Operator Algebra Seminar Note I

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Contents

Acknowledgement

This note has been written based on the first-year graduate seminar presented at the University of Tokyo in the 2023 Spring semester. Each seminar was delivered for 105 minutes. The main reference for this seminar was Brown-Ozawa, and detailed gaps were filled with the aids of other books such as Takesaki, Murphy, and Paulsen whenever required.

1 April 14

1.1 Completely positive maps

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *completely positive* (c.p.) if the inflation $\varphi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B}) : [a_{ij}] \mapsto [\varphi(a_{ij})]$ is positive for each $n \geq 1$.

Remark 1.2. For the positivity in matrix algebras, the following equivalent statements are useful.

(a) $[a_{ij}] \in M_n(\mathcal{A})$ is positive.

(b) $[a_{ij}] = [b_{ij}]^*[b_{ij}] = [b_{ji}^*][b_{ij}] = [\sum_k b_{ki}^* b_{kj}]$ for some $[b_{ij}] \in M_n(\mathcal{A})$.

(c) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for a faithful representation $\pi : \mathcal{A} \to B(H)$.

(d) $\sum_{i,j} \langle \pi(a_{ij})\xi_j, \xi_i \rangle_H \geq 0$ for $[\xi_i] \in H^n$, for every representation $\pi : \mathcal{A} \to B(H)$.

Example 1.3.

(a) A $*$-homomorphism is c.p.

(b) A state is c.p.

(c) A conjugation $B(\widehat{H}) \to B(H) : a \mapsto V^*aV$ is c.p. for every bounded linear $V : H \to \widehat{H}$.

(d) The transpose $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is not c.p.

(e) The convex combination, composition, restriction of c.p. maps is c.p.

Proof. (a) A $*$-homomorphism is positive, and its inflations are all $*$-homomorphisms.

(b) Let $\rho : \mathcal{A} \to \mathbb{C}$ be a state. If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(\mathcal{A})_+$, then we have for $[x_i] \in \ell_2^n$ that

$$\sum_{i,j} \langle \rho(a_{ij})x_j, x_i \rangle_{\mathbb{C}} = \sum_{i,j} \overline{x_i}\rho(a_{ij})x_j = \rho(\sum_{i,j,k} \overline{x_i} b_{ki}^* b_{kj} x_j) = \sum_k \rho((\sum_i b_{ki}x_i)^*(\sum_j b_{kj}x_j)) \geq 0.$$

(c) If $[a_{ij}] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(\widehat{H}))_+$, then we have for $[\xi_i] \in H^n$ that

$$\sum_{i,j} \langle V^* a_{ij} V\xi_j, \xi_i \rangle = \sum_{i,j,k} \langle b_{kj}V\xi_j, b_{ki}V\xi_i \rangle = \sum_k \langle \sum_j b_{kj}V\xi_j, \sum_i b_{ki}V\xi_i \rangle \geq 0.$$

(d) We have a counterexample for $M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The former has an eigenvalues $\{2, 0\}$, and the latter has $\{\pm 1\}$.

(e) Clear. □

Theorem 1.4 (Stinespring dilation)**.** *Let $\mathcal{A}$ be a unital C*-algebra and $\varphi : \mathcal{A} \to B(H)$ be a c.p. map. Then, there is a representation $\pi : \mathcal{A} \to B(\widehat{H})$ and a bounded linear operator $V : H \to \widehat{H}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & B(H) \\ B(\widehat{H}) \arrow[ur, "V^* \cdot V"'] & \end{tikzcd}$$

Proof. Define a sesquilinear form on the algebraic tensor product $\mathcal{A}\odot H$ as

$$\langle\sum_j a_j\otimes\xi_j,\sum_i b_i\otimes\eta_i\rangle:=\sum_{i,j}\langle\varphi(b_i^*a_j)\xi_j,\eta_i\rangle.$$

It is positive since

$$\sum_{i,j}\langle a_i^*a_j\xi_j,\xi_i\rangle=\sum_{i,j}\langle a_j\xi_j,a_i\xi_i\rangle=\|\sum_i a_i\xi_i\|^2\geq 0$$

implies

$$\langle\sum_j a_j\otimes\xi_j,\sum_i a_i\otimes\xi_i\rangle=\sum_{i,j}\langle\varphi(a_i^*a_j)\xi_j,\xi_i\rangle\geq 0.$$

Taking quotient by the left kernel N and completion, we obtain a hilbert space $\widehat{H}:=(\mathcal{A}\odot H/N)^-$.

Define $\pi:\mathcal{A}\to B(\widehat{H})$ such that

$$\pi(a)(b\otimes\xi+N):=ab\otimes+N,$$

and define $V:H\to\widehat{H}$ such that

$$V\xi:=1_{\mathcal{A}}\otimes\xi+N.$$

Then for any $\xi,\eta\in H$,

$$\langle V^*\pi(a)V\xi,\eta\rangle=\langle\pi(a)(1_{\mathcal{A}}\otimes\xi+N),1_{\mathcal{A}}\otimes\xi+N\rangle=\langle a_{\mathcal{A}}\otimes\xi+N,1_{\mathcal{A}}\otimes\xi+N\rangle=\langle\varphi(a)\xi,\eta\rangle.\quad\square$$

Remark 1.5.

(a) If φ is unital, then V is an isometry since $V^*V=V^*\pi(1)V=\varphi(1)=1$.

(b) If φ is unital and $H=\mathbb{C}$, then it is just the GNS-construction with the cyclic vector $V1_{\mathbb{C}}$.

(c) If $\varphi:\mathcal{A}\to\mathcal{B}$ is c.p., then by embedding $\mathcal{B}$ into $B(H)$ and applying the Stinespring dilation,

$$\|\varphi(a)\|=\|V^*\pi(a)V\|\leq\|V\|\|a\|\|V\|=\|a\|\|V^*V\|=\|a\|\|\varphi(1)\|$$

implies $\|\varphi\|\leq\|\varphi(1)\|$, hence $\|\varphi\|=\|\varphi(1)\|$.

(d) It has a physical meaning: a unital completely positive map is called quantum channel or quantum operation in quantum information theory. They are interpreted as an evolution in open quantum system, and taking $\widehat{H}$ means introducing a closed ambient system in which unitary evolution occurs.

Theorem 1.6 (Completely positive maps for matrix algebras)**.** *Let $\mathcal{A}$ be a C^*-algebra. Let $e_i\in\ell_2^n$ be standard orthonormal basis and let $e_{ij}=e_i\otimes e_j=|e_i\rangle\langle e_j|\in M_n(\mathbb{C})$ be unit matrix elements.*

(a) *There is a 1-1 correspondence*

$$\mathrm{CP}(M_n(\mathbb{C}),\mathcal{A})\to M_n(\mathcal{A})_+:\psi\mapsto[\psi(e_{ij})].$$

(b) *Let $\mathcal{A}$ be unital. There is a 1-1 correspondence*

$$\mathrm{CP}(\mathcal{A},M_n(\mathbb{C}))\to M_n(\mathcal{A})_+^*:\varphi\mapsto(\widehat{\varphi}:[a_{ij}]\mapsto\sum_{i,j}\langle\varphi(a_{ij})e_j,e_i\rangle).$$

Proof. (a) Fix $\mathcal{A}\to B(H)$ a faithful representation and just write $\mathcal{A}\subset B(H)$.

Suppose $\psi:M_n(\mathbb{C})\to\mathcal{A}$ is a c.p. map. Identify $M_n(\mathbb{C})=B(\ell_2^n)$. Since $[e_{ij}]\in M_n(B(\ell_2^n))_+$ is positive because

$$\sum_{i,j}\langle e_{ij}\xi_j,\xi_i\rangle=\sum_{i,j}\langle e_j,\xi_j\rangle\langle\xi_i,e_i\rangle=|\sum_i\langle e_i,\xi_i\rangle|^2\geq 0,\qquad\forall[\xi_i]\in(\ell_2^n)^n,$$

it follows that $[\psi(e_{ij})] \in M_n(\mathcal{A})_+$ by the complete positivity of ψ.

Conversely, let $[\psi(e_{ij})] = [\sum_k b_{ki}^* b_{kj}] \in M_n(B(H))_+$, For $T = [t_{ij}] \in M_n(\mathbb{C})$ and $\xi, \eta \in H$, write

$$\begin{aligned}
\langle \psi(T)\xi, \eta\rangle &= t_{ij}\langle \psi(e_{ij})\xi, \eta\rangle \\
&= t_{ij}\langle b_{kj}\xi, b_{ki}\eta\rangle \\
&= t_{ij}\delta_{kl}\langle b_{lj}\xi, b_{ki}\eta\rangle \\
&= \langle Te_j, e_i\rangle\langle e_l, e_k\rangle\langle b_{lj}\xi, b_{ki}\eta\rangle \\
&= \langle (T\otimes 1\otimes 1)(e_j\otimes e_l\otimes (b_{lj}\xi)), (e_i\otimes e_k\otimes (b_{ki}\eta))\rangle.
\end{aligned}$$

The summation symols are omitted in each row. Then, if we define

$$V : H \to \ell_2^n \otimes \ell_2^n \otimes H : \xi \mapsto \sum_{i,k} e_i \otimes e_k \otimes (b_{ki}\eta),$$

we have an expression

$$\langle \psi(T)\xi, \eta\rangle = \langle V^*(T\otimes 1\otimes 1)V\xi, \eta\rangle,$$

which implies that ψ is c.p. because $T \mapsto T \otimes 1_{\ell_2^n} \otimes 1_H$ is a $*$-homomorphism.

(b) Suppose $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ is a c.p. map. Then, $\widehat{\varphi}$ is positive since $[a_{ij}] \in M_n(\mathcal{A})_+$ implies

$$\widehat{\varphi}([a_{ij}]) = \sum_{i,j}\langle \varphi(a_{ij})e_j, e_i\rangle \geq 0.$$

Conversely, let $\widehat{\varphi} \in M_n(\mathcal{A})_+^*$. By the GNS-construction, we have a cyclic representation $\pi : M_n(\mathcal{A}) \to B(H)$ with a cyclic vector $\psi \in H$ such that

$$\widehat{\varphi}([a_{ij}]) = \langle \pi([a_{ij}])\psi, \psi\rangle.$$

For $\xi = \sum_j \xi_j e_j, \eta = \sum_i \eta_i e_i \in \ell_2^n$, write

$$\begin{aligned}
\langle \varphi(a)\xi, \eta\rangle &= \sum_{i,j}\langle \varphi(a)\xi_j e_j, \eta_i e_i\rangle = \sum_{i,j}\langle \varphi(\overline{\eta_i} a\xi_j)e_j, e_i\rangle \\
&= \widehat{\varphi}([\overline{\eta_i} a\xi_j]) = \langle \pi([\overline{\eta_i} a\xi_j])\psi, \psi\rangle = \langle \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}]^*[a][\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \psi\rangle \\
&= \langle \pi([a])\pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi, \pi([\delta_{ij}\eta_i 1_{\mathcal{A}}])\psi\rangle.
\end{aligned}$$

If we define

$$V : \ell_2^n \to H : \xi \mapsto \pi([\delta_{ij}\xi_j 1_{\mathcal{A}}])\psi,$$

then

$$\langle \varphi(a)\xi, \eta\rangle = \langle V^*\pi([a])V\xi, \eta\rangle,$$

so φ is c.p. since $\mathcal{A} \to M_n(\mathcal{A}) : a \mapsto [a]$ is a $*$-homomorphism. □

Theorem 1.7 (Arveson extension). *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras such that $1_{\mathcal{A}} \in \mathcal{B}$. Then, every c.p. map $\varphi : \mathcal{B} \to B(H)$ has an norm-preserving c.p. extension $\widetilde{\varphi} : \mathcal{A} \to B(H)$, i.e. $\|\widetilde{\varphi}\| = \|\varphi\|$.*

Proof. Let p_α be the net of projections of finite rank n_α in $B(H)$ with the image V_α, which strongly converges to id_H. Fix α temporarily and let $\varphi_\alpha := p_\alpha\varphi|_{V_\alpha} : \mathcal{B} \to B(V_\alpha)$. Choosing an any orthonormal basis of each V_α, we can rewrite as $\varphi_\alpha : \mathcal{B} \to M_{n_\alpha}(\mathbb{C})$. By the above theorem, we have the associated linear functional $\widehat{\varphi}_\alpha \in M_{n_\alpha}(\mathcal{B})$. Then, the Hahn-Banach extension provides an extension $(\widehat{\varphi}_\alpha)^\sim \in M_{n_\alpha}(\mathcal{A})$, and we can define $\widetilde{\varphi}_\alpha : \mathcal{A} \to M_{n_\alpha}(\mathbb{C})$ as the associated completely positive map. Via the identification $B(V_\alpha) = M_{n_\alpha}(\mathbb{C})$ we used to write $\varphi_\alpha : \mathcal{B} \to M_{n_\alpha}(\mathbb{C})$, we have $\widetilde{\varphi}_\alpha : \mathcal{A} \to B(V_\alpha)$. We can check $\widetilde{\varphi}_\alpha$ actually extends φ_α, i.e. $\widetilde{\varphi}_\alpha(b) = \varphi_\alpha(b)$ for $b \in \mathcal{B}$, by putting $[b\delta_{ik}\delta_{jl}]_{i,j} \in M_{n_\alpha}(\mathcal{B})$ and comparing matrix components for each k, l.

Since $\|\widetilde{\varphi}_\alpha\| = \|\widetilde{\varphi}_\alpha(1)\| = \|\varphi_\alpha(1)\| = \|\varphi_\alpha\| \le \|\varphi\|$, the net $\widetilde{\varphi}_\alpha$ is bounded in $B(\mathcal{A},B(H))$. The norm-closed unit ball is compact in the point-σ-weak topology $\sigma(B(\mathcal{A},B(H)),\mathcal{A}\odot L^1(H))$ because it is coarser than the weak* topology $\sigma(B(\mathcal{A},B(H)),\mathcal{A}\widehat{\otimes}_\pi L^1(H))$. By taking a convergent subnet, we have a limit point $\widetilde{\varphi}:\mathcal{A}\to B(H)$. It is easily seen to be completely positive and extend φ, and satisfies $\|\varphi\| = \|\varphi(1)\| = \|\widetilde{\varphi}(1)\| = \|\widetilde{\varphi}\|$. □

1.2 Enveloping von Neumann algebras

Theorem 1.8 (Sherman-Takeda)**.** *Let $\mathcal{A}$ be a C^*-algebra and $\pi:\mathcal{A}\to B(H)$ a faithful representation. Here we can obtain an linear map $\widetilde{\pi}:\mathcal{A}^{**}\to\pi(\mathcal{A})''$ by taking bitranspose for $\pi:\mathcal{A}\to(\pi(\mathcal{A})'',\sigma w)$.*

(a) *$\widetilde{\pi}$ is an isometric isomorhpism (w.r.t. norms), and is an homeomorphism (w.r.t. weak*-topologies)*

(b) *$\mathcal{A}^{**}$ enjoys a universal property in the sense that for every $*$-homomorphism $\varphi:\mathcal{A}\to\mathcal{M}$ to a von Neumann algebra $\mathcal{M}$, there exists a unique σ-weakly continuous extension $\widetilde{\varphi}:\mathcal{A}^{**}\to\mathcal{M}$ of φ.*

We will always see the bidual $\mathcal{A}^{**}$ as a von Neumann algebra.

Proof. (a) Consider

$$\pi:\mathcal{A}\to(\pi(\mathcal{A})'',\sigma w),\qquad \pi^*:\pi(\mathcal{A})''_*\to\mathcal{A}^*,\qquad \widetilde{\pi}:=\pi^{**}:\mathcal{A}^{**}\to\pi(\mathcal{A})'',$$

where $\pi(\mathcal{A})''_*$ denotes the set of σ-weakly continuous(=normal) linear functionals on $\pi(\mathcal{A})''$. Note that π is isometric and has dense range. It implies that π^* is surjective and injective. In fact, π^* is isometric because for $l\in\pi(\mathcal{A})''_*$ we have by the density that

$$\|\pi^*(l)\| = \sup_{\substack{\|a\|=1\\ a\in\mathcal{A}}} |l(\pi(a))| = \sup_{\substack{\|b\|=1\\ b\in\pi(\mathcal{A})''}} |l(b)| = \|l\|.$$

Then, the claim for π^{**} is now clear.

(b) We can define $\widetilde{\varphi}$ as the bitranspose of $\varphi:\mathcal{A}\to(\mathcal{M},\sigma w)$ as in the part (a), and it is a unique extension because $\mathcal{A}$ is σ-weakly dense in $\mathcal{A}^{**}$. □

Theorem 1.9 (Tomiyama)**.** *Let $\mathcal{B}\subset\mathcal{A}$ be C^*-algebras. Let $\varphi:\mathcal{A}\to\mathcal{B}$ be a* conditional expectation*, i.e. a contractive idempotent linear map.*

(a) *φ is B-bimodule map.*

(b) *φ is completely positive.*

Proof. Since each conclusion of (a) and (b) still holds for restriction, we may assume $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras by thinking of the bitranspose $\varphi^{**}:\mathcal{A}^{**}\to\mathcal{B}^{**}$.

(a) Since the linear span of projections is σ-weakly dense in a von Neumann algebra, we are enough to show $p\varphi(a)=\varphi(pa)$ and $\varphi(ap)=\varphi(a)p$ for any projection $p\in\mathcal{B}$.

Let $p\in\mathcal{B}$ be a projection and let $a\in\mathcal{A}$. Note that we have

$$p\varphi(a) = pp\varphi(a) = p\varphi(p\varphi(a))$$

and

$$(a-pa)^*(p\varphi(a-pa)) = (p\varphi(a-pa))^*(a-pa) = 0.$$

Then,

$$\begin{aligned}(1+t)^2\|p\varphi(a-pa)\|^2 &= \|p\varphi(a-pa)+tp\varphi(a-pa)\|^2\\ &= \|p\varphi((a-pa)+tp\varphi(a-pa))\|^2\\ &\le \|(a-pa)+tp\varphi(a-pa)\|^2\\ &= \|a-pa\|^2+t^2\|p\varphi(a-pa)\|^2\end{aligned}$$

implies $p\varphi(a-pa)=0$ by letting $t\to\infty$. Putting $1_{\mathcal{B}}-p$ and $1_{\mathcal{B}}$ instead of p, we obtain $(1_{\mathcal{B}}-p)\varphi(a-1_{\mathcal{B}}a+pa)=0$ and $\varphi(a-1_{\mathcal{B}}a)=0$, so

$$p\varphi(a)=p\varphi(pa)=\varphi(pa).$$

Similarly, we can show $\varphi(a-ap)p=0$ and $\varphi(ap)(1-p)=0$, we are done.

(b) Let $[a_{ij}]\in M_n(\mathcal{A})_+$. Let $\pi:\mathcal{B}\to B(H)$ be a cyclic representation with a cyclic vector ψ. Then, $[\xi_i]\in H^n$ can be replaced to $[\pi(b_i)\psi]$, so we can check the positivity of inflations φ_n as

$$\sum_{i,j}\langle\pi(\varphi(a_{ij}))\pi(b_j)\psi,\pi(b_i)\psi\rangle=\langle\pi(\varphi(\sum_{i,j}b_i^*a_{ij}b_j))\psi,\psi\rangle\geq 0,$$

because it follows $\sum_{i,j}b_i^*a_{ij}b_j\geq 0$ by the positivity of a_{ij} from

$$\langle\pi_{\mathcal{A}}(\sum_{i,j}b_i^*a_{ij}b_j)\xi,\xi\rangle=\sum_{i,j}\langle\pi_{\mathcal{A}}(a_{ij})\pi_{\mathcal{A}}(b_j)\xi,\pi_{\mathcal{A}}(b_i)\xi\rangle\geq 0,$$

where $\pi_{\mathcal{A}}$ is any representation of $\mathcal{A}$. □

Theorem 1.10 (Sakai)**.** *Suppose $\mathcal{A}$ is a C^*-algebra which admits a predual F.*

(a) *There is an injective $*$-homomorphism $\pi:\mathcal{A}\to\mathcal{A}^{**}$ with weakly* closed image.*

(b) *π is a topological embedding w.r.t. $\sigma(\mathcal{A},F)$ and $\sigma(\mathcal{A}^{**},\mathcal{A}^*)$.*

(c) *The predual F is unique in $\mathcal{A}^*$.*

In particular, there is a faithful representation $\mathcal{A}\to B(H)$ whose image is (σ-)weakly closed.

Proof. (a) By taking the adjoint for the inclusion $i:F\hookrightarrow\mathcal{A}^*$, we have a conditional expectation $\varepsilon:\mathcal{A}^{**}\twoheadrightarrow\mathcal{A}$. Its kernel is a $\mathcal{A}$-bimodule, and by the σ-weak density of $\mathcal{A}$ in $\mathcal{A}^{**}$ and the continuity of ε between weak* topologies, so it is in fact a $\mathcal{A}^{**}$-bimodule, which means it is a σ-weakly closed ideal of $\mathcal{A}^{**}$. Thus we have a central projection $z\in\mathcal{A}^{**}$ such that $\ker\varepsilon=(1-z)\mathcal{A}^{**}$.

Define $\pi:\mathcal{A}\to\mathcal{A}^{**}$ such that $\pi(a):=za$. It is clearly a $*$-homomorphism. The injectivity follows from $a=\varepsilon(a)=\varepsilon(za)$ for $a\in\mathcal{A}$. The image is weakly* closed because $\varepsilon(x-\varepsilon(x))=0$ implies $z(x-\varepsilon(x))=0$ for $x\in\mathcal{A}^{**}$ so that $z\mathcal{A}^{**}=z\mathcal{A}$.

(b) Since $\langle a,f\rangle=\langle\varepsilon(za),f\rangle=\langle za,f\rangle$ for $a\in\mathcal{A}$ and $f\in F$, in which the second equality holds by the definition of ε, it is enough to show $\sigma(z\mathcal{A},\mathcal{A}^*)=\sigma(z\mathcal{A},F)$.

For $l\in\mathcal{A}^*$, we claim there exists f such that $\langle za,l\rangle=\langle za,f\rangle$. Define $\widetilde{l}\in\mathcal{A}^*$ such that $\langle x,\widetilde{l}\rangle:=\langle zx,l\rangle$ for $x\in\mathcal{A}^{**}$. Then, $\langle zx,l\rangle=\langle z^2x,l\rangle=\langle zx,\widetilde{l}\rangle$ for $x\in\mathcal{A}^{**}$. Suppose $\widetilde{l}\notin F$. Because F is closed in $\mathcal{A}^*$, there is $x\in\mathcal{A}^{**}$ such that $\langle x,\widetilde{l}\rangle\neq 0$ and $\langle x,f\rangle=0$ for all $f\in F$ by the Hahn-Banach extension. Then, $0=\langle x,f\rangle=\langle x,i(f)\rangle=\langle\varepsilon(x),f\rangle$ implies $\varepsilon(x)=0$ so that $zx=0$, which leads a contradiction $\langle x,\widetilde{l}\rangle=\langle zx,l\rangle=0$, so we have $\widetilde{l}\in F$.

(c) If closed subspaces F_1 and F_2 of $\mathcal{A}^*$ are preduals of $\mathcal{A}$, then $\sigma(\mathcal{A},F_1)=\sigma(\mathcal{A},F_2)$ by the part (b). If $l\in F_1$, which is obviously continuous on $\sigma(\mathcal{A},F_1)$, and the continuity in $\sigma(\mathcal{A},F_2)$ implies that l is contained in a linear span of some finitely many elements of F_2, hence $F_1\subset F_2$. □

2 May 12

2.1 Nuclear maps

Definition 2.1. A linear map $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras is called *nuclear* if it is a limit of finite-rank contractive c.p.(c.c.p.) maps in the point-norm topology. Equivalently, by the following lemma, there is a net of pairs of c.c.p. maps $\varphi_\alpha : \mathcal{A} \to M_{n_\alpha}(\mathbb{C})$ and $\psi_\alpha : M_{n_\alpha}(\mathbb{C}) \to \mathcal{B}$ such that $\|\theta(a) - \psi_\alpha \circ \varphi_\alpha(a)\| \to 0$ for each $a \in \mathcal{A}$.

If $\mathcal{B}$ is a con Neumann algebra, θ is called *weakly nuclear* if it is a limit of finite-rank c.c.p. maps in the point-σ-weak topology.

Lemma 2.2. *A c.c.p. map $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras is of finite-rank iff there are c.c.p. maps $\varphi : \mathcal{A} \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to \mathcal{B}$ for some n such that $\theta = \psi \circ \varphi$. In Brown-Ozawa, a finite-rank c.c.p. map is called a factorable map.*

Proof. ($\Leftarrow$) Clear. ($\Rightarrow$) By the structure theorem of finite-dimensional C*-algebras, we have $\operatorname{im}\theta \cong \bigoplus_{i=1}^m M_{n_i}(\mathbb{C})$, so for $n = \sum_{i=1}^m n_i$ there is a unital embedding $\operatorname{im}\theta \hookrightarrow M_n(\mathbb{C})$ and conditional expectation $M_n(\mathbb{C}) \to \operatorname{im}\theta : T \mapsto \sum_{i=1}^m P_i T P_i$, where P_i denotes the projection on the image of $M_{n_i}(\mathbb{C})$. Now we are done. (In fact, such a conditional expectation also exists for unital subalgebras beetween von Neumann algebras.) □

Proposition 2.3 (Local property)**.** *Let $\theta : \mathcal{A} \to \mathcal{B}$ be a linear map between C*-algebras. If the restriction of θ on any finite-dimensional subspace of $\mathcal{A}$ is nuclear, then θ is nuclear.*

Proof. □

Proposition 2.4 (Weak approximations)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras, and $\mathcal{M} \subset B(H)$ a von Neumann algebra.*

(a) *$\theta : \mathcal{A} \to \mathcal{B}$ is nuclear if there is a net $\mathcal{A} \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{B}$ such that*

$$\lim_\alpha \langle \theta(a) - \psi_\alpha \circ \varphi_\alpha(a), l \rangle = 0 \qquad a \in \mathcal{A},\ l \in \mathcal{B}^*.$$

(b) *$\theta : \mathcal{A} \to \mathcal{M}$ is weakly nuclear if there is a net $\mathcal{A} \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{M}$ such that*

$$\lim_\alpha \langle (\theta(a) - \psi_\alpha \circ \varphi_\alpha(a))\xi, \xi \rangle = 0 \qquad a \in \mathcal{A},\ \xi \in H.$$

Proof. (a) By applying the Hahn-Banach extension for each $a \in \mathcal{A}$, we can show the closures of a convex set is same with respect to the point-norm topology and the point-$\sigma(\mathcal{B}, \mathcal{B}^*)$-topology. Thus it suffices to show that the set of finite-rank c.c.p. maps is convex.

Let $\mathcal{A} \xrightarrow{\psi_i} M_{n_i}(\mathbb{C}) \xrightarrow{\varphi_i} \mathcal{B}$ be c.c.p. maps for $i \in \{0, 1\}$. Then, we have a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "(1-t)\psi_0\circ\varphi_0 + t\psi_1\circ\varphi_1"] \arrow[d] & & \mathcal{B} \\
\mathcal{A} \oplus \mathcal{A} \arrow[r, "\varphi_0 \oplus \varphi_1"'] & M_{n_0}(\mathbb{C}) \oplus M_{n_1}(\mathbb{C}) \arrow[r, "((1-t)\psi_0)\oplus(t\psi_1)"'] & \mathcal{B} \oplus \mathcal{B} \arrow[u]
\end{tikzcd}$$

which is commutative, so we are done.

(b) Fix $a \in \mathcal{A}$. Note that are net is bounded. Since the unit ball is compact in σ-weak topology and hence in the weak operator topology, we are enough to verify the convergence of $\psi_\alpha \circ \varphi_\alpha$ in the weak operator topology. Using the polarization identity, the claim holds. □

nonunital technicalities

2.2 Examples of nuclear C*-algebras

C*-subalgebra of a nuclear C*-algebra may not be nuclear. C*-subalgebra of a exact C*-algebra is exact. injective limit of nuclear C*-algebras is nuclear. $M_n(\mathcal{A})$ is nuclear if $\mathcal{A}$ is nuclear.

Theorem 2.5 (Effros-Lance)**.** *If $\mathcal{A}^{**}$ is semidiscrete, then $\mathcal{A}$ is nuclear. (The converse also holds)*

Proof. Since the set of finite-rank c.c.p. maps is convex, and since the closures of a convex set are same in the norm and weak topologies on a Banach space, □

Theorem 2.6. *An abelian C*-algebra is nuclear.*

3 May 22