Measure Theory

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Contents

Part I

Chapter 1

Measures

1.1 Measures

1.1 (σ-algebras)**.** Let X be a set. A *σ-algebra* of sets on X is a collection $\mathcal{A} \subset \mathcal{P}(X)$ which is closed under countable unions and complements.

(a) generated by a set.

(b) countable and cocountable sets

(c) Borel

1.2 (Measures)**.** A *measurable space* or a *Borel space* is a pair $(X, \mathcal{A})$ of a set X and a σ-algebra $\mathcal{A}$ on X. Each element of $\mathcal{A}$ is called *measurable*. We often omit $\mathcal{A}$ to just write X for $(X, \mathcal{A})$ if there is no confusion.

Let $(X, \mathcal{A})$ be a measurable space. A *measure* on $(X, \mathcal{A})$ is a set function $\mu : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ that is *countably additive*: we have

$$\mu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i), \qquad (E_i)_{i=1}^{\infty} \subset \mathcal{A}.$$

Here the squared cup notation reads the disjoint union. A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where μ is a measure on $(X, \mathcal{A})$. Let μ be a measure on X.

(a) μ is monotone: for $E, F \in \mathcal{A}$ if $E \subset F$ then $\mu(E) \le \mu(F)$.

(b) μ is countably subadditive: for

(c) μ is continuous from below:

(d) μ is continuous from above:

1.3 (Complete measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. A *null set* is a measurable set N satisfying $\mu(N) = 0$, and a *full set* is a measurable set whose complement is a null set.

A *complete measure* is a measure such that every subset of a null set is measurable.

For a predicate P of points $x \in X$, we say P is true *almost everywhere* or *a.e.* on X if there is a full set $F \subset X$ such that $P(x)$ is true for all $x \in F$.

1.2 Carathéodory extension

1.4 (Outer measures)**.** Let X be a set. An *outer measure* on X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty] : \varnothing \mapsto 0$ which is monotone and countably subadditive.

(i) μ^* is *monotone*: we have

$$S_1 \subset S_2 \quad \Rightarrow \quad \mu^*(S_1) \le \mu^*(S_2), \qquad S_1, S_2 \in \mathcal{P}(X),$$

(ii) μ^* is *countably subadditive*: we have

$$\mu^*\Big(\bigcup_{i=1}^{\infty} S_i\Big) \le \sum_{i=1}^{\infty} \mu^*(S_i), \qquad (S_i)_{i=1}^{\infty} \subset \mathcal{P}(X).$$

Comparing the definition of measures, we can see the outer measures extend the domain to the power set, but loosen the countable additivity to monotone countable subadditivity.

(a) A set function $\mu^* : \mathcal{P}(X) \to [0, \infty] : \varnothing \mapsto 0$ is an outer measure if and only if μ^* is *monotonically countably subadditive*:

$$S \subset \bigcup_{i=1}^{\infty} S_i \quad \Rightarrow \quad \mu^*(S) \le \sum_{i=1}^{\infty} \mu^*(S_i), \qquad S \in \mathcal{P}(X),\ (S_i)_{i=1}^{\infty} \subset \mathcal{P}(X).$$

(b) For any $\varnothing \in \mathcal{A}_0 \subset \mathcal{P}(X)$, let $\mu_0 : \mathcal{A}_0 \to [0, \infty] : \varnothing \mapsto 0$ be a set function. We can associate an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by defining as

$$\mu^*(S) := \inf\left\{ \sum_{i=1}^{\infty} \mu_0(B_i) : S \subset \bigcup_{i=1}^{\infty} B_i,\ B_i \in \mathcal{A}_0 \right\},$$

where we use the convention $\inf \varnothing = \infty$.

Proof. □

1.5 (Carathéodory measurable sets)**.** Let μ^* be an outer measure on a set X. We want to construct a measure by restriction of μ^* on a properly defined σ-algebra. A subset $E \subset X$ is called *Carathéodory measurable* relative to μ^* if

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$$

for every $S \in \mathcal{P}(X)$. Let $\mathcal{A} \subset \mathcal{P}(X)$ be the collection of all Carathéodory measurable subsets relative to μ^*.

(a) $\mathcal{A}$ is an algebra and μ^* is finitely additive on $\mathcal{A}$.

(b) $\mathcal{A}$ is a σ-algebra and μ^* is countably additive on $\mathcal{A}$. That is, $\mu := \mu^*|_{\mathcal{A}}$ is a measure.

(c) The measure μ is complete.

Proof. □

1.6 (Carathéodory extension theorem)**.** The Carathéodory extension is a construction method for a measure extending a given set function μ_0 on $\mathcal{A}_0 \subset \mathcal{P}(X)$ for a set X. The idea is to restrict the outer measure μ^* associated to μ_0 in order to obtain a measure μ. We want to find a sufficient condition for μ to be a measure on a σ-algebra containing $\mathcal{A}_0$.

Let $\varnothing \in \mathcal{A}_0 \subset \mathcal{P}(X)$, and let $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ be a set function with $\mu_0(\varnothing) = 0$. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the associated outer measure of μ_0, and $\mu : \mathcal{A} \to [0, \infty]$ the measure defined by the restriction of μ^* on Carathéodory measurable subsets.

(a) μ^* extends μ_0 if μ_0 satisfies the monotone countable subadditivity: we have

$$A \subset \bigcup_{i=1}^{\infty} B_i \quad \Rightarrow \quad \mu_0(A) \le \sum_{i=1}^{\infty} \mu_0(B_i), \qquad A \in \mathcal{A}_0,\ (B_i)_{i=1}^{\infty} \subset \mathcal{A}_0$$

(b) μ extends μ_0 if μ_0 satisfies the following property in addition: for $B, A \in \mathcal{A}_0$ and any $\varepsilon > 0$, there are $(C_j)_{j=1}^\infty, (D_j)_{j=1}^\infty \subset \mathcal{A}_0$ such that

$$B \cap A \subset \bigcup_{j=1}^\infty C_j, \quad B \setminus A \subset \bigcup_{j=1}^\infty D_j, \quad \sum_{j=1}^\infty (\mu_0(C_j) + \mu_0(D_j)) < \mu_0(B) + \varepsilon.$$

Proof. (a) Fix $A \in \mathcal{A}_0$. Clearly $\mu^*(A) \le \mu_0(A)$. For the opposite direction, we may assume $\mu^*(A) < \infty$. By the finiteness of $\mu^*(A)$, for any $\varepsilon > 0$ we have $(B_i)_{i=1}^\infty \subset \mathcal{A}_0$ such that $A \subset \bigcup_{i=1}^\infty B_i$ and

$$\sum_{i=1}^\infty \mu_0(B_i) < \mu^*(A) + \varepsilon.$$

Therefore we have $\mu_0(A) < \mu^*(A) + \varepsilon$ by the assumption, and we get $\mu_0(A) \le \mu^*(A)$ by limiting $\varepsilon \to 0$.

(b) Fix $A \in \mathcal{A}_0$. It is enough to check the inequality $\mu^*(S \cap A) + \mu^*(S \setminus A) \le \mu^*(S)$ for $S \in \mathcal{P}(X)$ with $\mu^*(S) < \infty$. By the finiteness of $\mu^*(S)$, we have $(B_i)_{i=1}^\infty \subset \mathcal{B}$ such that $S \subset \bigcup_{i=1}^\infty B_i$. From the condition, we have $B_i \cap A \subset \bigcup_{j=1}^\infty C_{i,j}$ and $B_i \setminus A \subset \bigcup_{j=1}^\infty D_{i,j}$ satisfying

$$\begin{aligned} \mu^*(S \cap A) + \mu^*(S \setminus A) &\le \mu^*\Big(\bigcup_{j=1}^\infty (B_i \cap A)\Big) + \mu^*\Big(\bigcup_{j=1}^\infty (B_i \setminus A)\Big) \\ &\le \sum_{i,j=1}^\infty (\mu_0(C_{i,j}) + \mu_0(D_{i,j})) \\ &\le \sum_{i=1}^\infty (\mu_0(B_i) + 2^{-i}\varepsilon) \\ &< \mu^*(S) + \varepsilon. \end{aligned}$$

Therefore, A is Carathéodory measurable relative to μ^*, so the domain of μ contains the domain of μ_0. The values coincide by the part (a). □

1.7 (Uniqueness of extension of measures)**.** The Carathéodory extension also provides a uniqueness result for measure extensions. Let $\rho : \mathcal{B} \to [0, \infty] : \varnothing \mapsto 0$ be a set function, where $\varnothing \in \mathcal{B} \subset \mathcal{P}(X)$ for a set X. We say ρ is *σ-finite* if there is a cover $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ of X such that $\rho(B_i) < \infty$ for each i.

Let $\mathcal{A}$ be a σ-algebra containing $\mathcal{B}$. Let μ be a measure on $\mathcal{A}$, which extends ρ, given by the restriction of the outer measure μ^* associated to ρ. Let ν be another measure on $\mathcal{A}$ which extends ρ. Let $E \in \mathcal{A}$ and $\{E_i\}_{i=1}^\infty \subset \mathcal{A}$.

(a) $\nu(E) \le \mu(E)$.
(b) $\nu(E_i) = \mu(E_i)$ implies $\nu\Big(\bigcup_{i=1}^\infty E_i\Big) = \mu\Big(\bigcup_{i=1}^\infty E_i\Big)$.
(c) $\nu(E) = \mu(E)$ for $\mu(E) < \infty$.
(d) $\nu(E) = \mu(E)$ for $\mu(E) = \infty$, if ρ is σ-finite

Proof. (a) We may assume $\mu(E) < \infty$. By the definition of the outer measure, there is $\{B_i\}_{i=1}^\infty \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^\infty B_i$. Also, whenever $E \subset \bigcup_{i=1}^\infty B_i$ we have

$$\nu(E) \le \nu\Big(\bigcup_{i=1}^\infty B_i\Big) \le \sum_{i=1}^\infty \nu(B_i) = \sum_{i=1}^\infty \rho(B_i) = \sum_{i=1}^\infty \mu(B_i),$$

hence $\nu(E) \le \mu(E)$.

(b) In the light of the inclusion-exclusion principle, we have

$$\mu(E_i \cup E_j) = \mu(E_i) + \mu(E_j) - \mu(E_i \cap E_j) \le \nu(E_i) + \nu(E_j) - \nu(E_i \cap E_j) = \nu(E_i \cup E_j),$$

so that $\mu(E_i \cup E_j) = \nu(E_i \cap E_j)$. Applying it inductively, we have for every n that

$$\mu\Big(\bigcup_{i=1}^{n} B_i\Big) = \nu\Big(\bigcup_{i=1}^{n} B_i\Big),$$

and by limiting $n \to \infty$ the continuity from below gives

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i\Big).$$

(c) Because $\mu(E) < \infty$, for any $\varepsilon > 0$ we have a sequence $(B_i)_{i=1}^{\infty} \subset \mathcal{B}$ such that $E \subset \bigcup_{i=1}^{\infty} B_i$ and

$$\sum_{i=1}^{\infty} \rho(B_i) < \mu(E) + \varepsilon.$$

Applying the part (b) Then, we have

$$\mu(E) \le \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \nu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) + \nu(E)$$

and

$$\nu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) \le \mu\Big(\bigcup_{i=1}^{\infty} B_i \setminus E\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) - \mu(E) \le \sum_{i=1}^{\infty} \mu(B_i) - \mu(E) = \sum_{i=1}^{\infty} \rho(B_i) - \mu(E) < \varepsilon,$$

we get $\mu(E) < \nu(E) + \varepsilon$ and $\mu(E) \le \nu(E)$ by limiting $\varepsilon \to 0$.

(d) Let $\{B_i\}_{i=1}^{\infty} \subset \mathcal{B}$ be a cover of X such that $\rho(B_i) < \infty$. Define $E_1 := B_1$ and $E_n := B_n \setminus \bigcup_{i=1}^{n-1} B_i$ for $n \ge 2$ so that $\{E_i\}_{i=1}^{\infty}$ is a pairwise disjoint cover of X with

$$\mu(E \cap E_i) \le \mu(E_i) \le \mu(B_i) = \rho(B_i) < \infty$$

for each i, so we have by the part (c) that

$$\nu(E) = \sum_{i=1}^{\infty} \nu(E \cap E_i) = \sum_{i=1}^{\infty} \mu(E \cap E_i) = \mu(E). \qquad \square$$

1.3 Measures on the real line

1.8 (Borel σ-algebra).

1.9 (Distribution functions).

1.10 (Helly selection theorem).

1.11 (Vitali set).

Exercises

1.12 (Boolean algebras and rings).

1.13 (Cardinalities). infinite σ-algebra is $\ge \mathfrak{c}$.

1.14 (Semi-rings and semi-algebras). We will prove a simplified Carathéodory extension with respect to *semi-rings* and *semi-algebras*. Let $\mathcal{A} \subset \mathcal{P}(X)$ such that $\varnothing \in \mathcal{A}$. We say that $\mathcal{A}$ is a semi-ring if it is closed under finite intersections, and each relative complement is a finite union of elements of $\mathcal{A}$. We say that $\mathcal{A}$ is a semi-algebra

Let $\mathcal{A}$ be a semi-ring of sets over X. Suppose a set function $\rho : \mathcal{A} \to [0, \infty] : \varnothing \mapsto 0$ satisfies

(i) ρ is *disjointly countably subadditive*: we have

$$\rho\Big(\bigsqcup_{i=1}^{\infty} A_i\Big) \le \sum_{i=1}^{\infty} \rho(A_i)$$

for $(A_i)_{i=1}^{\infty} \subset \mathcal{A}$,

(ii) ρ is *finitely additive*: we have

$$\rho(A_1 \sqcup A_2) = \rho(A_1) + \rho(A_2)$$

for $A_1, A_2 \in \mathcal{A}$.

A set function satisfying the above conditions are occasionally called a *pre-measure*.

(a)

(b)

1.15 (Monotone class lemma)**.** A collection $\mathcal{C} \subset \mathcal{P}(X)$ is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Let H be a vector space closed under bounded monotone convergence. If $\operatorname{span}\{\mathbf{1}_A : A \in \mathcal{A}\} \subset H$ then $B^{\infty}(\sigma(\mathcal{A})) \subset H$.

1.16 (Steinhaus theorem)**.** Let λ denote the Lebesgue measure on $\mathbb{R}$ and let $\mathbb{E} \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.

(a) For any $0 < \alpha < 1$, there is an interval $I = (a, b)$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.

(b) $E - E = \{x - y : x, y \in E\}$ contains an open interval containing zero.

Proof. (a) We may assum $\lambda(E) < \infty$. Since λ is outer measure and $\lambda(E) \neq 0$, we have an open subset U of $\mathbb{R}$ such that $\lambda(U) < \alpha^{-1}\lambda(E)$. Because U is a countable disjoint union of open intervals $U = \bigsqcup_{i=1}^{\infty}(a_i, b_i)$, we have

$$\sum_{i=1}^{\infty} \lambda((a_i, b_i)) = \lambda(U) < \alpha^{-1}\lambda(E) = \alpha^{-1}\sum_{i=1}^{n} \lambda(E \cap (a_i, b_i)).$$

Therefore, there is i such that $\alpha\lambda((a_i, b_i)) < \lambda(E \cap (a_i, b_i))$. □

Problems

*1. Every Lebesgue measurable set in $\mathbb{R}$ of positive measure contains an arbitrarily long arithmetic progression.

Chapter 2

Measurable functions

2.1 Simple functions

2.1 (Measurability of pointwise limits)**.** Conversely, every measurable extended real-valued function is a pointwise limit of simple functions.

Proof. Let $f(x) = \lim_{n\to\infty} s_n(x)$.

□

2.2 Almost everywhere convergence

2.2 (Almost everywhere convergence)**.** Let (X,μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ be measurable functions. The set of convergence of the sequence f_n is defined as the set

$$\{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\},$$

and the set of divergence is defined as its complement. We say f_n converges to f *alomst everywhere* with respect to μ if the set of divergence is a null set in μ. We simply write

$$f_n \to f \text{ a.e.}$$

if f_n converges to f almost everywhere, and we frequently omit the measure μ if it has no confusion.

(a) If μ is complete and, if $f_n \to f$ a.e., then f is measurable.

2.3 (Borel-Cantelli lemma)**.** Let (X,μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ and $f : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Note that the set of divergence is given by

$$\bigcup_{\varepsilon>0}\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_n(x) - f(x)| \geq \varepsilon\}.$$

Each measurable set of the form

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

is sometimes called the *tail event*, coined in probability theory.

(a) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\{x : \limsup_{n\to\infty} |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

(b) $f_n \to f$ a.e. if and only if for each $\varepsilon > 0$ we have

$$\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \ge \varepsilon\}) = 0.$$

(c) $f_n \to f$ a.e. if for each $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) < \infty.$$

Proof. (b) The set of divergence of the sequence f_n is given by

$$\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \ge \tfrac{1}{m}\} = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}(X \setminus E_n^m).$$

(c) Since

$$\mu\Big(\bigcup_{i=1}^{\infty}\{x : |f_i(x) - f(x)| \ge \varepsilon\}\Big) \le \sum_{i=1}^{\infty}\mu(\{x : |f_i(x) - f(x)| \ge \varepsilon\}) < \infty,$$

we have by the continuity from above that

$$\begin{aligned}\mu(\limsup_{n\to\infty}\{x : |f_n(x) - f(x)| \ge \varepsilon\}) &= \mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \ge \varepsilon\}\Big) \\ &= \lim_{n\to\infty}\mu\Big(\bigcup_{i=n}^{\infty}\{x : |f_i(x) - f(x)| \ge \varepsilon\}\Big) \\ &\le \lim_{n\to\infty}\sum_{i=n}^{\infty}\mu(\{x : |f_i(x) - f(x)| \ge \varepsilon\}) = 0.\end{aligned}$$

□

2.4 (Convergence in measure). Let (X, μ) be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. We say f_n converges to a measurable function $f : X \to \overline{\mathbb{R}}$ *in measure* if for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) = 0.$$

(a) If $f_n \to f$ in measure, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

(b) If every subsequence f_{n_k} of f_n has a further subsequence $f_{n_{k_j}}$ such that $f_{n_{k_j}} \to f$ a.e., then $f_n \to f$ in measure.

Proof. (a) Since for each positive integer k we have $\mu(\{x : |f_n(x) - f(x)| \ge \frac{1}{k}\}) \to 0$ as $n \to \infty$, there exists n_k such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| \ge \tfrac{1}{k}\}) < \frac{1}{2^k}.$$

By the Borel-Cantelli lemma, we get

$$\mu(\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \ge \tfrac{1}{k}\}) = 0.$$

Then, for each $\varepsilon > 0$,

$$\begin{aligned}\limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \ge \varepsilon\} &= \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \ge \varepsilon\} \\ &\subset \bigcap_{k=\lceil\varepsilon^{-1}\rceil}^{\infty}\bigcup_{j=k}^{\infty}\{x : |f_{n_j}(x) - f(x)| \ge \tfrac{1}{k}\} \\ &= \limsup_{k\to\infty}\{x : |f_{n_k}(x) - f(x)| \ge \tfrac{1}{k}\}\end{aligned}$$

implies the limit superior of the tail events is a null set, hence $f_{n_k} \to f$ a.e.

(b) □

2.5 (Egorov theorem)**.** Egorov's theorem informally states that an almost everywhere convergent functional sequence is "almost" uniformly convergent. Through this famous theorem, we introduce a convenient "$\varepsilon/2^m$ argument", occasionally used throughout measure theory to construct a measurable set having a special property.

Let (X,μ) be a finite measure space and let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that $f_n \to f$ a.e. For each positive integer m, which indexes the tolerance $1/m$, consider an increasing sequence of measurable subsets

$$E_n^m := \bigcap_{i=n}^{\infty} \{x : |f_i(x) - f(x)| < \tfrac{1}{m}\}.$$

(a) E_n^m converges to a full set for each m.

(b) For every $\varepsilon > 0$ there is a measurable $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and for each m there is finite n satisfying $K \subset E_n^m$.

(c) For every $\varepsilon > 0$ there is a measurable $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on K.

Proof. (a) Recall that the a.e. convergence $f_n \to f$ means that for every fixed m the intersection

$$\bigcap_{n=1}^{\infty} (X \setminus E_n^m) = \limsup_n \{x : |f_n(x) - f(x)| \geq \tfrac{1}{m}\}$$

is a null set. Since $\mu(X) < \infty$, it is equivalent to E_n^m converges to a full set for each m by the continuity from above.

(b) For each m, we can find n_m such that

$$\mu(X \setminus E_{n_m}^m) < \frac{\varepsilon}{2^m}.$$

If we define

$$K := \bigcap_{m=1}^{\infty} E_{n_m}^m,$$

then it satisfies the second conclusion, and also have

$$\mu(X \setminus K) = \mu\Big(\bigcup_{m=1}^{\infty} (X \setminus E_{n_m}^m)\Big) \leq \sum_{m=1}^{\infty} \mu(X \setminus E_{n_m}^m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon.$$

(c) Fix $m > 0$. Since $n \geq n_m$ implies $K \subset E_{n_m}^m \subset E_n^m$, we have

$$n \geq n_m \quad \Rightarrow \quad \sup_{x \in K} |f_n(x) - f(x)| < \frac{1}{m}. \qquad \square$$

Exercises

2.6 (Cauchy's functional equation)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Cauchy's functional equation refers to the equation $f(x+y) = f(x) + f(y)$, satisfied for all $x, y \in \mathbb{R}$. Suppose f satisfies the Cauchy functional equation. We ask if f is linear, that is $f(x) = ax$ for all $x \in \mathbb{R}$, where $a := f(1)$.

(a) $f(x) = ax$ for all $x \in \mathbb{Q}$, but there is a nonlinear solution of Cauchy's functional equation.

(b) If f is conitnuous at a point, then f is linear.

(c) If f is Lebesgue measurable, then f is linear.

2.7 (Pointwise approximation by simple functions)**.** Let (X,μ) be a measure space and X a metric space with Borel measurable structure. By a *simple function* we mean a measurable function $s : X \to X$ of finite image.

(a) For each open set $U \subset X$ there is a sequence of open sets U_i such that $U = \bigcup_i U_i$ and $\overline{U}_i \subset U$. Let $f : X \to X$ be any function.

(b) If f is the pointwise limit of a sequence of measurable functions, then f is measurable.

(c) If f is measurable, then f is the pointwise limit of a sequence of simple functions, if X is separable.

*(d) The pointwise limit of a net of simple functions may not be measurable.

Proof. (b) Suppose a sequence $(f_n)_n$ of measurable functions converges pointwisely to a function f. For fixed open $U \subset X$ we claim

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} \liminf_{n\to\infty} f_n^{-1}(U_i).$$

If it is true, then $f^{-1}(U)$ is the countable set operation of measurable sets $f_n^{-1}(U_i)$. Let U_i be the sequence associated to U taken by the part (a).

($\subset$) If $\omega \in f^{-1}(U)$, then for some i we have $f(\omega) \in U_i$, so $f_n(\omega)$ is eventually in U_i, thus we have $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$.

($\supset$) If $\omega \in \liminf_{n\to\infty} f_n^{-1}(U_i)$ for some i, then $f_n(\omega)$ is eventually in U_i, so $f(\omega) \in \overline{U}_i \subset U$, thus we have $\omega \in f^{-1}(U)$.

(c) Suppose there is a increasing sequence of finite tagged partitions $\mathcal{P}_n \subset \mathcal{B}$ satisfying the following property: for each open-neighborhood pair (x, U) there is n and i such that $P_{n,i} \in \mathcal{P}_n$ and $x \in P_{n,i} \subset U$. We denote the tags by $t_{n,i} \in P_{n,i}$ for each $P_{n,i} \in \mathcal{P}_n$. Define

$$s_n(\omega) := t_{n,i} \quad \text{for} \quad f(\omega) \in P_{n,i}.$$

To show $s_n(\omega) \to f(\omega)$, fix an open $f(\omega) \in U \subset X$. Then, there is n_0 such that there is a sequence $(P_{n,i_n})_{n=n_0}^{\infty}$ satisfying $P_{n,i_n} \in \mathcal{P}_n$ and $f(\omega) \in P_{n,i_n} \subset U$. Then, for all $n \geq n_0$, we have for $f(\omega) \in P_{n,i_n}$ that $s_n(\omega) = t_{n,i_n} \in P_{n,i_n} \subset U$.

The existence of such sequence of partitions...

Another approach: mimicking Pettis measurability theorem. $\square$

Chapter 3

Lebesgue integral

3.1 Monotone convergence theorem

3.1 (Lebesgue integral of non-negative functions)**.** Let (X, μ) be a measure space. Let $f : X \to \mathbb{R}_{\geq 0}$ be a measurable function. The *Lebesgue integral* of f is defined by

$$\int f \, d\mu := \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, \ s \text{ simple} \right\}$$

3.2 (Monotone convergence theorem)**.** Let (X, μ) be a measure space. Let f_n and f be measurable functions $X \to \mathbb{R}_{\geq 0}$.

(a) $E \mapsto \int_E f \, d\mu$ is a measure if f is simple.

(b) $E \mapsto \int_E f \, d\mu$ is a measure even if f is not simple.

(c) If $f_n \uparrow f$ a.e., then $\int f_n \to \int f$.

Proof. (a) Clear from the linearity of the integral for simple functions.

(b) For $E_n \uparrow E$, we want to show the continuity from below, $\int_{E_n} f \to \int_E f$. Take $\varepsilon > 0$. We introduce a continuous bijection $\beta : [0, \infty] \to [0, 1] : t \mapsto t/(1+t)$ to avoid dividing the cases for infinity. By the definition of the Lebesgue integral, we have a simple function s such that $0 \leq s \leq f$ and

$$\beta(\textstyle\int_E f) - \beta(\int_E s) < \varepsilon,$$

whether or not $\int_E f$ diverges. Then,

$$\begin{aligned} \beta(\textstyle\int_E f) - \beta(\int_{E_n} f) &= [\beta(\textstyle\int_E f) - \beta(\int_E s)] + [\beta(\int_E s) - \beta(\int_{E_n} s)] + [\beta(\int_{E_n} s) - \beta(\int_{E_n} f)] \\ &< \qquad \varepsilon \qquad + [\beta(\textstyle\int_E s) - \beta(\int_{E_n} s)] + \qquad 0 \\ &\xrightarrow{n\to\infty} \varepsilon \end{aligned}$$

by the part (a). We are done by letting $\varepsilon \to 0$.

(c) Define $E_n := \{x : f(x) < (1+\varepsilon) f_n(x)\}$, which converges to a full set because $f_n \to f$ a.e. Since f is a measure, we can choose N such that

$$\beta(\textstyle\int_E f) - \beta(\int_{E_N} f) < \varepsilon.$$

With this N, we have

$$\beta(\textstyle\int_{E_N} f) - \beta(\int_{E_N} f_n) \leq \dfrac{\int_{E_N} f - \int_{E_N} f_n}{(1 + \int_{E_N} f)(1 + \int_{E_N} f_n)} < \varepsilon, \qquad n \geq N.$$

Then, we have for $n \geq N$ that

$$\begin{aligned}\beta(\textstyle\int_E f) - \beta(\int_E f_n) &= [\beta(\textstyle\int_E f) - \beta(\int_{E_N} f)] + [\beta(\int_{E_N} f) - \beta(\int_{E_N} f_n)] + [\beta(\int_{E_N} f_n) - \beta(\int_E f_n)] \\ &< 0 + \varepsilon + \varepsilon,\end{aligned}$$

so we are done by letting $n \to \infty$ and $\varepsilon \to 0$. $\square$

3.3 (Corollaries of monotone convergence theorem)**.** Fatou's lemma, linearity of the integral, $f \geq 0$ and $\int f = 0$ imply $f = 0$ a.e.

3.2 Dominated convergence theorem

3.4 (Lebesgue integral of complex-valued functions)**.**

3.5 (Bounded convergence theorem)**.** Semifinite measures

(a)

$$\sup_{g \leq f} \int g \, d\mu = \int f \, d\mu$$

where g runs through bounded measurable functions.

(b)

3.3 Product measures

3.6 (Fubini-Tonelli theorem)**.** Lebesgue measure on Euclidean spaces

Exercises

3.7 (Convergence of one-parameter family)**.**

If $\|f_n\|_{L^2([0,1])} \leq C$ and $f_n \to f$ almost everywhere, then $f_n \to f$ weakly.

$$\lim_{n\to\infty} \int_0^1 n^3 x^2 (1-x)^n \, dx = 2 \neq 0 = \int_0^1 \lim_{n\to\infty} n^3 x^2 (1-x)^n \, dx.$$

$$\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \, dx = \infty \neq 0 = \int_0^\infty \lim_{n\to\infty} n^2 e^{-nx} \, dx.$$

Part II

Chapter 4

Signed measures

4.1 Radon-Nikodym theorem

An integrable function as a measure σ-finite measures

Chapter 5

Borel measures

5.1 Continuous functions on metric spaces

partition of unity. Urysohn and Tietze.

5.1 (Regular Borel measures on metric spaces)**.** Let μ be a Borel measure on a metric space X. We say μ is *outer regular* if

$$\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\},$$

and say μ is *inner regular* if

$$\mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ closed}\},$$

for every Borel subset $E \subset X$. If μ is both outer and inner regular, we say μ is *regular*.

(a) Let E be σ-finite. Then, E is μ-regular if and only if for any $\varepsilon > 0$ there are open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon$.

(b) If μ is σ-finite, then the set of μ-regular subsets is a σ-algebra. (may be extended?)

(c) Every closed set is G_δ.

(d) Every finite Borel measure on X is regular.

Proof. $\square$

5.2 (Luzin's theorem)**.** Let μ be a regular Borel measure on a metric space X. Let $f : X \to \mathbb{R}$ be a Borel measurable function. Two proofs: direct and Egoroff.

(a) If $E \subset X$ is σ-finite, then there is a continuous g blabla

(b) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset X$ such that $f|_F : F \to \mathbb{R}$ is continuous and $\mu(X \setminus F) < \varepsilon$.

(c) If f vanishes outside a σ-finite set, then for any $\varepsilon > 0$ there is a closed set $F \subset X$ and continuous $g : X \to \mathbb{R}$ such that $f|_F = g|_F$ and $\mu(X \setminus F) < \varepsilon$.

(d) If f is further bounded, then g also can be taken to be bounded.

Proof. (a) Let $\varepsilon > 0$ and suppose $E \subset X$ is measurable with $\mu(E) < \infty$. Since E is σ-finite, we have open U and closed F such that $F \subset E \subset U$ and $\mu(U \setminus F) < \varepsilon/2$. By the Urysohn lemma, there is a continuous function $g : X \to [0,1]$ such that $g|_{U^c} = 0$ and $g|_F = 1$. Then,

$$\int |\mathbf{1}_E - g|\, d\mu = \int_{U \setminus F} |\mathbf{1}_E - g|\, d\mu \le 2\mu(U \setminus F) < \varepsilon.$$

(b) Since $\mathbb{R}$ is second countable, we have a base $(V_n)_{n=1}^\infty$ of $\mathbb{R}$. Since μ is σ-finite, for each n we can take open U_n and closed F_n such that

$$F_n \subset f^{-1}(V_n) \subset U_n$$

and $\mu(U_n \setminus F_n) < \varepsilon/2^n$. Define $F := \left(\bigcup_{n=1}^\infty (U_n \setminus F_n)\right)^c$ so that $\mu(X \setminus F) < \varepsilon$ and F is closed. Then,

$$\begin{aligned} U_n \cap F &= U_n \cap ((U_n^c \cup F_n) \cap F) \\ &= (U_n \cap (U_n^c \cup F_n)) \cap F \\ &= (\varnothing \cup (U_n \cap F_n)) \cap F \\ &\subset F_n \cap F \end{aligned}$$

proves $f^{-1}(V_n)$ is open in F for every n, hence the continuity of $f|_F$. (In fact, we require that X to be just a topological space.)

(b') We can alternatively use the part (a) and the Egoroff theorem. By the part (a), we can construct a sequence (f_n) of continuous functions $X \to \mathbb{R}$ such that $f_n \to f$ in L^1. By taking a subsequence, we may assume $f_n \to f$ pointwise. Assuming μ is finite, by the Egorov theorem, there is a measurable $A \subset X$ such that $f_n \to f$ uniformly on A and $\mu(X \setminus A) < \varepsilon/2$. Since μ is inner regular, we have closed $F \subset A$ such that $\mu(A \setminus F) < \varepsilon/2$, so that we have $\mu(X \setminus F) < \varepsilon$. Then, f is continuous on A, and of course on F.

□

Proposition 5.1.1. *A σ-finite Radon measure is regular.*

Proof. First we approximate Borel sets of finite measure, with compact sets. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U - E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned} \mu(K - V) &\geq \mu(K) - \mu(V) \\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U - E) + \frac{\varepsilon}{2}\right) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^\infty E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^\infty K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^\infty \mu(K_n) > \sum_{n=1}^\infty \left(\mu(E_n) - \frac{\varepsilon}{2^n}\right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. □

5.2 Locally compact spaces

compact closed set not containing infty
open open not containing infty
closed closed set containing infty

for a measure that "vanishes at infty" = tight
two definitions of inner regularity is equivalent.

inner regular on compact sets -> inner regular on closed sets
inner regular on compact sets + sigma finite -> tight

5.3 (One-point compactification).

5.4 (Regular Borel measures on locally compact metric spaces). sss

(a) $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

(b) If μ is σ-finite, then for any $\varepsilon > 0$ there is compact $K \subset X$ and continuous $g : X \to \mathbb{R}$ such that $f|_K = g|_K$ and $\mu(X \setminus K) < \varepsilon$.

5.5 (Tightness and inner regularity). We have a similar but confusing concept called tightness; we say a Borel measure μ on a topological space X is *tight* if for any $\varepsilon > 0$ there is a compact $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$.

History of Bourbaki's text.

(a)

5.3 Riesz-Markov-Kakutani representation theorem

5.6 (Radon measures). Let X be a locally compact metric space. A *Radon measure* is a Borel measure μ on X such that

(i) μ is outer regular for every Borel set: $\mu(E) = \inf\{\mu(U) : E \subset U,\, U \text{ open}\}$ for Borel $E \subset X$,

(ii) μ is inner regular for every open set: $\mu(U) = \sup\{\mu(K) : K \subset U,\, K \text{ compact}\}$ for open $U \subset X$,

(iii) μ is locally finite.

(a) A σ-finite Radon measure is regular.

(b) If every open subset of X is σ-compact, then a locally finite Borel measure is Radon.

(c) $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

5.7 (Riesz-Markov-Kakutani representation theorem for $C_0(X)$). Let X be a locally compact metric space. We want to establish the following one-to-one correspondence:

$$\begin{array}{ccc} \{\text{finite Radon measures on } X\} & \xrightarrow{\sim} & \{\text{positive linear functionals on } C_0(X)\} \\ \mu & \mapsto & (f \mapsto \int f\, d\mu). \end{array}$$

Let I a positive linear functional on $C_0(X)$. Let $\mathcal{T}$ be the set of all open subsets of X and $\mu_0 : \mathcal{T} \to [0, \infty]$ a set function defined such that

$$\mu_0(U) := \sup\{I(f) : f \in C_c(U, [0,1])\}, \qquad U \in \mathcal{T}.$$

Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be the associated outer measure defined by

$$\mu^*(S) := \inf\left\{\sum_{i=1}^{\infty} \mu_0(U_i) : S \subset \bigcup_{i=1}^{\infty} U_i,\ U_i \in \mathcal{T}\right\}, \qquad S \in \mathcal{P}(X),$$

and let $\mu := \mu^*|_{\mathcal{A}}$ be the restriction, where $\mathcal{A}$ is the σ-algebra of Carathéodory measurable subsets relative to μ^*.

(a) μ^* extends μ_0.

(b) μ extends μ_0.

(c) μ is a finite Radon measure.

(d) The correspondence is surjective.

(e) The correspondence is injective.

Proof. (a) It suffices to show that μ_0 satisfies monotonically countably subadditive. For an open set U and a countable open cover $\{U_i\}_{i=1}^{\infty}$ of U we claim that $\rho(U) \le \sum_{i=1}^{\infty} \rho(U_i)$.

Take any $f \in C_c(U, [0, 1])$ and find a finite subcover $\{U_{i_k}\}_{k=1}^{n}$ of $\{U_i\}$ together with a partition of unitiy $\{\chi_{i_k}\}$ subordinate to the open cover $\{U_{i_k} \cap \operatorname{supp} f\}_k$. Now we have $f\chi_{i_k} \in C_c(U_{i_k}, [0, 1])$ for each k, because then I is linear so that it preserves finite sum, we have

$$I(f) = \sum_{k=1}^{n} I(f\chi_{i_k}) \le \sum_{k=1}^{n} \mu_0(U_{i_k}) \le \sum_{i=1}^{\infty} \mu_0(U_i).$$

Since f is arbitrary, we are done.

(b) We claim $\mathcal{T} \subset \mathcal{A}$. It suffices to show $\mu^*(E \cap U) + \mu^*(E \setminus U) \le \mu^*(E)$ for any measurable E and open U. Take $\varepsilon > 0$. Since we may assume $\mu^*(E) < \infty$, there is a countable open cover $\{U_i\}_{i=1}^{\infty}$ of E such that

$$\sum_{i=1}^{\infty} \mu_0(U_i) < \mu^*(E) + \frac{\varepsilon}{3}.$$

Take $f_i \in C_c(U_i \cap U, [0, 1])$ such that

$$\mu_0(U_i \cap U) < I(f_i) + \frac{1}{3} \cdot \frac{\varepsilon}{2^i},$$

and take $g_i \in C_c(U_i \setminus \operatorname{supp} f_i, [0, 1])$ such that

$$\mu_0(U_i \setminus \operatorname{supp} f_i) < I(g_i) + \frac{1}{3} \cdot \frac{\varepsilon}{2^i}.$$

Then, since $f_i + g_i \in C_c(U_i, [0, 1])$, we have

$$\begin{aligned}
\mu^*(E \cap U) + \mu^*(E \setminus U) &\le \sum_{i=1}^{\infty} \mu_0(U_i \cap U) + \sum_{i=1}^{\infty} \mu_0(U_i \setminus U) \\
&< \sum_{i=1}^{\infty} I(f_i + g_i) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \sum_{i=1}^{\infty} \mu_0(U_i) + \frac{2}{3}\varepsilon \\
&\le \mu^*(E) + \varepsilon.
\end{aligned}$$

Limiting $\varepsilon \to 0$, we get the desired inequality.

(c) Since μ is a countably additive and $\mathcal{T}$ is closed under union, we can rewrite

$$\mu^*(S) = \inf\{\mu_0(U) : S \subset U \in \mathcal{T}\}, \qquad S \in \mathcal{P}(X),$$

hence μ is outer regular. Here now we claim for $f \in C_c(X,[0,1])$ and $0<a<1$ that

$$a\mu(f^{-1}((a,1])) \le I(f) \le \mu(\operatorname{supp} f).$$

If it is true, then the right inequality implies the inner regularity, and the left inequality together with the Urysohn lemma implies the local finiteness.

The right inequality directly follows from the definition of μ_0 and the outer regularity

$$I(f) \le \inf\{\mu_0(U) : \operatorname{supp} f \subset U \in \mathcal{T}\} = \mu(\operatorname{supp} f).$$

For the left, if $h \in C_c(f^{-1}((a,1]),[0,1])$, then the inequality $ah \le f$ implies

$$a\mu(f^{-1}((a,1])) = a\mu_0(f^{-1}((a,1])) \le aI(h) \le I(f).$$

(d) We will show $I(f) = \int f\,d\mu$ for $f \in C_c(X)$. Since $C_c(X)$ is the linear span of $C_c(X,[0,1])$, we may assume $f \in C_c(X,[0,1])$. For a fixed positive integer n and for each index $1 \le i \le n$, let $K_i := f^{-1}([i/n,1])$ and define

$$f_i(x) := \begin{cases} \frac{1}{n} & \text{if } x \in K_i, \\ f(x) - \frac{i-1}{n} & \text{if } x \in K_{i-1} \setminus K_i, \\ 0 & \text{if } x \in X \setminus K_{i-1}, \end{cases}$$

where $K_0 := \operatorname{supp} f$. Note that $f_i \in C_c(X,[0,n^{-1}])$ and $f = \sum_{i=1}^n f_i$. For $1 \le i \le n$ we have $\mu(K_i) < \infty$ because K_i is compact subsets contained in a locally compact Hausdorff space $U := f^{-1}((0,1])$. By the previous claim and the property of integral, we have

$$\frac{\mu(K_i)}{n} \le I(f_i), \qquad \frac{\mu(K_i)}{n} \le \int f_i\,d\mu, \qquad 1 \le i \le n$$

and

$$I(f_i) \le \frac{\mu(K_{i-1})}{n}, \qquad \int f_i\,d\mu \le \frac{\mu(K_{i-1})}{n}, \qquad 2 \le i \le n.$$

Then, using the above inequalities and $\mu(K_n) \ge 0$, we have

$$I(f) \le I(f_1) + \int f\,d\mu \quad \text{and} \quad \int f\,d\mu \le \int f_1\,d\mu + I(f).$$

Note that $f_1 = \min\{f, n^{-1}\}$ is a sequence of functions indexed by n. By the monotone convergence theorem, $\int f_1\,d\mu \to 0$ as $n \to \infty$. We now show $I(f_1)$ converges to zero. If we let $U := f^{-1}((0,1])$, then U is locally compact and $f_1 \in C_0(U) \subset C_c(X)$, and since a positive linear functional on $C_0(U)$ is bounded, we have $I(f_1) \le n^{-1}\|I\| \to 0$ as $n \to \infty$. ($\mu(K_0)$ is possibly infinite if X is not locally compact so that μ is not locally finite.)

(e) Let μ and ν be finite Radon measures on X such that

$$\int g\,d\mu = \int g\,d\nu$$

for all $g \in C(X)$. Let E be any measurable set. Since $\mu+\nu$ is a finite Radon measure, and by the Luzin theorem, we have a closed set F and $g \in C(X)$ with $0 \le g \le 1$ such that $\mathbf{1}_E|_F = g|_F$ and $(\mu+\nu)(X \setminus F) < \varepsilon/2$. Then,

$$\begin{aligned} |\mu(E) - \nu(E)| &= |\int \mathbf{1}_E\,d\mu - \int \mathbf{1}_E\,d\nu| \\ &\le \int_{X\setminus F} |\mathbf{1}_E - g|\,d\mu + \int_{X\setminus F} |g - \mathbf{1}_E|\,d\nu \\ &\le 2\mu(X \setminus F) + 2\nu(X \setminus F) < \varepsilon. \end{aligned}$$

By limiting $\varepsilon \to 0$, we have $\mu(E) = \nu(E)$. □

5.8 (Dual of continuous function spaces).

5.4 Hausdorff measures

Exercises

Chapter 6

Lebesgue spaces

6.1 L^p spaces

6.1 (Hölder inequality).

Proof.

$$\int fg \le C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q}$$

Take C such that

$$C^p \int \frac{|f|^p}{p} = \frac{1}{C^q} \int \frac{|g|^q}{q}.$$

Then,

$$C^p \int \frac{|f|^p}{p} + \frac{1}{C^q} \int \frac{|g|^q}{q} = 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \Big(\int |f|^p \Big)^{\frac{1}{p}} \Big(\int |g|^p \Big)^{\frac{1}{q}}.$$

Note that we can show that $1 \le 2p^{-\frac{1}{p}} q^{-\frac{1}{q}} \le 2$ and the minimum is attained only if $p = q = 2$, so this method does not provide the sharpest constant. $\square$

6.2 L^1 spaces

6.2 (Convolution?).

6.3 (Approximate identity?).

6.4 (Continuity of translation?).

6.3 L^2 spaces

6.4 L^∞ spaces

Part III

Distribution theory

Chapter 7

Test functions

Chapter 8

Distributions

Chapter 9

Linear operators

9.1 Boundedness

Translation and multiplication operators

9.1 (Bitranspose extension).

9.2 Kernels

9.2 (Schur test).

9.3 (Young's inequality of integral operators).

9.3 Convolution

9.4 (Approximation of identity). Fejér, Poisson, box?

9.5 (Summability methods).

Part IV

Fundamental theorem of calculus

Chapter 10

Absolute continuity

The space of weakly differentiable functions with respect to all variables $= W^{1,1}_{\mathrm{loc}}$.

10.1 (Product rule for weakly differentiable functions)**.** We want to show that if u, v, and uv are weakly differentiable with respect to x_i, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

(a) If u is weakly differentiable with respect to x_i and $v \in C^1$, then $\partial_{x_i}(uv) = \partial_{x_i}uv + u\partial_{x_i}v$.

10.2 (Interchange of differentiation and integration)**.** Let $f : X_x \times X_y \to \mathbb{R}$ be such that $\partial_{x_i}f$ is well-defined. Suppose f and $\partial_{x_i}f$ are locally integrable in x and integrable y.

Then,

$$\partial_{x_i}\int f(x,y)\,dy = \int \partial_{x_i}f(x,y)\,dy.$$

Do not think the Schwarz theorem as the condition for partial differentiation to commute. We should understand like this: if F is C^2 then the *classical* partial differentiation commute, and if F is not C^2 then the *classical* partial derivatives of order two or more are *meaningless* because it is not compatible with the generalized concept of differentiation.

10.1 Absolutely continuity

(a) f is $\mathrm{Lip}_{\mathrm{loc}}$ iff f' is L^∞_{loc}
(b) f is $\mathrm{AC}_{\mathrm{loc}}$ iff f' is L^1_{loc}

(a) f is Lip iff f' is L^∞
(b) f is AC iff f' is L^1
(c) f is BV iff f' is a finite regular Borel measure

10.3 (Absolute continuous measures)**.**

10.4 (Absolute continuous functions)**.**

10.5 (Functions of bounded variation)**.**

10.2 Interpolations

weak Lp, marcinkiewicz

Definition 10.2.1. Let f be a measurable function on a measure space (X, μ). The *distribution function* $\lambda_f : [0, \infty) \to [0, \infty)$ is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}) = \mu(|f| > \alpha).$$

Do not use $\mu(\{x : |f(x)| \geq \alpha\})$. The strict inequality implies the *lower semi-continuity* of λ_f.

For $p > 0$,

$$\begin{aligned}
\|f\|_{L^p}^p &= \int |f(x)|^p \, d\mu(x) \\
&= \int \int_0^{|f(x)|} p\alpha^{p-1} \, d\alpha \, d\mu(x) \\
&= \int_0^\infty \int_{|f(x)|>\alpha} p\alpha^{p-1} \, d\mu(x) \, d\alpha \\
&= p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^p \frac{d\alpha}{\alpha}.
\end{aligned}$$

Definition 10.2.2.

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0<\alpha<\infty} \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right].$$

Theorem 10.2.3. *For $p \geq 1$ we have $\|f\|_{p,\infty} \leq \|f\|_p$.*

Proof. By the Chebyshev inequality,

$$\sup_{0<\alpha<\infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha) \, d\alpha = \|f\|_{L^p}^p.$$

□

10.6 (Marcinkiewicz interpolation)**.** Let X be a σ-finite measure space and Y be a measure space. Let

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator $T : L^{p_0}(X) + L^{p_1}(X) \to M(Y)$ has two weak-type estimates

$$\|T\|_{L^{p_0}(X)\to L^{p_0,\infty}(Y)} < \infty \quad \text{and} \quad \|T\|_{L^{p_1}(X)\to L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X)\to L^p(Y)} < \infty.$$

Proof. Let $f \in L^p(X)$ and denote $f_h = \chi_{|f|>\alpha} f$ and $f_l = \chi_{|f|\leq\alpha} f$. It is easy to show $f_h \in L^{p_0}$ and $f_l \in L^{p_1}$. Then,

$$\begin{aligned}
\|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha) \frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^p \cdot \mu(|Tf_h| > \alpha) \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha) \frac{d\alpha}{\alpha} \\
&\leq \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0,\infty}}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}} \|Tf_l\|_{L^{p_1,\infty}}^{p_1} \frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^{p-p_0} \|f_h\|_{p_0}^{p_0} \frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{p_1}^{p_1} \frac{d\alpha}{\alpha} \\
&\sim \|f\|_p^p.
\end{aligned}$$

by (1) Fubini, (2) Sublinearlity, (3) Chebyshev, (4) Boundedness, (5) Fubini. □

10.7 (Hadamard's three line lemma)**.** Let f be a bounded holomorphic function on vertical unit strip $\{z : 0 < \operatorname{Re} z < 1\}$ which is continuously extended to the boundary. Then, for $0 < \theta < 1$ we have

$$\|f\|_{L^\infty(\mathrm{Re}=\theta)} \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta}.$$

Proof. Fix n and define

$$g_n(z) := \frac{f(z)}{\|f\|_{L^\infty(\mathrm{Re}=0)}^{1-z} \|f\|_{L^\infty(\mathrm{Re}=1)}^{z}} e^{-\frac{z(1-z)}{n}}.$$

Then,

$$|g_n(z)| \le e^{-\frac{(\operatorname{Im} z)^2}{n}}$$

for z in the strip. By the maximum principle,

$$|f(z)| \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta} e^{\frac{y^2}{n}}.$$

Letting $n \to \infty$, we are done. □

10.8 (Riesz-Thorin interpolation)**.** Let X, Y be σ-finite measure spaces. Let

$$\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \qquad \frac{1}{q_\theta} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \to q_\theta} \le \|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta}.$$

Proof. Note that

$$\|T\|_{p_\theta \to q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g\rangle|}{\|f\|_{p_\theta}\|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z\rangle = \int \overline{g_z(y)}\, Tf_z(y)\, dy,$$

where f_z and g_z are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z) + \frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have $f_\theta = f$ and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for $\operatorname{Re} z = x$.

Then,

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0} \|f_z\|_{p_0} \|g_z\|_{q'_0} = \|T\|_{p_0 \to q_0} \|f\|_{p_\theta}^{p_\theta/p_0} \|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for $\operatorname{Re} z = 0$, and

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_1 \to q_1} \|f_z\|_{p_1} \|g_z\|_{q'_1} = \|T\|_{p_1 \to q_1} \|f\|_{p_\theta}^{p_\theta/p_1} \|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for $\operatorname{Re} z = 1$. By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0 \to q_0}^{1-\theta} \|T\|_{p_1 \to q_1}^{\theta} \|f\|_{p_\theta} \|g\|_{q'_\theta}$$

for $\operatorname{Re} z = \theta$. Putting $z = \theta$ in the last inequality, we get the desired result. □

Chapter 11

Lebesgue differentiation theorem

11.1 Hardy-Littlewood maximal function

Let T_m be a net of linear operators. It seems to have two possible definitions of maximal functions:

$$T^*f := \sup_m |T_m f|$$

and

$$T^*f := \sup_{m,\ \varepsilon:|\varepsilon(x)|=1} |T_m(\varepsilon f)|.$$

11.1 (Hardy-Littlewood maximal function)**.** The Hardy-Littlewood maximal function is just the maximal function defined with the approximate identity by the box kernel.

11.2 (Weak type estimate)**.**

$$\|Mf\|_{1,\infty} \le 3^d \|f\|_{L^1(X)}.$$

(a) Proof by covering lemma.

Proof. (a) By the inner regularity of μ, there is a compact subset K of $\{|Mf| > \lambda\}$ such that

$$\mu(K) > \mu(\{|Mf| > \lambda\}) - \varepsilon.$$

For every $x \in K$, since $|Mf(x)| > \lambda$, we can choose an open ball B_x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \lambda$$

if and only if

$$\mu(B_x) < \frac{1}{\lambda} \int_{B_x} |f|.$$

With these balls, extract a finite open cover $\{B_i\}_i$ of K. Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection $\{B_k\}_k$ such that

$$K \subset \bigcup_i Bi \subset \bigcup_k 3B_k.$$

Therefore,

$$\mu(K) \le \sum_k 3^d \mu(B_k) \le \frac{3^d}{\lambda} \sum_k \int_{B_k} |f| \le \frac{3^d}{\lambda} \|f\|_1.$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of B_k's. $\square$

11.3 (Radially bounded approximate identity). If an approximate identity K_n is radially bounded, then its maximal function is dominated by the Hardy-Littlewood maximal function:

$$\sup_n |K_n * f(x)| \lesssim Mf(x)$$

for every n and x, hence has a weak type estimate.

11.4 (Almost everywhere convergence of operators). Suppose is T_m is a sequence of linear operators such that the maximal function T^*f is dominated by Mf. If $f \in L^1(X)$ and $T_m g \to g$ pointwise for $g \in C(X)$, then $T_m f \to f$ a.e.

Proof. Take $\varepsilon > 0$ and $g \in C(X)$ such that $\|f - g\|_{L^1(X)} < \varepsilon$. Since $T_m g(x) \to g(x)$ pointwise, we have

$$\begin{aligned}
&\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) \\
&\quad \le \mu(\{x : \limsup_m |T_m f(x) - T_m g(x)| > \tfrac{\lambda}{2}\}) + \mu(\{x : |g(x) - f(x)| > \tfrac{\lambda}{2}\}) \\
&\quad \le \mu(\{x : M(f-g)(x) > \tfrac{\lambda}{2}\}) + \frac{2}{\lambda}\|f - g\|_{L^1(X)} \\
&\quad \lesssim \frac{1}{\lambda}\varepsilon
\end{aligned}$$

for every $\lambda > 0$. Limiting $\varepsilon \to 0$, we get

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > \lambda\}) = 0$$

for every $\lambda > 0$, hence the continuity from below implies

$$\mu(\{x : \limsup_m |T_m f(x) - f(x)| > 0\}) = 0.$$

□

Definition 11.1.1.

$$f^*(x) := \lim_{r \to 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\, dy.$$

Theorem 11.1.2 (Lebesgue differentiation). $f^* = 0$ *a.e.*

Proof. Note that $f^* \le Mf + |f|$ implies

$$\|f^*\|_{1,\infty} \le \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that $g^* = 0$ for $g \in C_c$. Approximate using $f^* = (f - g)^*$. □

Exercises

11.5 (Doubling measure).