C*-Algebras

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Part I

Constructions

Chapter 1

Completely positive maps

1.1 Operator spaces

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1.1 (Choi-Effros characterization)**.**

1.2 (Von Neumann inequality)**.**

The set $M_n(A)^+$ is linearly spanned by elements of the form $[a_i^* a_j] \in M_n(A)$ for $[a_i] \in A^n$. A linear map $\varphi : A \to B$ is completely positive if

$$\varphi(a_i^* a_j)$$

1.3 (n-positive maps)**.** Let S be an operator space. Let A and B be C*-algebras.

(a) (Cauchy-Schwarz inequality) If $\varphi : A \to B$ is a 2-positive map, then $\lim_\alpha \|\varphi(e_\alpha)\| = \|\varphi\|$ for any approximate unit (e_α) of A, and

$$\varphi(a)^*\varphi(a) \le \|\varphi\|\varphi(a^*a), \qquad a \in A.$$

(b) (Multiplicative domain) Let $\varphi : A \to B$ be a 4-positive map with $\|\varphi\| = 1$. If $a \in A$ satisfies $\varphi(a)^*\varphi(a) = \varphi(a^*a)$, then $\varphi(b)\varphi(a) = \varphi(ba)$ for all $b \in A$. In particular, if $\varphi : B \to C$ is an extension of a $*$-homomorphism $\pi : A \to C$, then $\varphi(ab) = \pi(a)\varphi(b)$ and $\varphi(ba) = \varphi(b)\pi(a)$ for $a \in A$ and $b \in B$.

Proof. (a) It suffices to show

$$\varphi(a)^*\varphi(a) \le \lim_\alpha \|\varphi(e_\alpha)\|\varphi(a^*a), \qquad a \in A,$$

since

$$\frac{\|\varphi(a)\|^2}{\|a\|^2} \le \lim_\alpha \|\varphi(e_\alpha)\| \frac{\|\varphi(a^*a)\|}{\|a^*a\|}$$

implies $\|\varphi\|^2 \le \lim_\alpha \|\varphi(e_\alpha)\|\|\varphi\|$. Suppose B acts on a Hilbert space H non-degenerately and faithfully. Since φ is 2-positive, we have

$$\begin{pmatrix} \varphi(e_\alpha^2) & \varphi(e_\alpha a) \\ \varphi(a^* e_\alpha) & \varphi(a^*a) \end{pmatrix} = \varphi^{(2)}\left(\begin{pmatrix} e_\alpha^2 & e_\alpha a \\ a^* e_\alpha & a^*a \end{pmatrix}\right) = \varphi^{(2)}\left(\begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} e_\alpha & a \\ 0 & 0 \end{pmatrix}\right) \ge 0,$$

and it is equivalent to

$$\langle \varphi(e_\alpha^2)\xi, \xi\rangle + 2\,\mathrm{Re}\langle \varphi(e_\alpha a)\eta, \xi\rangle + \langle \varphi(a^*a)\eta, \eta\rangle \ge 0, \qquad \xi, \eta \in H, \quad a \in A.$$

We put $\xi := -(\|\varphi(e_\alpha)\|+\varepsilon)^{-1}\varphi(e_\alpha a)\eta$ for $\varepsilon>0$ to get

$$\varphi(e_\alpha a)^*\varphi(e_\alpha a) \le \varphi(e_\alpha a)^*[2-(\|\varphi(e_\alpha)\|+\varepsilon)^{-1}\varphi(e_\alpha^2)]\varphi(e_\alpha a) \le (\|\varphi(e_\alpha)\|+\varepsilon)\varphi(a^*a)$$

We have the desired inequality by taking limits for α and ε.

(b) Since the second inflation $\varphi^{(2)}$ is 2-positive, we may write the Cauchy-Schwarz inequality

$$\varphi^{(2)}\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right)^* \varphi^{(2)}\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) \le \varphi^{(2)}\left(\begin{pmatrix} a^*a & a^*b \\ b^*a & b^*b \end{pmatrix}\right),$$

so

$$\begin{pmatrix} 0 & \varphi(a^*b)-\varphi(a^*)\varphi(b) \\ \varphi(b^*a)-\varphi(b^*)\varphi(a) & \varphi(b^*b)-\varphi(b^*)\varphi(b) \end{pmatrix} \ge 0,$$

which implies $\varphi(b^*a)-\varphi(b^*)\varphi(a)=0$ for any $b\in A$.

Note that $\|\pi\|=1$ if π is not trivial. Using the above argument for a and a^*, we are done. □

1.4 (Russo-Dye theorem)**.** If $C(X)\to B$ is positive, then it is c.p.

1.5 (Completely positive maps for matrix algebras)**.** Let A be a C*-algebra.

(a) Choi matrix

(b) There is a one-to-one correspondence

$$\mathrm{CP}(M_n(\mathbb{C}),A)\to M_n(A)_+ : \varphi\mapsto[\varphi(e_{ij})].$$

(c) Let A be unital. There is a one-to-one correspondence

$$\mathrm{CP}(A,M_n(\mathbb{C}))\to M_n(A)_+^* : \varphi\mapsto(s_\varphi:[a_{ij}]\mapsto\sum_{i,j}\langle\varphi(a_{ij})e_j,e_i\rangle).$$

(d) The above correspondences are (maybe?) isometric if we endow the complete norm on CP.

Proof. (b)

□

1.3 Dilations and Extensions

A linear map $\varphi:A\to B(H)$ is completely positive if and only if

$$\sum_{i,j}\langle\varphi(a_i^*a_j)\xi_j,\xi_i\rangle\ge0,\qquad (a_i)\in A^n,\ (\xi_i)\in H^n.$$

1.6 (Stinespring dilation)**.** Let A be a C*-algebra and $\varphi:A\to B(H)$ be a c.p. map. A *Stinespring dilation* of φ is a pair (π,V) of a representation $\pi:A\to B(K)$ and a bounded linear operator $V:H\to K$ such that $\varphi(a)=V^*\pi(a)V$ for $a\in A$.

$$\begin{tikzcd} B(K) \arrow[dr, "V^*\cdot V"] & \\ A \arrow[u, "\pi"] \arrow[r, "\varphi"'] & B(H) \end{tikzcd}$$

(a) φ has a Stinespring dilation (π,V) such that $\overline{\pi(A)VH}=K$.

(b) For a non-degenerate Stinespring dilation (π,V) of φ, the operator V is an isometry if and only if $\sup_\alpha\varphi(e_\alpha)=1$.

Proof. (a) As we have done in the construction of the GNS representation, define a sesquilinear form on the algebraic tensor product $A\odot H$ such that

$$\langle a\otimes\xi, b\otimes\eta\rangle := \langle\varphi(b^*a)\xi,\eta\rangle, \qquad a\otimes\xi, b\otimes\eta\in A\odot H.$$

It is positive semi-definite since the complete positivity of φ implies

$$\langle\sum_j a_j\otimes\xi_j,\sum_i a_i\otimes\xi_i\rangle = \sum_{i,j}\langle\varphi(a_i^*a_j)\xi_j,\xi_i\rangle\ge 0, \qquad a_i\otimes\xi_i\in A\odot H.$$

Then, we obtain a Hilbert space $K:=\overline{A\odot H/N}$, where $N:=\{\eta\in A\odot H:\langle\eta,\eta\rangle=0\}$. The above construction of a Hilbert space is sometimes called the separation and completion.

Define $\pi: A\to B(K)$ such that

$$\pi(a)(b\otimes\eta+N) := ab\otimes\eta+N, \qquad a\in A,\quad b\otimes\eta+N\in K,$$

and $V: H\to K$ such that

$$\langle V\xi, b\otimes\eta+N\rangle := \langle\varphi(b^*)\xi,\eta\rangle, \qquad \xi\in H,\quad b\otimes\eta+N\in K.$$

The operator V is well-defined by the Cauchy-Schwarz inequality

$$\begin{aligned}|\langle\varphi(b^*)\xi,\eta\rangle|^2 = |\langle\xi,\varphi(b)\eta\rangle|^2 &\le \|\xi\|^2\langle\varphi(b^*)\varphi(b)\eta,\eta\rangle\\ &\le \|\xi\|^2\|\varphi\|\langle\varphi(b^*b)\eta,\eta\rangle = \|\xi\|^2\|\varphi\|\|b\otimes\eta+N\|^2.\end{aligned}$$

Then, we can check $\pi(a)V\xi = a\otimes\xi+N$ for $a\in A$ and $\xi\in H$ from

$$\begin{aligned}\langle\pi(a)V\xi, b\otimes\eta+N\rangle = \langle V\xi, a^*b\otimes\eta+N\rangle &= \langle\varphi(b^*a)\xi,\eta\rangle\\ &= \langle a\otimes\xi+N, b\otimes\eta+N\rangle, \qquad b\otimes\eta+N\in K,\end{aligned}$$

so it follows that $V^*\pi(a)V=\varphi(a)$ for $a\in A$ from

$$\langle V^*\pi(a)V\xi,\eta\rangle = \langle V\xi, a^*\otimes\eta+N\rangle = \langle\varphi(a)\xi,\eta\rangle, \qquad \xi,\eta\in H,$$

and the condition $\overline{\pi(A)VH}=K$.

□

1.7 (Voiculescu theorem)**.** Let A be a unital C*-algebra. Let $\pi: A\to B(K)$ be a faithful non-degenerate representation and $\varphi: A\to B(H)$ be a u.c.p. map. Suppose further that $\varphi|_{\pi^{-1}(K(K))}=0$.

When do we need the faithfulness of π? When do we need the unitality of φ? When do we need the separability of A?

(a) φ is weakly* approximated by vector states, if H is one-dimensional. (Glimm)

(b) φ is approximated by isometry conjugations in $L(A,B(H))$, if H is finite-dimensional. (?)

(c) φ is approximated by isometry conjugations in $\varphi+L(A,K(H))$, if H,K are separable.

Proof. (a) Hahn-Banach separation and Weyl-von Neumann theorem.

(b) correspondence for c.p. maps to matrix algebras.

(c) quasi-central approximate unit and block diagonal c.p. maps.

□

1.8 (Arveson extension)**.** Let $A\subset B$ be C*-algebras. Let $\varphi: A\to B(H)$ be a c.p. map and consider the following diagram:

$$\begin{tikzcd} B \arrow[dr, dashed, "\widetilde{\varphi}"] & \\ A \arrow[u] \arrow[r, "\varphi"'] & B(H). \end{tikzcd}$$

(a) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if B is unital and $1_B \in A$.

(b) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is unital and $B = A \oplus \mathbb{C}$.

(c) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ exists if $\mathcal{A}$ is non-unital and $B = \widetilde{\mathcal{A}}$.

(d) The norm preserving c.p. extension $\widetilde{\varphi}$ of φ always exists.

1.9 (Representation extension)**.** Let I be a left ideal of a C*-algebra B. For a representation $\pi : I \to B(H)$, there is a representation $\widetilde{\pi} : B \to B(H)$ which extends π. If π is non-degenerate, the extension is unique and $\pi(e_\alpha b) \to \widetilde{\pi}(b)$ and $\pi(be_\alpha) \to \widetilde{\pi}(b)$ strongly for $b \in B$, where e_i is an approximate unit of I. The same holds for Hilbert module representations.

Proof. We may assume π is non-degenerate by replacing H to $\overline{\pi(I)H}$. Define $\widetilde{\pi} : B \to B(H)$ such that

$$\widetilde{\pi}(b)(\pi(a)\xi) := \pi(ba)\xi, \qquad a \in I,\ \xi \in H.$$

The well-definedness is from

$$\|\pi(ba)\xi\|^2 = \langle \pi(a^*b^*ba)\xi, \xi\rangle \le \|b\|^2 \langle \pi(a^*a)\xi, \xi\rangle = \|b\|^2 \|\pi(a)\xi\|^2.$$

It is clearly a $*$-homomorphism and extends π.

For the uniqueness, if π is non-degenerate and $\widetilde{\pi}$ is a $*$-homomorphism which extends π, then

$$\widetilde{\pi}(b)(\pi(a)\xi) = \widetilde{\pi}(b)\widetilde{\pi}(a)\xi = \widetilde{\pi}(ba)\xi = \pi(ba)\xi,$$

which is unique by the density of $\pi(I)H$ in H. □

extension of representations for ideals
unique extension of c.p. maps for hereditary subalgebras.

1.4 Tensor products

1.10 (Maximal tensor products)**.** Let A and B be C*-algebras.

(a) (restrictions) A commuting pair of $*$-homomorphisms $\pi : A \to B(H)$ and $\pi' : B \to B(H)$ corresponds to a $*$-homomorphism $\Pi : A \odot B \to B(H)$ via the relation $\Pi(a \otimes b) = \pi(a)\pi'(b)$.

(b) $A \odot B$ admits a $*$-representation and every norms induced from these $*$-representations are uniformly bounded. So, we can define a maximal tensor norm on $A \odot B$.

(c) $a \otimes - : B \to A \odot B$ is a bounded linear map for each $a \in A$ with respect to any C*-norm on $A \odot B$. [BO, 3.2.5]

1.11 (Minimal tensor product)**.** spatiality

1.12 (Takesaki theorem)**.**

Tensors with $M_n(\mathbb{C})$, $C_0(X)$.

1.13 (Haagerup tensor product)**.**

Trick

Exercises

1.14. Let A be a hereditary C*-subalgebra of a C*-algebra B and let $b \in B_+$. If for any $\varepsilon > 0$ there is $a \in A_+$ such that $b - a \leq \varepsilon$, then $b \in A$.

Proof. For $a \in A_+$ satisfying $b \leq a + \varepsilon \leq (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^2$, define

$$a_\varepsilon := a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} \in A.$$

Then,

$$\|b^{\frac{1}{2}} - b^{\frac{1}{2}} a^{\frac{1}{2}}(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\|^2 = \varepsilon\|(a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1} b (a^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})^{-1}\| \leq \varepsilon.$$

Thus $a_\varepsilon \to b$ in norm as $\varepsilon \to 0$. $\square$

Chapter 2

Hilbert modules

2.1 Hilbert modules

2.1 (Banach modules)**.** Let A be a Banach algebra. A *Banach A-module* is a Banach space E which is a A-module such that the action is bounded.

(a) (Cohen factorization theorem) If A has a left approximate unit, then AE is closed in E.

Proof. Suppose $\xi \in \overline{AE}$. We will construct a sequence a_n in the unitization $\widetilde{A}$ such that $a_n^{-1}\xi$ and a_n are both Cauchy in E and $\widetilde{A}$ respectively, but the limit of a_n is in A. In order for this, we first need to check $a_n^{-1} \in \widetilde{A}\setminus A$ can act on E, which is easy anyway.

Let $a_0 = 1 \in \widetilde{A}$ and suppose we have defined $a_n \in \widetilde{A}$ such that $\|1 - a_n\| \le 1 - 2^{-n}$. Since $\xi \in \overline{AE}$, we have $b\eta \in AE$ such that $\|\xi - b\eta\| < 2^{-(3n+1)}$. Since A has an approximate unit, we have $e_n \in A$ such that $\|e_n\| \le 1$, $\|1 - e_n\| \le 1$ (really?), and $\|(1-e_n)a_n^{-1}b\|\|\eta\| < 2^{-(2n+1)}$. Now inductively define

$$a_{n+1} := a_n - 2^{-(n+1)}(1-e_n) \in \widetilde{A}.$$

Since $\|1 - a_{n+1}\| \le 1 - 2^{-(n+1)}$, every term in the sequence a_n is invertible such that $\|a_n^{-1}\| \le 2^n$.

Then, we can check a_n converges to an element of A because

$$a_n = a_0 + \sum_{k=1}^{n} 2^{-k}(1 - e_{k-1}) \to \sum_{k=1}^{\infty} 2^{-k} e_{k-1}.$$

We can also check that $a_n^{-1}\xi$ is Cauchy because the identity

$$a_{n+1}^{-1} - a_n^{-1} = a_{n+1}^{-1}(a_n - a_{n+1})a_n^{-1} = 2^{-(n+1)}a_{n+1}^{-1}(1-e)a_n^{-1}$$

is applied to get

$$\begin{aligned} \|(a_{n+1}^{-1} - a_n^{-1})\xi\| &\le \|a_{n+1}^{-1} - a_n^{-1}\|\|\xi - b\eta\| + \|(a_{n+1}^{-1} - a_n^{-1})b\|\|\eta\| \\ &\le 2^{-(n+1)}\|a_{n+1}^{-1}\|\|a_n^{-1}\|\|\xi - b\eta\| + 2^{-(n+1)}\|a_{n+1}^{-1}\|\|(1-e)a_n^{-1}b\|\|\eta\| \\ &\le 2^{-(n+1)} \cdot 2^{n+1} \cdot 2^n \cdot 2^{-(3n+1)} + 2^{-(n+1)} \cdot 2^{n+1} \cdot 2^{-(2n+1)} \\ &\le 2^{-(2n+1)} + 2^{-(2n+1)} = 2^{-2n}. \end{aligned}$$

It implies that there is $\zeta \in E$ such that $a_n^{-1}\xi \to \zeta$ and $\|a_n^{-1}\xi - \zeta\| \le 2^{-(2n-1)}$.

Then,

$$\|\xi - a\zeta\| \le \|a_n\|\|a_n^{-1}\xi - \zeta\| + \|a_n - a\|\|\zeta\| \le 2^{-(n-1)} + 2^{-n}\|\zeta\|$$

deduces that $\xi = a\zeta$. $\square$

2.2 (Finsler modules)**.** Let A be a C*-algebra.

2.3 (Hilbert modules)**.** Let B be a C*-algebra. A *right Hilbert B-module* or simply a *Hilbert B-module* is a right module E over the complex algebra B which is not involutive, together with a map $\langle -,-\rangle : E \times E \to B$ such that for $\xi, \eta \in E$ and $b \in B$ we have

(i) $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$,

(ii) $\langle \eta, \xi b \rangle = \langle \eta, \xi \rangle b$,

(iii) $\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle$,

and E is Banach with respect to the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$. The map $\langle -,-\rangle$ is called the *B-valued inner product*. It is a non-commutative analogue of Hermitian bundles. Even though the complex scalars act on E from right in the rigorous sense, we will frequently write the scalar multiplication at left.

(a) The right action by b is bounded and the norm is coincides with B. It does not preserve the involutions and is not adjointable in general.

(b) The right action is always non-degenerate. In particular, it follows that $\xi 1 = \xi$ for $\xi \in E$ if A is unital.

(c) The right action is faithful if and only if E is full, i.e. the ideal $\langle E, E \rangle$ of A is dense in A.

(d) Examples: B itself, B^n, $\ell^2(\mathbb{N}, B)$, etc.

(e) direct sum, tensor product, localization

Proof. (c) Consider the approximate unit e_i of $\langle E, E \rangle$. Then, we can show $\xi e_i \to \xi$ in E for each $\xi \in E$, so EB is dense in E. □

2.4 (Adjointable and compact operators)**.** Let E and F be Hilbert B-modules over a C*-algebra B. An operator $T : E \to F$ is called an *adjointable operator* if there is an operator $T^* : F \to E$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in E$ and $\eta \in F$, and called *compact* if it is a norm limit of adjointable operators of the form $\theta_{\eta,\xi} : E \to F$ with $\xi \in E$ and $\eta \in F$, where $\theta_{\eta,\xi} := \eta\langle \xi, -\rangle$, which has an adjoint $\theta_{\xi,\eta}$ The Banach spaces of all adjointable and compact operators $E \to F$ are denoted by $B(E,F)$ and $K(E,F)$ respectively, and these will not be used in the sense of Banach spaces.

(a) An adjointable operator is a bounded B-module map.

(b) $K(E)$ is a closed essential ideal of a C*-algebra $B(E)$.

(c)

Proof. The B-linearity is clear. The boundedness follows from the uniform boundedness principle. □

2.5 (Weak topologies for Hilbert modules)**.** Let E and F be Hilbert B-modules for a C*-algebra B. The *strict topology* refers to the strong* operator topology of $B(E)$.

On the trivial Hilbert B-module B, $b_i \to 0$ strictly iff $b_i, b_i^* \to 0$ weakly. If B is unital, the strict topology on B and the norm topology coincide. An adjointable operator is weakly continuous.

On Hilbert modules:

- polarization identity? OK,
$$\langle \eta, \xi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \xi + i^k \eta, \xi + i^k \eta \rangle$$
- unbounded adjointable operators and spectral theory?
- polar decomposition? especially for unbounded adjointable operators?

- bounded sesquilinear form?
- Riesz representation? OK for adjointable operator $l : E \to B$, there is $\eta := l^*1$ (The classical Riesz representation states that every bounded linear functional is automatically adjointable in the sense of Hilbert $\mathbb{C}$-modules)
- alaoglu?
- uniform boundedness principle?
-

2.6 (Multiplier algebra)**.** Four descriptions for a multiplier algebra: double centralizers vs essential ideal vs multipliers in von Neumann algebra vs Hilbert module

1. Let B be a C*-algebra. A *double centralizer* of B is a pair (L, R) of bounded linear maps on A such that $aL(b) = R(a)b$ for all $a, b \in B$. The *multiplier algebra* $M(B)$ of B is defined to be the set of all double centralizers of B. There is another characterization of $M(B)$ as the set of adjointable operators to itself. Even if the notation $B(B)$ may cause confusion, we can write $M(B)$ to avoid this.

2. An ideal I of B is called an *essential* if it is a full Hilbert B-submodule of B.

Every C*-algebra A is a correspondence over $M(A)$.

(a) $\|\pi(a - e_\alpha a)\xi\|^2$

(b) If a_α are unitary, the convergences in the strict topology and the weak topology(how to define this?) coincide.

(c) If a_α are increasing, the convergences in the strict topology and the weak topology(how to define this?) coincide.

(d) $M(K(E)) \cong B(E)$.

(e) $M(C_0(\Omega)) \cong C_b(\Omega)$.

Proof. First we claim $C_0(\Omega)$ is an essential ideal of $C_b(\Omega)$. Since $C_b(\Omega) \cong C(\beta\Omega)$, and since closed ideals of $C(\beta\Omega)$ are corresponded to open subsets of $\beta\Omega$, $C_0(\Omega) \cap J$ is not trivial for every closed ideal J of $C_b(\Omega)$.

Now we have an injective $*$-homomorphism $C_b(\Omega) \to M(C_0(\Omega))$, for which we want to show the surjectivity. Let $g \in M(C_0(\Omega))_+$. □

characterization in an inclusion into a von Neumann algebra.
relations between Hilbert $B(H)$-modules and representations

2.7. C*-algebras together with a non-degenerate representation $C_0(X) \to Z(M(A))$.

2.8 (Dauns-Hoffman theorem)**.**

2.9.

(a) $\mathrm{LCH}_{\mathrm{prop}}$ is equivalent to $\mathrm{CC^*Alg}_{\mathrm{nondeg}}$.

(b) CH_* is equivalent to CC*Alg.

(c) LCH is equivalent to $\mathrm{CC^*Alg}_{\mathrm{mor}}$.

? ...( ) 2: , .. ?

2.2 C*-correspondences

2.10 (C*-correspondences)**.** Let A and B be C*-algebras. A *C*-correspondence*, *C*-bimodule*, or just simply a *correspondence* over A and B, or from A to B, is a Hilbert B-module E together with a $*$-homomorphism $\varphi : A \to B(E)$, called the *left action*. We say E is *faithful* or *non-degenerate* if the left action is faithful or non-degenerate, respectively.

(a) If $\varphi : A \to M(B)$ is a unital completely positive map, then we can construct a natural correspondence E from A to B by mimicking the GNS construction on $A \odot B$.

(b) If $\varphi : A \to M(B)$ is a non-degenerate $*$-homomorphism, $\varphi \in \mathrm{Mor}(A, B)$ in other words, then we can associate a canonical A-B-correspondence B such that the left action is realized with φ. More precisely, $\iota : E \to B : a \otimes b \mapsto \varphi(a)b$ provides a well-defined linear isomorphism (surjectivity follows from the density of $\varphi(A)B$ in B and the Cohen factorization theorem) and the two actions on E is described by $\iota(a\xi b) = \varphi(a)\iota(\xi)b$.

2.11 (Pimsner construction)**.** C*-correspondences over A can be interpreted as a generalized automorphism on A, and the Pimsner construction defines a new C*-algebra generated by the generalized cyclic action provided by a C*-correspondence. Let E be a C*-correspondence over a C*-algebra A. Let B be a C*-algebra and see it as a trivial C*-correspondence over B. A *Toeplitz representation* of E on B is a pair (π, τ) of a $*$-homomorphism $\pi : A \to B$ and a linear map $\tau : E \to B$ such that

$$\pi(\langle \xi, \eta \rangle) = \tau(\xi)^* \tau(\eta), \qquad \tau(\varphi(a)\xi) = \pi(a)\tau(\xi).$$

We define the *Katsura ideal*

$$J(E) := \varphi^{-1}(K(E)) \cap \varphi^{-1}(0)^{\perp}.$$

We say a Toeplitz representation of E is *covariant* if

$$\psi(\varphi(a)) = \pi(a), \qquad a \in J(E).$$

(a) Let $(A, \mathbb{Z}, \alpha)$ be a C*-dynamical system and consider the canonical C*-correspondence A over A with the left action $\varphi := \alpha_1 \in \mathrm{Aut}(A) \subset \mathrm{Mor}(A)$. This correspondence is full, faithful, and non-degenerate. Note that also we have $J(A) = \varphi^{-1}(A) \cap A = A$. If (π, τ) is an any representation of this C*-correspondence A on B, then

How can we decribe representations of C*-correspondence A with left action $\varphi \in \mathrm{Aut}(A)$ in terms of covariant representations of the C*-dynamical system $(A, \mathbb{Z}, \alpha)$ with $\alpha_n = \varphi^n$?

as a morphism sub and quotient, direct sum, tensor product,

Toeplitz-Cuntz Toeplitz-Pimsner Cuntz-Pimsner Cuntz-Krieger

Subproduct systems

2.3 Morita equivalence

Induced representations?

Chapter 3

Constructions

3.1 Categorical constructions

inverse limits: direct sum, direct product, restricted direct sum, locally C*-algebras.

Infinite direct sums and direct products are ill-behaved in the category of C*-algebras. An infinite direct sum must be interpreted as complete Hausdorff spaces, not a pointed compact Hausdorff space. For example, after adding a base point, the spectrum of $\bigoplus_{i=1}^{\infty} C_0(\mathbb{R})$ corresponds to the Hawaiian earing, and the spectrum of $\prod_{i=1}^{\infty} C_0(\mathbb{R})$ corresponds to the Stone-Čech compactification of the infinite wedge of circles. We cannot describe the infinite wedge of circles in terms of C*-algebras, so we need locally C*-algebras.

direct limits: filtered limits, tensor products, free products, amalgamated free products.

3.1 (Locally C*-algebras)**.** A *locally C*-algebra* is a complete topological $*$-algebra whose topology is generated by C*-semi-norms. We adopt the convention that a *homomorphism* between locally C*-algebras means a continuous $*$-homomorphism.

(a) A topological $*$-algebra is a locally C*-algebra if and only if it is an inverse limit of unital C*-algebras.

Proof. (a) Let A be a locally C*-algebra. The set of continuous C*-seminorms on A is a directed set. Construct an inverse system... Since every C*-algebra is a maximal ideal of a unital C*-algebra of codimension one, we may assume that the objects in this inverse system is unital... Also, elements of A are represented by coherent sequences. □

3.2 Crossed products

3.2 (Group algebras)**.** Let G be a locally compact group.

type I, subhomogeneous
crystallographic discrete heisenberg free groups projectionless of $C_r^*(F_2)$

3.3 (Enveloping C*-algeberas)**.** Let A be a $*$-algebra. A *C*-norm* is an submultiplicative norm satisfying the C*-identity. Does A have enough $*$-representations?

(a) A complete C*-norm is unique if it exists.

(b) For every C*-norm α on A, there is a $*$-isometry $\pi : A \to B(H)$.

(c) For maximal C*-norm, there is a universal property. The maximal C*-norm can be obtained by running through cyclic representations.

3.4 (C*-dynamical system). Let G be a locally compact group. A *C*-dynamical system* or a *G-C*-algebra* is a C*-algebra A together with a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ that is continuous in the point-norm topology. We will often write a triple (A, G, α) instead of A to refer to a C*-dynamical system.

(a) There is an equivalence between categories of locally compact transformation groups and C*-dynamical system on abelian C*-algebras.

On $U(H)$, the strict topology and the strong operator topology are equal. Therefore, we have three topologies to consider: strong, weak, and σ-weak.

3.5 (Covariant representation). Let G be a locally compact group.

A *covariant representation* of a C*-dynamical system (A, G, α) is a G-equivariant $*$-homomorphism $\pi : (A, G, \alpha) \to (B(H), G, \beta)$ for a C*-dynamical system $(B(H), G, \beta)$, where a Hilbert space H.

(a) There exists a unitary representation $u : G \to B(H)$ such that $\pi(\alpha_s a) = u_s \pi(a) u_s^*$.

(b) (Integrated form) There is a one-to-one correspondence between covariant representations of (A, G, α) and $*$-representations of $L^1(G, A)$. (non-degenerate)

Note that we have a homeomorphism $\mathrm{Aut}(K(H)) \cong PU(H)$ between the point-norm topology and the strong operator topology.

$\mathbb{Z}$-action, Homeo-action, left multiplication of subgroup induced representation regular representation $(C_0(G), G, \lambda) \to (B(L^2(G)), G, \lambda)$.

commutative case

3.3 Graph algebras

3.4 Groupoid algebras

Part II

Properties

Chapter 4

Approximation properties

4.1 Nuclearity and exactness

finite dimensional[BO, 3.3.2], abelian, AF permanence properties

4.1 (Completely positive approximation property)**.** Let A be a C*-algebra. We say A has the *completely positive approximation property* if the identity is contained in the point-norm, or equivalently the point-weak closure of $\mathcal{F}$ in $L(A)$.

(a) If A has the completely positive approximation property, then A is nuclear.

(b) If A is nuclear, then A has the completely positive approximation property.

Proof. (b)

Let $E \subset A$ and $F \subset A^*$ be finite subsets and fix $\varepsilon > 0$. We want to find completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon, \qquad a \in E,\ l \in F.$$

To implement the approximation, we would like to regard a bounded linear operator on A as a state of a tensor product of C*-algebras, which maps $\theta \in L(A)$ to the linear functional characterized by $a \otimes l \mapsto l(\theta(a))$. However, since A^* is not a C*-algebra, we embed A^* locally in $B(H)$ through the Radon-Nikodym type result. Let $\pi : A \to B(H)$ be the cyclic representation obtained from a positive linear functional that dominates F and Ω the cyclic vector such that there is a linear map $\pi' : F \to \pi(A)'$ satisfying

$$l(a) = \langle \pi(a)\pi'(l)\Omega, \Omega\rangle, \qquad a \in E,\ l \in F.$$

Now the duality of A and F is embodied in the tensor product representation

$$\pi \times i : A \otimes_{\max} \pi(A)' \to B(H)$$

together with a cyclic vector $\Omega \in H$. Here the nuclearity is used to write $A \otimes_{\max} \pi(A)' = A \otimes_{\min} \pi(A)'$.

If we take any faithful representation $\rho : A \to B(K)$, then we obtain a faithful representation

$$\rho \otimes i : A \otimes_{\min} \pi(A)' \to B(K \otimes H).$$

By the Hahn-Banach separation, the state $(\pi \times i)^*\omega_\Omega$ on $A \otimes_{\min} \pi(A)'$ can be approximated weakly* by convex combinations of vector states in $B(K \otimes H)$. In particular, by the density of $\pi(A)\Omega$ in H, we have algebraic tensors $(t_k)_{k=1}^m \subset K \odot \pi(A)\Omega$ such that

$$\left|\omega_\Omega((\pi \times i)(a \otimes \pi'(l))) - \sum_{k=1}^m \lambda_k \omega_{t_k}((\rho \otimes i)(a \otimes \pi'(l)))\right| < \varepsilon \tag{†}$$

for all $a \in E$ and $l \in F$, where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda = 1$.

If we write each element $t \in K \odot \pi(A)\Omega$ as

$$t = \sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega, \qquad \eta_i \in K,\ b_i \in A,$$

then

$$\begin{aligned}\omega_t((\rho \otimes i)(a \otimes \pi'(l))) &= \left\langle (\rho(a) \otimes \pi'(l))\Big(\sum_{j=1}^n \eta_j \otimes \pi(b_j)\Omega\Big), \Big(\sum_{i=1}^n \eta_i \otimes \pi(b_i)\Omega\Big) \right\rangle \\ &= \sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i\rangle \langle \pi'(l)\pi(b_i^* b_j)\Omega, \Omega\rangle \\ &= l\Big(\sum_{i,j=1}^n \langle \rho(a)\eta_j, \eta_i\rangle b_i^* b_j\Big).\end{aligned}$$

If we define completely positive maps $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ for each τ such that

$$\varphi(a) := [\langle \rho(a)\eta_j, \eta_i\rangle], \quad \psi([\delta_{ik}\delta_{jl}]) := b_k^* b_l,$$

then we have $\omega_t(a \otimes \pi'(l)) = l(\psi \circ \varphi(a))$. We may assume φ and ψ are contractive by adjusting their norms.

Since $\mu(a \otimes \pi'(l)) = l(a)$ and since the completely positive contractions which factor through a matrix algebra form a convex set, we have completely positive contractions $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ such that the inequality (†) is rewritten as

$$|l(a) - l(\psi \circ \varphi(a))| < \varepsilon,$$

so we are done. □

The set $\mathcal{F}$ of factorable maps is a convex set of $L(A)$. Note that we have an embedding

$$L(A) \hookrightarrow L(A, A^{**}) = \varprojlim_F L(A, F^*).$$

We have a continuous bijection

$$(A \widehat{\otimes}_\pi F)^* \to L(A, F^*).$$

If we let $M := \pi(A)'' \subset B(H)$ be the GNS representation for F, then the Radon-Nikodym theorem on commutant gives rise to a continuous map

$$(A \mathbin{\widehat{\otimes}_\pi} M')^* \to (A \mathbin{\widehat{\otimes}_\pi} F)^*.$$

$$\begin{tikzcd} B(K \otimes \pi(A)\Omega)^* \arrow[d, two heads] & B(\pi(A)\Omega)^* \arrow[d, two heads] & \\ (A \otimes_{\min} M')^* \arrow[r, tail] & (A \otimes_{\max} M')^* \arrow[r, tail] & (A \mathbin{\widehat{\otimes}_\pi} M')^* \end{tikzcd}$$

The first map is in fact surjective by the nuclearity.

quotients of nuclear local reflexivity

4.2. A C*-algebra C is called *injective* every completely positive map $\varphi : A \to C$ from a C*-subalgebra A of a C*-algebra B is extended to a completely positive map $\widetilde{\varphi} : B \to C$. A von Neumann algebra is called injective if it is injective as a C*-algebra. (operator subsystem? unital?)

The C*-algebra $B(H)$ is injective, and its image of completely positive idempotent is injective. A von Neumann algebra on M on H is injective if and only if there is a conditional expectation $B(H) \to M$.

A^{**} semi-discrete -> A nuclear is done by four step approximation

The reverse implication follows from A is nuclear -> A' is injective -> A'' is injective -> A'' is semi-discrete.

Let A be nuclear. Note $A^{**} = I^{**} \oplus (A/I)^{**}$. Since A^{**} is semi-discrete, $(A/I)^{**}$ is semi-discrete. Therefore, A/I is nuclear.

a separable C*-algebra is nuclear if and only if every factor representation is hyperfinite.

Extension properties weak expectation property relatively weakly injective maximal tensor product inclusion problem

excision: Akemann-Anderson-Pedersen

4.2 Quasi-diagonality

4.3 (Weyl-von Neumann theorem)**.** A self-adjoint bounded operator is quasi-diagonal.

4.4 (Glimm lemma)**.** If a state ω of $B(H)$ vanishes on $K(H)$, then it is a weak* limit of vector states.

4.5 (Voiculescu theorem)**.**

4.6 (Quasi-diagonal algebras)**.** An operator $a \in B(H)$ is called *quasi-diagonal* if there is a net of projections $p_i \in B(H)$ such that $[p_i, a] \to 0$ in norm and $p_i \uparrow \mathrm{id}_H$ strongly. A C*-algebra is called *quasi-diagonal* if it admits a faithful representation whose image is quasi-diagonal.

faithful non-degenerate essential representations of a quasi-diagonal C*-algebra are all quasi-diagonal
locally quasi-diagonal

4.3 AF-embeddability

Chapter 5

Amenability

5.1 Amenable groups

5.2 Amenable actions

crossed products Z_2-grading Connes-Feldman-Weiss Anantharaman-Delaroche Gromov boundaries approximately central structure? dynamical Kirchberg-Phillips

stably finite dynamical Elliott program

Ornstein-Weiss-Rokhlin lemma

5.3 Exact groups

Exact groups

5.4 Other properties

Kazdahn property (T) factorization property Haagerrup property

Kaplansky conjecture

A state τ on A is called an *amenable trace* if there is a state ω of $B(H)$ such that ω extends τ and $\omega(uxu^*) = \omega(x)$ for $x \in B(H)$ and $u \in U(A)$. It is automatically tracial. The amenability of a trace does not depend on the choice of faithful representation of A, using the Arveson extension and the multiplicaitve domain.

For a discrete group Γ, $C_r^*(\Gamma)$ is amenable if and only if has an amenable tracial state. Note that a mean is a state of $\ell^\infty(\Gamma)$, which may not be normal.

Chapter 6

Simplicity

Furstenburg boundary

Part III

Invariants

Chapter 7

Operator K-theory

7.1 Zeroth K-groups

Three pictures: projections of $M_n(A)$(standard), projections of $A \otimes K(H)$(recall that $K(H)$ is AF and hence nuclear), algebraically finitely generated projective Hilbert modules over A.

7.1 (Equivalences of projections)**.** Let A be a unital C*-algebra. Let p and q be projections in A. Recall that they are called *Murray-von Neumann equivalent* or just equivalent, denoted $p \sim q$, if $p = v^*v$ and $q = vv^*$ for some $v \in A$, *unitarily equivalent*, denoted by $p \sim_u q$, if $p = u^*qu$ for some $u \in U(A)$, and *homotopic*, denoted by $p \sim_h q$, if there is a continuous path in $P(A)$ connecting them.

(a) If $p \sim_h q$, then $p \sim_u q$, and if $p \sim_u q$, then $p \sim q$.

(b) If $p \sim q$, then $p \oplus 0 \sim_u q \oplus 0$ in $M_2(A)$.

(c) If $p \sim_u q$, then $p \oplus 0 \sim_h q \oplus 0$ in $M_2(A)$.

Almost projection: if $\|a^2 - a\| < \varepsilon$, then $\|p - a\| < 2\varepsilon$ for some $p \in A$.
If $p \in A = \operatorname{colim}_i A_i$, then $\|p_i - p\| < \varepsilon$ for some $p_i \in A_i$.

7.2 (Properties of $K(H)$)**.** Let H be a separable Hilbert space.

7.3 (Definition of zeroth K-group)**.** Let A be a unital C*-algebra. Define $V(A) := \bigcup_{n=1}^{\infty} P(M_n(A))/\sim$. It gives a functor from the category of unital C*-algebras to the category of ordered abelian monoid with cancellation property. If A is unital, we define $K_0(A) := G(V(A))$, the Grothendieck group of the monoid $V(A)$. Its elements can be described by $[p]-[p_n]$.

(a) $V(M_n(\mathbb{C})) \cong \mathbb{Z}_{\geq 0}$ because two projections are equivalent iff they have same range dimensions, so $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

(b) $V(K(H)) \cong \mathbb{Z}_{\geq 0} = \mathrm{Card}_{<\omega}$, $V(B(H)) \cong \mathrm{Card}_{\leq \dim H}$, $V(Q(H)) \cong \{0\} \cup (\mathrm{Card}_{\geq \omega} \cap \mathrm{Card}_{\leq \dim H})$, so $K_0(B(H)) \cong K_0(Q(H)) \cong 0$. (Weyl-von Neumann theorem: self-adjoint elements of $Q(H)$ with same spectrum are unitarily equivalent)

(c) $K_0(C(S^2)) \cong \mathbb{Z}^2$.

(d) For a II$_1$ factor M, $K_0(M) \cong \mathbb{R}$.

(e) $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

7.4 (Relative K-theory)**.** We want to discuss the exactness of K-theory. For this, we have to consider pairs of C*-algebras. We define a *pair* of C*-algebras as a surjective $*$-homomorphism between unital C*-algebras. Let $\pi : A \to B$ is a pair of C*-algebras. Then, $K_0(A, B)$ can be concretely described or defined by the set of equivalence classes of (p, q, v), where p and q are projections in $M_\infty(A)$ and $v \in M_\infty(A)$

satisfies $\pi(p) = \pi(v^*v)$ and $\pi(q) = \pi(vv^*)$. In fact, we can show $K_0(A,B)$ only depends on the kernel $I := \ker\pi$. It is called the excision theorem. For a general non-unital C*-algebra I, it is well-defined that

$$K_0(I) := K_0(A, A/I),$$

where A is any unitization of I. We can show that if I is unital, then it is naturally isomorphic to the original without-base-point definition of K-theory.(for example, $K_0(A) \cong K_0(A \oplus \mathbb{C}, \mathbb{C})$ for unital A) In particular, since $K_1(\mathbb{C}) = 0$, the six-term exact sequence implies that $K_0(I) \cong \ker(K_0(\widetilde{I}) \to K_0(\mathbb{C}))$, and since $0 \to I \to \widetilde{I} \to \mathbb{C} \to 0$ splits, we have $K_0(I) \oplus \mathbb{Z} \cong K_0(\widetilde{I})$. A generally non-unital C*-algebra is the non-commutative analogue of the pointed quotient of compact pairs.

Even if A and B are non-unital, one can check the followings are exact:

$$K_0(I) \to K_0(A) \to K_0(B)$$

$[p,q,v] \mapsto [p]-[q] \mapsto ...$

When we consider exact sequences, we may think every algebra A in K-theory as a pair (B,C) such that $B/C \cong A$! If the algebra A is unital, then it is also possible to think it as a space without base point, as in the definition of $K_0(A)$. The basic way to think is to consider non-unital C*-algebras A and $K_0(A)$ as the paired or pointed version. But we do not need the tilde.

$$\{\text{pair of spaces}\} \twoheadrightarrow \{\text{pointed spaces}\} \hookleftarrow \{\text{spaces}\}$$

$$\{\text{pair of C}^*\text{-algebras}\} \twoheadrightarrow \{\text{C}^*\text{-algebras}\} \hookleftarrow \{\text{unital C}^*\text{-algebras}\}$$

The first map is quotient. The second map is adjoining a new point for space, the inclusion for algebras. The first two categories are indistinguishable in generalized cohomology theoreis or homotopy theories. We do not have to introduce the notation $\widetilde{K}$, because we basically consider the unital algebra $C(X)$ not as a pointed space (X, x_0) (like in topology), but as $(X \cup *, *)$, i.e. $K(C_0(X)) = \widetilde{K}(X \cup *, *)$ for compact or non-compact X.

As $K_0 : \text{C}^*\text{Alg} \to \text{grAb}$, K_0 satisfies the axioms for cohomology theories

- functoriality
- homotopy invariance
- FINITE product-preserving*
- half-exactness
- long exactness

with additional properties

- lax symmetric monoidal functor
- filtered colimit-preserving
- $\mathbb{K}$-stable
- partial order
- ring axioms for K_0 only on commutatives

Here we only consider finite product-preserving because the infinite direct product does not mean the infinite wedge sum in the category of C-algebras. We need to consider locally C*-algebras.

7.5 (Homotopy of $*$-homomorphisms). Let A, B be C*-algebras. Two $*$-homomorphisms in $\mathrm{Mor}(A,B)$ are said to be *homotopic* if they are connected by a path in $\mathrm{Mor}(A,B)$ that is continuous with the point-norm topology.

(a) For pointed compact Hausdorff spaces $(X,x_0),(Y,y_0)$, two pointed maps $\varphi_0,\varphi_1 : X \to Y$ are homotopic if and only if $\varphi_0^*,\varphi_1^* : C_0(Y\setminus\{y_0\}) \to C_0(X\setminus\{x_0\})$ are homotopic.

Proof. (a) Suppose φ_0 and φ_1 are connected by a homotopy φ_t. Fixing $g \in C_0(Y)$ and $t_0 \in I$, we want to show

$$\lim_{t\to t_0}\sup_{x\in X}|g(\varphi_t(x)) - g(\varphi_{t_0}(x))| = 0.$$

Since the function g is uniformly continuous, with respect to an arbitrarily chosen uniformity on Y, so that there is an entourage $E \subset Y \times Y$ such that $(y,y') \in E\circ E$ implies $|g(y)-g(y')| < \varepsilon$. Using compactness we have a finite sequence $(y_i)_{i=1}^n \subset Y$ such that for every y there is y_i satisfying $(y,y') \in E$. Then, $\varphi^{-1}(E[y_i])$ is a finite open cover of $X\times I$, so we have δ such that $|t-t_0| < \delta$ implies for any $x \in X$ the existence of i satisfying $(\varphi_t(x),y_i) \in E$ and $(\varphi_{t_0}(x),y_i) \in E$, which deduces the desired inequality.

Conversely, suppose φ_0^* and φ_1^* are connected by a homotopy φ_t^*. By taking dual, we can induce $\varphi_t : X \to Y$ such that $g(\varphi_t(x)) = (\varphi_t^* g)(x)$ for each $g \in C(Y)$ from φ_t^* via the embedding $X \to M(X)$ by Dirac measures. Let V be an open neighborhood of $\varphi_{t_0}(x_0)$ and take $g \in C(Y)$ such that $g(\varphi_{t_0}(x_0)) = 1$ and $g(y) = 0$ for $y \notin V$. Now we have an open neighborhood U of x_0 such that $x \in U$ implies $|(\varphi_{t_0}^* g)(x) - (\varphi_{t_0}^* g)(x_0)| < \frac{1}{2}$. Also we have $\delta > 0$ such that $|t-t_0| < \delta$ implies $\|\varphi_t^* g - \varphi_{t_0}^* g\| < \frac{1}{2}$. Therefore, $(x,t) \in U \times (t_0-\delta, t_0+\delta)$ implies $g(\varphi_t(x)) > 0$, hence $\varphi_t(x) \in V$, which means $X \times I \to Y : (x,t) \mapsto \varphi_t(x)$ is continuous. $\square$

$$K_0(\mathbb{C}) = \mathbb{Z}, \quad K_0(C_0(\mathbb{R})) = 0, \quad K_1(C_0(\mathbb{R})) = K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$$
$$K^0(*) = \mathbb{Z}, \quad K^0(S^1) = \mathbb{Z}, \quad K^1(S^1) = K^0(S^2) = \mathbb{Z}[x]/(x-1)^2$$

Let X be a locally compact Hausdorff space, and $(X_+,*) = (X \sqcup \{*\},*)$ be the associated pointed compact Hausdorff space. Then, the K-theory with compact supports has

$$K_0(X) = K_0(X_+,*) = \widetilde{K}_0(X_+) = K^0(C_0(X)).$$

7.2 First K-groups

K_1 satisfies long exactness(triangulated structure), bott periodicity, ring structure?

$K(\mathbb{C}) \cong \mathbb{Z}[\beta^{\pm 1}]$.

$$CB := \{f \in B \otimes C([0,1]) : f(0) = 0\}, \qquad C_\varphi := \{(a,f) \in A \oplus CB : f(1) = \varphi(a)\}.$$

The mapping cone can be defined by an exact sequence

$$0 \to C_\varphi \to M_\varphi \to B \to 0,$$

or alternatively by the pullback

$$\begin{tikzcd} C_\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & CB \arrow[d, "f\mapsto f(1)"] \\ A \arrow[r] & B. \end{tikzcd}$$

The suspension can defined by an exact sequence

$$0 \to \Sigma B \to CB \to B \to 0,$$

or alternatively by the pullback

$$\begin{tikzcd} \Sigma B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & CB \arrow[d, "f\mapsto f(1)"] \\ CB \arrow[r] & B. \end{tikzcd}$$

We can see that CB is contractible, and ΣB is homotopic to the pullback $C_\varphi \oplus_A CA$.
distinguished triangle

$$\Sigma B \to C_\varphi \to A \xrightarrow{\varphi} B$$

Do not forget to describe the induced maps for K-groups!
$K_{-1}(A) := K_0(\Sigma A)$.
local Banach algebras

7.6 (Pimsner-Voiculescu exact sequence)**.**

Connes-Thom isomorphism

7.3 Cuntz semigroup

nuclear dimension

Chapter 8

KK-theory

8.1 Kasparov picture

- Kasparov stabilization theorem
- Kasparov-Stinespring theorem
- Kasparov-Voiculescu theorem
- Kasparov-Weyl-von Neumann theorem
- Kasparov technical theorem

8.1 (Group equivariant correspondences). Let G be a locally compact group. Let (A,α) and (B,β) be G-C*-algebras. An *equivariant correspondence* from (A,α) to (B,β) is a correspondence E from A to B together with a strongly continiuous map $u : G \to L(E)$ satisfying

$$u_s(a\xi b) = \alpha_s(a)u_s(\xi)\beta_s(b), \qquad \beta_s(\langle\eta,\xi\rangle) = \langle u_s\eta, u_s\xi\rangle,$$

for $a \in A$, $b \in B$, $s \in G$, and $\xi,\eta \in E$. It generalizes covariant representations of A and equivariant Hilbert modules over B. The map u is called a *group action* on E of G, and it is not in general B-linear unless the action β on B is trivial. For an equivariant correspondence (E,u) from (A,α) to (B,β), the adjoint action $\operatorname{Ad} u$ acts continuously on $K(E)$ and strictly continuously on $B(E)$.

(a) If E is a super-correspondence from A to B, then $(L^2(G)\otimes E, \lambda\otimes 1)$ is naturally an equivariant super-correspondence from (A,α) to (B,β). If E is faithful, non-degenerate, and full, then so is $L^2(G)\otimes E$, respectively.

(b) interior tensor product and coalgebra structure from the group...

Proof. (a) Define the super-correspondence $L^2(G)\otimes E$ from A to B with the natural grading, such that the left action of A, the right action of B, and the B-valued inner product is defined by

$$(a\xi b)(t) := \alpha_t^{-1}(a)\xi(t)\beta_t^{-1}(b), \qquad \langle\eta,\xi\rangle := \int_G \beta_t(\langle\eta(t),\xi(t)\rangle)\,dt,$$

for $a \in A$, $b \in B$, $t \in G$, and $\xi,\eta \in C_c(G,E)$. The group action on $L^2(G)\otimes E$ by G is given by $\lambda\otimes 1$.

We can check the above three structures preserve the grading and are all equivariant.

(Faithfulness) Suppose $a\xi = 0$ for all $\xi \in L^2(G)\otimes E$. Then, for $f\otimes\xi_0 \in C_c(G)\otimes E$,

$$0 = (a(f\otimes\xi_0))(t) = f(t)\otimes(\alpha_t^{-1}(a)\xi_0)$$

implies $f(e)\otimes(a\xi_0) = 0$ by putting $t = e$, so $a\xi_0 = 0$ and $a = 0$.

(Fullness) Because the a Hilbert module is full iff the right action is faithful, we can prove it in a similar way to faithfulness of the left action.

(Non-degeneracy) If $e_i \in A$ is a quasi-central approximate unit such that $\alpha_t(e_i) - e_i \to 0$ in A compactly on G (it can be shown whithout the condition that A is σ-unital, Lemma 2.12 of Ozawa), then

$$(e_i\xi - \xi)(t) = (\alpha_t^{-1}(e_i) - 1)\xi(t) = (\alpha_t^{-1}(e_i) - e_i)\xi(t) + (e_i - 1)\xi(t)$$

$$\begin{aligned}
|\xi - e_i\xi|^2 &= \int_G \beta_t(|((1-e_i)\xi)(t)|^2)\,dt \\
&= \int_G \beta_t(|(1-\alpha_t^{-1}(e_i))\xi(t)|^2)\,dt \\
&\le 2\int_G \beta_t(|(1-e_i)\xi(t)|^2 + |(e_i - \alpha_t^{-1}(e_i))\xi(t)|^2)\,dt \to 0
\end{aligned}$$

taking compact set outside which we have $\|\xi\| < \varepsilon$.

□

8.2 (Correspondences over commutative C*-algebras)**.** Let X be a locally compact Hausdorff space. Let A and B be $C_0(X)$-C*-algebras.

For equivariant versions, we do not require the compatibility of G and $C_0(X)$ on E, which is satisfied automatically.

(a)
(b) For a $C_0(X)$-C*-algebra A, there exists a faithful non-degenerate correspondence E from A to some $C_0(X)$-W*-algebra B.
(c) tensor products of G-C*-algebras

Proof. (b) We will choose $B = C_0(X)^{**}$. ($C_0(X)^{**}$ is not a $C_0(X)$-algebra...) Fix a state ω on A. Since $C_0(X)^{**} \subset Z(A^{**})$, there is a conditional expectation $\varphi : A^{**} \to C_0(X)^{**}$, which factors through $\omega^{**} = \omega^{**}\varphi$ because $C_0(X)^{**} \subset Z(A^{**})$ is unital. Since φ is completely positive, the Stinespring construction on $A \odot C_0(X)$ gives rise to a C*-correspondence E_ω from A to $C_0(X)^{**}$. Define $E := \bigoplus_{\omega\in S(A)} E_\omega$. If $a \in A$ acts trivially on E, which means $\varphi(a^*a) = 0$ and $\omega(a^*a) = 0$. Thus A acts failfully on E.

□

8.3 (Kasparov stabilization theorem)**.** Let G be a locally compact group. Let (B,β) be a G-C*-algebra. Let (E,u) be an equivariant Hilbert module over (B,β). Let $H_B := \ell^2 \otimes L^2(G) \otimes B$. If E is countably generated, then there is a equivariant B-linear isomteric isomorphism $E \to H_B \oplus E$.

(a) non-equivaraint version.
(b) equivariant version.

Proof. (a) The Hilbert B-module E is countably generated if and only if there is a dense range adjointable operator

$$\ell^2 \otimes B \to E.$$

(b) Let $H_E := \ell^2 \otimes L^2(G) \otimes E$.

We have

$$\begin{aligned}
H_B &= \ell^2 \otimes L^2(G) \otimes B \\
&= \ell^2 \otimes L^2(G) \otimes \ell^2 \otimes B \\
&= \ell^2 \otimes L^2(G) \otimes (E_0 \oplus (\ell^2 \otimes B)) \\
&= (\ell^2 \otimes L^2(G) \otimes E_0) \oplus (\ell^2 \otimes L^2(G) \otimes \ell^2 \otimes B)) \\
&= (\ell^2 \otimes L^2(G) \otimes E) \oplus H_B \\
&= H_E \oplus H_B,
\end{aligned}$$

where all the identities mean equivariant isometric B-linear isomorphisms.

Since G is compact, we have an equivariant linear isometry $\mathbb{C} \to L^2(G)$. It gives rise to direct sums $L^2(G) = \mathbb{C} \oplus \mathbb{C}^\perp$, and we get $L^2(G) \otimes E = E \oplus E^\perp$ by tensoring, that is, E is complemented Hilbert B-submodule of $L^2(G)$. We have

$$\begin{aligned}
E \oplus H_E &= E \oplus (\ell^2 \otimes L^2(G) \otimes E) \\
&= E \oplus (\ell^2 \otimes (E \oplus E^\perp)) \\
&= E \oplus (\ell^2 \otimes E) \oplus (\ell^2 \otimes E^\perp) \\
&= ((\mathbb{C} \oplus \ell^2) \otimes E) \oplus (\ell^2 \otimes E^\perp) \\
&= (\ell^2 \otimes E) \oplus (\ell^2 \otimes E^\perp) \\
&= \ell^2 \otimes (E \oplus E^\perp) \\
&= \ell^2 \otimes L^2(G) \otimes E \\
&= H_E.
\end{aligned}$$

Therefore,

$$H_B = H_E \oplus H_B = E \oplus H_E \oplus H_B = E \oplus H_B.$$

□

8.4 (Quasi-central approximate units). Let A be a σ-unital C*-algebra. Let Y be a locally compact σ-compact Hausdorff subset contained in a faithful representation $B(H)$ of A. Then, there is an increasing sequential approximate unit e_i for A such that $[y, e_i] \to 0$ in A compactly on Y.

Proof. Let e_i be an approximate unit of A. Take any compact $K \subset Y$. Let Λ be the algebraic convex closure of e_i. Define a bounded linear operator

$$L : A \to C(K,A) : a \mapsto (y \mapsto [y,a]).$$

Our goal is to show the closure $L\Lambda$ in $C(K,A)$ contains zero. Suppose not so that there is $l \in C(K,A)^*$ such that

$$0 < \inf_{v \in \Lambda} \operatorname{Re} l(Lv).$$

We claim that $Le_i \to 0$ weakly in $C(K,A)$. We can show that it converges in

$$\sigma(A \otimes C(K), A^* \odot \operatorname{span} \mathrm{PS}(C(K))).$$

To enhance the convergence, we need to introduce vector measures and require for an approximate unit to be a sequence for applying the bounded convergence theorem!!!! I think we can show this using the measure topology (maybe). □

8.5 (Kasparov technical theorem). Let G be a locally compact σ-compact group. Let J and A_1 be σ-unital G-C*-algebras such that $A_1 \subset M(J)$. Suppose

(i) Δ is a norm separable subset of $M(J)$ such that $[\Delta, A_1] \subset A_1$,

(ii) G, a locally compact σ-compact group, acts on A_1 so that $GA_1 \subset A_1$,

(iii) A_2 is a σ-unital graded C*-subalgebra of $M(J)$ such that $A_1A_2 \subset J$,

(iv) φ is a bounded function $G \to M(J)$ such that $\varphi(G)A_1, A_1\varphi(G) \subset J$ and $g \mapsto \varphi(g)a, a\varphi(g)$ are norm continuous for every $a \in A_1 + J$.

Then, there is $M_2 \in M(J)$ with $0 \le M \le 1$ such that $(1 - M_2)A_1 \subset J$ and

(i) $[\Delta, M_2] \subset J$,

(ii) $GM_2 - M_2 \subset J$,

(iii) $M_2A_2 \subset J$,

(iv) $\varphi(G)M_2, M_2\varphi(G) \subset J$ and $g \mapsto \varphi(g)M_2, M_2\varphi(g)$ are norm continuous.

Proof. □

If G acts on a C*-algebra A, then $\alpha_s^{**} : A^{**} \to A^{**}$ satisfies $\alpha_s^{**} L^\infty(\omega) = L^\infty(\alpha_s^* \omega)$. We can construct a unitary $u_s : L^2(\omega) \to L^2(\alpha_s^*\omega)$? Radon-Nikodym derivatives are mapped by this unitary?

8.6 (Kasparov cycles)**.** Let G be a locally compact σ-compact group and X a locally compact σ-compact Hausdorff space. Let (A, α) and (B, α) be σ-unital G-$C_0(X)$-C*-algebras. A *Kasparov cycle* or *Kasparov module* from (A, α) to (B, β) is a countably generated super-correspondence (E, u) from (A, α) to (B, β) which is G-equivariant over $C_0(X)$, together with an odd adjointable operator $F \in B(E)$ such that

(i) $[F, A] \subset K(E)$ and $FA \subset B(E)$ is G-continuous,

(ii) $(F - F^*)A \cup (F^2 - 1)A \subset K(E)$ and $GF - F \subset K(E)$.

The sets of unitary equivalence classes of Kasparov cycles and the unitary equivalence classes of degenerate Kasparov cycles are denoted by $E^G(A, B)$ and $D^G(A, B)$, where the G-$C_0(X)$-structures are usually omitted in notation.

(a) $E^G(\mathbb{C}, \mathbb{C})$

(b) $E^G(\mathbb{C}, B)$,

(c) $E^G(A, \mathbb{C})$, Fredholm module

8.7 (KK-groups)**.** Let (E, F) be a Kasparov cycle from (A, α) to $(B \otimes C([0, 1]), \beta \otimes \mathrm{id})$. Then, for each $t \in [0, 1]$, we can restrict it to a Kasparov cycle $(E_t, F_t) := (E \otimes_{B \otimes C([0,1])} B, F \otimes 1)$ from (A, α) to (B, β) as follows. First, we check that E_t is a countably generated super-correspondence over G and $C_0(X)$. If we introduce the evaluation maps

$$\mathrm{ev}_t := \mathrm{id} \otimes \delta_t : B \otimes C([0, 1]) \to B,$$
$$\mathrm{ev}_t : E \to E_t : \xi(b \otimes 1) \mapsto \xi b, \qquad \mathrm{ev}_t : B(E) \to B(E_t) : T \mapsto T \otimes 1,$$

which all commute the structures given by G and $C_0(X)$, then the first defines the left action used in the interior tensor product $E_t = E \otimes_{B \otimes C([0,1])} B$, the second is well-defined because the right action is non-degenerate and

$$\|\xi \otimes b\|^2 \le \cdots \le \|\xi(b \otimes 1)\|^2,$$

and the third maps compact operators to compact operators. Then, the two-sided actions and inner product is given by

$$a(\xi(t))b = (a\xi b)(t), \qquad \langle \eta(t), \xi(t) \rangle = \langle \eta, \xi \rangle(t), \qquad T(t)\xi(t) = (T\xi)(t),$$

hence E_t is a super-correspondence from A to B. Second, we check that F_t is a Fredholm operator.

$$[F_t, a], \quad (F_t - F_t^*)a, \quad (F_t^2 - 1)a$$

and group action continuity.

Then, (E, F) is called a *homotopy* between (E_0, F_0) and (E_1, F_1).

compact perturbation,

The set of homotopy classes of Kasparov cycles is denoted by $KK^G(A, B)$, where the actions α and β are usually omitted in notation. The set theoretic issue does not occur because we only consider countably generated correspondences.

(a) $KK^G(A, B)$ is an abelian group given by direct sum.

(b) KK^G is a homotopy invariant bivariant functor.

(c) KK^G preserves finite products. (infinite direct sum for the first argument)

Proof. (a) well-definedness

associativity: clear

identity: clear

inverse: two homotopies; rotation from the sum with opposite to degenerate, trivial homotopy from degenerate to zero.

$C([0, 1], B) \subset B \otimes C([0, 1])$

$C([0, 1], (E \oplus -E) \otimes_B B) \subset (E \oplus -E) \otimes_B (B \otimes C([0, 1]))$

$C([0, 1], B((E \oplus -E) \otimes_B B)) \subset B((E \oplus -E) \otimes_B (B \otimes C([0, 1])))$

We need to check the rotational matrix satisfies the Fredholm conditions

commutativity: clear

(b)

(c)

Suppose $\varphi_0, \varphi_1 : A \rightrightarrows A'$ are homotopic. We calim $\varphi_0^*, \varphi_1^* : KK^G(A', B) \rightrightarrows KK^G(A, B)$ are equal.

Suppose $\psi_0, \psi_1 : B \rightrightarrows B'$ are homotopic. We will show $\psi_{0*}, \psi_{1*} : KK^G(A, B) \rightrightarrows KK^G(A, B')$.

□

8.8 (Kasparov product). Let E_1 be a Hilbert module over B, and E_2 be a super-correspondence from B to C. Let $E_{12} := E_1 \otimes_B E_2$. For $F_2 \in B(E_2)$, we say $F_{12} \in B(E_{12})$ is a F_2*-connection* for E_1 if

$$F_{12} T_{\xi_1} - T_{\xi_1} F_2 \in K(E_2 \oplus E_{12}), \quad T_{\xi_1}^* F_{12} - F_2 T_{\xi_1}^* \in K(E_{12} \oplus E_2), \qquad \xi_1 \in E_1.$$

(How about G-$C_0(X)$-equivariant version?)

Let (A, α), (B, β), and (C, γ) be G-C*-algebras. Let (E_1, F_1) and (E_2, F_2) be Kasparov cycles from (A, α) to (B, β) and from (B, β) to (C, γ), and let $E_{12} := E_1 \otimes_B E_2$. We say a Kasparov cycle (E_{12}, F_{12}) is a *Kasparov product* if F_{12} is a F_2-connection for E_1 and $a^*[1 \otimes F_2, F_{12}]a \geq 0$ in $Q(E_{12})$ for all $a \in A$.

(a)

(half and long exactness?) (extension of k theory and k homology?) (direct sum, pullback, interior tensor product, pushout, exterior tensor product?)

cap product ring structure, $R(G)$-module structures

inverses equivariant imprimitivity bimodules

8.9 (Examples of Kasparov cycles). For a complete Riemannian manifold M, $(L^2(\Lambda T^*M), m, D(1 + D^*D)^{-\frac{1}{2}})$, where $D := d + d^*$ is the Hodge-Dirac operator and D^*D is the Laplace-de Rham operator, is a Kasparov module from $C_0(M)$ to $\mathbb{C}$.

8.2 Extension theory

K-homology: dual algebras, extension theory.

8.10 (Weyl-von Neumann theorem)**.** Let A be a C*-algebra. We say $a, b \in A$ are called *approximately unitarily equivalent*, denoted $a \sim_a b$, if $\mathrm{Ad}\, U(A)(a)$ and $\mathrm{Ad}\, U(A)(b)$ have same closures.

$\pi(U(H)) \subset U(Q(H))$ is proper.

essentially unitarily equivalent: same orbit in $Q(H)$ by $\pi(U(H))$.

If same spectrum in $Q(H)$, then they are essentially unitarily equivalent. We can prove this by the Weyl-von Neumann theorem.

Weyl-von Neumann: every bounded self-adjoint operator on a separable Hilbert space is an arbitrarily small compact perturbation of a diagonal operator($\sigma = \sigma_p$).

8.3 Cuntz-Thomsen picture

stable uniqueness theorem(Lin or Dadarlat-Eilers)

Chapter 9

Part IV

Classification

Chapter 10

Simple nuclear algebras

10.1 AF-algebras

Glimm's classification of UHF algebras Bratteli diagram Elliott's intertwining argument

Separable AF-algebras are classified by pointed ordered K_0.

10.2 Kirchberg-Phillips theorem

10.3 Classifiability

Jiang-Su stability Universal coefficient theorem

Toms-Winter conjecture strongly self-absorbing nuclear dimension

successful in Kirchberg algebras

https://arxiv.org/pdf/2307.06480.pdf

Elliott classification problem Kirchberg-Phillipes theorem

operator K-theory and its pairing with traces

$\mathcal{Z}$-stability, Rosenberg-Schochet universal coefficient theorem

Connes-Haagerup classification of injective factors

Kirchberg: unital simple separable $\mathcal{Z}$-stable algebra is either purely infinte or stably finite. Haagerup, Blackadar, Handelman: unital simple stably finite algebra has a trace.

Glimm: uniformly hyperfinite algebras Murray-von Neumann: hyperfinite II_1 factors

10.4 Inclusions

Chapter 11

Continuous fields

11.1 Fell bundles

11.1 (Banach bundles)**.** A *Banach bundle*, introduced by Fell, which is possibly not locally trivial, is a continuous open surjection $\pi : E \to X$ between topological spaces together with Banach space structure on each fiber $\pi^{-1}(x)$ such that:

(i) the addition $\{(e,e') : \pi(e) = \pi(e')\} \subset E \times E \to E : (e,e') \mapsto e + e'$ is continuous,

(ii) the scalar multiplication $\mathbb{C} \times E \to E : (\lambda, e) \mapsto \lambda e$ is continuous,

(iii) the norm $E \to \mathbb{R}_{\geq 0} : e \mapsto \|e\|$ is continuous,

(iv) the family of subsets
$$\{e \in B : \pi(e) \in U,\ \|e\| < r\}_{U \in N(x), r>0}$$
forms a neighborhood basis of $0 \in \pi^{-1}(x)$ in E.

The forth condition is equivalent to that if $\|e_i\| \to 0$ and $\pi(e_i) \to x$ then $e_i \to 0_x \in \pi^{-1}(x)$.

(a) For a Banach bundle $E \to X$, if X is locally compact Hausdorff and every fiber E_x shares a same finite dimension, then the bundle is locally trivial.

11.2 (Continuous fields of Banach spaces)**.**

span of $a[D,b]$ completion of the span of the gradient of test functions, dual of Borel time-dependent vector field,

For discussion of tangent vectors: sufficiently many absolutely continuous curves?

compact metric space

11.3 (Hilbert bundles)**.** A *Hilbert bundle* is a Banach bundle whose norm function satisfies the parallelogram law.

(a) On a compact X, there is an equivalence between the category of Hilbert $C(X)$-modules and the category of Hilbert bundles over X.

(b) On a compact X, there is an equivalence between the category of algebraically finitely generated Hilbert $C(X)$-modules and the category of classical locally trivial finite-rank complex vector bundle over X. It is due to that finitely generatedness implies the projectivity and the Serre-Swan theorem.

11.2 Dixmier-Douady theory

Fell's condition

A C*-algebra A is called *continuous trace* if the set of all $a \in \mathcal{A}$ such that $\widehat{A} \to \mathbb{R}_{\geq 0} : \pi \mapsto \mathrm{tr}(\pi(a^* a))$ is continuous is dense in A.

Dadarlat-Pennig theory

Coactions and Fell bundles

11.3 C*-dynamics

Izumi-Matui Rokhlin property Evans-Kishimoto intertwining argument dynamical Kirchberg-Phillips Tikusis-White-Winter