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LETTER TO THE EDITOR

The application of dispersion relations (Hilbert transforms) to phase retrieval

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Abstract. A method is given for the retrieval of phase from amplitude information using the Hilbert transform, without the need for evaluating Blaschke factors.

The connection between causality and dispersion relations is well known (Toll 1956). Dispersion relations connect, for example, the real part of a function to an integral involving the imaginary part. Such relations occur in the form of the Kramers–Kronig relation in optics (Loudon 1973), in particle scattering (van Kampen 1953, Hilgevoord 1960, Roman 1965), in electron optics (Misell *et al* 1974), and in communication theory (Kuo and Freeny 1962).

This integral relation may be defined by the Hilbert transform for a function $\gamma(t)$:

$$\mathrm{Im}\,[\gamma(t)] = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\,[\gamma(t')]}{t'-t}\,\mathrm{d}t' \tag{1}$$

$$\mathrm{Re}\,[\gamma(t)] = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Im}\,[\gamma(t')]}{t'-t}\,\mathrm{d}t' \tag{2}$$

where P denotes the Cauchy principal value.

It has been shown (Titchmarsh 1948) that equations (1) and (2) follow if the Fourier transform of $\gamma(t)$ vanishes for negative values of its argument, and $\gamma(t)$ is the limit as $\tau \to t$ of an analytic function $\gamma(\tau)$, where $\tau = t + \mathrm{i}s$, which is regular for $s > 0$.

Equations (1) and (2) have been used to determine the phase $\phi(t)$ of $\gamma(t)$ when an estimate of $\mathrm{Re}\,[\gamma(t)]$ (or $\mathrm{Im}\,[\gamma(t)]$) is available (Misell *et al* 1974). In general only $|\gamma(t)|$ is measured experimentally, and it is of value to explore the direct determination of $\phi(t)$ from $|\gamma(t)|$ (Page 1955, Toll 1956, Peřina 1971).

The minimal phase $\bar{\phi}(t)$ may be determined from the equation

$$\bar{\phi}(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\ln|\gamma(t')|}{t'-t}\,\mathrm{d}t' \tag{3}$$

(see, for example, Peřina 1971). In general there are two sources of additional phase factors to $\bar{\phi}(t)$: (i) the product of Blaschke phase factors arising from the zeros of $\gamma(\tau)$ in the upper half-plane ($s > 0$); (ii) an additive phase factor, linearly dependent on t (van Kampen 1953, Toll 1956), which reflects the fact that only $|\gamma(t)|$ is known on the real axis. This second factor follows since $|\gamma(t)|\,|G(t)|$ is indistinguishable from $|\gamma(t)|$, if $G(\tau)$ is any bounded analytic function in the upper half-plane and $|G(\tau)| = 1$, almost everywhere on the real axis. The purpose of this letter is to show that the Blaschke factors (see Roman and Marathay 1963, Nussenzveig 1967), which arise because of the singularities in $\ln|\gamma(\tau)|$ lying within the contour used to evaluate equation (3), can be removed. To this end we consider the effect of adding a complex constant C to $\gamma(\tau)$ to prevent the function $\gamma(\tau)+C$ having complex zeros within the contour. We examine later the choice of C and the physical validity of the method.

Since

$$P\int_{-\infty}^{+\infty}\frac{\mathrm{d}t'}{t'-t}=\lim_{\substack{\epsilon\to 0\\ A\to\infty}}\left(\int_{-A}^{t-\epsilon}\frac{\mathrm{d}t'}{t'-t}+\int_{t+\epsilon}^{A}\frac{\mathrm{d}t'}{t-t'}\right)=0$$

we may write from equations (1) and (2)

$$\mathrm{Im}\,[\gamma(t)+C]=\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\mathrm{Re}\,[\gamma(t')+C]}{t'-t}\,\mathrm{d}t'+\mathrm{Im}\,(C) \tag{4}$$

and

$$\mathrm{Re}\,[\gamma(t)+C]=-\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\,[\gamma(t')+C]}{t'-t}\,\mathrm{d}t'+\mathrm{Re}\,(C). \tag{5}$$

These expressions are similar to those written by Hilgevoord (1960) and Papoulis (1962).

Define $g(\tau)=\gamma(\tau)+C$, where $|g(\tau)|$ has no zeros in the upper half-plane ($s>0$). We now investigate if the phase $\alpha(t)$ of $g(t)$ may be determined from an equation similar to (3). Since $|g(\tau)|$ has no zeros for $s>0$, there will be no Blaschke factors to be added to $\alpha(t)$. We consider later the other source of ambiguity in the solution for $\alpha(t)$, namely the term linear in t.

The modulus and phase of $g(\tau)=|g(\tau)|\exp[\mathrm{i}\alpha(\tau)]$ are defined by

$$|g(\tau)|=(\{\mathrm{Re}\,[g(\tau)]\}^2+\{\mathrm{Im}\,[g(\tau)]\}^2)^{1/2} \tag{6}$$

and

$$\tan\alpha(\tau)=\frac{\mathrm{Im}\,[g(\tau)]}{\mathrm{Re}\,[g(\tau)]}. \tag{7}$$

From (5) and (7) we have

$$\frac{\mathrm{Im}\,[g(t)]}{\tan\alpha(t)}=-\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\,[g(t')]}{t'-t}\,\mathrm{d}t'+\mathrm{Re}\,(C). \tag{8}$$

To solve this equation we use a method similar to the one followed by Peřina (1971).

Firstly we define the function $\psi(\tau)$ as

$$\psi(\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\,[g(\tau')]}{\tau'-\tau}\,\mathrm{d}\tau'-\tfrac{1}{2}\,\mathrm{Re}\,(C),\qquad \mathrm{Im}\,(\tau)\neq 0. \tag{9}$$

We use the identity

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\left(\frac{1}{\tau\mp\mathrm{i}\epsilon}\right)=P\,\frac{1}{\tau}\pm\pi\mathrm{i}\delta\,(\tau) \tag{10}$$

in equation (9) and obtain

$$\psi^{+}(\tau)=\frac{1}{2\pi}P\int_{-\infty}^{+\infty}\frac{\mathrm{Im}\,[g(\tau')]}{\tau'-\tau}\,\mathrm{d}\tau'+\tfrac{1}{2}\,\mathrm{Im}\,[g(\tau)]-\tfrac{1}{2}\mathrm{Re}\,(C)=-\tfrac{1}{2}g^{*}(\tau) \tag{11}$$

and

$$\psi^{-}(\tau)=-\tfrac{1}{2}g(\tau) \tag{12}$$

where * denotes complex conjugate.

Now

$$\psi^+(\tau)-\psi^-(\tau)=\mathrm{i}\,\mathrm{Im}\,[g(\tau)]$$

and

$$\psi^+(\tau)+\psi^-(\tau)=-\mathrm{Re}\,[g(\tau)]$$

where $\psi^+(\tau)$ and $\psi^-(\tau)$ are the limits of $\psi(\tau)$ as $\tau \to t$ from the $s>0$ and $s<0$ half-planes respectively. It follows that

$$\ln\left(\frac{|g(\tau)|^2}{g^2(\tau)}\right)=2\mathrm{i}\alpha(\tau)$$

and since from (11) and (12) $|g(\tau)|^2/g^2(\tau)=\psi^+(\tau)/\psi^-(\tau)$, then

$$\ln\psi^+(\tau)-\ln\psi^-(\tau)=2\mathrm{i}\alpha(\tau).$$

Since $\psi(\tau)$ tends to a constant for $|\tau|\to\infty$ along the real axis ($|\gamma(\tau)|$ tends to zero as $|\tau|\to\infty$), the solution of this equation is (Muskhelishvili 1953)

$$\ln\psi(\tau)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{2\mathrm{i}\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'+K$$

where K is a constant. That is

$$\psi(\tau)=\exp\left(\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'+K\right). \tag{13}$$

Using the identity (10) in equation (13) we obtain

$$\psi^+(\tau)=\exp\left(\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'+\mathrm{i}\alpha(\tau)+K\right)$$

and

$$\psi^-(\tau)=\exp\left(\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'-\mathrm{i}\alpha(\tau)+K\right).$$

Thus

$$\mathrm{Im}\,[g(\tau)]=2\exp K\exp\left(\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'\right)\sin\alpha(\tau)$$

and

$$\mathrm{Re}\,[g(\tau)]=2\exp K\exp\left(\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'\right)\cos\alpha(\tau).$$

Substituting these relationships in equation (6) gives

$$|g(\tau)|=2\exp K\exp\left(\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'\right)$$

or

$$\ln\left(\frac{|g(\tau)|}{2\exp K}\right)=\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\alpha(\tau')}{\tau'-\tau}\,\mathrm{d}\tau'.$$

It then follows (Titchmarsh 1948) that the inverse formula also holds:

$$\alpha(\tau)=-\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{\ln|g(\tau')|}{\tau'-\tau}\,\mathrm{d}\tau'+\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{(\ln 2+K)}{\tau'-\tau}\,\mathrm{d}\tau'. \tag{14}$$

The second term in equation (14) is zero, and the relation between $|g(\tau)|$ and $\alpha(\tau)$ as $\tau \to t$ takes the same form as equation (3). The validity of the method depends upon the choice of C in such a way that $|g(\tau)|=|\gamma(\tau)+C|>0$ for all $\tau=t+\mathrm{i}s$, $s>0$. A zero in $|\gamma(\tau)|$ implies that both the real and imaginary parts of $\gamma(\tau)$ are simultaneously zero, and hence we may choose C to be real. A suitable choice would be $C>|\gamma(\tau)|_{\max}$. The phase $\phi(t)$ of $\gamma(t)$ is given by

$$\tan \phi(t)=\frac{|g(t)| \sin \alpha(t)}{|g(t)| \cos \alpha(t)-C}. \tag{15}$$

A possible application of this approach is in bright-field optics. In this case, image formation involves the interference of a background wave and a scattered wave $|\gamma(t)| \exp[-\mathrm{i}\phi(t)]$. The choice of the constant which represents this background defines the origin for $\phi(t)$. Under normal experimental conditions $|C| \gg |\gamma(t)|$, and consequently equations (14) and (15) may be applied directly.

We are still left with the ambiguity of the term linearly dependent on t. In any practical situation it should be possible to determine this term from the phase solution far beyond the bounds of the region corresponding to the object, where the actual phase distribution may be expected to be constant. Thus the linear term should be easily detectable and a correction applied.

The validity of the derivation of the phase $\phi(t)$ of $\gamma(t)$ via the phase $\alpha(t)$ of $g(t)$ has been tested numerically. The specific application of this method to the determination of the phase of the image wavefunction in electron microscopy has been considered (Greenaway and Misell 1974, to be published).

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