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Dynamics of group formation in collective motion of organisms

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A mathematical description of the collective motion of organisms using a density–velocity model is presented. This model consists of a system of nonlinear parabolic equations, a forced Burgers equation for velocity and a diffusion–convection equation for density. The motion is mainly due to forces resulting from the differences between local density levels and a prescribed density level.

The existence of a global attractor for a 1D density–velocity model is proved by asymptotic analysis to demonstrate different patterns in the attractors for density. The theoretical results are supplemented with numerical results. These patterns correspond to movements of collective organized groups of organisms such as fish schools and bird flocks.

Keywords: attractors; Burgers equation; diffusion–convection equation; collective organization.

Introduction

A characteristic feature of living systems is that they move and this movement depends on external environmental factors like temperature, food availability, predator risk and internal genetic factors. Large systems of homogeneous organisms such as fish, birds or bacteria demonstrate the collective organization of such systems. Such collective organization serves, for example, the purpose of reducing the risk of being eaten or making search for food easier (Partridge, 1982; Giske *et al.*, 1998; Parrish & Edelstein-Keshet, 1999; Stöcker, 1999; Krause *et al.*, 2000).

Synchronized and polarized swimming groups which remain together are termed 'schools' (Niwa, 1996). Hundreds of fish swim in unison, more like a single organism than a collection of individuals, and the distance between individuals is roughly uniform (Hunter, 1966; Partridge, 1982). Each fish uses its eyes and its lateral lines to adapt its velocity and position to the velocity and position of all neighbours. The preferred distance to neighbours is observed in many populations of fish (Partridge, 1982; Toner & Tu, 1995; Krause *et al.*, 2000). The phenomenon of schooling is apparently based on the same behaviour pattern for each member of the school, in contrast to mammal herds with a leader (Stöcker, 1999).

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Orientation processes of animals have received much attention in mathematical modelling. Individual-based models use stochastic differential equations (Niwa, 1996), difference equations (Toner & Tu, 1998; Vicsek *et al.*, 1999; Czirok & Vicsek, 2000; Hubbard *et al.*, 2004) and cellular automata with a hexagonal grid (Stöcker, 1999). Partial differential equations are traditionally used to describe the density of spatially distributed populations. The effect of alignment on the schooling behaviour is described in Mogilner & Edelstein-Keshet (1995); Mogilner *et al.* (1996); Edelstein-Keshet *et al.* (1998); Lutscher (2002). A non-local model for grouping and its comparison with some local models is presented in Mogilner & Edelstein-Keshet (1999).

In this paper, we restrict our attention to the analysis and simulation of spatial dynamics for grouping patterns. For the description of these processes we use density–velocity analysis, because the dynamics of individual organisms become very complicated when their number is large.

We consider a density–velocity model, where the motion of the organisms is mainly due to the forces resulting from the differences between local density levels and a prescribed density level. In the analysis presented below attention is restricted to forces acting on a short time scale that describe the interaction between single organisms in large groups of organisms and give rise to grouping of elements in such structures. These forces depend essentially on the state of the living system.

1. Problem formulation

The organisms are modelled as a fluid accelerated by internal and external forces. We adopt the following diffusion–advection equation,

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = \mu \Delta \rho, \tag{1.1}$$

to describe the time evolution of the density of organisms $\rho(t, \mathbf{x})$. Here the term $\mu \Delta \rho$ describes random motion with corresponding diffusion coefficient μ, and the term $\operatorname{div}(\rho \mathbf{u})$ is the advection with velocity $\mathbf{u}$. This equation has been extensively used in the traditional population models for description of spatially distributed populations (Okubo, 1986; Murray, 1989; Grindrod, 1991; Mogilner & Edelstein-Keshet, 1999; Flierl *et al.*, 1999; Okubo & Levin, 2001).

We model the time evolution in the velocity field $\mathbf{u}(t, \mathbf{x})$ with the forced Burgers equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \mathbf{F}. \tag{1.2}$$

There has been a strong interest in the Burgers equation since the 1980s. One of the main reasons is that it can actually be integrated explicitly. The Burgers equation has a lot in common with the Navier–Stokes equation: the same type of advective nonlinearity, and the presence of a diffusion term, as well as many invariance and conservation laws (Frisch & Bec, 2001). It can be viewed as a special case of the Navier–Stokes equation (Woyczyński, 1998).

The Burgers equation has found many applications in cosmology, nonlinear acoustics and other nonlinear wave problems. It was applied to the problem of cell motion with chemotactic and aggregation factors in Kowalczyk *et al.* (2004). These factors were introduced as gradient forces for the velocity of cells.

We view the diffusion term $\nu\Delta\mathbf{u}$ in the Burgers equation as an approximation to an averaging or alignment force for velocity

$$\frac{1}{\tau}\left[\frac{1}{|\Omega_\varepsilon(\mathbf{x})|}\int_{\Omega_\varepsilon(\mathrm{x})}\mathbf{u}(t,\mathbf{y})\,\mathrm{d}\mathbf{y}-\mathbf{u}(t,\mathbf{x})\right]\to\frac{1}{\tau}\cdot\frac{\varepsilon^2}{2(n+2)}\Delta\mathbf{u}(t,\mathbf{x}),$$

as $\varepsilon\to 0$, where $\Omega_\varepsilon(\mathbf{x})$ denotes a disc of 'influence' with radius ε around the point $\mathbf{x}$, τ is a time constant for the alignment of the velocity at $\mathbf{x}$ to the average velocity within $\Omega_\varepsilon(\mathbf{x})$, and n is the space dimension. Alternatively, we might have replaced

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\mathbf{u}\,\mathrm{d}y$$

by

$$\frac{\int_{\Omega_\varepsilon}\rho\mathbf{u}\,\mathrm{d}y}{\int_{\Omega_\varepsilon}\rho\,\mathrm{d}y}$$

resulting in $\Delta\mathbf{u}$ being replaced by $\Delta\mathbf{u}+\dfrac{2}{\rho}\nabla\rho\cdot\nabla\mathbf{u}$ as in Czirok & Vicsek (2000), but in this paper we restrict the attention to the former expression.

$\mathbf{F}$ denotes a locomotory force that is composed of the internal force $-\nabla P$ representing the interaction between particles in the population and the external force $\mathbf{f}$. The internal force depends on the state of the particle system: that is, on the mutual distribution of particles. At the same time, the external forces do not depend directly on mutual distribution of particles, but they depend on numerous external factors, e.g. food resources, distribution of predators, migration directions and physical factors such as currents and temperature fields.

Considering the nature of the internal forces in more detail, it has been noted in Okubo (1986), Partridge (1982) and Niwa (1996) that, in the case of real particles such as fish or birds and their movement in groups of similar organisms, there exists an optimal distance between the neighbours in such groups. Different factors could be the reason for such a distribution of real particles: e.g. making the search for food easier, reducing the risk of being eaten and reducing the energetic expenditures for movement. So, if the distance between neighbouring particles is larger than the optimal distance, then the particles try to reduce this distance and if the distance is smaller then they try to increase this distance.

This property of moving organisms leads us to consider a density approach to particles with some prescribed density related to the optimal distance between particles. When the density of particles is less than the prescribed density, then the internal force acts along the density gradient, and against the density gradient if it is greater than the prescribed level. When density is very small compared to the prescribed level, the distance between neighbouring particles is too large for interaction, and the internal force tends to zero.

Thus, the internal force is assumed to be of the form

$$-\nabla P=-P'(\rho)\nabla\rho, \tag{1.3}$$

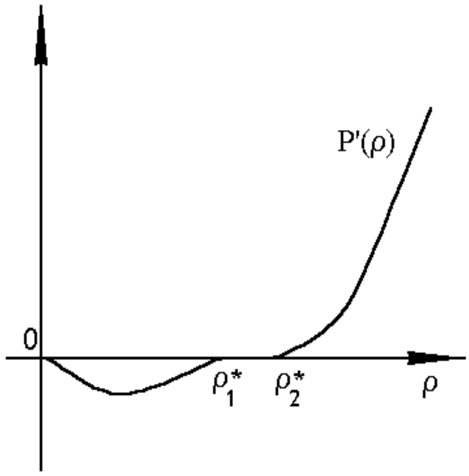


FIG. 1. The coefficient $P'(\rho)$ in the internal force. $P'(\rho) = 0$ on the interval $[\rho_1^*, \rho_2^*]$.

where the sign of the coefficient $P'(\rho)$ is positive if ρ is greater than some prescribed density ρ_2^* and negative if it is less than some density ρ_1^* $(0 < \rho_1^* \leqslant \rho_2^*)$. It is natural that $P'(0) = 0$. On the interval $[\rho_1^*, \rho_2^*]$, $P'(\rho) = 0$, but this interval could be compressed to one point ρ^*, $\rho^* > 0$ (see Fig. 1).

Generalizing the previous assumptions and (1.1)–(1.3), we write the density–velocity model in the form

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = \mu \Delta \rho, \tag{1.4}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla P + \mathbf{f}. \tag{1.5}$$

For the bounded spatial domain Ω the equation for velocity is supplemented with the no-flux boundary condition,

$$\mathbf{u} \cdot \vartheta = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where ϑ is outward normal vector at the points of the boundary $\partial\Omega$, and, in order to maintain mass conservation for all time t,

$$\int_\Omega \rho(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} = \bar{\rho} = \text{constant},$$

we impose the Neumann condition for density

$$\frac{\partial \rho}{\partial \vartheta} = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

The periodic conditions in the space arguments for density and velocity ensure mass conservation within a rectangular subdivision.

The main aim of this paper is to show how, with such a model for ρ and $\mathbf{u}$, the density will tend to limiting values that reflect the pattern of groups of moving organisms with some prescribed density level. We do this by asymptotic analysis of dynamical systems associated with such models. The theoretical analysis is restricted to the 1D case. We view it as a first step to a more general analysis in two or three dimensions while bringing out some of the important features of the movement. The analysis is supplemented by numerical results illustrating these features both in 1D and 2D cases.

2. 1D problem. Attractors

The time evolution of the density ρ and the velocity u of a collective motion of organisms in 1D space driven by an external force f is governed by the following system of parabolic equations:

$$\rho_t + (\rho u)_x = \mu \rho_{xx}, \tag{2.1}$$

$$u_t + u \cdot u_x + (P(\rho))_x = \nu u_{xx} + f. \tag{2.2}$$

The unknowns $\rho = \rho(t, x)$ and $u = u(t, x)$ are functions of time $t \in I \subset \mathbb{R}^1$ and position $x \in \Omega \subset \mathbb{R}^1$, $\Omega = (0, L)$. Equations (2.1) and (2.2) are supplemented with initial and boundary conditions. Two problems corresponding to different boundary conditions will be considered.

PROBLEM 2.1 (2.1) and (2.2) hold in $D = (0, \infty) \times \Omega$ with boundary conditions

$$\rho_x(t, 0) = \rho_x(t, L) = 0, \tag{2.3}$$

$$u(t, 0) = u(t, L) = 0, \tag{2.4}$$

for $t \in (0, \infty)$, and initial conditions

$$\rho(0, x) = \rho_0(x), \tag{2.5}$$

$$u(0, x) = u_0(x), \tag{2.6}$$

for $x \in (0, L)$.

PROBLEM 2.2 (2.1) and (2.2) hold in D with periodic boundary conditions

$$\rho(t, 0) = \rho(t, L), \qquad \rho_x(t, 0) = \rho_x(t, L), \tag{2.7}$$

$$u(t, 0) = u(t, L), \qquad u_x(t, 0) = u_x(t, L), \tag{2.8}$$

for $t \in (0, \infty)$, and initial conditions (2.5), (2.6).

For the mathematical setting of this problem we introduce the following spaces:

- a Hilbert space $H = H_1 \times H_2$, where $H_i = L^2(\Omega)$ $(i = 1, 2)$, endowed with the norm $\|\cdot\|_H = (\|\cdot\|_{H_1}^2 + \theta\|\cdot\|_{H_2}^2)^{1/2}$, where $\|\cdot\|_{H_i} = \|\cdot\|_{L^2(\Omega)}$ $(i = 1, 2)$, $\Omega = (0, L)$, $L > 0$, $\theta > 0$;
- a Hilbert space $V = V_1 \times V_2$ endowed with the norm $\|\cdot\|_V = (\|\cdot\|_{V_1}^2 + \theta\|\cdot\|_{V_2}^2)^{1/2}$, $\|\cdot\|_{V_i} = \|\cdot\|_{H^1(\Omega)}$ $(i = 1, 2)$, $\theta > 0$, where, for Problem 2.1, $V_1 = H^1(\Omega)$ and $V_2 = H_0^1(\Omega)$, and for the periodic case (Problem 2.2), $V_i = H_{per}^1(\Omega)$, $(i = 1, 2)$;

- metric spaces H^a and $\mathcal{H}^\alpha$, endowed with the norm $\|\cdot\|_H$ (Temam, 1997, p.157), where

$$H^a = \left\{(\phi, v) \in H : \frac{1}{L}\int_0^L \phi(x)\,\mathrm{d}x = a\right\}, \qquad \mathcal{H}^\alpha = \bigcup_{0 \leqslant a \leqslant \alpha} H^a;$$

- metric spaces $H^{a,b}$ and $\mathcal{H}^{\alpha,\beta}$, endowed with the norm $\|\cdot\|_H$, where

$$H^{a,b} = \left\{(\phi, v) \in H^a : \frac{1}{L}\int_0^L v(x)\,\mathrm{d}x = b,\right\}, \qquad \mathcal{H}^{\alpha,\beta} = \bigcup_{0 \leqslant a \leqslant \alpha,\ |b| \leqslant \beta} H^{a,b}.$$

Note that the space V is compactly embedded in the space H and the sets H^a, $\mathcal{H}^\alpha$, $H^{a,b}$ and $\mathcal{H}^{\alpha,\beta}$ are convex.

The following assumptions are made concerning Problems 2.1 and 2.2.

ASSUMPTION 2.1 $\lim_{\rho\to\infty} \dfrac{P(\rho)}{\rho^2} = P_\infty > 0$; $|\rho^2 P_\infty - P(\rho)| \leqslant P_1|\rho| + P_0$ and $\mu\nu > \dfrac{c_1^P P_1^2}{8P_\infty}$, where ν, μ, P_∞, P_0, P_1 are positive constants and c_1^P the Poincaré constant depending only on Ω.

ASSUMPTION 2.2 $P'(\cdot) \in L^\infty_{loc}(\mathbb{R})$ and $|P'(\rho)| \leqslant P_3\rho^2 + P_2$, where P_2, P_3 are positive constants.

Because we are interested only in non-negative solutions for ρ, we can restrict our attention to the case when $P(-\rho) = P(\rho)$ for simplicity.

ASSUMPTION 2.3 Let $f \in L^2(\Omega)$. Moreover, for Problem 2.2 we assume that the integral of f over Ω is equal to zero, i.e.

$$\int_0^L f(x)\,\mathrm{d}x = 0.$$

Assumptions 2.1–2.3 guarantee the global well-posedness of solutions for Problems 2.1–2.2.

PROPOSITION 2.1 (Babak *et al.*, 2003) Let Assumptions 2.1–2.3 hold and the initial data $w_0 = (\rho_0, u_0)$ given in H. There exists a unique solution $w = (\rho, u)$ of Problem 2.1 satisfying

$$w \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V), \qquad \forall T : 0 < T < \infty.$$

The mapping $w_0 \to w(t)$ is continuous in H, and for every fixed $\varepsilon > 0$ and T: $w \in L^\infty(\varepsilon, T; V) \cap L^2(\varepsilon, T; H^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)))$. If, furthermore, w_0 is given in V, then $w \in \mathcal{C}([0, T]; V) \cap L^2(0, T; H^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)))$.

Note that the interpretation of the first part of Proposition 2.1 in terms of the boundary value condition for ρ in Problem 2.1 is that (Dautray & Lions, 1988)

$$\rho_x = 0 \quad \text{in } H^{-1/2}(\Gamma), \qquad \Gamma = \{0, L\}.$$

A similar statement can be formulated for Problem 2.2. The proofs of well-posedness theorems for similar problems are sketched in Temam (1997).

From (2.1) and boundary conditions (2.3) and (2.4) for Problem 2.1, it follows that the total mass on the interval $[0, L]$ is constant:

$$\int_0^L \rho(t,x)\,\mathrm{d}x = \int_0^L \rho_0(x)\,\mathrm{d}x = \bar{\rho}. \tag{2.9}$$

Analogous properties for density ρ and velocity u in Problem 2.2 can be obtained:

$$\begin{aligned} \int_0^L \rho(t,x)\,\mathrm{d}x &= \int_0^L \rho_0(x)\,\mathrm{d}x = \bar{\rho}, \\ \int_0^L u(t,x)\,\mathrm{d}x &= \int_0^L u_0(x)\,\mathrm{d}x = \bar{u}. \end{aligned} \tag{2.10}$$

Now, we can formulate the main theoretical results of this work (see Appendix A for some basic definitions and results and Appendix B for proofs of the theorems).

THEOREM 2.1 Let Assumption 2.1–2.3 hold. Then, for every $\alpha \geqslant 0$, the semigroup $S(t)$ associated with Problem 2.1 maps $\mathcal{H}^\alpha$ into itself. It possesses a maximal attractor $\mathcal{A}^\alpha$ in $\mathcal{H}^\alpha$ that is bounded in V, compact and connected in H. Moreover, the semigroup $S(t)$ maps H^α into itself and possesses in H^α a maximal attractor A^α that is compact and connected.

THEOREM 2.2 Let Assumption 2.1–2.3 hold. Then, for every $\alpha \geqslant 0$ and $\beta \geqslant 0$, the semigroup $S(t)$ associated with Problem 2.2 maps $\mathcal{H}^{\alpha,\beta}$ into itself. It possesses a maximal attractor $\mathcal{A}^{\alpha,\beta}$ in $\mathcal{H}^{\alpha,\beta}$ that is bounded in V, compact and connected in H. Moreover, the semigroup $S(t)$ maps $H^{\alpha,\beta}$ into itself and possesses in $H^{\alpha,\beta}$ a maximal attractor $A^{\alpha,\beta}$ that is compact and connected.

REMARK The theorems assert the existence of compact attractors on level sets $\mathcal{H}^\alpha$ and $\mathcal{H}^{\alpha,\beta}$, i.e. that each $\mathcal{H}^\alpha$ has a unique global—in $\mathcal{H}^\alpha$—compact attractor of the system that attracts all bounded subsets of $\mathcal{H}^\alpha$. However, the attractors are not attractors in the usual sense in H since they are not attractors in any open subsets of H. This is a consequence of conservation of mass property in Problem 2.1, and the conservation of mass and velocity in Problem 2.2.

3. Numerical simulations

The asymptotic analysis demonstrates the existence of an attractor for Problems 2.1 and 2.2. However it is more difficult to examine the properties of these attractors. In this section we consider numerical approximations of different attractors for the 1D problems (Problems 2.1 and 2.2) in the case of no external forces. The pressure term $P'(\rho)$ in (2.2) is chosen as in Section 1, see Fig. 1. In our numerical experiments for the 1D case we set $L = 1,\ \nu = 0{\cdot}03,\ \mu = 0{\cdot}001$ and

$$P'(\rho) = \begin{cases} 0 & 0 \leqslant \rho \leqslant \varepsilon\rho^*, \\ m_1(\rho - \varepsilon\rho^*)(\rho - \rho^*) & \varepsilon\rho^* < \rho \leqslant \rho^*, \\ m_2(\rho - \varepsilon\rho^*)(\rho - \rho^*) & \rho^* < \rho \leqslant 2\rho^*, \\ m_2\frac{2-\varepsilon}{2}\rho^*\rho & 2\rho^* < \rho, \end{cases} \tag{3.1}$$

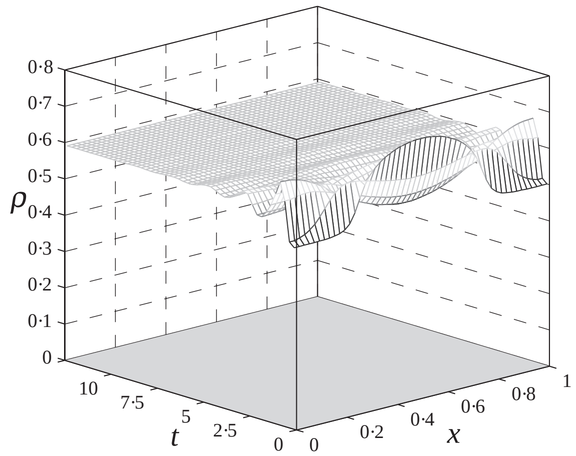


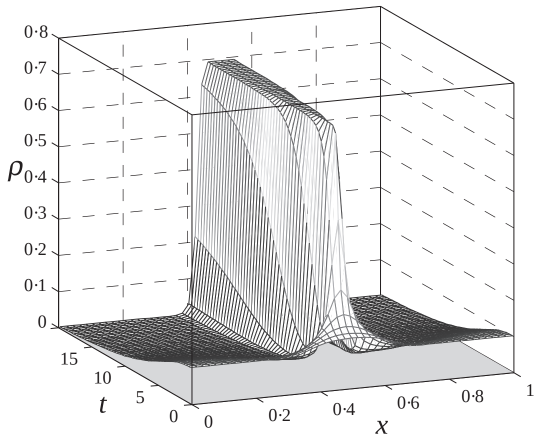
FIG. 2. The evolution of density. The density approaches a stationary state attractor as time increases.

with $\rho^* = 0{\cdot}5$, $\varepsilon = 0{\cdot}1$, $m_1 = 10$, and $m_2 = 10$. It is easy to check that all Assumptions 2.1–2.3 are fulfilled for such values of the coefficients with the following values of the pressure term constants: $P_\infty = m_2 \dfrac{2-\varepsilon}{4}\rho^* = 2{\cdot}375$, $P_0 = P_1 = 0$, $P_2 = (P_\infty)^2 < 6$ and $P_3 = 1$.

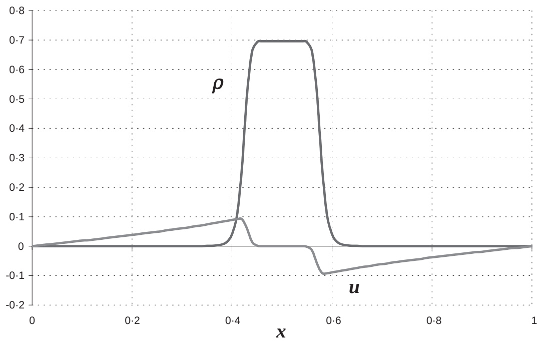
(a) Problem 2.1. For the boundary value problem we get two different types of attractors: a trivial attractor, i.e. the stationary state $(0, \bar{\rho})$, and a non-trivial attractor. The trivial attractor appears in a natural way when the initial condition is a small perturbation of the stationary state with high average density. In this situation the solution will be attracted to the stationary state as is to be expected from the stochastic character of the movement in systems of organisms (see Fig. 2).

The size of the basin of the stationary state attractor essentially depends on the diffusion coefficients ν and μ and on the pressure term $P'(\rho)$. From numerical experiments we see that by increasing the parameters ν and μ the basin of the stationary state attractor increases. This property relates to the stochastic and the alignment components in the movement of the particles as reflected in the diffusion coefficients. As the noise in the movement increases, the structure of motion depends less on the interaction between particles. However, by increasing the intensity of the pressure term, i.e. coefficients m_1 or m_2 in expression (3.1), the basin of the stationary state attractor decreases. This is equivalent to increase in the interaction between moving particles.

If the value of $\bar{\rho}$ is large enough then the attractors for the density have a form similar to a step function. The levels of such an attractor are at zero and at some positive constant. These two levels are connected by continuous functions. The shape of the slopes and the non-zero level of the non-trivial step attractors depend on the diffusion coefficients ν and μ and on the pressure term $P'(\rho)$. Increasing the diffusion coefficients increases the stochastic and alignment effect in the moving system of particles; this leads to prevalence in this effect in the interaction between particles, and the shape of the groups is weaker. It is



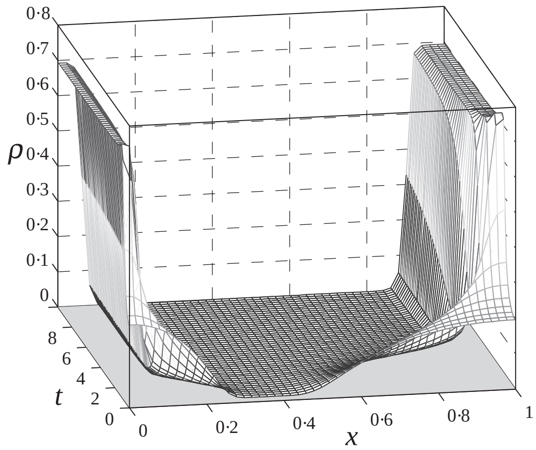
(a)



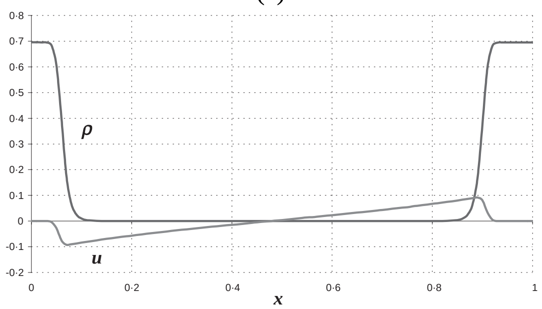
(b)

FIG. 3. The attractor for Problem 2.1 when the initial distribution of the organisms is far from the boundaries. The density approaches its attractor with two connected constant levels. The non-zero constant level of the density attractor is located in the interior of the space domain. (a) Three-dimensional plot of the evolution of density; (b) the attractor for density. Also shown is the corresponding attracting velocity profile.

confirmed by the numerical experiments that the slopes of the non-trivial attractor decrease with increasing diffusion parameters. On the other hand the pressure term influences the height of the non-zero level in the non-trivial attractor. It is further shown by the numerical experiments that the height of the non-zero level does not vary for different initial conditions provided the values of the diffusion coefficients and the pressure term remain unchanged. This height of the non-zero level is determined by the equilibrium between the forces which spread particles and those which try to attract particles to each other. According to the numerical experiments, increasing the force $-P'(\rho)\rho_x$ in the interval (ρ^*, ∞), i.e. the component in the pressure term which depends on the value of m_2, decreases the height of the non-zero level, and increasing the force in the interval $(0, \rho^*)$, i.e. the coefficient m_1 in the pressure term, increases the height of the non-zero level. The value of the prescribed level ρ^* in the pressure term also affects the height of the non-zero level; increasing this parameter also increases the height of the non-zero level.



(a)



(b)

FIG. 4. The attractor for Problem 2.1 when the initial distribution of the organisms is close to the boundaries. The density approaches the attractor with two connected constant levels of density. The non-zero constant level of the density attractor is located at both sides of the space domain. (a) Three-dimensional plot of the evolution of density; (b) the attractor for density. Also shown is the corresponding attracting velocity profile.

These density shapes correspond to those encountered in large aggregations of, for example, fish, birds and micro-organisms (Czirok & Vicsek, 2000; Othmer & Stevens, 1997; Parrish & Edelstein-Keshet, 1999; Partridge, 1982). Different types of such attractors are shown in Figs 3 and 4. Also shown in Figs 3(b) and 4(b) are the corresponding velocity attractors. The direction of the velocity is towards the aggregation and the movement of mass into it is balanced by the diffusion in density. The velocity is zero inside the aggregation where the density is constant and non-zero. The different results are caused by differences in the initial data but all other parameters are the same.

The attractor in Fig. 3 corresponds to a group of organisms in the case when the distance between the organisms and boundaries is large. This figure thus describes the movement of organisms in a homogeneous environment.

In Fig. 4 we have patterns which correspond to a group distribution near the boundaries. It is easy to see that the non-zero level of the density attractors is the same as in Fig. 3, so we get groups with the same density.

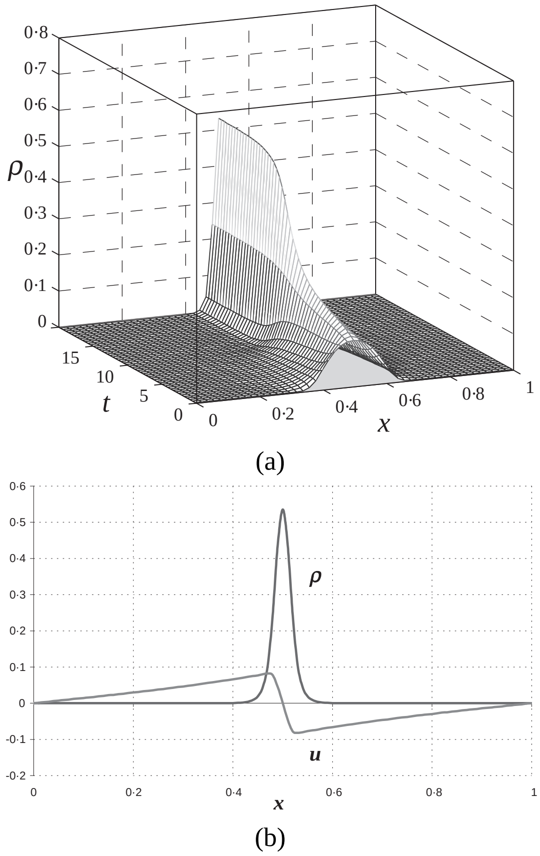
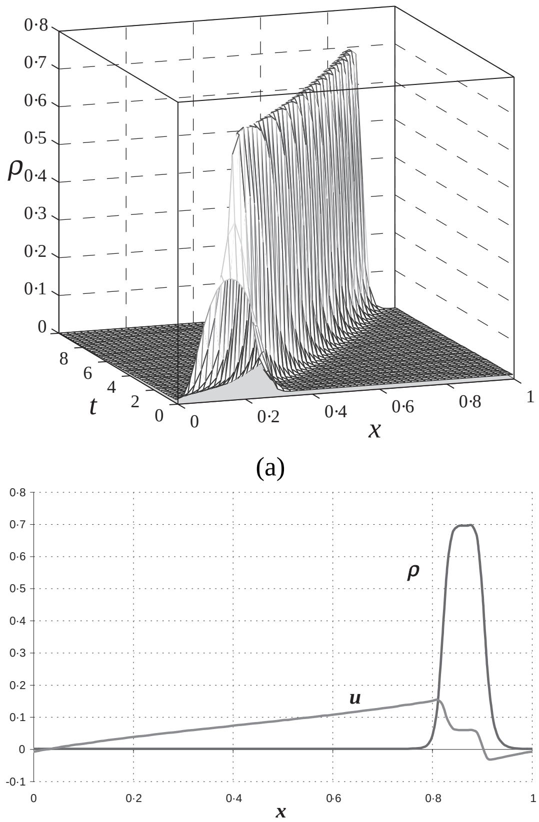


FIG. 5. The attractor for Problem 2.1 when the total mass is small, thus the density approaches the attractor with the density peak below the non-zero constant level in Figs 3 and 4. (a) Three-dimensional plot of the evolution of density; (b) the attractor for density. Also shown is the corresponding attracting velocity profile.

Figure 5 shows an interesting example of an attractor which occurs if the average density $\bar{\rho}/L$ of organisms is not as high as in the previous cases. The maximum value of the density attractor is lower than in the previous cases. This attractor can correspond to a small group of organisms. It is natural that the maximum value is lower, because in this case the number of organisms is small, and the diffusion in density prevents the formation of higher density groups. Note that in this case there is no interval of zero velocity within the aggregation.

(b) Problem 2.2. For the periodic problem we have shown the existence of the trivial attractor which consists of the solutions $(\bar{u}, \bar{\rho})$, where $\bar{u}$ = constant. It is easy to get this kind of attractor by numerical methods if the initial conditions are a small perturbation of this stationary state.

Another kind of attractor is similar to the one shown in Fig. 3. From the homogeneous



(a)

(b)

FIG. 6. Travelling wave attractor for Problem 2.2 with non-zero average velocity. The shape of density attractor is similar to the one in Fig. 3, but it moves with constant speed $|\bar{u}|/L$. (a) Three-dimensional plot of the evolution of density; (b) the attractor for density. Also shown is the corresponding attracting velocity profile.

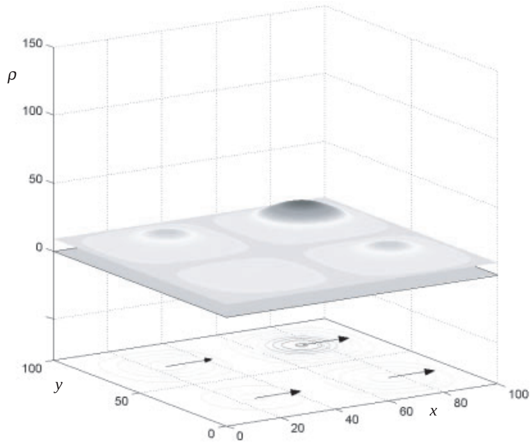
equation for velocity we can obtain the following equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_x^{x+1} u(t, y)\mathrm{d}y = 0$$

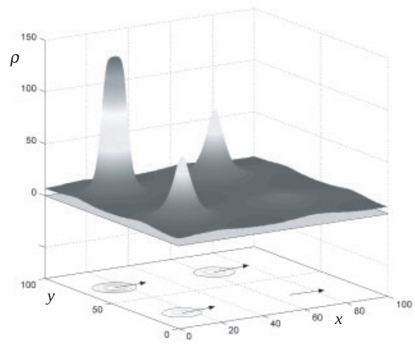
implying that

$$\int_x^{x+1} u(t, y)\mathrm{d}y = \text{constant}.$$

The numerical approximations show the drift of the solution for density with a steady speed that depends on the constant in this expression. Thus, we get a travelling wave as the attractor, as shown in Fig. 6. The velocity profile shown in Fig. 6(b) travels with the same steady speed. The shapes of the attractors again correspond to the shapes of moving groups of organisms.



(a)

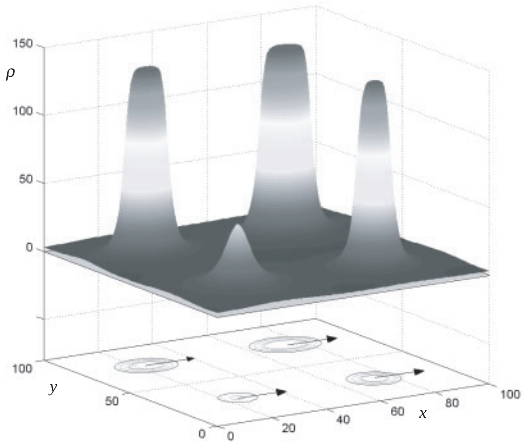


(b)

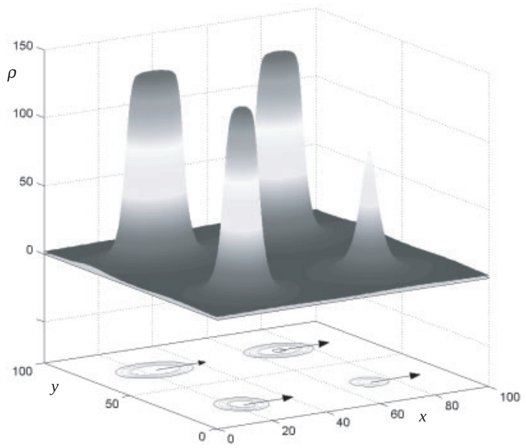
FIG. 7. Evolution of the density in 2D space for the system (1.4)–(1.5) with periodic boundary conditions and non-zero average velocity. (a) Initial density distribution, adding slight perturbations to uniform density, and constant velocity along the x-axis; (b) early stage of group formation due to the internal force. Note that the density patterns are periodically shifted in the x-direction. *cont.*

(c) Two-dimensional problem. The analysis of the problem of grouping due to an internal force in the 1D case should be considered as the first step towards a more comprehensive and realistic study in two dimensions. The 1D analysis gives an asymptotic behaviour of density which corresponds to the shapes of moving groups of organisms. By a numerical approximation, similar dynamical behaviour of density can be observed for the 2D case.

We present a numerical example for the system (1.4)–(1.5) in a rectangular 2D domain with periodic boundary conditions for density and velocity and no external force. In the



(c)



(d)

FIG. 7. *cont.* Evolution of the density in 2D space for the system (1.4)–(1.5) with periodic boundary conditions and non-zero average velocity. (c) Three fully formed groups with the same maximum level of density and one small group with the lower peak due to small total mass forming it. The density group patterns are similar to the ones in the 1D cases; (d) the same density patterns preserved at a later stage except of being shifted periodically in the x–direction.

numerical experiment we set $L_x = L_y = 100$, $\nu = 150$, $\mu = 150$ and

$$P'(\rho) = \begin{cases} -m_1\rho & 0 \leqslant \rho \leqslant \varepsilon\rho^*, \\ -m_1\varepsilon\rho^* & \varepsilon\rho^* < \rho \leqslant (1-\varepsilon)\rho^*, \\ -m_1(\rho^* - \rho) & (1-\varepsilon)\rho^* < \rho \leqslant \rho^*, \\ m_2(\rho - \rho^*) & \rho^* < \rho \leqslant 2\rho^*, \\ \frac{m_2}{2}\rho & 2\rho^* < \rho, \end{cases}$$

with $\rho^* = 100$, $\varepsilon = 0{\cdot}1$ and $m_1 = m_2 = 10$.

Starting from a slightly non-homogeneous density distribution in the rectangular domain at time $t = 0$ with constant non-zero initial velocity $m = (0, 40)$, see Fig. 7(a), the evolution of the density is presented in Figs 7(b)–(d).

The density distribution in Fig. 7(b) exhibits the early stage of group formation at time $t = 0{\cdot}9971$, with Figs 7(c) and (d) illustrating the further evolution of the density towards stable group patterns at times $t = 2{\cdot}3701$ and $t = 3{\cdot}5348$. As in the 1D cases, the density tends to step-like functions with two levels: the zero level and a non-zero level. This non-zero level is the same for three of the groups shown in Fig. 7, but the fourth group has a lower peak due to the small total mass.

4. Discussion

The objective of the work described here is to study the formation of fish schools. The phenomenon of school formation has been studied in a number of papers, from both empirical and theoretical points of view, as well as by modelling and numerical simulations (e.g. Niwa, 1996; Okubo, 1986; Partridge, 1982; Stöcker, 1999; Vicsek *et al.*, 1999). Both individual-based and continuous-density models have been considered. Individual-based models are a useful tool for studying relatively small groups, but become impractical when the number of individuals approaches the sizes of real fish schools. The collective behaviour exhibited by fish schools can only be properly understood by considering large numbers of school members interacting amongst themselves.

Here we have used a continuous density–velocity model to study large formations of organisms and the formation of fish schools. The forced Burgers equation describing the time evolution of velocity has a force that is composed of an internal force representing the interaction between particles or fish and an external force, as well as a diffusion term that may viewed as an alignment force.

The central idea of this paper is to model the internal force as 'pressure' resulting from the behaviour of individuals trying to achieve and maintain a prescribed level of density. Thus, the internal force acts in the direction of the density gradient or in the opposite direction depending on the local density. The force is in the direction of density gradient if the density is less than the prescribed density level and it is opposite to the direction of density gradient if the density is greater than this level. This formulation of the internal force is based on the observation that in fish groups the distance to the neighbour is more or less uniform (Partridge, 1982).

The external force incorporates environmental effects, such as temperature gradient, chemical gradient, as well as food and predator densities (Flierl *et al.*, 1999; Grünbaum & Okubo, 1994; Kowalczyk *et al.*, 2004). The external force may also lead to group fragmentation in a non-homogenous environment or when predators are present in the surroundings. The mass conservation equation with advection and diffusion in the flux transport describes the dynamics of the density. A derivation of such diffusion and advection terms from assumptions on individual behaviour can be found in a number of papers (e.g. Murray, 1989; McComb, 1992; Woyczyński, 1998). The external force is taken to be zero in the numerical experiments described in this paper.

The theoretical analysis of the 1D density–velocity model demonstrates the existence of global attractors on metric subspaces of the Hilbert space of square integrable functions $H = L^2(\Omega) \times L^2(\Omega)$. Two types of boundary conditions are considered: no-flux boundary condition for velocity and the Neumann condition for density (Problem 2.1), and periodical boundary conditions (Problem 2.2). Both of these boundary value problems can be considered as arising in a natural way from the behaviour of moving organisms. The conservation of mass (2.9) in the absence of recruitment and mortality is ensured by these boundary conditions; moreover, for the periodic boundary conditions the conservation of velocity (2.10) is guaranteed under some restrictions on the external force f (Assumption 2.3).

The theoretical result about the existence of attractors is restricted to metric subspaces of the Hilbert space H. These metric subspaces are constructed as level sets $\mathcal{H}^{\alpha}$ for Problem 2.1 and level sets $\mathcal{H}^{\alpha,\beta}$ for Problem 2.2 (Temam, 1997). The attractors on these level sets are not attractors in any open subset of H. However, the attractors on the whole space H, $H = \bigcup_{\alpha \geqslant 0} \mathcal{H}^{\alpha}$ or $H = \bigcup_{\alpha \geqslant 0, \beta \geqslant 0} \mathcal{H}^{\alpha,\beta}$, can be constructed using the attractors on the level sets, but these attractors are non-compact.

The sufficient Assumptions 2.1–2.3 for the existence of the attractors give an essential restriction on the behaviour of the pressure term $P'(\rho)$ when the density tends to infinity as well as a restriction on the diffusion coefficients μ and ν. Since the size of the individual fish defines an upper bound on density of fish, the pressure term can be chosen in an appropriate way above this upper limit on density ensuring that the restrictions on the diffusion coefficients are satisfied, i.e. the diffusion coefficients can be chosen to be arbitrary small positive values, by an appropriate choice of the pressure term above the upper bound on density.

As demonstrated by the numerical solutions of the equations, the shape of the density function belonging to the attractor (Figs 2–7) corresponds well to observed shapes of groups of organisms such as fish, birds or insects (i.e. local uniform density with zero density outside).

The factor giving rise to the density shapes illustrated in Figs 2–7 is the internal pressure force, $-P'(\rho)\nabla\rho$ in the Burgers equation for the velocity (1.5). The pressure term $P'(\rho)$ governs the height of the non-zero level in the non-trivial attractor for density through the location of the zero pressure interval $[\rho_1^*, \rho_2^*]$ or the zero pressure point ρ^*, see Fig. 1. On the other hand, the slopes of the non-trivial attractor increase with increasing magnitude of the pressure term.

The shape of the attractor depends on the values of the diffusion coefficients for velocity, ν, and density, μ. It is confirmed by the numerical experiments that the slopes of the density attractor decrease with increasing diffusion coefficients. The latter coefficient is related to the stochastic effects in the particle motion; thus, increasing the individual noise will result in a weaker individual attraction or repulsion and the particles will therefore scatter over a larger domain in less coherent patterns. The other 'diffusion' coefficient, ν, is related to the strength of the alignment interaction, i.e. $\nu = \frac{1}{\tau} \cdot \frac{\varepsilon^2}{2(n+4)}$, where ε is the interaction radius of the n-dimensional ball and τ is characteristic reaction time. A strong alignment force—i.e. a large ν—should make the density distribution more uniform since the alignment interaction will act over a larger area and spread faster through the domain. This is in fact borne out by the numerical experiments.

A 1D model for a fish school is obviously too restrictive. The analysis of the simplified 1D models should therefore be regarded as a starting point for more elaborate and more realistic studies in two and three dimensions. Numerical experiments of the 1D models show grouping behaviour of individuals and a theoretical foundation for this behaviour is given by the existence of the attractor. The preliminary numerical experiments of 2D models reveal the same type of grouping behaviour as in the 1D case.

Acknowledgements

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Appendix A. Definitions (dynamical systems, attractors)

We recall some definitions and results about attractors (Temam, 1997; Chueshov, 2002). For this we introduce the semigroup of continuous operators on a complete metric space X: that is, a one-parameter family, $S(t)$, $0 \leqslant t < \infty$, of continuous operators from X into X, such that

$$\left.\begin{array}{l} S(0) = I, \quad (I \text{ is the indentity operator on } X); \\ S(t+\tau) = S(t)S(\tau) \quad \text{for every } t, \tau \geqslant 0. \end{array}\right\} \text{the semigroup property.}$$

DEFINITION A.1 An attractor of the semigroup $\{S(t)\}$ is a set $\mathcal{A} \subset X$ that has the following properties:

(i) $\mathcal{A}$ is an invariant set ($S(t)\mathcal{A} = \mathcal{A}, \forall t \geqslant 0$),

(ii) $\mathcal{A}$ possesses an open neighbourhood $\mathcal{U}$ such that for every u_0 in $\mathcal{U}$, $S(t)u_0$ converges to $\mathcal{A}$ as $t \to \infty$:

$$\operatorname{dist}(S(t)u_0, \mathcal{A}) \to 0 \qquad \text{as} \quad t \to \infty.$$

The distance in (ii) is understood to be a distance from a point to a set

$$\operatorname{dist}(u, A) = \inf_{v \in A} d(u, v),$$

$d(x, y)$ denoting the distance from u to v in X. If $\mathcal{A}$ is an attractor, the largest open set $\mathcal{U}$ that satisfies (ii) is called *the basin of attraction* of $\mathcal{A}$.

DEFINITION A.2 We say that $\mathcal{A} \subset X$ is a global attractor for the semigroup $\{S(t)\}_{t \geqslant 0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of X.

DEFINITION A.3 Let $\mathcal{B}$ be a subset of X and $\mathcal{U}$ an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is absorbing in $\mathcal{U}$ if for any bounded set $\mathcal{B}_0$ in $\mathcal{U}$ there exists $t_0(\mathcal{B}_0)$ such that $S(t)\mathcal{B}_0 \subset \mathcal{B}$ for every $t \geqslant t_0(\mathcal{B}_0)$.

DEFINITION A.4 The operators $S(t)$ are uniformly compact in X for large t if for every bounded set $\mathcal{B} \subset X$ there exists t_0 such that $\cup_{t \geqslant t_0} S(t)\mathcal{B}$ is relatively compact in X.

THEOREM A.1 (Chueshov, 2002; Temam, 1997) Assume that the continuous operators $S(t)$ $(t \geqslant 0)$ given on the metric space X satisfy the semigroup property, moreover, they are uniformly compact for large t. Also assume that there exists a bounded set $\mathcal{B}$ of X such that $\mathcal{B}$ is absorbing in X.

Then the ω-limit set of $\mathcal{B}$,

$$\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S(t)\mathcal{B}},$$

is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$.

Furthermore, if X is a convex set of a Banach space H, and the mapping $t \to S(t)u_0$ is continuous from $[0, +\infty)$ into X, for every u_0 in X, then $\mathcal{A}$ is connected.

Appendix B. Proofs of Theorems 2.1 and 2.2

Proof. In order to prove these theorems we shall use Theorem A.1. For this purpose it will be necessary to prove the existence of absorbing sets in $\mathcal{H}^\alpha$ with fixed $\alpha \geqslant 0$ for Problem 2.1 or in $\mathcal{H}^{\alpha,\beta}$ with fixed $\alpha \geqslant 0$, $\beta \geqslant 0$ for Problem 2.2 and the uniform compactness of $S(t)$ for large t in the spaces $\mathcal{H}^\alpha$ and $\mathcal{H}^{\alpha,\beta}$, respectively. Note, that for Problem 2.1 the semigroup $S(t)$ maps the set $\mathcal{H}^\alpha$ into itself, and for Problem 2.2 it maps the set $\mathcal{H}^{\alpha,\beta}$ into itself. We shall prove Theorem 2.1 and make remarks about differences in the proof of Theorem 2.2.

Let us denote by $\|\cdot\|_p$ $(1 < p)$ the norm on $L^p(\Omega)$, by $\|\cdot\|$ the norm on $L^2(\Omega)$ and by $\|\cdot\|_\infty$ the norm on $L^\infty(\Omega)$.

We shall make use of the Poincaré inequalities

$$\|v\|^2 \leqslant c_0^P \|v_x\|^2, \quad v \in H_0^1(\Omega); \qquad \|v\|^2 \leqslant c_1^P \|v_x\|^2 + \frac{1}{L}\left(\int_0^L v \,\mathrm{d}x\right)^2, \quad v \in H^1(\Omega),$$

the Agmon inequality

$$\|v\|_\infty^2 \leqslant c^A \|v\| \cdot \|v_x\|, \quad v \in H^1(\Omega),$$

where constants c_0^P, c_1^P and c^A depend only on Ω, and the following lemma.

Uniform Gronwall lemma. Let g, q, y be three positive locally integrable functions on $(t_0, +\infty)$ such that y' is locally integrable on $(t_0, +\infty)$, and which satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant gy + q, \qquad \text{for} \quad t \geqslant t_0,$$

$$\int_t^{t+r} g(s)\mathrm{d}s \leqslant a_1, \qquad \int_t^{t+r} q(s)\mathrm{d}s \leqslant a_2, \qquad \int_t^{t+r} y(s)\mathrm{d}s \leqslant a_3, \qquad \text{for} \quad t \geqslant t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \leqslant \Big(\frac{a_3}{r} + a_2\Big)\exp(a_1), \quad \forall t \geqslant t_0.$$

(a) The existence of an absorbing set. We multiply (2.1) by 2ρ, and (2.2) by $2u$, and integrate over the domain Ω; using integration by parts and boundary value conditions (2.3), (2.4) or periodic conditions (2.7), (2.8), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|^2 + 2\mu\|\rho_x(t)\|^2 = -\int_0^L \rho^2 \cdot u_x \,\mathrm{d}x; \tag{B.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + 2\nu\|u_x(t)\|^2 = \int_0^L 2P(\rho)\cdot u_x \,\mathrm{d}x + 2\int_0^L f\cdot u \,\mathrm{d}x. \tag{B.2}$$

Add expression (B.2) to (B.1) multiplied by positive constant h:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + h\cdot\|\rho(t)\|^2) + (2\nu\cdot\|u_x(t)\|^2 + 2\mu h\cdot\|\rho_x(t)\|^2) = \int_0^L (2P(\rho) - h\cdot\rho^2)\cdot u_x \,\mathrm{d}x + 2\int_0^L f\cdot u\,\mathrm{d}x. \tag{B.3}$$

Using Assumptions 2.3 and 2.1, taking $h = 2P_\infty$, and applying the Poincaré inequalities, we obtain the following estimates:

$$\begin{aligned}\int_0^L (2P(\rho) - h\rho^2)u_x \,\mathrm{d}x &= \int_0^L 2(P(\rho) - P_\infty\rho^2)u_x\,\mathrm{d}x \leqslant \int_0^L 2\big|P_1|\rho| + P_0\big|\cdot|u_x|\,\mathrm{d}x \\ &\leqslant \int_0^L 2P_1|\rho|\cdot|u_x| + 2P_0|u_x|\,\mathrm{d}x \leqslant (\tau_1+\tau_2)\|u_x\|^2 + \frac{P_1^2}{\tau_1}\|\rho\|^2 + \frac{P_0^2 L}{\tau_2} \\ &\leqslant (\tau_1+\tau_2)\|u_x\|^2 + \frac{c_1^P P_1^2}{\tau_1}\|\rho_x\|^2 + \frac{P_1^2\bar\rho^2}{\tau_1 L} + \frac{P_0^2 L}{\tau_2}, \qquad \forall \tau_1, \tau_2 > 0\end{aligned}$$

and

$$2\int_0^L f\cdot u\,\mathrm{d}x \leqslant 2\|f\|\cdot\|u\| \leqslant 2\sqrt{c_0^P}\|f\|\cdot\|u_x\| \leqslant \tau_3\|u_x\|^2 + \frac{c_0^P}{\tau_3}\|f\|^2, \qquad \forall\tau_3 > 0.$$

After substitution into (B.3) for any positive constants τ_1, τ_2 and τ_3, we get that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + 2P_\infty\|\rho(t)\|^2) + (2\nu - \tau_1 - \tau_2 - \tau_3)\|u_x(t)\|^2 \\ &\quad + \Big(4\mu P_\infty - \frac{c_1^P P_1^2}{\tau_1}\Big)\|\rho_x(t)\|^2 \leqslant \frac{c_0^P}{\tau_3}\|f\|^2 + \frac{P_0^2 L}{\tau_2} + \frac{P_1^2}{\tau_1 L}\bar\rho^2 = F_0(\bar\rho).\end{aligned}$$

Note that, for Problem 2.2, we get the same estimates, except that $F_0(\bar{\rho})$ has to be replaced by

$$F_0(\bar{\rho}, \bar{u}) = \frac{c_1^P}{\tau_3}\|f\|^2 + \frac{P_0^2 L}{\tau_2} + \frac{P_1^2}{\tau_1 L}\bar{\rho}^2 + \frac{\tau_3}{c_1^P L}\bar{u}^2.$$

By Assumption 2.1, there exist positive constants τ_1, τ_2 and τ_3, such that $k_1 = 2\nu - \tau_1 - \tau_2 - \tau_3 > 0$ and $k_2 = 4\mu P_\infty - \dfrac{c_1^P P_1^2}{\tau_1} > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + 2P_\infty \cdot \|\rho(t)\|^2) + k_1 \cdot \|u_x(t)\|^2 + k_2 \cdot \|\rho_x(t)\|^2 \leqslant F_0. \tag{B.4}$$

Due to the Poincaré inequalities for Problem 2.1 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + 2P_\infty\|\rho(t)\|^2) + \frac{k_1}{c_0^P}\|u(t)\|^2 + \frac{k_2}{c_1^P}\|\rho(t)\|^2 \leqslant F_0(\bar{\rho}) + \frac{k_2}{c_1^P L}\bar{\rho}^2 = F_1(\bar{\rho}),$$

and for Problem 2.2,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + 2P_\infty\|\rho(t)\|^2) &+ \frac{k_1}{c_1^P}\|u(t)\|^2 + \frac{k_2}{c_1^P}\|\rho(t)\|^2 \\ &\leqslant F_0(\bar{\rho},\bar{u}) + \frac{k_1}{c_1^P L}\bar{u}^2 + \frac{k_2}{c_1^P L}\bar{\rho}^2 = F_1(\bar{\rho},\bar{u}).\end{aligned}$$

Let $k = \min\left\{\dfrac{k_1}{c_0^P}, \dfrac{k_2}{2c_1^P P_\infty}\right\}$ for Problem 2.1, and $k = \min\left\{\dfrac{k_1}{c_1^P}, \dfrac{k_2}{2c_1^P P_\infty}\right\}$ for Problem 2.2. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|^2 + 2P_\infty \cdot \|\rho(t)\|^2) + k(\|u(t)\|^2 + 2P_\infty \cdot \|\rho(t)\|^2) \leqslant F_1.$$

Using the classical Gronwall lemma, we obtain

$$\|u(t)\|^2 + 2P_\infty \cdot \|\rho(t)\|^2 \leqslant (\|u_0\|^2 + 2P_\infty \cdot \|\rho_0\|^2)\mathrm{e}^{-k\cdot t} + \frac{F_1}{k}[1 - \mathrm{e}^{-k\cdot t}]. \tag{B.5}$$

Thus, taking $\theta = 1/(2P_\infty)$ in the norm of the space H, we get

$$\limsup_{t\to\infty}\|w(t)\|_H^2 = \limsup_{t\to\infty}(\|\rho(t)\|^2 + \frac{1}{2P_\infty}\|u(t)\|^2) \leqslant K_0^2, \quad \text{where } K_0^2 = \frac{F_1}{2kP_\infty}. \tag{B.6}$$

We infer from (B.5) that for each fixed $\alpha \geqslant 0$ the balls $\mathcal{B}(K') = \mathcal{B}^\alpha(K') = B_H(K') \cap \mathcal{H}^\alpha$ of $\mathcal{H}^\alpha$ with centre in the origin and radius K' ($K' \geqslant K_0 = K_0(\alpha)$) are absorbing for the semigroup $S(t)$, which maps $\mathcal{H}^\alpha$ into itself. Here $B_H(K')$ is the ball of H with the same centre and radius. Since any bounded set $\mathcal{B}$ of $\mathcal{H}^\alpha$ is included in some ball $\mathcal{B}^\alpha(K)$, it is easy to deduce from (B.5) that for any fixed K' such that $K' > K_0$ there exists $t_0(\alpha, K', \mathcal{B})$ such that $S(t)\mathcal{B} \subset \mathcal{B}(K') = \mathcal{B}^\alpha(K')$ for $t > t_0 = t_0(\alpha, K', \mathcal{B})$, namely

$$\|w(t)\|_H^2 = \|\rho(t)\|^2 + \frac{1}{2P_\infty}\|u(t)\|^2 \leqslant K'^2, \quad \forall t > t_0(\alpha, K', \mathcal{B}) = \frac{1}{k}\ln\frac{K^2}{K'^2 - K_0^2}. \tag{B.7}$$

We infer from (B.4), after integrating with respect to t, that $\forall r > 0$,

$$\int_t^{t+r} (k_1\|u_x(s)\|^2 + k_2\|\rho_x(s)\|^2)\,\mathrm{d}s \leqslant rF_0 + 2P_\infty(\|\rho(t)\|^2 + \frac{1}{2P_\infty}\|u(t)\|^2). \tag{B.8}$$

Using (B.6), we conclude that for $k_3 = \min\left\{k_1, \dfrac{k_2}{2P_\infty}\right\}$,

$$\limsup_{t\to\infty} \int_t^{t+r} \|\rho_x(s)\|^2 + \frac{1}{2P_\infty}\|u_x(s)\|^2\,\mathrm{d}s \leqslant \frac{rF_0}{2k_3P_\infty} + \frac{K_0^2}{k_3}, \tag{B.9}$$

and if $w_0 = (\rho_0, u_0) \in \mathcal{B} \subset \mathcal{B}(K) = \mathcal{B}^\alpha(K)$, $K \geqslant K_0 = K_0(\alpha)$ and $t > t_0(\alpha, K', \mathcal{B})$, then

$$\int_t^{t+r} \|\rho_x(s)\|^2 + \frac{1}{2P_\infty}\|u_x(s)\|^2\,\mathrm{d}s \leqslant \frac{rF_0}{2k_3P_\infty} + \frac{K'^2}{k_3} = K''^2. \tag{B.10}$$

We shall make use of this estimate in the second part of the proof.

Note that in the case of Problem 2.2 the space $\mathcal{H}^\alpha$ is replaced by $\mathcal{H}^{\alpha,\beta}$, for each fixed $\alpha \geqslant 0$ and $\beta \geqslant 0$ the semigroup $S(t)$ maps $\mathcal{H}^{\alpha,\beta}$ into itself, $K_0 = K_0(\alpha,\beta)$, $\mathcal{B}(K) = B^{\alpha,\beta}(K) = B_H(K) \cap \mathcal{H}^{\alpha,\beta}$ and $t_0 = t_0(\alpha, \beta, K', \mathcal{B})$.

(b) Uniform compactness for large t. The space V is compactly embedded in H. Thus, for each $\alpha \geqslant 0$ the restriction of the compact embedding operator from V into H to the domain $V \cap \mathcal{H}^\alpha$ is a compact operator from $V \cap \mathcal{H}^\alpha$ into $\mathcal{H}^\alpha = H \cap \mathcal{H}^\alpha$. Therefore, the semigroup of the operators $S(t)$ possess uniform compactness for large t in the space $\mathcal{H}^\alpha$ if it is bounded in $V \cap \mathcal{H}^\alpha$ for large t.

In order to show the uniform compactness for the semigroup of operators $S(t)$ we introduce another energy-type equation by taking the scalar product of (2.1) and (2.2) with $(-2)\rho_{xx}$ and $(-2)u_{xx}$, respectively, and integrating over the domain Ω

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_x(t)\|^2 + 2\mu\|\rho_{xx}(t)\|^2 &= 2\int_0^L u\cdot\rho_x\cdot\rho_{xx}\,\mathrm{d}x + 2\int_0^L \rho\cdot u_x\cdot\rho_{xx}\,\mathrm{d}x;\\
\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(t)\|^2 + 2\nu\|u_{xx}(t)\|^2 &= 2\int_0^L u\cdot u_x\cdot u_{xx}\,\mathrm{d}x\\
&\quad + 2\int_0^L P'(\rho)\rho_x\cdot u_{xx}\,\mathrm{d}x - 2\int_0^L f\cdot u_{xx}\,\mathrm{d}x.
\end{aligned}$$

Adding the first equation to the second equation multiplied by $\theta = \dfrac{1}{2P_\infty}$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho_x(t)\|^2 + \theta\cdot\|u_x(t)\|^2) + 2\mu\|\rho_{xx}(t)\|^2 + 2\nu\theta\|u_{xx}(t)\|^2\\
&\quad = 2\theta\int_0^L (u\cdot u_x + P'(\rho)\rho_x - f)\cdot u_{xx}\,\mathrm{d}x + 2\int_0^L u\cdot\rho_x\cdot\rho_{xx} + \rho\cdot u_x\cdot\rho_{xx}\,\mathrm{d}x.
\end{aligned} \tag{B.11}$$

Using the Agmon, Young and Cauchy-Schwarz inequalities, we obtain

$$2\left|\int_0^L u\cdot u_x\cdot u_{xx}\,\mathrm{d}x\right| \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|u\cdot u_x\|^2 \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|u_x\|^2\cdot\|u\|_\infty^2$$

$$\leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3c^A}{\nu}\|u_x\|^3\cdot\|u\| \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{9c^A}{4\nu}\|u_x\|^4+\frac{3c^A}{4\nu}\|u\|^4;$$

$$2\left|\int_0^L u\cdot \rho_x\cdot \rho_{xx}\,\mathrm{d}x\right| \leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2}{\mu}\|u\cdot \rho_x\|^2 \leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2}{\mu}\|\rho_x\|^2\cdot\|u\|_\infty^2$$

$$\leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2c^A}{\mu}\|\rho_x\|^2\|u_x\|\cdot\|u\| \leqslant \frac{\mu}{2}\|u_{xx}\|^2+\frac{c^A}{\mu}\|\rho_x\|^2\|u_x\|^2+\frac{c^A}{\mu}\|\rho_x\|^2\|u\|^2;$$

$$2\left|\int_0^L \rho\cdot u_x\cdot \rho_{xx}\,\mathrm{d}x\right| \leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2}{\mu}\|\rho\cdot u_x\|^2 \leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2}{\mu}\|u_x\|^2\|\rho\|_\infty^2$$

$$\leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{2c^A}{\mu}\|u_x\|^2\|\rho_x\|\cdot\|\rho\| \leqslant \frac{\mu}{2}\|\rho_{xx}\|^2+\frac{c^A}{\mu}\|u_x\|^2\|\rho_x\|^2+\frac{c^A}{\mu}\|u_x\|^2\|\rho\|^2;$$

$$2\int_0^L f\cdot u_{xx}\,\mathrm{d}x \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|f\|^2.$$

From Assumption 2.2 we have

$$2\int_0^L P'(\rho)\rho_x\cdot u_{xx}\,\mathrm{d}x \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|P'(\rho)\rho_x\|^2 \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|\rho_x\|^2\cdot\|P'(\rho)\|_\infty^2$$

$$\leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{3}{\nu}\|\rho_x\|^2\cdot\|P'(\rho)\|_\infty^2 \leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{6}{\nu}\|\rho_x\|^2(P_3\|\rho\|_\infty^4+P_2)$$

$$\leqslant \frac{\nu}{3}\|u_{xx}\|^2+\frac{6}{\nu}\|\rho_x\|^2(P_3(c^A)^2\|\rho_x\|^2\cdot\|\rho\|^2+P_2).$$

After substitution of these inequalities into (B.11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho_x(t)\|^2+\theta\|u_x(t)\|^2)+\mu\|\rho_{xx}(t)\|^2+\nu\theta\|u_{xx}(t)\|^2 \leqslant \frac{3\theta\|f\|^2}{\nu}+\frac{3c^A\theta}{4\nu}\|u\|^4$$
$$+\frac{9c^A\theta}{4\nu}\|u_x\|^4+\frac{2c^A}{\mu}\|\rho_x\|^2\cdot\|u_x\|^2+\frac{c^A}{\mu}\|\rho_x\|^2\cdot\|u\|^2+\frac{c^A}{\mu}\|u_x\|^2\cdot\|\rho\|^2$$
$$+\frac{6\theta}{\nu}\|\rho_x\|^2((c^A)^2P_3\|\rho_x\|^2\cdot\|\rho\|^2+P_2). \tag{B.12}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho_x\|^2+\theta\|u_x\|^2) \leqslant C_0+C_1\theta^2\|u\|^4+(\|\rho_x\|^2+\theta\|u_x\|^2)$$
$$\times\,(C_2+C_3(\|\rho\|^2+\theta\|u\|^2)+C_4\theta\|u_x\|^2+C_5\|\rho_x\|^2\cdot\|\rho\|^2), \tag{B.13}$$

where C_i, $i=1,\dots,6$ depend only on ν, μ, θ and c^A, and C_0 depends on ν, θ, $\|f\|$.

Assuming that $w_0=(\rho_0,u_0)$ belongs to a bounded set $\mathcal{B}\subset\mathcal{B}^\alpha(K)$ of $\mathcal{H}^\alpha$, taking $t>t_0$, t_0 as in (B.7), and applying the uniform Gronwall lemma to (B.13) with

$$y(t)=\|\rho_x(t)\|^2+\theta\|u_x(t)\|^2;\qquad q(t)=C_0+C_1\theta^2\|u(t)\|^4;$$
$$g(t)=C_2+C_3(\|\rho(t)\|^2+\theta\|u(t)\|^2)+C_4\theta\|u_x(t)\|^2+C_5\|\rho_x(t)\|^2\cdot\|\rho(t)\|^2,$$

we obtain

$$\|\rho_x(t)\|^2 + \theta\|u_x(t)\|^2 \leqslant \left(\frac{a_3}{r} + a_2\right)\exp(a_1) = K''' \qquad \forall t \geqslant t_0 + r. \tag{B.14}$$

Here $K''' = K'''(\alpha, K')$ with

$$a_1 = r(C_2 + C_3K'^2) + C_4K''^2 + C_5K'^2K''^2; \quad a_2 = r(C_0 + C_1K'^4); \quad a_3 = K''^2,$$

where $K'' = K''(\alpha, K')$ is defined in (B.10).

Let $\mathcal{B}_1(K') = \mathcal{B}_1^\alpha(K') = \mathcal{B}^\alpha(K') \cap \{(\phi, v) \in V : \|\phi_x\|^2 + \theta\|v_x\|^2 \leqslant K'''\}$. Then the above results show that the image if any bounded set $\mathcal{B}$ of $\mathcal{H}^\alpha$ by $S(t)$ is included in $\mathcal{B}_1(K')$ when $t \geqslant t_0 + r$. At the same time this shows the existence of an absorbing set in $V \cap \mathcal{H}^\alpha$ and the fact that $S(t)$ is uniformly compact in $\mathcal{H}^\alpha$ for large t.

Note that in the case of Problem 2.2 we get the existence of an absorbing set in $V \cap \mathcal{H}^{\alpha,\beta}$ and the uniform compactness of $S(t)$ in $\mathcal{H}^{\alpha,\beta}$ for large t.

Now, all the assumptions of Theorem A.1 are satisfied for the metric space $X = \mathcal{H}^\alpha$ and we deduce from this theorem the existence of a maximal attractor $\mathcal{A}^\alpha$ in $\mathcal{H}^\alpha$ for Problem 2.1 which is compact and connected. Analogously, there exists a maximal compact connected attractor $\mathcal{A}^{\alpha,\beta}$ in $\mathcal{H}^{\alpha,\beta}$ for Problem 2.2.

Furthermore, the semigroup $S(t)$ maps H^α ($\alpha \geqslant 0$) into itself for Problem 2.1, as well as $H^{\alpha,\beta}$ into itself for Problem 2.2. Thus, the existence of attractors A^α, $A^{\alpha,\beta}$ in the spaces H^α and $H^{\alpha,\beta}$, respectively, can be easily established by analogy. $\square$