



Newtonian Dynamical Approach to Fish Schooling

HIRO-SATO NIWA

National Research Institute of Fisheries Engineering, Hasaki, Ibaraki 314-04, Japan

(Received on 1 June 1994, Accepted in revised form on 9 February 1996)

Schools are groups of fish engaging in cohesive movements with parallel orientation. Schooling fish perform a well-organized collective motion by some kind of social interaction even in the absence of leaders or external stimuli. In this paper, a Newtonian model is developed for schooling behavior to study the self-organizing (i.e., decentralized control) mechanism of a school in a setting of collective motion of autonomous individuals, that is, within a Lagrangian (i.e., individual-based) framework of N -fish motion dynamics. The school is modeled as an interacting particle system with behavioral and environmental stochasticity, such as gas molecules following the rules of Newton's mechanics. Two equations for the school mean and the variance of swimming velocity are derived from the Lagrangian model which has a great degree of freedom ($N \gg 1$). These two reduced equations of motion of the system illustrate a feed-forward control structure underlying the self-organization of the school. On the basis of these equations, the transient process in which the system approaches the stationary polarized schooling is investigated. Features of the temporal evolution of the system from the motionless state of the non-polarized aggregation to the onset of the polarized school are extracted by calculating the following: transient fluctuation of the individual velocity in a school, transient fluctuation of the centroid velocity of a school, two-time correlation of initial fluctuation of the centroid velocity, and temporal evolution of the mean square velocity of the center of school, which are regulated by individual stochastic movement in a school and nonlinearity of a system. Experimental observations of the transient behavior of schooling are consistent with the theoretical predictions. It is expected that the transient fluctuation of the centroid velocity shows the enhancement at around the onset-time of schooling structure. The fluctuation enhancement is found to be the essential mechanism for the onset of schooling.

© 1996 Academic Press Limited

1. Introduction

The grouping of fish, called shoaling or schooling, is one of the most familiar forms of animal social behavior. Groups of fish which remain together for social reasons are termed "shoals." Defined as a social group of fish the term "shoal" has no implications for structure or function. Synchronized and polarized swimming groups are called "schools." The constituent individuals in a school swim with similar orientation. On the other hand, fish shoaling, in which neither the synchronization nor the polarization is observed or identified, is termed "swarming" (Clutter, 1969). The individuals in a swarm have a lesser degree of polarization. Schooling is therefore one of the behaviors exhibited by fish in shoals (Pitcher & Parrish, 1993), and schools have a

structure measured in polarity and synchrony. One of most striking behaviors of a school is its synchronization. Hundreds of small fish glide in unison, more like a single organism than a collection of individuals, and the distance between individuals is roughly uniform (Hunter, 1966; van Olst & Hunter, 1970). Fish do not need leaders or external stimuli to form a polarized structure or to synchronize their swimming. Each fish, having established its position, uses its eyes and its lateral lines simultaneously to measure the speed of all the other fish in a school (Pitcher, 1979; Partridge, 1982). While strong correlations are observed between individual velocity and average velocity of the entire school, the correlation between the velocity fluctuation of a particular fish and those of other fish in a school is not strong (Aoki, 1980; Partridge, 1980).

The results suggest that many schools have no leader, i.e., their speed and heading are not closely related to those of any single member. The individual adjusts its own velocity to match a group average. Thus, in a sense, the entire school is the leader and the individuals are the followers. The phenomena of polarity and synchrony of the entire school generated only by cooperative interactions among constituents lead to the question of “self-organization” (Haken, 1983).

In this paper, the question to be addressed is how patterns or structures in a school of fish arise spontaneously in a setting of collective motion. Mechanisms underlying self-organization in a fish school will be revealed in the study of the average values of behavioral properties for schools of individual fish involved in random motion. It is assumed that individual fish can be regarded as particles having autonomous locomotion and inbuilt response with respect to each other which is the social behavior such as attraction or repulsion. The analysis starts with simplifying assumptions:

- (i) All fish in a shoal are equal and individual differences between members are neglected.
- (ii) The inbuilt response works as a two-body interaction, the internal force acting on one fish results from another fish in a shoal (called the “social force”).
- (iii) The total social force produced on one fish by a number of other fish placed around it is the vector sum of the individual two-body forces.

This is clearly a simplification of real fish behavior, however, it allows for analytical progress which in the end may be illuminating the dynamics of schooling. It should be noted that no command structure exists between individuals and a shoal, that is, between *hierarchical levels*. In this setting, we focus on the dynamics of self-organization: when can such a system organize itself into a macroscopic state with a well-defined structure?

When modeling the schooling behavior, we are interested in the centroid velocity of a shoal, that is, the *macroscopic* quantity. However, we are not particularly interested in each individual velocity in a shoal, that is, the *microscopic* states of the system, because it is clear that a large number of different microscopic states will give the same macroscopic state. The schooling behavior is, therefore, treated with the methods of *statistical mechanics*, which is the study of the average values of physical properties for an ensemble of systems composed of the original system with all possible initial conditions, i.e., “Gibbs ensemble.” It is then very important to examine the

variance of motion of each individual, i.e., the fluctuation around the average properties of the behavior, motivating the following questions: when the statistical fluctuation or the variance of swimming speed in a shoal is varied, how do shoals change their structure? Even if fluctuation of individual motion in a shoal increases, do systems always adjust their collective behavior to the individual properties on the average? Can we find some drastic modifications of their collective behavior depending on the variance of individual swimming speeds?

2. Dynamics of Fish Swimming

From a purely mechanical standpoint, a fish can swim forward by pushing its environmental water backward; the surroundings in turn react to provide thrust to the fish. Performance (i.e., speed and acceleration) depends on the balance between thrust and drag. Because the physical process of swimming can be described by the form in application of *Newton’s mechanics*, the equation of motion of a fish is given by *Newton’s second law of motion* (Okubo, 1980, 1986; Alexander, 1992):

$$\text{mass} \times \text{acceleration} = \text{force},$$

or, after dividing it by mass, the vector equation reads

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v}) + \boldsymbol{\eta}(t), \quad (1)$$

where \mathbf{v} denotes the swimming velocity, whose components are v_x, v_y, v_z . Here, the three-dimensional motion of a free system (i.e., in the absence of external forces like currents or gravitation) is considered. Since the swimming behavior fluctuates, the force is assumed to be split up into two parts: first, a systematic part $\mathbf{f}(\mathbf{v})$ representing the “locomotory force” whose components are f_x, f_y, f_z , and second, a fluctuating part $\boldsymbol{\eta}(t)$ representing the “the fluctuating force” which causes the stochastic behavior of fish. The locomotory force will be given as a function of the swimming velocity. The locomotory force $\mathbf{f}(\mathbf{v})$ is composed of the forward thrust and the hydrodynamic drag which is approximately proportional to the square velocity for the Reynolds number \mathcal{R} much greater than one. Hence the faster a fish swims, the greater the friction force becomes. Thus, in steady swimming, the fish moves at the equilibrium speed v_m such that the propulsion just equals the resistance. The fluctuating force $\boldsymbol{\eta}(t)$ may occur, because the turbulent flow arises around a fish at $\mathcal{R} \gg 1$. Besides physical origin, the fluctuating behavior of fish must be of biological origin. Therefore $\boldsymbol{\eta}(t)$ is supposed to

be regulated via the physiological state of fish, depending upon the environmental conditions.

As for the fluctuating force $\boldsymbol{\eta}(t)$, the following principal assumptions are made for mathematical simplification:

- (i) $\boldsymbol{\eta}(t)$ is independent of \mathbf{v} .
- (ii) $\boldsymbol{\eta}(t)$ varies extremely rapidly compared to the variation of \mathbf{v} .

The second assumption implies that time intervals of duration Δt exist such that during Δt the variations in \mathbf{v} which are to be expected are very small indeed while during the same interval $\boldsymbol{\eta}(t)$ may undergo several fluctuations. Alternatively, it may be said that though $\mathbf{v}(t)$ and $\mathbf{v}(t + \Delta t)$ are expected to differ by a negligible amount, no correlation between $\boldsymbol{\eta}(t)$ and $\boldsymbol{\eta}(t + \Delta t)$ exists. This allows us to pass to the idealization of the Gaussian white noise. Thus we have a δ -correlated fluctuating force such that

$$\langle \boldsymbol{\eta}(t) \rangle = 0, \quad \langle \boldsymbol{\eta}(t) \boldsymbol{\eta}^T(t') \rangle = 2\epsilon \delta(t - t') \mathbf{I} \quad (2)$$

where the parameter ϵ denotes the strength of the fluctuating force, \mathbf{I} is 3 by 3 unit matrix, and the bracket $\langle \dots \rangle$ denotes the Gibbs' ensemble average.

The problem is to solve the stochastic differential equation (1) subject to the restriction on $\boldsymbol{\eta}(t)$. But "solving" a stochastic differential equation like eqn (1) is not the same thing as solving any ordinary differential equation. For one thing, eqn (1) involves the fluctuation $\boldsymbol{\eta}(t)$ which has only statistically defined properties. Consequently, "solving" the Langevin's equation (1) has to be understood rather in the sense of specifying a probability distribution $P(\mathbf{v}, t)$ which gives the probability of finding the fish at a given velocity \mathbf{v} at time t . The attention is fixed more on the determination of the systematic evolution, i.e., the mean path $\langle |\mathbf{v}(t)| \rangle$, and the fluctuation around it, i.e., the variance $\langle \{ |\mathbf{v}(t)| - \langle |\mathbf{v}(t)| \rangle \}^2 \rangle$ which depends upon the statistical properties of $\boldsymbol{\eta}(t)$. The temporal evolution of the probability distribution $P(\mathbf{v}, t)$ is governed by the Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(\mathbf{v}, t) = - \sum_{\mu \in \{x, y, z\}} \frac{\partial}{\partial v_\mu} (f_\mu P) + \epsilon \sum_{\mu \in \{x, y, z\}} \frac{\partial^2 P}{\partial v_\mu^2}, \quad (3)$$

which can be derived from the Langevin's equation (1) (Haken, 1983).

The explicit form of \mathbf{f} is determined as follows. The maximum of the stationary solution $P_{\text{st}}(\mathbf{v})$ is assumed to lie at $|\mathbf{v}| = v_m$. The minimum of $P_{\text{st}}(\mathbf{v})$ is assumed to lie at $|\mathbf{v}| = 0$, because the zero speed is an equilibrium state at which thrust and drag are zero

and this state is supposed to be unstable. Therefore differentiations of $P_{\text{st}}(\mathbf{v})$ with respect to $v \equiv |\mathbf{v}|$ read

$$\begin{aligned} \left. \frac{\partial P_{\text{st}}}{\partial v} \right|_{v=0} &= \left. \frac{\partial P_{\text{st}}}{\partial v} \right|_{v=v_m} = 0, \\ \left. \frac{\partial^2 P_{\text{st}}}{\partial v^2} \right|_{v=0} &> 0, \quad \left. \frac{\partial^2 P_{\text{st}}}{\partial v^2} \right|_{v=v_m} < 0. \end{aligned} \quad (4)$$

It is also assumed that $P_{\text{st}}(\mathbf{v})$ is rotationally symmetric around $\mathbf{v} = 0$, i.e., $P_{\text{st}}(\mathbf{v}) = P_{\text{st}}(\mathbf{v}')$ for $|\mathbf{v}| = |\mathbf{v}'|$. The "Landau theory" of the second order phase transition is very suggestive for finding an approximate stationary solution of the Fokker-Planck equation (3) (Landau & Lifshitz, 1980; Haken, 1983). To study the behavior of $P_{\text{st}}(\mathbf{v})$, the following procedure is performed:

- (i) $P_{\text{st}}(\mathbf{v})$ is written in the form

$$P_{\text{st}}(\mathbf{v}) = \mathcal{N} \exp\{-U(\mathbf{v})/\epsilon\}, \quad (5)$$

where \mathcal{N} is a normalization factor.

- (ii) $U(\mathbf{v})$ is expanded into a Taylor series around $\mathbf{v} = 0$ up to the fourth order, i.e.,

$$\begin{aligned} U(\mathbf{v}) &= U(0) + \sum_{\mu} U_{\mu} v_{\mu} + \frac{1}{2!} \sum_{\mu\nu} U_{\mu\nu} v_{\mu} v_{\nu} \\ &+ \frac{1}{3!} \sum_{\mu\nu\lambda} U_{\mu\nu\lambda} v_{\mu} v_{\nu} v_{\lambda} + \frac{1}{4!} \sum_{\mu\nu\lambda\kappa} U_{\mu\nu\lambda\kappa} v_{\mu} v_{\nu} v_{\lambda} v_{\kappa}, \end{aligned} \quad (6)$$

where the subscripts of U indicate differentiation of U with respect to v_{μ} , v_{ν} , ... at $v = 0$, and $\mu, \nu, \dots \in \{x, y, z\}$.

- (iii) It is required that $U(\mathbf{v})$ is invariant under all transformations of \mathbf{v} which leave the physical problem invariant.

By requirement (iii), relations among the coefficients U_{μ} , $U_{\mu\nu}$, ... can be established so that the number of these expansion parameters can be considerably reduced. Since the problem is completely symmetric with respect to the inversion $\mathbf{v} \rightarrow -\mathbf{v}$, inserting the postulate $P_{\text{st}}(\mathbf{v}) = P_{\text{st}}(-\mathbf{v})$ and thus $U(\mathbf{v}) = U(-\mathbf{v})$ into eqn (6) yields $U_{\mu} = 0$ and $U_{\mu\nu\lambda} = 0$, so that only even powers of \mathbf{v} are retained. Because of the rotational symmetry around $v = 0$, there results

$$U(\mathbf{v}) = U(0) - \frac{\kappa}{2} \sum_{\mu} v_{\mu}^2 + \frac{\kappa\beta}{4} \sum_{\mu\nu} v_{\mu}^2 v_{\nu}^2, \quad (7)$$

where parameters κ and β are positive constants. This approximate stationary solution of the Fokker-

Planck equation implies that the drift coefficient $\mathbf{f}(\mathbf{v})$ fulfils the so-called potential condition:

$$f_\mu(\mathbf{v}) = -\frac{\partial U(\mathbf{v})}{\partial v_\mu}. \quad (8)$$

In this way the explicit form of the locomotory force was obtained.

The stochastic dynamical equation governing the locomotory movement of a fish which corresponds to the Fokker–Planck equation (3) is therefore obtained in the form

$$\frac{d\mathbf{v}}{dt} = \kappa(1 - \beta\mathbf{v}^2)\mathbf{v} + \boldsymbol{\eta}(t), \quad (9)$$

where the relaxation constant κ represents the rapidity attaining equilibrium, and the parameter β^{-1} is the square of the most probable speed (i.e., equilibrium speed) in steady swimming, $\beta^{-1} \equiv v_m^2$. Equation (9) can also be derived by using the theory of optimal swimming (Niwa, 1994). Equation (9) is solved in Appendix A, where the stationary solutions are found to be:

$$\langle \mathbf{v}^2 \rangle = \frac{1}{\beta} + 3 \frac{\epsilon}{\kappa}, \quad (10)$$

for the mean square velocity, and

$$\sigma_0^2 \equiv \langle \mathbf{v}^2 \rangle - \langle |\mathbf{v}| \rangle^2 = 3 \frac{\epsilon}{\kappa}, \quad (11)$$

for the variance.

The velocity distribution in steady swimming is then given by

$$P_{st}(\mathbf{v}) = \mathcal{N} \exp \left[-\frac{\kappa}{\epsilon} \left\{ -\frac{1}{2} \mathbf{v}^2 + \frac{\beta}{4} (\mathbf{v}^2)^2 \right\} \right]. \quad (12)$$

Aoki (1980) gave the speed frequency distribution of *Gnathopogon elongatus elongatus* in a shoal of eight fish by experiments. Figure 1 shows the velocity distribution for individuals in the *Gnathopogon* shoal fitted to eqn (12). The resulting distribution (12) has an encouraging consistency with fish movement data. Note that this distribution of swimming fish is dissimilar to “Maxwellian velocity distribution” of gas kinetics (Landau & Lifshitz, 1980). It is interesting that the speed frequency distribution of swarming midges *Anarete pritchardi* fits well to Maxwellian distribution (Okubo & Chiang, 1974).

3. Dynamic Model for Fish Schooling

Consider a shoal composed of N fish. The schools usually consist of individuals of the same size range (Breder, 1976; Aoki & Inagaki, 1988), and their

swimming abilities are nearly equal to one another (Wu, 1977). The school is supposed to be generated and maintained by social interaction between constituents. Specific emphasis is on the interpretation of the schooling processes in Newtonian term “force,” which is exerted on the individuals and governs their motion. The relation of *cause* and *effect* in neurophysiological or ethological processes of social behavior is assumed to be reduced to the laws of Newton. The social force experienced by one fish (i) is assumed to be produced by another fish (j) in a shoal, which is expressed as \mathbf{f}_{ij} . This force, however, is not produced solely by the action of fish j on fish i , but the *interaction* between them in a shoal of three or more fish (Partridge, 1982). To organize into the school structure with a high degree of polarization, fish must possess parallel movement to neighbors as well as cohesive movement (Shaw, 1969; Sakai, 1973; Aoki, 1982). Hence, the social interaction is assumed to be divided into two components: $\mathbf{f}_{ij}^{(g)}$ for the “grouping force” (the fish j attracts the fish i), and $\mathbf{f}_{ij}^{(p)}$ for the “arrayal force” (the fish i adjusts its speed and heading to match those of the fish j). Therefore, for a free system (i.e., no external forces), the equation of motion for the fish i in the shoal is expressed as

$$\frac{d\mathbf{v}_i}{dt} = \kappa(1 - \beta\mathbf{v}_i^2)\mathbf{v}_i + \sum_{j=1}^N \mathbf{f}_{ij}^{(g)} + \sum_{j=1}^N \mathbf{f}_{ij}^{(p)} + \boldsymbol{\eta}_i(t), \quad (13)$$

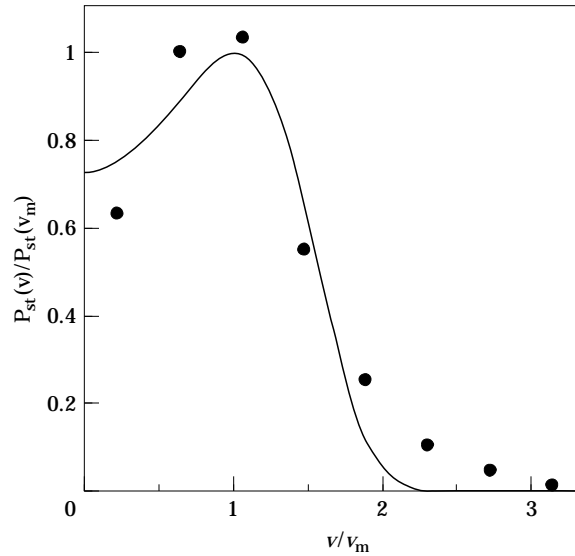


FIG. 1. Velocity distribution for individuals in the *Gnathopogon* shoal composed of eight fish swimming in two-dimensions. The probability distribution of finding fish at a given swimming velocity \mathbf{v} in a given direction is shown. Data from Aoki (1980; fig. 22) are fitted to eqn (12), where κ/ϵ is estimated at $5.6 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^2$, and β at $4.4 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^2$. Swimming velocity $v \equiv |\mathbf{v}|$ and velocity distribution $P_{st}(v)$ are normalized with the most probable steady speed, $v_m = 4.8 \text{ cm sec}^{-1}$, and the maximum value $P_{st}(v_m)$, respectively.

providing that the fish mixes the influences of members. For both the grouping force and the arrayal force, the sum of the social interaction over all pairs of fish in a shoal vanishes, i.e.,

$$\sum_{i,j=1}^N \mathbf{f}_{ij}^{(g)} = 0 \quad \text{and} \quad \sum_{i,j=1}^N \mathbf{f}_{ij}^{(p)} = 0, \quad (14)$$

because the social interaction is the internal force and is not included in the total driving force of a shoal, $\sum_{i=1}^N d\mathbf{v}_i/dt$. The social force does not accelerate the motion of whole system.

As for the grouping and arrayal forces, the following assumptions are made:

- (i) The grouping force is independent of individual swimming velocities.
- (ii) The arrayal force acting on the fish i is a (linear) velocity-matching force expressed as

$$\sum_{j=1}^N \mathbf{f}_{ij}^{(p)} = J(\mathbf{V} - \mathbf{v}_i), \quad (15)$$

where \mathbf{V} is the centroid velocity of the shoal, i.e., the group-average velocity

$$\mathbf{V} \equiv \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i, \quad (16)$$

and J is the coefficient of arrayal interaction.

Two neighboring fish tend to swim parallel with each other, equalizing their velocities. Each fish effectively interacts with a mean field produced by all the other fish, that is, with the average velocity of the entire shoal (Aoki, 1980; Partridge, 1980).

The fluctuating force $\boldsymbol{\eta}_i(t)$ will have a certain influence on the collective motion of fish shoaling. Thus, the system is considered to be coupled to an environment as a noise source. It is assumed that the Gaussian fluctuating forces acting on each fish are independent of each other:

$$\langle \boldsymbol{\eta}_i(t) \boldsymbol{\eta}_j^T(t') \rangle = 2\epsilon \delta_{ij} \delta(t - t') \mathbf{I}. \quad (17)$$

Now that the movement of individual fish with small relaxation constant κ is easily affected by surrounding fish or environment, if $\kappa \ll J$, individual swimming behavior in a shoal is strongly influenced by surrounding companions. In the following sections the schooling behavior of a system which consists of a great number of fish (i.e., $N \gg 1$) is investigated under the condition that $\kappa \ll J$, which may hold for highly organized species.

4. Analysis of Collective Properties

In order to investigate global properties of a system, let us consider the behavior of the *macro-variable*, i.e., the centroid velocity of the shoal, $\mathbf{V}(t)$, which describes the collective properties of the movement of the shoal. Summing eqn (13) over all i 's in the shoal yields

$$\frac{d\mathbf{V}}{dt} = \kappa \left(1 - \frac{5}{3} \beta \sigma^2(t) \right) \mathbf{V} - \kappa \beta \mathbf{V}^2 \mathbf{V} + \bar{\boldsymbol{\eta}}(t), \quad (18)$$

where the variance $\sigma^2(t) \equiv \langle \{ \mathbf{v}_i(t) - \mathbf{V}(t) \}^2 \rangle$. The detail of derivation is shown in Appendix B. Here the parameter β is regarded as the “nonlinearity” of the system. The fluctuating force $\bar{\boldsymbol{\eta}}(t) \equiv \sum_{i=1}^N \boldsymbol{\eta}_i(t)/N$ is Gaussian white, i.e.,

$$\langle \bar{\boldsymbol{\eta}}(t) \bar{\boldsymbol{\eta}}^T(t') \rangle = \frac{2\epsilon}{N} \delta(t - t') \mathbf{I}. \quad (19)$$

Now that we are concerned in this case for small ϵ , the calculation up to the first order of ϵ is sufficient for the microvariables which exhibit the movement of the constituents, and up to the first order of ϵ/N for the macrovariables which exhibit the collective property. The higher order terms of ϵ or ϵ/N are negligible. It should be noted that $\sigma^2(t)$ arises from the interplay between the individual fluctuating behavior and the nonlinearity of the system.

As is shown in Appendix C, the equation describing the temporal evolution of the variance is given by

$$\frac{d\sigma^2}{dt} = -2 \left(J - \kappa + \frac{5}{3} \kappa \beta \mathbf{V}^2 \right) \sigma^2 + 6\epsilon. \quad (20)$$

We now make a very important assumption that *fast* and *slow* relaxing variables are identified, which is quite typical for many cooperative systems (Haken, 1983). In general, the microvariables vary and achieve a stationary state much more rapidly than the macrovariables. It is assumed that the individual relaxation time around which the locomotory movement of the individual comes to an equilibrium, t_i ($\sim \kappa^{-1}$), is much smaller than the relaxation time inherent in the collective motion of the school, t_0 (which will be given by the onset-time of schooling in the next section), i.e.,

$$t_i \ll t_0. \quad (21)$$

This “adiabatic hypothesis” that the macrovariable

\mathbf{V} changes much more slowly than the microvariable σ^2 allows us to put

$$\frac{d\sigma^2}{dt} = 0, \quad (22)$$

so that the differential equation (20) reduces to a simple algebraic equation, which is solved by

$$\sigma^2(t) = \frac{3\epsilon}{J - \kappa + 5\kappa\beta\mathbf{V}^2(t)/3}. \quad (23)$$

Equation (23) tells us that the microvariable $\sigma^2(t)$ obeys instantaneously the macrovariable $\mathbf{V}^2(t)$.

Consequently we have the *asymptotic form* of the equation governing the collective motion of schooling by the adiabatic elimination of fast relaxing variable $\sigma^2(t)$ from eqn (18), i.e.,

$$\frac{d\mathbf{V}}{dt} = \kappa \left(1 - 5 \frac{\beta\epsilon}{J} \right) \mathbf{V} - \kappa\beta\mathbf{V}^2\mathbf{V} + \bar{\boldsymbol{\eta}}(t), \quad (24)$$

calculating up to the first order of κ/J . If the stationary solution σ^2 of eqn (20) is larger than $3/5\beta$, the adiabatic hypothesis [eqn (21)] fails. Even in this case, however, eqn (24) clearly holds if the system is near the stationary state. The typical features of eqn (24) are: a linear term, $\kappa(1 - 5\beta\epsilon/J)\mathbf{V}$, the coefficient of which is a growing rate of the system and changes its sign at a certain *threshold* of the size of the fluctuation, $\epsilon_c \equiv J/5\beta$, and a nonlinear term, $\kappa\beta\mathbf{V}^2\mathbf{V}$, which serves for a stabilization of the system.

The stationary distribution of the centroid velocity is given by

$$P_{st}(\mathbf{V}) = \mathcal{N} \exp \left[\left\{ \frac{\kappa}{2} \left(1 - 5 \frac{\beta\epsilon}{J} \right) \mathbf{V}^2 - \frac{\kappa\beta}{4} (\mathbf{V}^2)^2 \right\} / \left(\frac{\epsilon}{N} \right) \right], \quad (25)$$

where \mathcal{N} is a normalization factor (Haken, 1983; Niwa, 1994). The stable states of the movement correspond to the extreme value of $P_{st}(\mathbf{V})$. It is clear that we have two completely different situations across the *critical point* ϵ_c . In the case

(1) $\epsilon > \epsilon_c$, i.e., $1 - 5\beta\epsilon/J < 0$,
 $P_{st}(\mathbf{V})$ takes a maximum at

$$|\mathbf{V}_0| \equiv 0, \quad \text{stable}. \quad (26)$$

The linear term in eqn (24) is then regarded as the restoring force by which the system remains at the stable state \mathbf{V}_0 . At elevated fluctuation ϵ , the individuals in a shoal swim in random directions, and shoaling corresponds to swarming. Hence the center

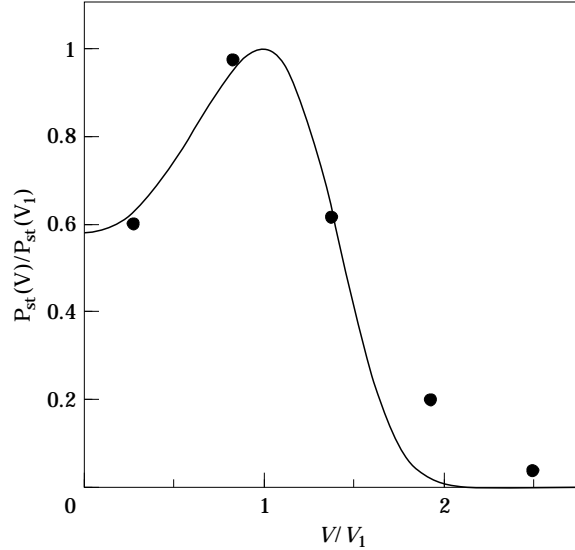


FIG. 2. Stationary distribution of the centroid velocity of the *Gnathopogon* school ($N = 8$) in two-dimensions. The probability distribution of finding the school at a given centroid velocity \mathbf{V} in a given direction is shown. Data from Inagaki *et al.* (1976) are fitted to eqn (25), where $\kappa N(1 - 5\beta\epsilon/J)/\epsilon$ is estimated at $0.66 \text{ cm}^{-2} \text{ sec}^2$, and $\kappa\beta N/\epsilon$ at $0.20 \text{ cm}^{-2} \text{ sec}^2$. Centroid velocity $V \equiv |\mathbf{V}|$ and velocity distribution $P_{st}(V)$ are normalized with the most probable steady speed, $V_1 \equiv |\mathbf{V}_1| = 1.8 \text{ cm sec}^{-1}$, and the maximum value $P_{st}(V_1)$, respectively.

of the mass of a shoal hardly moves and the non-polarized shoal performs an amoebic movement owing to the fluctuating force. The quantity $\langle \mathbf{V}^2 \rangle$ then expresses the fluctuation of the system. Whereas in the case

(2) $\epsilon < \epsilon_c$, i.e., $1 - 5\beta\epsilon/J > 0$,

two types of extrema are found: $P_{st}(\mathbf{V})$ takes a minimum at \mathbf{V}_0 which is evidently unstable, and takes maximum at

$$|\mathbf{V}_1| \equiv \sqrt{\frac{1}{\beta} \left(1 - 5 \frac{\beta\epsilon}{J} \right)}, \quad \text{stable}. \quad (27)$$

Then the variance in steady swimming is given by

$$\sigma^2 = 3 \frac{\epsilon}{J}. \quad (28)$$

It is then seen that fish as a whole tend to perform a rectilinear movement, forming a polarized school structure, and that the arrayal interaction suppresses the fluctuating movement of swimming fish in a school by comparing eqn (28) with eqn (11). The experimental data from Inagaki *et al.* (1976) are satisfactorily fitted to eqn (25), as shown in Fig. 2.

The transitions between the different aggregate states, called “phases,” are quite abrupt. The change from one phase to the other is called “phase transition.” Since a shoal consists of many fish and

they are able to swim autonomously, the shoal has many degrees of freedom. In a swarming phase at a higher variance to the individual velocity above the critical value, i.e., $\sigma^2 > \sigma_c^2 \equiv 3\epsilon_c/J$, fish swim quite independently. In schooling phase at lower variance below the critical value, i.e., $\sigma^2 < \sigma_c^2$, fish eventually concentrate locomotion from many degrees of freedom into a single degree of freedom (i.e., motion in one direction). Their motion is then highly correlated. This phase transition can be seen in the excellent photographs of a minnow shoal (*Phoxinus phoxinus*) by Partridge (1980): when the swarming minnow shoal was threatened by a predator, the shoal instantaneously went from a non-polarized state to a highly polarized state.

When the growing rate of the system, $\kappa(1 - 5\beta\epsilon/J)$, passes from negative values to positive values, the stable points are exchanged: it is now observed the transition of the zero solution \mathbf{V}_0 to the non-zero solution \mathbf{V}_1 . Though the problem described by eqn (24) is completely symmetric with respect to the inversion $\mathbf{V} \rightarrow -\mathbf{V}$, the symmetry is now broken by the actually realized solution. The whole phenomenon may be thus described as a “symmetry breaking instability” (Haken, 1983). Without fluctuating forces, the system could sit anywhere in a (marginal) stable position. Fluctuating forces drive the system round the stable position, completely analogous to Brownian movement. In the stationary state, the system may be found with equal probability in every stable position, that is, the symmetry is restored.

We now inquire into features of shoaling behavior near the critical point ϵ_c . In the limit $\kappa(1 - 5\beta\epsilon/J) \rightarrow -0$, the stable position \mathbf{V}_0 becomes unstable, i.e., the restoring force becomes smaller and smaller in the neighborhood of \mathbf{V}_0 . It is then observed that the system with non-zero velocity \mathbf{V} falls down more and more slowly into its stable state (i.e., zero velocity). Anomalous fluctuations then arise in the system, i.e., the fluctuation of the system, $\langle \mathbf{V}^2 \rangle$, becomes considerable, because the restoring force acts only via higher power of \mathbf{V} . Crossing over the critical point, the system accomplishes the transformation of shoaling for many degrees of freedom (individuals darting this way and that, i.e., swarming phase) into few degrees of freedom (individuals darting in parallel to each other, i.e., schooling phase). These phenomena in the neighborhood of $\epsilon = \epsilon_c$ may be observed in the ocean at dawn. Herring shoals are polarized in the daytime but form non-polarized, loose aggregation at night (Buerkle, 1983). Anchovy, a typical schooling species, forms swarms at night (Aoki & Inagaki, 1988).

5. Formation Processes of Schooling Structure

Let us now consider the transient behavior of fish schooling. The governing eqn (18) is investigated in the case for $\epsilon < \epsilon_c$, where the system has the *globally* stable states at \mathbf{V}_1 . Equation (18) for the movement of a shoal is intrinsically homogeneous, i.e., \mathbf{V}_0 must be a solution. Since the microstate is supposed to be equilibrium, the mean square velocity of individuals in the shoal is given by the stationary solution (10). Letting the system be initially inactive, i.e., $\mathbf{V}(t) = \mathbf{V}_0$ at $t = 0$, we obtain

$$\langle \mathbf{V}^2(0) \rangle = \frac{9}{2} \frac{\epsilon/N}{\kappa(1 + 15\beta\epsilon/2\kappa)} \quad (29)$$

(see Appendix D). Since the variance of individual swimming velocity in the school is written as $\sigma^2(t) \approx \langle \mathbf{v}^2 \rangle - \langle \mathbf{V}^2(t) \rangle$, the growing rate of the system is initially negative: $\kappa(1 - 5\beta\sigma^2(0)/3) = -2\kappa/3 + \mathcal{O}(\epsilon) < 0$. Therefore the state \mathbf{V}_0 is locally stable, i.e., when the system is slightly displaced the system returns to this stable point owing to the restoring force. It is natural that the initially inactive system is locally stable because fish have a tendency to continue to move with present velocity owing to *Newton's first law*, i.e., the law of *inertia*. If there is no arrayal interaction between members of a shoal, the system in which individuals dart completely at random will not undergo a change in its state of motion. Then the shoal which is initially inactive remains at rest forever and no self-organization takes place because of the negative growing rate.

In the initial time-region of the schooling process the center of the shoal performs Brownian movement owing to the fluctuating forces. The fluctuation of the system,

$$\sigma_v^2(t) \equiv \langle \mathbf{V}^2(t) \rangle - \langle |\mathbf{V}(t)| \rangle^2, \quad (30)$$

can then be expressed as $\langle \mathbf{V}^2(t) \rangle$. The initial fluctuation of the system, $\sigma_v^2(0) = \langle \mathbf{V}^2(0) \rangle$, is of order ϵ/N . Individuals in a shoal adjust their headings to the centroid velocity $\mathbf{V}(t)$ fluctuating around \mathbf{V}_0 because of the arrayal interaction. Then the mutually correlated motion of individuals appears out of a completely random state of the system. The arrayal force derives the originally inactive system at \mathbf{V}_0 to enhance the initial fluctuation of the system $\langle \mathbf{V}^2(t) \rangle$. When lowering the variance $\sigma^2(t)$ with enhancing the initial fluctuation of the system, the system may remain at \mathbf{V}_0 . Evidently, the system stays at \mathbf{V}_0 before the fluctuation of the system grows in amplitude, i.e., $\langle \mathbf{V}^2(t) \rangle > 2/5\beta + 3\epsilon/\kappa$. When lowering $\sigma^2(t)$ further, the system eventually reaches a marginal situation with the state becoming unstable, called the

“embryonic state.” The system then reaches the region where the growing rate is positive, i.e., the variance $\sigma^2(t)$ is below the marginal value $\sigma_c^2 \equiv 3/5\beta$. Note that, in this initial time-region, if the two-time correlation of the centroid velocity weakens at a faster rate than the mutual correlation between members emerges in the originally inactive system, the system will not reach the embryonic state. The initial fluctuation is dumped and the system loses its memory (i.e., its statistical dependence on past movement) after the duration T , i.e., $\langle \mathbf{V}(T) \cdot \mathbf{V}(0) \rangle \approx 0$. However, it takes the inactive system the time t_e to come into the embryonic state, and it is anticipated that the time t_e is of order of (the relaxation constant of parallel movement) $^{-1}$, i.e., $t_e \sim J^{-1}$. Therefore, the initially inactive system becomes asymptotically unstable providing that

$$T > t_e. \quad (31)$$

By calculating the Gibbs’ ensemble average of the product of the centroid velocities $\mathbf{V}(0)$ and $\mathbf{V}(t)$ on the supposition of a stationary process, the two-time correlation coefficient is obtained:

$$\frac{\langle \mathbf{V}(t) \cdot \mathbf{V}(0) \rangle}{\langle \mathbf{V}^2(0) \rangle} = \exp \left\{ -\frac{2}{3} \kappa \left(1 + \frac{15}{2} \frac{\beta \epsilon}{\kappa} \right) t \right\} \quad (32)$$

(see Appendix E). The initial fluctuation exponentially decreases and the correlation time is given by

$$T \equiv \frac{3}{2\kappa} \left(1 + \frac{15}{2} \frac{\beta \epsilon}{\kappa} \right)^{-1}, \quad (33)$$

which is of order κ^{-1} . Here we are concerned with the case that $J \gg \kappa$. Therefore, the occurrence condition of embryonic state [eqn (31)] holds. Moreover, the adiabatic hypothesis [eqn (21)] is also confirmed because the relaxation constant (i.e., growing rate) of the system becomes smaller and smaller near the embryonic state. Analysing the experimental data (Aoki, 1980) shows this characteristic feature of initial fluctuation as depicted in Fig. 3.

Now we inquire into the dynamics of the system in the vicinity of the marginal point where $\kappa(1 - 5\beta\sigma^2(t)/3) \approx 0$, and analyse the initial processes of schooling. Around the time t_e we can put

$$\frac{d\mathbf{V}}{dt} \approx 0. \quad (34)$$

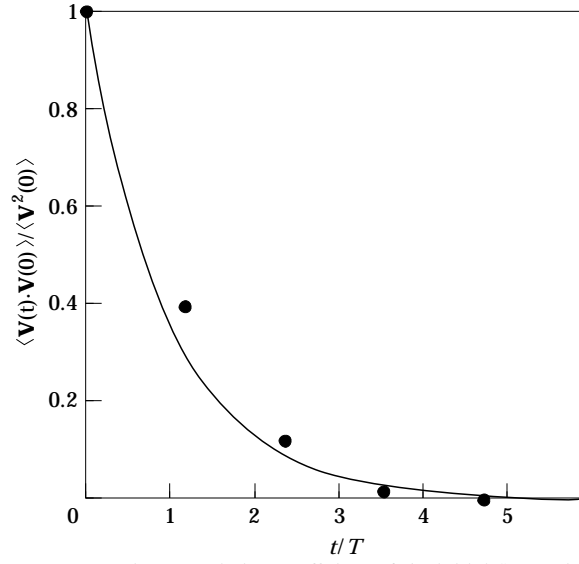


FIG. 3. Two-time correlation coefficient of the initial fluctuation of the *Gnathopogon* shoal ($n = 8$) in two-dimensions. Data from Aoki (1980; table. 10) are satisfactorily fitted to the theoretical curve based on eqn (32) with the correlation time, $T = 1.7$ sec. Time t is normalized with T .

Considering the timescale inherent in the microvariable $\sigma^2(t)$, we obtain the self-contained equation:

$$\frac{d\sigma^2}{dt} = -2J \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \{ \sigma^2(0) - \sigma^2(t) \} \right) \times \sigma^2(t) + 6\epsilon \quad (35)$$

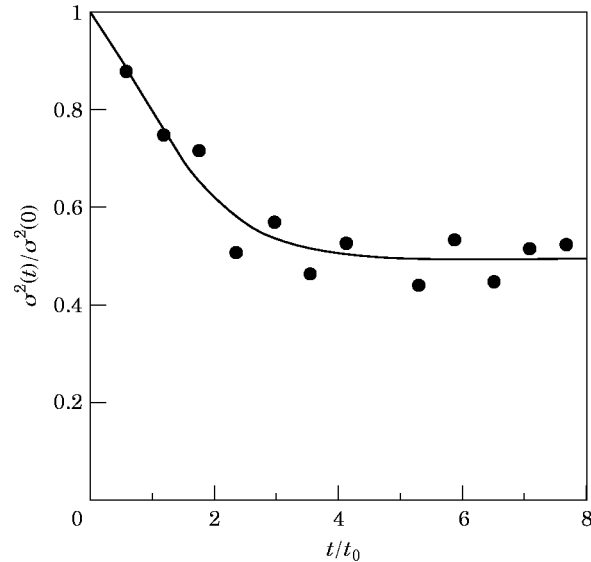


FIG. 4. Temporal evolution of variance of individual velocity in the *Gnathopogon* shoal ($N = 8$) in two-dimensions. The curve is based on eqn (36), and circles are data from Aoki (1980; fig. 44). Time t and variance $\sigma^2(t)$ are normalized with the scaling time (42), $t_0 = 0.85$ sec, and the initial variance, $\sigma^2(0) = 26.0 \text{ cm}^2 \text{ sec}^{-2}$, respectively. The stationary value of the variance is $\sigma^2 = 12.9 \text{ cm}^2 \text{ sec}^{-2}$.

(see Appendix F). The marginal solution is found by including terms up to first order in ϵ and κ/J :

$$\sigma^2(t) = \frac{3 \frac{\epsilon}{J} + \left[\sigma^2(0) \left(1 + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right) - 3 \frac{\epsilon}{J} \right] \exp \left[-2Jt \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right) \right]}{1 + \frac{\kappa}{J} \beta \sigma^2(0) \exp \left[-2Jt \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right) \right]}. \quad (36)$$

By substituting $\sigma^2(t_e) = \sigma_e^2$ into eqn (36), the time t_e is estimated at

$$t_e = -\frac{1}{2J} \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right)^{-1} \times \ln \left[\frac{3}{5} \frac{1 - 5\beta\epsilon/J}{\beta\sigma^2(0)} \left(1 + 2 \frac{\kappa}{J} + \frac{5\epsilon}{J\sigma^2(0)} \right)^{-1} \right], \quad (37)$$

which confirms the occurrence condition of embryonic state [eqn (31)]. The time t_e is independent of the shoal size N because the uniform system is now investigated in which individuals interact with a mean field. The time t_e is regulated chiefly by the arrayal interaction. The situation is thus reached where the system jumps to a new stable state, $\mathbf{V} = \mathbf{V}_1$, i.e., the onset of the polarized schooling structure. The growth of the initial fluctuation of the system is

definitely needed to generate the embryonic state. Aoki (1980) observed this relaxation phenomenon of the variance of individual velocity in experiments. The data can be fitted quite well to the predicted functional form (36) (Fig. 4).

Now we investigate a global feature of the transient behavior of the system on the basis of the asymptotical form (24) for the governing equation. Suzuki (1978, 1981) proposed several methods to treat these transient nonlinear phenomena. Niwa (1994) analysed the nonlinear Langevin's equation (24) with the simplest dynamical molecular field treatment which replace the nonlinear term $\mathbf{V}^2(t)\mathbf{V}(t)$ by $\langle \mathbf{V}^2(t) \rangle \mathbf{V}(t)$, where the average $\langle \mathbf{V}^2(t) \rangle$ is determined self-consistently using the solution of the linearized Langevin's equation. As is shown in Appendix G, the following equation of self-consistency is obtained:

$$\frac{d}{dt} \langle \mathbf{V}^2(t) \rangle = 2\kappa \left(1 - 5 \frac{\beta\epsilon}{J} \right) \langle \mathbf{V}^2(t) \rangle - 2\kappa\beta \langle \mathbf{V}^2(t) \rangle^2 + \frac{6\epsilon}{N}. \quad (38)$$

The solution to this equation takes the asymptotic form

$$\langle \mathbf{V}^2(t) \rangle = \langle \mathbf{V}^2 \rangle_{\text{st}} \frac{\tau}{1 + \tau}, \quad (39)$$

where

$$\tau \equiv \frac{1}{\mathbf{V}_1^2} \left\{ \langle \mathbf{V}^2(0) \rangle + \frac{3\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right\} \times \exp \left[2\kappa \left(1 - 5 \frac{\beta\epsilon}{J} \right) t \right], \quad (40)$$

and the stationary solution

$$\langle \mathbf{V}^2 \rangle_{\text{st}} = \mathbf{V}_1^2 + \frac{3\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)}. \quad (41)$$

A very interesting feature is seen in eqn (39), i.e., the average $\langle \mathbf{V}^2(t) \rangle$ is expressed only by the so-called “scaling” variable τ with which the time is scaled. Figure 5 shows the comparison between theory and observation for transient behavior of self-organization of the polarized school structure.

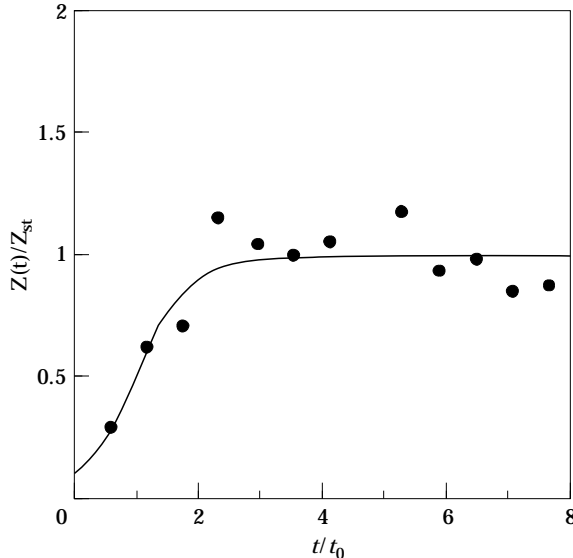


FIG. 5. Transient behavior of formation of polarized school structure; the mean square velocity of the center of *Gnathopogon* shoal ($N = 8$) in two-dimensions, $Z(t) \equiv \langle \mathbf{V}^2(t) \rangle$. Data from Aoki (1980; fig. 44) are fitted to eqn (39). Time t and mean square velocity $Z(t)$ are normalized with the onset-time of the schooling structure, $t_0 = 0.85$ sec, and the stationary value of the mean square velocity, $Z_{\text{st}} = 22.0 \text{ cm}^2 \text{ sec}^{-2}$, respectively. The initial fluctuation is estimated at $Z(0) = 2.1 \text{ cm}^2 \text{ sec}^{-2}$. The experimental points exhibit the square of mean centroid-velocity $\langle |\mathbf{V}(t)| \rangle^2$, instead of the mean square centroid-velocity $\langle \mathbf{V}^2(t) \rangle$.

The characteristic time for the onset of schooling structure is given by $\tau = 1$, i.e., the “onset-time” is given by

$$t_0 \equiv -\frac{1}{2\kappa(1 - 5\beta\epsilon/J)} \times \ln \left[\frac{1}{\mathbf{V}_1^2} \left\{ \langle \mathbf{V}^2(0) \rangle + \frac{3\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right\} \right]. \quad (42)$$

The onset-time t_0 has a qualitative meaning around which the nature of fluctuation changes from microscopic order of magnitude to macroscopic one.

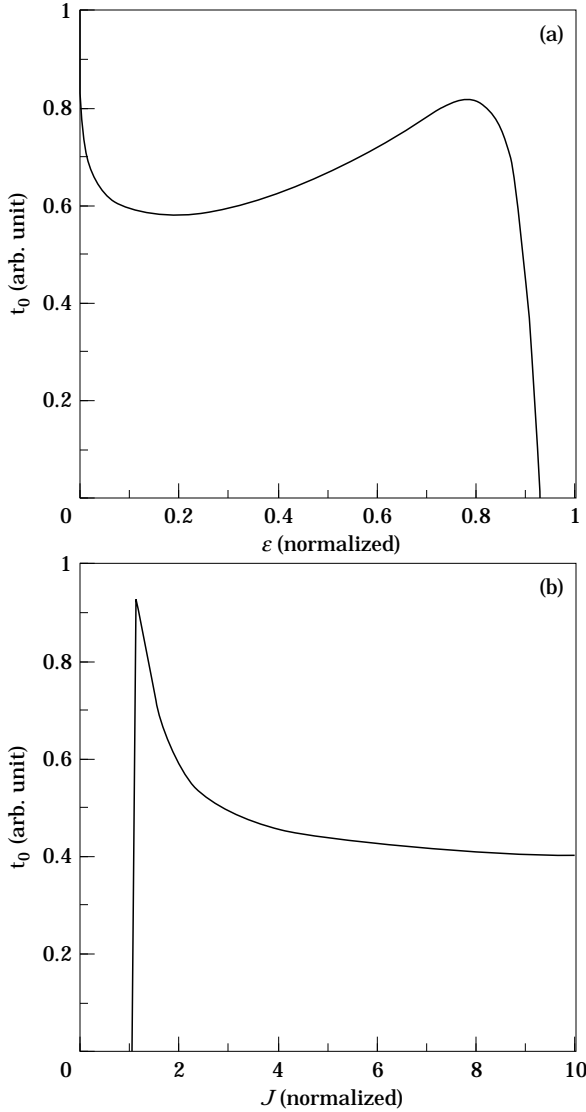


FIG. 6. (a) ϵ -dependence of t_0 (schematic). The size of the fluctuation ϵ is normalized with the critical value ϵ_c . (b) J -dependence of t_0 (schematic). The coefficient of arrayal interaction J is normalized with the fixed value $5\beta\epsilon$. β^{-1} -dependence of t_0 , i.e., t_0 vs. β^{-1} , can also be seen in (b) by replacing J -axis by β^{-1} -axis which is normalized with the fixed value $5\epsilon/J$. The onset-time t_0 is expressed in an arbitrary unit.

It gives a characteristic time for the formation process of the polarized school structure. Figure 6(a) shows the dependence of the onset-time upon the strength of the fluctuating force. Here it can be seen that in the neighborhood of $\epsilon = 0$ the onset-time t_0 is of order

$$t_0 \sim \ln(\epsilon_c/\epsilon). \quad (43)$$

However, when raising ϵ further, t_0 becomes of order

$$t_0 \sim (1 - \epsilon/\epsilon_c)^{-1}. \quad (44)$$

Accordingly a remarkable feature of the schooling process is discovered: t_0 takes a minimum value when fish tune the randomness of the movement, ϵ , to the particular degree. Figure 6(b) exhibits the dependence of the onset-time upon the arrayal interaction as well as upon the nonlinearity of the system, because the horizontal axis reads $J/5\beta\epsilon$ with fixing ϵ . Here it can be seen that the onset-time t_0 becomes smaller as the interaction coefficient J becomes larger or as the nonlinearity β becomes smaller, i.e.,

$$t_0 \sim \frac{1}{2\kappa(1 - 5\beta\epsilon/J)}. \quad (45)$$

In Fig. 6 it is seen that t_0 tends to zero in the vicinity of the marginal value, i.e., $5\beta\epsilon/J \approx 1$: this is why the magnitude of $\langle \mathbf{V}^2 \rangle_{st}$ becomes very small (of order ϵ/N) in this parameter-region. Although the onset-time is very small, i.e., $t_0 \approx 0$, the polarized schooling structure will therefore never be observed. In addition, the dependence of t_0 upon the school size N is given by

$$t_0 \sim \ln N. \quad (46)$$

The school size suppresses the fluctuating movement of the school. Thus t_0 becomes larger and larger as the size N . Therefore the onset-time t_0 is much larger than the individual relaxation time t_i ($\equiv -\ln(3\beta\epsilon/\kappa)/2\kappa$) for fish species of an intense schooling habit, providing that $N \gg 1$. Note that this is consistent with the adiabatic hypothesis [eqn (21)].

Without fluctuations the system at \mathbf{V}_0 would never realize the state at \mathbf{V}_1 which is still more stable, because the onset-time t_0 becomes larger as the behavioral stochasticity ϵ becomes smaller. Therefore in order to self-organize the school a certain initial push or randomly repeated pushes must be provided, which is achieved by the fluctuating force $\tilde{\mathbf{q}}(t)$. The fluctuating force drives the system at the asymptotically unstable state \mathbf{V}_0 to enhance the initial fluctuation of the system, $\langle \mathbf{V}^2(t) \rangle$, and drives the system to explore still more stable states. The system therefore falls down into the new globally stable state, and self-organization of the polarized school takes place. The whole range of times in the transient

behavior from the non-polarized state to the onset of the polarized schooling is divided into three regions:

- (1) The initial time-region in which the system is in the asymptotically unstable state and $\langle \mathbf{V}^2(t) \rangle$ expresses the fluctuation of the system. Here the linear approximation is valid in eqn (24) as shown in Appendix D.
- (2) The intermediate region in which the order of magnitude of $\langle \mathbf{V}^2(t) \rangle$ becomes quite different from the initial value of order ϵ/N . The nonlinear term in eqn (24) then becomes substantial.
- (3) The final region in which the system approaches the stationary state. At the onset of the schooling structure the fluctuation of the system, $\sigma_V^2(t)$, begins to decrease and it tends to the order of ϵ/N .

The enhancement of the fluctuation of the centroid velocity is therefore regarded as the essential mechanism of the self-organization of the polarized school.

In the previous paragraphs the enhancement of the fluctuation of the system were briefly discussed. Let us now investigate in detail a global feature of the transient fluctuation of the system on the basis of the asymptotic form (24) for the governing equation. The variance of the centroid velocity, $\sigma_V^2(t)$, is found to satisfy the following evolution equation:

$$\frac{d}{dt} \sigma_V^2(t) = 2 \left[\kappa \left(1 - 5 \frac{\beta \epsilon}{J} \right) - 3\kappa \beta \mathbf{V}_m^2(t) \right] \sigma_V^2(t) + \frac{2\epsilon}{N}, \quad (47)$$

where $\mathbf{V}_m(t)$ denotes one of the most probable paths of the system in the $\mathbf{V} - t$ space, i.e., $|\mathbf{V}_m(t)| = \langle |\mathbf{V}(t)| \rangle$. If a deviation of the initial system from the asymptotically unstable point (i.e., $\mathbf{V}_0 \equiv 0$) is small, eqn (47) is solved as the following asymptotic form:

$$\sigma_V^2(t) = \left[\sigma_V^2(0) + \frac{\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right] \frac{\dot{\mathbf{V}}_m^2(t)}{\dot{\mathbf{V}}_m^2(0)}, \quad (48)$$

where $\dot{\mathbf{V}}_m(t) \equiv d\mathbf{V}_m(t)/dt$ [see Appendix H for the derivation of eqns (47) and (48)]. The stationary solution is given by

$$\sigma_V^2 = \frac{\epsilon/N}{2\kappa(1 - 5\beta\epsilon/J)}. \quad (49)$$

$\sigma_V^2(t)$ is proportional to the square of the acceleration $\dot{\mathbf{V}}_m(t)$ normalized by the initial acceleration $\dot{\mathbf{V}}_m(0)$.

Therefore, the transient fluctuation of the system shows a large enhancement in the intermediate time-region of the schooling process. It is quite natural that the fluctuation of the system is enhanced in the time-region at which the change of the centroid velocity becomes rapid, as shown in Fig. 7. The variance takes the maximum value

$$\sigma_{V,\max}^2 = \frac{4}{27} \left[\sigma_V^2(0) + \frac{\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right] \frac{\mathbf{V}_1^2}{\langle \mathbf{V}^2(0) \rangle}, \quad (50)$$

at $\tau = 1/2$ (i.e., $\mathbf{V}_m(t_m) = \mathbf{V}_1/3$) or at

$$t_m \equiv t_0 - \frac{\ln 2}{2\kappa(1 - 5\beta\epsilon/J)}. \quad (51)$$

The maximum value of the variance is proportional to $\langle \mathbf{V}^2(0) \rangle^{-1}$. Therefore the fluctuation enhancement around the time t_m increases as the initial system approaches the asymptotically unstable point \mathbf{V}_0 .

This enhancement of fluctuation is derived by linearizing eqn (24) around the asymptotically

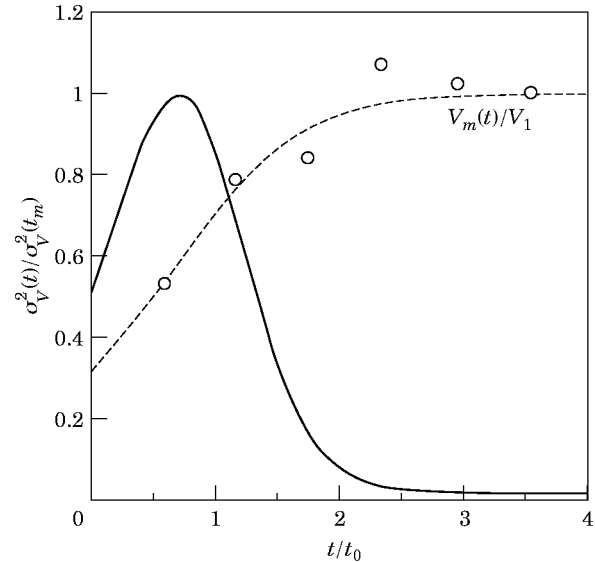


FIG. 7. Fluctuation-velocity relation in the *Gnathopogon* school ($n = 8$). Theoretical expectation of the fluctuation enhancement (solid line) is calculated from eqn (H.6) in Appendix H by using the Aoki's data (Aoki, 1980; fig. 44) of the centroid speed of the *Gnathopogon* school in two-dimensions (open circle), which are fitted to the most probable path of the centroid speed (broken line) given by eqn (H.2) in Appendix H. The change of the centroid velocity becomes most rapid at time $t_m = 0.58$ sec. The variance takes the maximum value at t_m . Time t , variance $\sigma_V^2(t)$, and centroid speed $V_m(t)$ are normalized with the scaling time, $t_0 = 0.85$ sec, the maximum value of the variance, $\sigma_V^2(t_m) = 2.1 \text{ cm}^2 \text{ sec}^{-2}$, and the most probable steady state, $V_1 = 4.7 \text{ cm sec}^{-1}$, respectively. The initial variance is $\sigma_V^2(0) = 1.1 \text{ cm}^2 \text{ sec}^{-2}$. The initial speed is $V_m(0) = 1.5 \text{ cm sec}^{-1}$.

unstable point \mathbf{V}_0 as eqn (D.1) in Appendix D and by simplifying eqn (47) to

$$\frac{d}{dt} \sigma_{v,\text{lin}}^2(t) = 2\kappa \left(1 - 5 \frac{\beta\epsilon}{J}\right) \sigma_{v,\text{lin}}^2(t) + \frac{2\epsilon}{N}, \quad (52)$$

near the asymptotically unstable point. The solution of eqn (52) is given by

$$\sigma_{v,\text{lin}}^2(t) = \left\{ \sigma_v^2(0) + \frac{\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right\} \times \exp \left[2\kappa \left(1 - 5 \frac{\beta\epsilon}{J}\right) t \right] - \frac{\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)}. \quad (53)$$

Therefore, the linear variance becomes larger and larger, as time t increases. However, the “saturation effect” cannot be explained in this linear approximation. It comes from the “nonlinear effect” neglected in the above approximation [eqn (52)].

The mechanism for the anomalous enhancement of the fluctuation of the system is thus found to be the synergism, i.e., the cooperative effect of initial fluctuation, fluctuating force, and nonlinearity of the system. More detail of application of the present theory of the fluctuation–velocity relation to the analysis of the transient enhancement of the fluctuation in the self-organization of the school will be reported elsewhere.

6. Summary and Discussion

In this paper, the mechanistic interpretation of schooling phenomena has been discussed. Newtonian equations of motion for shoaling fish have been assumed to be given by the stochastic differential equation (13). General solutions of these nonlinear stochastic equations are hopeless. There is, however, a class of interesting phenomena in which the number of degrees of freedom of eqn (13) can be tremendously reduced, i.e., when the system consists of a great number of fish ($N \gg 1$), the microscopic irregular motion of each individual gives rise to a regularity of motion of the total fish shoal. It was shown that eqn (13) can be considerably simplified to the nonlinear stochastic dynamic equation (18) for the centroid velocity of the shoal, which describes the collective properties of shoaling. Furthermore, eqn (18) asymptotically reads the nonlinear Langevin’s equation (24), which is identical with the equation for the electric field strength of the laser theory (Haken, 1983). I have theoretically studied the self-organization of school formation under the condition that $N \gg 1$, and have revealed that the

random motion of fish and the nonlinearity of the system play crucial roles in schooling. The main conclusions obtained through this study are as follows:

(i) The school exhibits the *hierarchical* structure [eqns (18) and (20)]. The shoal receives *selective* information, i.e., the fluctuation (or variance) of the individual velocity in the shoal, σ^2 , *from below* i.e., from constituent elements, and in turn, it exercises *efferent feed-forward control commands*, i.e., the square velocity of the center of the shoal, \mathbf{V}^2 , on the dynamics of the individual level (Nicolis, 1986). This hierarchical structure is constructed by the nonlinearity of the system.

(ii) The fluctuation σ^2 has the critical value σ_c^2 . The characteristic of schooling (i.e., polarity and synchrony) disappear when $\sigma^2 > \sigma_c^2$. Then, even though an inherent swimming speed of constituents in a shoal is large, fish cannot organize the polarized school without external forces. Here it can be seen that the fluctuation of the individual velocity deeply modifies the collective property of the shoal. This phenomenon is called a “noise-induced transition” (Horsthemke & Lefever, 1983). This “polarization–non-polarization transition” of the school is similar to that of laser light in which atomic dipoles oscillate in phase, though each atom is excited from the outside by pumping energy by shining light on it completely at random (Haken, 1983).

(iii) The fluctuation of the centroid velocity is essential for the onset of schooling. The dynamic features of schooling are as follows: In the initial time-region of schooling process the centroid velocity of a shoal, $\mathbf{V}(t)$, fluctuates around the locally stable point, $\mathbf{V}_0 \equiv 0$, where the fluctuation of the system is expressed as $\langle \mathbf{V}^2(t) \rangle$; in the intermediate time-region the fluctuation of the system increases and the order of magnitude of $\langle \mathbf{V}^2(t) \rangle$ becomes quite different from the initial values; in the final region $\mathbf{V}(t)$ approaches the globally stable state and the onset of schooling structure is observed. Fluctuations of the system put out a feeler to realize a new state. The theoretically predicted transient process from initial non-polarized aggregation to polarized school agrees with what was observed in computer simulations (Sakai, 1973) and in experiments (Aoki, 1980) as follows: First, individuals do not move as a whole but gather; some individuals are accidentally removed from shoaling companions; then the other individuals are quickly attracted toward them and the fish as a whole gradually perform a rectilinear movement; a polarized structure is thus self-organized. Hara (1985) observed these transient behavior of schooling in the ocean: Japanese sardine shoals (*Sardinops melanosticta*) which were performing amoebic movements at the

beginning of observations were gradually drawn up in order and thus the shoals performed steadily rectilinear movement.

The theoretically predicted results have been compared with the experimental data on the *Gnathopogon* shoal of eight fish (Inagaki *et al.*, 1976; Aoki, 1980). The theoretical analysis is based on a large shoal. While the shoal size of eight may not be large enough for this to be true, the experimental data have confirmed the theoretical prediction of the analysis developed here. Eight fish may constitute a shoal large enough to exhibit the characteristic self-organization of school (Partridge, 1982).

In this investigation on schooling, it was supposed that each individual efficiently interacts with an average velocity of the entire shoal to adjust their swimming velocities to match the average velocity. For large shoals, however, the average which is most strongly correlated may not be the simple arithmetic mean of the speeds and headings of the members of the shoal. A fish is naturally more strongly influenced by its close neighbors than the distant members of the shoal. The contribution of each fish to the average may be inversely proportional either to the square of the inter-fish distance owing to vision, or to the cube owing to the lateral line's sensitivity to water displacement (Partridge, 1980, 1982). Fish utilize the water current sense for behavioral decision making (Pitcher *et al.*, 1976). Fish detect the delicate movement of water by filamentous sensory hairs lying inside the lateral line organs. The effective stimulus to a neuromast is not the pressure variation or the fluid motions of the propagating sound wave, but the water displacement or the medium flow disturbance around the fish. Hence, the lateral lines operate predominantly in the close, rather than the distant, zone around vibrating bodies (Harris & van Bergeijk, 1962; Kalmijn, 1988). The water current which is produced by neighboring fish therefore mediates the arrayal interaction. Thus, it would be supposed that the arrayal force works in the vicinity of the neighboring fish especially in the initial time-region of the schooling process where the school has not been organized. The fish then interacts with the near field, $\mathbf{V}(\mathbf{x})$, which is defined as the velocity arithmetically averaged over bodies in the vicinity of the fish located at \mathbf{x} . In this paper we were concerned with the mean field, i.e., the simple average velocity $\mathbf{V}(t)$, in order to discuss the temporal evolution of schooling structure. On the other hand, in order to discuss the schooling process of spatio-temporal structure the behavior of the near field $\mathbf{V}(\mathbf{x}, t)$ should be investigated. In the first approximation, the spatio-temporal dependent function $\mathbf{V}(\mathbf{x}, t)$ may obey eqn (24). In the second

approximation, the coupling between the neighboring near fields is taken into account, which derives the diffusion-like equation governing the motion of continuously extended system, i.e., $\mathbf{V}(\mathbf{x}, t)$ (Niwa, 1992). It will enable us to discuss the critical features of synchrony between individuals in a school by calculating the two-point correlation function $\langle \mathbf{V}(\mathbf{x}', t') \mathbf{V}^T(\mathbf{x}, t) \rangle$.

In this paper a specific form of the grouping force which provides the cohesion of fish in a shoal was not given. The statistical distribution of the inter-fish distance in a shoal implies the functional form of grouping force. Figure 8 shows the frequency distribution of the distance to the nearest neighbor in a *Gnathopogon* shoal. A mechanistic explanation of this graph is that the attractive interaction depends linearly upon the inter-fish distance (harmonic-spring attraction), and that the repulsive interaction nearly disappears at distances beyond one body length (hard-core repulsion). If the attraction depends on vision, individuals in a shoal may interact only with near

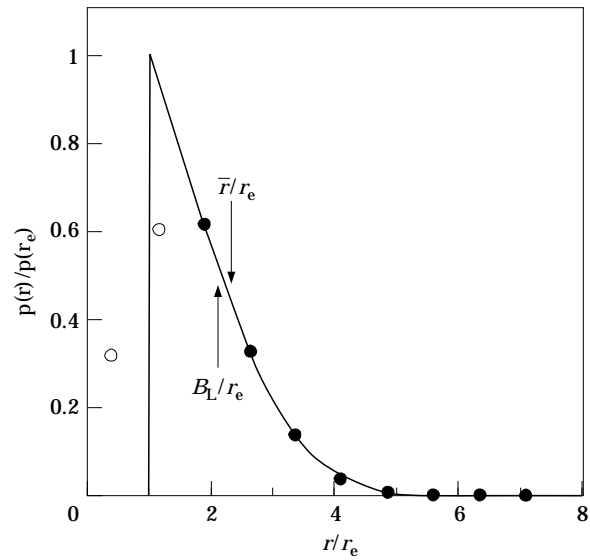


FIG. 8. Distribution for inter-fish distance to the nearest neighbor in the *Gnathopogon* shoal ($n = 8$) in two-dimensions. Circles are data from Aoki (1980; fig. 7). The grouping force experienced by neighboring fish is assumed to be composed of the harmonic-spring attraction and hard-core repulsion. The first two open circles signify that the fish has its own core region which other fish cannot come into. Therefore, data given by closed circles are fitted to a form for the distribution: $p(r) = \mathcal{N} \exp(-r^2/b^2)$, which is proportional to the probability of finding the nearest neighbor at given point \mathbf{r} from reference fish. Here the parameter b and the average distance to the nearest neighbor \bar{r} are estimated at 6.1 cm and 6.2 cm, respectively. The mean body length was $B_L = 5.6$ cm. It can be seen that \bar{r} and B_L are almost equal. Inter-fish distance $r \equiv |\mathbf{r}|$ and distribution $p(r)$ are normalized with the core radius r_e , and $p(r_e)$, respectively. The core radius r_e is defined by using a normalization condition: $\sum_k p_{\text{obs}}(r_k \leq r < r_{k+1}) 2\pi r_k \Delta r_k = \int_{r_e}^{\infty} p(r) 2\pi r dr$, where $p_{\text{obs}}(r)$ is the observed frequency, and $\Delta r_k \equiv r_{k+1} - r_k$. The core radius r_e is estimated at 2.7 cm.

neighbors. This means that the grouping force effectively works in the range of the average distance to the nearest neighbor, \bar{r} (short-range attraction). For a large shoal, $N \gg 1$, the dimensional size of the shoal, R_S , is much larger than the effective range of attraction or the mean inter-fish distance, i.e., $R_S \gg \bar{r}$. Therefore, a variation of the number-density distribution of fish in a shoal is supposed to be sufficiently slow except near the periphery of shoal. Moreover, if a shoal is maintained by short-range attractions and hard-core repulsions, it is expected that the number density of fish in a shoal does not depend upon the school size N . In Fig. 9 it is shown that the time-averaged number-density varies very slowly in the interior of the core of the *Gnathopogon* shoal.

Parr (1927) discussed the maintenance of fish shoal in terms of the balance of two counteracting forces, attraction and repulsion, between individual fish. Breder (1954) considered the constant attractive force which is independent of inter-fish distance and the soft-core repulsive force which depends inversely upon the square of the distance. Niwa (1994) analysed the collective motion of schooling with this Parr–Breder model, and obtained the governing equations of schooling which are equivalent to eqns (18) and (20). Niwa showed that the resultant grouping force $\sum_{j=1}^N \mathbf{f}_{ij}^{(g)}$ produces, as the average, a centrally attractive force acting on a fish i dependent only upon the distance from the center of a shoal. Then it is expected that the number-density distribution of fish in a shoal is peaked

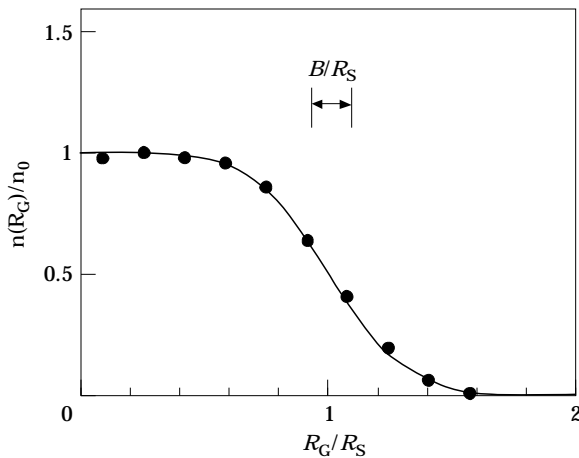


FIG. 9. Time-averaged number-density vs. distance from the center of the *Gnathopogon* shoal ($N = 8$) in two-dimensions; constant density in the interior of the shoal and rapidly varying density in a surface layer of thickness B . Providing the shoal is isotropic, data from Aoki (1980; fig. 16) are fitted to a form for the density distribution: $n(\mathbf{R}_G) = n_0 / [\exp\{(|\mathbf{R}_G| - R_S)/B\} + 1]$. n_0 is the number of fish per square centimeter inside the core of the shoal. Distance from the center, $R_G \equiv |\mathbf{R}_G|$, and number-density $n(R_G)$ are normalized with the shoal radius, $R_S = 12.1$ cm, and the core density, $n_0 = 1.0 \times 10^{-2}$ cm $^{-2}$, respectively. The thickness is estimated at $B = 1.9$ cm.

at a single attractive center just as the number-density distribution of swarming midges *Anarete pritchardi* which is similar to “Gaussian distribution” (Okubo & Chiang, 1974). Moreover, from Parr–Breder’s description of the attractive-repulsive interaction with the infinite-range attractions and the soft-core repulsions, it was predicted that the average number density of fish in the interior of the core of a shoal depends on the shoal size N . However, the number densities of herring shoals (*Clupea harengus*) observed by Misund (1991) do not exhibit a peaked pattern and are almost independent of the dimensional size of the shoal. A more detailed analysis of the attraction-repulsion problem (taking into account both short-range attractions and hard-core repulsions) is required in order to explain the observations of packing density.

Anchovy forms either compactly packed schools or thinly aggregated shoals, depending on their activities or environmental conditions. While the anchovy shoals take the form of compactly packed schools in daytime, they disperse at night, and the nearest neighbor distance in the shoal shows great variation compared with daytime (Aoki & Inagaki, 1988). For northern anchovy shoals (*Engraulis mordax*), the estimated packing densities at night were much smaller than those in daytime; the reduction amounted to a factor of 1/460 to 1/132 (Graves, 1976). This drastic conformational change of shoals suggests the phase transition which is analogous to the “liquid-vapor phase transition.” This “compactness-dispersal transition” is another important feature of fish shoaling, which seems to relate closely to the polarization–non-polarization transition. Symons (1971) studied spacing and density in shoaling threespine stickleback (*Gasterosteus aculeatus*) and mummichog (*Fundulus heteroclitus*). Symons showed that fish were usually spaced regularly when the average distance between nearest neighbors was less than one fish length. At a greater distance, i.e., lower density spacing between fish was found to be either random or in a state of loose aggregation. Partridge (1980) observed that a degree of polarization in minnow schools (*Phoxinus phoxinus*) became higher, as the number of fish in a school increased and as the density became higher. Their observations suggest that *critical density* is necessary for the occurrence of schooling. To elucidate the relation between packing density and schooling behavior will be one of the most important tasks imposed on further mechanistic investigations of schooling.

This work was partially supported by the Grant-in-Aid for Special Project Research (Bio-Cosmos Program) from the Ministry of Agriculture, Forestry and Fisheries of

Japan. This work is indebted to SALS System (Version 2.4) of SALS group (Nakagawa & Oyanagi, 1980) for statistical analysis of the experimental data with the least squares fitting, and to the Computer Center for Agriculture, Forestry and Fisheries Research for the data processing. I thank Tracy Monson for improving the English in the manuscript.

REFERENCES

- ALEXANDER, R. McN. (ed.) (1992). *Mechanics of Animal Locomotion. Advances in Comparative and Environmental Physiology*, Vol. 11. Berlin: Springer-Verlag.
- AOKI, I. (1980). An analysis of the schooling behavior of fish: internal organization and communication process. *Bull. Ocean Res. Inst. Univ. Tokyo*, No. 12, 1–65.
- AOKI, I. (1982). A simulation on the schooling mechanism in fish. *Bull. Jap. Soc. Sci. Fish.* **48**, 1081–1088.
- AOKI, I. & INAGAKI, T. (1988). Photographic observations on the behavior of Japanese anchovy *Engraulis japonica* at night in the sea. *Mar. Ecol. Prog. Ser.* **43**, 213–221.
- BREder, C. M., JR. (1954). Equation descriptive of fish schools and other animal aggregations. *Ecology* **35**, 361–370.
- BREder, C. M., JR. (1976). Fish schools as operational structures. *Fish. Bull.* **74**, 471–502.
- BUERKLE, U. (1983). First look at herring distributions with a bottom referencing underwater towed instrumentation vehicle 'Brutiv'. *FAO Fish. Rep.* No. 300, 125–130.
- CLUTTER, R. I. (1969). The microdistribution and social behaviour of some pelagic mysid shrimps. *J. exp. mar. Biol. Ecol.* **3**, 125–155.
- GRAVES, J. (1976). Photographic method for measuring spacing and density within pelagic fish schools at sea. *Fish. Bull.* **75**, 230–234.
- HAKEN, H. (1983). *Synergetics. An Introduction: Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology*, 3rd edn. Berlin: Springer-Verlag.
- HARA, I. (1985). Moving direction of Japanese sardine school on the basis of aerial surveys. *Bull. Jap. Soc. Sci. Fish.* **51**, 1939–1945.
- HARRIS, G. G. & VAN BERGEUK, W. A. (1962). Evidence that the lateral-line organ responds to near-field displacements of sound sources in water. *J. Acoust. Soc. Amer.* **34**, 1831–1841.
- HORSTHEMKE, W. & LEFEVER, R. (1983). *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology*. Berlin: Springer-Verlag.
- HUNTER, J. R. (1966). Procedure for analysis of schooling behavior. *J. Fish. Res. Bd. Canada* **23**, 547–562.
- INAGAKI, T., SAKAMOTO, W., AOKI, I. & KUROKI, T. (1976). Studies on the schooling behavior of fish. III. Mutual relationship between speed and form in schooling behavior. *Bull. Jap. Soc. Sci. Fish.* **42**, 629–635.
- KALMIJN, A. J. (1988). Hydrodynamic and acoustic field detection. In: *Sensory Biology of Aquatic Animals* (Atema, J., Fay, R. R., Popper, A. N. & Tavolga, W. N., eds) pp. 83–130. New York: Springer-Verlag.
- LANDAU, L. D. & LIFSHITZ, E. M. (1980). *Statistical Physics*, Part 1, 3rd edn. (Course of Theoretical Physics, Vol. 5) Oxford: Pergamon Press.
- MISUND, O. A. (1991). *Swimming Behaviour of Schools Related to Fish Capture and Acoustic Abundance Estimation*. Ph.D. Thesis, Department of Fisheries and Marine Biology, University of Bergen, Bergen, Norway.
- NAKAGAWA, T. & OYANAGI, Y. (1980). Program system SALS for nonlinear least squares fitting in experimental sciences. In: *Recent Developments in Statistical Inference and Data Analysis* (Matusita, K., ed.) pp. 221–225. Amsterdam: North Holland Publ. Co.
- NICOLIS, J. S. (1986). *Dynamics of Hierarchical Systems: An Evolutionary Approach*. Berlin: Springer-Verlag.
- NIWA, H.-S. (1992). Equation for swimming formation of a fish school. *Bull. Soc. Sci. Form Japan* **7**(2), 44–47 (in Japanese).
- NIWA, H.-S. (1994). Self-organizing dynamic model of fish schooling. *J. theor. Biol.* **171**, 123–136.
- OKUBO, A. (1980). *Diffusion and Ecological Problem: Mathematical Models*. Berlin: Springer-Verlag.
- OKUBO, A. (1986). Dynamical aspects of animal grouping: swarms, schools, flocks and herds. *Adv. Biophys.* **22**, 1–94.
- OKUBO, A. & CHIANG, H. C. (1974). An analysis of the kinematics of swarming of *Anarete pritchardi* Kim (Diptera: Cecidomyiidae). *Res. Popul. Ecol.* **16**, 1–42.
- PARR, A. E. (1927). *A Contribution to the Theoretical Analysis of the Schooling Behavior of Fishes*. Occasional Papers of the Bingham Oceanographic Collection, No. 1.
- PARTRIDGE, B. L. (1980). The effect of school size or the structure and dynamics of minnow schools. *Anim. Behav.* **28**, 68–77.
- PARTRIDGE, B. L. (1982). The structure and function of fish schools. *Sci. Am.* **246**, 90–99.
- PITCHER, T. J. (1979). Sensory information and the organization of behavior in a shoaling cyprinid fish. *Anim. Behav.* **27**, 126–149.
- PITCHER, T. J. & PARRISH, J. K. (1993). Functions of shoaling behavior in teleosts. In: *Behavior of Teleost Fishes*, 2nd edn. (Pitcher, T. J., ed.) pp. 363–439. London: Chapman & Hall.
- PITCHER, T. J., PARTRIDGE, B. L. & WARDLE, C. S. (1976). A blind fish can school. *Science* **194**, 963–965.
- SAKAI, S. (1973). A model for group structure and its behavior. *Biophysics* **13**, 82–90 (in Japanese).
- SHOW, E. (1969). Some new thought on the schooling of fishes. *FAO Fish. Rep.* No. 62, 217–231.
- SUZUKI, M. (1978). Theory of instability, nonlinear brownian motion and formation of macroscopic order. *Phys. Lett.* **67A**, 339–341.
- SUZUKI, M. (1981). Passage from an initial unstable state to a final stable state. *Adv. Chem. Phys.* **46**, 195–278.
- SYMONS, P. E. K. (1971). Spacing and density in schooling threespine sticklebacks (*Gasterosteus aculeatus*) and mummichog (*Fundulus heteroclitus*). *J. Fish. Res. Bd. Canada* **28**, 999–1004.
- VAN OLST, J. C. & HUNTER, J. R. (1970). Some aspects of the organization of fish schools. *J. Fish. Res. Bd. Canada* **27**, 1225–1238.
- WU, T. Y. (1977). Introduction to the scaling of aquatic animal locomotion. In: *Scale Effects in Animal Locomotion* (Pedley, T. J., ed.) pp. 203–232. London: Academic Press.

APPENDIX A

Linearizing eqn (9) in the main text by replacing the nonlinear term $\mathbf{v}^2(t)\mathbf{v}(t)$ by $\langle \mathbf{v}^2(t) \rangle \mathbf{v}(t)$, yields

$$\frac{d\mathbf{v}}{dt} = \kappa(1 - \beta \langle \mathbf{v}^2 \rangle) \mathbf{v} + \boldsymbol{\eta}(t), \quad (\text{A.1})$$

where the average $\langle \mathbf{v}^2(t) \rangle$ is determined self-consistently. The formal solution is given by

$$\begin{aligned} \mathbf{v}(t) = \exp \left\{ \int_0^t \kappa(1 - \beta \langle \mathbf{v}^2(t') \rangle) dt' \right\} \\ \times \left[\int_0^t \boldsymbol{\eta}(t') \exp \left\{ - \int_0^{t'} \kappa(1 - \beta \langle \mathbf{v}^2(t'') \rangle) dt'' \right\} dt' + \mathbf{v}(0) \right]. \end{aligned} \quad (\text{A.2})$$

Let us calculate the quantity $\langle \mathbf{v}(t) \cdot \boldsymbol{\eta}(t) \rangle$. Since the averaging process over the Gibbs ensemble has nothing to do with the integration, the sequence of averaging and integration may be exchanged, and the

averaging may be performed first. Exploiting eqn (2) in the text, we then have

$$\begin{aligned} \langle \mathbf{v}(t) \cdot \boldsymbol{\eta}(t) \rangle &= \exp \left\{ \int_0^t \kappa (1 - \beta \langle \mathbf{v}^2(t') \rangle) dt' \right\} \\ &\times \int_0^t 6\epsilon \delta(t - t') \exp \left\{ - \int_0^{t'} \kappa (1 - \beta \langle \mathbf{v}^2(t'') \rangle) dt'' \right\} dt'. \end{aligned} \quad (\text{A.3})$$

Owing to the δ -function, the integral can be immediately evaluated as

$$\langle \mathbf{v}(t) \cdot \boldsymbol{\eta}(t) \rangle = 3\epsilon. \quad (\text{A.4})$$

Multiplying by \mathbf{v} and averaging over the Gibbs ensemble yield the following equation of self-consistency:

$$\frac{d}{dt} \langle \mathbf{v}^2(t) \rangle = 2\kappa(1 - \beta \langle \mathbf{v}^2(t) \rangle) \langle \mathbf{v}^2(t) \rangle + 6\epsilon. \quad (\text{A.5})$$

The stationary solution is then given by eqn (10) in the text. Replacing the mean value by the most probable value of stationary velocity, i.e., $\langle |\mathbf{v}| \rangle = v_m$, yields eqn (11) in the text.

APPENDIX B

The velocity \mathbf{v}_i is split up into the following two terms:

$$\mathbf{v}_i \equiv \mathbf{V} + \boldsymbol{\xi}_i, \quad (\text{B.1})$$

where $\boldsymbol{\xi}_i$ denotes fluctuations around the group average $\mathbf{V} \equiv \sum_{i=1}^N \mathbf{v}_i / N$. Inserting eqn (B.1) into eqn (13) in the text and averaging eqn (13) over all i 's yield

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= \kappa \mathbf{V} - \kappa \beta \left\{ \mathbf{V}^2 \mathbf{V} + \frac{1}{N} \sum_{i=1}^N \boldsymbol{\xi}_i^2 \mathbf{V} + \frac{2}{N} \sum_{i=1}^N (\mathbf{V} \cdot \boldsymbol{\xi}_i) \boldsymbol{\xi}_i \right\} \\ &+ \frac{1}{N} \sum_{i=1}^N \boldsymbol{\eta}_i(t), \end{aligned} \quad (\text{B.2})$$

where eqn (14) in the text was used. Since $N \gg 1$, $\sum_{i=1}^N \boldsymbol{\xi}_i^2 / N = \langle (\mathbf{v}_i - \mathbf{V})^2 \rangle$ holds. Because a problem is invariant with respect to the inversion $\mathbf{v}_i \rightarrow -\mathbf{v}_i$, $\sum_{i=1}^N (\mathbf{V} \cdot \boldsymbol{\xi}_i) \boldsymbol{\xi}_i / N = \sigma^2 \mathbf{V} / 3$ holds. Thus eqn (18) in the text is obtained.

APPENDIX C

Subtracting eqn (18) from eqn (13) in the text yields

$$\begin{aligned} \frac{d\boldsymbol{\xi}_i}{dt} &= \kappa(1 - \beta \mathbf{V}^2) \boldsymbol{\xi}_i - 2\kappa \beta (\mathbf{V} \cdot \boldsymbol{\xi}_i) \\ &+ \sum_{j=1}^N \mathbf{f}_{ij}^{(g)} - J \boldsymbol{\xi}_i + \boldsymbol{\eta}_i(t) - \bar{\boldsymbol{\eta}}(t) + \mathcal{O}(\boldsymbol{\xi}_i^2). \end{aligned} \quad (\text{C.1})$$

Multiplying eqn (C.1) by $\boldsymbol{\xi}_i$ gives

$$\begin{aligned} \frac{d}{dt} \boldsymbol{\xi}_i^2 &= 2\kappa(1 - \beta \mathbf{V}^2) \boldsymbol{\xi}_i^2 - 4\kappa \beta (\mathbf{V} \cdot \boldsymbol{\xi}_i)^2 \\ &+ \boldsymbol{\xi}_i \cdot \sum_{j=1}^N \mathbf{f}_{ij}^{(g)} - 2J \boldsymbol{\xi}_i^2 + 2\boldsymbol{\xi}_i \cdot \{\boldsymbol{\eta}_i(t) - \bar{\boldsymbol{\eta}}(t)\}, \end{aligned} \quad (\text{C.2})$$

where $\mathcal{O}(\epsilon^2)$ was neglected. Averaging eqn (C.2) over all i 's eqn (20) in the text is obtained. Here, $\sum_{i=1}^N (\mathbf{V} \cdot \boldsymbol{\xi}_i)^2 / N = \mathbf{V}^2 \sigma^2 / 3$ holds because of the rotational symmetry of the system; $\sum_{i=1}^N \boldsymbol{\xi}_i \cdot \sum_{j=1}^N \mathbf{f}_{ij}^{(g)} = 0$ holds because of the assumption that the grouping force is independent of the individual velocities; and by exploiting eqn (A.4) we have $\sum_{i=1}^N \boldsymbol{\xi}_i \cdot [\boldsymbol{\eta}_i(t) - \bar{\boldsymbol{\eta}}(t)] / N = 3\epsilon$ for large N .

APPENDIX D

In the initial time-region of the schooling process, since $|\mathbf{V}| \approx 0$, eqn (18) in the text reads

$$\frac{d\mathbf{V}}{dt} = \kappa \left(1 - \frac{5}{3} \beta \langle \mathbf{v}^2 \rangle \right) \mathbf{V} + \bar{\boldsymbol{\eta}}(t). \quad (\text{D.1})$$

Then multiplying by \mathbf{V} and averaging over the Gibbs ensemble yield the following equation in the same way as discussed in Appendix A:

$$\frac{d}{dt} \langle \mathbf{V}^2 \rangle = -\frac{4}{3} \kappa \left(1 + \frac{15}{2} \frac{\beta \epsilon}{\kappa} \right) \langle \mathbf{V}^2 \rangle + 6 \frac{\epsilon}{N}, \quad (\text{D.2})$$

where eqn (10) in the text was used. The stationary solution (29) in the text is then obtained.

APPENDIX E

The differential equation (D.1) is formally solved as follows

$$\begin{aligned} \mathbf{V}(t) &= \exp \left\{ \int_0^t \kappa \left(1 - \frac{5}{3} \beta \langle \mathbf{v}^2 \rangle \right) dt' \right\} \\ &\times \left[\int_0^t \bar{\boldsymbol{\eta}}(t') \exp \left\{ - \int_0^{t'} \kappa \left(1 - \frac{5}{3} \beta \langle \mathbf{v}^2 \rangle \right) dt'' \right\} dt' + \mathbf{V}(0) \right]. \end{aligned} \quad (\text{E.1})$$

By exploiting eqns (10) and (19) in the text, the two-time correlation function (32) in the text is immediately obtained.

APPENDIX F

Let us consider eqn (18) in the text in the marginal time-region of $t \approx t_c$. Multiplying the linearized

equation (replacing the nonlinear term $\mathbf{V}^2\mathbf{V}$ by $\langle \mathbf{V}^2 \rangle \mathbf{V}$) by \mathbf{V} and averaging over the Gibbs ensemble yield

$$\kappa \left(1 - \frac{5}{3} \beta \sigma^2(t) \right) \langle \mathbf{V}^2 \rangle - \kappa \beta \langle \mathbf{V}^2 \rangle^2 + 3 \frac{\epsilon}{N} = 0, \quad (\text{F.1})$$

where $\langle \mathbf{V}(t) \cdot \tilde{\mathbf{\eta}}(t) \rangle = 3\epsilon/N$ because of eqn (A.4). Equation (F.1) is solved by

$$\begin{aligned} \langle \mathbf{V}^2(t) \rangle &= \frac{1}{\beta} \left(1 - \frac{5}{3} \beta \sigma^2(t) \right) + \frac{3\epsilon}{\kappa N \langle \mathbf{V}^2(0) \rangle} \\ &= \frac{5}{3} \{ \sigma^2(0) - \sigma^2(t) \}, \end{aligned} \quad (\text{F.2})$$

where $\mathcal{O}(\epsilon^2)$ was neglected. Inserting eqn (F.2) into eqn (20) in the text yields eqn (35). For small ϵ and small κ/J eqn (35) in the text reduces to

$$\begin{aligned} \frac{d\sigma^2(t)}{dt} &= -\frac{50}{9} \kappa \beta \left\{ \frac{9}{25} \frac{J}{\kappa \beta} \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right) \right. \\ &\quad \left. - \sigma^2(t) \right\} \left\{ \sigma^2(t) - \frac{3\epsilon}{J} \left(1 - \frac{\kappa}{J} + \frac{25}{9} \frac{\kappa}{J} \beta \sigma^2(0) \right)^{-1} \right\}, \end{aligned} \quad (\text{F.3})$$

which has the solution (36) in the text.

APPENDIX G

For the linearized equation

$$\frac{d\mathbf{V}}{dt} = \left\{ \kappa \left(1 - 5 \frac{\beta \epsilon}{J} \right) - \kappa \beta \langle \mathbf{V}^2(t) \rangle \right\} \mathbf{V} + \tilde{\mathbf{\eta}}(t), \quad (\text{G.1})$$

the equation of self-consistency (38) in the text follows immediately. For small ϵ/N and large t , eqn (38) in the text reduces to the following equation:

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{V}^2(t) \rangle &= 2\kappa \beta \{ \langle \mathbf{V}^2 \rangle_{\text{st}} - \langle \mathbf{V}^2(t) \rangle \} \\ &\quad \times \left\{ \langle \mathbf{V}^2(t) \rangle + \frac{3\epsilon/N}{\kappa(1 - 5\beta\epsilon/J)} \right\}, \end{aligned} \quad (\text{G.2})$$

which is solved in the form (39) in the text.

APPENDIX H

The temporal evolution of $\mathbf{V}_m(t)$ is given by

$$\frac{d\mathbf{V}_m}{dt} = \kappa \left(1 - 5 \frac{\beta \epsilon}{J} \right) \mathbf{V}_m - \kappa \beta \mathbf{V}_m^2 \mathbf{V}_m, \quad (\text{H.1})$$

the solution of which takes the following form:

$$\begin{aligned} \mathbf{V}_m(t) &= \mathbf{V}_1 \left[1 + \left(\frac{\mathbf{V}_1^2}{\mathbf{V}_m^2(0)} - 1 \right) \right. \\ &\quad \left. \times \exp \left\{ -2\kappa \left(1 - 5 \frac{\beta \epsilon}{J} \right) t \right\} \right]^{-1/2}. \end{aligned} \quad (\text{H.2})$$

Letting $V(t) \equiv |\mathbf{V}(t)|$ and $\tilde{\eta}(t)$ be Gaussian white satisfying the relation

$$\langle \tilde{\eta}(t) \rangle = 0, \quad \langle \tilde{\eta}(t) \tilde{\eta}(t') \rangle = 2\epsilon \delta(t - t')/N, \quad (\text{H.3})$$

the equation of motion is given by

$$\frac{dV}{dt} = F(V(t)) + \tilde{\eta}(t), \quad (\text{H.4})$$

where $F(V(t)) \equiv \kappa(1 - 5\beta\epsilon/J)V(t) - \kappa\beta V^3(t)$. It is easily shown that a deviation of the system from the most probable path, $y(t) \equiv V(t) - V_m(t)$, satisfies the equation

$$\begin{aligned} \frac{d}{dt} y(t) &= \kappa \left(1 - 5 \frac{\beta \epsilon}{J} \right) y(t) \\ &\quad - 3\kappa \beta V_m^2(t) y(t) + \tilde{\eta}(t), \end{aligned} \quad (\text{H.5})$$

where $V_m(t) \equiv |\mathbf{V}_m(t)|$. The temporal evolution of the variance, $\sigma_V^2(t) = \langle y^2(t) \rangle$, is then given by eqn (47) in the text. Equation (47) is solved formally as

$$\begin{aligned} \sigma_V^2(t) &= \sigma_V^2(0) \left\{ \frac{F(V_m(t))}{F(V_m(0))} \right\}^2 \\ &\quad + \{F(V_m(t))\}^2 \int_{V_m(0)}^{V_m(t)} \frac{2\epsilon/N}{\{F(V)\}^3} dV \end{aligned} \quad (\text{H.6})$$

(Suzuki, 1981). If $\langle \mathbf{V}^2(0) \rangle$ is small, the variance takes the following asymptotic form:

$$\begin{aligned} \sigma_V^2(t) &\approx \sigma_{V,\text{as}}^2(t) \equiv \frac{\sigma_V^2(0) + \epsilon/\kappa N(1 - 5\beta\epsilon/J)}{\langle \mathbf{V}^2(0) \rangle} \\ &\quad \times \left\{ \frac{F(V_m(t))}{\kappa(1 - 5\beta\epsilon/J)} \right\}^2. \end{aligned} \quad (\text{H.7})$$

For the proof of this formula, we put

$$R(\delta_V) \equiv \frac{\sigma_V^2(t) - \sigma_{V,\text{as}}^2(t)}{\sigma_{V,\text{as}}^2(t)}, \quad (\text{H.8})$$

where $\delta_V \equiv \sqrt{\langle \mathbf{V}^2(0) \rangle}$. It is easily shown that $R(\delta_V) \rightarrow 0$ as $\delta_V \rightarrow 0$. This means that $\sigma_V^2(t) \approx \sigma_{V,\text{as}}^2(t)$ for a small δ_V . The transient fluctuation can also be expressed more intuitively. By substituting $\dot{V}_m = F(V_m)$ into eqn (H.7) and using the property that $\dot{V}_m(0) = F(\delta_V) = \kappa(1 - 5\beta\epsilon/J)\delta_V + \mathcal{O}(\delta_V^2)$, $\sigma_V^2(t)$ is written as eqn (48) in the text.