Topological Algebras

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Contents

Part I

Topological vector spaces

Chapter 1

Locally convex spaces

1.1 Category of locally convex spaces

complete locally convex space

bornology, tensor products,

1.1 (Bilinear forms on topological vector spaces)**.** We will distringuish embeddings and topological embeddings.

Topologies on the space of operators $L(E,F)$.

1.2 (Topological tensor products)**.** Let E and F be locally convex spaces. The *projective tensor product* of E and F is a locally convex space which is universal among the jointly continuous bilinear operators from $E\times F$ to a locally convex space.

We can also describe it with semi-norms. We have

$$B_{\mathrm{jnt}}(E,F)\cong(E\widehat{\otimes}_\pi F)^*.$$

$$(E\widehat{\otimes}_\pi F)^*_\sigma\cong L_?(E_?,F^*_?)$$

Induced topology on $E\odot F$ from the space of separately continuous bilinear forms on $E^*_\sigma\times F^*_\sigma$ with the topology of uniform convergence on products of equicontinuous subsets of E^* and F^*.

σ: on finite sets τ: on weakly compact sets β: on weakly bounded sets ε: on equi-continuous sets

A subset of E^*_σ is equicontinuous iff it is contained in the polar of a neighborhood of E. A subset is polar of finite sets iff

The topology of uniform convergence on $\mathcal{G}=$ The topology generated by polars of $\mathcal{G}$.

E_ε is the original topology

Note that we have

$$X\otimes Y\cong B_{\mathrm{jnt}}(X^*_\sigma,Y^*_\sigma)\subset B_{\mathrm{sep}}(X^*_\sigma,Y^*_\sigma).$$

The space $B_{\mathrm{sep}}(X^*_\sigma,Y^*_\sigma)$ of separately continuous bilinear forms, which has a natural topology of uniform convergence on the products of equicontinuous sets in X^*_σ and Y^*_σ, and this topology is complete if and only if X and Y are complete. The induced topology on $X\otimes Y$ is called the *injective tensor product* topology. We have $C^k(\Omega,E)\cong C^k(\Omega)\widehat{\otimes}_\varepsilon E$ if E is complete.

Note that the projective tensor product reflects the original topologies of locally convex spaces, while the injective tensor product only depends on the dual pair structure.

The dual of $X\widehat{\otimes}_\pi Y\to X\widehat{\otimes}_\varepsilon Y$ defines an injection $J(X,Y)\to B_{\mathrm{jnt}}(X,Y)$. A bilinear form in $J(X,Y)$ is called to be *integral*.

1.3. $L(E)$ is a topological algebra

1.2 Vector-valued functions

1.4 (Vector-valued measurable functions)**.** Let (X,μ) and (Y,ν) be localizable measure spaces. Let (E,E^*) be a dual pair.

Define vector valued Lebesgue spaces as the completion? Weakly measurable functions?

(a) $L^1(X,E)$ and $L^1(X)\otimes E$: E is ... and $\otimes$ is ...

(b) $L^2(X,E)$ and $L^2(X)\otimes E$ if E is a Hilbert space and $\otimes$ is the Hilbert space tensor product.

(c) $L^\infty(X,E)$ and $L^\infty(X)\otimes E$ if E is ... and $\otimes$ is ...

(d) $\mu : L^1(X)\otimes E\to E\subset E^{**}$ is well-defined if E is ... and μ is ...

(e) What is the relation between the product measurability and the Bochner measurability.

(f) $L^p(X,L^q(Y))=L^p(X)\otimes L^q(Y)$ if $\otimes$ is ...

(g) $L^p(X,L^p(Y))=L^p(X\times Y)$?

- weakly integrable: $L^1(X)\otimes E\to(E^*)^\#$.
- Dunford integrable: $L^1(X)\otimes E\to E^{**}$.
- Pettis integrable: $L^1(X)\otimes E\to E$.
- Bochner integrable: $L^1(X)\otimes_\pi E\to E$.
- For a Pettis integrable function, if we check it is strongly measurable using the Pettis measurability theorem and bound it with L^1 norm, then it becomes Bochner integrable.
- If E is normed so that V^* is Fréchet, then weakly integrability implies the Dunford integrability.

Proof. □

1.5 (Vector-valued continuous functions)**.** Let X be a locally compact Hausdorff space and (E,E^*) be a dual pair. Suppose

(i) the closed convex hull of a compact subset is compact in E_σ,

(ii) E is closed in the strong bidual E^{**}_β.

An example is the case when E is a Banach space. The weak dual pair (E,E^*) satisfies the assumption by the Krein-Šmulian theorem and the completeness of E. The weak* dual pair (E^*,E) also satisfies the assumption by the fact that the closed convex hull of a bounded set is bounded, and the norm topology and $\beta(E^*,E_\beta)$ on E^* coincide by the Goldstine theorem. In particular, for $F\subset E^*$, the Banach space E is closed in the strong bidual for the dual pair (E,F) if and only if the closed unit ball $F_1=F\cap E_1^*$ is weakly* dense in the closed ball E_1^*.

We want to construct a canonical element of $L(C_b(X,E_\sigma),L_\sigma(M(X)_\sigma,E_\sigma))$.

(a) $C(X,E_\sigma)$ and $C(X)\otimes E_\sigma$. (X compact Hausdorff)

(b) $C_b(X,E_\sigma)\to L(M(\beta X)_\sigma,E_\sigma)\to L(M(X)_\sigma,E_\sigma)$ if (E,E^*) satisfies the two properties.

(c) $C_b(X,E_\sigma)\to L(L^1(X)_\beta,E_\tau)$

(d) the boundedly completeness?

(e) $C_b(X,E_\tau)\to L(M(X)_\sigma,E_\tau)$?

Proof. (a) Consider a common dense subset $C(X)\odot E$. For $f\in C(X,E_\sigma)$ and for a fixed finite sequence ξ_j^* in E^* and $\varepsilon>0$, taking U_{x_i} at each $x_i\in X$ such that $\max_j|\langle f(x_i)-f(x),\xi_j^*\rangle|<\varepsilon$ for $x\in U_{x_i}$, then the partition of unity constructs a function $\sum_k f(x_k)\chi_k\in C(X)\odot E$ such that

$$\max_j\|\langle f-\sum_k f(x_k)\chi_k,\xi_j^*\rangle\|=\sup_{x\in X}\sum_k\chi_k(x)\max_j|\langle f(x)-f(x_k),\xi_j^*\rangle|<\sup_{x\in X}\sum_k\chi_k(x)\varepsilon=\varepsilon,$$

so the algebraic tensor is dense in $C(X,E_\sigma)$.

$$[f]_{\mu_j,\xi_j^*}=\int\langle f(x),\xi_j^*\rangle\,d\mu_j(x).$$

$$ih\partial_t=H(h)$$

propagator $e^{-itH/h}$

(b) First we have $C_b(X,E_\sigma)\to L(E_\beta^*,C_b(X)_\beta):f\mapsto(\xi^*\mapsto\langle f(\cdot),\xi^*\rangle)$ because for a net $\xi_i^*\in E^*$ such that $\xi_i^*\to0$ in E_β^* the weak boundedness of $f(X)\subset E_\sigma$ implies

$$\|\langle f(\cdot),\xi_i^*\rangle\|_{C_b(X)_\beta}=\sup_{x\in X}|\langle f(x),\xi_i^*\rangle|\to0,\qquad f\in C_b(X,E_\sigma).$$

On the other hand, for any compact subset $K\subset X$ we have $C_b(X,E_\sigma)\to L(E_\tau^*,C(K)_\beta)$ because for a net ξ_i^* such that $\xi_i^*\to0$ in E_τ^* the compactness of the closed convex hull of the compact set $f(K)$ in E_σ implies that

$$\|\langle f(\cdot),\xi_i^*\rangle\|_{C(K)_\beta}=\sup_{x\in K}|\langle f(x),\xi_i^*\rangle|\to0.$$

Consider

$$L(E_\beta^*,C_b(X)_\beta)\to L(E_\beta^*,C_b(X)_\sigma)\to L(M(\beta X)_\beta,E_\beta^{**})\to L(M(X)_\beta,E_\beta^{**})$$

and

$$L(E_\tau^*,C(K)_\beta)=L(E_\sigma^*,C(K)_\sigma)\to L(M(K)_\beta,E_\beta).$$

Note that

$$C_b(X,E_\sigma)\to L\Big(\operatorname*{colim}_K M(K)_\beta,E_\beta\Big)\subset L(M(X)_\beta,E_\beta^{**}).$$

Since $\operatorname{colim}_K M(K)$ is strongly dense in $M(X)_\beta$, where K runs through all compact subsets of X, and since E is closed in E_β^{**}, ...

(c) Fix $\xi\in C_b(X,E_\sigma)$. ☐

1.6. Let (E,E^*) and (F,F^*) be dual pairs. We prove $L(E_\sigma^*,F_\sigma)=L(E_\tau^*,F_\alpha)$ as sets, where α is any dual topology.

(a) If $T\in L(E_\tau^*,F_\sigma)$, then $T^*\in L(F_\sigma^*,E_\sigma)$, and in particular $T^*\in L(F_\tau^*,E_\sigma)$.

(b) If $T\in L(E_\sigma^*,F_\sigma)$, then $T^*\in L(F_\tau^*,E_\tau)$, and in particular $T^*\in L(F_\tau^*,E_\sigma)$.

(c) If $T\in L(E_\sigma^*,F_\sigma)$, then $T^*\in L(F_\beta^*,E_\beta)$.

Proof. (a) If $\xi_i^*\to0$ in E_σ^*, then $T^*\xi_i^*\to0$ in F_σ since

$$|\langle\eta^*,T^*\xi_i^*\rangle|=|\langle T\eta^*,\xi_i^*\rangle|\to0,\qquad\eta^*\in F^*.$$

(b) If $\eta_i^*\to0$ in F_τ^*, then $T^*\eta_i^*\to0$ in E_τ since T preserves compact sets so that

$$\sup_{\xi^*\in C^*}|\langle T^*\eta_i^*,\xi^*\rangle|=\sup_{\xi^*\in C^*}|\langle\eta_i^*,T\xi^*\rangle|\to0.$$

(c) If $\xi_i^*\to0$ in E_β^*, then $T^*\xi_i^*\to0$ in F_β since T preserves bounded sets so that

$$\sup_{\eta^*\in B^*}|\langle\eta^*,T^*\xi_i^*\rangle|=\sup_{\eta^*\in B^*}|\langle T\eta^*,\xi_i^*\rangle|\to0.$$

☐

1.7 (Vector-valued differentiable functions).

1.8 (Vector-valued distributions).

1.9 (Locally compact group actions). Let G be a locally compact group and let (E, E^*) be a dual pair. Let $\alpha : G \to L_\sigma(E_\sigma)$ be a continuous bounded action.

(a) $\alpha : M(\beta G)_\sigma \to L_\sigma(E_\sigma)$.

(b) $\alpha : L^1(G)_\tau \to L_\sigma(E_\tau)$ if

(c) $\alpha^* : G \times E^*_\sigma \to E^*_\sigma$ preserves compactness if E_τ is barrelled, and E^*_σ has the Heine-Borel property.

(d) $\alpha : G \to L_\sigma(E_\tau)$ if (a) and (b) are satisfied. (if $E = A$, then a point-weakly continuous action is point-norm continuous, and if $E = M$, then a point-σ-weakly continuous action is point-σ-strongly continuous)

Proof. (a) If $(x, x^*) \in E \times E^*$, then $(s \mapsto \langle \alpha_s(x), x^* \rangle) \in C_b(G)$ defines a continuous linear functional on $M(\beta G)$. Thus, $\operatorname{span} G \subset M(\beta G)_\sigma \to L_\sigma(E_\sigma)$ can be extended by the continuity.

(b)

For a bounded set $B^* \in L^\infty(G)$,

$f \mapsto \langle \alpha_f(x), x^* \rangle$ is a linear functional on $L^1(G)$ with norm....?

Let $f_n \to 0$ in $L^1(G)_\tau$.

$$|\langle \alpha_{f_n}(x), x^* \rangle|$$

(c) Suppose s_i and x^*_i are nets in compact subsets of G and E^*_σ. We may assume $s_i \to e$ in G and $x^*_i \to 0$ in E^*_σ. We will show that we can take a subnet such that for each $x \in E$ we have

$$|\langle x, \alpha^*_{s_i}(x^*_i) \rangle| = |\langle \alpha_{s_i}(x), x^*_i \rangle| \le |\langle \alpha_{s_i}(x) - x, x^*_i \rangle|$$

converges.

For some neighborhood U of zero in E_τ, $\sup_{x \in U, x^* \in C^*, s \in K} |\langle \alpha_s(x), x^* \rangle| \le 1$?

$\alpha_K(U)$ is bounded in E_τ?

(d) We claim that $E_0 = E$, where

$$E_0 := \{x \in E : \lim_{s \to e} \alpha_s(x) = x \text{ in } E_\tau\}.$$

We first see that E_0 is closed in E_τ. Let $x_i \in E_0$ be a net such that $x_i \to x$ in E_τ. Fix $\varepsilon > 0$ and weakly compact convex set $C^* \subset E^*_\sigma$. Since the set $\alpha^*_K(C^*)$ is relatively compact in E^*_σ by the part (c), the convergence $x_i \to x$ in the Mackey topology implies that the limit $s \to e$ gives

$$\begin{aligned}\sup_{x^* \in C^*} |\langle \alpha_s(x) - x, x^* \rangle| &\le \sup_{x^* \in C^*} |\langle x - x_i, \alpha^*_s(x^*) \rangle| + \sup_{x^* \in C^*} |\langle \alpha_s(x_i) - x_i, x^* \rangle| + \sup_{x^* \in C^*} |\langle x_i - x, x^* \rangle| \\ &\to \varepsilon + 0 + \varepsilon,\end{aligned}$$

hence we have the claim $x \in E_0$ by letting $\varepsilon \to 0$.

Now it suffices to show E_0 is dense in E_σ by the Hahn-Banach separation and the fact that the Mackey topology is a dual topology. Since we have a continuous linear map $\alpha : M(\beta G)_\sigma \to L(E_\sigma)_\sigma$ by the part (a), if we take a net $e_i \in C_c(G)$ such that $e_i \to \delta_0$ weakly* in $M(\beta G)_\sigma$, then for any $x \in E$, the net $\alpha_{e_i}(x)$ belongs to E_0 by the uniform continuity of each e_i and the part (b), and it has the convergence $\alpha_{e_i}(x) \to x$ in E_σ, so we are done.

□

1.10. Let G be a compact Lie group for which the Chevalley complexification can be made.

1.3 Direct limit

distribution theory LF,LB spaces

1.4 Differentiable spaces

Chapter 2

Fréchet spaces

Chapter 3

Banach spaces

3.1 Universal properties

Notation

$L(X,Y)$	the set of bounded linear operators from X to Y
$B(X,Y)$	the set of bounded bilinear forms on $X \times Y$
$F(X,Y)$	the set of continuous finite-rank linear operators from X to Y
B_X	closed unit ball of a normed space X
S_X	unit sphere of a normed space X
$X \otimes Y$	algebraic tensor product of X and Y
X^*	continuous dual space
$X^\#$	algebraic dual space

3.1 (Algebraic tensor product of vector spaces)**.** Let X and Y be vector spaces. The *algebraic tensor product* is a vector space $X \otimes Y$ with a bilinear map $\otimes : X \times Y \to X \otimes Y$ such that the following universal property: for any vector space Z and any bilinear map $\sigma : X \times Y \to Z$, there exists a unique linear map $\widetilde{\sigma} : X \otimes Y \to Z$ such that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "\sigma"'] & X \otimes Y \arrow[d, dashed, "\widetilde{\sigma}"] \\ & Z \end{tikzcd}$$

is commutative.

(a) The tensor product $X \otimes Y$ always exists.

(b) We have linear maps $L(X,Z) \otimes L(Y,W) \to L(X \otimes Y, Z \otimes W)$ and $B(L(X,Z), L(Y,Z)) \to L(X \otimes Y, Z)$.

(c) Every element $t \in X \otimes Y$ is represented as $t = \sum_{i=1}^n x_i \otimes y_i$ such that $\{x_i\}$ is linearly indpendent. In this case, if $t = 0$ then $y_i = 0$ for all i.

Proof. (a) Let T be the set of formal linear combinations of $X \times Y$, that is, an element of T has the form $\sum_{i=1}^n a_i \cdot (x_i, y_i)$ for $x_i \in X$, $y_i \in Y$, and scalars a_i. Define $T_0 \subset T$ to be a linear space spanned by the elements of the following four types:

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(ax, y) - a(x, y), \qquad (x, ay) - a(x, y).$$

Then, the quotient space T/T_0 satisfies the universal property with the bilinear map $X \times Y \to T/T_0 : (x, y) \mapsto (x, y) + T_0$. □

3.2 (Algebraic tensor product of involutive algebras)**.**

3.2 Banach spaces

3.3 (Subcross norms).

3.4 (Injective tensor products). Let X and Y be Banach spaces. Define the *injective norm* ε on $X \otimes Y$ such that

$$\varepsilon\left(\sum_{i=1}^{n} x_i \otimes y_i\right) := \sup_{\substack{x^* \in B_{X^*} \\ y^* \in B_{Y^*}}} \left|\sum_{i=1}^{n} \langle x_i, x^* \rangle \langle y_i, y^* \rangle\right|.$$

We denote by $X \otimes_\varepsilon Y$ the algebraic tensor product with the injective norm, and by $X \widehat{\otimes}_\varepsilon Y$ its completion.

(a) $X \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F((X^*, w^*), (Y, w))$.

(b) $X^* \otimes_\varepsilon Y$ is naturally isometrically isomorphic to $F(X, Y)$.

3.5 (Projective tensor products). Let X and Y be Banach spaces. Define the *projective norm* π on $X \otimes Y$ such that

$$\pi(t) := \inf\left\{\sum_{i=1}^{n} \|x_i\| \|y_i\| : t = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

We denote by $X \otimes_\pi Y$ the algebraic tensor product with the projective norm, and by $X \widehat{\otimes}_\pi Y$ its completion.

(a) There are natural isometric isomorphisms $(X \otimes_\pi Y)^* \cong B(X, Y) \cong L(X, Y^*)$.

(b)

3.6 (Hilbert space tensor product). Let $\varphi : H \otimes K \to L(H^*, K)$. Then, $\lambda(\xi) = \|\varphi(\xi)\|$, $\gamma(\xi) = \operatorname{tr}(|\varphi(\xi)|)$, so $H \widehat{\otimes}_\lambda K \cong K(H^*, K)$ and $H \widehat{\otimes}_\gamma K \cong L^1(H^*, K)$.

3.7 (Nuclear operators).

$$X^* \otimes_\pi Y \to X^* \otimes_\varepsilon Y \xrightarrow{\sim} F(X, Y) \xrightarrow{1} K(X, Y)$$

defines

$$J : X^* \widehat{\otimes}_\pi Y \to K(X, Y).$$

Define $N(X, Y) := \operatorname{im} J$.

3.8 (Grothendieck theorem). Let Y^* be an RNP space. Then, there is an isometric isomorphism $(X \widehat{\otimes}_\varepsilon Y)^* \cong N(X, Y^*)$.

3.3 Approximation property

3.9 (Approximation property of locally convex spaces).

3.10 (Approximation property of Banach spaces).

3.11 (Approximation property of dual Banach spaces).

3.12 (Mazur's goose). (a) If X has a Schauder basis, then it has the approximation property.

3.4 Nuclear spaces

Part II

Topological algebras

Chapter 4

Locally convex algebras

Chapter 5

Fréchet algebras

For a Fréchet algebra A,

Chapter 6

Banach algebras