \\ &= \langle\, \varphi_2(0)\;\, I \circ \varphi_1(0) \,\rangle_{\rm UHP} = (-1)^{\varphi_1 \varphi_2} \langle\, I \circ \varphi_1(0)\;\, \varphi_2(0) \,\rangle_{\rm UHP} \\ &= (-1)^{\varphi_1 \varphi_2} \langle\, \varphi_2, \varphi_1 \,\rangle\,. \end{aligned} \tag{2·91}$$

For a proof of (2·90) based on the expression (2·88), we can use the conformal transformation

$$h \circ I \circ h^{-1}(z) = -\frac{1}{z}\,, \tag{2·92}$$

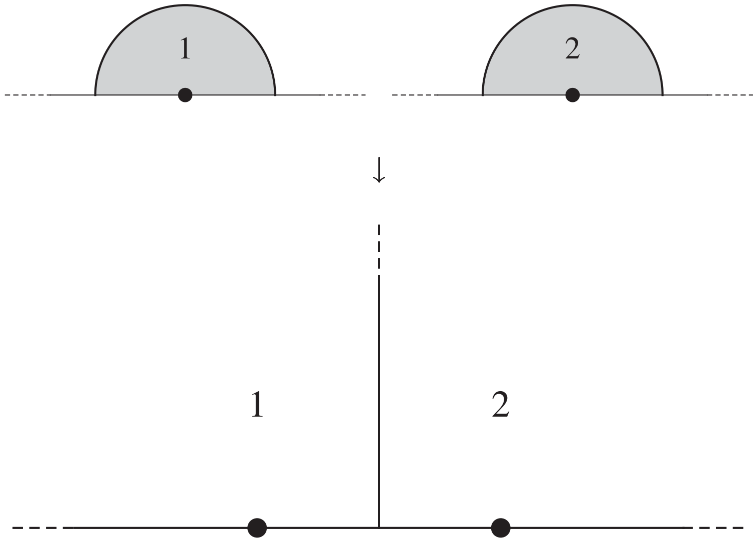

Fig. 5. The representation (2·88) of the BPZ inner product. The half unit disk 1 is mapped to the left half of the upper half-plane, and the half unit disk 2 is mapped to the right half of the upper half-plane. The two operators $h \circ I \circ \varphi_1(0)$ and $h \circ \varphi_2(0)$ in (2·88) are both within this coordinate patch.

which coincides with $I(z)$ and maps the upper half-plane to itself, to find

$$\begin{aligned}\langle \varphi_2, \varphi_1 \rangle &= \langle h \circ I \circ \varphi_2(0) \; h \circ \varphi_1(0) \rangle_{\rm UHP} \\ &= \langle h \circ I \circ h^{-1} \circ h \circ I \circ \varphi_2(0) \; h \circ I \circ h^{-1} \circ h \circ \varphi_1(0) \rangle_{\rm UHP} \\ &= \langle h \circ \varphi_2(0) \; h \circ I \circ \varphi_1(0) \rangle_{\rm UHP} = (-1)^{\varphi_1 \varphi_2} \langle h \circ I \circ \varphi_1(0) \; h \circ \varphi_2(0) \rangle_{\rm UHP} \\ &= (-1)^{\varphi_1 \varphi_2} \langle \varphi_2, \varphi_1 \rangle \,. \end{aligned} \tag{2·93}$$

We explained that the degrees of freedom of string field theory is a set of infinite spacetime component fields, and these component fields are incorporated into the string field as coefficients in front of complete states of the boundary CFT. How can we extract these component fields from the string field? We explained the string field using the analogy with the matrix field of the $SU(2)$ gauge fields:

$$A_\mu(x) = \frac{1}{2} \sum_{a=1}^{3} A_\mu^a(x) \, \sigma^a \,. \tag{2·94}$$

The component fields $A_\mu^a(x)$ with $a = 1, 2, 3$ can be extracted from the matrix field $A_\mu(x)$ by multiplying it with σ^a and taking the trace:

$$A_\mu^a(x) = \operatorname{tr} \sigma^a A_\mu(x) \,. \tag{2·95}$$

As can be understood from the analogy between $\operatorname{tr} AB$ for matrices and $\langle A, B \rangle$ for string fields, component fields of the string field can be extracted from BPZ inner products of the string field with states in the Fock space. Since the dynamical degrees of freedom are these component fields, a string field Ψ can be specified by giving $\langle\, \varphi, \Psi \,\rangle$ for all φ in the Fock space. In what follows we use φ to denote a generic state in the Fock space and $\varphi(0)$ to denote its corresponding operator in the state-operator correspondence, and we will often define a string field Ψ by giving $\langle\, \varphi, \Psi \,\rangle$. If we need more states in the Fock space, we use φ_1, φ_2, φ_3, ... and denote their corresponding operators in the state-operator correspondence by $\varphi_1(0)$, $\varphi_2(0)$, $\varphi_3(0)$, ...

It would be helpful to illustrate the definition of the BPZ inner product with an example. Consider $\langle\, T, Q_B T \,\rangle$ with $T = c_1|0\rangle$. The operator corresponding to T is $c(0)$. The operator corresponding to $Q_B T$ can be calculated as follows. Since

$$\begin{aligned} Q_B \cdot c(w) &= \oint \frac{dz}{2\pi i}\, j_B(z)\, c(w) = \oint \frac{dz}{2\pi i}\, :bc\partial c:(z)\, c(w) \\ &= \oint \frac{dz}{2\pi i}\, \frac{1}{z-w}\, c\partial c(z) = c\partial c(w)\,, \end{aligned} \tag{2·96}$$

where the contour of the integral encircles the point w counterclockwise, the operator corresponding to $Q_B T$ is $c\partial c(0)$. The BPZ inner product $\langle\, T, Q_B T \,\rangle$ can be expressed as

$$\langle\, T, Q_B T \,\rangle = \langle\, h \circ I \circ c(0)\ h \circ c\partial c(0) \,\rangle_{\rm UHP}\,. \tag{2·97}$$

There are two ways to proceed from here. The first one is to calculate the conformal transformations of $c(0)$ and $c\partial c(0)$. The operator $c(z)$ is a primary field of weight -1. How about the operator $c\partial c(z)$? Instead of calculating the conformal transformation of $c\partial c(0)$ directly, let us take the second way. Since the BRST operator is invariant under conformal transformations, we have the following general relation:

$$f \circ (Q_B \cdot \varphi(\xi)) = f \circ \left[\oint \frac{dz}{2\pi i}\, j_B(z)\, \varphi(\xi) \right] = \oint \frac{dz'}{2\pi i}\, j_B(z')\, f \circ \varphi(\xi) \tag{2·98}$$

with $z' = f(z)$. Namely, the BRST transformation and the conformal transformation commute:

$$f \circ (Q_B \cdot \varphi(\xi)) = Q_B \cdot (f \circ \varphi(\xi))\,. \tag{2·99}$$

Applying this to the BPZ inner product, we find

$$\langle\, T, Q_B T \,\rangle = \langle\, h\circ I\circ c(0)\ h\circ(Q_B\cdot c(0)) \,\rangle_{\rm UHP} = \langle\, h\circ I\circ c(0)\ Q_B\cdot(h\circ c(0)) \,\rangle_{\rm UHP}\,. \tag{2·100}$$

Since

$$h \circ c(0) = \frac{1}{2}\, c(1)\,, \quad h \circ I \circ c(0) = \frac{1}{2}\, c(-1) \tag{2·101}$$

and thus

$$Q_B \cdot (h \circ c(0)) = \frac{1}{2}\, Q_B \cdot c(1) = \frac{1}{2}\, c\partial c(1)\,, \tag{2·102}$$

the BPZ inner product is given by

$$\langle\, T, Q_B T \,\rangle = \frac{1}{4} \langle\, c(-1)\ c\partial c(1) \,\rangle_{\rm UHP}\,. \tag{2·103}$$

Actually, the relation (2·99) implies that the BRST transformation of a primary field of weight h is a primary field of weight h. Therefore, the operator $c\partial c(z)$, which is the BRST transformation of $c(z)$, is a primary field of weight -1. We can thus reproduce the expression (2·103) by calculating the conformal transformation of $c\partial c(0)$ directly. We normalize the correlation function as

$$\langle\, c(z_1)\, c(z_2)\, c(z_3)\,\rangle_{\mathrm{UHP},\, density} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\,. \tag{2·104}$$

Here and in what follows we use the subscript *density* to denote quantities divided by the spacetime volume factor. The BPZ inner product (2·103) is

$$\langle\, T, Q_B T\,\rangle_{density} = -1\,. \tag{2·105}$$

Based on the definition (2·85) of the BPZ inner product, let us derive the formulas of the BPZ conjugation for the operators α_n^μ, b_n, and c_n in (2·22). Consider a primary field $\mathcal{O}(z)$ of weight h. It is expanded as

$$\mathcal{O}(z) = \sum_{n=-\infty}^{\infty} \frac{\mathcal{O}_n}{z^{n+h}}\,, \tag{2·106}$$

and the mode $\mathcal{O}_n$ is given by

$$\mathcal{O}_n = \oint \frac{dz}{2\pi i}\, z^{n+h-1}\, \mathcal{O}(z)\,, \tag{2·107}$$

where the contour of the integral runs along the unit circle counterclockwise. Let us assume, for simplicity, that $\mathcal{O}(z)$ is Grassmann even, and consider the BPZ inner product $\langle\, \varphi_1, \mathcal{O}_n \varphi_2\,\rangle$. Under the conformal transformation

$$z' = -\frac{1}{z}\,, \tag{2·108}$$

the mode $\mathcal{O}_n$ transforms as

$$\begin{aligned} \mathcal{O}_n = \oint \frac{dz}{2\pi i}\, z^{n+h-1}\, \mathcal{O}(z) &\to -\oint \frac{dz'}{2\pi i}\, \frac{dz}{dz'} \left(-\frac{1}{z'}\right)^{n+h-1} \left(\frac{dz}{dz'}\right)^{-h} \mathcal{O}(z') \\ &= (-1)^{n+h} \oint \frac{dz'}{2\pi i}\, z'^{\,-n+h-1}\, \mathcal{O}(z') = (-1)^{n+h} \mathcal{O}_{-n}\,, \end{aligned} \tag{2·109}$$

where the first minus sign came from reversing the direction of the contour integral. We therefore find

$$\langle\, \varphi_1, \mathcal{O}_n \varphi_2\,\rangle = (-1)^{n+h} \langle\, \mathcal{O}_{-n} \varphi_1, \varphi_2\,\rangle\,, \tag{2·110}$$

or

$$\mathcal{O}_n^\star = (-1)^{n+h} \mathcal{O}_{-n}\,. \tag{2·111}$$

Since α_n^μ, b_n, and c_n are modes of primary fields of weights 1, 2, and -1, respectively, we reproduce the formulas in (2·22). Furthermore, the BRST operator is the zero mode of the BRST current, which is a primary field of weight 1, so that we confirm that the BRST operator is BPZ odd: $Q_B^\star = -Q_B$.

2.5. *Star product*

Let us next provide the definition of the star product $\langle\, \varphi_1, \varphi_2 * \varphi_3 \,\rangle$ for arbitrary states φ_1, φ_2, and φ_3 in the Fock space. As in the case of the BPZ inner product, we prepare the states φ_1, φ_2, and φ_3 in the upper half-plane using the state-operator correspondence. Then the inner product $\langle\, \varphi_1, \varphi_2 * \varphi_3 \,\rangle$ is defined by

$$\langle\, \varphi_1, \varphi_2 * \varphi_3 \,\rangle = \langle\, f_1 \circ \varphi_1(0)\ f_2 \circ \varphi_2(0)\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP}\,, \tag{2·112}$$

where

$$f_1(\xi) = \tan\left[\frac{2}{3}\Big(\arctan\xi - \frac{\pi}{2}\Big)\right], \quad f_2(\xi) = \tan\left(\frac{2}{3}\arctan\xi\right), \\ f_3(\xi) = \tan\left[\frac{2}{3}\Big(\arctan\xi + \frac{\pi}{2}\Big)\right]. \tag{2·113}$$

See Fig. 6.

Let us prove the important properties (2·33) based on the definition (2·112). The first one

$$\langle\, \varphi_1 * \varphi_2, \varphi_3 \,\rangle = \langle\, \varphi_1, \varphi_2 * \varphi_3 \,\rangle \tag{2·114}$$

can be shown using the conformal transformation

$$\tilde f(z) = \tan\left(\arctan z + \frac{\pi}{3}\right) = \frac{z+\sqrt{3}}{1-\sqrt{3}\,z}\,, \tag{2·115}$$

which maps the upper half-plane to itself. We can confirm that

$$\tilde f \circ f_1(\xi) = f_2(\xi)\,, \qquad \tilde f \circ f_2(\xi) = f_3(\xi)\,, \qquad \tilde f \circ f_3(\xi) = f_1(\xi)\,. \tag{2·116}$$

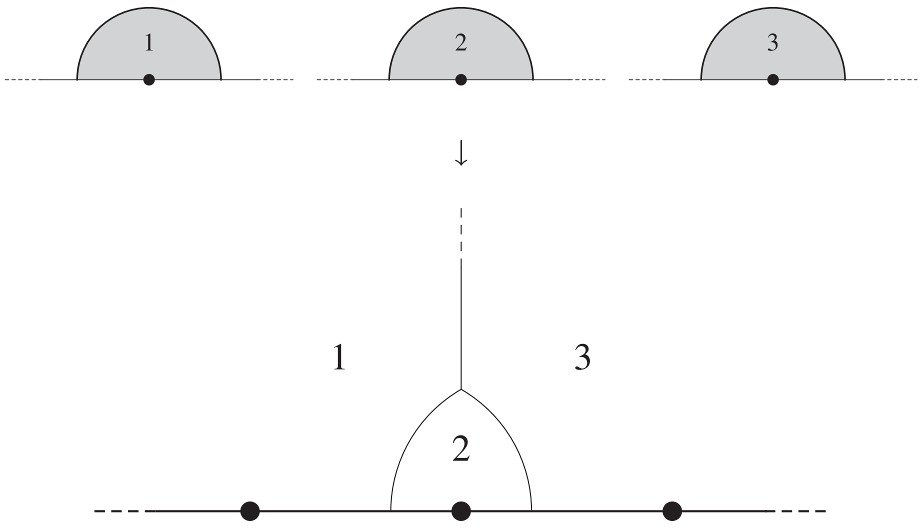

Fig. 6. The definition (2·112) of the star product. The three half unit disks 1, 2, and 3 are mapped to the left, center, and right regions, respectively, to form an upper half-plane.

We therefore find

$$\begin{aligned}\langle\, \varphi_1, \varphi_2 * \varphi_3 \,\rangle &= \langle\, \tilde{f} \circ f_1 \circ \varphi_1(0)\ \tilde{f} \circ f_2 \circ \varphi_2(0)\ \tilde{f} \circ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &= \langle\, f_2 \circ \varphi_1(0)\ f_3 \circ \varphi_2(0)\ f_1 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &= (-1)^{(\varphi_1+\varphi_2)\,\varphi_3} \langle\, f_1 \circ \varphi_3(0)\ f_2 \circ \varphi_1(0)\ f_3 \circ \varphi_2(0) \,\rangle_{\rm UHP} \\ &= (-1)^{(\varphi_1+\varphi_2)\,\varphi_3} \langle\, \varphi_3, \varphi_1 * \varphi_2 \,\rangle = \langle\, \varphi_1 * \varphi_2, \varphi_3 \,\rangle\,. \end{aligned} \tag{2·117}$$

The derivation property of the BRST operator with respect to the star product also follows from the definition easily. Consider the inner product $\langle\, Q_B\varphi_1, \varphi_2 * \varphi_3 \,\rangle$. Since $Q_B^\star = -Q_B$, we have

$$\langle\, Q_B\varphi_1, \varphi_2 * \varphi_3 \,\rangle = -(-1)^{\varphi_1} \langle\, \varphi_1, Q_B(\varphi_2 * \varphi_3) \,\rangle\,. \tag{2·118}$$

On the other hand, it follows from the definition of the star produce that

$$\begin{aligned}\langle\, Q_B\varphi_1, \varphi_2 * \varphi_3 \,\rangle &= \langle\, f_1 \circ (Q_B \cdot \varphi_1(0))\ f_2 \circ \varphi_2(0)\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &= \langle\, Q_B \cdot (f_1 \circ \varphi_1(0))\ f_2 \circ \varphi_2(0)\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP}\,. \end{aligned} \tag{2·119}$$

The contour of the integral for the BRST operator can be deformed as in Fig. 7. Changing the direction of the contours, we find

$$\begin{aligned}&\langle\, Q_B \cdot (f_1 \circ \varphi_1(0))\ f_2 \circ \varphi_2(0)\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &= -(-1)^{\varphi_1} \langle\, f_1 \circ \varphi_1(0)\ Q_B \cdot (f_2 \circ \varphi_2(0))\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &\quad - (-1)^{\varphi_1}(-1)^{\varphi_2} \langle\, f_1 \circ \varphi_1(0)\ f_2 \circ \varphi_2(0)\ Q_B \cdot (f_3 \circ \varphi_3(0)) \,\rangle_{\rm UHP} \\ &= -(-1)^{\varphi_1} \langle\, f_1 \circ \varphi_1(0)\ f_2 \circ (Q_B \cdot \varphi_2(0))\ f_3 \circ \varphi_3(0) \,\rangle_{\rm UHP} \\ &\quad - (-1)^{\varphi_1}(-1)^{\varphi_2} \langle\, f_1 \circ \varphi_1(0)\ f_2 \circ \varphi_2(0)\ f_3 \circ (Q_B \cdot \varphi_3(0)) \,\rangle_{\rm UHP} \\ &= -(-1)^{\varphi_1} \langle\, \varphi_1, (Q_B\varphi_2) * \varphi_3 \,\rangle - (-1)^{\varphi_1}(-1)^{\varphi_2} \langle\, \varphi_1, \varphi_2 * (Q_B\varphi_3) \,\rangle\,. \end{aligned} \tag{2·120}$$

We have thus derived

$$Q_B(\varphi_2 * \varphi_3) = (Q_B\varphi_2) * \varphi_3 + (-1)^{\varphi_2} \varphi_2 * (Q_B\varphi_3)\,. \tag{2·121}$$

It would again be helpful to illustrate the definition (2·112) of the star product by an example. Let us calculate $\langle\, T, T * T \,\rangle$ with $T = c_1|0\rangle$. Since

$$f_1 \circ c(0) = \frac{3}{8}\, c(\,-\sqrt{3}\,)\,, \quad f_2 \circ c(0) = \frac{3}{2}\, c(0)\,, \quad f_3 \circ c(0) = \frac{3}{8}\, c(\,\sqrt{3}\,)\,, \tag{2·122}$$

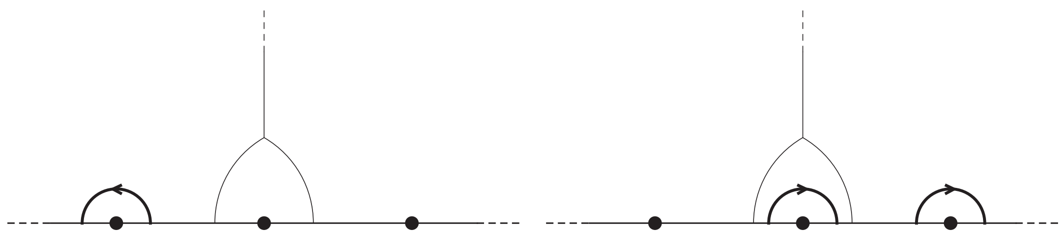

Fig. 7. The deformation of the contour used in (2·120). The contour in the left figure can be deformed to the sum of the two contours in the right figure.

we find

$$\langle\, T, T * T \,\rangle = \frac{27}{128} \langle\, c(\,-\sqrt{3}\,)\, c(0)\, c(\,\sqrt{3}\,)\,\rangle_{\rm UHP}\,. \tag{2·123}$$

Using (2·104), it is evaluated as

$$\langle\, T, T * T \,\rangle_{density} = -\frac{81\sqrt{3}}{64}\,. \tag{2·124}$$

It should be noted that there is another convention for the star product. It is given by

$$\langle\, \varphi_1, \varphi_2 \, * \, \varphi_3 \,\rangle = \langle\, f_3 \circ \varphi_3(0)\ f_2 \circ \varphi_2(0)\ f_1 \circ \varphi_1(0)\,\rangle_{\rm UHP}\,. \quad \text{(different convention)} \tag{2·125}$$

Namely, the operators in the correlation function are ordered in the opposite way. While both conventions (2·112) and (2·125) are consistent, let us explain why we choose the convention (2·112). As we have seen, the three operators $f_1 \circ \varphi_1(0)$, $f_2 \circ \varphi_2(0)$, and $f_3 \circ \varphi_3(0)$ are inserted at $-\sqrt{3}$, 0, and $\sqrt{3}$, respectively. In the standard way of writing the real axis, the operators are therefore located along the real axis in the order $f_1 \circ \varphi_1(0)\ f_2 \circ \varphi_2(0)\ f_3 \circ \varphi_3(0)$. In the convention (2·112), the operators are ordered in the same way inside the correlation function. We think that using the convention (2·112) helps us reduce sign mistakes when we handle Grassmann-odd operators.

§3. Tachyon condensation

3.1. *Approximate solutions in level truncation*

Consider an ordinary field theory of a scalar field ϕ with its potential $V(\phi)$ shown in Fig. 8. The static configuration $\phi = 0$ is a solution to the equation of motion, but it is an unstable background of ϕ, and there is a tachyonic mode around the background. In such cases, we look for a solution $\phi = \phi_*$ to the equation of motion satisfying

$$\left.\frac{dV(\phi)}{d\phi}\right|_{\phi=\phi_*} = 0 \quad \text{with} \quad \left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\phi_*} > 0\,, \tag{3·1}$$

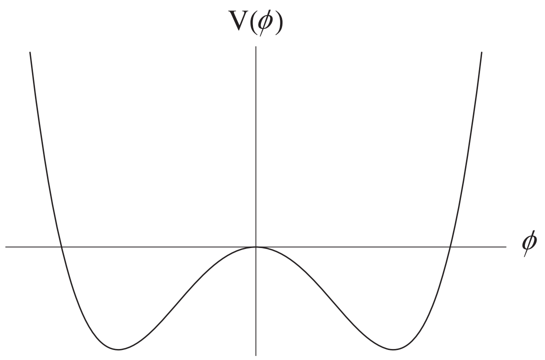

Fig. 8. An example of $V(\phi)$.

and we expand ϕ around the background ϕ_* as $\phi = \phi_* + \delta\phi$.

In string theory, open strings are excitations on solitonic extended objects called D-branes. The background we have considered corresponds to the D25-brane, and there is a tachyonic mode around the background. Sen conjectured that the instability associated with this open string tachyon corresponds to the decay of the D-brane and there exists a background where the unstable D-brane disappeared. This conjecture makes a quantitative prediction that the difference of the energy density between the two backgrounds is given by the energy density of the D-brane. However, string theory is defined only perturbatively, and we can only investigate an infinitesimal neighborhood around a background. We are not able to draw a potential like the one in Fig. 8, as it is an off-shell quantity. On the other hand, we can in principle discuss the relationship between backgrounds using string field theory. When we formulate open string field theory around the background with the unstable D-brane, we expect a classical solution to the equation of motion corresponding to the background without the D-brane. The problem of finding this solution is usually referred to as *tachyon condensation*, and it is conceptually the same as finding a nontrivial solution in the scalar field theory with the potential shown in Fig. 8. In the case of string field theory, however, there are infinitely many fields and finding a nontrivial solution is technically complicated. The expected background without the D-brane is often called *tachyon vacuum*, and the corresponding solution of open string field theory is called *tachyon vacuum solution*.

Sen and Zwiebach tested the quantitative conjecture using open string field theory.[5)] They used the approximation scheme called *level truncation*. In level truncation, component fields of the string field are truncated to a finite number of fields. At level 0, the zero mode of the tachyon t is the only degree of freedom:

$$\Psi = t\, c_1 |0\rangle\,. \tag{3·2}$$

The tachyon potential is given by

$$V(t) = \frac{1}{\alpha'^3 g_T^2} \left[\frac{1}{2}\, t^2 \, \langle\, T, Q_B T\, \rangle + \frac{1}{3}\, t^3 \, \langle\, T, T * T\, \rangle \right]_{density}, \tag{3·3}$$

where $T = c_1|0\rangle$. We have already calculated these terms in (2·105) and (2·124), and so the potential at level 0 is

$$V(t) = \frac{1}{\alpha'^3 g_T^2} \left(-\frac{1}{2}\, t^2 - \frac{27\sqrt{3}}{64}\, t^3 \right). \tag{3·4}$$

The D25-brane tension T_{25} is given by[*)]

$$T_{25} = \frac{1}{2\pi^2 \alpha'^3 g_T^2}\,. \tag{3·5}$$

The tachyon potential normalized by the D25-brane tension is thus

$$\frac{V(t)}{T_{25}} = 2\pi^2 \left(-\frac{1}{2}\, t^2 - \frac{27\sqrt{3}}{64}\, t^3 \right). \tag{3·6}$$

[*)] The expression of T_{25} in terms of g_T can be found, for example, in Appendix A of Ref. 6).

This potential has a critical point at

$$t = t_* = -\frac{64}{81\sqrt{3}}\,, \tag{3·7}$$

and the value of the potential at the critical point is

$$\frac{V(t_*)}{T_{25}} = -\frac{4096\pi^2}{59049} \simeq -0.68\,. \tag{3·8}$$

This is about 68% of the value predicted by Sen's conjecture.

At level 2, the string field is truncated to

$$\Psi = t\, c_1|0\rangle + u\, c_{-1}|0\rangle + v\, L^{(m)}_{-2} c_1|0\rangle\,, \tag{3·9}$$

where t, u, and v are dynamical degrees of freedom. The potential as a function of t, u, and v is given by

$$\begin{aligned}\frac{V(t,u,v)}{T_{25}} = 2\pi^2 \biggl(& -\frac{1}{2}\,t^2 - \frac{27\sqrt{3}}{64}\,t^3 - \frac{1}{2}\,u^2 + \frac{1}{2}\,v^2 \\ & -\frac{33}{64}\sqrt{3}\,t^2 u + \frac{15}{64}\sqrt{39}\,t^2 v - \frac{19}{64\sqrt{3}}\,tu^2 - \frac{581}{192\sqrt{3}}\,tv^2 + \frac{55}{96}\sqrt{\frac{13}{3}}\,tuv \\ & -\frac{1}{64\sqrt{3}}\,u^3 + \frac{20951}{5184\sqrt{39}}\,v^3 + \frac{95}{1728}\sqrt{\frac{13}{3}}\,u^2 v - \frac{6391}{5184\sqrt{3}}\,uv^2 \biggr)\,. \end{aligned} \tag{3·10}$$

The equations of motion,

$$\frac{\partial V(t,u,v)}{\partial t} = 0\,, \qquad \frac{\partial V(t,u,v)}{\partial u} = 0\,, \qquad \frac{\partial V(t,u,v)}{\partial v} = 0\,, \tag{3·11}$$

can be solved numerically, and we find a critical point at

$$t = t_* \simeq -0.544\,, \qquad u = u_* \simeq -0.190\,, \qquad v = v_* \simeq -0.202\,. \tag{3·12}$$

The value of the potential at the critical point is

$$\frac{V(t_*, u_*, v_*)}{T_{25}} \simeq -0.959\,. \tag{3·13}$$

This is about 96% of the value predicted by Sen's conjecture.

The calculation up to level 18 was performed by Gaiotto and Rastelli[7] and the results up to level 12 are summarized in Table I.[*] As can be seen from the table, the value of the potential at the critical point normalized by the D25-brane tension is very close to the value -1 predicted by Sen's conjecture. We can think of the result of this analysis as evidence for Sen's conjecture, but we can also think of it as evidence that open string field theory can describe nonperturbative phenomena such as tachyon condensation.

While the improvement of the numerical solution is impressive, it is still an approximate solution. As we mentioned in the introduction, an analytic solution was constructed by Schnabl,[1] and it is for this problem of tachyon condensation. In order to understand his construction, we need to develop the CFT description of string field theory further.

[*] The current world record is level 26.[8]

Table I. The value of the potential at the critical point.

level	V/T_{25}
4	-0.987822
6	-0.995177
8	-0.997930
10	-0.999183
12	-0.999822

3.2. *Sliver frame*

The coordinate of the upper half-plane is useful and we can use the state-operator correspondence to represent a state. To describe the star product, there is a more convenient coordinate called the *sliver frame* introduced by Rastelli and Zwiebach.[9] The coordinate z of the sliver frame is related to the coordinate ξ of the upper half-plane by

$$z = f(\xi) = \frac{2}{\pi} \arctan \xi \,. \tag{3·14}$$

The right half of the open string $e^{i\theta}$ with $0 \le \theta \le \pi/2$ in the upper half-plane is mapped to the semi-infinite line with $\Re z = 1/2$ and $\Im z \ge 0$ in the sliver frame. The left half of the open string $e^{i\theta}$ with $\pi/2 \le \theta \le \pi$ in the upper half-plane is mapped to the semi-infinite line with $\Re z = -1/2$, $\Im z \ge 0$ in the sliver frame. The region $|\xi| \le 1$ with $\Im \xi \ge 0$ is mapped to the semi-infinite strip $-1/2 \le \Re z \le 1/2$, $\Im z \ge 0$. See Fig. 9. Since either of the right half or the left half of the open string is mapped to a semi-infinite line, the gluing of half open strings can be done simply by translation.

Let us recalculate $\langle\, T, Q_B T \,\rangle$ and $\langle\, T, T * T \,\rangle$ with $T = c_1|0\rangle$ in the sliver frame. The vacuum state $|0\rangle$ is represented in the sliver frame by the region $-1/2 \le \Re z \le 1/2$, $\Im z \ge 0$ without any operator insertions. Since

$$f \circ c(0) = \frac{1}{f'(0)}\, c(f(0)) = \frac{\pi}{2}\, c(0)\,, \tag{3·15}$$

the state $T = c_1|0\rangle$ is represented in the same region with an insertion of $\frac{\pi}{2}\, c(0)$.

Fig. 9. The sliver frame. The half unit disk of the upper half-plane of ξ in the left figure is mapped to the semi-infinite strip in the sliver frame with the coordinate z by the conformal transformation (3·14).

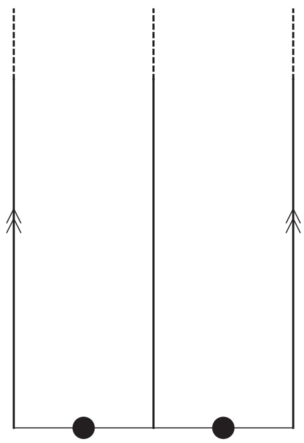

Fig. 10. The BPZ inner product in the sliver frame. The left semi-infinite line and the right semi-infinite line are identified by translation. BPZ inner products are represented by two-point functions on this semi-infinite cylinder.

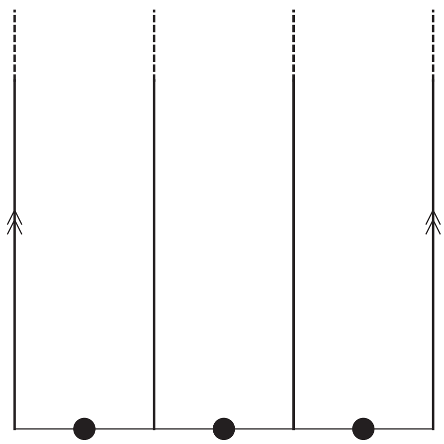

Fig. 11. The star product in the sliver frame. The left semi-infinite line and the right semi-infinite line are identified by translation. Star products are represented by three-point functions on this semi-infinite cylinder.

The state $Q_B T$ is also represented in the same region with an insertion of $\frac{\pi}{2}\, c\partial c(0)$. In the BPZ inner product $\langle\, T, Q_B T\,\rangle$, the region for $Q_B T$ can be translated to the region $1/2 \leq \Re z \leq 3/2$, $\Im z \geq 0$ with the insertion $\frac{\pi}{2}\, c\partial c(1)$, and the semi-infinite line $\Re z = -1/2$, $\Im z \geq 0$ of the state T and the semi-infinite line $\Re z = 3/2$, $\Im z \geq 0$ of the state $Q_B T$ are identified by translation. The resulting surface is a semi-infinite cylinder. See Fig. 10. Similarly, the inner product $\langle\, T, T * T\,\rangle$ can be represented by the region $-1/2 \leq \Re \leq 5/2$, $\Im z \geq 0$ with the semi-infinite line $\Re z = -1/2$, $\Im z \geq 0$ and the semi-infinite line $\Re z = 5/2$, $\Im z \geq 0$ are identified by translation, and the operators $\frac{\pi}{2}\, c(0)$, $\frac{\pi}{2}\, c(1)$ and $\frac{\pi}{2}\, c(2)$ are inserted. See Fig. 11. We denote by C_n the semi-infinite cylinder constructed from the upper half-plane of z by the identification $z \sim z+n$, and we represent it in the region $-1/2 \le \Re \le n-1/2$ unless otherwise specified. The formula for correlation functions on C_n can be derived from the formula (2·104) for the upper half-plane. The surface C_n can be mapped to the upper half-plane by

$$z' = g_n(z) = f^{-1}\Big(\frac{2z}{n}\Big) = \tan\frac{\pi\, z}{n}\,, \tag{3·16}$$

so we have

$$\begin{aligned}\langle\, c(z_1)\, c(z_2)\, c(z_3)\,\rangle_{C_n} &= \langle\, g_n\circ c(z_1)\;\, g_n\circ c(z_2)\;\, g_n\circ c(z_3)\,\rangle_{g_n\circ C_n}\\ &= \frac{1}{g_n'(z_1)\, g_n'(z_2)\, g_n'(z_3)}\,\langle\, c(g_n(z_1))\, c(g_n(z_2))\, c(g_n(z_3))\,\rangle_{\rm UHP}\,.\end{aligned} \tag{3·17}$$

From this we obtain the formula for correlation functions on C_n given by

$$\langle\, c(z_1)\, c(z_2)\, c(z_3)\,\rangle_{C_n,\, density} = \Big(\frac{n}{\pi}\Big)^3 \sin\frac{\pi(z_1-z_2)}{n}\,\sin\frac{\pi(z_1-z_3)}{n}\,\sin\frac{\pi(z_2-z_3)}{n}\,. \tag{3·18}$$

In the case of $\langle\, T, Q_B T\,\rangle$, we use the formula with $n=2$ and find

$$\langle\, T, Q_B T\,\rangle_{density} = -1\,. \tag{3·19}$$

In the case of $\langle\, T, T*T\,\rangle$, we use the formula with $n=3$ and find

$$\langle\, T, T*T\,\rangle_{density} = -\frac{81\sqrt{3}}{64}\,. \tag{3·20}$$

We have thus reproduced the previous results (2·105) and (2·124).

3.3. *Wedge state*

Let us now introduce a class of states called *wedge states*, which play an important role in the development of analytic methods in open string field theory. First consider the star product $|0\rangle * |0\rangle$. As we explained before, the star product $|0\rangle * |0\rangle$ is specified by $\langle\, \varphi, 0*0\,\rangle$ for an arbitrary state φ in the Fock space, where we represented $|0\rangle * |0\rangle$ by $0*0$ in the BPZ inner product. Using the sliver frame, the inner product $\langle\, \varphi, 0*0\,\rangle$ is given by a one-point function of $f\circ\varphi(0)$ on the semi-infinite cylinder C_3:

$$\langle\, \varphi, 0*0\,\rangle = \langle\, f\circ\varphi(0)\,\rangle_{C_3}\,. \tag{3·21}$$

Similarly, the star product $|0\rangle * |0\rangle * \ldots * |0\rangle$ from n vacuum states is specified by $\langle\, \varphi, 0*0*\ldots*0\,\rangle$ for an arbitrary state φ in the Fock space, where we represented $|0\rangle * |0\rangle * \ldots * |0\rangle$ by $0*0*\ldots*0$ in the BPZ inner product. The inner product $\langle\, \varphi, 0*0*\ldots*0\,\rangle$ is given by a one-point function of $f\circ\varphi(0)$ on the semi-infinite cylinder C_{n+1}:

$$\langle\, \varphi,\, \underbrace{0*0*\ldots*0}_{n}\,\rangle = \langle\, f\circ\varphi(0)\,\rangle_{C_{n+1}}\,. \tag{3·22}$$

Let us generalize these star products. The wedge state W_α for non-negative real number α is defined by

$$\langle\, \varphi, W_\alpha\,\rangle = \langle\, f\circ\varphi(0)\,\rangle_{C_{\alpha+1}} \tag{3·23}$$

for an arbitrary state φ in the Fock space. In this notation, the vacuum state $|0\rangle$ and the star product $|0\rangle * |0\rangle$ are W_1 and W_2, respectively. Let us consider star products of wedge states. The star product $W_\alpha * W_\beta$ is specified by $\langle\, \varphi, W_\alpha * W_\beta\,\rangle$, which is evaluated as follows:

$$\langle\, \varphi, W_\alpha * W_\beta\,\rangle = \langle\, f \circ \varphi(0)\,\rangle_{C_{\alpha+\beta+1}} = \langle\, \varphi, W_{\alpha+\beta}\,\rangle\,. \tag{3·24}$$

We therefore find

$$W_\alpha * W_\beta = W_{\alpha+\beta}\,. \tag{3·25}$$

Namely, wedge states are closed under the star multiplication. We also consider wedge states with operator insertions. We call this class of states *wedge-based states.* Wedge-based states are closed under the star multiplication and closed also under the BRST transformation. Therefore, the space of wedge-based states is a good place to look for solutions to the equation of motion: $Q_B\Psi + \Psi * \Psi = 0$.

It is interesting to consider the limit $\alpha \to \infty$ of the wedge state W_α. The inner product $\langle\, \varphi, W_\alpha\,\rangle$ in the limit $\alpha \to \infty$ is given by the one-point function of $f \circ \varphi(0)$ in the upper half-plane and is thus well defined and finite. The state obtained in the limit $\alpha \to \infty$ of W_α is called the *sliver state* and denoted by W_∞. If we formally take the limit $\alpha \to \infty$ and $\beta \to \infty$ in (3·25), we obtain

$$W_\infty * W_\infty = W_\infty\,. \quad \text{(formal)} \tag{3·26}$$

Namely, the sliver state W_∞ is formally a *star-algebra projector*. We keep saying that the sliver limit is formal because in general the sliver limit is not uniquely determined when multiple sliver states are involved. For example, the inner product $\langle\, \varphi_1, W_\infty * \varphi_2 * W_\infty\,\rangle$ for states φ_1 and φ_2 in the Fock space should be understood as the limit $\alpha \to \infty$ and $\beta \to \infty$ of $\langle\, \varphi_1, W_\alpha * \varphi_2 * W_\beta\,\rangle$, but the result depends on how we send α and β to infinity. Similarly, the inner product $\langle\, \varphi_1, W_\infty * \varphi_2 * W_\infty * W_\infty\,\rangle$, which is expected to coincide with $\langle\, \varphi_1, W_\infty * \varphi_2 * W_\infty\,\rangle$ from the relation (3·26), is also ambiguous and whether this coincides with $\langle\, \varphi_1, W_\infty * \varphi_2 * W_\infty\,\rangle$ depends on how we take the sliver limit.

3.4. *Derivatives of the wedge state*

Let us next consider the derivative of the wedge state W_α with respect to the parameter α:

$$W'_\alpha = \frac{dW_\alpha}{d\alpha} = \lim_{\beta\to 0} \frac{W_{\alpha+\beta} - W_\alpha}{\beta}\,. \tag{3·27}$$

Since $\langle\, \varphi, W_{\alpha+\beta}\,\rangle$ and $\langle\, \varphi, W_\alpha\,\rangle$ are represented on different surfaces, it is convenient to perform the conformal transformation $g(z)$ given by

$$g(z) = \frac{\alpha+1}{\alpha+\beta+1}\, z \tag{3·28}$$

such that the surface for $\langle\, \varphi, W_{\alpha+\beta}\,\rangle$ is independent of β:

$$\langle\, \varphi, W_{\alpha+\beta}\,\rangle = \langle\, f \circ \varphi(0)\,\rangle_{C_{\alpha+\beta+1}} = \langle\, g \circ f \circ \varphi(0)\,\rangle_{C_{\alpha+1}}\,. \tag{3·29}$$

Since

$$g(z) = \frac{\alpha+1}{\alpha+\beta+1}\, z = z - \frac{\beta}{\alpha+1}\, z + O(\beta^2)\,, \tag{3·30}$$

the transformation with infinitesimal β is generated by the energy-momentum tensor as follows:

$$\begin{aligned} \langle\, \varphi, W_{\alpha+\beta}\,\rangle &= \langle\, g\circ f\circ \varphi(0)\,\rangle_{C_{\alpha+1}} \\ &= \langle\, f\circ\varphi(0)\,\rangle_{C_{\alpha+1}} - \frac{\beta}{\alpha+1}\,\langle \oint \frac{dz}{2\pi i}\, z\, T(z)\, f\circ\varphi(0)\,\rangle_{C_{\alpha+1}} + O(\beta^2)\,, \end{aligned} \tag{3·31}$$

where the contour of the integral encircles the origin counterclockwise. The derivative of the wedge state W'_α is thus given by

$$\langle\, \varphi, W'_\alpha\,\rangle = -\frac{1}{\alpha+1}\,\langle \oint \frac{dz}{2\pi i}\, z\, T(z)\, f\circ\varphi(0)\,\rangle_{C_{\alpha+1}}\,. \tag{3·32}$$

Let us now deform the contour of the integral. We denote by V_α the vertical contour along the infinite line with $\Re z = \alpha$ in the complex z plane with the direction from $i\,\infty$ to $-i\,\infty$. Namely, it runs downwards. The contour of the integral in (3·32) can be deformed as follows:

$$\begin{aligned} \oint \frac{dz}{2\pi i}\, z\, T(z) &= \int_{-V_\epsilon} \frac{dz}{2\pi i}\, z\, T(z) + \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, T(z) \\ &= \int_{-V_{\alpha+1/2-\epsilon}} \frac{dz_+}{2\pi i}\, z_+\, T(z_+) + \int_{V_{-1/2+\epsilon}} \frac{dz_-}{2\pi i}\, z_-\, T(z_-)\,, \end{aligned} \tag{3·33}$$

where ϵ is an infinitesimal positive real number. Let us further deform the contour $V_{-1/2+\epsilon}$ in the coordinate z_- to $V_{-1/2-\epsilon}$, which is identified with $V_{\alpha+1/2-\epsilon}$ in the coordinate z_+ by the conformal transformation $z_+ = z_- + \alpha + 1$. We therefore have

$$\begin{aligned} &\int_{-V_{\alpha+1/2-\epsilon}} \frac{dz_+}{2\pi i}\, z_+\, T(z_+) + \int_{V_{-1/2+\epsilon}} \frac{dz_-}{2\pi i}\, z_-\, T(z_-) \\ &= \int_{-V_{\alpha+1/2-\epsilon}} \frac{dz_+}{2\pi i}\, z_+\, T(z_+) + \int_{V_{\alpha+1/2-\epsilon}} \frac{dz_+}{2\pi i}\, (z_+ - \alpha - 1)\, T(z_+) \\ &= -(\alpha+1) \int_{V_{\alpha+1/2-\epsilon}} \frac{dz_+}{2\pi i}\, T(z_+)\,. \end{aligned} \tag{3·34}$$

The upshot is

$$\langle\, \varphi, W'_\alpha\,\rangle = \langle\, f\circ\varphi(0) \int_{V_\gamma} \frac{dz}{2\pi i}\, T(z)\,\rangle_{C_{\alpha+1}}\,, \tag{3·35}$$

where γ can be any value in the range $1/2 < \gamma < \alpha + 1/2$. We have learned that taking a derivative with respect to α corresponds to an insertion of the line integral of the energy-momentum tensor given by

$$\mathcal{L} \equiv \int_{V_\gamma} \frac{dz}{2\pi i}\, T(z) \tag{3·36}$$

with γ in the region of the wedge state. We write

$$\langle \varphi, W'_\alpha \rangle = \langle f \circ \varphi(0) \, \mathcal{L} \rangle_{C_{\alpha+1}} \tag{3·37}$$

with the understanding that the location of the line integral $\mathcal{L}$ is to the right of the operator $f \circ \varphi(0)$, although in this case we could bring $\mathcal{L}$ to the left of $f \circ \varphi(0)$ using the identification $z \sim z + \alpha + 1$. This can be generalized to higher derivatives:

$$\langle \varphi, W^{(n)}_\alpha \rangle = \langle f \circ \varphi(0) \int_{V_{\gamma_1}} \frac{dz}{2\pi i} T(z) \int_{V_{\gamma_2}} \frac{dz}{2\pi i} T(z) \ldots \int_{V_{\gamma_n}} \frac{dz}{2\pi i} T(z) \rangle_{C_{\alpha+1}} \,, \tag{3·38}$$

where γ_i can be any value in the range $1/2 < \gamma_i < \alpha + 1/2$. It is not difficult to confirm that insertions of the line integral commute using the OPE of the energy-momentum tensor with itself. We therefore write (3·38) as follows:

$$\langle \varphi, W^{(n)}_\alpha \rangle = \langle f \circ \varphi(0) \underbrace{\mathcal{L}\mathcal{L} \ldots \mathcal{L}}_{n} \rangle_{C_{\alpha+1}} = \langle f \circ \varphi(0) \, \mathcal{L}^n \rangle_{C_{\alpha+1}} \,. \tag{3·39}$$

We can also write $W_{\alpha+\beta}$ as

$$\begin{aligned} \langle \varphi, W_{\alpha+\beta} \rangle &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle f \circ \varphi(0) \int_{V_{\gamma_1}} \frac{dz}{2\pi i} T(z) \int_{V_{\gamma_2}} \frac{dz}{2\pi i} T(z) \ldots \int_{V_{\gamma_n}} \frac{dz}{2\pi i} T(z) \rangle_{C_{\alpha+1}} \\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle f \circ \varphi(0) \underbrace{\mathcal{L}\mathcal{L} \ldots \mathcal{L}}_{n} \rangle_{C_{\alpha+1}} = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle f \circ \varphi(0) \, \mathcal{L}^n \rangle_{C_{\alpha+1}} \,. \end{aligned} \tag{3·40}$$

The correspondence between taking a derivative with respect to α and an insertion of the line integral of the energy-momentum tensor is exactly what we expected. In the strip coordinate, the operator e^{-tL_0} generates a rectangular surface. Taking a derivative with respect to t corresponds to an insertion of $-L_0$:

$$\partial_t \, e^{-tL_0} = -L_0 \, e^{-tL_0} \,. \tag{3·41}$$

The line integral $\mathcal{L}$ is locally the same as $-L_0$ in the strip coordinate. There is, however, an important difference. The operator $-L_0$ is a line integral of the energy-momentum tensor from one boundary to the other boundary. The operator $\mathcal{L}$ is a line integral of the energy-momentum tensor from the open string midpoint to the boundary before using the doubling trick. In the coordinate ξ of the upper half-plane, the integral is written as

$$\int \frac{dz}{2\pi i} T(z) \to \int \frac{d\xi}{2\pi i} \left(\frac{dz}{d\xi} \right)^{-1} T(\xi) = \frac{\pi}{2} \int \frac{d\xi}{2\pi i} \, (\xi^2 + 1) \, T(\xi) \,. \tag{3·42}$$

Namely, the operator $\mathcal{L}$ is a half integral of

$$\frac{\pi}{2} (L_1 + L_{-1}) \equiv \frac{\pi}{2} K_1 \,. \tag{3·43}$$

When an operator $\mathcal{O}$ is defined by an integral along the unit circle counterclockwise,

$$\mathcal{O} = \oint \frac{d\xi}{2\pi i} \, \phi(\xi) \,, \tag{3·44}$$

where $\phi(\xi)$ can be a field or a field multiplied by a function of ξ, we define the right half integral $\mathcal{O}^R$ and the left half integral $\mathcal{O}^L$ by

$$\mathcal{O}^R = \int_{C_R} \frac{d\xi}{2\pi i}\, \phi(\xi)\,, \qquad \mathcal{O}^L = \int_{C_L} \frac{d\xi}{2\pi i}\, \phi(\xi)\,, \tag{3·45}$$

where the contour C_R runs along the right half of the unit circle from $-i$ to i counterclockwise and the contour C_L runs along the left half of the unit circle from i to $-i$ counterclockwise.*) Using this notation, the state W_1' can be written as

$$W_1' = \frac{\pi}{2}\, K_1^L \, |0\rangle \tag{3·46}$$

or as

$$W_1' = -\frac{\pi}{2}\, K_1^R \, |0\rangle\,. \tag{3·47}$$

3.5. *Eigenstates of the operator L*

The space of wedge-based states is closed under star multiplication and under the BRST transformation so that it is a good subspace of string fields to look for solutions to the equation of motion. Schnabl[1)] found a much smaller subspace closed under star multiplication and under the BRST transformation and introduced a basis of the space based on eigenvalues of the operator L, which is the generator of dilatation in the sliver space:**)

$$L = \oint \frac{dz}{2\pi i}\, z\, T(z) = \oint \frac{d\xi}{2\pi i}\, \frac{f(\xi)}{f'(\xi)}\, T(\xi) = L_0 + \frac{2}{3}\, L_2 - \frac{2}{15}\, L_4 + \ldots\,. \tag{3·48}$$

The vacuum state $|0\rangle$ is an eigenstate of L, and its eigenvalue is 0:

$$L\, |0\rangle = 0\,. \tag{3·49}$$

Next consider the vacuum state $|0\rangle$ with an insertion of $\mathcal{L}$, which is W_1'. Since

$$\int_{V_\gamma} \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\, z\, T(z)\, T(w) = \int_{V_\gamma} \frac{dw}{2\pi i}\, T(w)\,, \tag{3·50}$$

where the contour of the integral over z encircles the point w counterclockwise, the state W_1' is an eigenstate of L, and its eigenvalue is 1:

$$L\, W_1' = W_1'\,. \tag{3·51}$$

Similarly, we find that $W_1^{(n)}$ is an eigenstate of L with its eigenvalue n:

$$L\, W_1^{(n)} = n\, W_1^{(n)}\,. \tag{3·52}$$

Schnabl also considered insertions of the line integral $\mathcal{B}$, which is obtained from $\mathcal{L}$ by replacing the energy-momentum tensor with the b ghost:

$$\mathcal{B} \equiv \int_{V_\gamma} \frac{dz}{2\pi i}\, b(z)\,. \tag{3·53}$$

*) In Ref. 1) Schnabl's $\mathcal{O}^L$ is our $\mathcal{O}^R$, and Schnabl's $\mathcal{O}^R$ is our $\mathcal{O}^L$.

**) The operator L here is denoted by $\mathcal{L}_0$ in Ref. 1).

The vacuum state $|0\rangle$ with an insertion of $\mathcal{B}$ whose contraction with φ given by

$$\langle\, f \circ \varphi(0)\, \mathcal{B}\, \rangle_{C_2} = \langle\, f \circ \varphi(0) \int_{V_\gamma} \frac{dz}{2\pi i}\, b(z)\, \rangle_{C_2}\,, \tag{3·54}$$

where $1/2 < \gamma < 3/2$, is an eigenstate of L with its eigenvalue 1 because

$$\int_{V_\gamma} \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\, z\, T(z)\, b(w) = \int_{V_\gamma} \frac{dw}{2\pi i}\, b(w)\,, \tag{3·55}$$

where the contour of the integral over z encircles the point w counterclockwise. If we define the line integral B_1 by

$$B_1 \equiv b_1 + b_{-1} = \oint \frac{d\xi}{2\pi i}\, (\xi^2 + 1)\, b(\xi)\,, \tag{3·56}$$

the vacuum state $|0\rangle$ with an insertion of $\mathcal{B}$ can be written using the half integral B_1^L as

$$\frac{\pi}{2}\, B_1^L\, |0\rangle \tag{3·57}$$

or using B_1^R as

$$-\frac{\pi}{2}\, B_1^R\, |0\rangle\,. \tag{3·58}$$

We can confirm from the OPE of the energy-momentum tensor and the b ghost that

$$\int_{V_\gamma} \frac{dz}{2\pi i}\, T(z) \int_{V_{\gamma+\epsilon}} \frac{dw}{2\pi i}\, b(w) = \int_{V_{\gamma-\epsilon}} \frac{dw}{2\pi i}\, b(w) \int_{V_\gamma} \frac{dz}{2\pi i}\, T(z)\,, \tag{3·59}$$

where $\epsilon > 0$. We write

$$\mathcal{L}\, \mathcal{B} = \mathcal{B}\, \mathcal{L} \tag{3·60}$$

with the understanding that the contour of the right integral is to the right of the contour of the left integral. We also write it as the commutation relation:

$$[\, \mathcal{B}, \mathcal{L}\,] = 0\,. \tag{3·61}$$

Another important property is that

$$\int_{V_{\gamma_1}} \frac{dz_1}{2\pi i}\, b(z_1) \int_{V_{\gamma_2}} \frac{dz_2}{2\pi i}\, b(z_2) = 0 \tag{3·62}$$

when there are no operators in the region $\gamma_1 < \Re z < \gamma_2$. We write

$$\mathcal{B}^2 = 0\,. \tag{3·63}$$

Finally, consider local insertions of the c ghost and its derivatives on the boundary. For example, the state defined by the contraction with φ given by

$$\langle\, f \circ \varphi(0)\, c\partial^2 c\partial^5 c(1)\, \rangle_{C_2} \tag{3·64}$$

is an eigenstate of L with its eigenvalue $2+5-3=4$, where the positive contribution $2+5$ comes from the number of derivatives and the negative contribution -3 comes from the number of c ghosts.

We have found that a set of L eigenstates in the form

$$\langle\, f\circ\varphi(0)\,\prod_i \partial^{n_i} c(1)\,\mathcal{L}^m\,\rangle_{C_2} \tag{3·65}$$

or

$$\langle\, f\circ\varphi(0)\,\prod_i \partial^{n_i} c(1)\,\mathcal{L}^m\,\mathcal{B}\,\rangle_{C_2}\,. \tag{3·66}$$

This is the basis of the states introduced in Ref. 1).*) States in this class are closed under the BRST transformation and under the star multiplication. First, the BRST operator acts on the ingredients of the states as follows:

$$Q_B\cdot\prod_i \partial^{n_i} c(1)=\prod_j \partial^{n'_j} c(1)\,,\qquad Q_B\cdot\mathcal{B}=\mathcal{L}\,,\qquad Q_B\cdot\mathcal{L}=0\,. \tag{3·67}$$

Thus the set of states is closed under the BRST transformation. Note also that the eigenvalue of L does not change under the BRST transformation because $[\,Q_B, L\,]=0$. Let us next consider the star product of two states in the class (3·66):

$$\langle\, f\circ\varphi(0)\,\prod_i \partial^{n_i} c(1)\,\mathcal{L}^m\,\mathcal{B}\,\prod_j \partial^{k_j} c(2)\,\mathcal{L}^l\,\mathcal{B}\,\rangle_{C_3}\,. \tag{3·68}$$

We first move the line integrals $\mathcal{L}^m\,\mathcal{B}$ in the region $1<\Re z<2$ to the right of the operators at $z=2$. For the line integral $\mathcal{B}$, we have

$$\int_{V_{z-\epsilon}}\frac{dw}{2\pi i}\,b(w)\,c(z)=\oint\frac{dw}{2\pi i}\,b(w)\,c(z)-c(z)\int_{V_{z+\epsilon}}\frac{dw}{2\pi i}\,b(w)\,, \tag{3·69}$$

where $\epsilon>0$ and the contour of the integral of the first term on the right-hand side encircles the point z counterclockwise. We therefore find

$$\int_{V_{z-\epsilon}}\frac{dw}{2\pi i}\,b(w)\,c(z)=1-c(z)\int_{V_{z+\epsilon}}\frac{dw}{2\pi i}\,b(w)\,, \tag{3·70}$$

which can be written as

$$\mathcal{B}\,c(z)=1-c(z)\,\mathcal{B} \tag{3·71}$$

or as the anticommutation relation in the form

$$\{\,\mathcal{B}, c(z)\,\}=1\,. \tag{3·72}$$

Taking derivatives with respect to z, we also find

$$\mathcal{B}\,\partial^n c(z)+\partial^n c(z)\,\mathcal{B}=0\quad\text{for}\quad n\geq 1\,. \tag{3·73}$$

*) To be more precise, states in this set do not in general satisfy the reality condition of the string field,[10] and the set of states introduced in Ref. 1) is a refined set of states such that the reality condition is satisfied. In what follows we neglect the reality condition to simplify the presentation.

For the line integral $\mathcal{L}$, we have

$$\int_{V_{z-\epsilon}} \frac{dw}{2\pi i}\, T(w)\, \varphi(z) = \oint \frac{dw}{2\pi i}\, T(w)\, \varphi(z) + \varphi(z) \int_{V_{z+\epsilon}} \frac{dw}{2\pi i}\, T(w)\,, \tag{3·74}$$

where $\varphi(z)$ is an arbitrary operator and the contour of the integral of the first term on the right-hand side encircles the point z counterclockwise. We therefore find

$$\int_{V_{z-\epsilon}} \frac{dw}{2\pi i}\, T(w)\, \varphi(z) = \partial\varphi(z) + \varphi(z) \int_{V_{z+\epsilon}} \frac{dw}{2\pi i}\, T(w)\,, \tag{3·75}$$

which can be written as

$$\mathcal{L}\, \varphi(z) = \partial\varphi(z) + \varphi(z)\, \mathcal{L} \tag{3·76}$$

or as the commutation relation given by

$$[\, \mathcal{L}, \varphi(z)\,] = \partial\varphi(z)\,. \tag{3·77}$$

After moving the line integrals, we have states which can be expanded in the states of the form

$$\langle\, f \circ \varphi(0) \prod_i \partial^{n_i} c(1) \prod_j \partial^{m_j} c(2)\, \mathcal{L}^k\, \mathcal{B}\, \rangle_{C_3}\,. \tag{3·78}$$

Note that $[\, \mathcal{B}, \mathcal{L}\,] = 0$ and $\mathcal{B}^2 = 0$ so that there are no states with two insertions of $\mathcal{B}$. Similarly, the star product of two states in the class (3·65) can be transformed to the form expanded in the states of the form

$$\langle\, f \circ \varphi(0) \prod_i \partial^{n_i} c(1) \prod_j \partial^{m_j} c(2)\, \mathcal{L}^k\, \rangle_{C_3}\,, \tag{3·79}$$

and the star product of a state in the class (3·65) and a state in the class (3·66) can be transformed to the form expanded in the states of the forms (3·78) and (3·79). We can then expand the c ghosts around $z = 1$ to obtain states of the form

$$\langle\, f \circ \varphi(0) \prod_i \partial^{n_i} c(1)\, \mathcal{L}^k\, \rangle_{C_3} \tag{3·80}$$

or of the form

$$\langle\, f \circ \varphi(0) \prod_i \partial^{n_i} c(1)\, \mathcal{L}^k\, \mathcal{B}\, \rangle_{C_3}\,. \tag{3·81}$$

Finally, we use the formula (3·40) to express the states based on the surface C_3 in terms of states based on the surface C_2, which are states in the classes (3·65) and (3·66). We have thus shown that the set of states in the classes (3·65) and (3·66) is closed under the star multiplication.

Furthermore, the basis of the states (3·65) and (3·66) has the following property. Let us denote the set of states in the basis by $\{\phi_i\}$ and the eigenvalue of L for ϕ_i by h_i. We have shown that the star product of two states ϕ_1 and ϕ_2 in the basis can be expanded as

$$\phi_1 * \phi_2 = \sum_i c_i\, \phi_i\,, \tag{3·82}$$

where c_i's are nonvanishing coefficients. By carefully investigating each step of the proof, we find that the eigenvalue h_i of a state which appears on the right-hand side cannot be less than the sum of the eigenvalues h_1 and h_2:

$$h_i \geq h_1 + h_2 \,. \tag{3·83}$$

This allows us to solve the equation of motion recursively.

In the sector of ghost number 1, the lowest eigenvalue of L is -1, and the corresponding eigenstate $\phi^{(-1)}$ can be expressed by its contraction with φ given by

$$\langle\, \varphi, \phi^{(-1)} \,\rangle = \langle\, f \circ \varphi(0)\, c(1) \,\rangle_{C_2} \,. \tag{3·84}$$

Namely,

$$\phi^{(-1)} = \frac{2}{\pi}\, c_1 |0\rangle \,. \tag{3·85}$$

There are three states in the basis with eigenvalue 0, which we call $\phi_1^{(0)}$, $\phi_2^{(0)}$, and $\phi_3^{(0)}$. Their inner products with φ are given by

$$\begin{aligned} &\langle\, \varphi, \phi_1^{(0)} \,\rangle = \langle\, f \circ \varphi(0)\, \partial c(1) \,\rangle_{C_2} \,, \qquad \langle\, \varphi, \phi_2^{(0)} \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L} \,\rangle_{C_2} \,, \\ &\langle\, \varphi, \phi_3^{(0)} \,\rangle = \langle\, f \circ \varphi(0)\, c\partial c(1)\, \mathcal{B} \,\rangle_{C_2} \,. \end{aligned} \tag{3·86}$$

3.6. *Schnabl gauge*

While we can now start looking for solutions in this subspace of string fields, Schnabl further restricted the space of states by imposing a gauge condition. It is an analog of the Siegel gauge condition $b_0 \Psi = 0$ in the sliver frame and is given by

$$B \Psi = 0 \,, \tag{3·87}$$

where*)

$$B = \oint \frac{dz}{2\pi i}\, z\, b(z) = \oint \frac{d\xi}{2\pi i}\, \frac{f(\xi)}{f'(\xi)}\, b(\xi) = b_0 + \frac{2}{3}\, b_2 - \frac{2}{15}\, b_4 + \ldots \,. \tag{3·88}$$

This gauge condition is called the *Schnabl gauge* condition. The eigenstate $\phi^{(-1)}$ with eigenvalue -1 satisfies the Schnabl gauge condition:

$$B\phi^{(-1)} = 0 \,. \tag{3·89}$$

While it is obvious from the mode expansion of B and $\phi^{(-1)} \propto c_1|0\rangle$, it can be shown using the CFT expression as follows:

$$\langle\, \varphi, B\phi^{(-1)} \,\rangle = \langle\, f \circ \varphi(0)\, B \cdot c(1) \,\rangle_{C_2} = \langle\, f \circ \varphi(0) \oint \frac{dz}{2\pi i}\, (z-1)\, b(z)\, c(1) \,\rangle_{C_2} = 0 \,. \tag{3·90}$$

Let us next calculate $B\phi_1^{(0)}$, $B\phi_2^{(0)}$, and $B\phi_3^{(0)}$. The first one is

$$\langle\, \varphi, B\phi_1^{(0)} \,\rangle = \langle\, f \circ \varphi(0) \oint \frac{dz}{2\pi i}\, (z-1)\, b(z)\, \partial c(1) \,\rangle_{C_2} = \langle\, f \circ \varphi(0) \,\rangle_{C_2} \,. \tag{3·91}$$

*) The operator B here is denoted by $\mathcal{B}_0$ in Ref. 1).

Namely, $B\phi_1^{(0)} = |0\rangle$. For $B\phi_2^{(0)}$, we find

$$\begin{aligned}
\langle \varphi, B\phi_2^{(0)} \rangle &= \langle\, f \circ \varphi(0) \oint \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c(1)\,\mathcal{L}\,\rangle_{C_2} \\
&= \langle\, f \circ \varphi(0) \left[\int_{V_{1/2}} \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c(1)\,\mathcal{L} + c(1)\,\mathcal{L} \int_{V_{3/2}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \right] \rangle_{C_2} \\
&= \langle\, f \circ \varphi(0) \left[\int_{V_{1-\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c(1) + c(1) \int_{V_{1+\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \right] \mathcal{L}\,\rangle_{C_2} \\
&\quad - \langle\, f \circ \varphi(0)\,c(1)\,\mathcal{B}\,\rangle_{C_2} \\
&= -\langle\, f \circ \varphi(0)\,c(1)\,\mathcal{B}\,\rangle_{C_2}\,,
\end{aligned} \tag{3·92}$$

where we have used

$$\begin{aligned}
\mathcal{L} \int_{V_{\gamma+\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z) &= \int_{V_\gamma} \frac{dw}{2\pi i}\,T(w) \int_{V_{\gamma+\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \\
&= \int_{V_{\gamma-\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \int_{V_\gamma} \frac{dw}{2\pi i}\,T(w) - \int_{V_\gamma} \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\,(z-1)\,b(z)\,T(w) \\
&= \int_{V_{\gamma-\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z)\,\mathcal{L} - \mathcal{B}\,.
\end{aligned} \tag{3·93}$$

For $B\phi_3^{(0)}$, we find

$$\begin{aligned}
\langle \varphi, B\phi_3^{(0)} \rangle &= \langle\, f \circ \varphi(0) \oint \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c\partial c(1)\,\mathcal{B}\,\rangle_{C_2} \\
&= \langle\, f \circ \varphi(0) \left[\int_{V_{1/2}} \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c\partial c(1)\,\mathcal{B} + c\partial c(1)\,\mathcal{B} \int_{V_{3/2}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \right] \rangle_{C_2} \\
&= \langle\, f \circ \varphi(0) \left[\int_{V_{1-\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z)\,c\partial c(1) - c\partial c(1) \int_{V_{1+\epsilon}} \frac{dz}{2\pi i}\,(z-1)\,b(z) \right] \mathcal{B}\,\rangle_{C_2} \\
&= -\,\langle\, f \circ \varphi(0)\,c(1)\,\mathcal{B}\,\rangle_{C_2}\,.
\end{aligned} \tag{3·94}$$

Therefore, the linear combination $\phi_2^{(0)} - \phi_3^{(0)}$ satisfies the Schnabl gauge condition:

$$B\,(\phi_2^{(0)} - \phi_3^{(0)}) = 0\,. \tag{3·95}$$

In fact, the states $\phi^{(-1)}$ and $\phi_2^{(0)} - \phi_3^{(0)}$ can be written as

$$\phi^{(-1)} = \psi_0\,, \qquad \phi_2^{(0)} - \phi_3^{(0)} = \psi'_0 \tag{3·96}$$

using ψ_n labeled by a real parameter n defined by

$$\begin{aligned}
\langle \varphi, \psi_n \rangle &= \langle\, f \circ \varphi(0)\,c(1)\,\mathcal{B}\,c(n+1)\,\rangle_{C_{n+2}} \\
&= \langle\, f \circ \varphi(0)\,c(1) \int_{V_\gamma} \frac{dz}{2\pi i}\,b(z)\,c(n+1)\,\rangle_{C_{n+2}}
\end{aligned} \tag{3·97}$$

with $1 < \gamma < n+1$, and its derivative ψ_n' with respect to n:

$$\psi_n' \equiv \frac{d\psi_n}{dn}\,. \tag{3·98}$$

The string field ψ_n is an important ingredient of the analytic solution by Schnabl.

We will show shortly that ψ_n satisfies the Schnabl gauge condition for any n, but let us first prove the relations in (3·96). The string field ψ_0 should be understood as the limit $n \to 0$ of ψ_n. In the limit, the three operators $c(1)\,\mathcal{B}\,c(n+1)$ collide. In taking the limit, we first move the line integral $\mathcal{B}$ to the right of $c(n+1)$ using the anticommutation relation (3·72). The inner product $\langle\, \varphi, \psi_n \,\rangle$ can then be written as

$$\langle\, \varphi, \psi_n \,\rangle = \langle\, f \circ \varphi(0)\, c(1) \,\rangle_{C_{n+2}} - \langle\, f \circ \varphi(0)\, c(1)\, c(n+1)\, \mathcal{B} \,\rangle_{C_{n+2}}\,. \tag{3·99}$$

It is now straightforward to take the limit $n \to 0$. The second term on the right-hand side vanishes in the limit because two c ghosts collide. We thus have

$$\langle\, \varphi, \psi_0 \,\rangle = \lim_{n\to 0} \langle\, \varphi, \psi_n \,\rangle = \langle\, f \circ \varphi(0)\, c(1) \,\rangle_{C_2}\,, \tag{3·100}$$

or

$$\psi_0 = \lim_{n\to 0} \psi_n = \frac{2}{\pi}\, c_1 |0\rangle\,. \tag{3·101}$$

We have thus shown the first relation in (3·96).

To prove the second relation in (3·96), let us express $\langle\, \varphi, \psi_n' \,\rangle$ in the CFT language. This can be done in a way analogous to that we did for $\langle\, \varphi, W_\alpha' \,\rangle$, and the final result is

$$\langle\, \varphi, \psi_n' \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(n+1) \,\rangle_{C_{n+2}}\,. \tag{3·102}$$

In the limit $n \to 0$, the four operators $c(1)\,\mathcal{L}\,\mathcal{B}\,c(n+1)$ collide. We first move $\mathcal{B}$ to the right of $c(n+1)$ to obtain

$$\langle\, \varphi, \psi_n' \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L} \,\rangle_{C_{n+2}} - \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, c(n+1)\, \mathcal{B} \,\rangle_{C_{n+2}}\,. \tag{3·103}$$

We then use the commutation relation (3·77) to find

$$\begin{aligned}\langle\, \varphi, \psi_n' \,\rangle = {}& \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L} \,\rangle_{C_{n+2}} - \langle\, f \circ \varphi(0)\, c(1)\, \partial c(n+1)\, \mathcal{B} \,\rangle_{C_{n+2}} \\ & - \langle\, f \circ \varphi(0)\, c(1)\, c(n+1)\, \mathcal{L}\, \mathcal{B} \,\rangle_{C_{n+2}}\,.\end{aligned} \tag{3·104}$$

The third term on the right-hand side vanishes in the limit $n \to 0$ because two c ghosts collide. We thus have

$$\langle\, \varphi, \psi_0' \,\rangle = \lim_{n\to 0} \langle\, \varphi, \psi_n' \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L} \,\rangle_{C_2} - \langle\, f \circ \varphi(0)\, c\partial c(1)\, \mathcal{B} \,\rangle_{C_2}\,. \tag{3·105}$$

The second relation in (3·96) has now been shown.

Let us move to the proof that ψ_n satisfies the Schnabl gauge condition:

$$B\, \psi_n = 0 \tag{3·106}$$

for any n. It then follows that

$$B\, \psi_0 = 0\,, \quad B\, \psi_0' = 0\,, \quad \ldots\,, \quad B\, \psi_0^{(n)} = 0\,, \quad \ldots\,. \tag{3·107}$$

What we need to prove is that $\langle\, \varphi, B\,\psi_n \,\rangle = 0$. We know how the operator B acts on a state φ in the Fock space. In the sliver frame, the state $B\varphi$ corresponds to the operator

$$\oint \frac{dz}{2\pi i}\, z\, b(z)\, f\circ\varphi(0)\,, \tag{3·108}$$

where the contour of the integral encircles the origin counterclockwise. On the other hand, we do not know how B acts on general wedge-based states. We therefore use the relation

$$\langle\, \varphi, B\,\psi_n \,\rangle = -\langle\, B^\star\varphi, \psi_n \,\rangle \tag{3·109}$$

and consider $B^\star\varphi$. Here the minus sign on the right-hand side comes from the anticommutation of $B^\star$ and φ because φ has to carry ghost number 3 for the inner product to be nonvanishing and thus φ is Grassmann odd. The BPZ conjugation is defined in the upper half-plane of ξ using the conformal map

$$\xi' = I(\xi) = -\frac{1}{\xi}\,. \tag{3·110}$$

In the sliver frame z, we have to use the map

$$z' = f(\xi') = f\circ I(\xi) = f\circ I\circ f^{-1}(z)\,. \tag{3·111}$$

The operator B in the sliver frame z can be decomposed into two pieces:

$$B = -\int_{V_{1/2}} \frac{dz}{2\pi i}\, z\, b(z) + \int_{V_{-1/2}} \frac{dz}{2\pi i}\, z\, b(z)\,. \tag{3·112}$$

For the first piece, the coordinates z' and z are related as $z' = z - 1$. The BPZ conjugation of the first piece is thus given by

$$-\int_{V_{1/2}} \frac{dz}{2\pi i}\, z\, b(z) \to -\int_{V_{-1/2}} \frac{dz'}{2\pi i}\, (z'+1)\, b(z')\,. \tag{3·113}$$

For the second piece, the coordinates z' and z are related as $z' = z + 1$. The BPZ conjugation of the second piece is thus given by

$$\int_{V_{-1/2}} \frac{dz}{2\pi i}\, z\, b(z) \to \int_{V_{1/2}} \frac{dz'}{2\pi i}\, (z'-1)\, b(z')\,. \tag{3·114}$$

Combining the two pieces, we obtain the following expression for the inner product $\langle\, B^\star\varphi, \psi_n \,\rangle$:

$$\begin{aligned}
\langle\, B^\star\varphi, \psi_n \,\rangle &= -\langle\, f\circ\varphi(0) \int_{V_{1/2}} \frac{dz}{2\pi i}\, (z-1)\, b(z)\, c(1)\, \mathcal{B}\, c(n+1) \,\rangle_{C_{n+2}} \\
&\quad -\langle \int_{V_{-1/2}} \frac{dz}{2\pi i}\, (z+1)\, b(z)\, f\circ\varphi(0)\, c(1)\, \mathcal{B}\, c(n+1) \,\rangle_{C_{n+2}} \\
&= -\langle\, f\circ\varphi(0) \int_{V_{1/2}} \frac{dz}{2\pi i}\, (z-1)\, b(z)\, c(1)\, \mathcal{B}\, c(n+1) \,\rangle_{C_{n+2}} \\
&\quad -\langle\, f\circ\varphi(0)\, c(1)\, \mathcal{B}\, c(n+1) \int_{V_{n+3/2}} \frac{dz}{2\pi i}\, (z-n-1)\, b(z) \,\rangle_{C_{n+2}}\,,
\end{aligned} \tag{3·115}$$

where the minus sign of the term with the line integral along $V_{1/2}$ is from the anticommutation of $b(z)$ and $f \circ \varphi(0)$ and we used the identification $z \sim z+n+2$ of the surface C_{n+2}. An important point of this expression is that the integrand of the line integral along $V_{1/2}$ has a zero at the location of the c ghost insertion $c(1)$ and the integrand of the line integral along $V_{n+3/2}$ has a zero at the location of the c ghost insertion $c(n+1)$. Therefore, the pole of the OPE between the b and c ghosts is canceled for each case and we can simply anticommute the line integral and the c ghost to find

$$\begin{aligned}\langle B^\star \varphi, \psi_n \rangle = & \langle f \circ \varphi(0)\, c(1) \int_{V_{1+n/2}} \frac{dz}{2\pi i} (z-1)\, b(z)\, \mathcal{B}\, c(n+1) \rangle_{C_{n+2}} \\ & + \langle f \circ \varphi(0)\, c(1)\, \mathcal{B} \int_{V_{1+n/2}} \frac{dz}{2\pi i} (z-n-1)\, b(z)\, c(n+1) \rangle_{C_{n+2}} . \end{aligned} \tag{3·116}$$

Now the part of the line integral at $\Re z = 1 + n/2$ proportional to $\mathcal{B}$ vanishes because $\mathcal{B}^2 = 0$,

$$\begin{aligned}\langle B^\star \varphi, \psi_n \rangle = & \langle f \circ \varphi(0)\, c(1) \int_{V_{1+n/2}} \frac{dz}{2\pi i}\, z\, b(z)\, \mathcal{B}\, c(n+1) \rangle_{C_{n+2}} \\ & + \langle f \circ \varphi(0)\, c(1)\, \mathcal{B} \int_{V_{1+n/2}} \frac{dz}{2\pi i}\, z\, b(z)\, c(n+1) \rangle_{C_{n+2}} , \end{aligned} \tag{3·117}$$

and the remaining part of the line integral at $\Re z = 1 + n/2$ anticommutes with $\mathcal{B}$ to find

$$\langle B^\star \varphi, \psi_n \rangle = 0 . \tag{3·118}$$

We have thus shown that ψ_n satisfies the Schnabl gauge condition for any n.

3.7. *Schnabl's analytic solution*

The space of string fields where we look for solutions to the equation of motion has been narrowed down to the form

$$\alpha_0\, \psi_0 + \alpha_1\, \psi_0' + \alpha_2\, \psi_0'' + \alpha_3\, \psi_0^{(3)} + \ldots + \alpha_n\, \psi_0^{(n)} + \ldots , \tag{3·119}$$

and we know that the coefficients α_i can be determined recursively. Schnabl calculated them to find

$$\begin{aligned}\Psi = & -\psi_0 + \frac{1}{2}\psi_0' - \frac{1}{6}\frac{\psi_0''}{2!} + \frac{1}{30}\frac{\psi_0^{(4)}}{4!} - \frac{1}{42}\frac{\psi_0^{(6)}}{6!} + \frac{1}{30}\frac{\psi_0^{(8)}}{8!} \\ & - \frac{5}{66}\frac{\psi_0^{(10)}}{10!} + \frac{691}{2730}\frac{\psi_0^{(12)}}{12!} - \frac{7}{6}\frac{\psi_0^{(14)}}{14!} + \ldots . \end{aligned} \tag{3·120}$$

What would be the general form of the coefficient α_i? Schnabl inferred that the coefficient α_i can be written using the Bernoulli number B_n as follows:

$$\Psi = -\sum_{n=0}^{\infty} \frac{B_n}{n!}\, \psi_0^{(n)} , \tag{3·121}$$

where the Bernoulli number B_n is defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n\, x^n}{n!}\,. \tag{3·122}$$

We will later prove that Ψ in the form (3·121) formally satisfies the equation of motion, but let us continue the route Schnabl followed in Ref. 1). Schnabl used the Euler-Maclaurin sum formula

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} \Big[\, f^{(k)}(b) - f^{(k)}(a) \,\Big] = \sum_{n=a}^{b-1} f'(n)\,, \tag{3·123}$$

which is

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} \Big[\, \psi^{(k)}_{N+1} - \psi^{(k)}_0 \,\Big] = \sum_{n=0}^{N} \psi'_n \tag{3·124}$$

for $f(n) = \psi_n$, $a = 0$, and $b = N + 1$, to turn the solution to the form

$$\Psi = -\sum_{k=0}^{\infty} \frac{B_k}{k!}\, \psi^{(k)}_0 = \sum_{n=0}^{N} \psi'_n - \sum_{k=0}^{\infty} \frac{B_k}{k!}\, \psi^{(k)}_{N+1}\,. \tag{3·125}$$

Then Schnabl defined the solution in the limit given by

$$\Psi = \lim_{N\to\infty} \left[\, \sum_{n=0}^{N} \psi'_n - \psi_N \,\right]. \tag{3·126}$$

Here derivatives $\psi^{(k)}_{N+1}$ with $k > 0$ are dropped because they vanish in the limit $N \to \infty$. In fact, the string field ψ_N, which is retained, also vanishes in the limit $N \to \infty$ in the following sense:

$$\lim_{N\to\infty} \langle\, \varphi, \psi_N \,\rangle = 0 \tag{3·127}$$

for any φ in the Fock space. However, the string field ψ_N gives a finite contribution in the limit $N \to \infty$ in the calculation of the energy of the solution, while derivatives $\psi^{(k)}_N$ with $k > 0$ do not. The term with ψ_N is often called the *phantom piece*. It would be important to understand why the inner product $\langle\, \varphi, \psi_N \,\rangle$ vanishes in the limit $N \to \infty$. The inner product $\langle\, \varphi, \psi_N \,\rangle$ is given by

$$\langle\, \varphi, \psi_N \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{B}\, c(N+1) \,\rangle_{C_{N+2}} = \langle\, c(-1) f \circ \varphi(0)\, c(1)\, \mathcal{B} \,\rangle_{C_{N+2}}\,, \tag{3·128}$$

where we used the identification $z \sim z + N + 2$ of the surface C_{N+2} to bring $c(N+1)$ to $c(-1)$ so that the surface C_{N+2} of the last expression of the inner product should be understood to be represented, for example, in the region $-(N+2)/2 \le \Re z \le (N+2)/2$. The operators $c(-1) f \circ \varphi(0)\, c(1)$ carrying ghost number 4 can be expanded around the origin, and the expansion generically takes the form

$$c(-1) f \circ \varphi(0)\, c(1) \propto c\partial c \partial^2 c \partial^3 c(0) + \dots\,, \tag{3·129}$$

where $c\partial c\partial^2 c\partial^3 c$ is the operator carrying the lowest weight 2 in the sector of ghost number 4 and the dots represent operators whose weight is larger than 2. The N dependence of the leading term is given by

$$\langle\, c\partial c\partial^2 c\partial^3 c(0)\, \mathcal{B}\,\rangle_{C_{N+2}} = \frac{1}{(N+2)^3}\,\langle\, c\partial c\partial^2 c\partial^3 c(0)\, \mathcal{B}\,\rangle_{C_1}\,, \qquad (3\cdot130)$$

where we used the conformal transformation

$$z' = \frac{1}{N+2}\, z\,. \qquad (3\cdot131)$$

We therefore find that

$$\langle\, \varphi, \psi_N\,\rangle \sim O\Big(\frac{1}{N^3}\Big) \qquad (3\cdot132)$$

for a generic state φ. If the leading term $c\partial c\partial^2 c\partial^3 c(0)$ happens to be absent in the OPE of $c(-1)\, f\circ\varphi(0)\, c(1)$, the inner product vanishes faster than $1/N^3$. We also see that derivatives $\psi_N^{(k)}$ vanish faster than ψ_N:

$$\langle\, \varphi, \psi_N^{(k)}\,\rangle \sim O\Big(\frac{1}{N^{k+3}}\Big) \qquad (3\cdot133)$$

for a generic state φ.

Since the phantom piece ψ_N formally vanishes in the limit $N\to\infty$, we can write the solution as

$$\Psi = \sum_{n=0}^{\infty} \psi'_n. \quad \text{(formal)} \qquad (3\cdot134)$$

We can prove that Ψ in this form, again formally, satisfies the equation of motion from

$$Q_B \psi'_0 = 0\,, \qquad Q_B \psi'_{n+1} = -\sum_{m=0}^{n} \psi'_m * \psi'_{n-m}\,. \qquad (3\cdot135)$$

These relations do not involve any limit and they hold without any subtlety.

Let us prove these relations. The first relation states that ψ'_0 is BRST closed. It is convenient to use the expression (3·105). The BRST transformation of ψ'_0 is given by

$$\begin{aligned}\langle\, \varphi, Q_B\psi'_0\,\rangle &= \langle\, f\circ\varphi(0)\,(Q_B\cdot c(1))\,\mathcal{L}\,\rangle_{C_2} - \langle\, f\circ\varphi(0)\, c(1)\, Q_B\cdot\mathcal{L}\,\rangle_{C_2}\\ &\quad - \langle\, f\circ\varphi(0)\,(Q_B\cdot c\partial c(1))\,\mathcal{B}\,\rangle_{C_2} - \langle\, f\circ\varphi(0)\, c\partial c(1)\, Q_B\cdot\mathcal{B}\,\rangle_{C_2}\,.\end{aligned} \qquad (3\cdot136)$$

Since

$$Q_B\cdot c(z) = c\partial c(z)\,, \quad Q_B\cdot c\partial c(z) = 0\,, \quad Q_B\cdot\mathcal{B} = \mathcal{L}\,, \quad Q_B\cdot\mathcal{L} = 0\,, \qquad (3\cdot137)$$

we have

$$\langle\, \varphi, Q_B\psi'_0\,\rangle = \langle\, f\circ\varphi(0)\, c\partial c(1)\,\mathcal{L}\,\rangle_{C_2} - \langle\, f\circ\varphi(0)\, c\partial c(1)\,\mathcal{L}\,\rangle_{C_2} = 0\,. \qquad (3\cdot138)$$

In fact, we can write

$$\langle\, \varphi, \psi'_0\,\rangle = -\,\langle\, f\circ\varphi(0)\; Q_B\cdot(\, c(1)\,\mathcal{B}\,)\,\rangle_{C_2}\,. \qquad (3\cdot139)$$

The string field ψ'_0 is thus BRST exact. Since

$$Q_B \cdot (\, c(1)\, \mathcal{B}\,) = Q_B \cdot (\, 1 - \mathcal{B}\, c(1)\,) = \, - Q_B \cdot (\, \mathcal{B}\, c(1)\,)\,, \qquad (3\cdot140)$$

the inner product $\langle\, \varphi, \psi'_0 \,\rangle$ can also be written as

$$\langle\, \varphi, \psi'_0 \,\rangle = \langle\, f \circ \varphi(0)\; Q_B \cdot (\, \mathcal{B}\, c(1)\,)\,\rangle_{C_2}\,. \qquad (3\cdot141)$$

In terms of the half integrals B_1^R and B_1^L of B_1 in (3·56), ψ'_0 can thus be written as

$$\psi'_0 = \, - Q_B B_1^R c_1 |0\rangle \qquad (3\cdot142)$$

or as

$$\psi'_0 = Q_B B_1^L c_1 |0\rangle\,. \qquad (3\cdot143)$$

Let us next prove the second relation in (3·135). The BRST transformation of ψ'_{n+1} is

$$\begin{aligned}\langle\, \varphi, Q_B \psi'_{n+1} \,\rangle = {} & \langle\, f \circ \varphi(0)\, c\partial c(1)\, \mathcal{L}\, \mathcal{B}\, c(n+2)\,\rangle_{C_{n+3}} \\ & - \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}^2\, c(n+2)\,\rangle_{C_{n+3}} \\ & + \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c\partial c(n+2)\,\rangle_{C_{n+3}}\,. \end{aligned} \qquad (3\cdot144)$$

The star product $\psi'_m * \psi'_{n-m}$ with $0 < m < n$ is

$$\langle\, \varphi, \psi'_m * \psi'_{n-m} \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(m+1)\, c(m+2)\, \mathcal{B}\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}}\,. \quad (3\cdot145)$$

Let us calculate the part $\mathcal{B}\, c(m+1)\, c(m+2)\, \mathcal{B}$ in the middle. When there is one c ghost between two line integrals of $\mathcal{B}$, we find

$$\mathcal{B}\, c(z)\, \mathcal{B} = \mathcal{B}\,. \qquad (3\cdot146)$$

This follows from $\{\, \mathcal{B}, c(z)\,\} = 1$ and $\mathcal{B}^2 = 0$:

$$\mathcal{B}\, c(z)\, \mathcal{B} = \mathcal{B}\,[\, 1 - \mathcal{B}\, c(z)\,] = \mathcal{B}\,. \qquad (3\cdot147)$$

When there are two c ghosts between two line integrals of $\mathcal{B}$, we find

$$\begin{aligned}\mathcal{B}\, c(z_1)\, c(z_2)\, \mathcal{B} & = \mathcal{B}\, c(z_1)\,[\, 1 - \mathcal{B}\, c(z_2)\,] \\ & = \mathcal{B}\, c(z_1) - \mathcal{B}\, c(z_1)\, \mathcal{B}\, c(z_2) = \mathcal{B}\, c(z_1) - \mathcal{B}\, c(z_2)\,. \end{aligned} \qquad (3\cdot148)$$

Using this formula, the inner product $\langle\, \varphi, \psi'_m * \psi'_{n-m} \,\rangle$ can be written as

$$\begin{aligned}\langle\, \varphi, \psi'_m * \psi'_{n-m} \,\rangle = {} & \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(m+1)\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}} \\ & - \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(m+2)\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}}\,. \end{aligned} \qquad (3\cdot149)$$

Note the simple dependence on m. The relation also holds in the limit $m \to 0$ and in the limit $m \to n$. Using the results we obtained in deriving (3·105), we have

$$\begin{aligned}\langle\, \varphi, \psi'_0 * \psi'_n \,\rangle = {} & \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}^2\, c(n+2)\,\rangle_{C_{n+1}} \\ & - \langle\, f \circ \varphi(0)\, c\partial c(1)\, \mathcal{B}\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}} \\ & - \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(2)\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}}\,, \\ \langle\, \varphi, \psi'_n * \psi'_0 \,\rangle = {} & \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(n+1)\, \mathcal{L}\, c(n+2)\,\rangle_{C_{n+1}} \\ & - \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c\partial c(n+2)\,\rangle_{C_{n+1}}\,. \end{aligned} \qquad (3\cdot150)$$

Therefore, we find

$$\sum_{m=0}^{n} \langle\, \varphi, \psi'_m * \psi'_{n-m} \,\rangle = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}^2\, c(n+2) \,\rangle_{C_{n+1}} \\ -\langle\, f \circ \varphi(0)\, c\partial c(1)\, \mathcal{B}\, \mathcal{L}\, c(n+2) \,\rangle_{C_{n+1}} \\ -\langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c\partial c(n+2) \,\rangle_{C_{n+1}}\,. \qquad (3{\cdot}151)$$

We have thus shown the second relation in (3·135).

We have actually shown that

$$Q_B\, \Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0 \qquad (3{\cdot}152)$$

for

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1}\, \psi'_n \qquad (3{\cdot}153)$$

with formally any λ. We have a one-parameter family of solutions labeled by λ. It is believed, however, that Ψ_λ is a pure gauge for $|\lambda| < 1$. Let us first rewrite Ψ_λ in a pure-gauge form. We will mostly omit the star symbol in the rest of this section. We have already shown that ψ'_0 is BRST exact:

$$\psi'_0 = Q_B\, \Phi\,, \qquad (3{\cdot}154)$$

where Φ is defined by

$$\langle\, \varphi, \Phi \,\rangle = \langle\, f \circ \varphi(0)\, \mathcal{B}\, c(1) \,\rangle_{C_2}\,. \qquad (3{\cdot}155)$$

Star products Φ^n are given by

$$\langle\, \varphi, \Phi^n \,\rangle = \langle\, f \circ \varphi(0)\, \mathcal{B}\, c(1)\, \mathcal{B}\, c(2)\, \ldots\, \mathcal{B}\, c(n) \,\rangle_{C_{n+1}} = \langle\, f \circ \varphi(0)\, \mathcal{B}\, c(n) \,\rangle_{C_{n+1}}\,, \quad (3{\cdot}156)$$

and $(Q_B \Phi)\, \Phi^n$ are

$$\langle\, \varphi, (Q_B \Phi)\, \Phi^n \,\rangle = \lim_{\epsilon \to 0} \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(1+\epsilon)\, \mathcal{B}\, c(n) \,\rangle_{C_{n+1}} \\ = \langle\, f \circ \varphi(0)\, c(1)\, \mathcal{L}\, \mathcal{B}\, c(n) \,\rangle_{C_{n+1}}\,. \qquad (3{\cdot}157)$$

We find

$$\psi'_n = (Q_B \Phi)\, \Phi^n\,. \qquad (3{\cdot}158)$$

The solution Ψ_λ can be written in terms of Φ. We have

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1}\, \psi'_n = \sum_{n=0}^{\infty} \lambda^{n+1}\, (Q_B \Phi)\, \Phi^n = \lambda\, (Q_B \Phi)\, \frac{1}{1 - \lambda \Phi}\,. \qquad (3{\cdot}159)$$

This is a pure-gauge form

$$e^{-\Lambda}\, (\, Q_B e^{\Lambda}\,) = -(\, Q_B e^{-\Lambda}\,)\, e^{\Lambda} = \lambda\, (Q_B \Phi)\, \frac{1}{1 - \lambda \Phi} \qquad (3{\cdot}160)$$

under the identification

$$e^{\Lambda} = \frac{1}{1 - \lambda \Phi}\,. \qquad (3{\cdot}161)$$

3.8. *KBc subalgebra*

What we have derived so far can be summarized in a set of simple algebraic relations of three string fields K, B, and c, which is often called *KBc subalgebra*.[11] Let us start with the wedge state W_α. As we have seen in (3·40), the wedge state is generated by the line integral $\mathcal{L}$. We therefore write the wedge state W_α as

$$W_\alpha = e^{\alpha K}\,, \tag{3·162}$$

where the string field K is the wedge state W_0 of vanishing width with an insertion of the line integral $\mathcal{L}$:

$$\langle\, \varphi, K\,\rangle = \langle\, f\circ\varphi(0)\,\mathcal{L}\,\rangle_{C_1}\,. \tag{3·163}$$

In particular, the vacuum state $|0\rangle$ is written as

$$|0\rangle = e^{K}\,. \tag{3·164}$$

The algebra of the wedge state is now manifest:

$$e^{\alpha K}e^{\beta K} = e^{(\alpha+\beta)K}\,. \tag{3·165}$$

The zero-momentum tachyon state ψ_0 can be represented as

$$\psi_0 = e^{K/2}ce^{K/2}\,, \tag{3·166}$$

where the string field c is the wedge state W_0 of vanishing width with a local insertion of a c ghost on the boundary:

$$\langle\, \varphi, c\,\rangle = \langle\, f\circ\varphi(0)\, c\Big(\frac{1}{2}\Big)\,\rangle_{C_1}\,. \tag{3·167}$$

The string field ψ_n is written as

$$\psi_n = e^{K/2}cBe^{nK}ce^{K/2}\,, \tag{3·168}$$

where the string field B is the wedge state W_0 of vanishing width with an insertion of the line integral $\mathcal{B}$:

$$\langle\, \varphi, B\,\rangle = \langle\, f\circ\varphi(0)\,\mathcal{B}\,\rangle_{C_1}\,. \tag{3·169}$$

The string field ψ'_n is then

$$\psi'_n = e^{K/2}cBKe^{nK}ce^{K/2}\,. \tag{3·170}$$

The string fields K, B, and c satisfy the following relations:

$$\begin{aligned} &[\,K,B\,]=0\,, \qquad \{\,B,c\,\}=1\,, \qquad c^2=0\,, \qquad B^2=0\,,\\ &Q_B B = K\,, \qquad Q_B K = 0\,, \qquad Q_B c = cKc\,. \end{aligned} \tag{3·171}$$

This set of relations is called KBc subalgebra. The commutation relation $[\,K,B\,]=0$ follows from (3·61), and the anticommutation relation $\{\,B,c\,\}=1$ comes from (3·72). The relation $c^2=0$ represents the property of the c ghost that $c(z)\,c(z+\epsilon)$ vanishes in the limit $\epsilon\to 0$, and the relation $B^2=0$ is from (3·63). The BRST transformations of B and K simply follow from $Q_B \cdot \mathcal{B}$ and $Q_B \cdot \mathcal{L}$, respectively. As we calculated in (2·96), the BRST transformation of $c(z)$ is $c\partial c(z)$, and $\partial c(z)$ can be realized as $\mathcal{L}\, c(z) - c(z)\, \mathcal{L}$, which is a special case of the relation (3·77). We therefore find that $Q_B c = c\,[\,K, c\,] = cKc - c^2 K = cKc$.

Let us prove that Ψ_λ satisfies the equation of motion only using KBc subalgebra. The string field Ψ_λ can be written as

$$\Psi_\lambda = \lambda\, e^{K/2} c\, \frac{BK}{1-\lambda\, e^K}\, c e^{K/2} = f(K)\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K) \tag{3·172}$$

with

$$f(K) = \sqrt{\lambda}\, e^{K/2}\,. \tag{3·173}$$

The BRST transformation of Ψ_λ is

$$\begin{aligned} Q_B \Psi_\lambda &= Q_B \left[\, f(K)\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K) \right] \\ &= f(K)\, cKc\, \frac{BK}{1-f(K)^2}\, c\, f(K) \\ &\quad - f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K) + f(K)\, c\, \frac{BK}{1-f(K)^2}\, cKc\, f(K)\,. \end{aligned} \tag{3·174}$$

The product Ψ_λ^2 is

$$\begin{aligned} \Psi_\lambda^2 &= f(K)\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K)^2\, c\, \frac{BK}{1-f(K)^2}\, c\, f(K) \\ &= f(K)\, c\, \frac{K}{1-f(K)^2}\, Bc\, f(K)^2\, cB\, \frac{K}{1-f(K)^2}\, c\, f(K)\,. \end{aligned} \tag{3·175}$$

Since

$$\begin{aligned} Bc\, f(K)^2\, cB &= Bc\, f(K)^2 - Bc\, f(K)^2\, Bc = Bc\, f(K)^2 - BcB\, f(K)^2\, c \\ &= Bc\, f(K)^2 - B\, f(K)^2\, c = Bc\, f(K)^2 - f(K)^2\, Bc \\ &= [\, Bc, f(K)^2\,] = -[\, Bc, 1 - f(K)^2\,]\,, \end{aligned} \tag{3·176}$$

we find

$$\begin{aligned} \Psi_\lambda^2 &= -f(K)\, c\, \frac{K}{1-f(K)^2}\, [\, Bc, 1-f(K)^2\,]\, \frac{K}{1-f(K)^2}\, c\, f(K) \\ &= -f(K)\, c\, \frac{K}{1-f(K)^2}\, BcKc\, f(K) + f(K)\, cKBc\, \frac{K}{1-f(K)^2}\, c\, f(K) \\ &= -f(K)\, c\, \frac{K}{1-f(K)^2}\, BcKc\, f(K) + f(K)\, cK(1-cB)\, \frac{K}{1-f(K)^2}\, c\, f(K) \\ &= -f(K)\, c\, \frac{BK}{1-f(K)^2}\, cKc\, f(K) \\ &\quad + f(K)\, c\, \frac{K^2}{1-f(K)^2}\, c\, f(K) - f(K)\, cKc\, \frac{BK}{1-f(K)^2}\, c\, f(K)\,. \end{aligned} \tag{3·177}$$

We have thus shown that $Q_B \Psi_\lambda + \Psi_\lambda^2 = 0$.

Note that we have never used the explicit form of $f(K)$ in the proof. We have actually shown that Ψ given by

$$\Psi = f(K)\, c \,\frac{BK}{1 - f(K)^2}\, c\, f(K) \tag{3·178}$$

satisfies the equation of motion $Q_B \Psi + \Psi^2 = 0$ formally for any $f(K)$. This string field Ψ can also be written as a pure-gauge form. The string field Φ defined by (3·155) can be generalized to

$$\Phi = f(K)\, Bc\, f(K)\,. \tag{3·179}$$

Star products Φ^n are

$$\Phi^n = f(K)\, Bc\, f(K)^2\, Bc\, f(K)^2 \ldots\, Bc\, f(K) = f(K)^{2n-1}\, Bc\, f(K)\,, \tag{3·180}$$

and $(Q_B\Phi)\,\Phi^n$ are

$$\begin{aligned}(Q_B\Phi)\,\Phi^n &= Q_B\,[\, f(K)\, Bc\, f(K)\,]\, f(K)^{2(n-1)+1}\, Bc\, f(K)\\ &= f(K)\, cKBc\, f(K)^{2n}\, Bc\, f(K) = f(K)\, cBK\, f(K)^{2n}\, c\, f(K)\,.\end{aligned} \tag{3·181}$$

Therefore, Ψ in (3·178) can be written as the following pure-gauge form:

$$\Psi = f(K)\, c \,\frac{BK}{1 - f(K)^2}\, c\, f(K) = (Q_B\Phi)\,\frac{1}{1-\Phi} \tag{3·182}$$

with Φ defined in (3·179).

Let us come back to Ψ_λ in (3·172) with $f(K) = \sqrt{\lambda}\, e^{K/2}$. An important difference between the case $|\lambda| < 1$ and the case $\lambda = 1$ can be seen from this form of Ψ_λ. If we expand the factor $K/(1 - f(K)^2)$ in the middle of Ψ_λ in powers of K, we find

$$\frac{K}{1 - f(K)^2} = \frac{K}{1 - \lambda\, e^K} = O(K) \quad \text{for} \quad |\lambda| < 1\,. \tag{3·183}$$

When $\lambda = 1$, the order of K changes:

$$\frac{K}{1 - e^K} = O(K^0)\,. \tag{3·184}$$

The expansion of the factor $K/(1 - \lambda\, e^K)$ in K corresponds to the expansion with respect to the eigenvalue of L, and when $\lambda = 1$, we find that the solution acquires the zero-momentum tachyon term ψ_0. In fact, since

$$\frac{K}{1 - e^K} = -\sum_{n=0}^{\infty} \frac{B_n\, K^n}{n!}\,, \tag{3·185}$$

we recover the tachyon vacuum solution (3·121) in terms of the Bernoulli number B_n:

$$\Psi_{\lambda=1} = e^{K/2} c\, \frac{BK}{1 - e^K}\, c e^{K/2} = -\sum_{n=0}^{\infty} \frac{B_n}{n!}\, e^{K/2} cBK^n c e^{K/2} = -\sum_{n=0}^{\infty} \frac{B_n}{n!}\, \psi_0^{(n)}\,. \tag{3·186}$$

On the other hand, the factor $1/(1-e^K)$ can also be expanded in e^K. Then we formally have

$$\frac{K}{1-e^K} = \sum_{n=0}^{\infty} K e^{nK}\,, \tag{3·187}$$

but it turned out that there is an issue of convergence with this sum. If we truncate the sum up to the term $n = N$, we find

$$\sum_{n=0}^{N} K e^{nK} = \frac{K(1-e^{(N+1)K})}{1-e^K}\,, \tag{3·188}$$

so $K/(1-e^K)$ can be written as

$$\frac{K}{1-e^K} = \sum_{n=0}^{N} K e^{nK} + \frac{K}{1-e^K}\, e^{(N+1)K}\,. \tag{3·189}$$

In the limit $N \to \infty$, component fields of the string field e^{NK} are finite and they define the sliver state W_∞, as we discussed before. This means that component fields of $K^n e^{NK}$ vanish for any positive integer n in the limit $N \to \infty$. We thus define $K/(1-e^K)$ as follows:

$$\lim_{N\to\infty} \left[\sum_{n=0}^{N} K e^{nK} - e^{NK} \right]. \tag{3·190}$$

This corresponds to the form of the tachyon vacuum solution (3·126) with the phantom piece.

It is sometimes convenient to introduce the trace notation

$$\operatorname{tr} AB \equiv \langle\, A, B \,\rangle_{density} \tag{3·191}$$

for BPZ inner products following the analogy we mentioned before. BPZ inner products of two states which consist of K, B, and c can be reduced using the KBc subalgebra to a superposition of traces of the form $\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K}$ and of the form $\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K} ce^{t_4 K} B$. In fact, the former $\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K}$ is a special case of the latter $\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K} ce^{t_4 K} B$ with $t_4 = 0$. In terms of correlation functions, they can be expressed as

$$\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K} = \langle\, c(0)\, c(t_1)\, c(t_1+t_2) \,\rangle_{C_n,\, density} \quad \text{with} \quad n = t_1+t_2+t_3\,, \tag{3·192}$$

and

$$\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K} ce^{t_4 K} B = \langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \mathcal{B} \,\rangle_{C_n,\, density} \quad \text{with} \quad n = t_1+t_2+t_3+t_4\,. \tag{3·193}$$

Using (3·18) we find

$$\operatorname{tr} ce^{t_1 K} ce^{t_2 K} ce^{t_3 K} = -\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n}\, \sin\frac{\pi t_2}{n}\, \sin\frac{\pi t_3}{n} \quad \text{with} \quad n = t_1+t_2+t_3\,. \tag{3·194}$$

To calculate the correlation function in (3·193), let us consider the correlation function given by

$$\langle \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \tag{3·195}$$

with

$$n = t_1 + t_2 + t_3 + t_4\,. \tag{3·196}$$

We evaluate this in two ways. First, the b-ghost integral can be moved to the right as follows:

$$\begin{aligned}
&\langle \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&= \langle \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(0) \right] c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - \langle\, c(0) \int_{V_{\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&= \langle \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(0) \right] c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - \langle\, c(0) \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1) \right] c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad + \langle\, c(0)\, c(t_1) \int_{V_{t_1+\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n}\,.
\end{aligned} \tag{3·197}$$

We further continue the procedure to obtain

$$\begin{aligned}
&\langle \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&= \langle \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(0) \right] c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - \langle\, c(0) \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1) \right] c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad + \langle\, c(0)\, c(t_1) \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1+t_2) \right] c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - \langle\, c(0)\, c(t_1)\, c(t_1+t_2) \left[\oint \frac{dz}{2\pi i}\, z\, b(z)\, c(t_1+t_2+t_3) \right] \rangle_{C_n} \\
&\quad + \langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3) \int_{V_{t_1+t_2+t_3+\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, \rangle_{C_n}\,.
\end{aligned} \tag{3·198}$$

We therefore find

$$
\begin{aligned}
&\langle \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&= -\, t_1 \,\langle\, c(0)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad + (t_1+t_2)\,\langle\, c(0)\, c(t_1)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - (t_1+t_2+t_3)\,\langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, \rangle_{C_n} \\
&\quad + \langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3) \int_{V_{t_1+t_2+t_3+\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, \rangle_{C_n} \,.
\end{aligned}
\tag{3·199}
$$

Second, using the identification $z \sim z+n$ of the surface C_n, the correlation function can be transformed to

$$
\begin{aligned}
&\langle \int_{V_{-\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&= \langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3) \int_{V_{t_1+t_2+t_3+t_4-\epsilon}} \frac{dz}{2\pi i}\, (z-n)\, b(z)\, \rangle_{C_n} \\
&= \langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3) \int_{V_{t_1+t_2+t_3+\epsilon}} \frac{dz}{2\pi i}\, z\, b(z)\, \rangle_{C_n} \\
&\quad - n\,\langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \mathcal{B}\, \rangle_{C_n} \,.
\end{aligned}
\tag{3·200}
$$

We therefore find that

$$
\begin{aligned}
&\langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \mathcal{B}\, \rangle_{C_n} \\
&= \frac{t_1}{n}\,\langle\, c(0)\, c(t_1+t_2)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad - \frac{t_1+t_2}{n}\,\langle\, c(0)\, c(t_1)\, c(t_1+t_2+t_3)\, \rangle_{C_n} \\
&\quad + \frac{t_1+t_2+t_3}{n}\,\langle\, c(0)\, c(t_1)\, c(t_1+t_2)\, \rangle_{C_n} \,.
\end{aligned}
\tag{3·201}
$$

The trace in (3·193) is thus given by

$$
\begin{aligned}
&\mathrm{tr}\, c e^{t_1 K} c e^{t_2 K} c e^{t_3 K} c e^{t_4 K} B \\
&= -\,\frac{t_1}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi(t_1+t_2)}{n}\, \sin\frac{\pi t_3}{n}\, \sin\frac{\pi t_4}{n} \\
&\quad + \frac{t_1+t_2}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n}\, \sin\frac{\pi(t_2+t_3)}{n}\, \sin\frac{\pi t_4}{n} \\
&\quad - \frac{t_1+t_2+t_3}{n}\left(\frac{n}{\pi}\right)^3 \sin\frac{\pi t_1}{n}\, \sin\frac{\pi t_2}{n}\, \sin\frac{\pi(t_3+t_4)}{n}
\end{aligned}
\tag{3·202}
$$

with

$$
n = t_1+t_2+t_3+t_4 \,. \tag{3·203}
$$

We can in principle calculate any trace of string fields made of K, B, and c using the formulas (3·194) and (3·202). The kinetic energy of the tachyon vacuum solution in the form (3·126) with the phantom piece was calculated in Ref. 1). The calculation is technically complicated so that we only present the result, and we will calculate the energy of a simpler solution for tachyon vacuum later. We have

$$\begin{aligned}
\mathcal{K}_2 &= \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{N} \langle\, \psi'_n, Q_B \psi'_m \,\rangle_{density} = \frac{1}{2} - \frac{1}{\pi^2}\,, \\
\mathcal{K}_1 &= \lim_{N\to\infty} \sum_{m=0}^{N} \langle\, \psi_N, Q_B \psi'_m \,\rangle_{density} = \frac{1}{2} + \frac{1}{\pi^2}\,, \\
\mathcal{K}_0 &= \lim_{N\to\infty} \langle\, \psi_N, Q_B \psi_N \,\rangle_{density} = \frac{1}{2} + \frac{1}{\pi^2}\,,
\end{aligned} \tag{3·204}$$

so that the kinetic term of the solution is evaluated as

$$\langle\, \Psi, Q_B \Psi \,\rangle_{density} = \mathcal{K}_2 - 2\,\mathcal{K}_1 + \mathcal{K}_0 = -\frac{3}{\pi^2}\,. \tag{3·205}$$

The cubic term was evaluated in Refs. 11) and 12). It turned out that the phantom piece ψ_N is necessary for the equation of motion contracted with the solution

$$\langle\, \Psi, Q_B \Psi \,\rangle + \langle\, \Psi, \Psi^2 \,\rangle = 0 \tag{3·206}$$

to be satisfied. The energy density of the solution normalized by the D25-brane tension T_{25} is thus

$$\begin{aligned}
&\frac{1}{T_{25}}\,\frac{1}{\alpha'^3 g_T^2} \left[\frac{1}{2} \langle\, \Psi, Q_B \Psi \,\rangle_{density} + \frac{1}{3} \langle\, \Psi, \Psi * \Psi \,\rangle_{density} \right] \\
&= 2\pi^2 \left[\frac{1}{2} \langle\, \Psi, Q_B \Psi \,\rangle_{density} + \frac{1}{3} \langle\, \Psi, \Psi * \Psi \,\rangle_{density} \right] = \frac{\pi^2}{3} \langle\, \Psi, Q_B \Psi \,\rangle_{density} = -1\,.
\end{aligned} \tag{3·207}$$

The value of the energy density predicted by Sen's conjecture is thus reproduced analytically.

3.9. *Phantomless solution*

Later Erler and Schnabl found a phantomless solution for the tachyon vacuum.[13)] They chose the function $f(K)$ to be

$$f(K) = \frac{1}{\sqrt{1-K}}\,. \tag{3·208}$$

Since

$$\frac{K}{1-f(K)^2} = K - 1 \tag{3·209}$$

for this choice of $f(K)$, the solution Ψ is

$$\Psi = \frac{1}{\sqrt{1-K}}\, cB(K-1)c\, \frac{1}{\sqrt{1-K}}\,. \tag{3·210}$$

If Ψ satisfies the equation of motion, $\widetilde{\Psi} = U(K)^{-1}\, \Psi\, U(K)$ also satisfies the equation of motion because the BRST transformation of any function of K vanishes. This can also be considered as a gauge transformation analogous to a homogeneous gauge transformation in ordinary gauge theory. Choosing $U(K) = 1/\sqrt{1-K}$, we can simplify the solution as

$$\widetilde{\Psi} = U(K)^{-1}\,\Psi\,U(K) = cB(K-1)c\,\frac{1}{1-K}\,. \tag{3·211}$$

While Ψ satisfies the reality condition, $\widetilde{\Psi}$ does not. The factors $1/\sqrt{1-K}$ and $1/(1-K)$ should be understood as being defined using the Laplace transformation:

$$\frac{1}{\sqrt{1-K}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt\,\frac{e^{-t}}{\sqrt{t}}\,e^{tK}\,, \qquad \frac{1}{1-K} = \int_0^\infty dt\,e^{-(1-K)\,t} = \int_0^\infty dt\,e^{-t}\,e^{tK}\,. \tag{3·212}$$

Then these can be interpreted as superpositions of wedge states.

The calculation of the energy is much simpler than that of the solution in Schnabl gauge. The potential $\widehat{V}(\Psi)$ normalized by the D-brane tension is

$$\widehat{V}(\Psi) \equiv \frac{V(\Psi)}{T_{25}} = 2\pi^2\,\mathrm{tr}\left[\,\frac{1}{2}\,\Psi\,Q_B\Psi + \frac{1}{3}\,\Psi^3\,\right]. \tag{3·213}$$

Obviously, $\widehat{V}(\Psi) = \widehat{V}(\widetilde{\Psi})$. The solution $\widetilde{\Psi}$ can be written as

$$\widetilde{\Psi} = cB(K-1)c\,\frac{1}{1-K} = [\,Q_B(Bc)\,]\,\frac{1}{1-K} - c\,\frac{1}{1-K}\,. \tag{3·214}$$

The BRST transformation of $\widetilde{\Psi}$ is

$$Q_B\widetilde{\Psi} = -(Q_Bc)\,\frac{1}{1-K} = -cKc\,\frac{1}{1-K}\,, \tag{3·215}$$

and $\widetilde{\Psi}\,Q_B\widetilde{\Psi}$ can be written as

$$\begin{aligned}\widetilde{\Psi}\,Q_B\widetilde{\Psi} &= -[\,Q_B(Bc)\,]\,\frac{1}{1-K}\,(Q_Bc)\,\frac{1}{1-K} + c\,\frac{1}{1-K}\,cKc\,\frac{1}{1-K} \\ &= -Q_B\left[\,Bc\,\frac{1}{1-K}\,(Q_Bc)\,\frac{1}{1-K}\,\right] + c\,\frac{1}{1-K}\,cKc\,\frac{1}{1-K}\,.\end{aligned} \tag{3·216}$$

The first term is BRST exact and thus does not contribute to the energy. The normalized potential is

$$\widehat{V}(\Psi) = \frac{\pi^2}{3}\,\mathrm{tr}\,\widetilde{\Psi}\,Q_B\widetilde{\Psi} = \frac{\pi^2}{3}\,\mathrm{tr}\,c\,\frac{1}{1-K}\,cKc\,\frac{1}{1-K}\,. \tag{3·217}$$

Since

$$\mathrm{tr}\,ce^{t_1K}cKce^{t_2K} = -\Big(\frac{t_1+t_2}{\pi}\Big)^2\sin\frac{\pi\,t_1}{t_1+t_2}\,\sin\frac{\pi\,t_2}{t_1+t_2} = -\Big(\frac{t_1+t_2}{\pi}\Big)^2\sin^2\frac{\pi\,t_1}{t_1+t_2}\,, \tag{3·218}$$

we have

$$\widehat{V}(\Psi) = -\frac{\pi^2}{3}\int_0^\infty dt_1\int_0^\infty dt_2\,e^{-t_1}\,e^{-t_2}\Big(\frac{t_1+t_2}{\pi}\Big)^2\sin^2\frac{\pi\,t_1}{t_1+t_2}\,. \tag{3·219}$$

Changing the variables as

$$u = t_1 + t_2\,, \qquad v = \frac{t_1}{t_1 + t_2}\,, \tag{3·220}$$

we evaluate the normalized potential as follows:

$$\widehat{V}(\Psi) = -\frac{1}{3}\int_0^\infty du\, u^3\, e^{-u} \int_0^1 dv\, \sin^2 \pi\, v = -\frac{1}{3}\,\Gamma(4) \times \frac{1}{2} = -1\,. \tag{3·221}$$

We have thus reproduced the value predicted by Sen's conjecture.

§4. Analytic solutions for marginal deformations

4.1. *Marginal deformations with regular operator products*

Using string field theory, we can potentially discuss various backgrounds of string theory. In the current construction of string field theory, however, we need to choose one consistent background to formulate the theory, and other backgrounds are expected to be described by classical solutions.

In the case of the open string, a consistent background corresponds to a boundary CFT with $c = 26$. Deformations of the boundary condition which preserve the conformal invariance are called *marginal deformations*, and we expect a one-parameter family of solutions to the equation of motion of open string field theory for each marginal deformation.

The deformation of the boundary CFT

$$S_{\rm BCFT} \to S_{\rm BCFT} + \lambda \int_{\partial\Sigma} dt\, V(t)\,, \tag{4·1}$$

where we denoted the boundary of the world-sheet by $\partial\Sigma$, is marginal to $O(\lambda)$ if V is a primary field of weight 1. For example, the operator $V(t)$ given by

$$V(t) = \frac{i}{\sqrt{2\alpha'}}\,\partial_t X^\mu(t) \tag{4·2}$$

corresponds to turning on a constant mode of the gauge field on the D-brane, and the operator $V(t)$ given by

$$V(t) = \frac{i}{\sqrt{2\alpha'}}\,\partial_\perp X^\alpha(t)\,, \tag{4·3}$$

where $\partial_\perp$ is the derivative normal to the boundary, corresponds to a transverse coordinate of the D-brane. A more nontrivial example is

$$V(t) = \sqrt{2} : \cos\frac{X^\mu(t)}{\sqrt{\alpha'}} :\,, \tag{4·4}$$

which deforms the Neumann boundary condition to the Dirichlet boundary condition. For example, the D25-brane is deformed into a periodic array of D24-branes at a particular value of the deformation parameter λ.

For any primary field V of weight 1 in the matter sector, the operator cV is BRST closed:

$$Q_B \cdot cV(t) = \oint \frac{dz}{2\pi i}\, j_B(z)\, cV(t) = 0\,, \tag{4·5}$$

where the contour of the integral encircles the point t counterclockwise. Therefore, the string field $\Psi^{(1)}$ given by

$$\langle\, \varphi, \Psi^{(1)}\,\rangle = \langle\, f \circ \varphi(0)\, cV(1)\,\rangle_{C_2} \tag{4·6}$$

satisfies the linearized equation of motion:

$$Q_B\, \Psi^{(1)} = 0\,. \tag{4·7}$$

When the deformation is exactly marginal, we expect a solution of the form

$$\Psi = \sum_{n=1}^{\infty} \lambda^n\, \Psi^{(n)} \tag{4·8}$$

to the equation of motion $Q_B\,\Psi + \Psi * \Psi = 0$. Since the equation of motion has to be satisfied for any λ, we have the following equations of motion for $\Psi^{(n)}$:

$$\begin{aligned} Q_B\, \Psi^{(1)} &= 0\,, \\ Q_B\, \Psi^{(2)} &= -\Psi^{(1)} * \Psi^{(1)}\,, \\ &\vdots \\ Q_B\, \Psi^{(n)} &= -\sum_{m=1}^{n-1} \Psi^{(m)} * \Psi^{(n-m)}\,. \end{aligned} \tag{4·9}$$

Formally, the equation of motion for $\Psi^{(2)}$ can be solved by

$$\Psi^{(2)} = -\frac{b_0}{L_0}\left[\, \Psi^{(1)} * \Psi^{(1)}\,\right]. \tag{4·10}$$

Combined actions of b_0/L_0 and the star multiplication, however, are in general complicated.

When operator products of the marginal operator

$$V(t_1)\, V(t_2)\, \ldots\, V(t_n) \tag{4·11}$$

are regular, analytic solutions were constructed in Refs. 14) and 15) . The basic idea is to replace b_0/L_0 by the corresponding operator B/L in the sliver frame:

$$\Psi^{(2)} = -\frac{B}{L}\left[\, \Psi^{(1)} * \Psi^{(1)}\,\right]. \tag{4·12}$$

We can write this as

$$\Psi^{(2)} = -\int_0^{\infty} dT\, B\, e^{-TL}\left[\, \Psi^{(1)} * \Psi^{(1)}\,\right], \tag{4·13}$$

although there is some subtlety to be discussed later. After a few nontrivial steps, $\Psi^{(2)}$ can be expressed as

$$\langle\, \varphi, \Psi^{(2)} \,\rangle = \int_0^1 dt \, \langle\, f \circ \varphi(0)\, cV(1)\, \mathcal{B}\, cV(1+t)\,\rangle_{C_{2+t}} \,. \tag{4·14}$$

Instead of explaining the derivation of this expression, let us confirm that $\Psi^{(2)}$ satisfies the equation of motion, which is in fact much easier. The BRST transformation of $\Psi^{(2)}$ is given by

$$\begin{aligned}\langle\, \varphi, Q_B\, \Psi^{(2)} \,\rangle &= -\int_0^1 dt \, \langle\, f \circ \varphi(0)\, cV(1)\, \mathcal{L}\, cV(1+t)\,\rangle_{C_{2+t}} \\ &= -\int_0^1 dt \, \frac{\partial}{\partial t} \langle\, f \circ \varphi(0)\, cV(1)\, cV(1+t)\,\rangle_{C_{2+t}} \,. \end{aligned} \tag{4·15}$$

There are two surface terms. At $t=1$, we find

$$-\,\langle\, f \circ \varphi(0)\, cV(1)\, cV(2)\,\rangle_{C_3} = -\,\langle\, \varphi, \Psi^{(1)} * \Psi^{(1)} \,\rangle \,. \tag{4·16}$$

The other surface term at $t=0$ vanishes when

$$\lim_{\epsilon\to 0} cV(0)\, cV(\epsilon) = 0 \,. \tag{4·17}$$

Therefore, if

$$\lim_{\epsilon\to 0} V(0)\, V(\epsilon) \tag{4·18}$$

is finite or vanishing, $\Psi^{(2)}$ is finite and satisfies the equation of motion $Q_B\, \Psi^{(2)} = -\,\Psi^{(1)} * \Psi^{(1)}$. It is also easy to see that $\Psi^{(2)}$ satisfies the Schnabl gauge condition because it takes the form of an integral of a state whose ghost part is ψ_t.

Once we understand how $\Psi^{(2)}$ in the form of (4·14) satisfies the equation of motion, it is easy to construct $\Psi^{(n)}$ satisfying the equation of motion. It is given by

$$\begin{aligned}\langle\, \varphi, \Psi^{(n)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \, \langle\, f \circ \varphi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1) \\ \times \mathcal{B}\, cV(1+t_1+t_2) \ldots \mathcal{B}\, cV(1+t_1+t_2+\ldots+t_{n-1})\,\rangle_{C_{2+t_1+t_2+\ldots+t_{n-1}}} \,. \end{aligned} \tag{4·19}$$

Introducing the length parameters

$$\ell_i \equiv \sum_{k=1}^{i} t_k \,, \tag{4·20}$$

the solution can be written more compactly as

$$\langle\, \varphi, \Psi^{(n)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \, \langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \,\Big] \,\rangle_{C_{2+\ell_{n-1}}} \,. \tag{4·21}$$

Let us now prove that the equation of motion is satisfied for $\Psi^{(n)}$. It is straightforward to generalize the calculation of $\langle\, \phi, Q_B \Psi^{(2)}\,\rangle$ to that of $\langle\, \phi, Q_B \Psi^{(n)}\,\rangle$. Since the operator cV is BRST closed, the BRST operator acts only on the insertions of $\mathcal{B}$:

$$\langle\, \varphi, Q_B \Psi^{(n)}\,\rangle = -\sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \,\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big] \\ \times \mathcal{L}\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \,\rangle_{C_{2+\ell_{n-1}}} . \tag{4·22}$$

An insertion of $\mathcal{L}$ between $cV(1+\ell_{j-1})$ and $cV(1+\ell_j)$ corresponds to taking a derivative with respect to t_j. When operator products of V are regular, we have

$$\begin{aligned}\langle\, \varphi, Q_B \Psi^{(n)}\,\rangle &= -\sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \,\frac{\partial}{\partial t_j} \langle\, f \circ \phi(0)\, cV(1) \\ &\quad \times \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big]\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \,\rangle_{C_{2+\ell_{n-1}}} \\ &= -\sum_{j=1}^{n-1} \int_0^1 dt_1 \ldots \int_0^1 dt_{j-1} \int_0^1 dt_{j+1} \ldots \int_0^1 dt_{n-1} \,\langle\, f \circ \phi(0)\, cV(1) \\ &\quad \times \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big]\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \,\rangle_{C_{2+\ell_{n-1}}} \Big|_{t_j=1} \\ &= -\sum_{j=1}^{n-1} \langle\, \varphi, \Psi^{(j)} * \Psi^{(n-j)}\,\rangle .\end{aligned} \tag{4·23}$$

The equation of motion is thus satisfied.

An important example of marginal operators which satisfy the regularity condition is

$$V(t) = \,:\exp\Big[\, \frac{1}{\sqrt{\alpha'}}\, X^0(t) \Big] : \,. \tag{4·24}$$

This describes a time-dependent decay process of an unstable D-brane, which is usually referred to as *rolling tachyon*. Actually, the deformation by

$$\lambda \int_{\partial \Sigma} dt : \exp\Big[\, \frac{1}{\sqrt{\alpha'}}\, X^0(t) \Big] : \tag{4·25}$$

does not generate a one-parameter family of physically distinct solutions because changing the absolute value of λ without changing the sign can be absorbed by a shift in the origin of the time coordinate. However, the construction of the solution can be applied to this case without any problem. It is an exact time-dependent solution incorporating all α' corrections. By the way, it is difficult to deal with time-dependent solutions in string theory, but in string field theory it is conceptually the same problem as that in ordinary field theory. This can be considered as one of the advantages of string field theory.

While the rolling tachyon is an interesting example of $V(t)$ with regular operator products, the marginal operator in general has singular operator products. Since it is a primary field of weight 1, its typical OPE is

$$V(t_1)\,V(t_2) \sim \frac{1}{(t_1 - t_2)^2}\,. \tag{4·26}$$

In this case, the construction we described has to be modified. One possible approach was presented in Ref. 15). First, the integral of t_i is regularized as follows:

$$\int_0^1 dt_i \to \int_\epsilon^1 dt_i\,. \tag{4·27}$$

In (4·13), this corresponds to regularizing the integral for large T. Namely, the representation of B/L used in (4·13) is not well defined for $V(t)$ with the OPE (4·26). The equation of motion is no longer satisfied because of the regularization, but we can add counterterms such that the equation of motion is satisfied and the solution is finite in the limit $\epsilon \to 0$. In Ref. 15) $\Psi^{(2)}$ and $\Psi^{(3)}$ were constructed in this way. It was also found that there exists an obstruction in constructing a solution when the deformation is not exactly marginal. It is also interesting to note that while the solution when operator products of the marginal operator are regular satisfies the Schnabl gauge condition, the solution for the singular case does not satisfy the Schnabl gauge condition. In fact, there does not seem to exist a solution in Schnabl gauge.

4.2. *Solutions for general marginal deformations*

The solution $\Psi^{(n)}$ for marginal deformations with regular operator products in the preceding subsection consists of n unintegrated vertex operators, $n-1$ integrals of moduli, and $n-1$ corresponding b-ghost insertions. In attempting to generalize the solution for the singular case, one source of difficulties came from the moduli integrals. The solution $\Psi^{(n)}$ is represented on wedge states of various lengths, which makes the regularization complicated. A different approach to the construction of solutions for marginal deformations with singular operator products was presented in Ref. 16). The new solution $\Psi^{(n)}$ consists of one unintegrated vertex operator and $n-1$ integrated vertex operators. The moduli integral is absent and the solution $\Psi^{(n)}$ is represented on a single surface C_{n+1}.

The *unintegrated vertex operator* cV is BRST closed:

$$Q_B \cdot cV(t) = 0\,. \tag{4·28}$$

The *integrated vertex operator*

$$V(a,b) = \int_a^b dt\,V(t) \tag{4·29}$$

is BRST closed up to boundary terms:

$$Q_B \cdot V(a,b) = Q_B \cdot \int_a^b dt\,V(t) = \int_a^b dt\,\partial_t\big[\,cV(t)\,\big] = cV(b) - cV(a)\,. \tag{4·30}$$

It follows only from these basic properties of unintegrated and integrated vertex operators that $\Psi_L^{(2)}$ given by

$$\langle\, \varphi, \Psi_L^{(2)}\,\rangle = \langle\, f\circ\varphi(0) \int_1^2 dt\, cV(1)\, V(t)\,\rangle_{C_3} \tag{4·31}$$

satisfies the equation of motion. In fact, the BRST transformation of $\Psi_L^{(2)}$ can be calculated from

$$Q_B \cdot \int_1^2 dt\, cV(1)\, V(t) = -\int_1^2 dt\, cV(1)\, \partial_t \big[\, cV(t)\,\big] = -\, cV(1)\, cV(2) \tag{4·32}$$

when the operator product $V(1)\, V(t)$ is regular in the limit $t \to 1$, and we confirm that $Q_B \Psi_L^{(2)} = -\,\Psi^{(1)} * \Psi^{(1)}$. It is also easy to verify that $\Psi_L^{(3)}$ given by

$$\begin{aligned}\langle\, \varphi, \Psi_L^{(3)}\,\rangle &= \langle\, f\circ\varphi(0)\, \frac{1}{2}\, cV(1) \Big[\int_1^3 dt\, V(t) \Big]^2 \rangle_{C_4} \\ &\quad - \langle\, f\circ\varphi(0)\, \frac{1}{2}\, cV(1) \Big[\int_2^3 dt\, V(t) \Big]^2 \rangle_{C_4}\end{aligned} \tag{4·33}$$

satisfies the equation of motion $Q_B \Psi_L^{(3)} = -\Psi^{(1)} * \Psi_L^{(2)} - \Psi_L^{(2)} * \Psi^{(1)}$ when operator products of the marginal operator are regular. The solution $\Psi_L^{(3)}$ can also be written as

$$\langle\, \varphi, \Psi_L^{(3)}\,\rangle = \langle\, f\circ\varphi(0) \int_1^2 dt_1 \int_{t_1}^3 dt_2\, cV(1)\, V(t_1)\, V(t_2)\,\rangle_{C_4}\,, \tag{4·34}$$

and this form of $\Psi_L^{(3)}$ generalizes to $\Psi_L^{(n)}$ given by

$$\begin{aligned}\langle\, \varphi, \Psi_L^{(n)}\,\rangle = \langle\, f\circ\varphi(0) \int_1^2 dt_1 \int_{t_1}^3 dt_2 \int_{t_2}^4 dt_3 \ldots \int_{t_{n-2}}^n dt_{n-1}\, cV(1) \\ \times V(t_1)\, V(t_2) \ldots V(t_{n-1}) \rangle_{C_{n+1}}\,.\end{aligned} \tag{4·35}$$

The integration region is complicated, but it can be decomposed and the solution can be constructed only from operators of the forms $V(a,b)^m$ and $cV(a)\, V(a,b)^m$ with appropriate values of the parameters a, b, and m. These are the operators which appear when we expand $e^{\lambda V(a,b)}$ and $\lambda\, cV(a)\, e^{\lambda V(a,b)}$ in λ as

$$e^{\lambda V(a,b)} = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}\, V(a,b)^m\,, \quad \lambda\, cV(a)\, e^{\lambda V(a,b)} = \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!}\, cV(a)\, V(a,b)^m\,, \tag{4·36}$$

and we only need the relations

$$\begin{aligned} Q_B \cdot e^{\lambda V(a,b)} &= e^{\lambda V(a,b)}\, \lambda\, cV(b) - \lambda\, cV(a)\, e^{\lambda V(a,b)}\,, \\ Q_B \cdot \big[\, \lambda\, cV(a)\, e^{\lambda V(a,b)}\,\big] &= -\, \lambda\, cV(a)\, e^{\lambda V(a,b)}\, \lambda\, cV(b)\end{aligned} \tag{4·37}$$

in proving that $\Psi_L^{(n)}$ satisfies the equation of motion.

These are the relations we expect when the operator $e^{\lambda V(a,b)}$ implements the change of the boundary condition for the region between a and b. The effect of the BRST transformation localizes to the points where the boundary condition is changed, and it can be interpreted in terms of boundary condition changing operators. When operator products of the marginal operator $V(t)$ are singular, the operator $e^{\lambda V(a,b)}$ has to be properly renormalized, but there should be a renormalized operator $[\, e^{\lambda V(a,b)}\,]_r$ which implements the change of the boundary condition for the region between a and b, and there should be relations which generalize (4·37) and can be interpreted in terms of boundary condition changing operators. Then analytic solutions can be constructed systematically from $[\, e^{\lambda V(a,b)}\,]_r$.[16)]

§5. Future directions

When the first analytic solution was constructed by Schnabl,[1)] it was not clear whether it is limited to the particular problem of tachyon condensation or it can be more universal. Later analytic solutions for marginal deformations were constructed, and they provided a perspective that the development is more universal. Furthermore, the development does not depend on the bosonic nature of the problem, and there is no immediate obstruction to the extension to the superstring. In fact, analytic solutions for marginal deformations were extended to open superstring field theory. The development is not even restricted to the construction of classical solutions. As can be seen from the construction of the analytic solutions for marginal deformations, we have better control over the combination of the star multiplication and the propagator in Schnabl gauge. It is therefore promising to calculate scattering amplitudes analytically in Schnabl gauge. Four-point amplitudes in Schnabl gauge at the tree level were discussed in detail,[17)] and the generalization to one-loop amplitudes has also been explored.[18)]

We have seen that open string field theory is able to describe tachyon condensation. How far can we explore nonperturbative phenomena using open string field theory?

(i) *Can open string field theory describe other backgrounds?*

The current formulation of open string field theory requires a choice of one boundary CFT. The question is whether open string field theory formulated around the chosen boundary CFT describes backgrounds corresponding to other boundary CFT's. For backgrounds connected by marginal deformations, analytic solutions have been constructed to all orders in the deformation parameter. Whether or not the expansion has a finite radius of convergence would depend on a choice of the original boundary CFT and details of marginal deformations, and little has been understood so far.

In Siegel gauge, numerical solutions in level truncation have been constructed, where the deformation parameter is treated nonperturbatively.[19)] It turned out that the branch of the solutions truncates at a finite distance from the origin. While this is not an immediate problem, the correspondence between the space of boundary

CFT's and the space of solutions in open string field theory, if we assume that it exists, may not be so simple. It may also be related to the issue that the gauge condition may not be imposed globally in the configuration space of string fields.

For backgrounds which are not continuously connected to the original background, the situation is more complicated. Analytic solutions for relevant deformations were proposed in Ref. 20), but later it was pointed out[21)] that the equation of motion was not completely satisfied, while the energy and the coupling to closed strings were correctly reproduced. Since the violation of the equation of motion seems to be related to the assumption of nontrivial cohomology,[21)] this may not be a technical problem, and it might be related to a more fundamental issue of what the space of string fields is. On the other hand, it would still be possible that the problem comes from the particular ansatz and that we can find solutions in different gauges. In fact, numerical solutions for such relevant deformations were constructed in Siegel gauge.

(ii) *Can open string field theory describe creation of D-branes?*

We often say that D-branes are kind of solitonic objects in string theory, and open strings correspond to collective excitations on the soliton. In this perspective, the tachyon vacuum solution in open string field theory is remarkable because it corresponds to the description of the annihilation of the soliton in terms of its collective excitations. Then how about creation of the soliton? Namely, starting from open string field theory formulated around a background with one D-brane, can we construct solutions for multiple D-branes? Such multi-brane solutions were proposed by Murata and Schnabl,[22),23)] and it was demonstrated that the energy and the coupling to closed strings can be reproduced correctly. The solutions, however, contain some singularities and it is intensively discussed how to treat such singularities.

(iii) *Can open string field theory describe closed strings?*

If we succeed in quantizing open string field theory consistently and calculating on-shell loop amplitudes, there should be poles corresponding to closed strings. Then from the standard argument using unitarity there should be external closed strings as well in perturbation theory. Therefore, the question of whether open string field theory is able to describe closed strings is closely related to the question of whether open string field theory can be quantized consistently.

Imagine a closed string field theory on a background with one space-like dimension compactified on a circle. Truncation to closed string fields with vanishing winding number is classically consistent. For example, in perturbation theory of on-shell scattering amplitudes, closed strings with nonvanishing winding number never appear in tree-level Feynman diagrams when all external closed strings carry vanishing winding number. Such truncation, however, is not consistent in quantum theory. In loop amplitudes, pair creation of closed strings with nonvanishing winding number is possible, and we will not be able to forbid such processes consistently. We know in string perturbation theory that the inconsistency is related to the violation of modular invariance for such truncation. In the context of string field theory, the inconsistency would manifest itself as anomaly of the gauge invariance upon quantization.

Classically, open string field theory is a consistent gauge theory. The question is whether we can preserve the gauge invariance upon quantization without adding independent degrees of freedom such as closed string fields. This question has been discussed in open bosonic string field theory, but the quantization of the bosonic theory is necessarily formal because of the tachyons in the open string sector and in the closed string sector, and no definite conclusion has been obtained. We should consider quantization of open superstring field theory where tachyons are absent. In particular, as we mentioned earlier, the recent analytic developments can be extended to the superstring, and we may be able to carry out analytic calculations in Schnabl gauge at least at one loop. We should seriously consider quantization of open superstring field theory now!

With the analytic methods that have been developed so far, we are now able to explore these directions of research more concretely than before. We hope that further developments of string field theory, together with those of various other approaches, will help us uncover the nature of the nonperturbative theory underlying string theory.

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