Algebraic Number Theory

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L-functions

Riemann $\zeta(s)$
Dedekind $\zeta_K(s)$
Hasse-Weil $\zeta_X(s)$

3.1 Dirichlet L-functions

3.1 (Hecke character). Dirichlet character can be understood as a group homomorphism $\chi : \widehat{\mathbb{Z}}^\times \to \mathbb{C}$ of finite order, which means that there is n such that χ factors through $(\mathbb{Z}/n\mathbb{Z})^\times$.

In order to construct an L-function from a character, we need to extend a character as a function of ideals. We interpret $(\mathbb{Z}/n\mathbb{Z})^\times$ as the ray class group modulo $\mathfrak{m}$.

To extend the order of a character to possibly infinite cases, Hecke character is defined a character of an idele class group $C_K := \mathbb{A}_K^\times / K^\times$.

Dirichlet (Hecke) L-functions for ray-class characters $\chi : C_K \to \mathbb{C}$:

$$L(\chi, s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p} \text{ prime}} \frac{1}{1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s}}$$

Artin L-functions for a Galois representation $\rho : \mathrm{Gal}(L/K) \to GL_n(\mathbb{C})$:

$$L(\rho, s) = \prod_{\mathfrak{p} \text{ prime}} \frac{1}{\det(1 - \rho(\mathrm{Frob}_{\mathfrak{p}}) N(\mathfrak{p})^{-s})}$$

Elliptic curves $L(E, s)$
Modular forms $L(f, s)$