# Foundations of Calculus

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October 17, 2022

# Contents

# Part I

# Convergence

# Chapter 1

# Sequences

## 1.1 Limit of sequences

preserving inequalities limsup and liminf

## 1.2 Extended real numbers

**1.1** (Operations in the extended real numbers)**.** We can extend addition (except $\infty + (-\infty)$), subtraction, multiplication (except $\infty \times 0$), division (except dividing by zero).

**1.2** (Limits in the extended real numbers)**.**

## 1.3 Control of the error

sufficiently large asymptotic expressions

Approximate sequences and change of limits

**1.3** (Change of limits)**.**

$$|a_n - a| \le |a_n - b_{mn}| + |b_{mn} - b_m| + |b_m - a|$$

$$\lim_m \sup_n |a_n - b_{mn}| = 0$$

$$\lim_n |b_{mn} - b_m| = 0$$

$$a_n = b_{mn} + c_{mn} \le b_{mn} + \varepsilon$$

## 1.4 Bounded sequences

monotone convergence Bolzano-Weierstrass

## Exercises

**1.4.**

**1.5** (Newton method)**.**

## Problems

1. Show that every real sequence $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$.

# Chapter 2

# Series

## 2.1 Absolute convergence

**2.1** (Unconditional convergence).

## 2.2 Convergence tests

comparison limit comparison cauchy condensation integral....

ratio root

**2.2** (Abel transform).

$$A_k(B_k - B_{k-1}) + (A_k - A_{k-1})B_{k-1} = A_kB_k - A_{k-1}B_{k-1}$$

$$\sum_{m<k\le n} A_k b_k = A_nB_n - A_mB_m - \sum_{m<k\le n} a_k B_{k-1}.$$

abel test

**2.3** (Dirichlet test).

**2.4** (Mertens' theorem). If $\sum_{k=0}^\infty a_k$ converges to $A$ absolutely and $\sum_{k=0}^\infty b_k$ converges to $B$, then their Cauchy product $\sum_{k=0}^\infty c_k$ with $c_k := \sum_{l=0}^k a_l b_{k-l}$ converges to $AB$.

(a) We have

$$\lim_{m\to\infty} \sup_n \sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l = 0.$$

(b) We have for each $m$ that

$$\lim_{n\to\infty} \sum_{k=1}^m \sum_{l=n-k+1}^n a_k b_l = 0$$

*Proof.* Let

$$A_n := \sum_{k=0}^n a_k,\ B_n := \sum_{k=0}^n b_k, \quad \text{and} \quad C_n := \sum_{k=0}^n c_k.$$

As $m \to \infty$.

$$\Big|\sum_{k=m+1}^n \sum_{l=n-k+1}^n a_k b_l\Big| \le \sum_{k=m+1}^n |a_k| \Big|\sum_{l=n-k+1}^n b_l\Big| = \sum_{k=m+1}^n |a_k||B_n - B_{n-k}| \lesssim \sum_{k=m+1}^\infty |a_k| \to 0.$$

For fixed $m$, as $n \to \infty$,

$$\Big|\sum_{k=0}^{m} \sum_{l=n-k+1}^{n} a_k b_l\Big| \le \sum_{k=0}^{m} |a_k| \Big|\sum_{l=n-k+1}^{n} b_l\Big| = \sum_{k=0}^{m} |a_k||B_n - B_{n-k}| \to \sum_{k=0}^{m} |a_k||B - B| = 0.$$

We will prove

$$A_n B_n - C_n = \sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l \to 0$$

as $n \to \infty$. For $\varepsilon > 0$, take $m$ such that

$$|\sup_n \sum_{k=m+1}^{n} \sum_{l=n-k+1}^{n} a_k b_l| < \varepsilon.$$

Then for every $n$ we have

$$|\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l| \le \varepsilon + |\sum_{k=0}^{n} \sum_{l=n-k+1}^{n} a_k b_l|.$$

Taking limits $n \to \infty$ and $\varepsilon \to 0$ in order, we are done. □

## Exercises

**2.5** (Cesàro mean)**.**

**2.6** (Recursive sine sequence)**.** Let $a_{n+1} = \sin a_n$ and $a_n = 1$. We can use $\sin x = x - \frac{x^3}{6} + O(x^5)$.

$$a_n = \sqrt{3} n^{-\frac{1}{2}} - \frac{3\sqrt{3}}{20} n^{-\frac{3}{2}} + o(n^{-\frac{3}{2}}).$$

## Problems

1. If $a_n \to 0$, then $\frac{1}{n} \sum_{k=1}^{n} a_k \to 0$.
2. If $a_n \ge 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ also diverges.

# Chapter 3

# Metrics and norms

## 3.1 Metric spaces

**3.1** (Definition of metric spaces). Let $X$ be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(i) $d(x, y) = 0$ if and only if $x = y$, (nondegeneracy)

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$, (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

A pair $(X, d)$ of a set $X$ and a metric on $X$ is called a *metric space*. We often write it simply $X$.

(a) A normed space $X$ is a metric space with a metric defined by $d(x, y) := \|x - y\|$.

(b) A subset of a metric space is a metric space with a metric given by restriction.

**3.2** (System of open balls). A metric is often misunderstood as something that measures a distance between two points and belongs to the study of geoemtry. The main function of a metric is to make a system of small balls, sets of points whose distance from specified center points is less than fixed numbers. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly.

Note that taking either $\varepsilon$ or $\delta$ in analysis really means taking a ball of the very radius. Investigation of the distribution of open balls centered at a point is now an important problem.

Let $X$ be a metric space. A set of the form

$$\{y \in X : d(x, y) < \varepsilon\}$$

for $x \in X$ and $\varepsilon > 0$ is called an *open ball centered at $x$ with radius $\varepsilon$* and denoted by $B(x, \varepsilon)$ or $B_\varepsilon(x)$.

**3.3** (Convergence and continuity in metric spaces). Let $\{x_n\}_n$ be a sequence of points on a metric space $(X, d)$. We say that a point $x$ is a *limit* of the sequence or the sequence *converges to $x$* if for arbitrarily small ball $B(x, \varepsilon)$, we can find $n_0$ such that $x_n \in B(x, \varepsilon)$ for all $n > n_0$. If it is satisfied, then we write

$$\lim_{n \to \infty} x_n = x,$$

or simply $x_n \to x$ as $n \to \infty$. We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

A function $f : X \to Y$ between metric spaces is called *continuous at $x \in X$* if for any ball $B(f(x), \varepsilon) \subset Y$, there is a ball $B(x, \delta) \subset X$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. The function $f$ is called *continuous* if it is continuous at every point on $X$.

(a) A sequence $x_n$ in a metric space $X$ converges to $x \in X$ if and only if $d(x_n, x)$ converges to zero.

(b) Let $f : X \to Y$ be a function between two metric spaces. If there is a constant $C$ such that $d(x, y) \leq Cd(f(x), f(y))$ for all $x$ and $y$ in $X$, then $f$ is continuous. In this case, $f$ is particularly called *Lipschitz continuous* with the *Lipschitz constant* $C$.

## 3.2 Normed spaces

banach space

## 3.3 Open sets and closed sets

convergence, limit point

## 3.4 Compact sets

## 3.5 Connected sets

## Exercises

# Part II

# Real functions

# Chapter 4

# Continuous functions

## 4.1 Intermediate and extreme value theorems

## 4.2 Uniform convergence

*Proof.* Divide the error

$$|f(x_n) - f(x)| \leq |f(x_n) - f_m(x_n)| + |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|.$$

Using the uniform convergence, we can take $m$ such that $\|f_m - f\| < \varepsilon$, so we have

$$|f(x_n) - f(x)| < \varepsilon + |f_m(x_n) - f_m(x)| + \varepsilon.$$

Then, taking $\limsup_{n\to\infty}$ on the both-hand sides, we get

$$\limsup_{n\to\infty} |f(x_n) - f(x)| \leq \varepsilon + 0 + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ has been arbitrarily taken,

$$\lim_{n\to\infty} |f(x_n) - f(x)| = 0.$$

$\square$

## 4.3 Arzela-Ascoli theorem

## 4.4 Stone-Weierstrass theorem

## Exercises

## Problems

1. The set of local minima of a convex real function is connected.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The equation $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.
3. A continuous function that takes on no value more than twice takes on some value exactly once.
4. Let $f$ be a function that has the intermediate value property. If the preimage of every singleton is closed, then $f$ is continuous.

*5. If a sequence of real functions $f_n\colon [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$, then it has a uniformly convergent subsequence.

# Chapter 5

# Differentiable functions

## 5.1 Monotonicty and convexity

## 5.2 Mean value theorem

Darboux

## 5.3 Taylor theorem

## 5.4 Differentiable class

completeness

## Exercises

**5.1** (Variations on the mean value theorem). Let $f$ be a differentiable function on the unit closed interval.

(a) If $f(0) = 0$ there is $c$ such that $cf'(c) = f(c)$. (Flett)

(b) If $f(0) = 0$ there is $c$ such that $cf(c) = (1-c)f'(c)$.

**5.2** (Convergence rates of recursive sequences). If $a_{n+1} = a_n - f(a_n)$, $f(0) = 0$, $f(x) > 0$ for $0 < x < \varepsilon$, $f \in C^2$? then

$$f'(a_n) \sim \lim_{x\to 0+} \frac{f'(x)^2}{f''(x)f(x)} \frac{1}{n}.$$

*Proof.* $\square$

## Problems

1. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} f'(x) = b$, then $a = 0$.
2. Let $f$ be a real $C^2$ function with $f(0) = 0$ and $f''(0) \neq 0$. Defined a function $\xi$ such that $f(x) = xf'(\xi(x))$ with $|\xi| \le |x|$, we have $\xi'(0) = 1/2$.
3. Let $f$ be a $C^2$ function such that $f(0) = f(1) = 0$. We have $\|f\| \le \frac{1}{8}\|f''\|$.
4. A smooth function such that for each $x$ there is $n$ having the $n$th derivative vanish is a polynomial.

5. If a real $C^1$ function $f$ satisfies $f(x) \neq 0$ for $x$ such that $f'(x) = 0$, then in a bounded set there are only finite points at which $f$ vanishes.
6. Let a real function $f$ be differentiable. For $a < a' < b < b'$ there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b') - f(a') = f'(c')(b' - a')$.

# Chapter 6

# Analytic functions

## 6.1 Power series

uniform convergence and absolute convergence, abel theorem? differentiation convergence of radius sum, product, composition, reciprocal? closed under uniform convergence

## 6.2 Complex analytic functions

complex domain (real analytic iff its domain contains real line) convergence of radius, revisited identity theorem

## 6.3 Special functions

hypergeometric, bessel, gamma, zeta

## Exercises

# Part III

# Integration

# Chapter 7

# Riemann integral

## 7.1 Riemann integral

tagged partition

## 7.2 Henstock-Kurzweil intergral

bounded compact support <-> lebesgue

## 7.3 Improper integral

## 7.4 Fundamental theorem of calculus for continuous functions

### Exercises

**7.1.** Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

**7.2.** Find all $a > 0$ and $b > 0$ such that $\int_0^\infty x^{-b}|\tan x|^a\,dx$ converges.

### Problems

*1. If $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

## Chapter 8

# Integrable functions

### 8.1

# Chapter 9

# Part IV

# Multivariable Calculus

# Chapter 10

# Frećhet derivatives

## 10.1 Tangent spaces

**10.1** (Vector fields).

## 10.2 Inverse function theorem

# Chapter 11

# Differential forms

## 11.1 Multilinear algebra

**11.1** (Tensor product).

**11.2** (Wedge product).

**11.3** (One-forms).

**11.4** (Multiple integral). volume forms, stone weierstrass and fubini

## 11.2 Vector calculus

**11.5** (Exterior derivative).

**11.6** (Musical isomorphisms).

**11.7** (Inner product of differential forms). ONB

**11.8** (Hodge star operator). Identification of 2-forms and vector fields

**11.9** (Gradient, curl, and divergence).

**11.10** (Potentials).

**11.11** (Vector calculus identities).

## Exercises

**11.12** (Multivariable Taylor's theorem). Symmetric product

**11.13** (Vector analysis in two dimension).

**11.14** (Geometric algebra).

# Chapter 12

# Stokes theorems

## 12.1 Local coordinates

**12.1** (Spherical coordinates)**.** Let $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = 0,\ y \geq 0\}$.

$$(x, y, z) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)$$

for $(r, \theta, \varphi) \in (0, \infty) \times (0, \pi) \times (0, 2\pi)$. Orthonormal bases are

$$\left(\partial_r,\ \frac{1}{r}\partial_\theta,\ \frac{1}{r\sin\theta}\partial_\varphi\right),$$

$$(dr,\ r\,d\theta,\ r\sin\theta\,d\varphi),$$

$$(r^2 \sin\theta\,d\theta \wedge d\varphi,\ r\sin\theta\,d\varphi \wedge dr,\ r\,dr \wedge d\theta).$$

(a)

(b) The Laplacian is given by

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

*Proof.* Write $df$ in the orthonormal basis

$$\begin{aligned} df &= \frac{\partial f}{\partial r}\,dr + \frac{\partial f}{\partial\theta}\,d\theta + \frac{\partial f}{\partial\varphi}\,d\varphi \\ &= \left(\frac{\partial f}{\partial r}\right) dr + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) r\,d\theta + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right) r\sin\theta\,d\varphi. \end{aligned}$$

After taking the Hodge star operator

$$\begin{aligned} *df &= \left(\frac{\partial f}{\partial r}\right) r^2\sin\theta\,d\theta\wedge d\varphi + \left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) r\sin\theta\,d\varphi\wedge dr + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\right) r\,dr\wedge d\theta \\ &= r^2\sin\theta\frac{\partial f}{\partial r}\,d\theta\wedge d\varphi + \sin\theta\frac{\partial f}{\partial\theta}\,d\varphi\wedge dr + \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\,dr\wedge\theta, \end{aligned}$$

the differential is computed as

$$\begin{aligned} d*df &= d\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) d\theta\wedge d\varphi + d\left(\sin\theta\frac{\partial f}{\partial\theta}\right) d\varphi\wedge dr + d\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\right) dr\wedge\theta \\ &= \left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right] dr\wedge d\theta\wedge d\varphi, \end{aligned}$$

so that we have

$$\begin{aligned}\Delta f = *d*df &= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\end{aligned}$$

□

## 12.2 Integration on curves and surfaces

**12.2** (Line integral)**.**

**12.3** (Surface integral)**.**

## 12.3 Stokes theorems

**12.4** (Bump functions)**.**

**12.5** (Partition of unity)**.**

**12.6.**