Abstract Harmonic Analysis

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Contents

Part I

Fourier analysis on groups

Chapter 1

Locally compact groups

1.1 Haar measures

1.1 (Non-σ-finite measures). Following technical issues are important

(a) The Fubini theorem

(b) The Radon-Nikodym theorem

(c) The dual space of L^1 space

1.2 (Existence of the Haar measure).

1.2 Group algebras

1.3 (Modular functions).

1.4 (Convolution).

1.5 (Positive definite functions). Bochner theorem

1.6 (Fourier-Stieltjes algebra).

1.7 (GNS construction for locally compact groups). Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

$$\begin{tikzcd}
G \arrow[r] & M(G) & & \\
L_1(G) \arrow[ur, hook] \arrow[r, hook] \arrow[d, dashed, "*"] & C^*(G) \arrow[r, two heads] \arrow[d, dashed, "*"] & C_r^*(G) \arrow[r, hook] \arrow[d, dashed, "*"] & L(G) \arrow[d, dashed, "* \text{ with } \sigma w"] \\
L^\infty(G) & B(G) \arrow[l, hook'] & C_r^*(G)^* \arrow[l, hook'] & A(G) \arrow[l, hook'] \\
 & C_0(G) \arrow[ul, hook'] & &
\end{tikzcd}$$

1.8 (Uniformly continuous functions). G acts on $C_{lu}(G)$ and $L^1(G)$ continuously with respect to the point-norm topology. A function on G is left uniformly continuous if and only if it is written as $f * x$ for some $f \in L^1(G)$ and $x \in L^\infty(G)$.

1.3 Pontryagin duality

1.9 (Dual group).

1.10 (Fourier inversion theorem).

1.11 (Plancherel's theorem).

1.4 Structure theorems

1.5 Spectral synthesis

Chapter 2

Representation theory

2.1 (Schur's lemma).

2.2 (Operator-valued Fourier transform).

Since it is not easy to introduce the quantum dual of G for now, we cannot discuss $L^1(G)$ as the Fourier algebra, the predual of the quantum group von Neumann algebra. ($A(G) = L(G)_* = L^1(\widehat{G})$ and also is the closed linear span of matrix coefficients of the left regular representation.)

Chapter 3

Compact groups

3.1 Peter-Weyl theorem

3.2 Tannaka-Krein duality

3.3 Example of compact Lie groups

Chapter 4

Mackey machine

4.1 Example of non-compact Lie groups

Wigner classification

Chapter 5

Kac algebras

Part II

Topological quantum groups

Chapter 6

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Chapter 7

Locally compact quantum groups

7.1 Multiplicative unitaries

Part III

Tensor categories