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1 Potential from a source

Theorem. Let $d \geq 3$. A distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity if and only if there is a distribution $\rho \in \mathcal{D}'(\mathbb{R}^d)$ such that $u = \Phi * \rho$ and $\text{supp}(\rho) \subset \{0\}$, where Φ denotes the fundamental solution of Laplace's equation.

Proof. (\Rightarrow) Define a distribution ρ by

$$\langle \rho, \varphi \rangle := -\langle u, \Delta \varphi \rangle$$

for $\varphi \in C_c^\infty(\mathbb{R}^d)$. In other words, $\rho = -\Delta u$ in distributional sense. Then, ρ has the support contained in $\{0\}$ because if $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ then

$$\langle \rho, \varphi \rangle = -\langle u, \Delta \varphi \rangle = -\int u(x) \Delta \varphi(x) dx = -\int \Delta u(x) \varphi(x) dx = 0.$$

Therefore, we only need to verify $u = \Phi * \rho$ to complete the proof.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Be cautious that the argument

$$\langle \Phi * \rho, \varphi \rangle = \langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle = \langle u, \varphi \rangle$$

fails to provide a proof because the function $\Phi * \rho$ is not compactly supported so that we cannot deduce $\langle \rho, \Phi * \varphi \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$, and here we use the condition that u vanishes at infinity to justify the equality. Define a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{5}{4} \\ 0 & \text{if } |x| \geq \frac{7}{4} \end{cases}.$$

If we denote $\chi_r(x) := \chi(\frac{x}{r})$, then we have

$$\langle \rho, (\Phi \chi_r) * \varphi \rangle = -\langle u, \Delta((\Phi \chi_r) * \varphi) \rangle$$

by the definition of ρ . We have the limit of the left-hand side

$$\lim_{r \rightarrow \infty} \langle \rho, (\Phi \chi_r) * \varphi \rangle = \langle \rho, \Phi * \varphi \rangle$$

because

$$\begin{aligned} \text{supp}((\Phi(1 - \chi_r) * \varphi) &\subset \text{supp}(\Phi(1 - \chi_r)) + \text{supp}(\varphi) \\ &\subset \mathbb{R}^d \setminus B(0, 2R) + B(0, R) = \mathbb{R}^d \setminus B(0, R) \end{aligned}$$

for all $r > 2R$ so that the supports of $\Phi(1 - \chi_r) * \varphi$ and ρ are disjoint, where we define $R := \sup_{x \in \text{supp}(\varphi)} |x|$. However, the right-hand limit

$$-\lim_{r \rightarrow \infty} \langle u, \Delta((\Phi \chi_r) * \varphi) \rangle = -\langle u, \Delta(\Phi * \varphi) \rangle$$

is not a trivial result.

Assuming $\chi(x) = \chi(-x)$ without loss of generality, we have

$$\langle u, \Delta(\Phi(1 - \chi_r) * \varphi) \rangle = \langle u * \Delta(\Phi(1 - \chi_r)), \varphi \rangle.$$

Because

$$\Delta_y \left[\Phi(x - y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] = 0$$

for $|y| < R$ and $x \in \text{supp}(\varphi)$ if $r > 2R$, we can write

$$\langle u * \Delta(\Phi(1 - \chi_r)), \varphi \rangle = \int \varphi(x) \int u(y) \Delta_y \left[\Phi(x - y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy dx.$$

We compute

$$\begin{aligned}\Delta_y \left[\Phi(x-y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] &= 2\nabla\Phi(x-y) \cdot \frac{1}{r} \nabla\chi\left(\frac{x-y}{r}\right) - \Phi(x-y) \frac{1}{r^2} \Delta\chi\left(\frac{x-y}{r}\right) \\ &= -\frac{2}{\omega_d} \frac{x-y}{|x-y|^d} \cdot \frac{1}{r} \nabla\chi\left(\frac{x-y}{r}\right) - \frac{1}{(d-2)\omega_d} \frac{1}{|x-y|^{d-2}} \frac{1}{r^2} \Delta\chi\left(\frac{x-y}{r}\right).\end{aligned}$$

Then, since $\frac{5}{4}r \leq |x-y| \leq \frac{7}{4}r$ if $\nabla\chi\left(\frac{x-y}{r}\right) \neq 0$ and $\Delta\chi\left(\frac{x-y}{r}\right) \neq 0$, we obtain

$$\left| \Delta_y \left[\Phi(x-y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] \right| \leq C \frac{1}{r^d} \psi\left(\frac{x-y}{r}\right)$$

for some constant $C > 0$, where

$$\psi(y) := |\nabla\chi(y)| + |\Delta\chi(y)|.$$

For each $x \in \text{supp}(\varphi)$, since we have $\frac{5}{4}r \leq |x-y| \leq \frac{7}{4}r$ implies $r \leq |y| \leq 2r$ if $r > 4R$, it follows that

$$\begin{aligned}\left| \int u(y) \Delta_y \left[\Phi(x-y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy \right| &\leq C \int |u(y)| \frac{1}{r^d} \psi\left(\frac{x-y}{r}\right) dy \\ &\leq C \max_{r \leq |y| \leq 2r} u(y)\end{aligned}$$

converges to zero as $r \rightarrow \infty$. By the bounded convergence theorem, we can deduce

$$\lim_{r \rightarrow \infty} \int \varphi(x) \int u(y) \Delta_y \left[\Phi(x-y) \left(1 - \chi\left(\frac{x-y}{r}\right) \right) \right] dy dx = 0,$$

so we are done.

(\Leftarrow) Let $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$. Since

$$\langle \Phi * \rho, \Delta\varphi \rangle = \langle \rho, \Phi * (\Delta\varphi) \rangle = \langle \rho, \varphi \rangle = 0,$$

the distribution $\Phi * \rho$ on $\mathbb{R}^d \setminus \{0\}$ is weakly harmonic, and by Weyl's lemma for distributions, it is a smooth harmonic function on $\mathbb{R}^d \setminus \{0\}$.

Since ρ is supported at zero, we have a positive integer k and constants a_α such that

$$|\langle \rho, \varphi \rangle| \leq \sum_{|\alpha| \leq k} |a_\alpha D^\alpha \varphi(0)|$$

for $\varphi \in C^\infty(\mathbb{R}^d)$. Then, for non-zero $x \in \mathbb{R}^d$, by taking a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(y) = \begin{cases} 1 & \text{if } |y-x| \leq \frac{1}{3}|x| \\ 0 & \text{if } |y| \leq \frac{1}{3}|x| \end{cases},$$

we have

$$|\Phi * \rho(x)| = |(\Phi\chi) * \rho(x)| = |\langle \rho(x-y), \Phi(y)\chi(y) \rangle_y| \leq \sum_{|\alpha| \leq k} |a_\alpha D^\alpha \Phi(x)| = O(r^{2-d})$$

as $r \rightarrow \infty$. Therefore, $\Phi * \rho$ vanishes at infinity. \square

Lemma. Let ρ be a distribution on \mathbb{R}^d such that $\text{supp}(\rho) \subset \{0\}$. Then, there is a constant coefficient partial differential operator $P(D)$ such that $\rho = P(D)\delta$.

Corollary. Let $d \geq 3$. If a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a harmonic function on $\mathbb{R}^d \setminus \{0\}$ and vanishes at infinity, then there are an integer $k \geq 0$ and constants a_α such that

$$u(x) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha \Phi(x)$$

for $x \neq 0$, where Φ denotes the fundamental solution of Laplace's equation.

2 Unified error analysis

2.1 Approximation of Banach spaces

We follow closely Temam for the abstract error analysis. The word “approximation” in here can be replaced into “discretization”.

Definition 2.1 (Approximation). Let X be a Banach space. An *approximation* of X is an indexed family X_h of finite-dimensional normed spaces, with a *prolongation operator* $p_h \in B(X_h, X)$ and a *restriction operator* $r_h \in B(X, X_h)$ such that $r_h p_h = \text{id}_{X_h}$. The operator $\tau_h := p_h r_h$ is called the *truncation operator*.

$$\begin{array}{c} X \\ \downarrow \scriptstyle r_h \quad \uparrow \scriptstyle p_h \\ X_h \end{array}$$

Definition 2.2 (Errors). Let X_h be an approximation of a Banach space X . For $x \in X$ and $x_h \in X_h$, the quantities $SE(x_h, x) := \|p_h x_h - x\|$ and $DE(x_h, x) := \|x_h - r_h x\|_h$ are called the *strong error* and the *discrete error* of x and x_h . The quantity $TE(x) := \|x - \tau_h x\|$ is called the *truncation error*.

Definition 2.3 (Stable and convergent approximations). We say an approximation X_h is

- (a) *stable* if $\|p_h\| + \|r_h\| \lesssim 1$,
- (b) *convergent* if $TE = \|x - \tau_h x\|_h \rightarrow 0$.

Lemma 2.1. Let X_h be an approximation of a Banach space X . If X_h is stable and convergent, then the convergences in discrete error and strong error are equivalent:

$$\begin{aligned} DE &\leq \|r_h\| \cdot SE, \\ SE &\leq \|p_h\| \cdot DE + TE. \end{aligned}$$

Example 2.1.

2.2 Approximation of problems

A *well-posed problem* is an operator $L : \mathcal{X} \rightarrow \mathcal{Y}$ such that there is a continuous operator $L^{-1} : Y \rightarrow X$ satisfying $LL^{-1} = \text{id}_Y$, where $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ are embeddings. Say, consider the spaces \mathcal{X} and \mathcal{Y} as space of distributions. We will always assume $L : X \rightarrow Y$ is a right invertible (i.e. well-posed) linear operator between Banach spaces.

Definition 2.4 (Approximation). Let L be a well-posed linear problem. An *approximation* of L is an indexed family $L_h \in L(X_h, Y_h)$ of invertible linear operators, where X_h and Y_h are stable and convergent approximations of X and Y .

Note that we never use the prolongation operator for Y_h , but the restriction operator r_h for Y_h is taken to be routine. We also do not need to assume in fact the stability of r_h . The approximation X_h of X is where we should take subtly, and the art of numerical analysis begins with the choice of X_h . The following diagram does not commute, but *approximately* commute.

$$\begin{array}{ccc} X & \xrightarrow{L} & Y \\ \downarrow \scriptstyle r_h \quad \uparrow \scriptstyle p_h & & \downarrow \scriptstyle r_h \\ X_h & \xrightarrow{L_h} & Y_h \end{array}$$

Given an approximated problem L_h , we will always write $x_h := L_h^{-1} r_h L x$ the *approximate solution* for each $x \in X$.

Definition 2.5. Let L_h be an approximation of a well-posed linear problem L . We say L_h is

- (a) *stable* if $\|L_h^{-1}\| \lesssim 1$,
- (b) *consistent* if $CE = \|r_h Lx - L_h r_h x\|_h \rightarrow 0$,
- (c) *convergent* if $DE = \|x_h - r_h x\|_h \rightarrow 0$.

Theorem 2.2 (Lax equivalence). *Let L_h be an approximation of a well-posed linear problem L . If L_h is consistent, then it is stable if and only if it is convergent.*

Proof. (\Rightarrow) It is clear from

$$DE = \|x_h - r_h x\|_h \leq \|L_h^{-1}\| \|r_h Lx - L_h r_h x\|_h = \|L_h^{-1}\| \cdot CE.$$

Conversely, suppose L_h is convergent. For any $y_h \in Y_h$, since $r_h : Y \rightarrow Y_h$ and L are surjective, there is x such that $y_h = r_h Lx$ so that $L_h^{-1} y_h = x_h$ is convergent since

$$DE = \|x_h - r_h x\|_h \leq \|r_h\|_{X \rightarrow X_h} \cdot SE.$$

By the uniform boundedness principle, L_h^{-1} is uniformly bounded. □

Example 2.2.

2.3 Consistency analysis

The Taylor's theorem method for finite difference scheme.

2.4 Stability analysis

We must bound $\|L_h^{-1}\|$.

The von Neumann stability analysis for finite difference scheme.

2.5 Applications

Example 2.3. Consider

$$\begin{cases} u'(x) - u(x) = f(x) & \text{in } x \in (0, 1), \\ u(0) = c. \end{cases}$$

Let $X := C^1([0, 1])$, $Y := C([0, 1]) \times \mathbb{R}$, and $Au(x) := (u'(x) - u(x), u(0))$. Then it is well-posed since there is $E : Y \rightarrow X$ defined by

$$E(f, c)(x) := c + \int_0^x e^{-y} f(y) dy$$

satisfies

Example 2.4. Consider

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } x \in (0, 1)^2, \\ u(x) = 0 & \text{on } x \in \partial(0, 1)^2. \end{cases}$$

Let $X =, Y =, Au$

Example 2.5. Consider

$$\begin{cases} \partial u(t, x) = \Delta u(t, x) & \text{in } (t, x) \in (0, \infty) \times (0, 1), \\ u(0, x) = f(x) & \text{on } x \in [0, 1], \\ u(t, 0) = 0 & \text{on } t \in [0, \infty), \\ u(t, 1) = 0 & \text{on } t \in [0, \infty), \end{cases}$$

Let $X =, Y =, Au$

$$u_j^n, t = t_0 + nk, x = x_0 + jh$$

3 Vector valued integral

3.1 (Simple functions). Let (Ω, μ) be a measure space and X a topological space with the Borel measurable structure. By a *simple function* we mean a measurable function $s : \Omega \rightarrow X$ of finite image.

Suppose X satisfies the following property: for each open set $U \subset X$ there is a countable family $\{U_i\}_{i=1}^\infty$ of open sets such that $U = \bigcup_{i=1}^\infty U_i$ and $\overline{U_i} \subset U$. This is the “pointwise approximation condition” for the target space X . In particular, every metrizable space satisfy this condition.

- (a) The pointwise limit of a sequence of measurable functions is measurable.
- (b) A measurable function is the pointwise limit of a sequence of simple functions, if X is separable and metrizable.
- (c) The pointwise limit of a net of simple functions may not be measurable, even for real-valued functions; nets are not appropriate for measure theory.

Proof. (a) Suppose a sequence $(f_n)_n$ of measurable functions converges pointwisely to a function f . For fixed open $U \subset X$ we claim

$$f^{-1}(U) = \bigcup_{i=1}^\infty \liminf_{n \rightarrow \infty} f_n^{-1}(U_i).$$

If it is true, then $f^{-1}(U)$ is the countable set operation of measurable sets $f_n^{-1}(U_i)$. Let $\{U_i\}_i$ be the countable family associated to U taken by the “pointwise approximation condition”.

(\subset) If $\omega \in f^{-1}(U)$, then for some i we have $f(\omega) \in U_i$, so $f_n(\omega)$ is eventually in U_i , thus we have $\omega \in \liminf_{n \rightarrow \infty} f_n^{-1}(U_i)$.

(\supset) If $\omega \in \liminf_{n \rightarrow \infty} f_n^{-1}(U_i)$ for some i , then $f_n(\omega)$ is eventually in U_i , so $f(\omega) \in \overline{U_i} \subset U$, thus we have $\omega \in f^{-1}(U)$.

(b) Suppose there is a increasing sequence of finite tagged partitions $\mathcal{P}_n \subset \mathcal{B}$ satisfying the following property: for each open-neighborhood pair (x, U) there is n and i such that $P_{n,i} \in \mathcal{P}_n$ and $x \in P_{n,i} \subset U$. We denote the tags by $t_{n,i} \in P_{n,i}$ for each $P_{n,i} \in \mathcal{P}_n$. Define

$$s_n(\omega) := t_{n,i} \quad \text{for } f(\omega) \in P_{n,i}.$$

To show $s_n(\omega) \rightarrow f(\omega)$, fix an open $f(\omega) \in U \subset X$. Then, there is n_0 such that there is a sequence $(P_{n,i_n})_{n=n_0}^\infty$ satisfying $P_{n,i_n} \in \mathcal{P}_n$ and $f(\omega) \in P_{n,i_n} \subset U$. Then, for all $n \geq n_0$, we have for $f(\omega) \in P_{n,i_n}$ that $s_n(\omega) = t_{n,i_n} \in P_{n,i_n} \subset U$.

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem.

□

3.2 (Strong measurability). Let (Ω, μ) be a measure space and X a Banach space. A function $f : \Omega \rightarrow X$ is said to be *strongly measurable* or *Bochner measurable* if it is a pointwise limit of a sequence of simple functions.

- (a) A

3.3 (Pettis measurability theorem). Furthermore, not only measurable, weakly measurables are also approximated by simples if X is separable Banach. We introduce the notion of separably valued functions.

If we associate a complete measure, then the pointwise convergence is relaxed to the almost everywhere convergence....

3.4 (Bochner integral). Let (Ω, μ) be a measure space and X a Banach space.

if there is a net of simple functions $(s_\alpha)_{\alpha \in \mathcal{A}}$ such that

$$\int_{\Omega} \|f(\omega) - s_\alpha(\omega)\| d\mu \rightarrow 0$$

for $\alpha \in \mathcal{A}$.

4 Kinetic theory

4.1 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 4.1 (Velocity averaging). *Let L be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. Then,*

$$\left\| \int u \varphi dv \right\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}$$

for $\varphi \in C_c^\infty(\mathbb{R}_v^n)$,

Proof. Let $m(t, x) = \int u \varphi dv$. By Fourier transform with respect to t and x , we have

$$\widehat{u}(\tau, \xi, v) = \frac{1}{i} \frac{\widehat{Lu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v) \varphi(v) dv.$$

Fixing τ, ξ , decompose the integral and use Hölder's inequality to get

$$\begin{aligned} |\widehat{m}(\tau, \xi)| &\leq \int_{|\tau + v \cdot \xi| < \alpha} |\widehat{u} \varphi| dv + \int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\widehat{Lu} \varphi|}{|\tau + v \cdot \xi|} dv \\ &\leq \|\widehat{u}\|_{L_v^2}^{1/2} \left(\int_{|\tau + v \cdot \xi| < \alpha} |\varphi|^2 dv \right)^{1/2} + \|\widehat{Lu}\|_{L_v^2}^{1/2} \left(\int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2} dv \right)^{1/2}, \end{aligned}$$

where $\alpha > 0$ is an arbitrary constant that will be determined later. Let

$$I_s(\tau, \xi, \alpha) := \int_{|\tau + v \cdot \xi| < \alpha} |\varphi|^2 dv, \quad I_n(\tau, \xi, \alpha) := \int_{|\tau + v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|} dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \quad I_n \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

Define coordinates (v_1, v_2) on \mathbb{R}_v as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}, \quad v_2 := v - \frac{v \cdot \xi}{|\xi|^2} \xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = \left(v_1 - \frac{\tau}{|\xi|}\right)^2 + |v_2|^2 \quad \text{and} \quad \int dv = \iint dv_2 dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| \leq R$. If $\frac{|\tau| - \alpha}{|\xi|} > R$, then the region of integration vanishes so that $I_s = 0$. If $|\tau| \leq \alpha + R|\xi|$, then

$$\begin{aligned} I_s &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 dv_1 \\ &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}, |v_1| \leq R} \int_{|v_2| \leq R} dv_2 dv_1 \\ &\lesssim \min\left\{\frac{2\alpha}{|\xi|}, R\right\} \cdot R^{n-1} \\ &\approx \frac{1}{\sqrt{1 + \left(\frac{|\xi|}{\alpha}\right)^2}} \\ &\lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}. \end{aligned}$$

For the second integral, suppose that φ is supported on $|\nu| < R$ so that $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < R$. Then,

$$\begin{aligned} I_n &\lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|}| < R} \int_{|v_2| < R} \frac{1}{v_1^2 |\xi|^2} dv_2 dv_1 \\ &\simeq \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - R\} \leq v_1 < \frac{|\tau|}{|\xi|} + R} \frac{1}{v_1^2 |\xi|^2} dv_1 \\ &\simeq \frac{1}{|\xi|^2} \left(\frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - R\}} - \frac{1}{\frac{|\tau|}{|\xi|} + R} \right). \end{aligned}$$

If $\frac{|\tau|}{|\xi|} - R > \frac{\alpha}{|\xi|}$, then

$$I_n \lesssim \frac{2R}{\tau^2 - (R|\xi|)^2} < \frac{2R}{\alpha(|\tau| + R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

If $|\tau| \leq \alpha + R|\xi|$, then

$$I_n \lesssim \frac{1}{|\xi|} \frac{(|\tau| + R|\xi|) - \alpha}{\alpha(|\tau| + R|\xi|)} \leq \frac{2R}{\alpha(|\tau| + R|\xi|)} \simeq \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

To sum up, we have

$$|\widehat{m}(\tau, \xi)| \lesssim \frac{1}{(\tau^2 + |\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Lu}\|_{L_v^2}^{1/2}).$$

Letting $\alpha = \sqrt{\|\widehat{Lu}\|_{L_v^2} / \|\widehat{u}\|_{L_v^2}}$ and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2} \|\widehat{Lu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$ and Plancherel's theorem gives

$$\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}.$$

□

Corollary 4.2. *Let \mathcal{F} be a family of functions on $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. If \mathcal{F} and $L\mathcal{F}$ are bounded in $L_{t,x,v}^2$, then $\int \mathcal{F} \varphi dv$ is bounded in $H_{t,x}^{1/2}$.*

Theorem 4.3. *Let \mathcal{F} be a family of functions on $I_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. If \mathcal{F} is weakly relatively compact and $L\mathcal{F}$ is bounded in $L_{t,x,v}^1$, then $\int \mathcal{F} \varphi dv$ is relatively compact in $L_{t,x}^1$.*

5 Sturm-Liouville theory

5.1 Self-adjointness

Let $I = [a, b]$ and

$$\begin{aligned} L &= -\frac{1}{w(x)} \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right], \\ 0 &\leq p(x) \in C^\infty(I), \quad q(x) \in C^\infty(I), \quad 0 < w(x) \in C^\infty(I). \end{aligned}$$

We expect L to be self-adjoint. In this regard, our interest is elimination of the difference term

$$\langle f, Lg \rangle - \langle Lf, g \rangle = p(f'g - fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) - \frac{m^2}{1-x^2} \right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2} \left[\frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} \right) \right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

5.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that p does not vanish on the boundary of I that we should cancel $f'g - fg'|_a^b$. View the Sturm-Liouville operator L as a non-densely defined operator on the space $C^\infty(I)$ with inner product $\langle f, g \rangle = \int_I f g w$ with domain

$$V = \{u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0, \beta_0 u(b) + \beta_1 u'(b) = 0\},$$

the subspace for the *separated* Robin boundary condition.

Proposition 5.1. *The operator $L : V \rightarrow C^\infty(I)$ is self-adjoint when $C^\infty(I)$ has the inner product $\langle f, g \rangle = \int_I f g w$.*

We are interested in the eigenvalue problem of $L : V \rightarrow C^\infty(I)$ on V . Fortunately, if we choose a constant $z \in \mathbb{C} \setminus \mathbb{R}$, then $(L - z)^{-1} : C^\infty(I) \rightarrow V$ is well-defined.

Proposition 5.2. *If z is not an eigenvalue of L , then $L - z : V \rightarrow C^\infty(I)$ is bijective.*

Proof. The injectivity follows from the definition of eigenvalues. We may assume that L is injective by translation $q \mapsto q - \lambda$.

Suppose $f \in C^\infty(I)$. The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$\begin{aligned} -pu'' - p'u' - qu &= f w, \\ \alpha_0 u(a) + \alpha_1 u'(a) &= 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0. \end{aligned}$$

Let u_a, u_b be the unique solutions of the corresponding homogeneous equation with initial conditions

$$u_a(a) = -\alpha_1, \quad u'_a(a) = \alpha_0, \quad u_b(b) = -\beta_1, \quad u'_b(b) = \beta_0.$$

Then we can define $L^{-1} : C^\infty([0, 1]) \rightarrow D(L)$ by

$$L^{-1}f(x) := u_a(x) \int_x^b \frac{u_b}{W[u_a, u_b]} \frac{f}{(-p)} w + u_b(x) \int_a^x \frac{u_a}{W[u_a, u_b]} \frac{f}{(-p)} w,$$

where $W[u_a, u_b] := u_a u'_b - u_b u'_a$ denotes the Wronskian. This formula is derived from variation of parameters: we can compute c_a and c_b from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u'_a & u'_b \end{pmatrix} \begin{pmatrix} c'_a \\ c'_b \end{pmatrix} \implies L(c_a u_a + c_b u_b) = f.$$

Then, we can check that

$$L^{-1} L u = u$$

for $u \in D(L)$ by computation, which implies L is surjective. □

5.3 Legendre's equation

The Legendre equation is

$$(1-x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right).$$

Since $p(\pm 1) = 0$, the operator $L : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1])$ is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

l	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2} x \log \frac{1+x}{1-x}, \quad \dots \in L^2(-1, 1) \setminus C^\infty([-1, 1])$$

as eigenvectors of L , then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x) \left(Q'_0(x) f(x) - Q_0(x) f'(x) \right) \Big|_{-1}^1 \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for $f \in C^\infty([-1, 1])$.

5.4 Bessel's equation

The Bessel equation is

$$x^2 u'' + x u' + (k^2 x^2 - \nu^2) u = 0, \quad \text{on } (0, \infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x} \left[\frac{d}{dx} \left(x \frac{d}{dx} \right) - \nu^2 \frac{1}{x} \right].$$

6 Peetre's theorem

Lemma 6.1. Suppose a linear operator $L : C_c^\infty(M) \rightarrow C_c^\infty(M)$ satisfies

$$\text{supp}(Lu) \subset \text{supp}(u) \quad \text{for } u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m , we can find $u \in C_c^\infty(U \setminus \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C . We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x .

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U_i}.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m :

$$\begin{aligned} \sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\ &\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\ &< \infty. \end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction. □

7 Characteristic curve

Algorithm:

- (a) Establish the associated vector field by substituting $u \mapsto y$.
- (b) Find the integral curve.
- (c) Eliminate the auxiliary variables to get an algebraic equation.
- (d) Verify the computed solution is in fact the real solution.

Proposition 7.1. Suppose that there exists a smooth solution $u : \Omega \rightarrow \mathbb{R}_y$ of an initial value problem

$$\begin{cases} u_t + u^2 u_x = 0, (t, x) \in \Omega \subset \mathbb{R}_{t \geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, \text{ at } x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t, x)$.

Let $\gamma : I \rightarrow \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\tilde{\gamma} : \tilde{I} \rightarrow M$ as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\tilde{\gamma}(0) = \gamma(0)$. Since X and \tilde{X} coincide on M , the curve $\tilde{\gamma}$ is also an integral curve of X with $\tilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\tilde{I} \subset I$ and $\gamma(\theta) = \tilde{\gamma}(\theta)$ for all $\theta \in \tilde{I}$.

Since M is closed in E , the open interval $\tilde{I} = \gamma^{-1}(M)$ is closed in I , hence $\tilde{I} = I$ by the connectedness of I . □

Definition 7.1. The projection of the integral curve γ onto Ω is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X . Once we find a necessary condition of the form of algebraic equation, we can demonstrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u , we can solve the ODE: let $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$ be the integral curve of X such that $\gamma(0) = (0, \xi, \xi)$. Then, the system of ODEs

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi \end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad y(\theta) = \xi, \quad x(\theta) = \xi^2 \theta + \xi.$$

Therefore,

$$u(t, x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega = \{(t, x) : tx > -\frac{1}{4}\}.$$

7.1 Wave equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } t, x > 0, \\ u(0, x) &= g(x), \quad u_t(0, x) = h(x), \quad u_x(t, 0) = \alpha(t). \end{aligned}$$

Define $v := u_t - cu_x$. Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0, x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t, x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0, x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t, 0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter $\xi > 0$,

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t, x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi) \end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad x(\theta) = -c\theta + \xi, \quad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta') d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned} u(t, x) &= g(\xi) - \int_0^\theta v(s, \xi - cs) ds \\ &= g(x + ct) \\ &= \frac{3g(x + ct) - g(x - ct)}{2} - \int_0^t h(x + c(t - 2s)) ds \end{aligned}$$

7.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

- (a) Suppose $u(0, x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx -plane.
- (b) Suppose $u(0, x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx -plane.
- (c) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and “paste” the solution together.

8 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0, 1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every separable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem \rightarrow continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable strict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by \mathcal{T}_β coincides with \mathcal{T}_α)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- The integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x} : D' \rightarrow D'$ is continuous. They commute (Schwarz theorem holds).
- $D \rightarrow S \rightarrow L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

9 Ultrafilter

Definition 9.1. An *ultrafilter* is a synonym for maximal filter. If we say \mathcal{U} is an *ultrafilter* on a set A , then it means \mathcal{U} is a maximal filter as a directed subset of $\mathcal{P}(A)$.

existence of ultrafilter.

Theorem 9.1. Let \mathcal{U} be an ultrafilter on a set A and X be a compact space. For a function $f : A \rightarrow X$, the limit $\mathcal{U}\text{-}\lim f$ always exists.

Theorem 9.2. Let $X = \prod_{\alpha \in A} X_\alpha$ be a product space of compact spaces X_α . A net $f : \mathcal{D} \rightarrow X$ has a convergent subnet.

1. Use Tychonoff. Compactness and net compactness are equivalent. □
2. It is a proof without Tychonoff. Let \mathcal{U} be an ultrafilter on a set \mathcal{D} containing all $\uparrow d$. Define a directed set $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$ as $(d, U) \succ (d', U')$ for $U \subset U'$. Let $f : \mathcal{E} \rightarrow X$ be a subnet of $f : \mathcal{D} \rightarrow X$ defined by $f_{(d, U)} = f_d$.

By the previous theorem, $\mathcal{U}\text{-}\lim \pi_\alpha f_d \in X_\alpha$ exists for each α . Define $f \in X$ such that $\pi_\alpha f = \mathcal{U}\text{-}\lim \pi_\alpha f_d$. Let $G = \prod_\alpha G_\alpha \subset X$ be any open neighborhood of f . Then, $\pi_\alpha f \in G_\alpha$ and we have $G_\alpha = X_\alpha$ except finite. For α , we can take $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$ by definition of convergence with ultrafilter. Since $U_\alpha = \mathcal{D}$ except finites, we can take an upper bound $U_0 \in \mathcal{U}$ of $\{U_\alpha\}_\alpha$. Then, by taking any $d_0 \in U_0$, we have $f_{(d,U)} \in G$ for every $(d, U) \succ (d_0, U_0)$. This means $f = \lim_{\mathcal{E}} f_{(d,U)}$, so we can say $\lim_{\mathcal{E}} f_{(d,U)}$ exists. \square

10 Selected analysis problems

10.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \leq x - x_0 \leq x \leq 2^k$, $x - x_0 \in A_k$.

$$|A_k| \geq \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\ &\geq \sum_{k=1}^N (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^N \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &> \sum_{k=1}^N \frac{2^k}{2^{k+2}} \frac{1}{7k} \\ &= \frac{1}{28} \sum_{k=1}^N \frac{1}{k} \\ &\rightarrow \infty. \end{aligned}$$

\square

10.2. If $|xf'(x)| \leq M$ and $\frac{1}{x} \int_0^x f(y) dy \rightarrow L$, then $f(x) \rightarrow L$ as $x \rightarrow \infty$.

Solution. It is a kind of Tauberian theorems. Since for each fixed $\varepsilon > 0$ we have

$$\begin{aligned} |f(x) - \frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x f(y) dy| &\leq \frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x |f(x) - f(y)| dy \\ &\leq \frac{M}{\varepsilon x} \int_{(1-\varepsilon)x}^x \frac{x-y}{y} dy \\ &= M \left(\frac{1}{\varepsilon} \log \frac{1}{1-\varepsilon} - 1 \right) = O(\varepsilon) \end{aligned}$$

by the mean value theorem and

$$\frac{1}{\varepsilon x} \int_{(1-\varepsilon)x}^x f(y) dy = \frac{1}{\varepsilon x} \int_0^x f(y) dy - \frac{1}{\varepsilon x} \int_0^{(1-\varepsilon)x} f(y) dy \rightarrow \frac{1}{\varepsilon} L - \frac{1-\varepsilon}{\varepsilon} L = L$$

as $x \rightarrow \infty$, we get

$$\limsup_{x \rightarrow \infty} |f(x) - L| = O(\varepsilon),$$

so we are done. □

10.3. Let $f_n : [0, 1] \rightarrow [0, 1]$ be a sequence of functions such that $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$ for each $n \geq 1$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap [0, 1] \rightarrow [0, 1]$ pointwisely.

Let $n \geq 4$. Then, for $x \in [0, 1]$ there is $z \in [0, 1]$ such that $|x - z| = \frac{2}{n}$ so that

$$|f_n(x) - f_n(z)| \leq |x - z| = \frac{2}{n}.$$

Whenever $y \in [0, 1]$ satisfies $|x - y| \leq \frac{1}{n}$, then we have $|y - z| \geq |x - z| - |x - y| \geq \frac{1}{n}$, so we get

$$|f_n(y) - f_n(z)| \leq |y - z| \leq |y - x| + |x - z| \leq \frac{3}{n}.$$

Combining the two inequalities, we obtain

$$|x - y| \leq \frac{1}{n} \implies |f_n(x) - f_n(y)| \leq \frac{5}{n} \quad (1)$$

for $n \geq 4$.

Let $\varepsilon > 0$ and suppose $|x - y| \leq \frac{\varepsilon}{5}$. For every $n \geq \max\{\frac{10}{\varepsilon}, 4\}$, since $|x - y| \leq \frac{1}{n}$ implies by the inequality (1) that

$$|f_n(x) - f_n(y)| \leq \frac{5}{n} \leq \frac{\varepsilon}{2},$$

and since $|x - y| > \frac{1}{n}$ implies by the condition in the problem that

$$|f_n(x) - f_n(y)| \leq |x - y| \leq \frac{\varepsilon}{5} < \frac{\varepsilon}{2},$$

we have

$$|x - y| \leq \frac{\varepsilon}{5} \implies |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{2} \quad (2)$$

for all $n \geq \max\{\frac{10}{\varepsilon}, 4\}$.

For $\varepsilon > 0$, take $\delta := \varepsilon/5$ and fix x and y in $\mathbb{Q} \cap [0, 1]$ satisfying $|x - y| < \delta$. Then, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + \frac{\varepsilon}{2} + |f_n(y) - f(y)| \end{aligned}$$

for all $n \geq \max\{\frac{10}{\varepsilon}, 4\}$, and by limiting $n \rightarrow \infty$,

$$|f(x) - f(y)| \leq 0 + \frac{\varepsilon}{2} + 0 < \varepsilon.$$

Therefore, f is uniformly continuous on $\mathbb{Q} \cap [0, 1]$ so that it has a unique continuous extension on the whole $[0, 1]$. Let it denoted by the same notation f .

Finally, we are going to show $f_n \rightarrow f$ uniformly on $[0, 1]$. By the uniform continuity of f , for each $\varepsilon > 0$ we have $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}. \quad (3)$$

Take a finite subset $F \in \mathbb{Q} \cap [0, 1]$, such that for every x there is y satisfying $|x - y| < \min\{\frac{\varepsilon}{5}, \delta\}$. Then, by (2) and (3), we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(y)| + |f_n(y) - f(y)| + |f(y) - f(x)| \\ &< \frac{\varepsilon}{2} + \max_{z \in F} |f_n(z) - f(z)| + \frac{\varepsilon}{2} \end{aligned}$$

for all $n \geq \max\{\frac{10}{\varepsilon}, 4\}$. Therefore, by taking supremum for x and limiting $n \rightarrow \infty$ on it we have

$$\limsup_{n \rightarrow \infty} \|f_n - f\| \leq \varepsilon,$$

so we are done because ε is arbitrary. □

10.4. A measurable subset of \mathbb{R} with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

Solution. Let $E \subset \mathbb{R}$ be measurable with $\mu(E) > 0$. We may assume E is bounded so that we have $E \subset I$ for a closed bounded interval since \mathbb{R} is σ -compact. Let n be a positive integer arbitrarily taken. Then, we can find N such that $\sum_{k=1}^N \frac{1}{k} > (n-1) \frac{\mu(I)}{\mu(E)}$.

Assume that every point x in E is contained in at most $n-1$ sets among

$$E, \frac{1}{2}E, \frac{1}{3}E, \dots, \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k \in A} \frac{1}{k}E = \emptyset$$

for any subset $A \subset \{1, \dots, N\}$ with $|A| \geq n$. Define

$$E_A := \bigcap_{k \in A} \frac{1}{k}E \cap \bigcap_{k' \in A} \left(\frac{1}{k'}E\right)^c$$

for $A \subset \{1, \dots, N\}$. Then, $\mu(E_A) = 0$ for $|A| \geq n$.

Note that we have

$$\mu\left(\frac{1}{k}E\right) = \sum_{k \in A} \mu(E_A) = \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) = \sum_{k=1}^N \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A) = \sum_{|A| < n} |A| \mu(E_A)$$

by double counting, and since E_A are disjoint, we have

$$\sum_{|A| < n} |A| \mu(E_A) = (n-1) \sum_{0 < |A| < n} \mu(E_A) \leq (n-1) \mu(I),$$

hence a contradiction to

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) > (n-1) \mu(I).$$

Therefore, we may find an element x that belongs to $\frac{1}{k}E$ for $k \in A$, where $A \subset \{1, \dots, N\}$ with $|A| = n$. Then, $ax \in E$ for all $a \in A \subset \mathbb{Z}$. \square

11 Physics problem

11.1 Resonance

Let m, b, k, A, ω_d be positive real constants. Consider an underdamped oscillator with sinusoidal driving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0, \quad x'(0) = 0.$$

There are some observations:

- (a) The underdamping condition means $b^2 - 4mk < 0$ so that the roots of characteristic equation are imaginary.
- (b) The positivity of m, b implies the real part of solution that will be denoted by $-\beta = -\frac{b}{2m}$ is negative; it shows exponential decay of solutions.

(c) Introducing the natural frequency $\omega_n = \sqrt{k/m}$, we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega t.$$

(d) The complementary solution is computed as

$$x_c(t) = x_0 e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \rightarrow \infty} x_c(t) = 0.$$

(e) The condition $\beta > \omega_n$ is equivalent to that the oscillator is underdamped.

(f) Let m, k be fixed. Then, the solution x_c decays most fastly when b satisfied $b^2 = 4mk$, equivalently, $\beta = \omega_n$.

(g) When $\omega_d = \omega_n$ such that the amplitude of particular solution diverges.