Lebesgue Theory

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Contents

Part I

Measure theory

Chapter 1

Measures and σ-algebras

1.1 Definition of measures

1.2 The Carathéodory extension theorem

1.1 (Outer measures)**.** Let X be a set. An *outer measure* on X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ such that

(i) if $E \subset E'$, then $\mu^*(E) \le \mu^*(E')$, (monotonicity)

(ii) $\mu^*(\bigcup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu^*(E_i)$. (countable subadditivity)

(a) A function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ with $\mu^*(\varnothing) = 0$ is an outer measure if and only if $E \subset \bigcup_{i=1}^\infty E_i$ implies $\mu^*(E) \le \sum_{i=1}^\infty \mu^*(E_i)$.

(b) Let $\mathcal{A} \subset \mathcal{P}(X)$ such that $\varnothing \in \mathcal{A}$. If a function $\rho : \mathcal{A} \to [0,\infty]$ satisfies $\rho(\varnothing) = 0$, then we can associate an outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by defining as

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^\infty \rho(A_i) : E \subset \bigcup_{i=1}^\infty A_i,\ A_i \in \mathcal{A} \right\},$$

where we use the convention $\inf\varnothing = \infty$.

1.2 (Carathéodory measurability)**.** Let μ^* be an outer measure on a set X. A subset $A \subset X$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

e for every subset $E \subset X$. Let $\mathcal{M}$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{M}$ is an algebra and μ^* is finitely additive on $\mathcal{M}$.

(b) $\mathcal{M}$ is a σ-algebra and μ^* is countably additive on $\mathcal{M}$.

(c) The measure $\mu := \mu^*|_{\mathcal{M}} : \mathcal{M} \to [0, \infty]$ is complete.

1.3 (The Carathéodory extension theorem)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ be a semi-ring of sets on a set X and $\rho : \mathcal{A} \to [0, \infty]$ a function with $\rho(\varnothing) = 0$. If the function ρ satisfies

(i) $\rho(A) = \sum_{i=1}^{n} \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, (finite additivity)

(ii) $\rho(A) \leq \sum_{i=1}^{\infty} \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$,
((disjoint) countable subadditivity)

then it is called a *premeasure*. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the associated outer measure of ρ, and $\mu : \mathcal{M} \to [0, \infty]$ the measure defined from μ^* on Carathéodory measurable subsets. We call μ the *Carathéodory measure* constructed from ρ.

(a) If ρ is finitely additive, then $\mathcal{A} \subset \mathcal{M}$.

(b) If ρ is countably subadditive, then $\mu^*(A) = \rho(A)$ for every $A \in \mathcal{A}$.

(c) If ρ is a premeasure, then μ is an extension of ρ and called *Carathéodory extension* of ρ.

(d) In particular, a premeasure is a priori countably additive in the sense that $\rho(A) = \sum_{i=1}^{\infty} \rho(A_i)$ for $A \in \mathcal{A}$ a disjoint union of $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$.

1.4 (Uniqueness of extensions)**.** The Carathéodory extension theorem provides with a uniqueness theorem for measures.

Monotone class lemma: alternative direct proof method without using Carathéodory extension.

Chapter 2

Measures on the real line

Chapter 3

Measurable functions

Part II

Integration

Chapter 4

Lebesgue integration

4.1 Definition of Lebesgue integration

4.2 Convergence theorems

Stein: Egorov → BCT → Fatou → MCT → L1 is a measure
Stein: BCT + L1 is a measure → DCT
Folland: MCT → Fatou → DCT → BCT

4.1 (Egorov's theorem)**.** Let Ω be a finite measure space. Let $(f_n : \Omega \to \mathbb{R})_n$ be a sequence of a.e. convergent measurable functions. For $\varepsilon > 0$, there exists a measurable $E_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus E_\varepsilon) < \varepsilon$ and f_n uniformly convergent on E_ε.

Proof. Assume $f_n \to 0$. The set of convergence is

$$\bigcap_{k>0} \bigcup_{n_0>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\},$$

which is a full set. We want to get rid of the dependence on the point x of n_0 in the union $\bigcup_{n_0>0}$. Since

$$\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \frac{1}{k} \,\}$$

is increasing as $n_0 \to \infty$ to a full set for each $k > 0$, we can find $n_0(k,\varepsilon)$ such that

$$\mu(\bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\}) > \mu(\Omega) - \frac{\varepsilon}{2^k}.$$

Then,

$$\mu(\bigcap_{k>0} \bigcap_{n\geq n_0} \{\, x : |f_n(x)| < \tfrac{1}{k} \,\}) > \mu(\Omega) - \varepsilon.$$

If we define

$$E_\varepsilon := \bigcap_{k>0} \bigcap_{n\geq n_0} \{ x : |f_n(x)| < \tfrac{1}{k} \},$$

then for any $k > 0$ and $x \in E_\varepsilon$, and with the $n_0(k, \varepsilon)$ we have chosen, we have

$$n \geq n_0 \quad \Rightarrow \quad |f_n(x)| < \frac{1}{k}.$$

□

4.3 Modes of convergence

Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0, \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > \tfrac{1}{k},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n - f| > \tfrac{1}{k}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n - f| > \tfrac{1}{k}\}.\end{aligned}$$

Since for every k we have

$$\begin{aligned}\limsup_n \{x : |f_n - f| > \tfrac{1}{k}\} &\subset \limsup_{n>k} \{x : |f_n - f| > \tfrac{1}{n}\} \\ &= \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\},\end{aligned}$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n - f| > \tfrac{1}{n}\}.$$

4.2. Let (X, μ) be a measure space. Let f_n be a sequence of measurable functions. If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.

Proof. We can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) > \tfrac{1}{2^k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k} - f| > \tfrac{1}{k}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k} - f| > \tfrac{1}{k}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. □

Chapter 5

Product measures

5.1 The Fubini-Tonelli theorem

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Chapter 6

Lebesgue spaces

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6.2 L^2 spaces

6.3 The Riesz representation theorem

Part III

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Chapter 9

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9.2 Regular integral operators

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9.4 Weak L^p spaces

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Part IV

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Chapter 10

Weak derivatives

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\text{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : \Omega \to \mathbb{R}$ such that $f(x,y)$ and $\partial_{x_i}f(x,y)$ are both locally integrable in x and integrable y. Then,

$$\partial_{x_i}\int f(x,y)\,dy = \int \partial_{x_i}f(x,y)\,dy$$

where ∂_{x_i} denotes the weak partial derivative.

Chapter 11

Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}

(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞

(b) f is AC iff f' is L^1

(c) f is BV iff f' is a finite regular Borel measure

Chapter 12

The Lebesgue differentiation theorem