Abstract Harmonic Analysis

Ikhan Choi

November 10, 2022

Contents

Part I

Fourier analysis on groups

Chapter 1

Locally compact groups

1.1 Topological groups

1.2 Haar measures

1.1 (Non-σ-finite measures)**.** Following technical issues are important

(a) Positive linear functionals on C_c

(b) The Fubini theorem

(c) The Radon-Nikodym theorem

(d) The dual space of L^1 space

1.2 (Radon measures)**.** Let Ω be a locally compact Hausdorff space. A *Radon measure* is a Borel measure μ on Ω such that

(i) μ is outer regular for every Borel set: $\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\}$ for Borel E,

(ii) μ is inner regular for every open set: $\mu(U) = \inf\{\mu(K) : K \subset U,\ K \text{ compact}\}$ for open U,

(iii) μ is locally finite.

(a) A σ-finite Radon measure is regular.

(b) If every open subset of Ω is σ-compact, then a locally finite Borel measure is Radon.

(c) $C_c(\Omega)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

1.3 (Riesz-Markov-Kakutani representation theorem for C_c)**.** Let Ω be a locally compact Hausdorff space and consider the following map:

$$\begin{array}{ccc} \{\text{Radon measures on } \Omega\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_c(\Omega, \mathbb{R})\}, \\ \mu & \mapsto & (f \mapsto \int f\, d\mu). \end{array}$$

(a) a

1.4 (Existence of the Haar measure)**.**

1.3 Group algebras

1.5 (Modular functions)**.**

1.6 (Convolution)**.**

1.4 Structure theorems

Exercises

1.7.

Problems

1. Let Ω be a topological space. For every positive linear functional I on $C_c(\Omega,\mathbb{R})$, show that there exists a Borel measure μ on Ω such that $I(f)=\int f\,d\mu$ for all $f\in C_c(\Omega,\mathbb{R})$. (Hint: Consider the uncountable wedge sum of circles as an example.)

Solution. 1. The constructed Carathéodory measure μ on Ω is outer regular Borel measure, but we do not have local finiteness. Everything is same to when Ω is locally compact Hausdorff except that $\mu(\operatorname{supp} f)$ may be infinite. Now it is enough to show $I(\min\{f,\frac{1}{n}\})$ converges to zero as $n\to\infty$ for $f\in C_c(\Omega,[0,1])$.

Let $U:=f^{-1}((0,1])$. For $g\in C_0(U,[0,1])$, it clearly has compact support, and and it is also continuous because $g^{-1}((a,1])$ is open in U and $g^{-1}([a,1])$ is closed in K for any $0<a\leq 1$, so that we have $C_0(U)\subset C_c(X)$. We also have $f_1\in C_0(U)$ since $f_1^{-1}([\varepsilon,1])$ is a compact set in U for every $\varepsilon>0$. Therefore, I is a positive linear functional on $C_0(U)$. Assume that I is not bounded; there is no constant C such that $g\in C_0(U,[0,1])$ implies $I(g)\leq C$. Construct a sequence $(h_k)_{k=1}^\infty$ in $C_0(U,[0,1])$ such that $I(h_k)\geq 2^k$, and define $h:=\sum_{k=1}^\infty h_k/2^k$ so that $h\in C_0(U,[0,1])$. Then, $I(h)\geq\sum_{k=1}^m I(h_k)/2^k\geq m$ for every $m>0$, it contradicts to the assumption, which means that there is a constant C such that $I(g)\leq C$ for all $g\in C_0(U,[0,1])$, and it proves $I(f_1)\leq C/n\to 0$ as $n\to\infty$. Therefore, $I(f)=\int f\,d\mu$. □

Chapter 2

Pontryagin duality

2.1 Dual groups

2.2

2.3 Fourier inversion

2.1 (Positive definite functions).

2.2 (Bochner's theorem).

2.3 (Fourier inversion theorem).

2.4 (Plancherel's theorem).

Chapter 3

Spectral synthesis

3.1 Closed ideals of the colvolution algebra

Part II

Representation theory

Chapter 4

Unitary representations

4.1

4.1 (Schur's lemma).

4.2 Group C*-algerbas

4.2 (Operator-value Fourier transform).

4.3 Functions of positive type

4.3 (Functions of positive type).

4.4 (Fourier-Stieltjes algebra).

4.5 (GNS construction for locally compact groups). Let G be a locally compact group. By a state of $C^*(G)$, we could construct the GNS representation of G. An analog of GNS construction for $L^1(G)$ without completion is doable, when given a function of positive type on G, instead of a state.

Chapter 5

Compact groups

5.1 Peter-Weyl theorem

5.2 Tannaka-Krein duality

5.3 Example of compact Lie groups

Chapter 6

Mackey machine

6.1 Example of non-compact Lie groups

Wigner classification

Part III

Kac algebras

Chapter 7

Left Hilbert algebras

Part IV

Topological quantum groups

Chapter 8

Compact quantum groups

Chapter 9

Locally compact quantum groups

9.1 Multiplicative unitaries