Introduction to Operation Research & Combinatorics

Production Planning Project — IORC

Operation Research Project with CPLEX

Modeling and Sloving an Industrial Production Planning Problem by Mixed-Integer Linear Programming

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Energy Block's Production Planning

Question One

Model's Parameters

"Parameters:"

$M = 1$: One Extruding Machine

$p = 1 \dots P$: Tube of type p where $P = 6$

$F = 1$: One family of tubes will be used on extruding machine 1

$\wp_1 = \{1, \dots, 6\}$: Tube types included in family one

$s = 1 \dots S$: Shift s where S will be 14 for First Instance & 28 for the second

C_p: Maximum quantity extruding machine can produce in (kg / shift)

$f = 2360£$: Cost of material waste whenever we perform a changeover

h_p: Unit inventory holding cost in £$/(kg$. shift)

D_{ps}: The demand of product p to be satisfied at end of shift s

p_0: product for which the extruding machine 1 is setup at beginning of shift 1

$Istart_p$: Inventory balance at beginning of shift 1 for product p

Additional Info: Extruding machine produces only one type of tube per shift.

Decision Variables

"Decision Variables:"

- $x^{1 \times S}$: vector of length S with binary entries that indicates if a switch occurred at shift s

$$x_s = \begin{cases} 1 & \text{if switch from one type to another happened at shift } s \\ 0 & \text{otherwise} \end{cases}$$

- $y^{P \times S}$: a $P \times S$ matrix with binary entries that designate whether the extruding machine was working on type p at shift s

$$y_{ps} = \begin{cases} 1 & \text{if tube of type } p \text{ was produced at shift } s \\ 0 & \text{otherwise} \end{cases}$$

- $I^{P \times S}$: a $P \times S$ matrix with positive integer values that stands for the quantities of each tube of type p available in the Inventory at end of shift s (in kg)
- $Q^{P \times S}$: a $P \times S$ matrix with positive integer entries stands for quantities of tubes (in kg) of type p produced from extruding machine one at end of shift s

Mixed Integer Linear Programming formulation:

MILP Model

Objective Function

$$z = minimize \quad \sum_{s=1}^{S}(f \times x_s) + \sum_{s=1}^{S}\sum_{p=1}^{P}(h_p \times I_{ps})$$

Minimize the cost of material waste whenever we do a changeover and minimize the holding cost of quantities in kg of tubes that are stored in the inventory at end of each shift.

Subject to

$$\begin{cases} Q_{ps} = C_p \times y_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 1 \dots S \\ \qquad \text{Quantity produced of type } p \text{ at shift } s. \\ I_{p1} = Istart_p + (C_p \times y_{p1}) - D_{p1}, \quad \forall p = 1 \dots P \\ \qquad \text{Inventory Balance of tube type } p \text{ at end of shift 1.} \\ I_{ps} = I_{p(s-1)} + (C_p \times y_{ps}) - D_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \\ \qquad \text{Inventory balance of tube of type } p \text{ saved in inventory at end of shift } s. \end{cases}$$

Constraints continuation

$$
\begin{cases}
I_{ps} \geq 0, \quad \forall p = 1 \dots P \quad \forall s = 1 \dots S \\
\qquad\qquad \text{Sign constraint on each entry, and maintain demand satisfaction.} \\
\sum_{p=1}^{P} y_{ps} = 1, \quad \forall s = 1 \dots S \\
\qquad\qquad \text{Allow extruding machine to work on only one type per shift.} \\
x_1 + y_{p_0 1} \geq 1 \\
\qquad\qquad \text{Force } x_1 \text{ to be 1 if extruding machine did not produce } p_0 \text{ at shift 1.} \\
y_{ps} - y_{p(s-1)} \leq x_s, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \\
\qquad\qquad \text{First disjunctive constraint (i.e. if a switch happened force } x_s \text{ to be 1).} \\
y_{p(s-1)} - y_{ps} \leq x_s, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \\
\qquad\qquad \text{Second disjunctive constraint (i.e. if a switch happened force } x_s \text{ to be 1).} \\
x_s \in \{0,1\}, \ \forall s = 1 \dots S \ \ \& \ \ y_{ps} \in \{0,1\}, \ \forall p = 1 \dots P, \ \forall s = 1 \dots S \\
\qquad\qquad \text{Binary Constraints} \\
I_{ps} \ \ \& \ \ Q_{ps} \quad Integer, \ \forall p = 1 \dots P, \ \forall s = 1 \dots S \\
\qquad\qquad \text{Integrality Constraints} \qquad\qquad \text{** No need for sign constraint on } Q_{ps}
\end{cases}
$$

Resolution with CPLEX for Instance 1:

See File (" Question 1 _ OR Project ")

This file presents:

* An array with the shifts that will have a switch * Along with an up to date inventory balance for each type and for each shift * In addition to that the quantities that will be produced in each shift for the types that we will produce, we will interpret now the results to give a better insight for this production planning:

$z^* = 38354.7$£

z^* represents the total minimum cost we will pay for every quantity stored in the inventory at the end of each shift and the total minimum cost of material wastes we will come across due to changeover from one type to another in the same family during extruding phase to produce tube of types from family one that meets the demand forecasted.

Shifts with Changeover: (Shift 1, Shift 6, Shift 8, Shift 10, Shift 11, Shift 13)

The Type we will produce at each shift:

shift 1	shift 2	shift 3	shift 4	shift 5	shift 6	shift 7	shift 8	shift 9	shift 10
type 4	type 4	type 4	type 4	type 4	type 3	type 3	type 2	type 2	type 5

shift 11	shift 12	shift 13	shift 14
type 6	type 6	type 1	type 1

We can see the changeover happened at shift one is due to the fact that the extruding machine has been set up at a different type that we did not produce so we had to change settings the extruding machine was sat at.
We also know that the quantity produced from each type according to the above table is same as maximum capacity dedicated to each type per shift.

To have a track for the **inventory** balance in kg per shift:
Type 1: [899 899 634 634 634 634 634 634 369 104 104 104 241 378]
Type 2: [383 233 233 83 83 83 83 406 729 729 579 429 429 429]
Type 3: [342 192 192 192 192 436 680 680 680 680 680 680 380 380]
Type 4: [386 508 955 1077 1199 1199 1199 1049 1049 1049 724 399 399 249]
Type 5: [345 345 295 245 245 155 35 35 35 364 244 244 124 124]
Type 6: [787 787 552 552 552 552 552 552 317 317 668 549 549 314]

Resolution with CPLEX for Instance 2:

See File (" Question 1 _ OR Project ")
Now we interpret the results for a horizon of 2 weeks (i.e. 28 shifts), the way this file is constructed is similar to the file for Instance 1.

$z^* = 70561.41£$

The minimum possible total cost of wastes due to changeover between types from same family and the total holding price we will pay on excess products stored in the inventory.

Shifts with Changeover:
(Shift 3, Shift 4, Shift 6, Shift 8, Shift 9, Shift 11, Shift 13, Shift 16, Shift 18, Shift 20, Shift 22, Shift 23, Shift 24, Shift 27)

The Type we will produce at extruding machine 1 at each shift:

shift 1	shift 2	shift 3	shift 4	shift 5	shift 6	shift 7	shift 8	shift 9	shift 10
type 1	type 1	type 5	type 2	type 2	type 3	type 3	type 6	type 1	type 1

shift 11	shift 12	shift 13	shift 14	shift 15	shift 16	shift 17	shift 18	shift 19
type 4	type 4	type 6	type 6	type 6	type 5	type 5	type 1	type 1

shift 20	shift 21	shift 22	shift 23	shift 24	shift 25	shift 26	shift 27	shift 28
type 3	type 3	type 2	type 4	type 6	type 6	type 6	type 1	type 1

** The Red color here designates a changeover from one type to another during the respective shift.
We can witness that no changeover happened at shift one meaning that the extruding machine was sat up at the type we produced which is type one.

To have a track for the **inventory** balance in kg per shift:
Type 1: [190 592 592 327 327 327 327 327 464 743 743 743 478 213 213 213 213 350 752 752 752 752 487 364 364 99 236 373]
Type 2: [245 245 95 418 891 891 891 741 591 591 591 441 441 441 441 441 291 141 141 141 141 464 314 314 314 164 164 164]]
Type 3: [534 234 234 234 234 478 722 722 722 722 722 722 422 422 422 272 272 272 272 516 1060 760 760 760 760 760 460 460]
Type 4: [346 346 346 346 346 346 346 346 346 346 468 590 590 440 440 440 440 440 440 440 440 290 737 737 737 412 412 262]
Type 5: [68 68 397 397 397 397 157 157 157 157 37 37 37 37 37 246 575 575 455 455 335 335 335 335 215 215 95 95]
Type 6: [22 22 22 22 22 22 22 373 138 138 138 138 254 370 554 554 554 554 554 554 319 319 319 670 551 667 432 197]

Summary for this case study:

This production planning problem consists of several phases to produce an energy block that can be attached to a gas generator that helps in inflating the air bag in cars during accidents, we focus on the extruding phase of energy block production which is considered as the bottleneck phase for the whole production process that is to say the limitations and optimization you apply to this phase apply to all other phases.

In Question 1 we consider the simple case of having only one extruding machine producing 6 types of tubes where exactly one type is produced per shift the production unit is kg, we have two costs that shall be minimized one that stands for material waste or the money you lose when you perform switching from one type to another, the other cost is the holding cost which is the cost of saving the excess amount of tubes produced at the end of a shift and there is no demand on them, so due to low flexibility of the extruding machine we have to hold these excess amount and send it to the inventory for the upcoming demands and all these quantities stored in inventory at end of each shift you pay a holding price on them we are asked to find the best production planning for two time horizons.

Defining the Decision Variables:

As any production planning problem we are interested in the tube type we have to produce per shift and the quantity of tubes for each type stored in inventory at end of each shift, also since we have a material waste cost which we have to minimize we are implicitly minimizing the number of changeovers which requires a variable that looks like an array with length S with binary entries and finally an inventory balance variable that tells us the quantity of tubes available in inventory for each type.

Now we provide a well-detailed explanation for each of the above-mentioned variables:

- $x^{1\times S}$

  An array of length S that contains 0's and 1's (binary entries) that tells if a switch occurs at shift s this variable is needed to compute the material waste cost as this fixed charge is not applied for all shift whilst it is only added when we have a switch, the way this variable obtains its values will be explained afterwards.

- $y^{P\times S}$

  A matrix of dimension $P \times S$ that tells if the extruding machine is producing a tube of

type p at shift s where the entries are binary where 1 tells the extruding machine has to produce a tube of type p at shift s this variable is needed to compute the quantities produced for each type at each shift, also to know the quantities you will have in the inventory at end of each shift.
This variable will be used to define x_s using some disjunctive constraints

- $I^{P \times S}$

  An inventory balance constraint that tells us the quantity in kilogram available in the inventory at end of each shift, this variable is needed to minimize the holding cost of the quantities that will be stored in the inventory.

Defining the Constraints:

$$Q_{ps} = C_p \times y_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 1 \dots S \tag{1}$$

The first variable we wish to know its values is the quantity of each tube type we will produce in kg at each shift which will be the only product you produced times its maximum capacity which will be produced during its respective shift, therefore we simply perform matrix multiplication ($C_p \times y_{ps}$) since the columns of y_{ps} are zeros except one entry which is 1.

$$I_{p1} = Istart_p + (C_p \times y_{p1}) - D_{p1}, \quad \forall p = 1 \dots P \tag{2}$$

In order to compute the inventory balance we will exploit the recursion concept (i.e. recalling the previous balance to figure out the current balance) because the quantities in inventory are continuously updated either by producing some or demanding, therefore we need to find the inventory balance at shift one.

For this regard, to find your inventory balance at end of shift one, we need to know the total amount available (in kg) of each type before considering the demand ($Istart_p + Q_{p1}$) then you take off the demand of first shift (D_{p1}) and the result will be the amount you have in the inventory after a shift one

$$I_{ps} = I_{p(s-1)} + (C_p \times y_{ps}) - D_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \tag{3}$$

Since we have the first iteration for every p indeed we can perform recursion to figure out the quantities available in the inventory at end of each shift for every type p which is your inventory balance of the previous shift combined with the quantities produced at the respective shift s (which will be indeed zero for every product except the only product p that you will produce with its associated quantity C_p) (i.e. $I_{p(s-1)} + Q_{ps}$) then you take off the demand of the respective shift s indeed here in this s has to be greater than or equal to 2

$$I_{ps} \geq 0, \quad \forall p = 1 \ldots P \quad \forall s = 1 \ldots S \tag{4}$$

So far, we have defined I_{ps} for all values of p and s, now we need to put a constraint on the values this matrix can take.

Just like any production planning problem the demand has to be satisfied meaning that the quantity you have in the inventory plus the quantities you produce has to be greater than the demand for any type at any shift
(i.e. $Istart_p + (C_p \times y_{p1}) \geq D_{p1} \quad \& \quad I_{p(s-1)} + (C_p \times y_{ps}) \geq D_{ps} \ \ \forall s = 2 \ldots S) \qquad \forall p$
It is worth mentioning that there is no constraint that tells the extruding machine must produce only the maximum capacity of the product that it is working on for a shift s implicitly speaking this rule has been respected via the demand satisfaction constraint why? because the only variable in the above equation is a matrix of 0's and 1's that only tells if the extruding machine produces the type p or not at shift s hence the quantity of what is produced depends only on y_{ps} and by maintaining what you have in inventory plus what you produce times the maximum capacity makes things work properly up until now but this raise the attention to the next important constraint regarding y_{ps}

$$\sum_{p=1}^{P} y_{ps} = 1, \quad \forall s = 1 \ldots S \tag{5}$$

We can allow the extruding machine to produce exactly one type per shift therefore the summation over any column must yield 1 implying that exactly one type will be produced per shift.

Now we move to the next set of constraints that link y_{ps} with x_s where we will define explicitly how x_s is defined whenever a switch happened, for this regard let us illustrate the following observation:

let $y_{3\times3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ where columns represent the shifts and rows represent the tube type where we considered only three tubes just for the sake of simplicity as we can see that this matrix has exactly one entry with value 1 in each column, from this matrix we can see that at shift one you will work on type one and the same for shift two meaning that a switch did not happen at shift two (in other words to determine if a switch happened at a shift we basically compare it with the previous shift) and we can see that if the difference between the value of this y matrix at shift s and at shift $s-1$ is 0 for every single p indicates that no switch happened hence x_s will be 0 for this respective shift, unlike for shift three where we can see that the extruding machine switches from working on type one to type two therefore we can conclude that if there exists a p in which the difference between it's respective value in the y matrix for the shift s and shift $s-1$ was 1 or -1 (-1 occurs when extruding machine was producing type p at shift $s-1$ and switched to produce another type at shift s in such case the difference will be -1) so challenge is to build two constraints where one of them should hold and the other got violated (Disjunctive constraints) - you know we are not in an ideal world where there are no oddly cases that causes troubles otherwise something is going wrong -.

Hence we can find a relation between x_s and y_{ps} as follows:

$$x_s = \begin{cases} 0 & if \;\; y_{p(s-1)} - y_{ps} = 0 \quad \forall p \\ 1 & o.w. \end{cases}$$

since we use $(s-1)$ and we know that before the production starts, the extruding machine is set up to produce a specific type p_0 so the switch for the first shift has to be defined uniquely:

$$x_1 + y_{p_0 1} \geq 1 \tag{6}$$

The typical constraint we use to let one of the two binary variables take value one, this constraint tells us if your extruding machine produced type p_0 at shift one this means that $y_{p_0 1} = 1$ and hence x_1 is set free and as this is a minimization problem it will be 0, however, if the extruding machine did not produce type p_0 at shift one $y_{p_0 1} = 0$ and hence x_1 is forced to be 1 meaning that a switch happened.

Finally we arrive to the final two constraints that will hopefully manage to fully identify the x_s array where one of the constraints will be violated and the other will hold for the 3 cases we have (i.e. when the difference between y_{ps} & $y_{p(s-1)}$ is 1 or $-1 \longrightarrow x_s = 0$ and when the difference is 0 $\longrightarrow x_s = 0$)

$$y_{ps} - y_{p(s-1)} \leq x_s, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \tag{7}$$

$$y_{p(s-1)} - y_{ps} \leq x_s, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \tag{8}$$

Notice that x_s has to let the two equations hold simultaneously for every value of p these two equations are disjunctive constraints, to check their validity we consider the critical cases where at a specific p the first difference is 1 between shift s and the previous shift meaning that a switch happened hence x_s need to be 1 and when the difference is -1 which occurs at another p in the same shift where you used to produce at the previous shift and at shift s you are producing the type p and finally when you have the difference is 0 for all p which means no switch happened implying that x_s is 0

- when difference is 0 for all p at shift s:

  In this case for the two constraints x_s will be free it could be either 0 or 1 and here the minimization plays the role of making $x_s = 0$ for shift s

- When at a specific p saying p_t the first difference is 1 and there will be indeed another p where the difference is -1 at the same shift:

  when the first difference is 1 this means that we switch to produce the tube of type p_t at shift s and stop producing the other type in which the difference is -1, Now checking the validity of the constraints:

  $y_{p_t s} - y_{p_t(s-1)} = 1 \leq x_s$ here x_s is forced to be 1 for the type p_t at shift s and for the second constraint the difference will be -1 making x_s free implying that $x_s = 1$ and this value will indeed hold for the other tube types at the same shift as the difference for them will be 0 in both constraints making $x_s = 1$ hold and hence x_s will be 1 at shift s

This it all for the first Question of the Project, and indeed not to forget mentioning the search space for the decision variables

$$y \in \{0,1\}^{P \times S} \qquad x \in \{0,1\}^{S} \qquad I, Q \in \mathbb{Z}_{\geq 0}^{P \times S}$$

Question two

Model's Parameters

"Parameters:"

$m = 1 \ldots M$ where M is two extruding machines

$p = 1 \ldots P$ Tube of type p where $P = 12$

$Family1 = \{1, 2, 3, 4, 5, 6\}$ Tube types included in family one

$Family2 = \{7, 8, 9, 10, 11, 12\}$ Tube types included in family Two

$s = 1 \ldots S$ Shift s where S will be 14 for First Instance & 28 for the second

C_p Maximum quantity extruding machine can produce in (kg /shift)

$f = 2360£$ Cost of material waste whenever we perform a changeover between two types from the same family

$F = 3540£$ Cost of material waste whenever we perform a changeover between two tube types from different families

h_p Unit inventory holding cost is £/(kg . shift)

D_{ps} The demand of product p to be satisfied at end of shift s

$p_{0m} = \{1, 6\}$ Tube types the extruding machines 1 & 2 were sat up respectively at the beginning of shift 1

$Istart_p$ Inventory balance at beginning of shift 1 for product p

Decision Variables

"Decision Variables:"

- $x^{M \times S}$: an $M \times S$ matrix with binary entries that indicates if a switch from one tube type to another from the same family occurred at shift s on machine m.

$$x_{ms} = \begin{cases} 1 & if \text{ a switch occurred between types from same family at shift } s \\ & \text{on machine } m \\ 0 & \text{otherwise} \end{cases}$$

- $y^{M \times P \times S}$: an $M \times P \times S$ three-dimensional matrix with binary entries that designate whether the extruding machine was working on type p at shift s on machine m.

$$y_{mps} = \begin{cases} 1 & \text{if tube of type } p \text{ was produced at shift } s \text{ on machine } m. \\ 0 & \text{otherwise} \end{cases}$$

- $W^{1\times S}$: An array of length S with binary entries, related to machine 2 where it tells if a switch happened from a tube type to another type in another family.

$$W_s = \begin{cases} 1 & \text{if machine 2 have changeover at shift } s \text{ between different families.} \\ 0 & \text{otherwise} \end{cases}$$

- $I^{P\times S}$: a $P \times S$ matrix with positive integer values that stands for the quantities of each tube of type p available in the Inventory at end of shift s (in kg)
- t^P: an artificial array of length P where we assumed that extruding machine 2 produced the tube of type p_{02} at shift 1.

$$t_p = \begin{cases} 1 & if \;\; p \; = \; p_{02} \\ 0 & \text{otherwise} \end{cases}$$

Mixed Integer Linear Programming formulation:

MILP Model

Objective Function

$$z = minimize \qquad \sum_{s=1}^{S} F \times W_s + \sum_{m=1}^{M}\sum_{s=1}^{S} f \times x_{ms} + \sum_{s=1}^{S}\sum_{p=1}^{P} h_p \times I_{ps}$$

Minimize the cost of material waste whenever a changeover happened between two types of tubes in the same family for both extruding machines and the changeover cost waste between two types from different families on machine 2 and minimize the holding cost of quantities in kg of tubes stored in the inventory at the end of each shift.

MILP Constraints

Subject to

$$I_{p1} = Istart_p + \sum_{m=1}^{M} (C_p \times y_{mp1}) - D_{p1}, \quad \forall p = 1 \dots P$$

Inventory Balance at end of shift one

$$I_{ps} = I_{p(s-1)} + \sum_{m=1}^{M} (C_p \times y_{mps}) - D_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S$$

Inventory Balance at end of shift s for a product p

$$I_{ps} \geq 0, \qquad \forall p = 1 \dots P, \qquad \forall s = 1 \dots S$$

Maintain Demand Satisfaction at end of each shift for all products

$$\sum_{p \in Family1} y_{1ps} = 1 \qquad \forall s = 1..S$$

Restrict machine one to produce only one type of tubes at any shift

& only the types of family one tubes

$$\sum_{p \in Family2} y_{1ps} = 0 \qquad \forall s = 1..S$$

Don't let machine one produce any of family 2 tubes

$$\sum_{p=1}^{p} y_{2ps} = 1 \qquad \forall s = 1..S$$

Restrict machine 2 to produce only one type of tubes at shift s

This type can be from any family

$$x_{11} + y_{1p0_1 1} \geq 1$$

Force x_{11} to be 1 if machine 1 didn't produce type p_{0_1} at shift 1

$$t_{p0_2} = 1$$

Let artificial array take value 1 at type p_{0_2} meaning that we are

assuming machine 2 produced type p_{0_2} at shift 1

$$\sum_{p=1}^{P} t_p = 1$$

Make sure all the other values in this array are 0's

Constraints Continuation

$$\sum_{p \in Family1} (y_{2p1} - t_p) \leq W_1$$

Force W_1 to be one if switch between different families happened

$$\sum_{p \in Family2} (y_{2p1} - t_p) \leq W_1$$

Contingent Constraint for the previous one if the difference was -1

$$\sum_{p \in Family1} y_{2ps} \leq \sum_{p \in Family1} y_{2p(s-1)} + W_s \qquad \forall s = 2..S$$

Determine if switch happened between different families at shift s

$$\sum_{p \in Family2} y_{2ps} \leq \sum_{p \in Family2} y_{2p(s-1)} + W_s \qquad \forall s = 2..S$$

Contingent constraint with the previous one so that at least one hold in case of a switch

$$y_{2p1} - t_p \leq x_{21} + W_1 \qquad \forall p \ = \ 1..P$$

Determine if switch happened on the same family, make sure at most one of x_{21} and W_1 can be 1 if switch happened on machine 2

$$t_p \ - \ y_{2p1} \leq x_{21} + W_1 \qquad \forall p \ = \ 1..P$$

Contingent constraint to force x_{21} to be 1 in case of switch and W_1 is 0

$$y_{1ps} - y_{1p(s-1)} \leq x_{1s}, \qquad \forall p \in Family1, \ \ \forall s = 2..S$$

Force x_{1s} to be one if a switch on family one happened at shift s

$$y_{1p(s-1)} - y_{1ps} \leq x_{1s} \qquad \forall p \in Family1, \forall s = 2..S$$

Contingent constraint with the previous

$$y_{2ps} - y_{2p(s-1)} \leq x_{2s} + W_s \qquad \forall p = 1..P, \ \ \forall s = 2..S$$

Force x_{2s} to be 1 when switch on the same family happens at machine 2

$$y_{2p(s-1)} - y_{2ps} \leq x_{2s} + W_s \qquad \forall p = 1..P, \ \ \forall s = 2..S$$

Contingent constraint with the previous one

$$x_{ms} \in \{0,1\}, \ \ y_{mps} \in \{0,1\}, \ \ W_s \in \{0,1\}, \ \ t_p \in \{0,1\}, \ \ I_{ps} \in \mathbb{Z} \qquad , \forall \ m,p,s$$

Binary Constraint and Integrality Constraint

Resolution with CPLEX for Instance 1:

See File (" Question 2 _ OR Project ")

This file presents:

* A $2 \times S$ matrix with the shifts that will have a switch between types of the same family for machine one and machine two * Along with an up-to-date inventory balance for each type and for each shift * An array that tells if a switch between different families happened on machine two * And finally a matrix for each machine that gives the production plan (i.e. the type you have to produce at each shift for each machine), we will interpret now the results to give a better insight for this production planning:

$z^* = 81987.62$£

z^* represents the total minimum cost we will pay for every quantity stored in the inventory at the end of each shift and the total minimum cost of material wastes we will come across due to changeover from one type to another in the same family for both machines and changeover between different families for machine two, during extruding phase to produce a tube of types from family one and two that meets the demand forecasted.

Machine 1

Shifts with Changeover between types of same family:

(Shift 1, Shift 6, Shift 9, Shift 10, Shift 13)

Machine 2

Shifts with Changeover between types of same family:

(Shift 2, Shift 3, Shift 6, Shift 9, Shift 12, Shift 14)

Shifts with Changeover between types of different families:

(Shift 1, Shift 7, Shift 11)

The Type we will produce at each shift for each machine:

The Shift:	shift 1	shift 2	shift 3	shift 4	shift 5	shift 6	shift 7	shift 8	shift 9
Machine 1:	type 4	type 4	type 4	type 4	type 4	type 3	type 3	type 3	type 5
Machine 2:	type 10	type 11	type 12	type 12	type 12	type 9	type 2	type 2	type 1

shift 10	shift 11	shift 12	shift 13	shift 14
type 6	type 6	type 6	type 1	type 1
type 1	type 7	type 10	type 10	type 8

** Red color Designates that you have to perform a changeover between types of similar family for this shift.

** Blue color means a changeover between different families will happen at this shift.

Resolution with CPLEX for Instance 2:

See File (" Question 2 _ OR Project ")

$z^* = 141644.92$£

z^* total minimum cost for a horizon of 28 shifts, this is indeed the cost CPLEX provided after time limit of 10 minutes there could be better enhancement if it took much more time

Machine 1

Shifts with Changeover between types of same family:

(Shift 1, Shift 2, Shift 4, Shift 5, Shift 10, Shift 12, Shift 13, Shift 14, Shift 16, Shift 18, Shift 22, Shift 24, Shift 27)

Machine 2

Shifts with Changeover between types of same family:

(Shift 2, Shift 5, Shift 6, Shift 11, Shift 12, Shift 18, Shift 19, Shift 23, Shift 26)

Shifts with Changeover between types of different families:

(Shift 4, Shift 8, Shift 10, Shift 14, Shift 16, Shift 21, Shift 25)

The Type we will produce at each shift for each machine:

The Shift:	shift 1	shift 2	shift 3	shift 4	shift 5	shift 6	shift 7	shift 8	shift 9
Machine 1:	type 5	type 3	type 3	type 4	type 1	type 1	type 1	type 1	type 1
Machine 2:	type 6	type 2	type 2	type 12	type 7	type 11	type 11	type 4	type 4

shift 10	shift 11	shift 12	shift 13	shift 14	shift 15	shift 16	shift 17	shift 18	shift 19
type 6	type 6	type 5	type 3	type 1	type 1	type 2	type 2	type 4	type 4
type 7	type 8	type 10	type 10	type 6	type 6	type 11	type 11	type 12	type 7

shift 20	shift 21	shift 22	shift 23	shift 24	shift 25	shift 26	shift 27	shift 28
type 4	type 4	type 1	type 1	type 6	type 6	type 6	type 1	type 1
type 7	type 5	type 5	type 3	type 3	type 8	type 10	type 10	type 10

Summary for this case study:

The case of having two extruding machines working simultaneously where each produce one type per shift and restricting the first machine to produce only types of family one makes the constraints for the first machine more or less similar to the previous problem, but for the second machine that can switch from one type to another in same family or between different families makes it challenging to define the switch for each shift whether this switch was between families and or the same family, and indeed we want to minimize these switches that have a cost assigned to them as well as minimizing the holding cost for the quantities in kg available in the inventory at end of each shift

Defining the Decision Variables:

- y_{mps}

  Like any production planning we are interested in finding the plan we have to follow to make sure we are minimizing the cost and satisfying the demand for each shift, this variable helps us knowing the production plan for each machine and which tube type we have to produce at each shift and this variable plays the key role in this whole minimization problem from defining the switches to knowing the minimum holding cost possible we can obtain and how the inventory balance will be defined.

- I_{ps}

  Just like the previous question an inventory balance to maintain demand satisfaction and minimizing the objective function.

- W_s

  A new decision variable related to machine two that tells if switch between different families happened at a shift s this variable will help defining the switch among same family for machine two using contingent decisions.

- t_p

  Finding W_s is not easy so the trick followed is that we are assuming machine two has produced the tube type it was set at before shift 1 then we can decide if switch happened and whether this switch is between different families or same family.

- x_{ms}

  Each row of this matrix gives series of switches happened for each machine at shift s

Defining the Objective Function:

This problem is indeed a minimization problem where finding minimum possible cost is the objective, this cost is the number of switches between different families will happen at machine two $\sum_{s=1}^{S} F \times W_s$ adding to that the cost of material waste from switching between products of same family at both extruding machines that is: $\sum_{m=1}^{M}\sum_{s=1}^{S} f \times x_{ms}$ and finally adding the holding cost of each quantity was saved in the inventory during this production planning horizon which is: $\sum_{s=1}^{S}\sum_{p=1}^{P} h_p \times I_{ps}$

Defining the Constraints:

Ultimately the last section in this report will be the approach followed to define the constraints:

$$I_{p1} = Istart_p + \sum_{m=1}^{M}(C_p \times y_{mp1}) - D_{p1}, \quad \forall p = 1 \dots P \tag{9}$$

$$I_{ps} = I_{p(s-1)} + \sum_{m=1}^{M}(C_p \times y_{mps}) - D_{ps}, \quad \forall p = 1 \dots P \quad \forall s = 2 \dots S \tag{10}$$

Just like the previous question exploiting the recursion concept to define the inventory balance and let CPLEX perform faster.

$$I_{ps} \geq 0, \qquad \forall p = 1 \dots P, \qquad \forall s = 1 \dots S \tag{11}$$

Maintain demand satisfaction by ensuring the entries are greater than zero and hence $I_{p(s-1)} + \sum_{m=1}^{M}(C_p \times y_{mps}) \geq D_{ps}$ for all values of p and s it's worth mentioning that when we define this decision variable in CPLEX we define it to be " int+ " so this constraint will be redundant in CPLEX

$$\sum_{p \in Family1} y_{1ps} = 1 \qquad \forall s = 1..S \tag{12}$$

$$\sum_{p \in Family2} y_{1ps} = 0 \qquad \forall s = 1..S \tag{13}$$

since we have defined a production matrix for both machines using y_{mps} both tubes families are included in the matrix so be summing over family one for machine one ensures that machine one will only produce tubes of types from family one and it will only produce one tube type per shift and by making the some over the second family zero prevent machine one from preventing any of family two products.

$$\sum_{p=1}^{p} y_{2ps} = 1 \qquad \forall s = 1..S \tag{14}$$

Machine two can produce any type of tubes but only one type per shift.

$$t_{p0_2} = 1 \tag{15}$$

$$\sum_{p=1}^{P} t_p = 1 \tag{16}$$

Now moving to define the switches by starting with W_s and for this regard we will define t_p where this array takes value one at the tube type that machine two was sat up at before starting the production, the spirit of this move is to ease to process of defining the switches for shift one for both W_1 and x_{21} and the array t_p takes o's otherwise that's why we let the sum to be one so that only the index p_{02} will be one in this array.

$$\sum_{p \in Family1} (y_{2p1} - t_p) \leq W_1 \tag{17}$$

$$\sum_{p \in Family2} (y_{2p1} - t_p) \leq W_1 \tag{18}$$

We know that when a switch happens between different families the value one will be in family one and move to family two or vice versa therefore if we sum all the entries in the matrix y_{2p1} (which are the entries corresponding to whether a type from both families was produced in machine two at shift one) and this sum is along types of family one so if we produced a type from family one in machine two at shift one this sum will be one and if we subtract from the sum over family one in t_p and machine two was sat up to produce a product of family two then the sum over t_p where p is in family one will be zero and hence the difference will be one so W_1 will be one which is a contingent decision and for the second constraint the difference will be -1 and hence W_1 will be free so one of these two constraints will hold and if the switch was from family one to two these two constraints will hold and the second constraint will force W_1 to be one, but if no switch happened all if a switch among the same family happened the difference will remain zero making W_1 zero.

$$\sum_{p \in Family1} y_{2ps} \leq \sum_{p \in Family1} y_{2p(s-1)} + W_s \qquad \forall s = 2..S \tag{19}$$

$$\sum_{p \in Family2} y_{2ps} \leq \sum_{p \in Family2} y_{2p(s-1)} + W_s \qquad \forall s = 2..S \tag{20}$$

To define if a switch between different families happened we need the previous shift and since we have W_1 defined and we can define for the other shifts starting from second shift using the shift before and it is the same as the previous two constraints but instead of t_p we use $y_{2p(s-1)}$

$$x_{11} + y_{1p0_11} \geq 1 \tag{21}$$

Since machine one has only one possibility for a switch which is only a switch among same family so to determine x_{11} it just the exact way followed to define x_1 from question one, if machine one produced the type it sat up at which is p_{01} then no switch making x_{11} zero else force x_{11} to be one so in other words at least one of these two x_{11} and $y_{1p_{01}1}$ will be one.

$$y_{2p1} - t_p \leq x_{21} + W_1 \qquad \forall p \ = \ 1..P \tag{22}$$

$$t_p \ - \ y_{2p1} \leq x_{21} + W_1 \qquad \forall p \ = \ 1..P \tag{23}$$

After defining x_{11}, defining x_{21} was the real struggle because there two possibilities for a switch either among same family or between different families, that's why we defined W_s

before so that at most one of them can be one if a switch happened, and that is exactly what the two constraints tell us, if switch between different families happened W_1 will be one and hence x_{21} will be free and since minimization it will be zero, otherwise in case of switch among same family we will face a difference 1 at a specific type and -1 at another causing a difference of -1 and W_1 will be zero from it's constraints and x_{21} will be free following from the other constraints it will be forced to be one the main difference here is that we are taking the difference for each type without the summation just like the previous question but here W_1 comes to the rescue in helping for defining x_{21} then we can proceed to define x_{1s} and x_{2s}.

$$y_{1ps} - y_{1p(s-1)} \leq x_{1s}, \qquad \forall p \in Family1, \ \ \forall s = 2..S \tag{24}$$

$$y_{1p(s-1)} - y_{1ps} \leq x_{1s} \qquad \forall p \in Family1, \forall s = 2..S \tag{25}$$

Defining x_{1s} just like we defined x_s in Question one a contingent decision and two constraints so that one of them holds and indeed the values of p are only family 1 in fact it does not matter as the values in the y_{1ps} matrix from the second family are 0's.

$$y_{2ps} - y_{2p(s-1)} \leq x_{2s} + W_s \qquad \forall p = 1..P, \ \ \forall s = 2..S \tag{26}$$

$$y_{2p(s-1)} - y_{2ps} \leq x_{2s} + W_s \qquad \forall p = 1..P, \ \ \forall s = 2..S \tag{27}$$

The last two constraints concerning x_{2s} after we defined x_{21} we can follow the same approach for the other shifts depending on the previous shift and we add W_s this trick that will designate that at most one of these two W_s & x_{2s} can be one if a switch happened but if no switch all the differences will be zero making both of them zero, and with this constraint we can successfully and finally manage to define all the constraints of our problem and clearly define the decision variables and ultimately making it possible to minimize this production plan and finding the best plan to follow for whatever the horizon is just like the plans enclosed above.

$$x_{ms} \in \{0,1\}, \ \ y_{mps} \in \{0,1\}, \ \ W_s \in \{0,1\}, \ \ t_p \in \{0,1\}, \ \ I_{ps} \in \mathbb{Z} \qquad , \forall \ m, p, s \tag{28}$$

Indeed we don't forget to define the search space for our decision variables: $x \in \{0,1\}^{M \times S}$ & $y \in \{0,1\}^{M \times P \times S}$ & $W \in \{0,1\}^S$ & $t \in \{0,1\}^P$ & $I \in \mathbb{Z}_{\geq 0}^{P \times S}$