Pseudodifferential operators

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Solution of 1. By symmetry, it is enough to show there are $c > 0$ such that

$$|\partial^\alpha a(\zeta)| \lesssim (1+|\zeta|^2)^{(m-|\alpha|)/2}$$

for $\zeta = \xi + i\eta$ such that $|\xi| \geq c$ and $\eta = 0$. Let

$$\Omega_\varepsilon := \{\xi + i\eta \in \mathbb{C}^d : |\eta| < \varepsilon|\xi|\}, \qquad \varepsilon > 0.$$

Then, a is holomorphic on Ω_ε by condition. For $0 < \varepsilon' < \varepsilon$, there is a small $r = r(\varepsilon, \varepsilon') > 0$ such that $\zeta = (\zeta_1, \cdots, \zeta_d) \in \Omega_{\varepsilon'}$ and $|\zeta_1' - \zeta_1| \leq r|\zeta|$ imply $(\zeta_1', \zeta_2, \cdots, \zeta_d) \in \Omega_\varepsilon$. With this r, write the Cauchy integral formula with respect to the first argument as follows:

$$\partial_1 a(\zeta) = \frac{1}{2\pi i}\int_{|\zeta_1'-\zeta_1|=r|\zeta|} \frac{a(\zeta_1', \zeta_2, \cdots, \zeta_d)}{(\zeta_1'-\zeta_1)^2}\, d\zeta_1', \qquad \zeta \in \Omega_{\varepsilon'}.$$

Thus we have an estimate

$$|\partial_1 a(\zeta)| \leq \frac{1}{2\pi} r|\zeta| \frac{C(1+(1+r^2)|\zeta|^2)^{m/2}}{(r|\zeta|)^2} \lesssim \frac{(1+|\zeta|^2)^{m/2}}{|\zeta|}, \qquad \zeta \in \Omega_{\varepsilon'},$$

so now we obtain for some small $c > 0$ that

$$|\partial_1 a(\zeta)| \lesssim (1+|\zeta|^2)^{(m-1)/2}, \qquad \zeta \in \Omega_{\varepsilon'} \setminus B(0,c).$$

Since the index 1 can be changed into any indices, by taking a finite decreasing sequence of $\varepsilon' > 0$, we get our claim by the induction. □

Solution of 2. (1) The Schwartz kernel of the operator $\mathcal{F}a^w(x, D_x)\mathcal{F}^*$ is given by

$$\begin{aligned}
k(\xi,\eta) &= (2\pi)^{-2d}\int_{\mathbb{R}^{3d}} e^{i(-x\xi+(x-y)\zeta+y\eta)} a(\frac{x+y}{2}, \zeta)\, dx\, dy\, d\zeta \\
&= (2\pi)^{-2d} 2^d \int_{\mathbb{R}^{3d}} e^{i(-x\xi+2(x-z)\zeta+(2z-x)\eta)} a(z,\zeta)\, dx\, dz\, d\zeta \\
&= (2\pi)^{-2d} 2^d \int_{\mathbb{R}^{3d}} e^{-ix(\xi+\eta-2\zeta)-i2z(\zeta-\eta)} a(z,\zeta)\, dx\, dz\, d\zeta \\
&= (2\pi)^{-2d}(-1)^d \int_{\mathbb{R}^{3d}} e^{-i2z(\frac{\xi+\eta}{2}-\eta)} a(z, \frac{\xi+\eta}{2})\, dz \\
&= (2\pi)^{-2d}(-1)^d \int_{\mathbb{R}^{3d}} e^{i(\xi-\eta)(-z)} a(z, \frac{\xi+\eta}{2})\, dz \\
&= (2\pi)^{-2d} \int_{\mathbb{R}^{3d}} e^{i(\xi-\eta)z} a(-z, \frac{\xi+\eta}{2})\, dz,
\end{aligned}$$

which can be interpreted to be the kernel of the operator $a^w(-D_\xi, \xi)$. The integral formula for the kernel is justified for $a \in S^0_{0,0}$ when we do the above same computation with Schwartz test functions. □

Solution of 4. (1) We have $\operatorname{sing\,supp}(\delta) = \{0\}$. Since

$$\mathcal{F}(\chi\delta)(\xi) = \chi(0)\mathcal{F}(\delta)(\xi) = (2\pi)^{-\frac{d}{2}}\chi(0)$$

is constant for $\chi \in C_c^\infty(\mathbb{R}^d)$, we have $\mathrm{WF}(\delta) = \{(0,\xi) \in \mathbb{R}^{2d} : \xi \neq 0\}$.

(2) We have $\operatorname{sing\,supp}(\delta \otimes 1) = \{(0, x'') \in \mathbb{R}^p \times \mathbb{R}^q : x'' \in \mathbb{R}^q\}$. Since

$$\begin{aligned}\mathcal{F}(\chi(\delta \otimes 1))(\xi', \xi'') &= \mathcal{F}(\delta \otimes (\chi|_{\{0\}\times\mathbb{R}^q}))(\xi', \xi'') \\ &= \mathcal{F}(\delta)(\xi')\mathcal{F}(\chi|_{\{0\}\times\mathbb{R}^q})(\xi'') \\ &= (2\pi)^{-\frac{p}{2}}\mathcal{F}(\chi|_{\{0\}\times\mathbb{R}^q})(\xi'')\end{aligned}$$

is constant along the direction of ξ' for $\chi \in C_c^\infty(\mathbb{R}^p \times \mathbb{R}^q)$, we have $\mathrm{WF}(\delta \otimes 1) = \{(0, x'', \xi', 0) : \xi' \neq 0\}$.

(3) We have $\operatorname{sing\,supp}(\delta_{S^{d-1}}) = S^{d-2}$. For test functions $\varphi = \varphi(\xi)$, we have

$$\begin{aligned}\langle \mathcal{F}(\chi\delta_{S^{d-1}}), \varphi\rangle &= \chi(x)\delta_{S^{d-1}}(\mathcal{F}^*\varphi) \\ &= \int_{S^{d-1}} \chi(x)(2\pi)^{-\frac{d}{2}} \int e^{ix\xi}\varphi(\xi)\,d\xi\,d\sigma(x) \\ &= (2\pi)^{-\frac{d}{2}} \int \left(\int_{S^{d-1}} \chi(x)e^{ix\xi}\,d\sigma(x)\right)\varphi(\xi)\,d\xi\end{aligned}$$

implies

$$\mathcal{F}(\chi\delta_{S^{d-1}})(\xi) = (2\pi)^{-\frac{d}{2}} \int_{S^{d-1}} \chi(y)e^{iy\xi}\,d\sigma(y).$$

Without loss of generality, fix a point $x = (1, 0, \cdots, 0)$ in the singular support. If ξ is not parallel to x, then we can take $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi(x) \neq 0$ on a small support so that $\mathcal{F}(\chi\delta_{S^{d-1}})(\xi)$ vanishes out. Then, we have three possibilities for $\xi = (t, 0, \cdots, 0)$; $t < 0$, $t > 0$, $t \neq 0$. We have a further calculation with the assumption that ξ is parallel to x and the coordinate representation $y = (z, \sqrt{1-z^2}w) \in S^{d-1}$ for $z \in [-1, 1]$ and $w \in S^{d-2}$ as

$$\mathcal{F}(\chi\delta_{S^{d-1}})(\xi) = \int_{-1}^{1}\int_{S^{d-2}} \chi(z, \sqrt{1-z^2}w)e^{itz}(1-z^2)^{\frac{d-3}{2}}\,d\sigma(w)\,dz.$$

If we introduce a function $a(z) := (1-z^2)^{\frac{d-3}{2}} \int_{S^{d-2}} \chi(z, \sqrt{1-z^2}w)\,d\sigma(w)$ supported on $[-1, 1]$, then

$$\mathcal{F}(\chi\delta_{S^{d-1}})(\xi) = \mathcal{F}^*a(\xi).$$

Since

$$a(1-\varepsilon) \sim |S^{d-2}|\chi(1,0)(2\varepsilon)^{\frac{d-3}{2}}, \qquad a(-1+\varepsilon) \sim |S^{d-2}|\chi(-1,0)(2\varepsilon)^{\frac{d-3}{2}}$$

as $\varepsilon \to 0+$, the condition $\chi(1,0) \neq 0$ implies $\mathcal{F}^*a$ does not decay at sufficiently fast speed. It means the wave front set includes the both case of $t < 0$ and $t > 0$. Thus $\mathrm{WF}(\delta_{S^{d-1}}) = \{(x, tx) : t \neq 0\}$.

(4) We have $\operatorname{sing\,supp}((x+i0)^{-1}) = \{0\}$. Note that

$$\mathcal{F}((x+i0)^{-1})(\xi) = -i(2\pi)^{-\frac{1}{2}}\mathbf{1}_{(-\infty,0]}(\xi).$$

Therefore,

$$\mathcal{F}(\chi(x+i0)^{-1})(\xi) = \mathcal{F}(\chi) * \mathcal{F}((x+i0)^{-1})(\xi) = i(2\pi)^{-\frac{1}{2}}\int_{\xi}^{\infty} \mathcal{F}(\chi)(\eta)\,d\eta,$$

which implies

$$\lim_{\xi\to\infty} \mathcal{F}(\chi(x+i0)^{-1})(\xi) = 0, \qquad \lim_{\xi\to-\infty} \mathcal{F}(\chi(x+i0)^{-1})(\xi) = i\chi(0) \neq 0.$$

Therefore $\mathrm{WF}((x+i0)^{-1}) = \{(0,\xi) : \xi < 0\}$.

(5) We first claim the characteristic function $1 \otimes H$ of the upper half plane has the wave front set $\{(x,0,0,\eta) : x \in \mathbb{R}, \eta \neq 0\}$, where $H = \mathbf{1}_{[0,\infty)}$ denotes the Heaviside step function. Its singular support is clealy $\{(x,0) : x \in \mathbb{R}\}$. Fix a point, say $(0,0)$, in this singular support. Take $\chi(x,y) = \chi_1(x)\chi_2(y)$ with $\chi(0,0) \neq 0$ so that

$$\mathcal{F}(\chi(1 \otimes H))(\xi,\eta) = \mathcal{F}(\chi_1)(\xi)\mathcal{F}(\chi_2 H)(\eta).$$

Then $(0,\infty) \to \mathbb{C} : t \mapsto \mathcal{F}(\chi(1 \otimes H))(t\xi, t\eta)$ decay rapidly if $\xi \neq 0$, so

$$\mathrm{WF}(1 \otimes H) \subset \{(x,0,0,\eta) : x \in \mathbb{R}, \eta \neq 0\}.$$

The inclusion is in fact the equality because the wave front set is not empty and the symmetry implies that $(x,0,0,\eta) \in \mathrm{WF}(1 \otimes H)$ is equivalent to $(x,0,0,-\eta)\,\mathrm{WF}(1 \otimes H)$. Then, we can eaily extend the above argument to show

$$\mathrm{WF}(u) = \{(r,0,0,t), (r\cos\alpha, r\sin\alpha, -t\sin\alpha, t\cos\alpha) : r > 0, t \neq 0\} \cup \{(0,0,\xi,\eta) : (\xi,\eta) \in \mathbb{R}^2\}. \quad \square$$

Solution of 5. (1) Fix x. Since $\xi \mapsto a(x,\xi)$ is integrable, we have the explicit formula for the Schwartz kernel

$$k(x,y) = (2\pi)^{-d} \int e^{i(x-y)\xi} a(x,\xi)\, d\xi$$

of the operator $a(x,D)$ by applying the Fubini on

$$a(x,D)u(x) = (2\pi)^{-d} \iint e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi.$$

We claim that k is diagonally supported from the locality condition of $a(x,D)$. In other words, we will show $y = x$ if y satisfies $k(x,y) \neq 0$. If the claim is true, then because the Fourier transform maps an integrable function to a continuous function, $k(x,y)$ is continuous with respect to y so that $k(x,y) \equiv 0$.

Suppose y satisfies $k(x,y) \neq 0$, and take $u_n \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} u_n \subset B(y, n^{-1})$ and $u_n \geq 0$. Since the integral

$$a(x,D)u_n(x) = \int k(x,y) u_n(y)\, dy$$

eventually belongs to $\mathbb{C} \setminus \{0\}$ as n goes to infinity, we can deduce

$$x \in \operatorname{supp} a(x,D)u_n \subset \operatorname{supp} u_n,$$

which implies $x \in \{y\}$. So we are done.

(2) The integral by part provides

$$(\partial_\xi a)(x,D)u(y) = -iy\, a(x,D)u(y) + i a(x,D)(M_x u)(y),$$

which implies the locality as follows:

$$\operatorname{supp}(\partial_\xi a)(x,D)u \subset \operatorname{supp} a(x,D)u \cup \operatorname{supp} a(x,D)(M_x u) \subset \operatorname{supp} u \cap \operatorname{supp} M_x u = \operatorname{supp} u,$$

where M_x denotes the multiplication operator by the identity function. Then, the desired result follows from the induction.

(3) Note that $\partial_\xi^\alpha a \in S_{\rho,\delta}^{m-\rho|\alpha|}(\mathbb{R}^{2d})$. Since $\rho > 0$, if we take α such that $m - \rho|\alpha| < -d$, then $\partial_x^\alpha a \equiv 0$ by the part (1) and (2). Therefore, a is a polynomial in ξ so that $a(x,D)$ is a partial differerntial operator. $\square$

Solution of 6. (1) Note

$$\operatorname{Re}\psi(x) = B(x_1) - B(x_1)^2 + x_2^2.$$

Since

$$\frac{B(x_1) - B(x_1)^2}{B(x_1)} \to 1$$

as $x_1 \to 0$, we have $B(x_1) \lesssim B(x_1) - B(x_1)^2 \lesssim B(x_1)$ at $|x_1| \ll 1$.

(2) We can check

$$a^*(x,D) = D_1 - ib(x_1)D_2, \qquad a^*(x,D)\psi = 0, \qquad a^*(x,D)e^{-\mu\psi(x)} = 0.$$

□