1 April 14

1.1 Completely positive maps

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* (c.p.) if the inflation $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B}) : [a_{ij}] \mapsto [\varphi(a_{ij})]$ is positive for each $n \geq 1$.

Remark 1.2. For the positivity in matrix algebras, the following equivalent statements are useful.

(a) $[a_{ij}] \in M_n(\mathcal{A})$ is positive.

(b) $[a_{ij}] = [b_{ij}]^*[b_{ij}] = [b_{ji}^*][b_{ij}] = [\sum_k b_{ki}^* b_{kj}]$ for some $[b_{ij}] \in M_n(\mathcal{A})$.

(c) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for a faithful representation $\pi : \mathcal{A} \to B(H)$.

(d) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for every representation $\pi : \mathcal{A} \to B(H)$.

Example 1.3.

(a) A $*$-homomorphism is c.p.

(b) A state is c.p.

(c) A conjugation $B(\widehat{H}) \to B(H) : a \mapsto V^*aV$ is c.p. for every bounded linear $V : H \to \widehat{H}$.

(d) The transpose $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is not c.p.

(e) The convex combination, composition, restriction of c.p. maps is c.p.

Proof. (a) A $*$-homomorphism is positive, and its inflations are all $*$-homomorphisms.

(b) Let $\rho : \mathcal{A} \to \mathbb{C}$ be a state. If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(\mathcal{A})_+$, then we have for $[x_i] \in \ell_2^n$ that

$$\sum_{i,j} \langle \rho(a_{ij})x_j, x_i \rangle_{\mathbb{C}} = \sum_{i,j} \overline{x_i}\rho(a_{ij})x_j = \rho(\sum_{i,j,k} \overline{x_i} b_{ki}^* b_{kj} x_j) = \sum_k \rho((\sum_i b_{ki}x_i)^*(\sum_j b_{kj}x_j)) \geq 0.$$

(c) If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(\widehat{H}))_+$, then we have for $[\xi_i] \in H^n$ that

$$\sum_{i,j} \langle V^* a_{ij} V\xi_j, \xi_i \rangle = \sum_{i,j,k} \langle b_{kj}V\xi_j, b_{ki}V\xi_i \rangle = \sum_k \langle \sum_j b_{kj}V\xi_j, \sum_i b_{ki}V\xi_i \rangle \geq 0.$$

(d) We have a counterexample for $M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The former has an eigenvalues $\{2, 0\}$, and the latter has $\{\pm 1\}$.

(e) Clear. $\square$

Theorem 1.4 (Stinespring dilation)**.** *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \to B(H)$ be a c.p. map. Then, there is a representation $\pi : \mathcal{A} \to B(\widehat{H})$ and a bounded linear operator $V : H \to \widehat{H}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & B(H) \\ B(\widehat{H}) \arrow[ur, "V^* \cdot V"'] & \end{tikzcd}$$

Proof. Define a sesquilinear form on the algebraic tensor product $\mathcal{A} \odot H$ as

$$\langle \sum_j a_j \otimes \xi_j, \sum_i b_i \otimes \eta_i \rangle := \sum_{i,j} \langle \varphi(b_i^* a_j)\xi_j, \eta_i \rangle.$$

It is positive since

$$\sum_{i,j} \langle a_i^* a_j \xi_j, \xi_i \rangle = \sum_{i,j} \langle a_j \xi_j, a_i \xi_i \rangle = \| \sum_i a_i \xi_i \|^2 \geq 0$$

implies

$$\langle \sum_j a_j \otimes \xi_j, \sum_i a_i \otimes \xi_i \rangle = \sum_{i,j} \langle \varphi(a_i^* a_j)\xi_j, \xi_i \rangle \geq 0.$$

Taking quotient by the left kernel N and completion, we obtain a hilbert space $\widehat{H} := (\mathcal{A} \odot H/N)^-$.

Define $\pi : \mathcal{A} \to B(\widehat{H})$ such that

$$\pi(a)(b \otimes \xi + N) := ab \otimes +N,$$

and define $V : H \to \widehat{H}$ such that

$$V\xi := 1_{\mathcal{A}} \otimes \xi + N.$$

Then for any $\xi, \eta \in H$,

$$\langle V^* \pi(a) V \xi, \eta \rangle = \langle \pi(a)(1_{\mathcal{A}} \otimes \xi + N), 1_{\mathcal{A}} \otimes \xi + N \rangle = \langle a_{\mathcal{A}} \otimes \xi + N, 1_{\mathcal{A}} \otimes \xi + N \rangle = \langle \varphi(a)\xi, \eta \rangle. \qquad \square$$

Remark 1.5.

(a) If φ is unital, then V is an isometry since $V^*V = V^*\pi(1)V = \varphi(1) = 1$.

(b) If φ is unital and $H = \mathbb{C}$, then it is just the GNS-construction with the cyclic vector $V1_{\mathbb{C}}$.

(c) If $\varphi : \mathcal{A} \to \mathcal{B}$ is c.p., then by embedding $\mathcal{B}$ into $B(H)$ and applying the Stinespring dilation,

$$\|\varphi(a)\| = \|V^*\pi(a)V\| \leq \|V\|\|a\|\|V\| = \|a\|\|V^*V\| = \|a\|\|\varphi(1)\|$$

implies $\|\varphi\| \leq \|\varphi(1)\|$, hence $\|\varphi\| = \|\varphi(1)\|$.

(d) It has a physical meaning: a unital completely positive map is called quantum channel or quantum operation in quantum information theory. They are interpreted as an evolution in open quantum system, and taking $\widehat{H}$ means introducing a closed ambient system in which unitary evolution occurs.

Theorem 1.6 (Completely positive maps for matrix algebras). *Let $\mathcal{A}$ be a C^*-algebra. Let $e_i \in \ell_2^n$ be standard orthonormal basis and let $e_{ij} = e_i \otimes e_j = |e_i\rangle\langle e_j| \in M_n(\mathbb{C})$ be unit matrix elements.*

(a) *There is a 1-1 correspondence*

$$\mathrm{CP}(M_n(\mathbb{C}), \mathcal{A}) \to M_n(\mathcal{A})_+ : \psi \mapsto [\psi(e_{ij})].$$

(b) *Let $\mathcal{A}$ be unital. There is a 1-1 correspondence*

$$\mathrm{CP}(\mathcal{A}, M_n(\mathbb{C})) \to M_n(\mathcal{A})_+^* : \varphi \mapsto (\widehat{\varphi} : [a_{ij}] \mapsto \sum_{i,j} \langle \varphi(a_{ij})e_j, e_i \rangle).$$

Proof. (a) Fix $\mathcal{A} \to B(H)$ a faithful representation and just write $\mathcal{A} \subset B(H)$.

Suppose $\psi : M_n(\mathbb{C}) \to \mathcal{A}$ is a c.p. map. Identify $M_n(\mathbb{C}) = B(\ell_2^n)$. Since $[e_{ij}] \in M_n(B(\ell_2^n))_+$ is positive because

$$\sum_{i,j} \langle e_{ij}\xi_j, \xi_i \rangle = \sum_{i,j} \langle e_j, \xi_j \rangle \langle \xi_i, e_i \rangle = |\sum_i \langle e_i, \xi_i \rangle|^2 \geq 0, \qquad \forall [\xi_i] \in (\ell_2^n)^n,$$

it follows that $[\psi(e_{ij})] \in M_n(\mathcal{A})_+$ by the complete positivity of ψ.

Conversely, let $[\psi(e_{ij})] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(H))_+$, For $T = [t_{ij}] \in M_n(\mathbb{C})$ and $\xi, \eta \in H$, write

$$\begin{aligned}\langle \psi(T)\xi, \eta\rangle &= t_{ij}\langle \psi(e_{ij})\xi, \eta\rangle \\ &= t_{ij}\langle b_{kj}\xi, b_{ki}\eta\rangle \\ &= t_{ij}\delta_{kl}\langle b_{lj}\xi, b_{ki}\eta\rangle \\ &= \langle Te_j, e_i\rangle\langle e_l, e_k\rangle\langle b_{lj}\xi, b_{ki}\eta\rangle \\ &= \langle (T \otimes 1 \otimes 1)(e_j \otimes e_l \otimes (b_{lj}\xi)), (e_i \otimes e_k \otimes (b_{ki}\eta))\rangle.\end{aligned}$$

The summation symols are omitted in each row. Then, if we define

$$V : H \to \ell_2^n \otimes \ell_2^n \otimes H : \xi \mapsto \sum_{i,k} e_i \otimes e_k \otimes (b_{ki}\eta),$$

we have an expression

$$\langle \psi(T)\xi, \eta\rangle = \langle V^*(T \otimes 1 \otimes 1)V\xi, \eta\rangle,$$

which implies that ψ is c.p. because $T \mapsto T \otimes 1_{\ell_2^n} \otimes 1_H$ is a $*$-homomorphism.

(b) Suppose $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ is a c.p. map. Then, $\widehat{\varphi}$ is positive since $[a_{ij}] \in M_n(\mathcal{A})_+$ implies

$$\widehat{\varphi}([a_{ij}]) = \sum_{i,j}\langle \varphi(a_{ij})e_j, e_i\rangle \geq 0.$$

Conversely, let $\widehat{\varphi} \in M_n(\mathcal{A})_+^*$. By the GNS-construction, we have a cyclic representation $\pi : M_n(\mathcal{A}) \to B(H)$ with a cyclic vector $\psi \in H$ such that

$$\widehat{\varphi}([a_{ij}]) = \langle \pi([a_{ij}])\psi, \psi\rangle.$$

For $\xi = \sum_j \xi_j e_j, \eta = \sum_i \eta_i e_i \in \ell_2^n$, write

$$\begin{aligned}\langle \varphi(a)\xi, \eta\rangle &= \sum_{i,j}\langle \varphi(a)\xi_j e_j, \eta_i e_i\rangle = \sum_{i,j}\langle \varphi(\overline{\eta_i} a \xi_j)e_j, e_i\rangle \\ &= \widehat{\varphi}([\overline{\eta_i} a \xi_j]) = \langle \pi([\overline{\eta_i} a \xi_j])\psi, \psi\rangle = \langle \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}]^*[a][\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \psi\rangle \\ &= \langle \pi([a])\pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}])\psi\rangle.\end{aligned}$$

If we define

$$V : \ell_2^n \to H : \xi \mapsto \pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi,$$

then

$$\langle \varphi(a)\xi, \eta\rangle = \langle V^*\pi([a])V\xi, \eta\rangle,$$

so φ is c.p. since $\mathcal{A} \to M_n(\mathcal{A}) : a \mapsto [a]$ is a $*$-homomorphism. □

Theorem 1.7 (Arveson extension). *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras such that $1_{\mathcal{A}} \in \mathcal{B}$. Then, every c.p. map $\varphi : \mathcal{B} \to B(H)$ has an norm-preserving c.p. extension $\widetilde{\varphi} : \mathcal{A} \to B(H)$, i.e. $\|\widetilde{\varphi}\| = \|\varphi\|$.*

1.2 Enveloping von Neumann algebras

Theorem 1.8 (Sherman-Takeda). *Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to B(H)$ a faithful representation. Here we can obtain an linear map $\widetilde{\pi} : \mathcal{A}^{**} \to \pi(\mathcal{A})''$ by taking bitranspose for $\pi : \mathcal{A} \to (\pi(\mathcal{A})'', \sigma w)$.*

(a) *$\widetilde{\pi}$ is an isometric isomorhpism (w.r.t. norms), and is an homeomorphism (w.r.t. weak*-topologies)*

(b) *$\mathcal{A}^{**}$ enjoys a universal property in the sense that for every $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{M}$ to a von Neumann algebra $\mathcal{M}$, there exists a unique σ-weakly continuous extension $\widetilde{\varphi} : \mathcal{A}^{**} \to \mathcal{M}$ of φ.*

We will always see the bidual $\mathcal{A}^{**}$ as a von Neumann algebra.

Proof. (a) Consider

$$\pi : \mathcal{A} \to (\pi(\mathcal{A})'', \sigma w), \qquad \pi^* : \pi(\mathcal{A})''_* \to \mathcal{A}^*, \qquad \widetilde{\pi} := \pi^{**} : \mathcal{A}^{**} \to \pi(\mathcal{A})'',$$

where $\pi(\mathcal{A})''_*$ denotes the set of σ-weakly continuous(=normal) linear functionals on $\pi(\mathcal{A})''$. Note that π is isometric and has dense range. It implies that π^* is surjective and injective. In fact, π^* is isometric because for $l \in \pi(\mathcal{A})''_*$ we have by the density that

$$\|\pi^*(l)\| = \sup_{\substack{\|a\|=1 \\ a\in\mathcal{A}}} |l(\pi(a))| = \sup_{\substack{\|b\|=1 \\ b\in\pi(\mathcal{A})''}} |l(b)| = \|l\|.$$

Then, the claim for π^{**} is now clear.

(b) We can define $\widetilde{\varphi}$ as the bitranspose of $\varphi : \mathcal{A} \to (\mathcal{M}, \sigma w)$ as in the part (a), and it is a unique extension because $\mathcal{A}$ is σ-weakly dense in $\mathcal{A}^{**}$. □

Theorem 1.9 (Tomiyama)**.** *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a* conditional expectation*, i.e. a contractive idempotent linear map.*

(a) *φ is B-bimodule map.*

(b) *φ is completely positive.*

Proof. Since each conclusion of (a) and (b) still holds for restriction, we may assume $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras by thinking of the bitranspose $\varphi^{**} : \mathcal{A}^{**} \to \mathcal{B}^{**}$.

(a) Since the linear span of projections is σ-weakly dense in a von Neumann algebra, we are enough to show $p\varphi(a) = \varphi(pa)$ and $\varphi(ap) = \varphi(a)p$ for any projection $p \in \mathcal{B}$.

Let $p \in \mathcal{B}$ be a projection and let $a \in \mathcal{A}$. Note that we have

$$p\varphi(a) = pp\varphi(a) = p\varphi(p\varphi(a))$$

and

$$(a - pa)^*(p\varphi(a - pa)) = (p\varphi(a - pa))^*(a - pa) = 0.$$

Then,

$$\begin{aligned}
(1+t)^2\|p\varphi(a-pa)\|^2 &= \|p\varphi(a-pa) + tp\varphi(a-pa)\|^2 \\
&= \|p\varphi((a-pa) + tp\varphi(a-pa))\|^2 \\
&\le \|(a-pa) + tp\varphi(a-pa)\|^2 \\
&= \|a-pa\|^2 + t^2\|p\varphi(a-pa)\|^2
\end{aligned}$$

implies $p\varphi(a-pa) = 0$ by letting $t \to \infty$. Putting $1_{\mathcal{B}} - p$ and $1_{\mathcal{B}}$ instead of p, we obtain $(1_{\mathcal{B}} - p)\varphi(a - 1_{\mathcal{B}}a + pa) = 0$ and $\varphi(a - 1_{\mathcal{B}}a) = 0$, so

$$p\varphi(a) = p\varphi(pa) = \varphi(pa).$$

Similarly, we can show $\varphi(a - ap)p = 0$ and $\varphi(ap)(1 - p) = 0$, we are done.

(b) Let $[a_{ij}] \in M_n(\mathcal{A})_+$. Let $\pi : \mathcal{B} \to B(H)$ be a cyclic representation with a cyclic vector ψ. Then, $[\xi_i] \in H^n$ can be replaced to $[\pi(b_i)\psi]$, so we can check the positivity of inflations φ_n as

$$\sum_{i,j}\langle \pi(\varphi(a_{ij}))\pi(b_j)\psi, \pi(b_i)\psi\rangle = \langle \pi(\varphi(\sum_{i,j} b_i^* a_{ij} b_j))\psi, \psi\rangle \ge 0,$$

because it follows $\sum_{i,j} b_i^* a_{ij} b_j \ge 0$ by the positivity of a_{ij} from

$$\langle \pi_{\mathcal{A}}(\sum_{i,j} b_i^* a_{ij} b_j)\xi, \xi\rangle = \sum_{i,j}\langle \pi_{\mathcal{A}}(a_{ij})\pi_{\mathcal{A}}(b_j)\xi, \pi_{\mathcal{A}}(b_i)\xi\rangle \ge 0,$$

where $\pi_{\mathcal{A}}$ is any representation of $\mathcal{A}$. □

Theorem 1.10 (Sakai)**.** *Suppose $\mathcal{A}$ is a C^*-algebra which admits a predual F.*

(a) *There is an injective $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{A}^{**}$ with weakly* closed image.*

(b) *π is a topological embedding w.r.t. $\sigma(\mathcal{A}, F)$ and $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$.*

(c) *The predual F is unique in $\mathcal{A}^*$.*

(a) In particular, there is a faithful representation $\mathcal{A} \to B(H)$ whose image is (σ-)weakly closed.

Proof. By taking the adjoint for the embedding $F \hookrightarrow i\mathcal{A}^*$, we have a conditional expectation $\varepsilon : \mathcal{A}^{**} \twoheadrightarrow \mathcal{A}$. Its kernel is a $\mathcal{A}$-bimodule, and by the σ-weak density of $\mathcal{A}$ in $\mathcal{A}^{**}$ and the continuity of ε between weak* topologies, so is a $\mathcal{A}^{**}$-bimodule, which means it is a σ-weakly closed ideal of $\mathcal{A}^{**}$. Thus we have a central projection $z \in \mathcal{A}^{**}$ such that $\ker \varepsilon = (1-z)\mathcal{A}^{**}$.

Define $\pi : \mathcal{A} \to \mathcal{A}^{**}$ such that $\pi(a) := za$. It is clearly a $*$-homomorphism. The injectivity follows from $a = \varepsilon(a) = \varepsilon(za)$ for $a \in \mathcal{A}$. The image is weakly* closed because $\varepsilon(x - \varepsilon(x)) = 0$ implies $z(x - \varepsilon(x)) = 0$ for $x \in \mathcal{A}^{**}$ so that $z\mathcal{A}^{**} = z\mathcal{A}$.

(b) Since $\langle a, f \rangle = \langle \varepsilon(za), f \rangle = \langle za, f \rangle$ for $a \in \mathcal{A}$ and $f \in F$, in which the second equality holds by the definition of ε, it is enough to show $\sigma(z\mathcal{A}, \mathcal{A}^*) = \sigma(z\mathcal{A}, F)$.

For $l \in \mathcal{A}^*$, we claim there exists f such that $\langle za, l \rangle = \langle za, f \rangle$. Define $\widetilde{l} \in \mathcal{A}^*$ such that $\langle x, \widetilde{l} \rangle := \langle zx, l \rangle$ for $x \in \mathcal{A}^{**}$. Then, $\langle zx, l \rangle = \langle z^2 x, l \rangle = \langle zx, \widetilde{l} \rangle$ for $x \in \mathcal{A}^{**}$. Suppose $\widetilde{l} \notin F$. Because F is closed in $\mathcal{A}^*$, there is $x \in \mathcal{A}^{**}$ such that $\langle x, \widetilde{l} \rangle \neq 0$ and $\langle x, f \rangle = 0$ for all $f \in F$ by the Hahn-Banach extension. Then, $0 = \langle x, f \rangle = \langle x, i(f) \rangle = \langle \varepsilon(x), f \rangle$ implies $\varepsilon(x) = 0$ so that $zx = 0$, which leads a contradiction $\langle x, \widetilde{l} \rangle = \langle zx, l \rangle = 0$, so we have $\widetilde{l} \in F$.

(c) If closed subspaces F_1 and F_2 of $\mathcal{A}^*$ are preduals of $\mathcal{A}$, then $\sigma(\mathcal{A}, F_1) = \sigma(\mathcal{A}, F_2)$ by the part (b). If $l \in F_1$, which is obviously continuous on $\sigma(\mathcal{A}, F_1)$, and the continuity in $\sigma(\mathcal{A}, F_2)$ implies that l is contained in a linear span of some finitely many elements of F_2, hence $F_1 \subset F_2$. □

2 May 12

2.1 Nuclear maps

Definition 2.1. A linear map $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras is called *nuclear* if it is a limit of finite-rank c.c.p. maps in the point-norm topology. Equivalently, by the following lemma, there is a net of pairs of c.c.p. maps $\varphi_\alpha : \mathcal{A} \to M_{n_\alpha}(\mathbb{C})$ and $\psi_\alpha : M_{n_\alpha}(\mathbb{C}) \to \mathcal{B}$ such that $\|\theta(a) - \psi_\alpha \circ \varphi_\alpha(a)\| \to 0$ for each $a \in \mathcal{A}$.

If $\mathcal{B}$ is a con Neumann algebra, θ is called *weakly nuclear* if it is a limit of finite-rank c.c.p. maps in the point-σ-weak topology.

Lemma 2.2. *A c.c.p. map $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras is of finite-rank iff there are c.c.p. maps $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to \mathcal{B}$ for some n such that $\theta = \psi \circ \varphi$. In Brown-Ozawa, a finite-rank c.c.p. map is called a factorable map.*

Proof. ($\Leftarrow$) Clear. ($\Rightarrow$) By the structure theorem of finite-dimensional C*-algebras, we have $\operatorname{im}\theta \cong \bigoplus_{i=1}^m M_{n_i}(\mathbb{C})$, so for $n = \sum_{i=1}^m n_i$ there is a unital embedding $\operatorname{im}\theta \hookrightarrow M_n(\mathbb{C})$ and conditional expectation $M_n(\mathbb{C}) \to \operatorname{im}\theta : T \mapsto \sum_{i=1}^m P_i T P_i$, where P_i denotes the projection on the image of $M_{n_i}(\mathbb{C})$. Now we are done. (In fact, such a conditional expectation also exists for unital subalgebras beetween von Neumann algebras.) ☐

Proposition 2.3 (Local property)**.** *Let $\theta : \mathcal{A} \to \mathcal{B}$ be a linear map between C*-algebras. If the restriction of θ on any finite-dimensional subspace of $\mathcal{A}$ is nuclear, then θ is nuclear.*

Proof. ☐

Proposition 2.4 (Weak approximations)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, and $\mathcal{M} \subset B(H)$ a von Neumann algebra.*

(a) *$\theta : \mathcal{A} \to \mathcal{B}$ is nuclear if there is a net $\mathcal{A} \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{B}$ such that*

$$\lim_\alpha \langle \theta(a) - \psi_\alpha \circ \varphi_\alpha(a), l \rangle = 0 \qquad a \in \mathcal{A},\ l \in \mathcal{B}^*.$$

(b) *$\theta : \mathcal{A} \to \mathcal{M}$ is weakly nuclear if there is a net $\mathcal{A} \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{M}$ such that*

$$\lim_\alpha \langle (\theta(a) - \psi_\alpha \circ \varphi_\alpha(a))\xi, \xi \rangle = 0 \qquad a \in \mathcal{A},\ \xi \in H.$$

Proof. (a) By applying the Hahn-Banach extension for each $a \in \mathcal{A}$, we can show the closures of a convex set is same with respect to the point-norm topology and the point-$\sigma(\mathcal{B}, \mathcal{B}^*)$-topology. Thus it suffices to show that the set of finite-rank c.c.p. maps is convex.

Let $\mathcal{A} \xrightarrow{\psi_i} M_{n_i}(\mathbb{C}) \xrightarrow{\varphi_i} \mathcal{B}$ be c.c.p. maps for $i \in \{0, 1\}$. Then, we have a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "(1-t)\psi_0\circ\varphi_0 + t\psi_1\circ\varphi_1"] \arrow[d] & & \mathcal{B} \\
\mathcal{A} \oplus \mathcal{A} \arrow[r, "\varphi_0 \oplus \varphi_1"'] & M_{n_0}(\mathbb{C}) \oplus M_{n_1}(\mathbb{C}) \arrow[r, "((1-t)\psi_0)\oplus(t\psi_1)"'] & \mathcal{B} \oplus \mathcal{B} \arrow[u]
\end{tikzcd}$$

which is commutative, so we are done.

(b) Fix $a \in \mathcal{A}$. Note that are net is bounded. Since the unit ball is compact in σ-weak topology and hence in the weak operator topology, we are enough to verify the convergence of $\psi_\alpha \circ \varphi_\alpha$ in the weak operator topology. Using the polarization identity, the claim holds. ☐

nonunital technicalities

2.2 Examples of nuclear C*-algebras

C*-subalgebra of a nuclear C*-algebra may not be nuclear. C*-subalgebra of a exact C*-algebra is exact. injective limit of nuclear C*-algebras is nuclear. $M_n(\mathcal{A})$ is nuclear if $\mathcal{A}$ is nuclear.

Theorem 2.5 (Effros-Lance)**.** *If $\mathcal{A}^{**}$ is semidiscrete, then $\mathcal{A}$ is nuclear. (The converse also holds)*

Proof. Since the set of finite-rank c.c.p. maps is convex, and since the closures of a convex set are same in the norm and weak topologies on a Banach space, □

Theorem 2.6. *An abelian C*-algebra is nuclear.*

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