A note on complexity

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September 22, 2020

Abstract

We introduce a complexity measure and study its properties.

In this note, we consider the following decoding algorithm to map a n-qubit state $|\psi\rangle$ to $|0^n\rangle$. At each iteration, we choose a Pauli string P and apply the map $|\psi\rangle \to e^{iP\theta}|\psi\rangle$. We will choose θ such that the overlap of $e^{iP\theta}|\psi\rangle$ with $|0^n\rangle$ is maximized.

The optimal angle can be chosen in a straightforward way, yielding a new overlap:

$$f_{i+1}^2 = \frac{f_i^2 + |e_P|^2 + |f^2 - e_P^2|}{2}, \tag{1}$$

where f_i is the square root of the overlap at the ith iteration and e_P is defined as

$$e_P := \langle 0^n|P|\psi\rangle. \tag{2}$$

Without loss of generality, suppose $e_P = x + iy$, where $x, y \in \mathbb{R}$. One can prove the following bound:

$$\begin{aligned} f^2 + |e_P|^2 + |f^2 - e_P^2| &= f^2 + x^2 + y^2 + |f^2 - (x^2 - y^2) - 2xyi| \\ &= f^2 + x^2 + y^2 + |f^2 - x^2 + y^2| \\ &\geq 2(f^2 + y^2). \end{aligned} \tag{3}$$

Therefore,

$$f_{i+1}^2 \geq f_i^2 + (\mathrm{Im}[e_P])^2. \tag{4}$$

Moreover, by using the fact that $\mathbb{E}_P[POP] = \frac{\mathrm{Tr}[O]I}{d}$, where $\mathbb{E}_P$ is the average over all the Pauli strings, one can show that

$$\mathbb{E}_P[(\mathrm{Im}[e_P])^2] = \frac{1 - f_i^2}{2d}. \tag{5}$$

Therefore, if we choose P randomly, we get

$$(1 - f_i^2)\left(1 - \frac{1}{2d}\right) \geq 1 - f_{i+1}^2, \tag{6}$$

leading to the following bound:

$$1 - f_k^2 \leq (1 - f_0^2)\left(1 - \frac{1}{2d}\right)^k. \tag{7}$$

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Therefore, on average, to approxiamte the target state up to an error of ϵ, it suffices to apply this protocol $2d\ln(1/\epsilon)$ times.

Conjecture: I think in the large d limit Eq. (7) holds as an equality (up to $1/d^2$ correction in the large paranthesis). The intuition is that in Eq. (3), we can estimate the $-2xyi$ term. Typically, both x and y scales as $\sim 1/\sqrt{d}$ whereas f becomes order one at some point. In that regime, the effect of $2xyi$ gives an additive contribution to the final line in Eq. (7), which is $\sim x^2y^2 \sim 1/d^2$.

Let us make a few remarks. First, provided that our conjecture is correct, the expected runtime of $2d\ln(1/\epsilon)$ would be independent of the initial and the final state. Second, we can reduce the number of iterations by choosing the very best P. One can modify our argument a bit to show that in such cases we are always guaranteed to obtain an error of ϵ in $\sim 2d\ln(1/\epsilon)$ iterations. Perhaps this is the best one can do? It will be good to investigate this.