Von Neumann Algebras

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Contents

Part I

Chapter 1

Modular theory

1.1 Weights

1.1 (Weights)**.** Let M be a von Neumann algebra. A *weight* is an additive homogeneous function $\varphi : M^+ \to [0,\infty]$ in the sense that

$$\varphi(x+y) = \varphi(x) + \varphi(y), \qquad \varphi(tx) = t\varphi(x), \qquad x, y \in M^+,\ t \geq 0,$$

where we use the convention $0 \cdot \infty = 0$. Define

$$\mathfrak{N}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}, \qquad \mathfrak{A}_\varphi := \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi, \qquad \mathfrak{M}_\varphi := \mathfrak{N}_\varphi^* \mathfrak{N}_\varphi.$$

It follows easily that $\mathfrak{N}_\varphi$ is a left ideal of M equipped with a sesqui-linear form $\langle x, y\rangle := \varphi(y^*x)$ for $x, y \in \mathfrak{N}_\varphi$, $\mathfrak{A}_\varphi$ is a hereditary $*$-subalgebra of M, and $\mathfrak{M}_\varphi$ is a hereditary $*$-subalgebra of M equipped with a linear functional which extends φ by the polarization identity.

Let φ be a weight on M. We say φ is *faithful* if $\varphi(x) = 0$ implies $x = 0$ for $x \in M^+$, *semi-finite* if $\mathfrak{M}_\varphi$or equivalently $\mathfrak{N}_\varphi$ is σ-weakly dense in M, and *normal* if it is written as the supremum of some set of normal positive linear functionals.

(a) Every von Neumann algebra admits a faithful semi-finite normal weight.

(b) $\mathfrak{M}_\varphi^+ = \{x^*x : x \in \mathfrak{N}_\varphi\}$ and $\mathfrak{M}_\varphi = \{y^*x : x, y \in \mathfrak{N}_\varphi\}$.

Proof. (a) Let $\{\omega_i\}_{i\in I}$ be a maximal family of normal states on M such that the support projections $p_i := s(\omega_i)$ are mutually orthogonal, taken with the Zorn lemma. From the maximality, we have the σ-strong* limit $\sum_i p_i = 1$. Define a weight $\varphi : M^+ \to [0,\infty]$ by

$$\varphi(x) := \sum_i \omega_i(x), \qquad x \in M^+.$$

It is faithful since $\varphi(x^*x) = 0$ means $\omega_i(x^*x) = 0$ for all i, which implies $xp_i = 0$ and

$$\langle x\xi, \xi\rangle = \langle \sum_i p_i\xi, x^*\xi\rangle = \sum_i \langle p_i\xi, x^*\xi\rangle = \sum_i \langle xp_i\xi, \xi\rangle = 0.$$

It is semi-finite since an increasing net $p_J := \sum_{i\in J} p_i \in \mathfrak{M}_\varphi$ converges to the unit, where J runs through all finite subsets of I. It is normal by definition.

(b) Let $z := \sum_{i=1}^n y_i^* x_i \in \mathfrak{M}_\varphi^+$ for some $x_i, y_i \in \mathfrak{N}_\varphi$. The polarization writes

$$z = \frac{1}{4}\sum_{i=1}^n \sum_{k=0}^3 i^k |x_i + i^k y_i|^2$$

so it follows from $z^*=z$ that

$$z=\frac{1}{2}\sum_{i=1}^{n}(|x_i+y_i|^2-|x_i-y_i|^2)\le\frac{1}{2}\sum_{i=1}^{n}|x_i+y_i|^2.$$

Since $\varphi(z)<\infty$ by $x_i+y_i\in\mathfrak{N}_\varphi$, if $x:=z^{\frac{1}{2}}\in\mathfrak{N}_\varphi$, then $z=x^*x$.

Let $z:=\sum_{i=1}^n y_i^*x_i\in\mathfrak{M}_\varphi$ for some $x_i,y_i\in\mathfrak{M}_\varphi$. Let $x:=(\sum_{i=1}^n x_i^*x_i)^{\frac{1}{2}}\in\mathfrak{N}_\varphi$. Since $x_i^*x_i\le x^2$, we have $v_i\in M$ such that $x_i=v_ix$. If we let $y:=\sum_{i=1}^n v_i^*y_i\in\mathfrak{N}_\varphi$, then

$$z=\sum_{i=1}^{n}y_i^*x_i=\sum_{i=1}^{n}y_i^*v_ix=\Big(\sum_{i=1}^{n}v_i^*y_i\Big)^*x=y^*x.\qquad\square$$

1.2 (Semi-cyclic representations). Let M be a von Neumann algebra. A *semi-cyclic representation* is a representation $\pi:M\to B(H)$ together with a densely defined and densely ranged left M-linear operator $\Lambda:\operatorname{dom}\Lambda\subset M\to H$. We say that a representation of M is *semi-cyclic* if it admits such an operator Λ. Note that it implies that a semi-cyclic representation is always non-degenerate. Define

$$\mathcal{F}_\Lambda:=\{\omega\in M_*^+:\omega(x^*x)\le\|\Lambda(x)\|^2,\ x\in\operatorname{dom}\Lambda\},$$
$$\mathcal{G}_\Lambda:=\{\omega\in M_*^+:\omega(x^*x)<\|\Lambda(x)\|^2,\ x\in\operatorname{dom}\Lambda\}.$$

(a)

(b) If (π,Λ) is normal and a semi-finite normal weight φ on M satisfies that $\varphi(x^*x)=0$ whenever $\Lambda(x)=0$ for $x\in\operatorname{dom}\Lambda$, then there is a positive self-adjoint operator h affiliated with $\pi(M)'$ such that $\operatorname{ran}\Lambda\subset\operatorname{dom}h$ and $\varphi(x^*x)=\langle h\Lambda(x),\Lambda(x)\rangle$.

(c) Let $\widehat{\mathcal{F}}_\Lambda$ be the set of all semi-finite normal weights φ such that $\varphi(x^*x)\le\|\Lambda(x)\|^2$ for $x\in\operatorname{dom}\Lambda$. Then, the vector functional map $(\pi(M)')_1^+\to\widehat{\mathcal{F}}_\Lambda:h\mapsto\varphi_h$ defined such that $\varphi_h(x^*x):=\langle h\Lambda(x),\Lambda(x)\rangle$ for $x\in\operatorname{dom}\Lambda$ is bijective. The operator h is called the *Radon-Nikodym derivative* of $\varphi_h\in\mathcal{F}_\Lambda$ in the commutant with respect to Λ.

Proof. (b) Using the semi-finiteness of φ, take a net e_i in $\mathfrak{N}_\varphi$ such that $e_i\uparrow 1$, and let φ_i be a normal positive linear functional on M defined such that $\varphi_i(x^*x):=\varphi(e_ix^*xe_i)$ for $x\in M$. For each fixed $y\in\operatorname{dom}\Lambda$ and i, we have $\operatorname{ran}\Lambda\subset H\to\mathbb{C}:\Lambda(x)\mapsto\varphi_i(y^*x)$ which is well-defined and bounded linear functional on H by the assumption and

$$|\varphi_i(y^*x)|^2\lesssim|\varphi_i(y^*y)||\varphi_i(x^*x)|\lesssim\|\Lambda(y)\|^2\|\Lambda(x)\|^2,\qquad x\in\operatorname{dom}\Lambda.$$

(If φ_i is not contained in the weakly closed convex cone $\mathbb{R}^+\mathcal{F}_\Lambda$, then there is $x\in M^+$ such that $\varphi_i(x)\ne 0$ and $\omega(x)=0$ for any $\omega\in\mathcal{F}_\Lambda$. normality of Λ and the assumption imply $|\varphi_i(x^*x)|\lesssim\|\Lambda(x)\|^2$. positive Hahn-Banachh separation?)

By the Riesz representation theorem, we can define $h_i\in\pi(M)'$ such that $\varphi_i(x^*x)=\langle h_i\Lambda(x),\Lambda(x)\rangle$ for $x\in\operatorname{dom}\Lambda$. $\square$

Justification of positive self-adjoint operators by the limit of increasing positive bounded operators?

1.3 (Normal semi-cyclic representations). Let M be a von Neumann algebra. A semi-cyclic representation (π,Λ) of M is called *normal* if π is normal and Λ is σ-weakly closed.

For a weight φ on M, on the Hilbert space H_φ constructed by separation and completion of the sesqui-linear form on $\mathfrak{N}_\varphi$, define $\pi_\varphi:M\to B(H_\varphi)$ by the left multiplication and $\Lambda_\varphi:\operatorname{dom}\Lambda_\varphi\subset H_\varphi\to H_\varphi$ as the canonical map with $\operatorname{dom}\Lambda_\varphi:=\mathfrak{N}_\varphi$. Then, $(\pi_\varphi,\Lambda_\varphi)$ is clearly a semi-cyclic representation, called the *Gelfand-Naimark-Segal representation* of φ. Conversely, for a semi-cyclic representation (π,Λ) of M on a Hilbert space H, define $\varphi_\Lambda:=\sup\{\omega:\omega\in\mathcal{F}_\Lambda\}$. Then, φ_Λ is clearly a weight.

(a) If φ is normal, then $(\pi_\varphi,\Lambda_\varphi)$ is normal, and $\varphi=\varphi_{\Lambda_\varphi}$.

(b) If (π, Λ) is normal, then φ_Λ is normal, and there exists a unitary operator $u : H \to H_{\varphi_\Lambda}$ satisfying $\pi_{\varphi_\Lambda} = (\operatorname{Ad} u)\pi$ and $\Lambda_{\varphi_\Lambda} = u\Lambda$.

(c) For a normal φ, φ is faithful if and only if Λ_φ is injective, and φ is semi-finite if and only if Λ_φ is σ-weakly densely defined.

Proof. (a) We first show π_φ is normal. The proof is almost same as the normality of cyclic representation associated to normal states. Consider the adjoint $\pi_\varphi^* : B(H)_* \to M^*$. For a normal state $\omega \in B(H)_*$ of the form $\omega = \omega_{\Lambda_\varphi(x)}$ for some $x \in \mathfrak{N}_\varphi$, since

$$\pi_\varphi^*(\omega)(y) = \omega_{\Lambda_\varphi(x)}(\pi_\varphi(y)) = \langle \pi_\varphi(y)\Lambda_\varphi(x), \Lambda_\varphi(x)\rangle = \varphi(x^*yx), \qquad y \in M,$$

and since φ is order continuous, we can see that $\pi_\varphi^*(\omega)$ is also order continuous, which means that it is contained in M_*. Because the image of Λ_φ is dense, the linear span of states of the form $\omega_{\Lambda_\varphi(x)}$ with $x \in \mathfrak{N}_\varphi$ is norm-dense in $B(H)_*$ by the inequality

$$\|\omega_\xi - \omega_\eta\| \le \|\xi - \eta\|\|\xi + \eta\|, \qquad \xi, \eta \in H.$$

Since M_* is norm-closed M^*, so π_φ^* maps normal states of $B(H)$ to normal states M_*.

Next, we show Λ_φ is σ-weakly closed. Let $\Gamma(\Lambda_\varphi) := \{(x, \Lambda_\varphi(x)) : x \in \mathfrak{N}_\varphi\} \subset M \times H$ be the graph of Λ_φ. Note that for closedness of convex subsets of $M \times H$ we do not have to distinguish σ-weak topology from σ-strong* topology on M and weak topology from norm topology on H. Suppose a net $x_i \in \mathfrak{N}_\varphi$ satisfies $x_i \to x$ σ-strongly* in M and $\Lambda_\varphi(x_i) \to \xi$ stronlgy in H. Since φ is normal, we have

$$\varphi = \sup_{\omega \in \mathcal{F}_\varphi} \omega, \qquad \mathcal{F}_\varphi := \mathcal{F}_{\Lambda_\varphi} = \{\omega \in M_*^+ : \omega \le \varphi\}.$$

Then, it follows that $x \in \mathfrak{N}_\varphi$ from

$$\varphi(x^*x) = \sup_{\omega \in \mathcal{F}_\varphi} \omega(x^*x) = \sup_{\omega \in \mathcal{F}_\varphi} \lim_i \omega(x_i^*x_i) \le \lim_i \varphi(x_i^*x_i) = \lim_i \|\Lambda_\varphi(x_i)\|^2 = \|\xi\|^2 < \infty.$$

For fixed $\varepsilon > 0$ and any $y \in \mathfrak{N}_\varphi$, if we take $\omega \in \mathcal{F}_\varphi$ such that $\omega(y^*y) > \varphi(y^*y) - \varepsilon$ and let $h \in (M')_1^+$ be the Radon-Nikodym derivative of ω with respect to φ in the commutant, then

$$\begin{aligned}
\|h\Lambda(y) - \Lambda(y)\|^2 &= \langle h^2\Lambda(y), \Lambda(y)\rangle - 2\langle h\Lambda(y), \Lambda(y)\rangle + \langle \Lambda(y), \Lambda(y)\rangle \\
&\le -\langle h\Lambda(y), \Lambda(y)\rangle + \langle \Lambda(y), \Lambda(y)\rangle \\
&= -\omega(y^*y) + \varphi(y^*y) < \varepsilon
\end{aligned}$$

and the convergence $\Lambda(x_i) \to \xi$ in norm imply

$$\langle \xi - \Lambda_\varphi(x), \Lambda_\varphi(y)\rangle \approx \langle \xi - \Lambda_\varphi(x), h\Lambda_\varphi(y)\rangle \approx \langle \Lambda(x_i) - \Lambda(x), h\Lambda(y)\rangle = \omega(y^*(x_i - x)), \qquad \varepsilon \to 0,$$

and the σ-weak convergence $x_i \to x$ implies $\omega(y^*(x_i - x)) \to 0$, we can conclude that $\langle \xi - \Lambda_\varphi(x), \Lambda_\varphi(y)\rangle$ vanishes for all $y \in \mathfrak{N}_\varphi$. Since Λ_φ has dense image, we finally have $\xi = \Lambda_\varphi(x)$. Therefore, the graph of Λ_φ is σ-weakly closed.

For $\omega \in M_*^+$, we have $\omega \le \varphi$ if and only if $\omega(x^*x) \le \varphi(x^*x) = \|\Lambda_\varphi(x)\|^2$ for $x \in \operatorname{dom}\Lambda_\varphi$, which means that $\mathcal{F}_\varphi = \mathcal{F}_{\Lambda_\varphi}$ and $\varphi = \varphi_{\Lambda_\varphi}$.

(b) The weight φ_Λ is obviously normal by definition. Let $x \in \operatorname{dom}\Lambda$. Then, we have

$$\varphi_\Lambda(x^*x) = \sup_{\omega \in \mathcal{F}_\Lambda} \omega(x^*x) \le \|\Lambda(x)\|^2$$

by definition of φ_Λ and $\mathcal{F}_\Lambda$, so we get $x \in \mathfrak{N}_{\varphi_\Lambda} = \operatorname{dom}\Lambda_{\varphi_\Lambda}$ and $\|\Lambda_{\varphi_\Lambda}(x)\| \le \|\Lambda(x)\|$. Conversely, let $x \in \operatorname{dom}\Lambda_{\varphi_\Lambda}$. We have $\varphi_\Lambda(x^*x) = \|\Lambda_{\varphi_\Lambda}(x)\|^2 < \infty$. We claim that if $\varphi_\Lambda(x^*x) \le 1$, then $x \in \operatorname{dom}\Lambda$ and $\|\Lambda(x)\|^2 \le 1$. As a corollary, the inequality $\|\Lambda(x)\| \le \|\Lambda_{\varphi_\Lambda}(x)\|$ follows by scaling from this claim.

We prove the claim. Since the graph of Λ is σ-weakly closed by assumption, and since the projection $\operatorname{dom}\Lambda \times H_1 \to \operatorname{dom}\Lambda$ is a closed map due to the tube lemma and the weak compactness of H_1, the image

$$\{y : \|\Lambda(y)\| \le 1,\ y \in \operatorname{dom}\Lambda\} \subset M$$

of the graph of Λ under this projection is σ-weakly closed. Since the positive part of this set is also σ-weakly closed and the square root is strongly continuous, if we temporarily consider a sufficiently large representation of M in which every normal state is a vector state so that a strong and σ-strong topology coincide on M, we can conclude that the inverse image under the square root

$$C := \{y^*y : \|\Lambda(y)\| \le 1,\ y \in \operatorname{dom}\Lambda\} \subset M$$

is σ-weakly closed. Note that C is also convex, and $\omega \in M_*^+$ is contained in $\mathcal{F}_\Lambda$ if and only if

$$\sup_{y^*y\in C} \omega(y^*y) \le 1.$$

If $x^*x \notin C$, then, by the Hahn-Banach separation, there is $\omega \in M_*^{sa}$ such that

$$\sup_{y^*y\in C} \omega(y^*y) \le 1 < \omega(x^*x),$$

and since we may assume ω is positive(How can we see $\omega(1) = \|\omega\|$?), we have $\omega \in \mathcal{F}_\Lambda$. Thus, we have $\omega(x^*x) \le \varphi_\Lambda(x^*x) \le 1$ by definition of φ_Λ, which leads a contradiction, so we get $x^*x \in C$. Now there is $y \in \operatorname{dom}\Lambda$ satisfying $x^*x = y^*y$ so that there is $v \in M$ with $x = vy$, and because $\operatorname{dom}\Lambda$ is a left ideal of M, we finally have $x \in \operatorname{dom}\Lambda$, and $x^*x \in C$ says that $\|\Lambda(x)\| \le 1$, hence the claim follows.

In conclusion, we have $\operatorname{dom}\Lambda_{\varphi_\Lambda} = \operatorname{dom}\Lambda$ and $\|\Lambda_{\varphi_\Lambda}(x)\| = \|\Lambda(x)\|$ on it. Since the images of Λ and $\Lambda_{\varphi_\Lambda}$ are dense in H and H_{φ_Λ} respectively, we can define the unitary $u : H \to H_{\varphi_\Lambda}$ such that $u\Lambda(y) := \Lambda_{\varphi_\Lambda}(y)$, and hence that $u\pi(x)u^* = \pi_{\varphi_\Lambda}(x)$ because

$$u^*\pi_{\varphi_\Lambda}(x)u\Lambda(y) = u^*\pi_{\varphi_\Lambda}(x)\Lambda_{\varphi_\Lambda}(y) = u^*\Lambda_{\varphi_\Lambda}(xy) = \Lambda(xy) = \pi(x)\Lambda(y),$$

for $x, y \in \operatorname{dom}\Lambda_{\varphi_\Lambda}$.

(c) □

1.4 (Countability of von Neumann algebras)**.** Let M be a von Neumann algebra. A projection $p \in M$ is called *countably decomposable* if mutually orthogonal nonzero projections majorized by p are at most countable, and we say M is *countably decomposable* if the identity is.

(a) M is countably decomposable if and only if it admits a faithful normal state.

(b) M is countably decomposable if and only if M_1 is metrizable in the σ-strong topology.

(c) M has separable predual if and only if it is countably decomposable and countably generated.

(d) M has separable predual if and only if it faithfully acts on a separable Hilbert space.

(e) M has separable predual if and only if M_1 is metrizable in the σ-weak topology.

Proof. □

1.5 (Normal weights)**.** Let M be a von Neumann algebra, and φ be a weight on M.

(a) If φ is order continuous, then it is σ-weakly lower semi-continuous.

(b) If φ is σ-weakly lower semi-continuous, then it is normal.

Proof. Since the product topology of the σ-weak topology on $\mathfrak{N}_\varphi$ and the weak topology on H is the weak* topology on the dual Banach space $(M_* \oplus_1 H)^* \cong M \oplus_\infty H$, it suffices to show the graph is closed in this weak* topology. In the spirit of the Krein-Sm̌ulian theorem, consider the unit ball

$$\Gamma(\Lambda_\varphi)_1 = \{(x, \Lambda_\varphi(x)) : x \in \mathfrak{N}_\varphi,\ \|x\| \le 1,\ \|\Lambda_\varphi(x)\| \le 1\},$$

which is also convex. □

1.6 (Lower semi-continuous weights)**.** Let φ be a weight on a C*-algebra A. Definition of semi-finite weight. Definition of lower semi-continuous weight. Definition of semi-cyclic representation.

(a) A semi-finite lower semi-continuous weight φ is extended to the unique semi-finite normal weight of $\pi(A)''$.

(b) For a locally compact Hausdorff space X, a semi-finite lower semi-continuous weight of $C_0(X)$ is nothing but the positive linear functional of $C_c(X)$, the Radon measures.

1.2 Hilbert algebras

1.7 (Left and right Hilbert algebras)**.** A *left Hilbert algebra* is a complex inner product space A which is a $*$-algebra such that

(i) the left multiplication defines a non-degenerate $*$-homomorphism $L_0 : A \subset H \to B(H)$,

(ii) the involution defines a closable anti-linear operator $S_0 : A \subset H \to H$,

where H is the completion of A. The involution of a left Hilbert algebra is denoted by $\xi \mapsto \xi^\sharp$.
For $\eta \in H$, define $R(\eta) : A \subset H \to H$ and $F\eta : A \subset H \to \mathbb{C}$ such that

$$R(\eta)\xi := L_0(\xi)\eta, \quad F\eta(\xi) = \langle \eta, S_0\xi \rangle, \qquad \xi \in A.$$

If we introduce

$$\operatorname{dom} R := \{\eta \in H : R(\eta) \text{ is bounded}\}, \qquad \operatorname{dom} F := \{\eta \in H : F\eta \text{ is bounded}\},$$

we can let $R : \operatorname{dom} R \subset H \to B(H)$ and $F : \operatorname{dom} F \subset H \to H$ be a densely defined anti-homomorphism and a densely defined anti-linear operator. Let $A' := \operatorname{dom} R \cap \operatorname{dom} F$. We will show A' is dense in H.

Symmetrically for $\xi \in H$, after verifying A' is dense in H, we can define $L(\xi) : A' \subset H \to H$ and $S\xi : A' \subset H \to \mathbb{C}$ such that

$$L(\xi)\eta := R(\eta)\xi, \quad S\xi(\eta) = \langle \xi, F\eta \rangle, \qquad \eta \in A',$$

and by introducing

$$\operatorname{dom} L := \{\xi \in H : L(\xi) \text{ is bounded}\}, \qquad \operatorname{dom} S := \{\xi \in H : S\xi \text{ is bounded}\},$$

we recognize $L : \operatorname{dom} L \subset H \to B(H)$ and $S : \operatorname{dom} S \subset H \to H$ as a densely defined homomorphism and a densely defined anti-linear operator which densely extend L_0 and S_0. Let $A'' := \operatorname{dom} L \cap \operatorname{dom} S$. If $A = A''$, then we say A is *full*.

(a) A' is a $*$-algebra with the multiplication $(\eta, \zeta) \mapsto \eta\zeta := R(\zeta)\eta$ and the involution $\eta \mapsto \eta^\flat := F\eta$.

(b) A' is a right Hilbert algebra.

(c) L, S, R and F are injective and closed.

Proof. (a) The well-definedness of the multiplication and the involution on A' follows clearly by combining from (i) to (iv) in the below. The fact that the multiplication and the involution are really multiplication and involution on A' comes from (iv) and (i), and their compatibility follows from (iii).

(i) We have $FF\eta = \eta$ in H for $\eta \in \operatorname{dom} F$ by

$$FF\eta(\xi) = \langle F\eta, S\xi\rangle = \langle SS\xi, \eta\rangle = \langle \xi^{\sharp\sharp}, \eta\rangle = \langle \xi, \eta\rangle, \qquad \xi \in A.$$

In particular, if $\eta \in \operatorname{dom} F$, then $F\eta \in \operatorname{dom} F$.

(ii) We have $R(F\eta) = R(\eta)^*$ on A for $\eta \in \operatorname{dom} F$ by

$$\begin{aligned}\langle R(F\eta)\xi, \xi\rangle &= \langle L(\xi)F\eta, \xi\rangle = \langle F\eta, L(\xi)^*\xi\rangle = \langle SL(\xi)^*\xi, \eta\rangle = \langle (\xi^\sharp\xi)^\sharp, \eta\rangle \\ &= \langle \xi^\sharp\xi, \eta\rangle = \langle L(\xi)^*\xi, \eta\rangle = \langle \xi, L(\xi)\eta\rangle = \langle \xi, R(\eta)\xi\rangle = \langle R(\eta)^*\xi, \xi\rangle, \qquad \xi \in A.\end{aligned}$$

In particular, if $\eta \in \operatorname{dom} F$, then $F\eta \in \operatorname{dom} R$, and $R(\eta)$ is a closed densely defined operator.

(iii) We have $F(R(\eta)^*\zeta) = R(\zeta)^*\eta$ in H for $\eta, \zeta \in \operatorname{dom} R$ by

$$\begin{aligned}\langle F(R(\eta)^*\zeta), \xi\rangle &= \langle S\xi, R(\eta)^*\zeta\rangle = \langle R(\eta)S\xi, \zeta\rangle = \langle L(S\xi)\eta, \zeta\rangle = \langle L(\xi)^*\eta, \zeta\rangle \\ &= \langle \eta, L(\xi)\zeta\rangle = \langle \eta, R(\zeta)\xi\rangle = \langle R(\zeta)^*\eta, \xi\rangle, \qquad \xi \in A.\end{aligned}$$

In particular, if $\eta, \zeta \in \operatorname{dom} R$, then $F(\eta)^*\zeta \in \operatorname{dom} F$.

(iv) We have $R(R(\eta)^*\zeta) = R(\eta)^*R(\zeta)$ on A for $\eta \in \operatorname{dom} R$ and $\zeta \in H$ by

$$\begin{aligned}\langle R(R(\eta)^*\zeta)\xi, \xi\rangle &= \langle L(\xi)R(\eta)^*\zeta, \xi\rangle = \langle \zeta, R(\eta)L(\xi)^*\xi\rangle = \langle \zeta, L(L(\xi)^*\xi)\eta\rangle \\ &= \langle \zeta, L(\xi^\sharp\xi)\eta\rangle = \langle \zeta, L(\xi)^*L(\xi)\eta\rangle = \langle L(\xi)\zeta, L(\xi)\eta\rangle \\ &= \langle R(\zeta)\xi, R(\eta)\xi\rangle = \langle R(\eta)^*R(\zeta)\xi, \xi\rangle, \qquad \xi \in A.\end{aligned}$$

In particular, if $\eta, \zeta \in \operatorname{dom} R$, then $F(\eta)^*\zeta \in \operatorname{dom} R$.

(b) Since F has a densely defined adjoint S, it is closable, so we only need to show R is non-degenerate. For $\operatorname{dom} F$ is dense in H by the closability of S, it suffices to verify the inclusion $\operatorname{dom} F \subset \overline{A'^2}$ in order for density of $R(A')H$ in H. We fix $\eta \in \operatorname{dom} F$ and claim it is approximated by elements of A'^2. Since $R(\eta)$ is closed and densely defined by (ii) in the part (a), we can write down the polar decomposition

$$R(\eta) = vh = kv, \qquad h := |R(\eta)|, \quad k := |R(\eta)^*|.$$

To control the unboundedness of $R(\eta)$, we introduce $f \in C_c((0,\infty))^+$ to cutoff $R(\eta)$. Let $\acute{f}(t) := tf(t)$ and $\grave{f}(t) := t^{-1}f(t)$. Now we have $f(k) \in \operatorname{ran} R$ and $f(k)\eta \in \operatorname{dom} R$ because we have on A that

$$f(k) = f(vhv^*) = vf(h)v^* = v\grave{f}(h)hv^* = v\grave{f}(h)R(F\eta) = R\big(v\grave{f}(h)F\eta\big)$$

and

$$R(f(k)\eta) = f(k)R(\eta) = f(k)kv = \acute{f}(k)v,$$

which are bounded and extended to H. Applying the above for $f^{\frac{1}{3}} \in C_c((0,\infty))$,

$$f(k)\eta = (f(k)^{\frac{1}{3}})^3\eta \in (\operatorname{ran} R)^*(\operatorname{ran} R)(\operatorname{ran} R)^* \operatorname{dom} R.$$

Because $(\operatorname{ran} R)^* \operatorname{dom} R \subset A'$ and $(\operatorname{ran} R)^*(\operatorname{ran} R) \subset R(A')$ by (iii) and (iv) in the part (a), we have $f(k)\eta \in A'^2$. If we consider a non-decreasing net e_i in $C_c((0,\infty))$ such that $e_i \uparrow 1_{(0,\infty)}$, then we get

$$e_i(k)\eta \to 1_{(0,\infty)}(k)\eta = s(k)\eta = s_l(R(\eta))\eta = \eta \in \overline{A'^2},$$

where the last equality is due to $\eta \in \overline{L(A)\eta} = \overline{R(\eta)A}$ by the non-degeneracy of L.

(c) We have

$$L_0^*(\omega_{\eta,\zeta}) = R(\eta)^*\zeta, \qquad \eta \in \operatorname{dom} R,\ \zeta \in H.$$

Is $\omega_{\operatorname{dom} R, H}$ weakly dense in M_*? Is $(\operatorname{ran} R)H$ dense in H? Yes! ☐

1.8 (Semi-cyclic representations and Hilbert algebras)**.** Let M be a von Neumann algebra on a Hilbert space H.

(a) For a full left Hilbert algebra $A \subset H$ such that $\overline{A} = H$ and $L(A)'' = M$, if we define $\Lambda : \operatorname{dom}\Lambda \subset M \to H$ as the inverse of $L : \operatorname{dom} L \subset H \to M$, then Λ is a densely defined σ-weakly closed left M-linear operator of dense image such that $A = \Lambda((\operatorname{dom}\Lambda)^* \cap (\operatorname{dom}\Lambda))$.

(b) For a densely defined σ-weakly closed left M-linear operator $\Lambda : \operatorname{dom}\Lambda \subset M \to H$ of dense image, if we define $A := \Lambda((\operatorname{dom}\Lambda)^* \cap (\operatorname{dom}\Lambda))$, then A is a natural full left Hilbert algebra such that $\overline{A} = H$ and $L(A)'' = M$ such that Λ is the inverse of L.

Proof. (a) The domain and image of Λ is dense in M and H because the image and domain of L is dense in M and H. The graph of the Λ is σ-weakly closed since its inverse L is σ-weakly closed. To check the left M-linearity, let $x \in M$ and $\xi \in \operatorname{dom} L$. Since $\operatorname{dom}\Lambda$ is a σ-weakly dense $*$-subalgebra of M, it admits an approximate unit $e_i \in (\operatorname{dom}\Lambda)_1^+$ with $e_i \to 1$ σ-strongly*. Because $\operatorname{dom}\Lambda$ is hereditary, we have a net $e_i x e_i \in \operatorname{dom}\Lambda$ satisfying $e_i x e_i \to x$ σ-strongly*. Then, we have $e_i x e_i \xi \to x\xi$ and a σ-weak limit

$$L(e_i x e_i \xi) = L(L(\Lambda(e_i x e_i))\xi) = L(\Lambda(e_i x e_i)\xi) = L(\Lambda(e_i x e_i))L(\xi) = e_i x e_i L(\xi) \to xL(\xi),$$

so the closedness of L implies that $L(x\xi) = xL(\xi)$.

(b)

(c?d?) The main difficulty is the closability of $\Lambda(x) \mapsto \Lambda(x^*)$.

□

1.9 (Hilbert algebras by cyclic separating vectors)**.**

1.10 (Hilbert algebras by locally compact groups)**.**

1.11 (Modular operator and modular conjugation)**.** Let A be a left Hilbert algebra.

$$S = J\Delta^{\frac{1}{2}}.$$

Tomita algebra analytic elements

1.12 (Tomita-Takesaki commutation theorem)**.** Wick rotation The *imaginary time* is $\frac{i}{\hbar}t$. Let H be a self-adjoint operator of bounded from below. Consider

$$\Delta_\beta := e^{-\beta H}, \qquad u_t := e^{\frac{i}{\hbar}tH}.$$

Then, Δ is an invertible trace-class operator. The unitary operator u_t^* is called the *propagator*.

The one-parameter automorphism σ_t has the formal infinitesimal generator $\frac{i}{\hbar}\operatorname{ad} H = \frac{i}{\hbar}[H, -]$ with

$$\sigma_t := \operatorname{Ad} u_t = e^{\frac{i}{\hbar}tH} \cdot e^{-\frac{i}{\hbar}tH} = e^{\frac{i}{\hbar}t\operatorname{ad}_H}.$$

$$\begin{tikzcd}
& A \arrow[rr, "u_t=\Delta^{it}"] \arrow[dl, leftrightarrow, "J"'] & & A \arrow[dl, leftrightarrow, "J"] \arrow[dd, "\lambda"] \\
A' \arrow[rr, "\Delta^{it}"'] \arrow[dd, "\rho"'] & & A' \arrow[dd, "\rho"] & \\
& & & \lambda(A) \arrow[dl, leftrightarrow, "\mathrm{Ad}\,J"] \\
\rho(A') \arrow[rr, "\sigma_t=\mathrm{Ad}\,\Delta^{it}"'] & & \rho(A') &
\end{tikzcd}.$$

(a) Fourier inversion

(b) $(e^{-s} + \Delta)^{-1} : A' \to A \cap \operatorname{dom} F$.

(c) commutation theorem

1.3 Standard forms

1.13 (Standard forms)**.** For a subset P of a complex vector space H, the *dual cone* is the closed convex cone $P^\circ := \{\eta \in H : \langle \xi, \eta \rangle \geq 0,\ \xi \in P\}$, and we say P is a *self-dual cone* if $P = P^\circ$. Let M be a von Neumann algebra. A *standard form* (M,H,J,P) of M is a faithful unital normal representation $M \subset B(H)$ together with an anti-linear isometric involution $J : H \to H$ and a self-dual cone $P \subset H$ satisfying the following axioms:

(i) $JMJ = M'$,

(ii) $z' = z^*$ for $z \in Z(M)$,

(iii) $J\xi = \xi$ for $\xi \in P$,

(iv) $xx'P \subset P$ for $x \in M$,

where $x' := JxJ \in M'$ for $x \in M$.

(a) For projections $p \in M$ and $q := pp' = p'p \in B(H)$ in a standard form, then $q \neq 0$ if and only if $p \neq 0$, and (qMq, qH, qJq, qP) is also a standard form.

Proof. From $(JpJ) \in M'$, we have $pq = q = qp$ and $pxq = qxq = qxp$ for $x \in M$. Note also that q is a projection such that $Jq = pJp = qJ$. Consider a surjective linear map $pMp \to qMq : pxp \mapsto p'pxpp' = qxq$. It is well-defined because $pxp = 0$ implies $qxq = p'pxpp' = 0$, and is a $*$-homomorphism because

$$q(px_1p)(px_2p)q = qx_1px_2q = qx_1qx_2q = (qx_1q)(qx_2q), \qquad x_1, x_2 \in M.$$

It is also injective since if $qxq = 0$, then

$$p = Jp'J \leq Jz(p')J = z(p')^* = z(p')$$

implies $pH \subset z(p')H = \overline{M'p'H}$ and

$$pxpH \subset \overline{pxpM'p'H} = \overline{M'pxpp'H} = \overline{M'qxqH} = 0, \qquad x \in M.$$

Therefore, $pMp \to qMq$ is a $*$-isomorphism, hence $q \neq 0$ if and only if $p \neq 0$.

We can check that qJq is an anti-linear isometric involution on qH by $J(qH) = q(JH) = qH$ and $(qJq)^2 = q$, and qP is a self-dual cone by $qP = pp'P \subset P = P^\circ \subset (qP)^\circ$ and conversely the inequality $\langle q\xi, \eta \rangle = \langle q\xi, q\eta \rangle \geq 0$ for any $\eta \in P$ implies $q\xi \in P^\circ = P$ and $q\xi \in qP$ whenever $q\xi \in (qP)^\circ$. The first condition follows from

$$(qJq)(qMq)(qJq) = qJMJq = qM'q = q(p'M'p')q = q(p'Mp')'q = (qMq)',$$

and the second condition follows because $Z(qMq) = qMq \cap (qMq)' = qMq \cap qM'q = qZ(M)q$ so that an element of $Z(qMq)$ is equal to qzq for some $z \in Z(M)$, we have

$$(qJq)(qzq)(qJq) = qJzJq = qz^*q = (qzq)^*, \qquad qzq \in Z(qMq).$$

Since the third and forth axioms for standard forms are clear. $\square$

1.14 (Existence of standard forms)**.**

1.15 (Uniqueness of standard forms)**.**

1.16 (Unitary implementation)**.** Powers-Størmer inequality (see Bratelli-Robinson)

Aut(M) and the set of unitaries preserving standard form are not homeomorphic! (Takesaki seems to be wrong) But on locally compact subgroups, the topologies are same. For example, If $\alpha : G \to \mathrm{Aut}(M)$ is σ-weakly continuous, then $\alpha^* : G \to \mathrm{Aut}(M_*)$ is weakly continuous. By smearing argument on G, we can see $\alpha^* : G \to \mathrm{Aut}(M_*)$ is strongly continuous. Inserting the homeomorphism $P \cong M_*^+$, we have strongly continuous unitary representation on H.

1.4 Modular actions

1.17 (Kubo-Martin-Schwinger weights)**.**

1.18 (Centralizer of weights)**.**

1.19 (Commuting weights)**.** Radon-Nikodym derivative

1.20 (Conditional expectations)**.**

1.21 (Operator-valued weights)**.** Let M and N be von Neumann algebras. The *extended positive cone* $\widehat{N}^+$ of N is defined as the set of lower semi-continuous additive homogeneous functions $y : N_*^+ \to [0,\infty]$. An *operator-valued weight* is a additive homogeneous function $T : M^+ \to \widehat{N}^+$.

faithful semi-finite normal operator-valued weight...

Exercises

1.22 (Completely additive weights)**.** Let φ be a *completely additive* weight on a von Neumann algebra in the sense that for every orthogonal family $\{p_\alpha\}$ of projections we have $\varphi(\sum_\alpha p_\alpha) = \sum_\alpha \varphi(p_\alpha)$.

(a) A completely additive state on a von Neumann algebra is normal.

(b) A completely additive and lower semi-continuous weight on a commutative von Neumann algebra is normal.

Chapter 2

Non-commutative integral

2.1 Haagerup spaces

2.1 (Semi-finite and tracial von Neumann algebras)**.** Let M be a von Neumann algebra. We say M is *semi-finite* if it admits a faithful semi-finite normal trace, and *tracial* if it admits a faithful normal tracial state.

(a) regular representation and antilinear isometric involution J. $L(G) = \rho(G)'$

(b) M is semi-finite if and only if type III does not occur in the direct sum.

(c) A factor M has at most one tracial state, which is normal and faithful.

(d) A factor is tracial if and only if it is type II_1.

2.2 (Semi-finite traces)**.** Let M be a von Neumann algebra and τ is a trace. For a trace τ

(a) τ is semi-finite if and only if $x \in M^+$ has a net $x_\alpha \in L^1(M,\tau)^+$ such that $x_\alpha \uparrow x$ strongly.

(b) Let τ be normal and faithful. Then, τ is semi-finite if and only if

$$\tau(x) = \sup\{\, \tau(y) : y \leq x,\ y \in L^1(M,\tau)^+ \,\} \quad \text{for} \quad x \in M^+.$$

2.3 (Uniformly hyperfinite algebras)**.** Let A be a uniformly hyperfinite algebra.

(a) Every matrix algebra admits a unique tracial state.

(b) Every UHF algebra admits a unique tracial state.

(c) Every hyperfinite

measurable operators, unbounded operators affilated with M, noncommutative L^p spaces for semi-finite con Neumann algebras, noncommutative L^p space for general von Neumann algebras: by Haagerup(crossed product), and by Kosaki-Terp(complex interpolation).

On semi-finite von Neumann algebras, measurable operators are affiliated. On a finite von Neumann algebras, affiliated operators are measurable.

2.4 (Measurable operators)**.**

- density of $C(X)$ in $L^p(X,\mu)$
- Hölder inequality
- Radon-Nikodym
- Riesz representation

- Fubini
- maximality of L^∞ in $B(L^2)$

The sequentiality of a net is required for the relation between the almost everywhere convergence and the local convergence in measure. In particular, an almost everywhere convergent net might not converges locally in measure. Monotone, bounded, dominated convergence theorems are true for nets that converge locally in measure.

For a localizable measure space (X,μ), $L^0_{\mathrm{loc}}(\mu) \cong S(L^\infty(\mu))$.

Chapter 3

3.1 Commutative von Neumann algebras

3.1.1

3.1. For a normal weight φ of a commutative von Neumanna algebra M, φ is semi-finite if and only if for every projection $p \in M$ with $\varphi(p) = \infty$, there is another projection $q \in M$ such that $q \leq p$ and $0 < \varphi(q) < \infty$.

Proof. ($\Rightarrow$) Take $e \in M^+$ such that $0 < \varphi(ep) < \infty$. Approximate ep from below by the simple functions $s = \sum_i a_i p_i$ such that $0 < \varphi(ep) - \varepsilon < \varphi(s) \leq \varphi(ep) < \infty$ by the normality, where $a_i \geq 0$ and p_i are mutually orthogonal. Then, $q := \sum_i p_i$ satisfies the property.

($\Leftarrow$) Suppose $\mathfrak{m}$ is not σ-weakly dense in M. Its σ-weak closure is given by pMp for some projection $p \in M$. Then, $1 - p$ is a counterexample of the contradictory assumption. $\square$

Monotone convergence theorem states that a measure on a countably decomposable(?) enhanced measurable space X uniquely defines a 'countably' normal weight on the space of all measurable functions. Note that a 'countably' normal weight is normal on a countably decomposable von Neumann algebra.

separable commutative von Neumann algebra is generated by one self-adjoint element.

3.2 (semi-finite lower semi-continuous weights)**.** Let Ω be a locally compact Hausdorff space. There is a one-to-one correspondence between semi-finite lower semi-continuous weights of $C_0(\Omega)$ and positive linear functionals on $C_c(\Omega)$.

A semi-finite lower semi-continuous weight on a C*-algebra A is uniquely extended to a semi-finite normal weight on A^{**}, and to $\pi(A)''$, where π is the semi-cyclic representation.

Proof. Let φ be a positive linear functional on $C_c(\Omega)$. We can extend it to $\varphi : C_0(\Omega)^+ \to [0, \infty]$ by letting

$$\varphi(f) := \sup\{\varphi(g) : g \leq f,\ g \in C_c(\Omega)\}.$$

Since $C_0(\Omega) \cap L^1(\varphi)$ is dense in $C_0(\Omega)$ by compact truncation, φ is semi-finite. If $f_n \in C_0(\Omega)$ such that $\int |f_n| \leq 1$ and $f_n \to f$ uniformly, then taking compact $K \subset \Omega$ such that $\int_{K^c} |f| < \varepsilon$, we can prove $\int |f| \leq 1$, so φ is lower semi-continuous. By taking the restriction on $C_c(\Omega)$, we can reconstrcut the original linear functional.

Conversely, let φ be a semi-finite lower semi-continuous weight on $C_0(\Omega)$. If there is a point $\omega \in \Omega$ such that $\varphi(f) = \infty$ whenever $f \in C_0(\Omega)^+$ and $f(\omega) > 0$, then $\mathfrak{m} \subset C_0(\Omega \setminus \{\omega\})$, which contradicts to the assumption that φ is semi-finite. Then, using compactness, we can prove $C_c(\Omega) \subset \mathfrak{m}$. Now we can check $\varphi(f) = \sup\{\varphi(g) : g \leq f,\ g \in C_c(\Omega)\}$ by constructing an increasing net in $C_c(\Omega)$ that converges to f uniformly. $\square$

3.3. The set of projections is a complete orthomodular lattice. If M is commutative, then the set of projections is a complete boolean algebra.

- commutative ring - distributive lattice - coherent locale
- clean ring+α - boolean algebra - stone space
- - complete boolean algebra - stonean space
- commutative von Neumann algebra - localizable boolean algebra - hyperstonean space

A *frame* is a partially ordered set F that admits a finite meets and arbitrary joins, and for any $a \in F$ the map $F \to F : x \mapsto x \wedge a$ preserves suprema. A *locale* is an object of the opposite category of frames. An element of a locale is called *open*.

A locale is called *coherent* if the set of compact opens is closed under finite meets and every open is the join of compact opens, i.e. generates opens. It is known that a coherent locale is spatial.

(i) X is a coherent space.

(ii) X is a (compact) sober space such that the set of compact open subsets is closed under finite intersections and forms a base.

(iii) X is homeomorphic to the underlying space of an affine scheme.

A morphism of CohLoc is a compact open preserving local morphism. A morphism of DistLat is just a lattice morphism. We can consider the compact open functor $\mathrm{CohLoc} \to \mathrm{DistLat}^{\mathrm{op}}$ and the ideal functor $\mathrm{DistLat}^{\mathrm{op}} \to \mathrm{CohLoc}$. They form a categorical equivalence between the category of coherent locales and the opposite category of distributive lattices with lattice morphisms (i.e. preserving finite meets and joins).

A locale is called *Stone* if it is a coherent locale in which every open is the join of all subopens of it.

(i) X is a Stone space.

(ii) X is totally disconnected and compact Hausdorff.

(iii) X is a compact zero-dimesional sober space.

(iv) X is a compact zero-dimesional Hausdorff space.

(v) X is coherent and Hausdorff.

A morphism of StoneLoc is a compact open (clopen) preserving locale morphism. A morphism of BoolLat is just a lattice morphism.

A locale is called *Stonean* if it is a Stone locale in which the (unique) complement of any element is clopen. A morphism of StoneanLoc is an open locale morphism. A morphism of CpltBoolLat is a continuous lattice morphism.

A locale is called *Hyperstonean* if... A boolean lattice is called *localizable* if it is complete, and the identity is approximated by elements admitting a faithful continuous valuation on their compression. The category LBAlg admits small products, and the products are preserved by the forgetful functor $\mathrm{LBAlg} \to \mathrm{BAlg}$.

*

3.4 (Boolean algebra)**.** A *boolean ring* is a ring in which every element is idempotent, which is automatically commutative. A *boolean algebra* is a unital boolean ring. A *boolean lattice* is a complemented distributive lattice.

(a) There is a one-to-one correspondence between boolean rings and boolean lattices.

(b) The category of boolean algebras with unital homomorphisms and the category of Stone spaces with continuous maps are equivalent.

(c) The category of complete boolean algebras with order continuous unital homomorphisms and the category of Stonean spaces with open continuous maps are equivalent. In the Stonean space, the join and meet is realized as the closure of union and the interior of intersection, respectively.

3.5 (Measurable algebras)**.** For a boolean algebra, existences of sequential suprema and sequential infima are equivalent. A boolean algebra is called a *measurable algebra* if it is order σ-complete.

(a) (Loomis-Sikorski representation) Every measurable algebra $\mathcal{L}$ is realized as $\mathcal{M}/\mathcal{M}\cap\mathcal{N}$ from a enhanced measurable space $(X,\mathcal{M},\mathcal{N})$.

(b) (Dedekind completion) Every boolean algbera $\mathcal{L}$

Proof. (a) Let X be the Stone space of $\mathcal{L}$, $\mathcal{M}$ the set of clopen subsets, and $\mathcal{N}$ the set of meager sets. Then, $\mathcal{M}$ is a σ-algebra on X and $\mathcal{N}$ is a σ-ideal of X.

(b) complete extension of order continuous homomorphisms and universal property. regular open algebra of X. □

3.6 (Measure algebras)**.** A *measure* on a measurable algebra $\mathcal{L}$ is a completely additive monotone function $\mathcal{L}\to[0,\infty]$. A *measure algebra* is a measurable algebra together with a faithful measure.

Let (X,M,μ) be a measure space, which is not necessarily faithful. There is a canonically associated measure algebra $(\mathcal{M}/\mathcal{M}\cap\mathcal{N},\mu)$, which is faithful, where $\mathcal{N}:=\mu^{-1}(0)$.

3.7 (Localizable measure algebras)**.** For a measure space $(X,\mathcal{M},\mu)$, the completion always does not change the measure algebra, and the complete locally determined version

$$\widetilde{\mathcal{M}}:=\{E\subset X : E\cap A\in\mathcal{M}\triangle\mathcal{N},\ \mu(A)<\infty\},\qquad \widetilde{\mu}(E):=\sup\{\mu(E\cap A):\mu(A)<\infty\}$$

does not change the measure algebra when the measure space is localizble.

(a) Every localizable measure algebra is obtained from a compact decomposable measure space.

(b) A σ-finite measure space is compact decomposable.

- HSTop: hyperstonean spaces with open continuous maps,
- HSLoc: hyperstonean locales with open localic maps,
- LBAlg: localizable boolean lattices with continuous lattice homomorphisms,
- CW*Alg: commutative W* algebras with normal $*$-homomorphisms.

$$\mathrm{HSTop}\ \underset{sp}{\overset{top}{\rightleftarrows}}\ \mathrm{HSLoc}\ \underset{ideal}{\overset{clopen}{\rightleftarrows}}\ \mathrm{LBAlg}^{\mathrm{op}}=\mathrm{MLoc}\ \underset{proj}{\overset{L^\infty}{\rightleftarrows}}\ \mathrm{CW^*Alg}^{\mathrm{op}}$$

3.8. (a) Construction of projection lattice functor.

(b) Construction of L^∞ functor.

(c) Equivalence.

Proof. (b) Let L be a measurable locale. For $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, define $L^\infty(L,\mathbb{F})$ to be the set of all bounded localic maps $x:L\to\mathbb{F}$, which are given by the opposite of lattice homomorphism $x^{-1}:\mathrm{top}(\mathbb{F})\to L$ which preserves finite meets and arbitrary joins, and factors an open ball of $\mathbb{F}$. We can define a normed $*$-algebra structure on $L^\infty(L,\mathbb{F})$ such that

$$(x+y)^{-1}(U):=\bigvee_{U_x+U_y\subset U}(x^{-1}(U_x)\wedge y^{-1}(U_y)),\qquad (xy)^{-1}(U):=\bigvee_{U_xU_y\subset U}(x^{-1}(U_x)\wedge y^{-1}(U_y)),$$

$$(x^*)^{-1}(U):=x^{-1}(\{\overline{z}:z\in U\}),\qquad \|x\|=\inf\{\sup_{z\in U}|z|:x^{-1}(U)=1\in L\}.$$

Using the axioms of locales, for example that the meet with a single element preserves arbitrary joins, we can manually check that $L^\infty(L,\mathbb{F})$ is a commutative normed $*$-algebra, and in particular the C*-identity when $\mathbb{F}=\mathbb{C}$. Furthermore, since $L^\infty(L,\mathbb{C})$ is the complexification of $L^\infty(L,\mathbb{R})$, if we prove $L^\infty(L,\mathbb{R})$ has a predual, then the completeness with respect to norm follows automatically, so $L^\infty(L,\mathbb{C})$ becomes a C*-algebra with a predual, i.e. a von Neumann algebra.

Define $L^1(L,\mathbb{R})$ the real linear span of continuous valuations on L, equipped with the variation norm. Recall that a continuous valuation is a monotone function $\nu : L \to [0,\infty)$ such that $\nu(0)=0$ and $\nu(p)+\nu(q)=\nu(p\vee q)+\nu(p\wedge q)$, which preserves directed suprema. Note that $L^\infty(L,\mathbb{R})$.... □

σ-field is a unital σ-ring. σ-ideal is an ideal of a σ-ring which is a σ-ring. σ-ideal is sometimes called the measure class because it corresponds to an equivalence class of measures up to absolute continuity.

3.9 (Enhanced measurable spaces)**.** An *enhanced measurable space* is a measurable space (X,M) together with a σ-ideal N of M. A morphism between enhanced measurable spaces is a partial function $f : X_1 \to X_2$ on a conegligible set such that f^* induces a ring homomorphism $M_2/N_2 \to M_1/N_1$.

(a) Maharam's theoem: every enhanced measurable space is isomorphic to the disjoint union of $\{0,1\}^I$, where I is an aribitrary cardinality...?

(b) A σ-finite enhanced measurable space is isomorphic to a enhenced measurable space induced from a standard probability space...?

(c) For σ-finite enhanced measurable spaces, a $*$-homomorphism $L^\infty(X_2) \to L^\infty(X_1)$ induces a morphism $X_1 \to X_2$...?

Premaps:

Strict maps: an a.e. equivalence class of premaps. For each strict map with non-empty codomain, there is a everywhere defined representative.

Quotients on morphisms:

$$\mathrm{PreEMS} \to \mathrm{StrictEMS} \to \mathrm{EMS}.$$

Fully faithful functors:

$$\mathrm{REMS} \to \mathrm{CDEMS} \to \mathrm{DEMS} \to \mathrm{LEMS}, \mathrm{DEMS} \to \mathrm{LDEMS}.$$

The functor LEMS $\to$ LBAlg : $(X,M,N) \mapsto M/N$ is a well-defined essentially surjective functor, which is fully faithful on the full subcategory CDEMS.

We say a enhanced measurable space is *decomposable* or *strictly localizable* if it is isomorphic to the small coproduct of countably decomposable enhanced measurable spaces.

DEMS is a full subcategory of PreEMS, but not of EMS, and we embed it to LDEMS

3.1.2 Complete Boolean algebras

3.1.3 Measure algebras

Definition. An (complete) *enhanced measurable space* is a set X together with a σ-algebra $\mathcal{M}$ on X and a σ-ideal $\mathcal{N}$ of $\mathcal{P}(X)$.

Definition. A *measurable algebra* is another name of a σ-complete Boolean algebra. A *measure* on a measurable algebra $\mathcal{A}$ is a countably additive monotone function $\mu : \mathcal{A} \to [0,\infty]$. A *measure algbera* is a measurable algebra together with a measure. We consider continuous homomorphisms and measure-preserving continuous homomorphisms as morphisms of the categories of measurable algebras and measure algebras.

In the below diagrams, morphisms of each category are supposed to be as follows: negligible reflecting measurable maps between enhanced measurable spaces, continuous homomorphisms between measurable algebras, and unital normal $*$-homomorphisms between von Neumann algebras. The arrow $\twoheadrightarrow$ means an essentially surjective functor.

$$\begin{tikzcd}
\begin{array}{c}\text{measure space}\\ (X,\mathcal{M},\mu)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{measure algebra}\\ ((\mathcal{M}\triangle\mathcal{N})/\mathcal{N},\mu)\end{array} \arrow[d] & \\
\begin{array}{c}\text{enhanced measurable space}\\ (X,\mathcal{M},\mathcal{N})\end{array} \arrow[r, two heads] & \begin{array}{c}\text{measurable algebra}\\ (\mathcal{M}\triangle\mathcal{N})/\mathcal{N}\end{array} & \begin{array}{c}\text{commutative}\\ \text{von Neumann algebra}\end{array} \arrow[l, hook']
\end{tikzcd}$$

above functors are fully faithful? Essential surjectivity of the horizontal functors are by the Loomis-Sikorski representation theorem. It states that every σ-complete Boolean algebra is isomorphic to $\mathcal{M}/\mathcal{M}\cap\mathcal{N}$, where $(X,\mathcal{M},\mathcal{N})$ is an enhanced measurable space.

3.10 (Measure space).

3.1.4 Maharam classification

3.11 (Localizable measure algebras). A measure algbera $(\mathcal{A},\mu)$ is called *localizable* if $\mathcal{A}$ is complete and μ is semi-finite. A measurable algebra $\mathcal{A}$ is called *localizable* if it is complete and it admits a semi-finite measure.

$$\begin{tikzcd}
\begin{array}{c}\text{localizable}\\ \text{measure space}\\ (X,\mathcal{M},\mu)\end{array} \arrow[r, two heads] \arrow[d, two heads] & \begin{array}{c}\text{localizable}\\ \text{measure algebra}\\ ((\mathcal{M}\triangle\mathcal{N})/\mathcal{N},\mu)\end{array} \arrow[r, leftrightarrow] \arrow[d, two heads] & \begin{array}{c}\text{commutative}\\ \text{von Neumann algebra}\\ \text{with a f.s.n. weight}\end{array} \arrow[d, two heads] \\
\begin{array}{c}\text{localizable}\\ \text{enhanced measurable space}\\ (X,\mathcal{M},\mathcal{N})\end{array} \arrow[r, two heads] & \begin{array}{c}\text{localizable}\\ \text{measurable algebra}\\ (\mathcal{M}\triangle\mathcal{N})/\mathcal{N}\end{array} \arrow[r, leftrightarrow] & \begin{array}{c}\text{commutative}\\ \text{von Neumann algebra}\end{array}
\end{tikzcd}$$

(a) A σ-finite measure algebra is localizable.

3.12 (Maharam classification). Let $\mathcal{A}$ be an localizable measurable algebra. A *Maharam type* or just a *type* of $\mathcal{A}$ is the smallest cardinal $\tau(\mathcal{A})$ of any dense subset of $\mathcal{A}$. If $\tau(\mathcal{A}) = \tau(\mathcal{A}\wedge a)$ for all non-zero $a\in\mathcal{A}$, then we say $\mathcal{A}$ is *Maharam homogeneous*. For an infinite cardinal κ, the *Maharam component* of type κ is the supremum e_κ of any non-zero elements $a\in\mathcal{A}$ such that $A\wedge a$ is Maharam homogeneous of type κ.

A *cellularity* of a Boolean algebra $\mathcal{A}$ is the supremum $c(\mathcal{A})$ of the cardinalities of any disjoint subset of $\mathcal{A}\setminus\{0\}$. Note that the cellularity is either zero or infinite if $\mathcal{A}$ is atomless, and $\mathcal{A}\wedge e_\kappa$ is atomless if κ is infinite. We define a cellularity function $c:\mathrm{InfCard}\to\mathrm{InfCard}\cup\{0\}$ such that $c(\kappa) := c(\mathcal{A}\wedge e_\kappa)$.

(a)

(b) All Maharam homogeneous probability algebras of same type are isomorphic.

(c) A measure algebra $(\mathcal{B}_\kappa,\mu)$ from the measure space $(\{0,1\}^\kappa,\mu)$ is Maharam homogeneous probability algebra of type κ.

(d) A

A disjoint union and product of localizable measurable algebras is passed to the direct product and the tensor product. Every commutative von neumann algebra can be realized as L^∞ of the disjoint union, or equivalently, the direct product of L^∞, of countably decomposable enhanced measurable spaces $\{0,1\}^\kappa$. Every countably decomposable commutative von Neumann algebra is the tensor product of ℓ^∞'s.

The invariants of localizable measure algebras:

$$m : \mathrm{InfCard} \to \mathrm{Card}, \qquad n : (0,\infty) \to \mathrm{Card}.$$

The invariants of localizable measurable algebras:

$$c : \mathrm{InfCard} \to \mathrm{InfCard} \cup \{0\}, \qquad a \in \mathrm{Card},$$

where

$$c(\kappa) := \begin{cases} \omega & , 0 < m(\kappa) < \infty \\ m(\kappa) & , \text{ otherwise} \end{cases}, \qquad a := \sum_{r\in(0,\infty)} n(r).$$

We have $m(\kappa) \le c(\kappa)$ and $m(\kappa) < c(\kappa)$ only if $c(\kappa) = \omega$.

For an atomless commutative von Neumann algebra M (no minimal projections), countable decomposability says that there is a countable collection of cardinals S such that $c(\kappa) = \omega$ if $\kappa \in S$ and $c(\kappa) = 0$, i.e. the countable product of $L^\infty(\{0,1\}^\kappa)$, and separability says that $c(\omega) = \omega$ and $c(\kappa) = 0$ for $\kappa > \omega$, i.e. $L^\infty(\{0,1\}^\omega)$.

3.13 (σ-finite and standard measure spaces)**.** Let (X,μ) be a localizable measure space.

(a) (X,μ) is σ-finite if and only if $L^\infty(\mu)$ is countably decomposable.

(b) (X,μ) is standard if and only if $L^\infty(\mu)$ is separable.

3.14 (Radon measures)**.** semi-finite lower semi-continuous weight

3.1.5 Spectral theory

Spectral theory = representation theory of commutative operator algebras

3.15. Let X be a locally compact Hausdorff space and $\pi : C_0(X) \to B(H)$ be a representation. Then, we have $\pi(C_0(X)) \subset \pi(C_0(X))'$.

A *multiplication representation* of $C_0(X)$ refers to nothing but the associated semi-cyclic representation of some Radon measure of $C_0(X)$. If the Radon measure is finite, then the multiplication representation would be cyclic.

Proof. Let μ be a Radon measure on X and $m : C_0(X) \to B(L^2(\mu))$ be the multiplication representation. The representation m is normally extended to $L^\infty(\mu)$ such that $m(L^\infty(\mu)) = m(C_0(X))''$, and we have $L^1(\mu) \cong L^\infty(\mu)_*$. The semi-cyclic data Λ is given by the identity on $\operatorname{dom}\Lambda = L^2(\mu) \cap L^\infty(\mu)$. Since $(m(C_0(X))')_1^+$ each defines a semi-finite normal weight $\varphi \le \mu$, which gives $L^\infty(\mu)_1^+$, we have $m(C_0(X))' = L^\infty(\mu) = m(C_0(X))''$. (In other words, we use the two versions of Radon-Nikodym, commutant and centralizer.) Therefore, $L^\infty(\mu)$ is maximal abelian subalgebra of $B(L^2(\mu))$. $\square$

3.16. Multiplicity theory. Consdier a cyclic decomposition of any representation $\pi : C_0(X) \to B(H)$.

3.2 Tensor products and direct integrals

3.17. Let M and N be von Neumann algebras. Consider the embedding $M_* \odot N_* \subset (M \otimes_{\min} N)^*$. The closure $M_*\overline{\otimes}N_*$ is invariant under $M \otimes_{\min} N$, so it defines a von Neumann subalgebra of $(M \otimes_{\min} N)^{**}$ by a central projection. This is the tensor product von Neumann algebra of M and N.

3.18 (Dixmier's measurable fields)**.** Let (X,μ) be a localizable measure space. Suppose $\{H_s\}_{s\in X}$ is a family of Hilbert spaces, and define $F := \prod_{s\in X} H_s$ the section space. We say $\{H_s\}_{s\in X}$ is *measurable* if F has a linear subspace S such that

(i) $s \mapsto \|\xi_s\|$ is measurable for $\xi \in S$,

(ii) for $\eta \in F$ if $s \mapsto \langle \xi_s, \eta_s\rangle$ is measurable for all $\xi \in S$, then $\eta \in S$,

(iii) There is a sequence of ξ_n in S such that for each $s \in X$ the linear span $\xi_{n,s}$ is dense in H_s.

An element of S is called a *measurable vector field*. For a measurable vector field ξ, we say it is *square-integrable* if $\int \|\xi_s\|^2 \, d\mu(s) < \infty$. The space of square-integrable vector fields has a natural sesquilinear form, which gives rise to a Hilbert space by separation without completion. The obtained Hilbert space is called the *direct integral* of the field $\{H_s\}_{s\in X}$ of Hilbert spaces.

A field of bounded linear operators $\{T_s\}_{s\in X}$ is called *measurable* if it sends a measurable vector field to a measurable vector field.

An operator T in $B(H)$ is called *decomposable* if it is represented by a measurable field of bounded linear operators. A decomposable operator is called *diagonalizable* if it is represented by an element of $L^\infty(\mu)$.

A *measurable field of Hilbert spaces* is a faithful unital normal representation of a commutative von Neumann algebra together with a faithful semi-finite normal weight, i.e. $L^\infty(\mu) \subset B(H)$. An operator $T \in B(H)$ is called *diagonal* if $T \in L^\infty(\mu)$, and called *decomposable* if $T \in L^\infty(\mu)'$.

Tensor products? Approximation by continuous fields?

Every von Neumann algebra M admits a faithful unital normal representation $M \subset B(H)$ such that M' is commutative?

3.19 (Effros Borel structure)**.**

3.20 (Decomposition of states)**.**

3.3 Type

For a von Neumann algebra M, M^{sa} is a conditionally monotone complete poset, and $P(M)$ is a complete orthomodular lattice. For a commutative von Neumann algebra M, M^{sa} is a Dedekind complete Banach lattice, and $P(M)$ is a complete Boolean lattice.

For von Neumann algebras, we want to compare order topology, measure topology, and operator topologies. - commutative and non-commutative. - M, M^{sa}, M_1^{sa}. - closedness for convex sets or bounded sets. - continuity of linear operators and functionals.

For subsets: - bounded lattices - complemented lattices - orthocomplemented lattices - orthomodular lattices - complete orthomodular lattices <- vN - complete Boolean lattices <- CvN

- For functions: - distributive lattices - vector lattices - Dedekind complete vector lattice

* * *

abelian, finite, purely infinite, properly infinite projections

central projection = union of components central support = a kind of minimal union of components centrally orthogonal

3.21. We say a von Neumann algebra is of *type I* if every non-zero central projection has a non-zero abelian subprojection, *type II* if has no non-zero abelian projection and every non-zero central projection has a non-zero finite subprojection, and *type III* if it has no non-zero finite projection.

For a type II von Neumann algebra, it is called of *type II_1* if it is finite, and of *type II_∞* if it has no central finite projections.

A projection is called *semi-finite* if every non-zero central subprojection has a non-zero finite subprojection.

- type I = semi-abelian
- type II = purely non-abelian semi-finite
- type III= purely infinite

V.1.35. For a purely non-abelian von Neumann algebra, every projection is the sum of two equivalent orthogonal projections.

3.22 (Type I)**.** Let M be a von Neumann algebra of type I. Then, there are families $\{M_\kappa\}_\kappa$ and $\{H_\kappa\}_\kappa$ of commutative von Neumann algebras M_κ and Hilbert spaces H_κ satisfying $\dim H_\kappa = \kappa$, indexed by cardinals κ, such that

$$M \cong \bigoplus_{\kappa \in \mathrm{Card}} M_\kappa \overline{\otimes} B(H_\kappa).$$

If M is a factor, then $M \cong B(H)$ for a Hilbert space H.

finite: f.n. center-valued trace,
semi-finite: f.s.n. extended center-valued trace, 2.34
tensor product and types

Type I factors. It possess a minimal projection. It is isomorphic to the whole $B(H)$ for some Hilbert space. Therefore, it is classified by the cardinality of H.

Type II factors. No minimal projection, but there are non-zero finite projections so that every projection can be "halved" by two Murray-von Neumann equivalent projections.

In type II_1 factors, the identity is a finite projection Also, Murray and von Neumann showed there is a unique finite tracial state and the set of traces of projections is $[0,1]$. Examples of II_1 factors include crossed product, tensor product, free product, ultraproduct. Free probability theory attacks the free groups factors, which are type II_1.

In type II_∞ factors. There is a unique semifinite tracial state up to rescaling and the set of traces of projections is $[0,\infty]$.

In type III factors no non-zero finite projections exists. Classified the $\lambda \in [0,1]$ appeared in its Connes spectrum, they are denoted by III_λ. Tomita-Takesaki theory. It is represented as the crossed product of a type II_∞ factor and $\mathbb{R}$.

- Type $\mathrm{III}_{0<\lambda<1}$ factor: unique $N \rtimes_\alpha \mathbb{Z}$, N II_∞ factor,
- Type III_1 factor: unique $N \rtimes_\alpha \mathbb{R}$, N II_∞ factor,
- Type III_0 factor: one-to-one correpondence with nontransitive ergodic flows.

Amenability, equivalently hyperfiniteness is a very nice condition in von Neumann algebra theory. Group-measure space construction can construct them. There are unique hyperfinite type II_1 and II_∞ factors, and their property is well-known. Fundamental groups of type II factors, discrete group theory, Kazhdan's property (T) are used.

Tensor product factors such as Araki-Woods factors and Powers factors.
cyclic group actions implies the classification of injective factors.

- cyclic groups: Connes (II, III< 1), Haagerup (III_1),
- finite groups: Jones (II_1)
- discrete amenable groups: Ocneanu (II_1),
- property T:

- one-parameter:
- compact abelian: Takesaki duality?

Type I: Every automorphism of type I factor is inner. Cocycle conjugacy classes of actions of Γ on the injective type I factor $B(\ell^2)$ is correponded to $H^2(\Gamma, \mathbb{T})$.

approximately inner automorphisms centrally trivial automorphisms pointwise inner automorphisms

minimal action

Part II

Constructions

Chapter 4

Group actions

4.1 Crossed products

Fixed point algebra: it is equivalent to considering orbits of group action

(Pettis integral and one-parameter case is dealt with in functional analysis, on general dual pairs)

group algebras

Dual weights

4.1 (Convolution algebra of action)**.** Let (M,G,α) be a W*-dynamical system. Let $C_c(G,M)$ be the $*$-algebra of σ-strongly* continuous functions such that

$$f*g(s):=\int_G \alpha_t(f(st))g(t^{-1})\,dt, \qquad f^\sharp(s):=\Delta(s)^{-1}\alpha_{s^{-1}}(f(s^{-1})^*), \qquad f,g\in C_c(G,M).$$

Given a faithful semi-finite normal weight φ on M, we can define an inner product on $C_c(G,M)$, which gives rise to a left Hilbert algebra. The associated left von Neumann algebra is equal to the crossed product $G\ltimes_\alpha M$.

4.2 (Crossed products)**.** Let (M,G,α) be a W*-dynamical system. For a faithful unital normal representation $\pi:M\to B(H)$, we have an covariant representation

$$(M,\alpha)\xrightarrow{\pi_\alpha}(B(L^2(G)\otimes H),\operatorname{Ad}\lambda\otimes\mathrm{id}), \qquad (\pi_\alpha(x)\xi)(s):=\pi(\alpha_s^{-1}(x))\xi(s),$$

called the *regular representation* associated to π. Do we have an embedding $C_c(G,M)\subset B(L^2(G)\otimes H)$?

Suppose π is a priori covariant in the sense that there is a unitary representation $u:G\to U(H)$ such that we have $(M,\alpha)\to(B(H),\operatorname{Ad}u)$. If we introduce $(w\xi)(s):=u_s\xi(s)$, then we have

$$\begin{tikzcd}
(M,G,\alpha) \arrow[r, "\pi_\alpha"] \arrow[d, equal] & (B(L^2(G)\otimes H),G,\operatorname{Ad}\lambda\otimes\mathrm{id}) \arrow[d, "\operatorname{Ad}w"] & \lambda_s\otimes 1,\quad \pi_\alpha(x) \arrow[d, mapsto, "\operatorname{Ad}w"] \\
(M,G,\alpha) \arrow[r, "\pi\otimes 1"] & (B(L^2(G)\otimes H),G,\operatorname{Ad}\lambda\otimes\operatorname{Ad}u) & \lambda_s\otimes u_s,\quad 1\otimes x.
\end{tikzcd}$$

(a) $G\ltimes_\alpha M$ generated by $\{\lambda_s\otimes 1,\pi_\alpha(x): s\in G,\ x\in M\}$.

4.3 (Takesaki duality)**.** Heisenberg-Weyl commutation relation

Proof. Note that a W*-dynamical system admits a covariant representation. In $U(L^2(\widehat{G})\otimes L^2(G)\otimes H)$, for $x \in U(M) \subset U(H)$,

$$\begin{array}{c}
1 \otimes 1 \otimes x, \quad 1 \otimes \lambda_s \otimes u_s, \quad \lambda_p \otimes \mu_p \otimes 1 \\
\Big\downarrow {\scriptstyle \mathrm{Ad}\,\mathcal{F} \otimes \mathrm{id} \otimes \mathrm{id}} \\
1 \otimes 1 \otimes x, \quad 1 \otimes \lambda_s \otimes u_s, \quad \mu_p \otimes \mu_p \otimes 1 \\
\Big\downarrow {\scriptstyle \mathrm{Ad}\,W^* \otimes \mathrm{id}} \\
1 \otimes 1 \otimes x, \quad 1 \otimes \lambda_s \otimes u_s, \quad 1 \otimes \mu_p \otimes 1 \\
\Big\downarrow {\scriptstyle \mathrm{id} \otimes \mathrm{Ad}\,w^*} \\
1 \otimes \pi_\alpha(x), \quad 1 \otimes \lambda_s \otimes 1, \quad 1 \otimes \mu_p \otimes 1.
\end{array}$$

$\widehat{G} \ltimes (G \ltimes M) \cong \mathbb{C} \mathbin{\overline{\otimes}} B(H) \mathbin{\overline{\otimes}} M$. □

4.2 Spectral analysis

4.3 Classification of group actions

cyclic, discrete, abelian, flow

kahzdahn property T, compact

Rokhlin property

(kazhdan T, some properties like pointwise inner, etc.)

Chapter 5

Chapter 6

Part III

Factors

Chapter 7

Type III factors

7.1 Connes invariants

7.2 Flow of weights

Chapter 8

Amenable factors

Injectivity and semi-discreteness are compatible with direct sum.

8.1. (a) If M is injective, then M' is injective.

(b) If M is injective and semi-finite, then M is semi-discrete.

(c) If M is injective, then M is semi-discrete.

Proof. M_{II_∞} is the union of type II_1 corners? So we may assume M is of type II_1?

Let τ be a faithful normal tracial state on M. Since M is injective, τ is amenable. $\square$

Chapter 9

Type II factors

9.1. Let M be a von Neumann algebra. Since every σ-weakly closed ideal of M admits a unit z so that we have $zM, Mz \subset I \subset zIz \subset zMz$, and it implies z is a central projection of M. A von Neumann algebra M on H is called a *factor* if $M \cap M' = \mathbb{C}\,\mathrm{id}_H$, which is equivalent to that there are only two σ-weakly closed ideals of M. In a factor, every ideal of M is σ-weakly dense in M

9.1

9.2 (Crossed products)**.** A p.m.p. action $\Gamma \curvearrowright (X, \mu)$ gives

$$\alpha : \Gamma \to \mathrm{Aut}(L^\infty(X)),$$

which has the Koopman representation

$$\sigma : \Gamma \to B(L^2(X)).$$

Then, we have a injective $*$-homomorphism

$$C_c(\Gamma, L^\infty(X)) \to B(L^2(X) \otimes \ell^2(\Gamma)) = B(\ell^2(\Gamma, L^2(X))),$$

whose element $s \mapsto x_s$ is written in

$$\sum_{s \in \Gamma,\ fin} (x_s \otimes 1)(\sigma_s \otimes \lambda_s).$$

(a) $L(\Gamma)$ is a II_1 factor if and only if Γ is a i.c.c. group.

(b) $L^\infty(X)$ is a m.a.s.a. of $L^\infty(X) \rtimes \Gamma$ if and only if the p.m.p. action $\Gamma \curvearrowright X$ is free.

(c) $L^\infty(X) \rtimes \Gamma$ is a II_1 factor if and only if the p.m.p. action $\Gamma \curvearrowright X$ is ergodic.

9.2 Ergodic theory

9.3 Rigidity theory

9.4 Free probability

9.5

Existentially closed II_1 factors

Part IV

Subfactors

Chapter 10

Standard invariant

The way how quantum systems are decomposed. And has Galois analogy.

10.1 (Jones index theorem)**.** A *subfactor* of a factor M is a factor N containing 1_M.

Tensor categories and topological invariants of 3-folds. Ergodic flows.
Ocneanu's paragroups Popa's λ-lattices Jones' planar algebras Quantum entropy