Equations

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Initial equation, $t = 0$

$$u(x, t = 0) = \begin{cases} b & \text{if } \frac{-a}{2} \leq x \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

Migration, $t = 1$

$x < \frac{-a}{2}$:

$$u_1'(x, t = 1) = \frac{-b}{2}\left[e^{\frac{-1}{\sigma}(\frac{a}{2}-x)} - e^{\frac{-1}{\sigma}(\frac{-a}{2}-x)}\right]$$

$\frac{-a}{2} \leq x \leq \frac{a}{2}$:

$$u_2'(x, t = 1) = \frac{b}{2}\left[2 - e^{\frac{-1}{\sigma}(x+\frac{a}{2})} - e^{\frac{-1}{\sigma}(\frac{a}{2}-x)}\right]$$

$x > \frac{a}{2}$:

$$u_3'(x, t = 1) = \frac{b}{2}\left[e^{\frac{-1}{\sigma}(x-\frac{a}{2})} - e^{\frac{-1}{\sigma}(x+\frac{a}{2})}\right]$$

Adding the reaction, t=1

$x < \frac{-a}{2}$:

$$u_1(x, t = 1) = b(k\hat{u} - \frac{1}{2})\left[e^{\frac{-1}{\sigma}(\frac{a}{2}-x)} - e^{\frac{-1}{\sigma}(\frac{-a}{2}-x)}\right] + \frac{kb^2}{2}(1 + \hat{u})\left[e^{\frac{-1}{\sigma}(\frac{a}{2}-x)} - e^{\frac{-1}{\sigma}(\frac{-a}{2}-x)}\right]^2 + \frac{kb^3}{4}\left[e^{\frac{-1}{\sigma}(\frac{a}{2}-x)} - e^{\frac{-1}{\sigma}(\frac{-a}{2}-x)}\right]^3$$

$\frac{-a}{2} \leq x \leq \frac{a}{2}$**:**

$$u_2(x,t=1) = b\left[2k(b-\hat{u})(1-b)+1\right] + b\left[k\left[b(3b-2)+\hat{u}(1-2b)\right]-\frac{1}{2}\right]\left[e^{\frac{-1}{\sigma}(x+\frac{a}{2})} + e^{\frac{-1}{\sigma}(\frac{a}{2}-x)}\right]$$
$$-\left[\frac{kb^2}{2}(3b-\hat{u}-1)\right]\left[e^{\frac{-1}{\sigma}(x+\frac{a}{2})} + e^{\frac{-1}{\sigma}(\frac{a}{2}-x)}\right]^2 + \frac{kb^3}{4}\left[e^{\frac{-1}{\sigma}(x+\frac{a}{2})} + e^{\frac{-1}{\sigma}(\frac{a}{2}-x)}\right]^3$$

$x > \frac{a}{2}$**:**

$$u_3(x,t=1) = b(\frac{1}{2}-k\hat{u})\left[e^{\frac{-1}{\sigma}(x-\frac{a}{2})} - e^{\frac{-1}{\sigma}(x+\frac{a}{2})}\right] + \frac{kb^2}{2}(1+\hat{u})\left[e^{\frac{-1}{\sigma}(x-\frac{a}{2})} - e^{\frac{-1}{\sigma}(x+\frac{a}{2})}\right]^2$$
$$-\frac{kb^3}{4}\left[e^{\frac{-1}{\sigma}(x-\frac{a}{2})} - e^{\frac{-1}{\sigma}(x+\frac{a}{2})}\right]^3$$

Solving for θ_1, the AUC after $t = 1$

$$\int_0^{\frac{a}{2}} u_2(x,t=1)dx = \left[\frac{ab}{2}\right]\left[2k(b-\hat{u})(1-b)+1\right] + \left[\sigma b\right]\left[k\left[b(3b-2)+\hat{u}(1-2b)\right]-\frac{1}{2}\right]\left[1-e^{\frac{-a}{\sigma}}\right]$$
$$-\left[\frac{kb^2}{2}(3b-\hat{u}-1)\right]\left[ae^{\frac{-a}{\sigma}} + \frac{\sigma}{2}(1-e^{\frac{-2a}{\sigma}})\right] - \left[\frac{\sigma kb^3}{4}\right]\left[\frac{1}{3}(1-e^{\frac{-3a}{\sigma}}) + 3(e^{\frac{-a}{\sigma}} - e^{\frac{-2a}{\sigma}})\right]$$

$$\int_{\frac{a}{2}}^{\infty} u_3(x,t=1)dx = \left[\sigma b(\frac{1}{2}-k\hat{u})(1-e^{\frac{-a}{\sigma}})\right] + \left[\frac{\sigma kb^2}{2}(1+\hat{u})\right]\left[\frac{1}{2}(1+e^{\frac{-2a}{\sigma}}) - e^{\frac{-a}{\sigma}}\right]$$
$$-\left[\frac{\sigma kb^3}{4}\right]\left[\frac{1}{3}(1-e^{\frac{-3a}{\sigma}}) + e^{\frac{-2a}{\sigma}} - e^{\frac{-a}{2\sigma}}\right]$$

$$\theta_1 = 2\left[\int_0^{\frac{a}{2}} u_2(x,t=1)dx + \int_{\frac{a}{2}}^{\infty} u_3(x,t=1)dx\right]$$

$$\boldsymbol{\theta_1} = \left[ab\right]\left[2k(b-\hat{u})(1-b)+1\right] + \left[\sigma b^2 k\right]\left[\frac{25b}{6} - 3(1+\hat{u})\right]$$
$$+\left[e^{\frac{-a}{\sigma}}\right]\left[b^2k\right]\left[a(\hat{u}-3b+1) + \sigma(3(1+\hat{u}) - \frac{9b}{2})\right] + \left[e^{\frac{-2a}{\sigma}}\right]\left[\frac{5\sigma b^3 k}{2}\right]$$
$$+\left[e^{\frac{-3a}{\sigma}}\right]\left[\frac{\sigma b^3 k}{3}\right] + \left[e^{\frac{-a}{2\sigma}}\right]\left[\frac{\sigma b^3 k}{2}\right]$$