Computational Mathematics

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3.1 Elliptic equations

3.1 (1D Poisson equation)**.** Consider the following boundary value problem:

$$\begin{cases} -u''(x) = f(x), & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

We discretize it by $(u_j)_{j=0}^{N}$ such that $hN = 1$ and

$$\begin{cases} -\dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, & \text{for } j = 1, \cdots, N-1, \\ u_0 = u_N = 0. \end{cases}$$

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

eigenvalue problems

3.2 Parabolic equations

3.3 Hyperbolic equations

CFD

Chapter 4

Finite element methods

4.0.1 Approximation of Banach spaces

We follow closely Temam for the abstract error analysis. The word “approximation” in here can be replaced into “discretization”.

Definition 4.0.1 (Approximation)**.** Let X be a Banach space. An *approximation* of X is an indexed family X_h of finite-dimensional normed spaces, with a *prolongation operator* $p_h \in B(X_h, X)$ and a *restriction operator* $r_h : X \to X_h$). The operator $p_h r_h : X \to X$ is called the *truncation operator*.

$$\begin{tikzcd} X \arrow[d, shift right, "r_h"'] \\ X_h \arrow[u, shift right, bend right, "p_h"'] \end{tikzcd}$$

Definition 4.0.2 (Errors)**.** Let X_h be an approximation of a Banach space X. For $x \in X$ and $x_h \in X_h$, the quantities $E(x_h, x) := \|p_h x_h - x\|$ and $DE(x_h, x) := \|x_h - r_h x\|$ are called the *error* and the *discrete error* between x and x_h. The quantity $TE(x) := \|x - p_h r_h x\|$ is called the *truncation error*.

Definition 4.0.3 (Stable and convergent approximations)**.** We say an approximation X_h is

(a) *stable* if $\|p_h\| + \|r_h\| \lesssim 1$,

(b) *convergent* if $\|p_h r_h x - x\| \to 0$ for each $x \in X$.

Lemma 4.0.4. *Let X_h be an approximation of a Banach space X. If X_h is stable and convergent, then for each net $x_h \in X_h$ the discrete convergence implies the strong convergence.*

Proof. We have for each $x \in X$ that

$$DE = \|r_h\| \cdot E \quad \text{and} \quad E = \|p_h\| \cdot DE + TE. \qquad \square$$

Lemma 4.0.5. *Let X_h be an approximation of a Banach space X. If $\|p_h x\| \sim \|x\|$, then the stability of X_h follows from the convergence of X_h.*

Proof. It is by the uniform boundedness principle:

$$\|r_h x\| \lesssim \|p_h r_h x - x\| + \|x\|.$$

$\square$

In most cases we have $\|p_h x\| = \|x\|$, so for an approximation it is enough to verify the truncation error converges to zero.

4.0.2 Approxiamation of problems

A *well-posed problem* is an operator $L : \mathcal{X} \to \mathcal{Y}$ such that there is a continuous operator $L^{-1} : Y \to X$ satisfying $LL^{-1} = \mathrm{id}_Y$, where $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are embeddings. Say, consider the spaces $\mathcal{X}$ and $\mathcal{Y}$ as space of distributions. We will always assume $L : X \to Y$ is a right invertible(i.e. well-posed) linear operator between Banach spaces.

Definition 4.0.6 (Approximation). Let L be a well-posed linear problem. An *approximation* of L is an indexed family $L_h \in L(X_h, Y_h)$ of invertible linear operators, where X_h and Y_h are stable and convergent approximations of X and Y.

We also do not need to assume in fact the stability of r_h. The approximation X_h of X is where we should take subtly, and the art of numerical analysis begins with the choice of X_h. The following diagram does not commute, but *approximately* commute.

$$\begin{tikzcd} X \arrow[r, "L"] \arrow[d, dashed, bend right, "r_h"'] & Y \arrow[d, "r_h"] \\ X_h \arrow[r, "L_h"] \arrow[u, dashed, bend right, "p_h"'] & Y_h \end{tikzcd}$$

Definition 4.0.7. Let L_h be an approximation of a well-posed linear problem L. We say L_h is

(a) *consistent* if $CE = \|r_h Lx - L_h r_h x\| \to 0$ for each x,

(b) *stable* if $\|L_h^{-1}\| \lesssim 1$,

(c) *convergent* if $DE = \|L_h^{-1} r_h Lx - r_h x\| \to 0$ for each x.

Theorem 4.0.8 (Lax equivalence). *Let L_h be an approximation of a well-posed linear problem L. If L_h is consistent, then it is stable if and only if it is convergent.*

Proof. ($\Rightarrow$) It is clear from

$$DE = \|x_h - r_h x\| \le \|L_h^{-1}\| \|r_h Lx - L_h r_h x\| = \|L_h^{-1}\| \cdot CE.$$

($\Leftarrow$) If we show for the net of operators $p_h L_h^{-1} r_h : Y \to X$ that $p_h L_h^{-1} r_h y$ is bounded in X for each $y \in Y$, then by the uniform boundedness principle the operators $p_h L_h^{-1} r_h$ is uniformly bounded, and we obtain the stability from

$$\|L_h^{-1}\| = \|r_h p_h L_h^{-1} r_h p_h\| \le \|r_h\| \|p_h L_h^{-1} r_h\| \|p_h\|.$$

Since L is surjective by the well-posedness, there is $x \in X$ such that $Lx = y$. With this x we have

$$\|p_h L_h^{-1} r_h y - x\| \le \|p_h\| \cdot DE + TE \to 0,$$

so we are done.

□

4.0.3 Numerical analyses

For a numerical approximation, we can consider three analyses:

1. Consistency analysis,
2. Statbility analysis,
3. Error analysis.

Note that we have $DE \le \|L_h^{-1}\| \cdot CE$. If we have the estimate for the rate of the consistency error from the consistency analysis, and also if we have the bound of $\|L_h^{-1}\|$ in the stability analysis, we can easily obtian an *error estimate*. In this regard, the main difficulty is the former two.

Consistency analysis

Usually the Taylor's theorem is used in finite difference schemes.

Stability analysis

For the bound of $\|L_h^{-1}\|$, we have to make a *stability estimate*

$$\|x_h\| \lesssim \|L_h x_h\|.$$

We have some notes about uniqueness and existence: the injectivity of L_h^{-1} clearly follows from the above estimate, and the surjectivity is deduced thanks to the finite-dimensional nature of X_h and Y_h when their dimensions coincide.

Error analysis

In the Ritz-Galerkin approximation the discrete solution operator $p_h L_h^{-1} r_h L$ can be directly shown to be an orthogonal projection called the *Ritz projection*, which deduces an *a priori* convergence result before justifying proving consistency and stability.

4.0.4 Applications

Example 4.0.9. Consider

$$\begin{cases} u'(x) - u(x) = f(x) & \text{in } x \in (0,1), \\ u(0) = c. \end{cases}$$

Let $X := C^1([0,1])$, $Y := C([0,1]) \times \mathbb{R}$, and $Au(x) := (u'(x) - u(x), u(0))$. Then it is well-posed since there is $E : Y \to X$ defined by

$$E(f,c)(x) := c + \int_0^x e^{-y} f(y)\, dy$$

satisfies

Example 4.0.10. Consider

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } x \in (0,1)^2, \\ u(x) = 0 & \text{on } x \in \partial(0,1)^2. \end{cases}$$

Let $X =$, $Y =$, Au

Example 4.0.11. Consider

$$\begin{cases} \partial u(t,x) = \Delta u(t,x) & \text{in } (t,x) \in (0,\infty) \times (0,1), \\ u(0,x) = f(x) & \text{on } x \in [0,1], \\ u(t,0) = 0 & \text{on } t \in [0,\infty), \\ u(t,1) = 0 & \text{on } t \in [0,\infty), \end{cases}$$

Let $X =$, $Y =$, Au

u_j^n, $t = t_0 + nk$, $x = x_0 + jh$

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