Classical Geometry

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February 8, 2024

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Reparametrizations

Theorem 4.3.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 4.3.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 4.3.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y|_p = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 4.3.4. *Metric coefficients*

$$\begin{aligned} \langle \alpha_x, \alpha_x \rangle &=: g_{11} \qquad & \langle \alpha_x, \alpha_y \rangle &=: g_{12} \\ \langle \alpha_y, \alpha_x \rangle &=: g_{21} \qquad & \langle \alpha_y, \alpha_y \rangle &=: g_{22} \end{aligned}$$

Theorem 4.3.5 (Normal coordinates)**.** *...?*

Differentiation of tangent vectors

Definition 4.3.6. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x,y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x,y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 4.3.7 (Gauss formula, Γ^k_{ij}, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma^k_{ij}\}^2_{i,j,k=1}$ and $\{L_{ij}\}^2_{i,j=1}$ by the Gauss formula

$$\begin{aligned} \alpha_{xx} &=: \Gamma^1_{11}\alpha_x + \Gamma^2_{11}\alpha_y + L_{11}\,\nu, \qquad & \alpha_{xy} &=: \Gamma^1_{12}\alpha_x + \Gamma^2_{12}\alpha_y + L_{12}\,\nu, \\ \alpha_{yx} &=: \Gamma^1_{21}\alpha_x + \Gamma^2_{21}\alpha_y + L_{21}\,\nu, \qquad & \alpha_{yy} &=: \Gamma^1_{22}\alpha_x + \Gamma^2_{22}\alpha_y + L_{22}\,\nu. \end{aligned}$$

The *Christoffel symbols* refer to eight functions $\{\Gamma^k_{ij}\}^2_{i,j,k=1}$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i(\partial_i Y^k)\alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma^k_{ij}\right)\alpha_k + X^i Y^j L_{ij}\,\nu. \end{aligned}$$

Differentiation of normal vector

The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu\rangle = \frac{1}{2}\partial_X \langle \nu, \nu\rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 4.3.8. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 4.3.9. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y)\rangle = \langle X, -\partial_Y \nu\rangle = \langle \partial_Y X, \nu\rangle = \langle \partial_X Y, \nu\rangle = \langle \mathcal{S}(X), Y\rangle. \qquad \square$$

Theorem 4.3.10. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i\alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j\alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}^j_i X_i$. Then,

$$g_{ik} X^i \mathcal{S}^k_j Y^j = \langle X, \mathcal{S}(Y)\rangle = \langle \partial_X Y, \nu\rangle = X^i Y^j L_{ij}$$

implies $g_{ik}\,\mathcal{S}^k_j = L_{ij}$. $\square$

Chapter 5

Fundamental forms

5.1 Riemannian metrics

5.2 Gaussian curvatures

Theorema egregium surfaces of constant gaussian curvature

Definition 5.2.1. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$E := \langle \alpha_x, \alpha_x \rangle = g_{11}, \qquad F := \langle \alpha_x, \alpha_y \rangle = g_{12}, \qquad G := \langle \alpha_y, \alpha_y \rangle = g_{22},$$
$$L := \langle \alpha_{xx}, \nu \rangle = L_{11}, \qquad M := \langle \alpha_{xy}, \nu \rangle = L_{12}, \qquad N := \langle \alpha_{yy}, \nu \rangle = L_{22}.$$

Corollary 5.2.2. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\begin{aligned}\nu_x &= \frac{FM - GL}{EG - F^2}\alpha_x + \frac{FL - EM}{EG - F^2}\alpha_y,\\ \nu_y &= \frac{FN - GM}{EG - F^2}\alpha_x + \frac{FM - EN}{EG - F^2}\alpha_y.\end{aligned}$$

Theorem 5.2.3.

$$\Gamma^l_{ij} = \frac{1}{2}g^{kl}(g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2}(\log g)_x = \Gamma^1_{11}.$$

$$\nu_x \times \nu_y = K\sqrt{\det g}\ \nu.$$

$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$

$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y \rangle = \det\begin{pmatrix} \langle \nu_x, \alpha_x \rangle & \langle \nu_x, \alpha_y \rangle \\ \langle \nu_y, \alpha_x \rangle & \langle \nu_y, \alpha_y \rangle \end{pmatrix} = \det\begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K \det g$$

5.1 (Gaussian curvature formula). (a) In general,

$$K = \frac{LN - M^2}{EG - F^2}.$$

(b) For orthogonal coordinates such that $F \equiv 0$,

$$K = -\frac{1}{2\sqrt{\det g}}\left((\frac{1}{\sqrt{\det g}}E_y)_y + (\frac{1}{\sqrt{\det g}}G_x)_x \right).$$

(c) For $f(x,y,z)=0$,

$$K = -\frac{1}{|\nabla f|^4}\begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f) \end{vmatrix},$$

where ∇f denotes the gradient $\nabla f = (f_x, f_y, f_z)$.

(d) (Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then

$$K = -\tau^2.$$

(e) (Brioschi) E, F, G describes K.

Proof. (a) Clear.

(b) We have $GM = EM$ and

$$\nu_x = -\frac{L}{E}\alpha_x - \frac{M}{G}\alpha_y, \qquad \nu_y = -\frac{M}{E}\alpha_x - \frac{N}{G}\alpha_y.$$

$$\nu_x \times \nu_y = \frac{LN - M^2}{EG}\alpha_x \times \alpha_y$$

After curvature tensors...

□

5.2 (Computation of Gaussian curvatures). (a) (Monge's patch) For $(x, y, f(x,y))$,

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(b) (Surface of revolution). Let $\gamma(t) = (r(t), z(t))$ be a plane curve with $r(t) > 0$. If $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

(c) (Models of hyperbolic planes)

Proof. (b) Let

$$\alpha(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

be a parametrization of a surface of revolution. Then,

$$\begin{aligned}
\alpha_\theta &= (-r(t)\sin\theta, r(t)\cos\theta, 0) \\
\alpha_t &= (r'(t)\cos\theta, r'(t)\sin\theta, z'(t)) \\
\nu &= \frac{1}{\sqrt{r'(t)^2 + z'(t)^2}}(z'(t)\cos\theta, z'(t)\sin\theta, -r'(t)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{\theta\theta} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0) \\
\alpha_{\theta t} &= (-r'(t)\sin\theta, -r'(t)\cos\theta, 0) \\
\alpha_{tt} &= (r''(t)\cos\theta, r''(t)\sin\theta, z''(t)).
\end{aligned}$$

Thus we have

$$E = r(t)^2, \quad F = 0, \quad G = r'(t)^2 + z'(t)^2,$$

and

$$L = -\frac{r(t)z'(t)}{\sqrt{r'(t)^2 + z'(t)^2}}, \quad M = 0, \quad N = \frac{r''(t)z'(t) - r'(t)z''(t)}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

□

5.3 (Local isomorphism)**.** Surfaces of the same constant Gaussian curvature are locally isomorphic.

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

Chapter 6

Compact smooth surfaces

Part III

Riemann surfaces

Chapter 7

Riemann-Roch theorem

Chapter 8

Algebraic curves

multiplicities, Bezout theorem divisors, line bundles Embedding theorem euler characteristic (tangent line bundle degree 2-2g, canonical line bundle 2g-2) $L(D) := H^0(X, \mathcal{O}(D))$

Jacobian variety (moduli spaces....) Chow theorem

Chapter 9

Uniformization

Part IV

Topological surfaces

Chapter 10

Fundamental groups

10.1 Homotopy

10.1. A *homotopy of paths* is a continuous map $h : I \times I \to X$ such that $h(0,) = x_0$ and

(a) linear homotopy

(b) reparametrization

10.2. The fundamental group is a group composition

10.3 (Van Kampen theorem)**.**

10.2 Covering spaces

path lifting property universal covering

Chapter 11

Homology groups

11.1 Singular homology

11.2 Simplicial homology

11.3 Cellular homology

Chapter 12

Classification of surfaces

12.1 Combinatorial surfaces

triangulation orientability euler characteristic genus connected sum