POSITIVE HAHN-BANACH SEPARATIONS IN OPERATOR ALGEBRAS

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Abstract.

1. Lemmas

Lemma 1.1. *Let A be a σ-unital C^*-algebra with a strictly positive element $h \in A^+$. For $\omega \in A^{*sa}$, we have*

$$\|h^{\frac{1}{2}}\omega h^{\frac{1}{2}}\| = \inf\{(\omega_+ + \omega_-)(h) : \omega = \omega_+ - \omega_-,\ \omega_\pm \geq 0\}.$$

Let ω_i and ω be a net and an element in A^{*sa}. If $\omega_i \to \omega$ in h and the net ω_i is bounded, then $\omega_i \to \omega$ weakly in A^{*sa}. If $\omega_i \to \omega$ weakly* in A^{*sa} with $\omega_i \leq \omega$ for all i, then $\omega_i \to \omega$ in h.*

Proof. Let $\rho(\omega)$ be the right hand-side. For $\omega \in A^{*sa}$ and for each $\varepsilon > 0$, by definition of d, we can find $\omega_+, \omega_- \in A^{*+}$ such that $\omega_+(h) + \omega_-(h) < \rho(\omega) + \varepsilon$, so the limit $\varepsilon \to 0$ on the following estimate

$$\begin{aligned}|\omega(h^{\frac{1}{2}}ah^{\frac{1}{2}})| &= |\omega_+(h^{\frac{1}{2}}ah^{\frac{1}{2}}) - \omega_-(h^{\frac{1}{2}}ah^{\frac{1}{2}})| \\ &\leq \omega_+(h^{\frac{1}{2}}ah^{\frac{1}{2}}) + \omega_-(h^{\frac{1}{2}}ah^{\frac{1}{2}}) \\ &\leq \omega_+(h) + \omega_-(h) \\ &< \rho(\omega) + \varepsilon, \qquad a \in A_1^+\end{aligned}$$

gives the inequality $\|h^{\frac{1}{2}}\omega h^{\frac{1}{2}}\| \leq \rho(\omega)$.

If $\rho(\omega) = 0$, then since h is strictly positive so that every element of A can be approximated in norm by linear spans of elements of the form $h^{\frac{1}{2}}ah^{\frac{1}{2}}$ for $a \in A$, the inequality $\omega(h^{\frac{1}{2}}ah^{\frac{1}{2}}) = 0$ for a implies $\omega = 0$. For $\omega_1, \omega_2 \in A^{*sa}$ and arbitrarily fixed $\varepsilon > 0$, we can choose $\omega_{1+}, \omega_{1-}, \omega_{2+}, \omega_{2-} \in A^{*+}$ such that

$$\omega_1 = \omega_{1+} - \omega_{1-}, \qquad \omega_2 = \omega_{2+} - \omega_{2-},$$

and

$$(\omega_{1+} + \omega_{1-})(h) < \rho(\omega_1) + \varepsilon, \qquad (\omega_{2+} + \omega_{2-})(h) < \rho(\omega_2) + \varepsilon,$$

so we have

$$\rho(\omega_1 + \omega_2) \leq ((\omega_+ + \omega_+') + (\omega_- + \omega_-'))(h) < \rho(\omega_1) + \rho(\omega_2) + 2\varepsilon,$$

and the subadditivity follows when ε tends to zero. The homogeneity clear, so ρ is a norm on A^{*sa}.

The opposite direction....

□

- definition and properties of f_ε
- weak closedness and closedness
- relation between $\{\omega' \in M_*^+ : \omega' \leq \omega\}$ and $\{h \in \pi(M)'^+ : h \leq 1\}$

2

Definition 2.1 (Hereditary subsets)**.** Let E be a partially ordered real locally convex space such that its positive cone $E^+ := \{x \in E : x \geq 0\}$ is weakly closed. We say a subset $F \in E^+$ of positive elements is *hereditary* if $0 \leq x \leq y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. We define the *positive polar* of F as the positive part of the real polar

$$F^{\circ+} := \{x^* \in (E^*)^+ : \sup_{x \in F} x^*(x) \leq 1\}.$$

An example that is a non-hereditary closed convex subset of a C*-algebra is $\mathbb{C}1$ in any unital C*-algebra. A C*-subalgebra B of a C*-algbera A is a hereditary C*-algebra if and only if the positive cone B^+ is a hereditary subset of A^+.

Theorem 2.2 (Positive Hahn-Banach separation for von Neumann algebras)**.** *Let M be a von Neumann algebra.*

(1) *If F is a hereditary σ-weakly closed convex subset of M^+, then $F = F^{\circ+\circ+}$. In particular, if $x \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega \leq 1$ on F.*

(2) *If F_* is a hereditary weakly closed convex subset of M_*^+, then $F_* = F_*^{\circ+\circ+}$. In particular, if $\omega \in M_*^+ \setminus F_*$, then there is $x \in M^+$ such that $\omega(x) > 1$ and $x \leq 1$ on F_*.*

Proof. (1) Since the positive polar is represented as the real polar

$$F^{\circ+} = F^\circ \cap M_*^+ = F^\circ \cap (-M^+)^\circ = (F \cup -M^+)^\circ = (F - M^+)^\circ,$$

the positive bipolar can be written as $F^{\circ+\circ+} = (F - M^+)^{\circ\circ+} = \overline{F - M^+}^+$ by the usual bipolar theorem. Because $F = (F - M^+)^+ \subset \overline{F - M^+}^+$, it suffices to prove the opposite inclusion $\overline{F - M^+}^+ \subset F$.

Let $x \in \overline{F - M^+}^+$. Take a net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly, and take a net $y_i \in F$ such that $x_i \leq y_i$ for each i. Suppose we may assume that the net x_i is bounded. Define strongly continuous functions $f_\varepsilon : [-(2\varepsilon)^{-1}, \infty) \to \mathbb{R} : z \mapsto z(1 + \varepsilon z)^{-1}$ parametrized by $\varepsilon > 0$. Then, for sufficiently small ε so that the bounded net x_i has the spectra in $[-(2\varepsilon)^{-1}, \infty)$, we have $f_\varepsilon(x_i) \to f_\varepsilon(x)$ σ-strongly, and hence σ-weakly. On the other hand, by the hereditarity and the σ-weak compactness of F, we may asumme that the bounded net $f_\varepsilon(y_i) \in F$ converges σ-weakly to a point of F by taking a subnet. Then, we have $f_\varepsilon(x) \in F - M^+$ by

$$0 \leq f_\varepsilon(x) = \lim_i f_\varepsilon(x_i) \leq \lim_i f_\varepsilon(y_i) \in F,$$

thus we have $x \in F$ since $f_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. What remains is to prove the existence of a bounded net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly.

Define a convex set

$$G := \{x \in \overline{F - M^+} : \exists\, x_m \in F - M^+,\ -2x \leq x_m \uparrow x\} \subset M^{sa},$$

where x_m denotes a sequence. (In fact, it has no critical issue for allowing x_m to be uncountably indexed, contrary to the part (b) as we will see below.) Since we clearly have $F - M^+ \subset G$ and every non-decreasing net with supremum is bounded and σ-strongly convergent, it suffices to show that G, or equivalently the closed unit ball G_1 of G by the Krein-Smŭlian theorem, is σ-strongly closed. Let $x_i \in G_1$ be a net such that $x_i \to x$ σ-strongly. For each i, take a sequence $x_{im} \in F - M^+$ such that $-2x_i \leq x_{im} \uparrow x_i$ as $m \to \infty$, and also take $y_{im} \in F$ such that $x_{im} \leq y_{im}$. Since $\|x_{im}\| \leq 2\|x_i\| \leq 2$ is bounded, we can choose arbitrarily small $\varepsilon > 0$ such that $\sigma(x_{im}) \subset [-(2\varepsilon)^{-1}, \infty)$ for all i and m. Then, as diagonal nets indexed by the directed set of pairs (i,m), since $f_\varepsilon(x_{im})$ converges to $f_\varepsilon(x)$ σ-strongly and $f_\varepsilon(y_{im})$ is a bounded net for each $\varepsilon > 0$ so that we may assume that it is σ-weakly covergent by taking a subnet, we have $f_\varepsilon(x) \in F - M^+$ by

$$f_\varepsilon(x) = \lim_{(i,m)} f_\varepsilon(x_{im}) \le \lim_{(i,m)} f_\varepsilon(y_{im}) \in F,$$

where the limit is in the σ-weak sense. By taking ε as any decreasingly convergent sequence to zero, we have $x \in G$, hence the closedness of G.

(2) It suffices to prove $\overline{F_* - M_*^+}^+ \subset F_*$, so we begin our proof with fixing $\omega \in \overline{F_* - M_*^+}^+$. Suppose we have a sequence $\omega_m \in F_* - M_*^+$ such that $\omega_m \uparrow \omega$. (In fact, we only need a dominated net ω_i such that $\omega_i \to \omega$ weakly) Take a sequence $\varphi_m \in F_*$ with $m \ge 0$ such that $\omega_m \le \varphi_m$. For a normal positive linear functional $\bar\omega \in M_*^+$ such that

$$\bar\omega := \omega + \omega_{0-} + \sum_m 2^{-m} \frac{\varphi_m}{1 + \|\varphi_m\|},$$

where $\omega_0 = \omega_{0+} + \omega_{0-}$ is defined by the Jordan decomposition, consider the associated cyclic representation $\pi : M \to B(H)$ with the canonical cyclic vector Ω, and the corresponding Radon-Nikodym derivatives h, h_m, and k_m in $\pi(M)'$ of ω, ω_m, and φ_m respectively. The weak convergence $\omega_m \uparrow \omega$ and the boundedness of h_m implies we have $h_m \uparrow h$ weakly in $\pi(M)'$. Thus, for sufficiently small $\varepsilon > 0$ but fixed such that $\sigma(h_m) \subset [-(2\varepsilon)^{-1}, \infty)$ for all m, we can take a σ-weakly convergent subnet $f_\varepsilon(k_i)$ of a bounded sequence $f_\varepsilon(k_m)$ so that the strong limit $f_\varepsilon(h_i) \uparrow f_\varepsilon(h)$ has weak limits

$$0 \le \omega_{f_\varepsilon(h)} = \lim_i \omega_{f_\varepsilon(h_i)} \le \lim_i \omega_{f_\varepsilon(k_i)} \in F_*,$$

where we write $\omega_y(x) := \langle y\pi(x)\Omega, \Omega\rangle$ for $x \in M$ and $y \in \pi(M)'$. Therefore, we have $\omega_{f_\varepsilon(h)} \in F_*$ by the hereditarity of F_*, and the limit $\varepsilon \to 0$ proves that $\omega = \omega_h \in F_*$ by the closedness of F_*.

Now it is enough to prove the assumption that there is always a sequence $\omega_m \in F_* - M_*^+$ such that $\omega_m \uparrow \omega$ for every $\omega \in \overline{F_* - M_*^+}^+$. Define a convex subset of M_*^{sa}

$$G_* := \{\omega \in \overline{F_* - M_*^+} : \exists\, \omega_m \in F_* - M_*^+,\ \omega_m \uparrow \omega\} \subset M_*^{sa},$$

where ω_m denotes a sequence. It clearly follows that $F_* - M_*^+ \subset G_*$ by letting ω_m be a constant sequence, so we claim G_* is norm closed. Suppose $\omega_n \in G_*$ is a sequence such that $\omega_n \to \omega$ in norm. By modifying ω_n into $\omega_n - (\omega_n - \omega)_+ \in G_*$ and taking a rapidly convergent subsequence, we may assume $\omega_n \le \omega$ and $\|\omega - \omega_n\| \le 2^{-n}$ for all n. For each n, take a sequence $\omega_{nm} \in F_* - M_*^+$ indexed by m such that $\omega_{nm} \uparrow \omega_n$ as $m \to \infty$, and take $\varphi_{nm} \in F_*$ such that $\omega_{nm} \le \varphi_{nm}$. Define a normal positive linear functional $\bar\omega \in M_*^+$ such that

$$\bar\omega := \omega + \sum_n (\omega - \omega_n) + \sum_n 2^{-n} \frac{\omega_{n0-}}{1 + \|\omega_{n0-}\|} + \sum_{n,m} 2^{-n-m} \frac{\varphi_{nm}}{1 + \|\varphi_{nm}\|},$$

and let $\pi : M \to B(H)$ be the associated cyclic representation to $\bar\omega$. Observe that $-\sum_n(\omega - \omega_n) \le \omega_n \le \omega$ implies $|\omega_n| \le \bar\omega$. Consider the commutant Radon-Nikodym derivatives h, h_n, h_{nm}, and k_{nm} in $\pi(M)'$ of ω, ω_n, ω_{nm}, and φ_{nm}, respectively. Since $\omega_n \to \omega$ as $n \to \infty$ and $\omega_{nm} \uparrow \omega_n$ as $m \to \infty$ weakly in M_*, we have the weak convergence $h_n \to h$ and $h_{nm} \to h_n$ by the boundedness of $-1 \le h_n \le h$ and $-2^n \le h_{nm} \le h_n$. Note that the existence of a vector Ω separating the commutant implies that $\pi(M)'$ is σ-finite so that the strong topology on the bounded part can be metrized by a metric d. Applying the Mazur lemma, we can enhance the convergence so that $h_n \to h$ and $h_{nm} \to h_n$ in the strong topology by considering convex combinations, which can be taken as sequential by the metrizability of the strong topology. We may also suppose $d(h_{nm}, h_n) < m^{-1}$

by taking more rapidly convergent subsequences for each n so that we have the strong convergence $h_{nn} \to h$ of the diagonal sequence. By the uniform boundedness principle, h_{nn} is norm bounded.

For m fixed sufficiently large such that the spectra $\sigma(h_{nn})$ are contained in $[-m/2, \infty)$, the bounded sequence $f_{m^{-1}}(k_{nn})$ has a σ-weakly convergence subnet $f_{m^{-1}}(k_i)$, hence the weak limits

$$\omega_{f_{m^{-1}}(h)} = \lim_i \omega_{f_{m^{-1}}(h_i)} \leq \lim_i \omega_{f_{m^{-1}}(k_i)} \in F_*.$$

If we define $\omega_m := \omega_{f_{m^{-1}}(h)} \in F_* - M_*^+$, then $\omega_m \uparrow \omega_h = \omega$ weakly as $m \to \infty$, therefore we obtain $\omega \in G_*$. Finally we get $G_* = \overline{F_* - M_*^+}$ by the closedness of G_*, and this completes the proof. □

Theorem 2.3 (Positive Hahn-Banach separation for C*-algebras)**.** *Let A be a C*-algebra.*

(1) *If F is a hereditary weakly closed convex subset of A^+, then $F = F^{\circ+\circ+}$. In particular, if $a \in A^+ \setminus F$, then there is $\omega \in (A^*)^+$ such that $\omega(a) > 1$ and $\omega \leq 1$ on F.*

(2) *If F^* is a hereditary weakly* closed convex subset of $(A^*)^+$, then $F^* = (F^*)^{\circ+\circ+}$. In particular, if $\omega \in (A^*)^+ \setminus F^*$, then there is $a \in A^+$ such that $\omega(a) > 1$ and $a \leq 1$ on F^*.*

Proof. (1) We directly prove the separation result without laying over the arguments of positive bipolars. Let $a \in A^+ \setminus F$. Let F^{**} be the σ-weak closure of F in the universal von Neumann algebra A^{**}. We claim that F^{**} is hereditary subset of $(A^{**})^+$, Suppose $0 \leq x \leq y$ in A^{**} and $y \in F^{**}$. Then, there is $v \in A^{**}$ such that $x^{\frac{1}{2}} = vy^{\frac{1}{2}}$. Take bounded nets u_i in A and b_i in F such that $u_i \to v$ and $b_i \to y$ σ-strongly* in A^{**} using the Kaplansky density. We may assume the indices of these two nets are same. Since both the multiplication and the involution of a von Neumann algebra on bounded parts is continuous in the σ-strong* topology, and since the square root on a positive bounded interval is a strongly continuous function, we have

$$x = y^{\frac{1}{2}} v^* v y^{\frac{1}{2}} = \lim_i b_i^{\frac{1}{2}} u_i^* u_i b_i^{\frac{1}{2}},$$

so $x \in F^{**}$ because $b_i^{\frac{1}{2}} u_i^* u_i b_i^{\frac{1}{2}} \in F$. Thus, F^{**} is hereditary in $(A^{**})^+$.

Observe that we have $a \in (A^{**})^+ \setminus F^{**}$ because if $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, meaning that $a_i \to a$ weakly in A and $a \in F$ by the weak closedness of F in A. By Theorem ? (1), the positive Hahn-Banach separation for von Neumann algebras, there is $\omega \in (A^*)^+$ such that $\omega(a) > 1$ and $\omega \leq 1$ on F^{**}, so the inclusion $F \subset F^{**}$ leads the proof.

(2) As same as above, our goal is to prove $\overline{F^* - A^{*+}}^+ \subset F^*$, so take $\omega \in \overline{F^* - A^{*+}}^+$. Suppose ω can be approximated by a dominated net ω_i in $F^* - A^{*+}$ such that $\omega_i \to \omega$ weakly*, and take $\varphi_i \in F^*$ satisfying $\omega_i \leq \varphi_i$ for all i. Consider the Gelfand-Naimark-Segal representation $\pi : A \to B(H)$ corresponding to the dominating positive linear functional, with the canonical cyclic vector $\Omega \in H$. Then, associated to ω, ω_i, and φ_i, we can construct the commutant Radon-Nikodym derivatives h, h_i contained in $\pi(A)'$ and k_i the self-adjoint operators affiliated with $\pi(A)'$ obtained by the Friedrichs extension, respectively. We have

$$\omega(a^*a) = \langle h\pi(a)\Omega, \pi(a)\Omega\rangle, \quad \omega_i(a^*a) = \langle h_i\pi(a)\Omega, \pi(a)\Omega\rangle,$$

$$\varphi_i(a^*a) = \langle k_i\pi(a)\Omega, \pi(a)\Omega\rangle$$

for all $a \in A$. Since h_i is a bounded, $h_i \to h$ weakly in $B(H)$. Apply the Mazur theorem to assume $h_i \to h$ strongly in $B(H)$. We can take f_ε.

Now what remains is to prove the weak* closedness of

$$G^* := \{\omega \in \overline{F^* - A^{*+}} : \exists\, \omega_i \in F^* - A^{*+},\ \omega_i \uparrow \omega\} \subset A^{*sa}.$$

Suppose first A is σ-unital, and let h be a strictly positive element of A, with the metric d constructed in Lemma ?. In the spirit of the Krein-Šmulian theorem, let ω_i be a net in the closed unit ball G_1^* of G^* such that $\omega_i \to \omega$ weakly* in A^{*sa}.

We cannot modify ω_i to $\omega_i - (\omega_i - \omega)_+$.....

which still belongs to G_1^* and converges to ω but we have $\omega_i \le \omega$ for all i. By Lemma ?, we have $\omega_i \to \omega$ in d, so we can take a subsequence ω_n of ω_i such that $\omega_n \to \omega$ in d. For each n, since any weakly* convergent increasing net is convergent in d by Lemma ? and may be assumed to be bounded, we can find a sequence, not a general possibly uncountable net, ω_{nm} in $F^* - A^{*+}$ such that $\omega_{nm} \uparrow \omega_n$ as $m \to \infty$. Take φ_{nm} in F^* such that $\omega_{nm} \le \varphi_{nm}$ for each n and m.

Now we construct an appropriate representation to write these functionals in terms of commutant Radon-Nikodym derivatives. Take a further subsequence to assume $d(\omega_n, \omega) < 2^{-n}$ for all n. Since we still have $\omega_n \le \omega$, the partial sums in the series $\sum(\omega - \omega_n)$ define an increasing Cauchy sequence in d, so that $\psi := \sum_n(\omega - \omega_n)$ is a densely defined lower semi-continuous weight on A with domain containing a dense subalgebra $h^{\frac12}Ah^{\frac12}$ of A. Consider the Gelfand-Naimark-Segal representation $\pi : A \to B(H)$ corresponding to the densely defined lower semi-continuous weight $\omega + \psi$, together with a densely defined left A-linear map $\Lambda : \operatorname{dom}\Lambda \subset A \to H$ of dense range such that $(\omega+\psi)(a^*a) = \|\Lambda(a)\|^2$ for all a such that a^*a belongs to the domain of ψ. Associated to ω, ω_n, ω_{nm}, and φ_{nm}, the commutant Radon-Nikodym derivatives h, h_n, h_{nm}, and k_{nm} are defined.

Note that $-1 \le h_n \le h$ is a bounded sequence, and h_{nm} are bounded increasing sequences for each m, and k_{nm} are self-adjoint operators with $h_{nm} \le k_{nm}$ for every n and m. The boundedness implies that $h_n \to h$ as $n \to \infty$ and $h_{nm} \uparrow h_n$ as $m \to \infty$ for each n in the weak operator topology, We cannot extract the diagonal sequence because the strong operator topology is not metrizable.....

Now we consider a general C*-algebra A. Note that G^* is directed complete. If we consider the standard approximate unit e_i of A, then $\overline{e_iAe_i}$ defines an increasing family of σ-unital hereditary C*-subalgebras of A. Fix i and let $B := \overline{e_iAe_i}$. It suffices to extend blabla.... We will use the symbol i for other usage.

Let ω be a limit point of G^*.

$$G_B^* := \{\omega_B \in B^* :\}$$

Then, $\omega|_B \in G_B^*$.

...

If A is commutative...

Let

$$G^* := \{\omega : \exists\, \omega_i \in F^* - A^{*+} \text{ s.t. } \omega_i \uparrow \omega\}.$$

Let $\omega \in \overline{G_1^*}$.

Let ω_i be a net in G_1^* such that $\omega_i \to \omega$ and $\|\omega_i\| \le 1$.

Let ω_{ij} be a net in $F^* - A^{*+}$ such that $\|\omega_{ij}\| \lesssim_i 1$.

For example, if $\omega_{ij} \to \omega_i$ in d, then we have $\|\omega_{ij+}\| \le 1 + \varepsilon$ since $\omega_{ij+}(h) \approx \|h^{\frac12}\omega_{ij}h^{\frac12}\| \approx \|h^{\frac12}\omega_i h^{\frac12}\| \le \|h\|$.

Take a convergent subnet such that $\omega_{ij+} \to \omega_i'$.

Then, $\omega_{ij} \le \omega_{ij+}$ implies $\omega_i \le \omega_i'$.

Since A is commutative,

$$\omega_{ij} \in F^* - A^{*+} \;\Rightarrow\; \omega_{ij+} \in F^* \;\Rightarrow\; \omega_i' \in F^* \;\Rightarrow\; \omega_i \in F^* - A^{*+}.$$

Take a convergent subnet such that $\omega_{i+} \to \omega'$.

Then, $\omega_i \le \omega_{i+}$ implies $\omega \le \omega'$.

Since A is commutative,

$$\omega_i \in F^* - A^{*+} \;\Rightarrow\; \omega_{i+} \in F^* \;\Rightarrow\; \omega' \in F^* \;\Rightarrow\; \omega \in F^* - A^{*+}.$$

$\square$

Corollary 2.4. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{subadditive normal} \\ \text{weights of } M \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{hereditary closed} \\ \text{convex subsets of } M_*^+ \end{array} \right\} \\ \varphi & \mapsto & \{\omega \in M_*^+ : \omega \leq \varphi\} \end{array}$$