POSITIVE HAHN-BANACH SEPARATIONS IN OPERATOR ALGEBRAS

IKHAN CHOI

ABSTRACT.

1. INTRODUCTION

- definition and properties of $f_\varepsilon(t) := (1+\varepsilon t)^{-1}t$
- commutant Radon-Nikodym, relation between $\{\omega' \in M_*^+ : \omega' \le \omega\}$ and $\{h \in \pi(M)'^+ : h \le 1\}$, order preserving linear map
- Mazur lemma

Definition 1.1 (Hereditary subsets)**.** Let E be a partially ordered real vector space. We say a subset F of the positive cone E^+ is *hereditary* if $0 \le x \le y$ in E and $y \in F$ imply $x \in F$, or equivalently $F = (F - E^+)^+$, where $F - E^+$ is the set of all positive elements of E bounded above by an element of F. A $*$-subalgebra B of a $*$-algbera A is called *hereditary* if the positive cone B^+ is a hereditary subset of A^+. We define the *positive polar* of F as the positive part of the real polar

$$F^{r+} := \{x^* \in (E^*)^+ : \sup_{x \in F} x^*(x) \le 1\}.$$

An example that is a non-hereditary closed convex subset of a C*-algebra is $\mathbb{C}1$ in any unital C*-algebra.

2. POSITIVE HAHN-BANACH SEPARATION THEOREMS

Theorem 2.1 (Positive Hahn-Banach separation for von Neumann algebras)**.** *Let M be a von Neumann algebra.*

(1) *If F is a σ-weakly closed convex hereditary subset of M^+, then $F = F^{r+r+}$. In particular, if $x \in M^+ \setminus F$, then there is $\omega \in M_*^+$ such that $\omega(x) > 1$ and $\omega(x') \le 1$ for $x' \in F$.*

(2) *If F_* is a norm closed convex hereditary subset of M_*^+, then $F_* = F_*^{r+r+}$. In particular, if $\omega \in M_*^+ \setminus F_*$, then there is $x \in M^+$ such that $\omega(x) > 1$ and $\omega'(x) \le 1$ for $\omega' \in F_*$.*

Proof. (1) Since the positive polar is represented as the real polar

$$F^{r+} = F^r \cap M_*^+ = F^r \cap (-M^+)^r = (F \cup -M^+)^r = (F - M^+)^r,$$

the positive bipolar can be written as $F^{r+r+} = (F - M^+)^{rr+} = (\overline{F - M^+})^+$ by the usual real bipolar theorem, where the closure is for the σ-weak topology. Because $F = (F - M^+)^+ \subset (\overline{F - M^+})^+$, it suffices to prove the opposite inclusion $(\overline{F - M^+})^+ \subset F$.

Let $x \in (\overline{F - M^+})^+$. Take a net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly, and take a net $y_i \in F$ such that $x_i \le y_i$ for each i. Suppose we may assume that the net x_i is bounded. For sufficiently small ε so that the bounded net x_i has the spectra in $[-(2\varepsilon)^{-1}, \infty)$, we have $f_\varepsilon(x_i) \to f_\varepsilon(x)$ σ-strongly, and hence σ-weakly. On the other hand, by the hereditarity and the σ-weak compactness of F, we may asumme that the bounded net $f_\varepsilon(y_i) \in F$ converges σ-weakly to a point of F by taking a subnet. Then, we have $f_\varepsilon(x) \in F - M^+$ by

$$0 \le f_\varepsilon(x) = \lim_i f_\varepsilon(x_i) \le \lim_i f_\varepsilon(y_i) \in F,$$

thus we have $x \in F$ since $f_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. What remains is to prove the existence of a bounded net $x_i \in F - M^+$ such that $x_i \to x$ σ-strongly.

Define a convex set

$$G := \left\{ x \in \overline{F - M^+} : \begin{array}{c} \text{there is a sequence } x_m \in F - M^+ \\ \text{such that } -2x \le x_m \uparrow x \ \sigma\text{-weakly} \end{array} \right\} \subset M^{sa},$$

where x_m denotes a sequence. In fact, it has no critical issue on allowing x_m to be uncountably indexed. Since we clearly have $F - M^+ \subset G$ and every non-decreasing net with supremum is bounded and σ-strongly convergent, it suffices to show that G, or equivalently its intersection with the closed unit ball by the Krein-Smǔlian theorem, is σ-strongly closed. Let $x_i \in G$ be a net such that $\sup_i \|x_i\| \le 1$ and $x_i \to x$ σ-strongly. For each i, take a sequence $x_{im} \in F - M^+$ such that $-2x_i \le x_{im} \uparrow x_i$ σ-strongly as $m \to \infty$, and also take $y_{im} \in F$ such that $x_{im} \le y_{im}$. Since $\|x_{im}\| \le 2\|x_i\| \le 2$ is bounded, it implies that there is a bounded net x_j in $F - M^+$ such that $x_j \to x$ σ-strongly, and we can choose arbitrarily small $\varepsilon > 0$ such that $\sigma(x_j) \subset [-(2\varepsilon)^{-1}, \infty)$ for all j. Since $f_\varepsilon(x_j)$ converges to $f_\varepsilon(x)$ σ-strongly and $f_\varepsilon(y_j)$ is a bounded net for each $\varepsilon > 0$ so that we may assume that the net $f_\varepsilon(y_j)$ is σ-weakly covergent by taking a subnet, we have $f_\varepsilon(x) \in F - M^+$ by

$$f_\varepsilon(x) = \lim_j f_\varepsilon(x_j) \le \lim_j f_\varepsilon(y_j) \in F,$$

where the limits are in the σ-weak sense. By taking ε as any decreasingly convergent sequence to zero, we have $x \in G$, hence the closedness of G.

(2) It is enough to prove $(\overline{F_* - M_*^+})^+ \subset F_*$, where the closure is for the weak topology or equivalently in norm by the convexity of $F_* - M_*^+$, so we begin our proof by fixing $\omega \in (\overline{F_* - M_*^+})^+$. For a sequence $\omega_n \in F_* - M_*^+$ such that $\omega_n \to \omega$ in norm of M_*, we can take $\varphi_n \in F_*$ such that $\omega_n \le \varphi_n$ for all n. By modifying ω_n into $\omega_n - (\omega_n - \omega)_+ \in F_* - M_*^+$ and taking a rapidly convergent subsequence, we may assume $\omega_n \le \omega$ and $\|\omega - \omega_n\| \le 2^{-n}$ for all n. If we consider the Gelfand-Naimark-Segal representation $\pi : M \to B(H)$ associated to a positive normal linear functional

$$\widehat{\omega} := \sum_n (\omega - \omega_n) + \omega + \sum_n 2^{-n} \left(\frac{[\omega_n]}{1 + \|\omega_n\|} + \frac{\varphi_n}{1 + \|\varphi_n\|} \right)$$

on M with the canonical cyclic vector Ω, we can construct commutant Radon-Nikodym derivatives $h, h_n, k_n \in \pi(M)'$ of $\omega, \omega_n, \varphi_n$ with respect to $\widehat{\omega}$ respectively. Since $-1 \le h_n \le h$ is bounded, $h_n \to h$ in the weak operator topology of $\pi(M)'$. By the Mazur lemma, we can take a net h_i by convex combinations of h_n such that $h_i \to h$ strongly in $\pi(M)'$, and the corresponding linear functionals ω_i and φ_i satisfy $\omega_i \le \varphi_i$ with $\varphi_i \in F_*$ by the convexity of F_* so that $\omega_i \in F_* - M_*^+$. The net h_i can be taken to be a sequence in fact because $\pi(M)'$ is σ-finite by the existence of the separating vector Ω, but it is not necessary in here. For each i and $0 < \varepsilon < 1$, define

$$h_\varepsilon := f_\varepsilon(h), \quad h_{i,\varepsilon} := f_\varepsilon(h_i), \quad k_{i,\varepsilon} := f_\varepsilon(k_i)$$

in $\pi(M)'$, where the functional calculi are well-defined because $-1 \le h_i$ and $0 \le h, k_i$ for all i, and define k_ε as the σ-weak limit of the bounded net $k_{i,\varepsilon}$, which may be assumed to be σ-weakly convergent. Define $\omega_\varepsilon, \omega_{i,\varepsilon}, \varphi_{i,\varepsilon}, \varphi_\varepsilon$ as the corresponding normal linear functionals on M to $h_\varepsilon, h_{i,\varepsilon}, k_{i,\varepsilon}, k_\varepsilon$. Note that $\varphi_i \in F_*$. The hereditarity of F_* and $0 \le \varphi_{i,\varepsilon} \le \varphi_i$ imply $\varphi_{i,\varepsilon} \in F_*$, and the weak closedness of F_* and the weak convergence $\varphi_{i,\varepsilon} \to \varphi_\varepsilon$ in M_* imply $\varphi_\varepsilon \in F^*$. From $\omega_i \le \varphi_i$, we can deduce $0 \le \omega_\varepsilon \le \varphi_\varepsilon$ by considering the operator monotonicity f_ε and taking the weak limit on i. Thus again, the hereditarity of F_* implies $\omega_\varepsilon \in F^*$, and the weak closedness of F_* and the weak convergence $\omega_\varepsilon \to \omega$ in M_* imply $\omega \in F^*$. □

Theorem 2.2 (Positive Hahn-Banach separation for C*-algebras). *Let A be a C*-algebra.*

(1) *If F is a norm closed convex hereditary subset of A^+, then $F = F^{r+r+}$. In particular, if $a \in A^+ \setminus F$, then there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega(a') \le 1$ for $a' \in F$.*

(2) *If F^* is a weakly* closed convex hereditary subset of A^{*+}, then $F^* = (F^*)^{r+r+}$. In particular, if $\omega \in A^{*+} \setminus F^*$, then there is $a \in A^+$ such that $\omega(a) > 1$ and $\omega'(a) \le 1$ for $\omega' \in F^*$.*

Proof. (1) We directly prove the separation without invoking the arguments of positive bipolars. Denote by F^{**} the σ-weak closure of F in the universal von Neumann algebra A^{**}. We first show that F^{**} is hereditary subset of A^{**+}. Suppose $0 \le x \le y$ in A^{**} and $y \in F^{**}$. Then, there is $z \in A^{**}$ such that $x^{\frac{1}{2}} = zy^{\frac{1}{2}}$. Take bounded nets b_i in F and c_i in A such that $b_i \to y$ and $c_i \to z$ σ-strongly* in A^{**} using the Kaplansky density. We may assume the indices of these two nets are same. Since both the multiplication and the involution of a von Neumann algebra on bounded parts are continuous in the σ-strong* topology, and since the square root on a positive bounded interval is a strongly continuous function, we have the σ-strong* limit

$$x = y^{\frac{1}{2}} z^* z y^{\frac{1}{2}} = \lim_i b_i^{\frac{1}{2}} c_i^* c_i b_i^{\frac{1}{2}},$$

so we obtain $x \in F^{**}$ from $b_i^{\frac{1}{2}} c_i^* c_i b_i^{\frac{1}{2}} \in F$. Thus, F^{**} is hereditary in A^{**+}.

Let $a \in A^+ \setminus F$. Observe that we have $a \in A^{**+} \setminus F^{**}$ because if $a \in F^{**}$, then we have a net a_i in F such that $a_i \to a$ σ-weakly in A^{**}, meaning that $a_i \to a$ weakly in A and by the weak closedness of F in A we get a contradiction $a \in F^{**} \cap A = F$. By Theorem 2.1, there is $\omega \in A^{*+}$ such that $\omega(a) > 1$ and $\omega \le 1$ on $F \subset F^{**}$, so it completes the proof.

(2) As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, where the bar notation will always be used for the weak* topology throughout the whole proof. Let

$$G^* := \left\{ \omega \in A^{*sa} : \begin{array}{c} \text{for each } 0 < \varepsilon < (1 + \|\omega\|)^{-4} \text{ we have } \widehat{\omega}_\varepsilon \in A^{*+} \text{ and } \varphi_\varepsilon \in F^* \\ \text{satisfying the following five conditions:} \\ |\omega(a)| \le \varepsilon^{-\frac{1}{4}} \widehat{\omega}(a) \text{ for all } a \in A^+,\ \|\widehat{\omega}_\varepsilon\| \le 1,\ \|\varphi_\varepsilon\| \le \varepsilon^{-1}, \\ \omega_\varepsilon \le \varphi_\varepsilon, \text{ and } \omega_\varepsilon \to \omega \text{ weakly* in } A^* \text{ as } \varepsilon \to 0, \\ \text{where } \omega_\varepsilon := \theta_{\widehat{\omega}_\varepsilon}(f_\varepsilon(\theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega))) \end{array} \right\}.$$

Since the first condition that $|\omega(a)| \le \varepsilon^{-\frac{1}{4}} \widehat{\omega}(a)$ for all $a \in A^+$ implies $\|\theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega)\| \le \varepsilon^{-\frac{1}{4}} < \varepsilon^{-1}$, the functional ω_ε is well-defined. We claim $G^* = \overline{F^* - A^{*+}}$. If the claim is true, then $G^{*+} \subset F^*$ is clear because for $\omega \in G^{*+}$ we have $\omega_\varepsilon \in F^*$ and $\omega_\varepsilon \to \omega$ weakly* in A^*, so this completes the proof.

Since every element $\omega \in G^*$ has a net $\omega_\varepsilon - C\varepsilon^{\frac{1}{2}}\widehat{\omega}_\varepsilon \in F^* - A^{*+}$ converges to ω weakly* as $\varepsilon \to 0$, we have $G^* \subset \overline{F^* - A^{*+}}$. For the other direction, suppose first $\omega \in F^* - A^{*+}$ and take any $\varphi \in F^*$ such that $\omega \le \varphi$. For each $0 < \varepsilon < (1 + \|\omega\|)^{-4}$, let

$$\widehat{\omega}_\varepsilon := \frac{[\omega]}{1 + \|\omega\|} + \frac{\varphi}{(1 + \|\omega\|)(1 + \|\varphi\|)}, \qquad \varphi_\varepsilon := \theta_{\widehat{\omega}_\varepsilon}(f_\varepsilon(\theta_{\widehat{\omega}_\varepsilon}^{-1}(\varphi))).$$

Then, we have

$$|\omega(a)| \le [\omega](a) \le (1 + \|\omega\|)\widehat{\omega}_\varepsilon(a) \le \varepsilon^{-\frac{1}{4}}\widehat{\omega}(a), \qquad a \in A^+$$

and

$$\|\widehat{\omega}_\varepsilon\| \le \frac{\|\omega\|}{1 + \|\omega\|} + \frac{1}{1 + \|\omega\|} \cdot \frac{\|\varphi\|}{1 + \|\varphi\|} \le 1,$$

and if we denote by $\pi_\varepsilon : A^{**} \to B(H_\varepsilon)$ the Gelfand-Naimark-Segal representation associated to $\widehat{\omega}_\varepsilon$ together with the canonical cyclic vector $\Omega_\varepsilon \in H_\varepsilon$, then

$$\|\varphi_\varepsilon\| = \varphi_\varepsilon(1_{A^{**}}) = \langle f_\varepsilon(\theta_{\widehat{\omega}_\varepsilon}^{-1}(\varphi))\Omega_\varepsilon, \Omega_\varepsilon\rangle \le \varepsilon^{-1}\|\Omega_\varepsilon\|^2 = \varepsilon^{-1}\|\widehat{\omega}_\varepsilon\| \le \varepsilon^{-1}.$$

If we let $\omega_\varepsilon := \theta_{\widehat{\omega}_\varepsilon}(f_\varepsilon(\theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega)))$ as in the definition of G^*, then the positivity of $\theta_{\widehat{\omega}_\varepsilon}$ and the operator monotonicity of f_ε give $\omega_\varepsilon \le \varphi_\varepsilon$, and since $\widehat{\omega}_\varepsilon$ is independent of ε so that $f_\varepsilon(\theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega)) \to \theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega)$ weakly in $\pi_\varepsilon(A)'$ as $\varepsilon \to 0$, we have $\omega_\varepsilon \to \omega$ weakly* in A^*. These show that $F^* - A^{*+} \subset G^*$. Thus, it is enough to show G^* is weakly* closed to prove the claim. Let $\omega_i \in G^*$ be a net satisfying $\omega_i \to \omega$ weakly* in A^*, which may be assumed to be bounded by the Krein-Šmulian theorem. Let $\|\omega_i\| \le 1$ without loss of generality. For each $2^{-4} \le \varepsilon < (1 + \|\omega\|)^{-4}$, since we do not need to care about the last fifth convergence condition in this range of ε, we can define $\widehat{\omega}_\varepsilon$ and φ_ε as same as above in the proof of $F^* - A^{*+} \subset G^*$ to make them satisfy the first four conditions. For $0 < \varepsilon < 2^{-4} \le \inf_i(1 + \|\omega_i\|)^{-\frac{1}{4}}$, if we take $\widehat{\omega}_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ for each i following the definition of G^*, then since $\widehat{\omega}_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ are bounded nets for each ε, we may define $\widehat{\omega}_\varepsilon$ and φ_ε as weak* limits in A^* of $\widehat{\omega}_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ by taking a suitable subnet. The second and third conditions for ω automatically follow, and the weak* convergence $\omega_i \to \omega$ in A^* implies the first condition. Before the check for the fourth and fifth conditions, introduce the notations $h_{i,\varepsilon} := \theta_{\widehat{\omega}_{i,\varepsilon}}^{-1}(\omega_i) \in \pi_{i,\varepsilon}(A)'$ and $h_\varepsilon := \theta_{\widehat{\omega}_\varepsilon}^{-1}(\omega) \in \pi_\varepsilon(A)'$, where $\pi_{i,\varepsilon}$ and π_ε are the Gelfand-Naimark-Segal representations of $\widehat{\omega}_{i,\varepsilon}$ and $\widehat{\omega}_\varepsilon$ respectivelty. Since $\|h_{i,\varepsilon}\| \le \varepsilon^{-\frac{1}{4}}$ and $\|h_\varepsilon\| \le \varepsilon^{-\frac{1}{4}}$, for any $\varepsilon > 0$ and i, we have

$$h_{i,\varepsilon} - \varepsilon^{\frac{1}{2}} \le f_\varepsilon(h_{i,\varepsilon}) \le h_{i,\varepsilon}, \qquad h_\varepsilon - \varepsilon^{\frac{1}{2}} \le f_\varepsilon(h_\varepsilon) \le h_\varepsilon,$$

so

$$\begin{aligned} |(\omega_{i,\varepsilon} - \omega_\varepsilon)(a^*a)| &= |\langle f_\varepsilon(h_{i,\varepsilon})\pi_{i,\varepsilon}(a)\Omega, \pi_{i,\varepsilon}(a)\Omega\rangle - \langle f_\varepsilon(h_\varepsilon)\pi_\varepsilon(a)\Omega, \pi_\varepsilon(a)\Omega\rangle| \\ &\le |\langle h_{i,\varepsilon}\pi_{i,\varepsilon}(a)\Omega, \pi_{i,\varepsilon}(a)\Omega\rangle - \langle h_\varepsilon\pi_\varepsilon(a)\Omega, \pi_\varepsilon(a)\Omega\rangle| \\ &\quad + \varepsilon^{\frac{1}{2}}|\|\pi_{i,\varepsilon}(a)\Omega\|^2 - \|\pi_\varepsilon(a)\Omega\|^2| \\ &= |(\omega_i - \omega)(a^*a)| + \varepsilon^{\frac{1}{2}}|(\widehat{\omega}_{i,\varepsilon} - \widehat{\omega}_\varepsilon)(a^*a)|, \qquad a \in A. \end{aligned}$$

If we fix $a \in A^+$, then the fourth condition $\omega_\varepsilon \le \varphi_\varepsilon$ follows from the limit for i on

$$\omega_\varepsilon(a) \le \varphi_{i,\varepsilon}(a) + \widehat{\omega}_{i,\varepsilon}(a) + |(\omega_i - \omega)(a)| + \varepsilon^{\frac{1}{2}}|(\widehat{\omega}_{i,\varepsilon} - \widehat{\omega}_\varepsilon)(a)|.$$

For arbitrary $\delta > 0$, if we choose i such that $|(\omega_i - \omega)(a)| < \delta$ and $|(\widehat{\omega}_{i,\varepsilon} - \widehat{\omega}_\varepsilon)(a)| < \delta$, then taking the limit superior $\varepsilon \to 0$ on the inequality

$$|(\omega_\varepsilon - \omega)(a)| \le |(\omega_{i,\varepsilon} - \omega_i)(a)| + (2 + \varepsilon^{\frac{1}{2}})\delta,$$

so we obtain by letting $\delta \to 0$ the fifth condition, the weak* convergence $\omega_\varepsilon \to \omega$ in A^*. Therefore, $\omega \in G^*$ proves that G^* is weakly* closed, hence the claim $G^* = \overline{F^* - A^{*+}}$ follows. $\square$

Proof. As same as above, our goal is to prove $(\overline{F^* - A^{*+}})^+ \subset F^*$, where the closure notation will always be used for the weak* topology throughout the whole proof. We first prove it when A is commutative. On a commutative C*-algebra, the rectifier function $\mathbb{R} \to \mathbb{R} : t \mapsto \max\{0, t\}$ plays the role of an operator monotone function in the sense that if $\omega_1 \le \omega_2$ are functionals in A^{*sa} then we have $\omega_{1+} \le \omega_{2+}$ for the Jordan decompositions. In this case, we can prove $F^* - A^{*+}$ is weakly* closed. If $\omega_i \in F^* - A^{*+}$ is a bounded net such that $\omega_i \to \omega$ weakly* in A^*, then $\omega_{i+} \in F^*$ is a bounded net so that we may assume $\omega_{i+} \to \omega'$ weakly* in A^* by taking a subnet, and $\omega \le \omega' \in F^*$ implies $\omega \in F^* - A^{*+}$. By the Krein-Šmulian theorem, it completes the proof of $(\overline{F^* - A^{*+}})^+ \subset F^*$ provided that A is commutative.

Now we consider a general C*-algebra A. For any separable C*-subalgebra B of A, define

$$F_B^* := \overline{\{\omega|_B \in B^{*+} : \omega \in F^* - A^{*+}\}}^{\|\cdot\|},$$

which is clearly a norm closed and convex, and we can see that it is hereditary in B^{*+} by the positive Hahn-Banach extension. We first claim $(\overline{F_B^* - B^{*+}})^+ \subset F_B^*$. As a remark, we take a note that the claim implies that F_B^* is weakly* closed, and if A is separable itself, then the proof of the theorem follows by letting $B = A$. Note that the separability of B makes the weak* topology on any bounded part of B^{*sa} metrizable. Consider

$$G_B^* := \overline{F_B^* - B^{*+}}^{\|\cdot\|}.$$

By Theorem 2.1 (2), if we prove G_B^* is weakly* closed, then the claim $(\overline{F_B^* - B^{*+}})^+ \subset F_B^*$ easily follows. To this end, we take a sequence $\omega_{B,n} \in G_B^*$ such that $\omega_{B,n} \to \omega_B$ weakly* in B^* to use the Krein-Šmulian theorem and the separability of B. We may assume $\omega_{B,n} \in F_B^* - B^{*+}$.

$\omega_n \in F^* - A^{*+}$ such that $\omega_n|_B \to \omega|_B$ weakly* in B^*... How to bound φ_n...

there is $y \in B^{**+}$ such that $\omega(y) > 1$ and $\omega'(y) \le 1$ for $\omega' \in F^*$. Take bounded $b_m \in B^+$ such that $b_m \to y$. We may assume $\omega_n \in F^* - A^{*+}$ such that $|(\omega - \omega_n)(b_m)| < (m+n)^{-1}$ for all $m \le n$.

Do we have $\omega_n(b_n) \lesssim \omega_n(y)$?

—

Now let $\omega \in (\overline{F^* - A^{*+}})^+$. Take a net $\omega_i \in F^* - A^{*+}$ and $\varphi_i \in F^*$ such that $\omega_i \to \omega$ weakly* in A^* and $\omega_i \le \varphi_i$ for each i. For each separable C*-subalgebra B of A, we have $\varphi_i|_B \in F_B^*$ and $\omega_i|_B \in F_B^* - B^{*+}$ with the weak* convergence $\omega_i|_B \to \omega|_B$ in B^*, thus we have $\omega|_B \in (\overline{F_B^* - B^{*+}})^+ = F_B^*$ because B is separable. If we consider the increasing net of all separable C*-subalgebras $(B_j)_{j \in J}$ of A, then we have $\omega|_{B_j} \in F_{B_j}^*$ so that there is a net $\omega_{(j,\varepsilon)} \in F^* - A^{*+}$ based on the product directed set $\{(j,\varepsilon) : j \in J,\ \varepsilon > 0\}$ such that $\|\omega_{(j,\varepsilon)}|_{B_j} - \omega|_{B_j}\| < \varepsilon$ for each (j,ε). With this net, as an intermediate step, we will prove that ω belongs to the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$, where A_0^{**} denotes the set of all elements of A^{**} whose left or right support projection is σ-finite. Observing that the left and right support projections of an arbitrary element of a von Neumann algebra are Murray-von Neumann equivalent, we can see A_0^{**} is an algebraic ideal of

A^{**}. Let $x \in A_0^{**+}$ with $\|x\| \le 1$, and let p be the support projection of x. Since p is σ-finite so that on the σ-weakly closed left ideal $A^{**}p$ of A^{**} its bounded part is σ-strongly metrizable, we can take by the Kaplansky density theorem a sequence $b_n \in A^+$ such that $\|b_n\| \le 1$ for all n and $b_n p \to p$ σ-strongly in A^{**}. If we let r be a σ-finite projection of A^{**} such that $b_n p \in rA^{**}r$ for all n, then since the closed unit ball of $rA^{**}r$ is σ-strongly metrizable and $pb_n \to p$ σ-weakly because of the σ-weak continuity of the involution, we can retake with the Mazur lemma a sequence $b_n \in A^+$ by convex combinations such that we still have $\|b_n\| \le 1$ for all n and $pb_n \to p$ σ-strongly, which implies that $b_n p b_n \to p$ σ-weakly. Take a separable C*-subalgebra B of A such that $b_n \in B$ for all n, and let $q := 1_{B^{**}}$. Then, $b_n p b_n \le b_n^2 \le q$ implies $p \le q$ and $xq = x$. Since every separable C*-algebra admits a faithful state, q is σ-finite, so we have a sequence $c_n \in B$ such that $\|c_n\| \le 1$ and $c_n \to q$ σ-strongly. Using the Kaplansky density theorem and the σ-finiteness of q again, take a sequence $a_n \in A^+$ such that $\|a_n\| \le 1$ for all n and $a_n q \to x$ σ-strongly. Then, $a_n c_n \to x$ σ-strongly. If we choose j_0 such that $a_n c_n \in B_{j_0}$ for all n, then for each $j \succ j_0$ the last term in the inequality

$$|(\omega_{(j,\varepsilon)} - \omega)(x)| \le |(\omega_{(j,\varepsilon)} - \omega)(x - a_n c_n)| + |(\omega_{(j,\varepsilon)} - \omega)(a_n c_n)|$$

is uniformly estimated by ε because the sequence $a_n c_n \in B_j$ is uniformly bounded by one, so we obtain $\lim_{(j,\varepsilon)}(\omega_{(j,\varepsilon)} - \omega)(x) = 0$. This proves that ω is contained in the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$.

Suppose now $\omega \notin F^*$. Then, there exists $x \in A^{**+}$ such that $\omega(x^2) > 1$ and $\omega'(x^2) \le 1$ for all $\omega' \in F^*$ by Theorem 2.1 (2). Let $\{p_i\}_{i \in I}$ be a maximal orthogonal family of σ-finite projections of the von Neumann algebra A^{**} whose sum is the support projection of x. If we consider order-preserving bounded linear maps $\Gamma : c_0(I) \to A^{**}$ and $\Gamma^* : A^* \to \ell^1(I)$ given by

$$\Gamma((c_i)_{i\in I}) := \sum_i c_i x p_i x, \qquad \Gamma^*(\omega') := (\omega'(x p_i x))_{i \in I},$$

then these maps are in dual, and Γ is extended to the linear map $\Gamma^{**} : \ell^\infty(I) \to A^{**}$ continuous with respect to weak* topologies. We have $\Gamma(c_0(I)) \subset A_0^{**}$ due to the fact that each element of $c_0(I)$ has at most countably many non-zero components. Since ω is an element of the $\sigma(A^*, A_0^{**})$-closure of $F^* - A^{*+}$, we have $\Gamma^*(\omega) \in \overline{\Gamma^*(F^* - A^{*+})}$, where the closure is taken in the weak* topology of $\ell^1(I)$. Then, beacuse

$$\left(\overline{\Gamma^*(F^* - A^{*+})}\right)^+ \subset \left(\overline{\Gamma^*(F^*) - \ell^1(I)^+}\right)^+ \subset \left(\overline{\overline{\Gamma^*(F^*)} - \ell^1(I)^+}\right)^+ \subset \overline{\Gamma^*(F^*)},$$

where the last inclusion follows from that $c_0(I)$ is a commutative C*-algebra, we have $\Gamma^*(\omega) \in \overline{\Gamma^*(F^*)}$. For any $\delta > 0$, if we choose $c \in c_0(I)^+$ such that $c \le 1$ and $|\langle 1_{\ell^\infty(I)} - c, \Gamma^*(\omega)\rangle| < \delta$ using the Kaplansky density theorem, and choose $\omega' \in F^*$ such that $|\langle c, \Gamma^*(\omega) - \Gamma^*(\omega')\rangle| < \delta$, then we get a contradiction

$$\begin{aligned} 1 < \omega(x^2) = \langle 1_{\ell^\infty(I)}, \Gamma^*(\omega)\rangle &\approx_\delta \langle c, \Gamma^*(\omega)\rangle \\ &\approx_\delta \langle c, \Gamma^*(\omega')\rangle \le \langle 1_{\ell^\infty(I)}, \Gamma^*(\omega')\rangle = \omega'(x^2) \le 1, \end{aligned}$$

where the relation symbol $\approx_\delta$ means that the difference converges to zero as $\delta \to 0$. Therefore, we finally have $\omega \in F^*$. □

3. Applications to weight theory

The positive Hahn-Banach separation theorem implies a generalization of the Combes theorem on subadditive normal weights.

Corollary 3.1. *Let M be a von Neumann algebra. Then, there is a one-to-one correspondence*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{subadditive normal} \\ \text{weights of } M \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{hereditary closed} \\ \text{convex subsets of } M_*^+ \end{array} \right\} \\ \varphi & \longmapsto & \{\omega \in M_*^+ : \omega \leq \varphi\} \end{array}$$