Toric Geometry (Lent 2022)

1) Tori
2) Toric Varieties
3) Cones and fans
4) Binomial eqns and semigroups
5) Smoothness
6) Gluing
7) orbit-cone correspondence
8) Properness
9) Toric morphisms
10) Surface Singularities.
11) Divisors and line bundles
12) Linear systems and Projectivity.
13) Quotients and homogeneous co-ordinates.

~~14) Singular cohomology~~

14) Topology
15) Where do we go from here?

Toric Geometry Lectures

$\boxed{\text{L1 Start}}$

- Algebraic geometry is about a correspondence

  geometry (algebraic varieties and schemes) $\longleftrightarrow$ algebra (rings, polynomials).

- Works very well locally:

$$\begin{pmatrix}\text{affine}\\\text{schemes}\end{pmatrix} \cong \big(\text{rings}\big)$$

- But globally, gets tricky. Hard to calculate. Many unanswered questions.

- Toric geometry is about toric varieties.

  $\downarrow$

  A special class of varieties.

  Simpler than "arbitrary" varieties.

  But still very interesting, varied, beautiful

  $\downarrow$

  Excellent testing ground.

  Good source of examples to build intuition.

- we will see:

$$\begin{pmatrix}\text{toric}\\ \text{varieties}\end{pmatrix} \cong \begin{pmatrix}\text{rational polyhedral}\\ \text{fans}\end{pmatrix}$$

combinatorics, linear algebra.

~~§1 What makes a toric variety?~~

§1: Tori

- Def^n: Algebraic torus is:

$$(\mathbb{C}^\times)^n \qquad (\text{"complexification of } (S^1)^n\text{"})$$

- Is an algebraic (in fact affine) variety

$$(\mathbb{C}^\times)^n = \{z_0 \cdots z_n = 1\} \subseteq \mathbb{A}^{n+1}_{z_0 \ldots z_n}.$$

- In fact, have:

$$(\mathbb{C}^\times)^n_{t_1 \ldots t_n} = \operatorname{Spec} \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}].$$

How to see this?

1) $$\frac{\mathbb{C}[z_0, \ldots, z_n]}{(z_0 \cdots z_n = 1)} \xrightarrow{\cong} \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$$

$$\begin{aligned}
z_0 &\longmapsto t_1^{-1} \cdots t_n^{-1} \\
z_1 &\longmapsto t_1 \\
&\ \vdots \\
z_n &\longmapsto t_n
\end{aligned}$$

2) $(\mathbb{C}^\times)^n \subseteq \mathbb{C}^n$ complement of co-ordinate hyperplanes:

$$(\mathbb{C}^\times)^n = \mathbb{C}^n \setminus \bigcup_{i=1}^{n} V(t_i)$$

So have to localise co-ordinate ring by inverting $t_1, \dots, t_n$.

• Will usually write $T = (\mathbb{C}^\times)^n$.

~~**Defn:** A toric variety is a variety X containing a dense open torus~~

~~$T \subseteq X$~~

~~together with an action~~

~~$T \times X \longrightarrow X$~~

~~which extends the multiplic~~

• Algebraic tori are group schemes, i.e. there is a multiplication:

$$T \times T \longrightarrow T$$

which is also a morphism of varieties.

- This is honestly just:

$$(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$$
$$((t_1,\dots,t_n),(s_1,\dots,s_n)) \longmapsto (t_1 s_1, \dots, t_n s_n)$$

- Corresponds to <u>comultiplication</u>:

$$\mathbb{C}[t_1^{\pm},\dots,t_n^{\pm}] \longrightarrow \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}] \otimes_{\mathbb{C}} \mathbb{C}[y_1^{\pm},\dots,y_n^{\pm}]$$
$$t_i \longmapsto x_i \otimes y_i$$

- Tori very nice: simple and well-behaved

In fact: closed subvariety of an algebraic torus is called <u>very affine</u>.

§2: Toric varieties

- Toric varieties are <u>equivariant enlargements</u> of tori.

- Defn: A <u>toric variety</u> is a variety X together with an open dense torus $T \subseteq X$ and an <u>action</u>:

$$T \times X \longrightarrow X$$

which ~~extends to the~~ restricts to T to give the multiplication:

$$T \times T \longrightarrow T.$$

- E.g.: $(\mathbb{C}^\times)^n$

  $\mathbb{C}^n$

  Products thereof.

- E.g.: $\mathbb{P}^n_{z_0 \dots z_n}$

  - Torus is:

$$T = \{[z_0 : \dots : z_n] : z_i \neq 0 \ \forall i\}$$
$$= (\mathbb{C}^\times)^{n+1} / \mathbb{C}^\times \cong (\mathbb{C}^\times)^n.$$

  - Action is:

$$\underset{\in T}{[t_0 : \dots : t_n]} \cdot \underset{\in \mathbb{P}^n}{[z_0 : \dots : z_n]} = \underset{\in \mathbb{P}^n}{[t_0 z_0 : \dots : t_n z_n]}$$

Well-defined because all $t_i \neq 0$.

- If you want you can make this more explicit by breaking symmetry, e.g.

$$T = (\mathbb{C}^\times)^{n+1}_{t_0 \cdots t_n} / \mathbb{C}^\times$$

$$\cong (\mathbb{C}^\times)^n_{\frac{t_1}{t_0} \cdots \frac{t_n}{t_0}}$$

$$= (\mathbb{C}^\times)^n_{s_1 \cdots s_n}$$

Then action is:

$$(s_1, \_ \, s_n)[z_0 : z_1 : \cdots : z_n]$$

$$= [z_0 : s_1 z_1 : \cdots : s_n z_n]$$

$$= [t_0 z_0 : t_1 z_1 : \cdots : t_n z_n].$$

- Upshot: toric varieties have a lot of symmetry.

- They all have a dense "interior" which is the same.

Differ along the "boundary"

↳ Q: what's the "boundary" for $\mathbb{P}^n$?

- Note: a toric variety isn't just a variety. It's a package:

$$(X, T \subseteq X, T \times X \to X)$$

often given X, there are many choices for T and for the action.

we will always fix one. we will then drop it from the notation.

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§3: Cones and fans

- Fix $T \cong (\mathbb{C}^\times)^n$. Consider:

$$\boxed{N = \mathrm{Hom}_{GP}(\mathbb{C}^\times, T)}$$ (cocharacters or 1-parameter subgps)

- Turns out: N is a lattice:

$$N \cong \mathbb{Z}^n$$

- Given $u = (u_1, \_, u_n) \in \mathbb{Z}^n = N$, corresponding 1-PS is:

$$\left.\begin{aligned} \lambda_u : \mathbb{C}^\times &\longrightarrow (\mathbb{C}^\times)^n \\ t &\longmapsto (t^{u_1}, \_, t^{u_n}) \end{aligned}\right\}$$

- Now consider a toric variety: $(\mathbb{C}^\times)^n \subseteq X$.
- Given $u \in N$, get: $\mathbb{C}^\times_t \xrightarrow{\lambda_u} (\mathbb{C}^\times)^n \hookrightarrow X$.
- Can ask what $\boxed{t \to 0 \text{ limit in } X}$ is (if exists).
- E.g: $(\mathbb{C}^\times)^2 = \{[t_0, t_1, 1] : t_0 \neq 0, t_1 \neq 0\} \subseteq \mathbb{P}^2_{z_0 z_1 z_2}$.

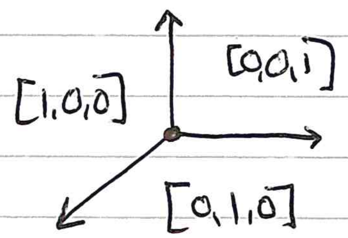
Choose $u = (u_0, u_1) \in \mathbb{Z}^2 = N$. Different cases:

- $u_0, u_1 > 0 \Rightarrow [t^{u_0}, t^{u_1}, 1] \xrightarrow{t \to 0} [0, 0, 1]$.
- $u_0 > 0, u_1 < 0 \Rightarrow [t^{u_0}, t^{u_1}, 1] = [t^{u_0 - u_1}, 1, t^{-u_1}]$
  $\xrightarrow{t \to 0} [0, 1, 0]$.

etc. etc.

↓

Partition $N = \mathbb{Z}^2$ into regions where choice of u gives same limit point. Get:



- This kind of picture has a name: it's a $\boxed{\text{fan}}$ (think of Japanese folding fans).

The fan tells us what we've added in (the limit points) to get from $(\mathbb{C}^\times)^n$ to X.

Turns out: can reconstruct X from its fan.

L1 End

L2 Start

- Defn: A rational polyhedral cone σ in N is a set:

$$\sigma \subseteq N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathbb{R}^n$$

such that there exist $v_1, \dots, v_k \in N$ such that:

$$\sigma = \operatorname{Cone}(v_1, \dots, v_k)$$
$$= \left\{ \sum_{i=1}^{k} c_i \cdot v_i \; : \; c_i \in \mathbb{R}_{\geq 0} \right\}.$$

- Note: $\sigma \subseteq N_{\mathbb{R}}$ but the v_i are lattice points (this is the "rational" part).

- A rational polyhedral cone is strictly convex if it doesn't contain a line.

- From now on by $\boxed{\text{"cone"}}$ we mean strictly convex rational polyhedral cone.

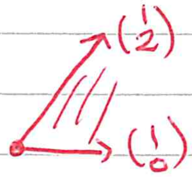
- A fan Σ in N is a collection of cones σ in N s.t.:

(i) if $\sigma \in \Sigma$ then every face of σ belongs to Σ also.

(ii) if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

- How to construct toric varieties from fans?

Start with a single cone σ.

- E.g: $\sigma =$  $\subseteq \mathbb{R}^2$.

- Belief: corresponding toric variety X_σ should be affine.

$\downarrow$

Goal: find ring of functions R on X_σ, so that $X_\sigma = \operatorname{Spec} R$.

- Start with functions on T. There are:

$$\mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}].$$

Want to work out which of these extend to X_σ. Whatever ring R we end up with, it'll be a subring:

$$R \subseteq \mathbb{C}[t_1^{\pm}, \dots, t_n^{\pm}].$$

- Focus just on monomials:

$$t_1^{b_1} \cdots t_n^{b_n} \qquad (b_1, \dots, b_n \in \mathbb{Z}).$$

- These are precisely the group homomorphisms:

$$T = (\mathbb{C}^\times)^n_{t_1 \cdots t_n} \longrightarrow \mathbb{C}^\times.$$

- They constitute the character lattice:

$$M = \mathrm{Hom}_{GP}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n$$

- M and N are dual. There's a canonical perfect pairing:

$$\langle \cdot, \cdot \rangle : N \times M \longrightarrow \mathbb{Z}.$$

- Remember the question: when does a given monomial extend from T to X_σ?

- New points in X_σ are precisely the limits of 1-PS:

$$\lim_{t \to 0} \lambda_u(t) \quad \text{for } u \in \sigma.$$

- Given monomial $\underline{t^m}$ for $m \in M$, have:

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\lambda_u"] \arrow[rr, bend right, "t \mapsto t^d"'] & T \arrow[r, "t^m"] & \mathbb{C}^\times \end{tikzcd}$$

where $d = \langle u, m \rangle$.

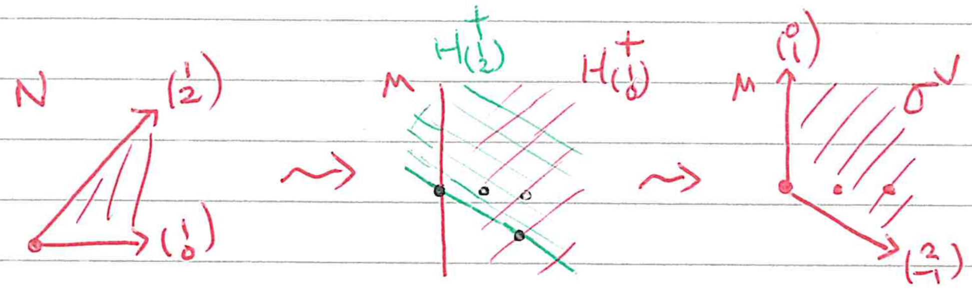
- $\Rightarrow$ function is well-defined (i.e. doesn't blow up to ∞) as $t \to 0$ iff:

$$\boxed{\langle u, m \rangle \geq 0}$$

- Conclusion: Monomials on T which extend to X_σ are precisely those contained in the dual cone:

$$\sigma^\vee = \{ m \in M_{\mathbb{R}} : \langle u, m \rangle \geq 0 \text{ for all } u \text{ in } \sigma \}$$
$$\subseteq M_{\mathbb{R}}.$$

- E.g.:



- $\sigma^\vee \subseteq M_{\mathbb{R}}$.

  Get monomials by picking out lattice points:

  $$\boxed{S_\sigma := \sigma^\vee \cap M}$$

- S_σ is a semigroup (or monoid): almost an abelian group, except inverses don't necessarily exist.

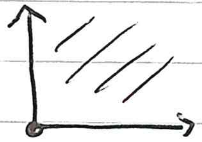
- This is our set of monomials.

  To get the whole ring, i.e. polynomials, take $\mathbb{C}$-linear combinations:

  $$\boxed{\mathbb{C}[S_\sigma] = \left\{ \sum_{i=1}^{r} c_i \cdot z^{m_i} : c_i \in \mathbb{C},\ m_i \in S_\sigma \right\}}$$

  Addition: formal.

  Multiplication: $z^{m_1} \cdot z^{m_2} = z^{m_1 + m_2}$

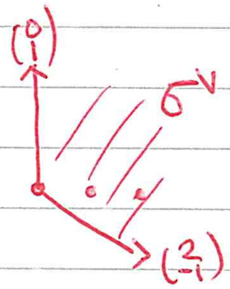
- **E.g.:** $\sigma^\vee =$  $= \mathbb{R}^2_{\geq 0} \subseteq \mathbb{R}^2 = M$.

  $\Rightarrow S_\sigma = \mathbb{N}^2$ generated by $\binom{1}{0}, \binom{0}{1}$.

  $\Rightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^2] = \mathbb{C}[x, y]$

  $\left(x = z^{\binom{1}{0}}, \ y = z^{\binom{0}{1}}\right)$

- **E.g.:** $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[t^{\pm}]$.

- **E.g.:** 

  $\Rightarrow S_\sigma$ generated by $\underset{a}{\binom{0}{1}}, \underset{b}{\binom{2}{-1}}, \underset{c}{\binom{1}{0}}$.

  Relation: $a + b = 2c$.

  $\Rightarrow S_\sigma = \mathbb{N}^3_{abc} / (a + b = 2c)$

  $\Rightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[x, y, z] / (xy = z^2)$.

- **Summary:**

  $\sigma \subseteq N_\mathbb{R} \rightsquigarrow \sigma^\vee \subseteq M_\mathbb{R}$

  $\rightsquigarrow S_\sigma = \sigma^\vee \cap M$ (semigroup of monomials which extend)

  $\rightsquigarrow X_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$.

$\boxed{\text{L2 end}}$

L3 Start

- How to see X_σ is a toric variety?

$$\downarrow$$

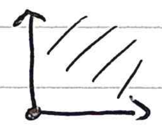
$S_\sigma \subseteq M$ (inclusion of semigroups).

$$\Rightarrow \mathbb{C}[S_\sigma] \subseteq \mathbb{C}[M]$$

$\uparrow$ localisation of $\mathbb{C}[S_\sigma]$.

$$\Rightarrow \boxed{\begin{matrix} \operatorname{Spec}\mathbb{C}[M] \\ \| \\ T \end{matrix}} \underset{\text{open}}{\subseteq} \begin{matrix} \operatorname{Spec}\mathbb{C}[S_\sigma] \\ \| \\ X_\sigma \end{matrix}$$

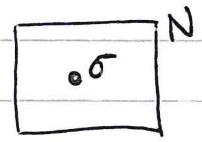
- __Exercise:__ Check how to define the action $T \times X_\sigma \to X_\sigma$.

- __E.g:__ $\sigma =$  $\Rightarrow X_\sigma = \mathbb{A}^2 = \mathbb{C}^2$

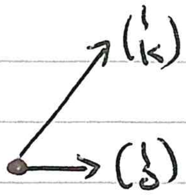
- __E.g:__ $\sigma = \mathbb{R}^n_{\geq 0} = \operatorname{Cone}\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right)$

$$\Rightarrow X_\sigma = \mathbb{A}^n = \mathbb{C}^n$$

- __E.g:__ $\sigma = 0 \subseteq \mathbb{R}^2 = N_\mathbb{R}$



$$\Rightarrow \sigma^\vee = M_\mathbb{R} \Rightarrow X_\sigma = (\mathbb{C}^\times)^2$$

- **Exercise**: $\sigma =$ 

$(k \geq 2)$

  Show that $X_\sigma = V(xy - z^k) \subseteq \mathbb{A}^3_{xyz}$.

- We have outlined a process:

$$\boxed{\sigma \text{ cone} \rightsquigarrow X_\sigma \text{ affine toric variety}}$$

- Q: Do all affine toric varieties arise in this way?

  A: All normal affine toric varieties arise in this way.

- Usually we are only interested in normal toric varieties, so the moral is that everything corresponds to a cone.

- To understand normality, need to detour and discuss one more way of producing toric varieties.

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§4: Binomial eqns and semigroups

- Start with $\mathbb{C}^n_{x_1 \ldots x_n} = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$.

- A monomial is a function of the form:

  $x_1^{k_1} \cdots x_n^{k_n}$ (depends on choice of co-ordinates $x_1, \ldots, x_n$.)

- A monomial ~~subscheme~~ ideal is an ideal generated by monomials:

$$I = (x_1^{k_1^1} \cdots x_n^{k_n^1}, \ldots, x_1^{k_1^d} \cdots x_n^{k_n^d})$$

$$\lhd \mathbb{C}[x_1, \ldots, x_n].$$

- A monomial subscheme is a subscheme defined by a monomial ideal:

$$X = V(I) = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/I.$$

- Is a union of co-ordinate subspaces (possibly non-reduced).

Interesting, but not what we're focusing on.

- A binomial is an equation involving two monomials:

$$x_1^{k_1} \cdots x_n^{k_n} = x_1^{l_1} \cdots x_n^{l_n}$$

$$\Leftrightarrow \quad x_1^{k_1} \cdots x_n^{k_n} - x_1^{l_1} \cdots x_n^{l_n} = 0$$

- A binomial subscheme is one whose ideal is generated by binomials.

- **Claim:** The co-ordinate ring of a binomial subscheme is isomorphic to $\mathbb{C}[S]$ for some semigroup S. (*)

- E.g.: $X = V(x^2 - y^3) \subseteq \mathbb{A}^2_{xy}$.

$$\Rightarrow \mathcal{O}_X = \mathbb{C}[x,y] / (x^2 - y^3).$$

Define $S = \mathbb{N}^2_{ab} / (2a = 3b)$

$$\Rightarrow \mathbb{C}[S] = \mathbb{C}[x,y] / (x^2 = y^3).$$

E.g.: $X = V(x^2 - y^2)$

$$S = \mathbb{N}^2_{ab} / (2a = 2b).$$

- Proof of claim: Given a binomial equation:

$$x_1^{k_1} \cdots x_n^{k_n} = x_1^{l_1} \cdots x_n^{l_n}$$

Take the "logarithm" to get:

$$(*) \quad k_1 a_1 + \cdots + k_n a_n = l_1 a_1 + \cdots + l_n a_n$$

where "$a_i = \log x_i$". Then let:

$$S = \mathbb{N}^n_{a_1 \cdots a_n} / (*). \qquad \square.$$

- Quick formulation:

$$\boxed{\mathbb{C}[\mathbb{N}^n]/I = \mathbb{C}[\mathbb{N}^n / \log I].}$$

- Notice monomials make sense in $\mathbb{C}[S]$. They're the image of $S \to \mathbb{C}[S]$.

  In $\mathbb{C}[\mathbb{N}^n]/I = \mathbb{C}[x_1, \dots, x_n]/I$, declare element to be monomial if one ~~(equivalently, any)~~ lift to $\mathbb{C}[x_1, \dots, x_n]$ is monomial. wrong!

- Conversely, if S is any finitely-generated semigroup then:

  $$S = \mathbb{N}^n / J$$

  binomial ideal.

  $$\Rightarrow \mathbb{C}[S] = \mathbb{C}[\mathbb{N}^n] / \exp J$$

  $\boxed{\text{So a scheme is binomial iff it's } \operatorname{Spec} \mathbb{C}[S] \text{ for } S \text{ a finitely-generated semigroup.}}$

- ~~Q: Is every semigroup S of the form $\mathbb{N}^n/J$ for [illegible] in $\mathbb{N}^n$?~~ ~~Answer~~

- With enough adjectives, we will see:

  $$\begin{pmatrix} \text{affine toric} \\ \text{varieties} \end{pmatrix} = \begin{pmatrix} \text{binomial} \\ \text{schemes} \end{pmatrix}.$$

L3 end

Aside on monomials

Def^n: consider a binomial co-ordinate ring:

$$\mathbb{C}[x_1,\ldots,x_n] \xrightarrow{\pi} \mathbb{C}[x_1,\ldots,x_n]/I$$

binomial ideal.

Then the monomials in $\mathbb{C}[x_1,\ldots,x_n]/I$ are precisely the images of monomials in $\mathbb{C}[x_1,\ldots,x_n]$.

I.e. $f \in \mathbb{C}[x_1,\ldots,x_n]/I$ is a monomial iff there exists a monomial $g \in \mathbb{C}[x_1,\ldots,x_n]$ such that $\pi(g) = f$.

Note: writing the binomial ring as:

$$\mathbb{C}[x_1,\ldots,x_n]/I = \mathbb{C}[\mathbb{N}^n/\log I].$$

The monomials are precisely the functions in the image of:

$$\mathbb{N}^n/\log I \subseteq \mathbb{C}[\mathbb{N}^n/\log I].$$

indexing set for monomials.

Note: The above definition works for any ring presented as:

$$\mathbb{C}[x_1,\ldots,x_n] \longrightarrow \mathbb{C}[x_1,\ldots,x_n]/I$$

I.e. I doesn't have to be binomial

What's special about binomial ideals is that you have a good indexing set, namely $\mathbb{N}^n/\log I$. So you know exactly when two monomials in $\mathbb{C}[\mathbb{N}^n]$ will map to the same function in $\mathbb{C}[\mathbb{N}^n/\log I]$.

E.g.: $R = \mathbb{C}[x,y,z]/(x^2+y-z^2, x^2-y-z^2)$.

The ideal is not binomial, but it contains binomial equations, e.g.:

$$(x^2+y-z^2)+(x^2-y-z^2) = 2x^2-2z^2$$

So $x^2=z^2$ in R. But we had no way to know this a priori, just by eyeballing the ideal!

[End of aside].

- S finitely-generated iff:
  $S = \mathbb{N}^n / J$
  for J some set of relations.

L4 start

- Given a semigroup S, the groupification is:
  $S^{gp} = \{ a-b : a,b \in S \} / \sim$
- If $S = \mathbb{N}^n/J$, then $S^{gp} = \mathbb{Z}^n/J$.
- There is always a natural map: $S \to S^{gp}$ satisfying universal property.

- Def: S integral iff $S \to S^{gp}$ is injective.

Eg: $S = \mathbb{N}^2 / \left(\binom{1}{1} = \binom{0}{0} \right)$ not integral.

- Defn: S torsion-free iff S^{gp} torsion-free, i.e. $S^{gp} \cong \mathbb{Z}^n$ for some n.

- Defn: S affine iff S finitely-generated, integral, torsion-free.

- Lemma: S affine $\iff S = \mathbb{N}m_1 + \dots + \mathbb{N}m_r$ for $m_1, \dots, m_r \in M$ some lattice

- ~~Eg: $S = \mathbb{N}^2/\left(\binom{1}{1} = \binom{1}{0}\right)$ not integral.~~

- ~~Def^n: S is torsion-free if S^{gp} is torsion-free, i.e. $S^{gp} \cong \mathbb{Z}^n$ for some n.~~

- Theorem: Affine toric varieties are (*) precisely $\operatorname{Spec} \mathbb{C}[S]$ for S affine. ~~finitely-generated, integral, torsion-free~~

  Proof: Let $M = S^{gp}$. Have $S \subseteq M$ so:

$$T = \operatorname{Spec} \mathbb{C}[M] \hookrightarrow \operatorname{Spec} \mathbb{C}[S]$$

So $\operatorname{Spec} \mathbb{C}[S]$ is an affine toric variety.

For the other direction, full proof is in [CLS, Th^m 1.1.17], but here's the idea.

Let $M = \operatorname{Hom}_{gp}(T, \mathbb{C}^\times)$ so that $T = \operatorname{Spec} \mathbb{C}[M]$. The inclusion $T \subseteq X$ induces an inclusion:

$$\mathcal{O}_X \subseteq \mathbb{C}[M].$$

Notice there are well-defined monomials z^m in $\mathbb{C}[M]$. Since $T \curvearrowright \mathcal{O}_X$ it follows (via some lemmas) that:

$$\mathcal{O}_X = \mathbb{C}[S] \text{ where } S = \{m \in M : \chi^m \in \mathcal{O}_X\}$$

$\square$.

- Let's recap. We have 3 competing definition of affine toric variety:

(affine variety $X \supseteq T$ with action $T \times X \to X$) ①	$\Leftrightarrow$ proved above	(Spec $\mathbb{C}[S]$ for S ~~f.g. integral torsion-free semigroup~~ affine semigroup) ②	$\overset{?}{\Leftrightarrow}$	(Spec $\mathbb{C}[\sigma^\vee \cap M]$ for σ cone in N) ③

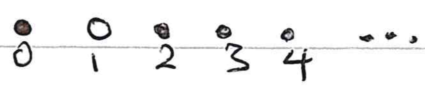
- How are ② and ③ related?
- Certainly $S_\sigma = \sigma^\vee \cap M$ is f.g., integral, torsion-free. (affine).

  So ③ is a special case of ②.

- Is there anything in ② which isn't in ③?

  A: Yes. Consider:

  $$S = \mathbb{N} \setminus \{1\} = 2\mathbb{N} + 3\mathbb{N} \subseteq \mathbb{N}.$$



  Then $S \neq \sigma^\vee \cap M$ for any σ.

But still S affine

- Conclusion: ② is slightly more general than ③:

  ① = ② $\supseteq$ ③ ← more special.

- But usually we restrict to toric varieties in ③.

  In fact, these have a characterisation.

- **Defn**: $S \subseteq S^{gp} = M$ is saturated iff:

$$\left.\begin{array}{l} m \in M \\ km \in S \quad (k \in \mathbb{N}, k \geq 1) \end{array}\right\} \Rightarrow m \in S.$$

- **Theorem**: Let S be an affine semigroup. Then the following are equivalent:

  (i) S is saturated.

  (ii) $S = S_\sigma = \sigma^\vee \cap M$ for σ a strictly convex rational polyhedral cone in $N = \mathrm{Hom}(S^{gp}, \mathbb{Z})$.

Proof: cf. [CLS, Thm 1.3.5 (b)$\Rightarrow$(c)] $\square$

- We saw every affine toric variety was Spec $\mathbb{C}[S]$.

What restriction occurs when we require S to be saturated? L4 End

L5 Start

- <u>Def^n</u>: An integral domain R is normal iff it is integrally closed in its field of fractions.

This means: if $f \in \operatorname{Frac} R$ is a root of a monic polynomial with coefficients in R, then we must have $f \in R$.

- <u>E.g.</u>: $R = \mathbb{Z}$, $\operatorname{Frac} R = \mathbb{Q}$.

- An affine scheme Spec$R$ is normal iff R is a normal integral domain.

- Think of as a local condition. It's some restriction on "how bad" the singularities are. All smooth schemes are normal. The slogan is:

"normal" = "smooth in codimension 1".

- ~~Lemma: Let S be f.g., torsion-free, integral. Then $\mathbb{C}[S]$ normal iff S saturated.~~

- Smooth in codimension 1

$$\Updownarrow$$

Codimension of $\operatorname{Sing} X \subseteq X$ is ≥ 2.

$$\Updownarrow$$

X smooth at the generic point of every hypersurface $Y \subseteq X$.

- cf. Serre's criterion for normality, which formalises this.

- Lemma: Let S be an ~~finitely generated torsion-free integral semigroup~~ affine semigroup. Then $\mathbb{C}[S]$ is normal iff S is saturated.
(one direction is a little tricky: cf. [CLS, Thm 1.3.5])

- Corollary: we have equalities:

(affine normal toric varieties)	=	($X_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ for σ a cone)	=	($\operatorname{Spec} \mathbb{C}[S]$ for S affine and saturated)
1'		3'		2'

KEY CORRESPONDENCE

- Note: From now on, when I say "toric variety" I mean "normal toric variety" unless otherwise stated.

~~§2: Local Properties of Toric Varieties~~

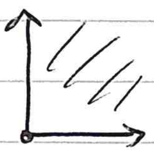
§5: Smoothness

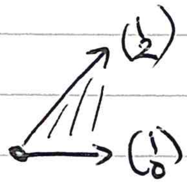
- From now on, we study normal toric varieties, i.e. those of the form

$$X_\Sigma$$

for Σ a fan in some lattice N.

- we've seen examples of smooth and non-smooth toric varieties:

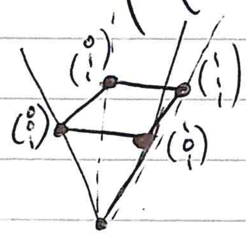
 $\rightsquigarrow \mathbb{C}^2$ (smooth)



$\rightsquigarrow V(xy-z^2) \subseteq \mathbb{C}^3_{xyz}$ (singular)

Another example:

$$\sigma = \operatorname{Cone}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$=$ 

Then $S_\sigma = \mathbb{N}^4/(e_1+e_2=e_3+e_4)$.

(cf. [Fulton, pp. 17-18] which uses co-ordinates better suited for computing.)

$\Rightarrow X_\sigma = V(x_1x_2 - x_3x_4) \subseteq \mathbb{C}^4_{x_1x_2x_3x_4}$.

- Lemma: Let σ be a cone in N:

  $\sigma = \text{Cone}(v_1, \dots, v_k)$.

  Suppose that $\{v_1, \dots, v_k\}$ forms part of a $\mathbb{Z}$-basis for N. Then:

  $X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$.

- Proof: We can choose a $\mathbb{Z}$-basis $e_1, \dots, e_n$ for N such that $e_1 = v_1, \dots, e_k = v_k$.

  Then $S_\sigma = \sigma^\vee \cap M$ is generated by:

  $e_1^*, \dots, e_k^*, \pm e_{k+1}^*, \dots, \pm e_n^*$

$$\Rightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[t_1, \dots, t_k, t_{k+1}^{\pm}, \dots, t_n^{\pm}]$$

$$= \mathbb{C}[t_1, \dots, t_k] \otimes_{\mathbb{C}} \mathbb{C}[t_{k+1}^{\pm}, \dots, t_n^{\pm}]. \quad \square$$

- Turns out: these are the only smooth affine toric varieties.

- **Theorem**: The following are equivalent:

  (1) X_σ is smooth

  (2) σ is generated by part of a $\mathbb{Z}$-basis for N.

  (3) $X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$ (where $k = \dim\sigma$).

- **Proof**: (2) $\Rightarrow$ (3) is the previous lemma, and (3) $\Rightarrow$ (1) is obvious.

  So remains to show (1) $\Rightarrow$ (2). For this, we need to pause and develop a bit more theory.

  [Proof paused.]

- **Lemma**: There is a bijective correspondence

$$\left\{\begin{array}{c}\text{closed points}\\ \text{of } X_\sigma\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{semigroup homomorphisms}\\ S_\sigma \to \mathbb{C}\end{array}\right\}$$

semigroup under multiplication

- **Proof**: we have:

$$\left\{\begin{array}{c}\text{closed pts}\\ \text{on } X_\sigma\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{scheme morphisms}\\ \operatorname{Spec}\mathbb{C} \to X_\sigma = \operatorname{Spec}\mathbb{C}[S_\sigma]\end{array}\right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{ring homomorphisms} \\ \mathbb{C}[S_\sigma] \to \mathbb{C} \end{array} \right\}$$

Exercise

$$\longleftrightarrow \left\{ \begin{array}{c} \text{semigroup homomorphisms} \\ S_\sigma \to \mathbb{C} \end{array} \right\}. \qquad \square$$

- Every affine toric variety comes equipped with a favourite point (like the origin in affine space).

- Defn: Let $x_\sigma \in X_\sigma$ be the closed point corresponding to the semigroup homomorphism:

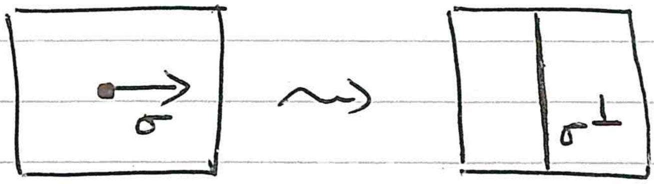
$$S_\sigma \longrightarrow \mathbb{C}$$

$$m \longmapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

- Note: Here $\sigma^\perp$ ("σ perp") is:

$$\sigma^\perp := \{ m \in M_\mathbb{R} : \langle u, m \rangle = 0 \ \forall u \in \sigma \}.$$

If σ is full-dimensional, $\sigma^\perp = 0$. Otherwise it's dual to the linear subspace generated by σ, e.g:



- **Lemma:** If σ is full-dimensional (so $\sigma^\perp = 0$) then $x_\sigma \in X_\sigma$ is the unique point fixed by the action of the torus T.

Proof: Exercise. $\square$

- One more thing we need before getting back to the proof.

- **Lemma:** we have:

  σ full-dimensional $\iff$ $\sigma^\vee$ strictly convex.

Proof: Exercise.

(see Fulton §1.2 for ideas) $\square$

L5 end

L6 start

- **Proof resumed:** we want to show:

  X_σ smooth $\implies$ σ generated by part of a $\mathbb{Z}$-basis for N.

First assume that σ is full-dimensional, so that $\sigma^\perp = 0$.

Let $M \lhd \mathbb{C}[S_\sigma]$ be the maximal ideal corresponding to $x_\sigma \in X_\sigma$, so:

$$M = \ker(\mathbb{C}[S_\sigma] \longrightarrow \mathbb{C})$$

Then M is generated by z^w for $w \in S_\sigma \setminus \{0\}$.

The Zariski cotangent space of X_σ at x_σ is M/M^2:

$$\Omega_{X_\sigma, x_\sigma} = M/M^2$$

And X_σ is smooth at x_σ iff $\dim_{\mathbb{C}}(M/M^2) = n$.

Now, M^2 is generated by monomials of the form $z^{\omega_1+\omega_2}$ where $\omega_1, \omega_2 \in S_\sigma \setminus \{0\}$.

Hence, M/M^2 is generated by monomials z^ω where ω cannot be written as a sum of two elements of $S_\sigma \setminus \{0\}$.

This is the case in particular for the generators of the rays of σ:

Since σ is full-dimensional, $\sigma^\vee$ is strictly convex. Hence (since $\sigma^\vee$ is also full-dimensional), there are at least n generators of rays of $\sigma^\vee$:

$w_1, \dots, w_n$

Since $\dim_{\mathbb{C}}(M/M^2) = n$ by the smoothness assumption, we conclude that these in fact generate S_σ:

$$S_\sigma = \mathbb{N}w_1 + \dots + \mathbb{N}w_n$$
$$= \{c_1 w_1 + \dots + c_n w_n : c_i \in \mathbb{N}\} \subseteq M$$

Since S_σ generates M as a group, it follows that $w_1, \dots, w_n$ is a $\mathbb{Z}$-basis for M.

Dualising, we find that the dual basis for N generates σ.

This completes the proof when σ is full-dimensional. The general case can be reduced to this case, and is left as an exercise. $\square$

- Corollary (of proof): X_σ is smooth/singular iff it's smooth/singular at x_σ.

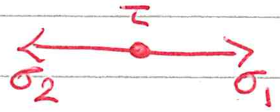
---

§6: Gluing

- So far, have focused on affine toric varieties.

They're affine because they come from a single cone.

General toric varieties are defined from fans by gluing together the affine toric varieties associated to their cones.

- E.g: $\Sigma =$ 

$\subseteq \mathbb{R} = N_{\mathbb{R}}$

- Note: Let $\tau \leq \sigma$ be a face. Then:

$\tau \leq \sigma \Rightarrow \sigma^\vee \subseteq \tau^\vee$

$\Rightarrow S_\sigma \subseteq S_\tau$

$\Rightarrow \mathbb{C}[S_\sigma] \subseteq \mathbb{C}[S_\tau]$

$\Rightarrow \operatorname{Spec} \mathbb{C}[S_\tau] \to \operatorname{Spec} \mathbb{C}[S_\sigma]$

- **Lemma:** Given $\tau \leq \sigma$ a face, the induced morphism of affine toric varieties:

$$X_\tau = \operatorname{Spec} \mathbb{C}[S_\tau] \longrightarrow \operatorname{Spec} \mathbb{C}[S_\sigma] = X_\sigma$$

is an open embedding.

- E.g.: $\tau = 0 \leq \mathbb{R}_{\geq 0} = \sigma_1$

$\Rightarrow \tau^\vee = M_\mathbb{R} = \mathbb{R}, \ \sigma_1^\vee = \mathbb{R}_{\geq 0}$

$\Rightarrow S_\tau = \mathbb{C}[t, t^{-1}] \supseteq S_{\sigma_1} = \mathbb{C}[t]$

$\Rightarrow X_\tau = \mathbb{C}_t^\times \subseteq \mathbb{C}_t = X_{\sigma_1}$

Proof of Lemma: We want to show that $\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[S_\tau]$ is a localisation, i.e.:

$$\mathbb{C}[S_\tau] = f^{-1} \cdot \mathbb{C}[S_\sigma] \quad \text{for some } f \in \mathbb{C}[S_\sigma].$$

Since $\tau \leq \sigma$ is a face, it can be obtained from σ by slicing with a supporting hyperplane.

A supporting hyperplane $H \subseteq N_\mathbb{R}$ is any hyperplane such that

σ lies entirely on one side.

Any $m \in \sigma^\vee \cap M$ defines a separating hyperplane:

$$m^\perp = \{ u \in N_{\mathbb{R}} : \langle u, m \rangle = 0 \}.$$

Choose any m_τ in the interior of $\sigma^\vee \cap \tau^\perp$. Then:

$$\boxed{\tau = \sigma \cap m_\tau^\perp} \qquad \text{(Exercise.)}$$

Claim: $S_\tau = S_\sigma + \mathbb{N} \cdot (-m_\tau)$

Proof: see [CLS, Prop 1.3.16] or [Fulton §1.3]

Do some examples to convince yourself. □.

Corollary of the claim is that $\mathbb{C}[S_\tau]$ is obtained from $\mathbb{C}[S_\sigma]$ by localising z^{m_τ}:

$$\mathbb{C}[S_\tau] = z^{-m_\tau} \cdot \mathbb{C}[S_\sigma].$$

It follows that $X_\sigma \to X_\tau$ is an open embedding:

$$X_\tau = X_\sigma \setminus V(z^{m_\tau}) \qquad \text{(Principal open set).} \quad \square$$

- <u>Construction of toric varieties</u>: Take a fan Σ in $N_{\mathbb{R}}$.

- Given cones $\sigma_1, \sigma_2 \in \Sigma$ which meet along a face τ, we have:

$$X_\tau \begin{array}{l} \overset{\text{open}}{\subseteq} X_{\sigma_1} \\ \underset{\text{open}}{\subseteq} X_{\sigma_2} \end{array}$$

- So we glue X_{σ_1} and X_{σ_2} along the common open set X_τ.

  Do this for all cones in Σ ("<u>colimit</u> of diagram of schemes").

  Get a variety X_Σ. L6 End

L7 Start

- Each $\sigma \in \Sigma$ contains 0 as a face, and $X_0 = T$. So have dense embedding:

  $T \subseteq X_\Sigma$.

  Action well-defined on each X_σ and compatible with face inclusions, hence glues to give action on X_Σ.

  So X_Σ is a toric variety $\square$

L7 Start

- **Theorem**: Every (normal, separated) toric variety comes from a fan:

$$\begin{pmatrix} \text{normal} \\ \text{separated} \\ \text{toric} \\ \text{varieties} \end{pmatrix} = \begin{pmatrix} \text{toric} \\ \text{varieties} \\ X_\Sigma \\ \text{for } \Sigma \text{ fan} \end{pmatrix}.$$

Proof: Detailed. Chase [CLS, Thm 3.1.7]. $\square$

- **E.g.**: $\sigma_2 \longleftarrow \overset{\tau}{\bullet} \longrightarrow \sigma_1$

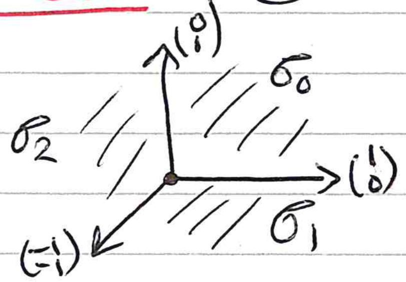
$$\begin{array}{ccccc} \mathbb{C}[S_{\sigma_2}] & \subseteq & \mathbb{C}[S_\tau] & \supseteq & \mathbb{C}[S_{\sigma_1}] \\ \| & & \| & & \| \\ \mathbb{C}[t^{-1}] & \subseteq & \mathbb{C}[t^{\pm 1}] & \supseteq & \mathbb{C}[t] \end{array}$$

$\Rightarrow$ Glue two copies of $\mathbb{C}$ along $\mathbb{C}^\times$, via $y = x^{-1}$

$\Rightarrow X_\Sigma = \mathbb{P}^1$.

~~Exercise:~~

- **Exercise**: Consider the fan:

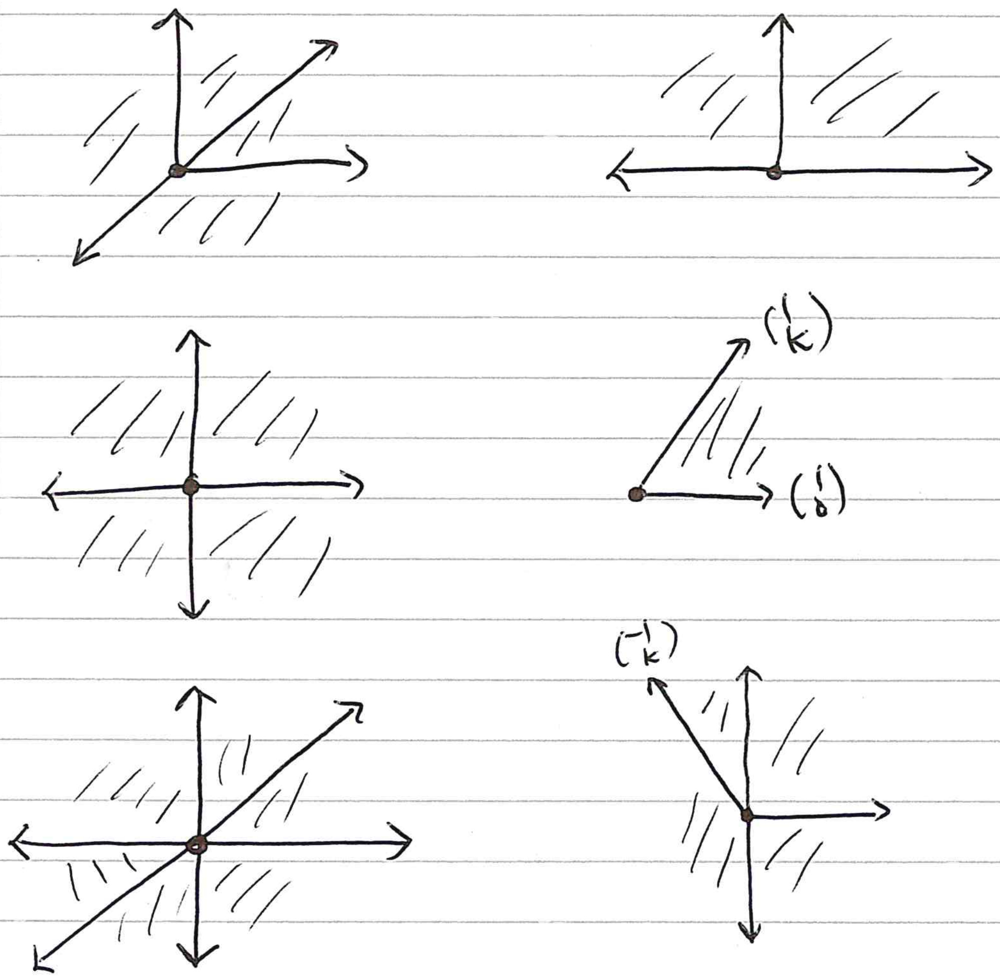


Show $X_\Sigma = \mathbb{P}^2$ with each $X_{\sigma_i} \cong \mathbb{C}^2$ giving one of the principal open sets

I <u>need</u> you to do this exercise.

If you're struggling, look at §1.1 of Fulton.

- More examples of fans:



By the end of this course, you should be able to look at these and <u>immediately</u> have some ideas about what the corresponding toric variety looks like.

§7: orbit-cone correspondence

- Provides a map for navigating the terrain of any toric variety.

- Given a toric variety $X = X_\Sigma$, the action $T \curvearrowright X_\Sigma$ has orbits:

$$T \cdot x = \{t \cdot x : t \in T\} \subseteq X_\Sigma$$

for $x \in X_\Sigma$ fixed. The variety X_Σ decomposes into a union of disjoint orbits. Some orbits are big, others small.

- E.g. $(\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2$. orbits are:
  - $(\mathbb{C}^\times)^2_{x_1 x_2} = \{(x_1, x_2) : x_1 \neq 0, x_2 \neq 0\} \subseteq \mathbb{C}^2_{x_1 x_2}$
  - $\mathbb{C}^\times_{x_2} = \{(0, x_2) : x_2 \neq 0\}$
  - $\mathbb{C}^\times_{x_1} = \{(x_1, 0) : x_1 \neq 0\}$
  - $\mathrm{Pt} = \{(0,0)\}$.

- E.g. More generally, orbits of $(\mathbb{C}^\times)^n \curvearrowright \mathbb{C}^n$ are:

$$\left\{(x_1, \ldots, x_n) : \begin{array}{l} x_i = 0 \text{ for } i \in I \\ x_i \neq 0 \text{ for } i \notin I \end{array}\right\}$$

for $I \subseteq \{1, \ldots, n\}$ any subset. / L7 End

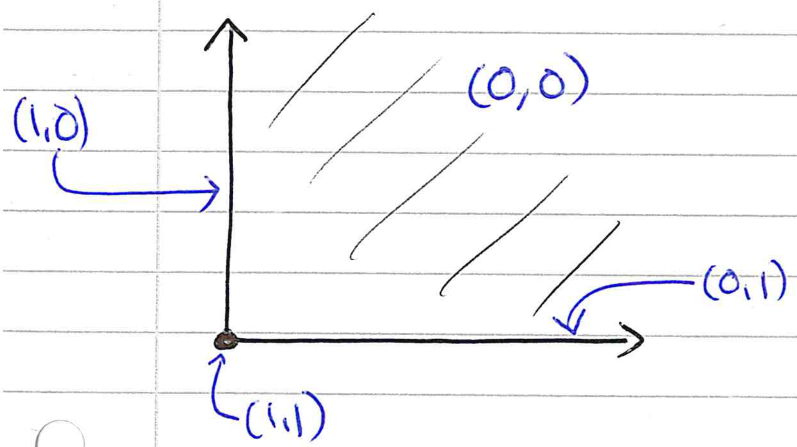
L8 Start

- <u>Theorem 7.1</u>: There is a bijective, size-reversing correspondence:

$$\boxed{\left\{\begin{matrix}\text{orbits of} \\ X_\Sigma\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{cones} \\ \sigma \in \Sigma\end{matrix}\right\}.}$$

- <u>E.g.</u>: Recall how we originally built the fan by considering limits of 1-parameter subgroups.

For $\mathbb{C}^2$ this gave us:



For each limit point x we can consider the orbit:

$$T \cdot x \subseteq \mathbb{C}^2.$$

Notice: all orbits distinct, and we get all of them this way.

- <u>Upshot</u>: Each orbit has a ~~unique~~ distinguished element which appears as the limit of a 1-parameter subgroup.

- <u>Lemma 7.2</u>: Fix $u \in \mathrm{RelInt}(\sigma) \cap N$, and consider the corresponding 1-parameter subgroup $\lambda_u : \mathbb{C}^{\times}_t \to T \subseteq X_\Sigma$. Then:

$$\lim_{t \to 0} \lambda_u(t) = x_\sigma \in \underset{\text{open}}{X_\sigma \subseteq} X_\Sigma$$

<u>Proof</u>: [CLS, Prop 3.2.2] $\square$

- <u>Def^n 7.3</u>: Given $\sigma \in \Sigma$ a cone, the <u>associated orbit</u> in X_Σ is:

$$O_\sigma = T \cdot x_\sigma \subseteq X_\sigma \subseteq X_\Sigma$$

- We will see that $O_\sigma \cong (\mathbb{C}^{\times})^{n - \dim \sigma}$. To get there, need to discuss non-full-dimensional cones more carefully.

- <u>Def^n 7.4</u>: Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone (not-necessarily full-dimensional). We let:

$$N_\sigma \subseteq N$$

denote the sublattice of N generated (as a group) by $\sigma \cap N$. We let:

$$N(\sigma) = N/N_\sigma$$

So we have a short exact sequence of lattices:

$$0 \to N_\sigma \to N \to N(\sigma) \to 0$$
$$N_\sigma \cong \mathbb{Z}^{\dim \sigma}, \quad N \cong \mathbb{Z}^n, \quad N(\sigma) \cong \mathbb{Z}^{n-\dim\sigma}$$

- **Note:** we have $\sigma \subseteq N_{\sigma,\mathbb{R}}$ full-dimensional. Call this σ' to distinguish it from $\sigma \subseteq N_{\mathbb{R}}$.

- **Prop 7.5:** There is a (non-canonical) isomorphism

  $$X_\sigma \cong X_{\sigma'} \times (\mathbb{C}^\times)^{n-\dim\sigma}$$

  Proof: we choose a splitting of the exact sequence above:

  $$N \cong N_\sigma \oplus N(\sigma).$$

  Then $\sigma = \sigma' \times \{0\}$, so:

  $$X_\sigma = X_{\sigma'} \times X_{\{0\}} = X_{\sigma'} \times (\mathbb{C}^\times)^{n-\dim\sigma} \qquad \square$$

~~Corollary: If $x \in X_\sigma$ then~~

- **Corollary 7.6:** Consider the distinguished point $x_\sigma \in X_\sigma$. Then $T \cdot x_\sigma \cong (\mathbb{C}^\times)^{n-\dim\sigma}$

  More invariantly: $T \cdot x_\sigma = N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^\times$

<u>Proof</u>: Identify $X_\sigma = X_{\sigma'} \times (\mathbb{C}^\times)^{n-\dim\sigma}$.
The torus similarly splits.

The first factor is $N_\sigma \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^{\dim\sigma}$ which acts trivially on X_σ, as we proved earlier.

The second factor is $N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^\times = (\mathbb{C}^\times)^{n-\dim\sigma}$ which acts transitively. The claim follows.

Invariantly, we have an exact sequence of tori:

$$0 \to N_\sigma \otimes \mathbb{C}^\times \to T \to N(\sigma) \otimes \mathbb{C}^\times \to 0$$

The subtorus $N_\sigma \otimes \mathbb{C}^\times \subseteq T$ acts trivially on X_σ, hence induces an action of the quotient $N(\sigma) \otimes \mathbb{C}^\times$ which is effective. □

• <u>Note</u>: $O_\sigma = N(\sigma) \otimes \mathbb{C}^\times = \operatorname{Spec} \mathbb{C}[\sigma^\perp \cap M]$.

Thus again, O_σ is a quotient of the big torus T. The inclusion $\sigma^\perp \cap M \subseteq M$ induces the quotient map.

Note $\underbrace{\sigma^\perp \cap M}_{M(\sigma)}$ is dual to $N(\sigma)$.

- Prop 7.7: Every T-orbit in X_Σ is of the form O_σ for a unique $\sigma \in \Sigma$.

  Proof: [CLS, Thm 3.2.6(a)]. □

- Prop 7.8: Given $\sigma, \tau \in \Sigma$ we have:

  $$O_\tau \subseteq \overline{O_\sigma} \iff \sigma \subseteq \tau.$$

  So the orbit-cone correspondence is order-reversing.

- Exercise: check this for $\mathbb{C}^2, \mathbb{P}^2$.

- Note: correspondence useful for studying topology of X_Σ.

---

§7b: Orbit closures

- Each orbit $O_\sigma \subseteq X_\Sigma$ is:

  - locally closed in X_Σ.
  - a torus, $O_\sigma \cong (\mathbb{C}^\times)^{n-\dim\sigma}$

- Defn 7.9: We denote the orbit closure:

  $$V(\sigma) = \overline{O_\sigma} \underset{\text{closed}}{\subseteq} X_\Sigma.$$

L8 End

- We will show: $V(\sigma)$ is a toric variety in its own right.

  Refer to $V(\sigma) \subseteq X_\Sigma$ as a (closed) toric stratum.

- Lemma 7.10: $V(\sigma)$ is a union of orbits. Precisely:

  $$V(\sigma) = \bigcup_{\tau \geq \sigma} O_\tau.$$

  Proof: Exercise. $\square$

- We use the preceding lemma to build the fan of $V(\sigma)$.

  Whatever $V(\sigma)$ ends up being, its dense torus will be O_σ, which has cocharacter lattice $N(\sigma)$.

- Def^n 7.11: Let $p: N \to N(\sigma) = N/N_\sigma$ be the quotient map. Given a cone $\tau \subseteq N_\mathbb{R}$ we denote its image under p by:

  $$\overline{\tau} = p(\tau) \subseteq N(\sigma)_\mathbb{R}.$$

  This is also a cone

- Defn 7.12: The Star of σ is the fan in $N(\sigma)$ consisting of:

$$\mathrm{Star}(\sigma) = \{\bar{\tau} \subseteq N(\sigma)_{\mathbb{R}} : \tau \geq \sigma\}.$$

(E.g.: $\bar{\sigma} = 0 \in \mathrm{Star}(\sigma)$.)

- Lemma 7.13: If $\tau \geq \sigma$ then $\bar{\tau}$ is strictly convex. Hence the above definition gives a well-defined fan.

- We now define a closed embedding:

$$X_{\mathrm{Star}(\sigma)} \hookrightarrow X_{\Sigma}$$

and verify that its image is $V(\sigma)$.

- $X_{\mathrm{Star}(\sigma)}$ has a covering by open affines $X_{\bar{\tau}}$. One checks:

$$\boxed{\bar{\tau}^{\vee} \cap M(\sigma) = \tau^{\vee} \cap \sigma^{\perp} \cap M} \quad \text{(Exercise)}$$

$$\Rightarrow X_{\bar{\tau}} = \mathrm{Spec}\,\mathbb{C}[\tau^{\vee} \cap \sigma^{\perp} \cap M].$$

- On the other hand X_{τ} is open in X_{Σ}, with:

$$X_{\tau} = \mathrm{Spec}\,\mathbb{C}[\tau^{\vee} \cap M].$$

- Def^n 7.14: The closed embedding:

$$X_{\overline{\tau}} \hookrightarrow X_\tau$$

is defined by the ring homomorphism:

$$\mathbb{C}[\tau^\vee \cap M] \longrightarrow \mathbb{C}[\tau^\vee \cap \sigma^\perp \cap M]$$

$$z^m \longmapsto \begin{cases} z^m & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

- Each of these gives:

$$X_{\overline{\tau}} \xhookrightarrow[\text{closed}]{} X_\tau \xhookrightarrow[\text{open}]{} X_\Sigma .$$

They agree on overlaps, hence glue to:

$$X_{\mathrm{Star}(\sigma)} \xhookrightarrow[\text{closed}]{} X_\Sigma .$$

- Exercise: Convince yourself that the image is $V(\sigma) = \overline{O_\sigma}$.

- Exercise: Do lots and lots of 2D and 3D examples!

- <u>Note</u>: The ring homomorphism on the previous page does <u>not</u> arise from a semigroup homomorphism:

$$\tau^\vee \cap M \longrightarrow \tau^\vee \cap \sigma^\perp \cap M.$$

Shortly we will discuss toric morphisms, and we will understand what this means geometrically.

- <u>Corollary 7.15:</u> $V(\tau) \subseteq V(\sigma) \iff \sigma \subseteq \tau$.

- <u>Corollary 7.16:</u> There is a bijective, <u>inclusion-reversing</u> correspondence:

$$\begin{pmatrix} \text{cones} \\ \sigma \in \Sigma \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{toric} \\ \text{strata} \\ V(\sigma) \subseteq X_\Sigma \end{pmatrix}.$$

- Toric varieties are <u>stratified spaces</u>.

A stratified space is one which can be written as a union of pairwise disjoint locally-closed subsets:

$$X = \bigcup_i O_i$$

with each $\overline{O_i}$ a union of some O_j

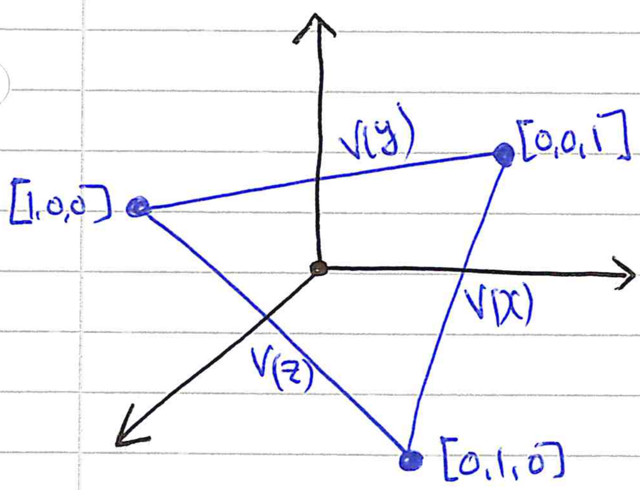
- In the case of toric varieties:

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} O_\sigma$$

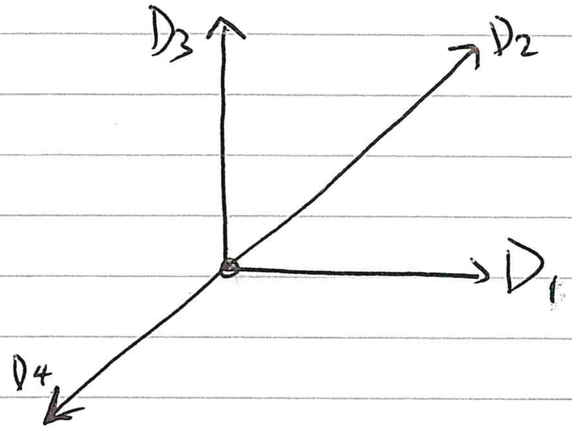
$$\overline{O_\sigma} = V(\sigma) = \bigcup_{\tau \geq \sigma} O_\tau$$

$$O_\sigma = V(\sigma) \setminus \bigcup_{\substack{\tau \geq \sigma \\ \tau \neq \sigma}} V(\tau).$$

- E.g.: For the fan of $\mathbb{P}^2$:



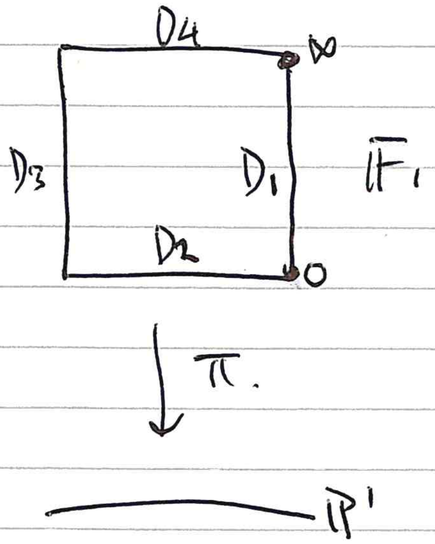
- E.g: For the fan of $\mathbb{F}_1$:



Have 4 toric hypersurfaces (each $\cong \mathbb{P}^1$) and 4 fixed points.

You know (Example Sheet 1) that $\mathbb{F}_1 \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle.

Later we'll see D_1 and D_3 are fibre classes, D_2 and D_4 are sections:



L9 End

L10 Start

§8: Properness (i.e. compactness)

- **Def^n**: The support of a fan is the union of all its cones:

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}.$$

- **Th^m 8.1**: X_Σ proper $\iff |\Sigma| = N_{\mathbb{R}}$.

- We recall the valuative criterion for properness.
- Take a DVR (discrete valuation ring) R with fraction field K.

Always think of the key example:

$$\boxed{R = \mathbb{C}[[t]], \quad K = \mathbb{C}((t)) = \left\{\sum_{k=-n}^{\infty} c_k t^k\right\}}$$

↑ Laurent series

Have $R \subseteq K$ which induces:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow[\text{open}]{} & \operatorname{Spec} R \\ \| & & \| \\ \Delta \setminus \{0\} & & \Delta \\ \text{punctured formal disc.} & & \text{formal disc} \end{array}$$

- Field K has a <u>valuation</u> which records asymptotics of functions on $\Delta\setminus\{0\} = \operatorname{Spec} K$ near 0.

$$v : K^\times \longrightarrow \mathbb{Z}$$

$$\sum c_k t^k \longmapsto \min\{k : c_k \neq 0\}$$

- <u>Valuative criterion for properness</u>: Let X be a separated scheme (of finite type). Then X is proper iff for all DVRs $(R, K, v : K^\times \to \mathbb{Z})$ the following diagram has a fill-in:

$$\begin{tikzcd}
\{\eta\} = \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X \\
\{0,\eta\} = \operatorname{Spec} R \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

I.e. any map $\Delta\setminus 0 \to X$ extends to $\Delta \to X$, i.e. limit at 0 exists.

- <u>Note</u>: By definition proper schemes are assumed separated. There is also a valuative criterion for separatedness, which says the fill-in is unique if it exists.

Taken together, the valuative criteria

for separatedness and properness say that the fill-in exists and is unique (i.e. limits exist – compactness – and are unique – Hausdorffness).

- Note: Toric varieties X_Σ arising from fans are always separated. In fact we have:

$$\left(\begin{array}{c}\text{normal}\\ \text{separated}\\ \text{toric}\\ \text{varieties}\end{array}\right) = \left(\begin{array}{c}\text{toric}\\ \text{varieties}\\ X_\Sigma \text{ for}\\ \Sigma \text{ a fan}\end{array}\right).$$

I will not prove this: you should take it on trust, or look at [CLS, Thm 3.1.5 ff.].

So we will take for granted that the uniqueness part of the valuative criterion is satisfied, and focus on the existence part.

Proof of theorem: First suppose $|\Sigma| \neq N_{\mathbb{R}}$. Then there is $u \in N$ which is not in the support of any cone of Σ.

By the interpretation of the fan in terms of limits of 1-parameter subgroups, it follows that

$\lambda_u : \mathbb{C}^* \longrightarrow T \hookrightarrow X_\Sigma$

has no limit as $t \to 0$. Hence the valuative criterion fails and X_Σ is not proper.

Conversely, suppose $|\Sigma| = N_\mathbb{R}$. Take a DVR $(R, K, \nu : K^* \to \mathbb{Z})$ and consider the lifting problem:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & X_\Sigma \\ \operatorname{Spec} R \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Because $T \subseteq X_\Sigma$ is dense, we can appeal to a weaker version of the valuative criterion, which allows us to assume that $\operatorname{Spec} K \to X_\Sigma$ factors through T.

Thus the lifting problem becomes:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\ \operatorname{Spec} R \arrow[r, dashed, "?"] & X_\Sigma \end{tikzcd}$$

The map $\operatorname{Spec} K \to T = \operatorname{Spec} \mathbb{C}[M]$ corresponds to a semigroup morphism:

$M \longrightarrow K$ semigroup under multiplication

Since M is a group, the image of every element must be invertible, so this map actually factors through the sub-semigroup $K^\times$:

$$M \longrightarrow K^\times.$$

Now we compose with the valuation $\nu : K^\times \to \mathbb{Z}$ to get:

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "u"'] & K^\times \arrow[r, "\nu"] & \mathbb{Z} \end{tikzcd}$$

Here $u \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = N$.

Since we are assuming $|\Sigma| = N_{\mathbb{R}}$, we have $u \in \sigma$ for some $\sigma \in \Sigma$. Consider the composition:

$$S_\sigma \hookrightarrow M \longrightarrow K$$

This corresponds to:

$$\mathrm{Spec}\, K \hookrightarrow T \hookrightarrow X_\sigma \ (\underset{\text{open}}{\hookrightarrow} X_\Sigma).$$

The morphism $\mathrm{Spec}\, K \to X_\sigma$ extends to $\mathrm{Spec}\, R \to X_\sigma$ iff the valuation:

$$S_\sigma \longrightarrow K^\times \xrightarrow{\nu} \mathbb{Z}$$

is non-negative on S_σ.

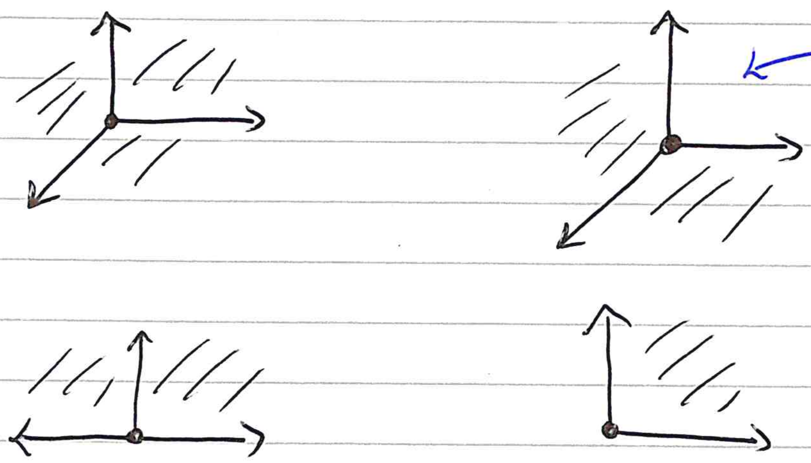
But the valuation is precisely u, and $u \in \sigma$ so u is non-negative on S_σ by definition of S_σ.

Thus the map does extend and the lifting/extension problem is solved.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\
\operatorname{Spec} R \arrow[r] \arrow[rr, bend right] & X_\sigma \arrow[r, hook] & X_\Sigma
\end{tikzcd}$$

□.

- <u>Examples to think about:</u>



this cone missing.

§9: Toric morphisms

- So far we have an equality of sets:

$$\begin{pmatrix} \text{normal} \\ \text{separated} \\ \text{toric} \\ \text{varieties} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{fans} \\ \Sigma \end{pmatrix}.$$

Want to enhance this to an equivalence of categories.

Need to specify what the morphisms are on each side. L10 End

L11 Start

- Def9.1: Let X and Y be toric varieties. A morphism

$$f: X \to Y$$

is toric iff it restricts to a group homomorphism on the tori:

$$f|_{T_X}: T_X \to T_Y.$$

- Lemma 9.2: Toric morphisms are equivariant, i.e. given $x \in X$ and $t \in T_X$:

$$f(t \cdot x) = f(t) \cdot f(x).$$

Proof: Equivariance is equivalent to the commutativity of the following square:

$$\begin{tikzcd} T_X \times X \arrow[r, "f \times f"] \arrow[d, "\text{action}"'] & T_Y \times Y \arrow[d, "\text{action}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The following square commutes because f restricts to a group homomorphism:

$$\begin{tikzcd} T_X \times T_X \arrow[r, "f \times f"] \arrow[d, "\text{mult.}"'] & T_Y \times T_Y \arrow[d, "\text{mult.}"] \\ T_X \arrow[r, "f"] & T_Y \end{tikzcd}$$

Since $T_X \subseteq X$, $T_Y \subseteq Y$ are dense, it follows by continuity that the first square commutes. $\square$

- Note: A toric morphism is simply a group homomorphism $T_X \to T_Y$ which extends to a map $X \to Y$.

The extension is unique if it exists, because $T_X \subseteq X$ is dense.

- So the data of a toric morphism is simply a choice of group homomorphism in:

$$\boxed{\mathrm{Hom}_{\mathrm{Alg.GP}}(T_X, T_Y) = \mathrm{Hom}(N_X, N_Y).}$$

There is a discrete choice!

- Question is, given $\varphi: N_X \to N_Y$ (essentially a matrix with $\mathbb{Z}$-coefficients), when does the corresponding map $T_X \to T_Y$ extend to $X \to Y$?

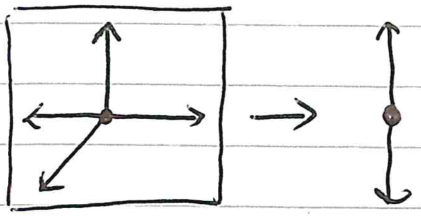
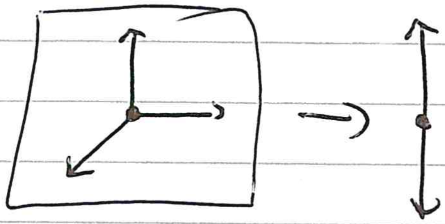
- ~~Defn: Called fans Σ_X in N_X and Σ_Y in N_Y and consider φ~~

- Defn 9.3: Given fans (Σ_X, N_X), (Σ_Y, N_Y), a morphism of fans:

$$\varphi: (\Sigma_X, N_X) \to (\Sigma_Y, N_Y)$$

is by definition a group homomorphism $\varphi: N_X \to N_Y$ such that:

(*) $\boxed{\text{for every cone } \sigma_X \in \Sigma_X \text{, there is a cone } \sigma_Y \in \Sigma_Y \text{ such that } \varphi(\sigma_X) \subseteq \sigma_Y.}$

- E.g.:  $\boxed{\text{is}}$ a fan map.
- E.g.:  $\boxed{\text{is not}}$ a fan map.

~~Thm $\varphi: N_X \to N_Y$ extends to a~~

- Thm 9.4: Let $\varphi: N_X \to N_Y$ be a lattice map and $\varphi_{\mathbb{C}^*}: T_X \to T_Y$ the corresponding homomorphism of tori.

  Then $\varphi_{\mathbb{C}^*}$ extends to a toric morphism $X \xrightarrow{f} Y$ iff φ is a fan morphism.

  Here there is a natural bijection:

$$(*) \quad \begin{pmatrix} \text{toric} \\ \text{morphisms} \\ X \xrightarrow{f} Y \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{fan} \\ \text{morphisms} \\ (\Sigma_X, N_X) \xrightarrow{\varphi} (\Sigma_Y, N_Y) \end{pmatrix}.$$

Proof: Suppose first that φ is a fan morphism.

Let $\sigma_X \in \Sigma_X$ be a cone, and suppose $\varphi(\sigma_X) \subseteq \sigma_Y$ for $\sigma_Y \in \Sigma_Y$.

Then $\varphi^\vee : M_Y \to M_X$ maps $S_{\sigma_Y} \to S_{\sigma_X}$.
So we get $\mathbb{C}[S_{\sigma_Y}] \to \mathbb{C}[S_{\sigma_X}]$ and:

$$X_{\sigma_X} \to X_{\sigma_Y} \to Y$$

The composite doesn't depend on the choice of σ_Y, and these patch together to give a map

$$X \xrightarrow{f} Y$$

which it is easy to show restricts to $\varphi_{\mathbb{C}^*} : T_X \to T_Y$.

Conversely suppose $f : X \to Y$ is a toric morphism extending $\varphi_{\mathbb{C}^*}$. Fix a cone $\sigma_X \in \Sigma_X$.

Recall there is a distinguished point $x_{\sigma_X} \in X_{\sigma_X} \subseteq X$, and that if $u \in \mathrm{Int}\,\sigma_X \cap N_X$ then:

$$\lim_{t \to 0} \lambda_u(t) = x_\sigma.$$

Since f is continuous we have:

$$\lim_{t \to 0} f(\lambda_u(t)) = f(x_\sigma).$$

But $\lambda_u(t) \in T_X$ so $f(\lambda_u(t)) = \varphi_{\mathbb{C}^*}(\lambda_u(t))$.
This is:

$$\varphi_{\mathbb{C}^*}(\lambda_u(t)) = \lambda_{\varphi(u)}(t).$$

(One way to think about this: $\varphi : N_X \to N_Y$ is the map given by post-composing 1-parameter subgroups $\mathbb{C}^* \to T_X$ with $\varphi_{\mathbb{C}^*} : T_X \to T_Y$.)

So we conclude:

$$\lim_{t \to 0} \lambda_{\varphi(u)}(t) = f(x_\sigma).$$

Since Σ_Y is obtained by partitioning N_Y into regions where the 1-parameter subgroups have the same limit, we conclude that there is a cone $\sigma_Y \in \Sigma_Y$ such that

$$\varphi(u) \in \mathrm{Int}\,\sigma_Y$$

for all $u \in \mathrm{Int}\,\sigma_X \cap N_X$. Thus:

$$\varphi(\mathrm{Int}\,\sigma_X) \subseteq \mathrm{Int}\,\sigma_Y$$

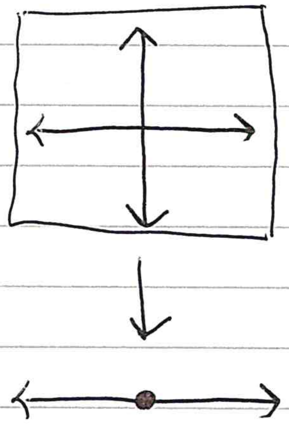
$$\Rightarrow \varphi(\sigma_X) \subseteq \sigma_Y.$$

So φ is a morphism of fans. □

• Upshot is an equivalence (covariant) of categories:

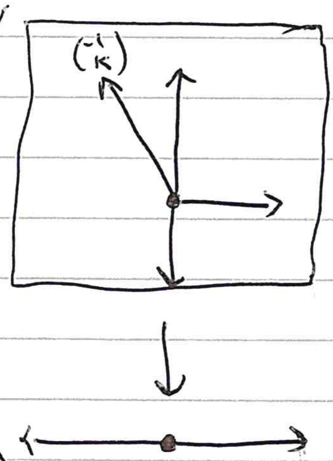
$$\left(\begin{array}{l}\text{normal, separated}\\ \text{toric varieties}\\ \text{with toric morphisms}\end{array}\right) \overset{*}{\longleftrightarrow} \left(\begin{array}{l}\text{fans with}\\ \text{fan morphisms}\end{array}\right)$$

- <u>Examples 1: Bundle Projections</u>
- Simplest example is:



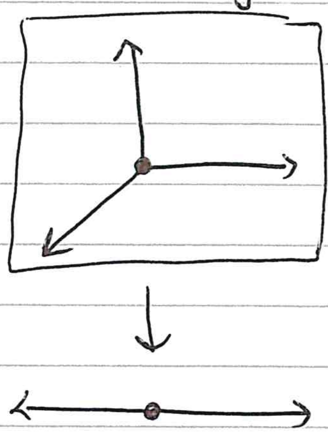
$$\begin{matrix}\mathbb{Z}^2 \\ \downarrow \pi_1 \\ \mathbb{Z}\end{matrix} \quad \longleftrightarrow \quad \begin{matrix}\mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow \\ \mathbb{P}^1\end{matrix}$$

- Generalises to Hirzebruch Surfaces (see Example Sheet 1):



$$\longleftrightarrow \quad \begin{matrix}\mathbb{F}_K \ \left(= \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(K) \oplus \mathcal{O}_{\mathbb{P}^1})\right) \\ \downarrow \\ \mathbb{P}^1\end{matrix}$$

- Following is not a fan map:



corresponds to the "morphism":

$$\left.\begin{array}{l} \mathbb{P}^2_{x_0 x_1 x_2} \longrightarrow \mathbb{P}^1_{y_1 y_2} \\ [x_0, x_1, x_2] \mapsto [x_1, x_2] \end{array}\right\}.$$

This is not well-defined at $[1,0,0]$.
It's a so-called rational map:

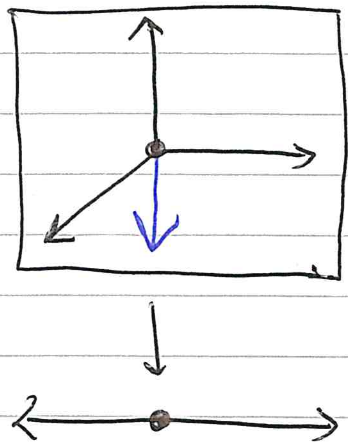
$$\mathbb{P}^2_{x_0 x_1 x_2} \dashrightarrow \mathbb{P}^1_{y_1 y_2}$$

can be resolved by blowing up the indeterminacy locus:

$$\begin{tikzcd} \mathrm{Bl}_{[1,0,0]}\mathbb{P}^2 \arrow[d] \arrow[dr] & \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1 \end{tikzcd}$$

well-defined everywhere

This corresponds to subdividing the problematic cone in $\mathbb{P}^2$:



This brings us on to the next class of toric morphisms. L11 End

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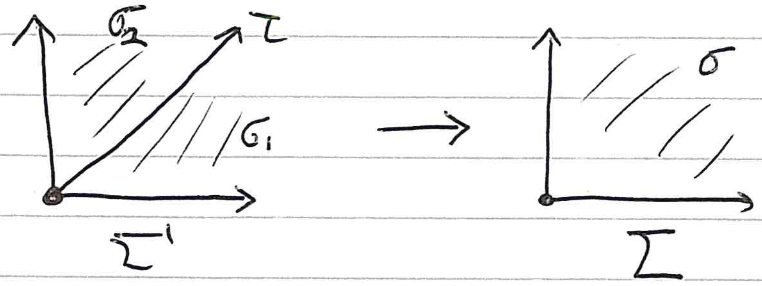
L12 Start

- ## Examples 2: Blowups

- Defn 9.5: A fan map $(\Sigma', N) \xrightarrow{\varphi} (\Sigma, N)$ where $\varphi = \mathrm{Id}_N : N \to N$ ~~is called a~~ ~~subdivision~~ and $|\Sigma'| = |\Sigma|$ is called a subdivision.

- Note: These will correspond to birational modifications: blowups etc.

- Consider the subdivision:



Have $X_\Sigma = \mathbb{C}^2$. Via the orbit-cone correspondence, we are intuitively replacing the point $V(\sigma) \subseteq X_\Sigma$ by a hypersurface $V(\tau) \cong \mathbb{P}^1 \subseteq X_{\Sigma'}$

- Lemma 9.6: Let $Y \xrightarrow{f} X$ be a toric morphism and $\sigma_Y \in \Sigma_Y$ a cone. Let $\sigma_X \in \Sigma_X$ be the minimal cone containing $\varphi(\sigma_Y)$. Then:

$$f(O_{\sigma_Y}) \subseteq O_{\sigma_X} \text{ and } f(V(\sigma_Y)) \subseteq V(\sigma_X).$$

Proof: Exercise.

- Prop 9.7: $X_{\Sigma'} = Bl_0 \mathbb{C}^2_{xy}$.

Proof: Let's recap how blowups work. We have $0 = V(I)$ where $I = (x, y)$. The appropriate graded ring is:

$$\bigoplus_{d \geq 0} I^d = I^0 \oplus I^1 \oplus I^2 \oplus \cdots$$
$$= \mathbb{C}[x,y] \oplus (x,y) \oplus (x^2, xy, y^2) \oplus \cdots$$

Notice: there are two "copies of x" in this ring: one in I^0 (degree 0) and one in I^1 (degree 1). To distinguish, we rename the degree 1 variable X (this is sometimes called a dummy variable). We have:

$$\bigoplus_{d \geq 0} I^d = \mathbb{C}[x,y] \oplus (X, Y) \oplus (X^2, XY, Y^2) \oplus \cdots$$

Now, as a module over $\mathbb{C}[x,y]$, the ideal I has two generators (X and Y) and a single relation:

$$xY = yX.$$

These in fact generate the relations in all I^d for $d \geq 2$. (This is because $O \in \mathbb{A}^2_{xy}$ is regularly embedded.)

So our graded ring ends up being:

$$\bigoplus_{d\geq 0} I^d = \mathbb{C}[\underbrace{\overset{0}{x}, \overset{0}{y}, \overset{1}{X}, \overset{1}{Y}}_{\text{degree of each variable.}}] / (xY - yX)$$

$$\Rightarrow \operatorname{Proj} \bigoplus_{d\geq 0} I^d = V(xY - yX) \subseteq \mathbb{A}^2_{xy} \times \mathbb{P}^1_{XY}.$$

Notice that on the affine open $Y \neq 0$ we have $x = y(X/Y)$ and hence:

$$(x, y) = (y)$$

Similarly on $X \neq 0$. Thus, the blowup principalises the ideal I (it's the universal thing which does this).

In Summary:

$$\mathrm{Bl}_0 \mathbb{C}^2_{xy} = V(xY - yX) \subseteq \mathbb{C}^2_{xy} \times \mathbb{P}^1_{XY}.$$

Now we turn to $X_{\Sigma'}$. We have:

$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[y, xy^{-1}]$, $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[x, yx^{-1}]$.

We embed each of these into 3-dimensional affine space as follows:

$$\left.\begin{array}{rcl} \mathbb{C}[x,y,s] & \longrightarrow & \mathbb{C}[y, xy^{-1}] \\ \\ x & \longmapsto & x \\ y & \longmapsto & y \\ s & \longmapsto & xy^{-1} \end{array}\right\} \leftrightarrow V(x-ys) \subseteq \mathbb{A}^3_{xys}$$

$$\left.\begin{array}{rcl} \mathbb{C}[x,y,s^{-1}] & \longrightarrow & \mathbb{C}[x, yx^{-1}] \\ \\ x & \longmapsto & x \\ y & \longmapsto & y \\ s^{-1} & \longmapsto & yx^{-1} \end{array}\right\} \leftrightarrow V(xs^{-1}-y) \subseteq \mathbb{A}^3_{xys}$$

Gluing the two copies of $\mathbb{A}^3$ together gives:

$\mathbb{C}^2_{xy} \times \mathbb{P}^1_{XY}$

where $X/Y = s$, $Y/X = s^{-1}$. Then the closed subschemes glue as well:

$$x = ys \iff xY = yX$$

$$\Rightarrow X_\Sigma = V(xY - yX) \subseteq \mathbb{C}^2_{xy} \times \mathbb{P}^1_{XY}.$$

So indeed $X_{\Sigma'} \cong \mathrm{Bl}_0 \mathbb{C}^2_{xy}$. Moreover the toric morphism $X_{\Sigma'} \to X_\Sigma$ is the blow down. $\square$

- Lemma 9.8: A toric morphism corresponding to a subdivision is always surjective and birational.

Proof: The map on dense tori is the identity, hence an isomorphism, hence the map is birational.

Since the image is dense, the map is surjective if it is proper. This is the previous properness theorem we proved, but for morphisms, and is the content of the next lemma. □

- Lemma 9.9: A toric morphism is proper iff $\varphi^{-1}(|\Sigma|) = |\Sigma'|$.

Proof: Adapt the proof of Thm 8.1. □

L12 End

L13 Start

- Examples 3: Change of lattice

- Let $N = 2\mathbb{Z} \subseteq \mathbb{Z} = N'$. There is an equality $N_{\mathbb{R}} = N'_{\mathbb{R}}$ and in both cases we take the positive ray:

~~we get:~~

$\cancel{\mathbb{C}[S_\sigma] = \mathbb{C}[t^2]}$

When we dualise we get:

$$M = \tfrac{1}{2}\mathbb{Z} \supseteq \mathbb{Z} = M'$$

$$\Rightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[t] \supseteq \mathbb{C}[t^2] = \mathbb{C}[S_{\sigma'}]$$

(where $t = z^{1/2} \in \mathbb{C}[M]$)

$$\Rightarrow X_\sigma = \mathbb{A}^1_t \longrightarrow \mathbb{A}^1_s = X_{\sigma'}$$
$$t^2 \leftarrow\!\shortmid s$$

So we get a ramified cover. Can view this as the quotient map:

$$\mathbb{A}^1_t \longrightarrow \mathbb{A}^1_t / \mu_2 = \mathbb{A}^1_{s=t^2}$$

where $\mu_2 = \{\pm 1\} \curvearrowright \mathbb{A}^1_t$.

Exercise: The toric morphisms $X_{\sigma_1} \xrightarrow{f} X_{\sigma_2}$ are precisely those induced by semigroup morphisms:

$$S_{\sigma_2} \longrightarrow S_{\sigma_1}.$$

Hence, they are precisely the morphisms for which monomials pull back to monomials:

$$f^*(z^m) = z^{\varphi^\vee(m)}.$$

§10: Surface singularities and cyclic quotients

- We will investigate and classify affine toric surfaces, otherwise known as "toric surface singularities".

- But first, we need a detour to discuss quotients in algebraic geometry.

- Setup: G a finite group
  $X = \operatorname{Spec} R$ affine scheme
  $G \curvearrowright X$

- Goal: Build quotient X/G (orbit space) as a scheme

- Defn 10.1: The affine quotient X/G is defined as:

$$X/G = \operatorname{Spec} R^G$$

ring of invariants:
$R^G = \{r \in R : gr = r \ \forall g \in G\}$

The quotient map

$$X \longrightarrow X/G$$

is the map corresponding to the ring inclusion $R^G \subseteq R$.

- It's the universal G-invariant map $X \to Z$. (See Harris: "Algebraic Geometry - A First Course" p. 123 ff.)

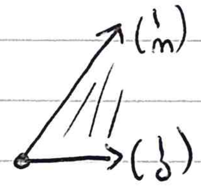
- <u>Note</u>: This only works because G is finite! E.g. for $\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1}$ we have:

$$\mathbb{C}[x_0, \dots, x_n]^{\mathbb{C}^*} = \mathbb{C}$$

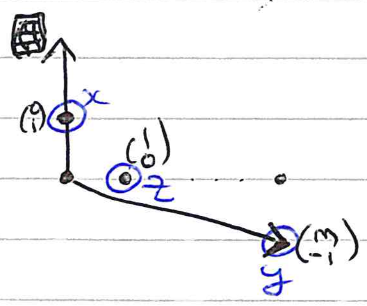
$$\Rightarrow \mathbb{C}^{n+1}/\mathbb{C}^* = \operatorname{Spec} \mathbb{C} = pt.$$

$\Rightarrow$ Don't get a sensible answer. To make it sensible, need to develop ~~[illegible]~~ geometric invariant theory (GIT). We might discuss this at the end of the course.

- <u>Example 10.2</u>: Consider:

$\sigma =$ 

We've already seen:

$\sigma^\vee =$ 

$$\Rightarrow \boxed{X_\sigma = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^m)}$$

This is called the <u>A_{m-1}-Singularity</u>.

- Consider the action $\mu_m \curvearrowright \mathbb{C}^2_{uv}$ given by:

$$\left.\begin{aligned} \xi(u) &= \xi \cdot u \\ \xi(v) &= \xi^{-1} \cdot v \end{aligned}\right\}$$

Here $\xi = e^{2\pi i/m} \in \mu_m$ is a primitive mth root of unity, generating $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$.

- <u>Lemma 10.3:</u> $\mathbb{C}^2/\mu_m \cong X_\sigma$.

Proof: Need to find the ring of invariants. Clearly, looking at the action, we see it's generated by monomials. In fact:

$$\mathbb{C}[u,v]^{\mu_m} = \mathbb{C}[u^m, uv, v^m] \subseteq \mathbb{C}[u,v].$$

But then:

$$\left.\begin{aligned} \frac{\mathbb{C}[x,y,z]}{(xy - z^m)} &\xrightarrow{\cong} \mathbb{C}[u^m, uv, v^m] \\ x &\longmapsto u^m \\ y &\longmapsto v^m \\ z &\longmapsto uv \end{aligned}\right\} \qquad \square$$

- Turns out: all toric surface singularities are quotients $\mathbb{C}^2/\mu_m$, with different actions $\mu_m \circlearrowright \mathbb{C}^2$.

- In fact, we claim that the quotient map

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2/\mu_m = X_\sigma$$

is a toric morphism. Let:

$$N' = \mathbb{Z}\cdot\binom{1}{0} + \mathbb{Z}\cdot\binom{1}{m} \subseteq N$$
$$= \mathbb{Z}\cdot\binom{1}{0} + \mathbb{Z}\cdot\binom{0}{m} \subseteq N$$

So $N' \cong \mathbb{Z}^2$ with $N/N' \cong \mathbb{Z}/m\mathbb{Z}$.

- The cone σ in $N_\mathbb{R}$ defines a cone σ' in $N'_\mathbb{R}$, and we have a fan map:

$$(\sigma', N') \to (\sigma, N).$$

- <u>Lemma 10.4</u>: The induced map is the quotient:

$$\begin{array}{ccc} X_{\sigma'} & \longrightarrow & X_\sigma \\ \wr\Vert & & \wr\Vert \\ \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\mu_m. \end{array}$$

<u>Proof</u>. The inclusion $N' = \mathbb{Z}\cdot\binom{1}{0} + \mathbb{Z}\cdot\binom{1}{m} \subseteq N$

dualises to $M \subseteq M'$.

If e_1, e_2 denotes the standard basis of N, with dual basis e_1^*, e_2^* of M, then M' has basis:

$$f_1^* = e_1^* - \frac{1}{m} e_2^*, \quad f_2^* = \frac{1}{m} e_2^* \quad \text{(Exercise)}$$

dual to the basis $f_1 = e_1$, $f_2 = e_1 + m e_2$ of N'. Then:

$$\left.\begin{aligned} \mathbb{C}[S_{\sigma'}] &= \mathbb{C}[t_1 \cdot t_2^{-1/m}, t_2^{1/m}] \\ \mathbb{C}[S_\sigma] &= \mathbb{C}[t_1, t_1^m t_2^{-1}, t_2] \end{aligned}\right\}$$

The map $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$ is:

$$\left.\begin{aligned} &\mathbb{C}[\underset{z}{t_1}, \underset{y}{t_1^m t_2^{-1}}, \underset{x}{t_2}] \longrightarrow \mathbb{C}[\underset{v}{t_1 t_2^{-1/m}}, \underset{u}{t_2^{1/m}}] \\ &\qquad x \longmapsto u^m \\ &\qquad y \longmapsto v^m \\ &\qquad z \longmapsto uv \end{aligned}\right\}$$

So precisely the inclusion of the ring of invariants:

$$\mathbb{C}[u^m, v^m, uv] \hookrightarrow \mathbb{C}[u, v]. \qquad \square$$

$\boxed{\text{L13 End}}$

L14 Start

- The A_{m-1}-singularity has a toric resolution of singularities.

- <u>Def^n 10.5:</u> Let X be an algebraic variety. A resolution of singularities is a proper birational morphism:

  $\pi : Y \to X$

  with Y smooth

- <u>Theorem (Hironaka)</u>: Every variety over a field of characteristic zero admits a resolution of singularities.

  <u>Proof</u>: Grothendieck summarised the proof in an address of 1971:

  "Aboutissement d'années d'efforts concentrés, elle est sans doute l'une des démonstrations les plus «dures» et les plus monumentales qu'on connaisse en mathématique."

  which roughly translates to:

  ~~"The outcome of years of concentrated~~
  "The result of years of concentrated effort, it is without a doubt one of the "hardest" and most monumental proofs known to mathematics."

(Apologies to any French speakers if I mangled the translati

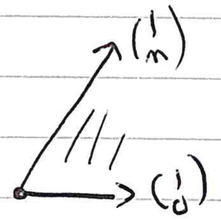
- Open question: Do resolutions of singularities always exist in characteristic $p>0$?

- As always, things are easier in toric geometry.

<u>Toric Resolution of Singularities</u>: Every toric variety admits a resolution of singularities, by a toric morphism.

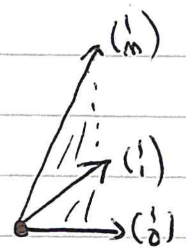
You will prove this in Example Sheet 2!!!

- Let's see a resolution of singularities of $X_\sigma = A_{m-1}$. Recall the cone was:



And $\det\begin{pmatrix} 1 & 1 \\ 0 & m \end{pmatrix} = m$ shows this is singular for $m \geq 2$.

- We <u>subdivide</u> σ by introducing $m-1$ new rays:

$\tilde{\Sigma} =$ 

- Have a map of fans:

  $(\tilde{\Sigma}, N) \to (\sigma, N)$.

  This is a subdivision, hence

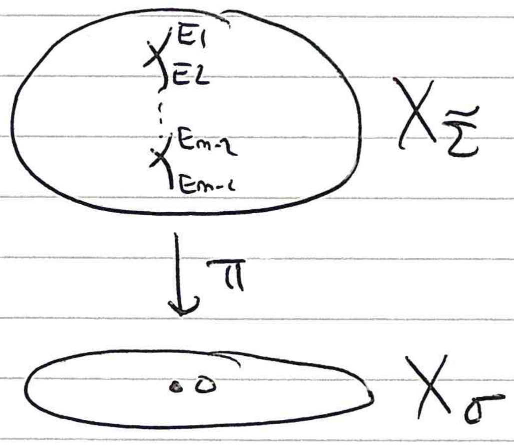
  $X_{\tilde{\Sigma}} \xrightarrow{\pi} X_\sigma$

  is proper and birational.

- Finally, $X_{\tilde{\Sigma}}$ is smooth: every cone has:

  $\det\begin{pmatrix} 1 & 1 \\ k-1 & k \end{pmatrix} = 1 \qquad (1 \leq k \leq m)$.

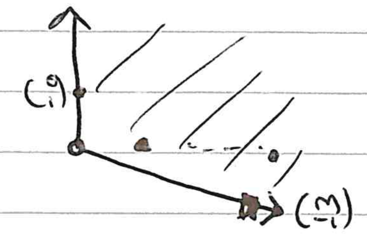
- By the orbit-cone correspondence, the exceptional locus is a chain of $m-1$ curves, each isomorphic to $\mathbb{P}^1$:



~~Example~~

~~Exercise~~

- **Example/Exercise 10.6:** Consider the fan:

$\sigma =$ 

Prove that $X_\sigma = \mathbb{C}^2/\mu_m$ where $\mu_m \curvearrowright \mathbb{C}^2_{uv}$ by

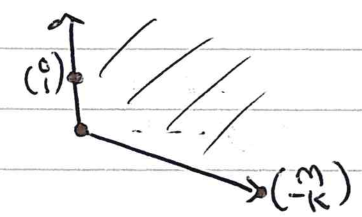
$$\left.\begin{aligned} \zeta(u) &= \zeta \cdot u \\ \zeta(v) &= \zeta \cdot v \end{aligned}\right\}$$ (compare to previous example).

Describe the quotient $\mathbb{C}^2 \to \mathbb{C}^2/\mu_m$ as a toric morphism.

Construct a toric resolution of singularities of X_σ.

- Generally, suppose $\sigma \subseteq \mathbb{R}^2 = N_{\mathbb{R}}$ is a 2D cone.

Proposition 10.7: After a change of $\mathbb{Z}$-basis, σ takes the form:

$\sigma =$ 

where:
- $m \geq 1$
- $\leq k \leq m-1$
- $\gcd(m,k) = 1$.

<u>Proof</u>: This is $\mathbb{Z}$-linear algebra.

Let $v_1, v_2 \in N$ be the primitive lattice points generating the rays.

We can choose a $\mathbb{Z}$-basis e_1, e_2 of N such that $v_2 = e_2$.

Writing $v_1 = me_1 + xe_2$, we can ensure $m > 0$ by replacing e_1 by $-e_1$ if necessary.

Finally, for $c \in \mathbb{Z}$ we have:

$$\det\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = 1 \Rightarrow \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in GL_2(\mathbb{Z})$$

And:

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & m \\ 1 & x \end{pmatrix} = \begin{pmatrix} 0 & m \\ 1 & cm+x \end{pmatrix}$$

So x can be replaced by $cm + x$ for any $c \in \mathbb{Z}$. So we replace x by $-k$ where $0 \le k \le m-1$.

So we have:

$$v_1 = me_1 - ke_2 = \begin{pmatrix} m \\ -k \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Finally $\gcd(m, k) = 1$ simply because v_1 is primitive. $\square$

- **Proposition 10.8**: Let σ be as above. Then

$$X_\sigma = \mathbb{C}^2/\mu_m$$

where $\mu_m \curvearrowright \mathbb{C}^2_{uv}$ via:

$$\left.\begin{array}{l} \zeta(u) = \zeta \cdot u \\ \zeta(v) = \zeta^k \cdot v \end{array}\right\}.$$

Proof: [Fulton, §2.2]. $\square$

- To round off this section, let's see how the above can be done in a co-ordinate free way, and generalised.

- **Defn 10.9**: A cone $\sigma \subseteq N_{\mathbb{R}}$ is simplicial iff it is generated by a subset of an $\mathbb{R}$-basis for $N_{\mathbb{R}}$.

(A fan is simplicial iff all its cones are)

- **Note**: Every cone in dimension ≤ 2 is simplicial. The cone over a square is a counterexample in dimension 3.

- **Theorem 10.10**: If σ is simplicial then

$$X_\sigma = \mathbb{C}^n/G$$

for some finite abelian group $G \curvearrowright \mathbb{C}^n$.

- **Proof:** Let $\sigma \subseteq N$ be simplicial, and let:

$$N' \subseteq N$$

be the lattice spanned by the primitive ray generators of σ.

Then σ induces a cone $\sigma' \subseteq N'_{\mathbb{R}}$ with $X_{\sigma'} = \mathbb{C}^n$

Let $G = N/N'$. This is a finite abelian group. We define an action:

$$\left.\begin{array}{l} G \curvearrowright \mathbb{C}[M'] \\ u \cdot z^{m'} = \exp(2\pi i \langle u, m' \rangle) \cdot z^{m'} \end{array}\right\}$$

Some remarks: here $u \in N$, $m' \in M'$. A priori we have $\langle u, m' \rangle \in \mathbb{Q}$ since $M \subseteq M'$. And $\langle u, m' \rangle \in \mathbb{Z}$ if $u \in N'$, so the action is well-defined on $G = N/N'$.

Also $\langle u, m' \rangle \in \mathbb{Z}$ if $m' \in M$, so:

$$\boxed{\cancel{\mathbb{C}[M']^G = \mathbb{C}[M]}}$$

$$\mathbb{C}[M] \subseteq \mathbb{C}[M']^G$$

But also conversely, since $\mathbb{C}[M']^G$ is generated by monomials:

$$\mathbb{C}[M] = \mathbb{C}[M']^G.$$

The action $G \curvearrowright \mathbb{C}[M']$ restricts to an action $G \curvearrowright \mathbb{C}[S_{\sigma'}]$, and:

$$\begin{aligned}\mathbb{C}[S_{\sigma'}]^G &= \mathbb{C}[S_{\sigma'}] \cap \mathbb{C}[M']^G \\ &= \mathbb{C}[S_{\sigma'}] \cap \mathbb{C}[M] \\ &= \mathbb{C}[S_\sigma].\end{aligned}$$

So indeed the fan map $(\sigma', N') \to (\sigma, N)$ induces:

$$\begin{array}{ccc} X_{\sigma'} & \longrightarrow & X_{\sigma'}/G = X_\sigma \\ \parallel & & \parallel \\ \mathbb{C}^n & \longrightarrow & \mathbb{C}^n/G. \end{array}$$

L14 End $\square$

L15 Start

§11: Divisors and line bundles

- Recall that given an algebraic variety X a hypersurface

$$Y \subseteq X$$

is a closed subvariety (in particular integral) of codimension 1.

- Note: $Y \subseteq X$ having codimension 1 is equivalent to the local ring

$\mathcal{O}_{X,Y} = \mathcal{O}_{X,\eta_Y}$

having Krull dimension 1 ($\eta_Y \in X$ the generic point of Y).

- Defn 11.1: A weil divisor on X is a finite formal $\mathbb{Z}$-linear sum of hypersurfaces:

$D = \sum_{i=1}^{k} a_i D_i$ $\quad$ $a_i \in \mathbb{Z}$, $D_i \subseteq X$ hypersurface

- Note: we say "weil" to distinguish from "cartier" divisors, which we will meet later. If X is smooth the two notions are the same, but in general they differ.

- The class group $Cl(X)$ is a sort of homology theory for weil divisors on X.

We need to decide when two divisors are "homologous".

- Idea: consider maps:

$f: X \longrightarrow \mathbb{P}^1$

Then $f^{-1}(0), f^{-1}(\infty)$ give divisors in X. we think of these as homologous:

$f^{-1}(0) \sim f^{-1}(\infty)$.

or equivalently (since we can add and subtract weil divisors):

$f^{-1}(0) - f^{-1}(\infty) \sim 0$

- Defn 11.2: Let $f \in H^0(X, K_X^*)$ be a global rational function on X. The corresponding principal divisor is:

$$\mathrm{div}(f) = \sum_{\substack{D \subseteq X \\ \text{hypersurface}}} \mathrm{ord}_f(D) \cdot D$$

Here $\mathrm{ord}_f(D) \in \mathbb{Z}$ is the order of vanishing of f along D. Negative vanishing order corresponds to having a pole.

- Formally, each $\mathcal{O}_{X,D}$ is normal and 1-dimensional, hence a DVR, and f determines an element in:

$(\mathrm{Frac}\, \mathcal{O}_{X,D})^*$

So $\text{ord}_f(D)$ is defined by applying the valuation ν to this element.

- Intuitively, in a neighbourhood of the generic point of D, f can be written as:

$$f = g \cdot s_D^K$$

where $V(s_D) = D$, $K \in \mathbb{Z}$ and g is invertible.

Then $\text{ord}_f(D) = K$. (s_D is the uniformiser of the DVR.)

- Principal divisors form a subgroup of weil divisors, since:

$$\left.\begin{aligned} \text{div}(1) &= 0 \\ \text{div}(fg) &= \text{div} f + \text{div}\, g. \end{aligned}\right\}$$

- Defn 11.3: The class group $Cl(X)$ of X is defined to be:

(*) $$\boxed{Cl(X) = \{\text{weil divisors}\} / \{\text{Principal divisors}\}}$$

It is an abelian group.

- Note: we also write $Cl(X) = A_{\dim X - 1}(X)$, the Chow homology group of codimension 1 cycles. There are $A_K(X)$ for all K.

- **Eg:** $Cl(\mathbb{P}^1) = \mathbb{Z}$
  $Cl(\mathbb{A}^1) = 0$

In general, very tricky to compute.

- **Lemma 11.4:** Let R be a UFD. Then $Cl(\operatorname{Spec} R) = 0$.

Proof: Let $T = \operatorname{Spec} R$ and $D \subseteq T$ a hypersurface, corresponding to a prime ideal $I \triangleleft R$, of height 1.

Here "height 1" means that if we have $0 \subseteq J \subseteq I$ for J prime, then either $J = 0$ or $J = I$. It is the algebraic expression of the fact that $D \subseteq T$ is codimension 1.

We claim that every prime ideal I of height 1 in a UFD is principal.

Let $r \in I \setminus \{0\}$ and write r as a product of irreducible elements:

$$r = \prod_{i=1}^{n} p_i$$

(This is always possible in a Noetherian ring) Since R is a UFD, we know that each p_i is in fact prime (and that the decomposition of r is essentially unique, though we won't use this).

Since $r \in I$ and I is prime, we have $p_i \in I$ for some $1 \leq i \leq n$. Then:

$0 \subseteq (p_i) \subseteq I$

Since (p_i) is prime and I has height 1, we conclude $I = (p_i)$ as claimed.

Coming back to $I = I(D) \lhd R$, if $I = (f)$ then $f \in R = H^0(T, \mathcal{O}_T)$ is in particular a rational function, and:

$\mathrm{div}(f) = D.$

Thus D is principal. But every Weil divisor is a sum of such D, hence is also principal. $\square$

- Corollary 11.4b: The class group of an algebraic torus $T \cong (\mathbb{C}^\times)^n$ is trivial.

Proof: $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ is a UFD, since it is a localisation of the UFD $\mathbb{C}[x_1, \dots, x_n]$. $\square$

- We will use the above corollary to prove that the class group of a toric variety is generated by so-called boundary divisors.

- Defn 11.5: Let X be a toric variety. A toric hypersurface (also called a boundary hypersurface) is a hypersurface

$$D \subseteq X$$

such that $T \curvearrowright X$ restricts to an action $T \curvearrowright D$, i.e. such that:

$$t \in T, \, p \in D \Rightarrow t(p) \in D.$$

- These are precisely the orbit closures $V(\tau) \subseteq X$ for $\tau \in \Sigma$ a ray (1D cone). We write these as:

$$D_\tau = V(\tau) \subseteq X.$$

- A toric weil divisor is a formal sum of toric hypersurfaces:

$$D = \sum_{\tau \in \Sigma(1)} a_\tau \cdot D_\tau. \qquad (a_\tau \in \mathbb{Z})$$

- Proposition 11.6: $Cl(X)$ is generated by the D_τ.

Proof: Let $\partial X = X \setminus T$. There is an excision sequence (cf. Fulton "Intersection Theory" §1.8):

$$A_{\dim X-1}(\partial X) \to Cl(X) \to Cl(T) \to 0$$

By Lemma 11.4 we have $Cl(T) = 0$, and hence a surjection:

$$A_{\dim X-1}(\partial X) \twoheadrightarrow Cl(X).$$

But $A_{\dim X-1}(\partial X) = \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z} \cdot D_\tau$. $\square$ L15 End

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L16 Start

- So the generators of $Cl(X)$ are visible in the fan. We now turn to the relations.

- Only need to focus on those $f \in H^0(X, K_X^*)$ with

$$|\operatorname{div} f| \subseteq \partial X$$

$$|\operatorname{div} f| := \bigcup_{\operatorname{div}_f(D) \neq 0} D \subseteq X$$

This means f has no zeros or poles on T, i.e.:

$$f|_T \in H^0(T, \mathcal{O}_T)^\times.$$

- Lemma 11.7: Let M be a lattice. Then:

$$\mathbb{C}[M]^\times = \{ \lambda \cdot z^m : \lambda \in \mathbb{C}^*, m \in M \}.$$

- Aside: This shows that $T = \operatorname{Spec}\mathbb{C}[M]$ has canonical co-ordinates (unlike e.g. affine space).

- Corollary 11.8: All relations between toric Weil divisors in $Cl(X)$ come from Laurent polynomials

$$\chi^m \in H^0(T, \mathcal{O}_T)^* \subseteq H^0(X, K_X^*)$$

for $m \in M$. Moreover:

$$\boxed{\operatorname{div}(\chi^m) = \sum_{\tau \in \Sigma(1)} \langle m, v_\tau \rangle \cdot D_\tau} \quad (*)$$

where $v_\tau \in \tau \cap N$ is the primitive lattice generator of τ.

Proof: The first part follows from the previous discussion.

The formula for $\operatorname{div}(\chi^m)$ reduces to a local calculation, and is left as an exercise. $\square$

- Combining Proposition 11.6 and Corollary 11.8 we obtain:

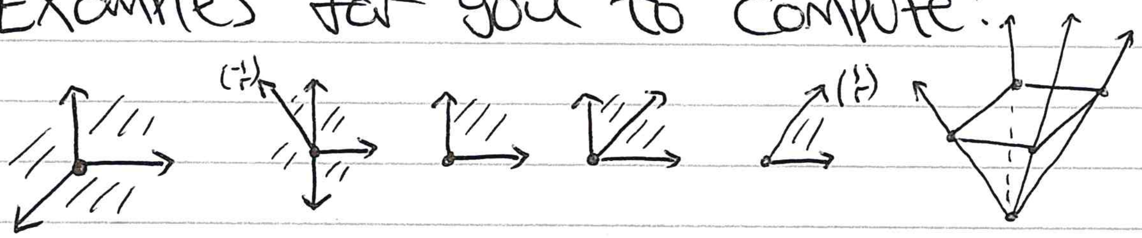
Theorem 11.9: For every toric variety X there is a short exact sequence:

for exactness here need ~~[illegible]~~, X_Σ ~~i.e. X_Σ has no torus factors~~ has no torus factors.

$$\boxed{0 \to M \to \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z} \cdot D_\tau \to Cl(X) \to 0}$$

where the first map is given by the fan-theoretic formula ⊛ on the previous page.

- This is amazing!
- Examples for you to compute:



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§11b: Cartier divisors

- Cartier divisors are Weil divisors which can locally be written as the vanishing/pole locus of a single function.

Defn 11.10: A Cartier divisor on X is a global section

$f \in H^0(X, K_X/\mathcal{O}_X^*)$

I.e. f is given by rational functions f_α on an open cover $\{U_\alpha\}$ which differ by units on overlaps.

- Example 11.10: Consider the 3D toric variety:

$$X = \mathrm{Spec}\,\mathbb{C}[x,y,z,w]/(xy - zw).$$

Then we have:

$$V(x) = V(x,z) \cup V(x,w) \subseteq X \quad \left(\begin{matrix}\text{union of}\\ \text{2 planes}\end{matrix}\right)$$

We can isolate one of the irreducible components of $V(x)$:

$$D_{xz} := V(x,z) \subseteq X.$$

Then $D_{xz} = \mathbb{A}^2_{yw}$ is a hypersurface (integral of codimension 1), hence is a Weil divisor.

But $D_{xz} \subseteq X$ cannot be written as the vanishing locus of a single function: we really need two equations to cut it out. This means that D_{xz} is not Cartier.

This all happened because X was singular.

- Example 11.11: Consider the A_1-singularity:

$$X = \mathrm{Spec}\,\mathbb{C}[x,y,z]/(xy - z^2).$$

Then we have the Weil divisor

(in fact, a toric Weil divisor):

$$D = V(x,z) \subseteq X.$$

This is also <u>not Cartier</u>. It looks as if $D = V(x)$, but this is not true. Indeed:

$$V(x) = V(x, z^2) \cong V(z^2) \subseteq \mathbb{A}^2_{yz}.$$

This is D with some additional non-reduced structure. We can write:

$$V(x) = 2 \cdot D.$$

~~later we will see th~~ So D is not Cartier but $2 \cdot D$ (whatever that means) is. This means that D is <u>$\mathbb{Q}$-Cartier</u>.

- <u>Defn 11.12</u>: A Cartier divisor on X is a global section:

$$\boxed{f \in H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times).}$$

assume X a normal var. otherwise, defn tricky.

I.e. f consists of the data of <u>rational functions</u> $\{f_\alpha\}$ on some open cover $\{U_\alpha\}$, which differ by <u>units</u> on pairwise overlaps.

(Here $\mathcal{K}_X^\times$ means "not-identically-zero" rational fns, not "non-vanishing".)

- **Def^n 11.13:** Given a Cartier divisor f, the associated Weil divisor is:

$$D_f = \sum_{\substack{D \subseteq X \\ \text{hypersurface}}} \mathrm{ord}_f(D) \cdot D.$$

well-defined

- **Note:** If $f \in H^0(X, K_X^\times)$, i.e. all the units can be taken to be $\underline{1}$, then

$$D_f = \mathrm{div}(f)$$

is a principal divisor, as in Def^n 11.2. But in general D_f will not be principal. In fact, the units on overlaps define transition functions for a line bundle:

$$\mathcal{O}(D_f) \in \mathrm{Pic}\, X.$$

And f can be viewed as a rational section of this line bundle:

$$f \in H^0(X, \mathcal{O}(D_f) \otimes K_X^\times).$$

L16 End

L17 Start

- **Lemma 11.14:**

$$\frac{\text{Cartier divisors}}{\text{Principal divisors}} = \mathrm{Pic}\, X$$

line bundles up to isom.

Proof: We have a short exact sequence:

$$0 \to \mathcal{O}_X^\times \to K_X^\times \to K_X^\times / \mathcal{O}_X^\times \to 0$$

Passing to the long exact cohomology sequence we obtain:

$$\begin{array}{ccccc} H^0(X, \mathcal{K}_X^\times) & \longrightarrow & H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) & \rightarrow & H^1(X, \mathcal{O}_X^\times) \\ \| & & \| & & \| \\ \text{Principal divisors} & & \text{Cartier divisors} & & \operatorname{Pic} X \end{array}$$

And $H^1(X, \mathcal{K}_X^\times) = 0$ because $\mathcal{K}_X^\times$ is flasque, so the final map is surjective. □

• <u>Note</u>: we have bijections:

$$\left\{ \begin{array}{c} \text{Cartier} \\ \text{divisors} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{line bundles with a rational} \\ \text{section} \end{array} \right\} / \sim$$

$$\left\{ \begin{array}{c} \text{classes of} \\ \text{Cartier divisors} \end{array} \right\} \longleftrightarrow \left\{ \text{line bundles} \right\} / \sim$$

Often, people say "divisor" when they really mean "divisor class", and use the words "divisor" and "line bundle" interchangeably.

Usually this is harmless, because many properties of Cartier divisors only depend on the class, i.e. only depend on the associated line bundle.

- **Defn 11.15**: A Cartier divisor f on a toric variety X is a toric Cartier divisor (or T-Cartier divisor) iff the associated Weil divisor D_f is toric.

- Note: Since X is normal and separated:

$$\left.\begin{array}{ccc} \{\text{Cartier divisors}\} & \longrightarrow & \{\text{Weil divisors}\} \\ f & \longmapsto & D_f \end{array}\right\}$$

is injective. So we can view Cartier divisors as a special class of Weil divisors.

- **Lemma 11.16**: Let $X = X_\sigma$ be an affine toric variety and let D be a T-Cartier divisor. Assume σ is full-dimensional. Then

$$D = \mathrm{div}(z^m)$$

for some $m \in M$.

Proof: (First assume $D \leq X_\sigma$ effective.) Let $I \lhd \mathbb{C}[S_\sigma]$ be the ideal of D. Since $T = \mathrm{Spec}\,\mathbb{C}[M]$ restricts to an action on D, it follows that the ideal I is graded, i.e.:

(*1) $$I = \bigoplus_{\substack{m \in S_\sigma \\ \text{s.t. } z^m \in I}} \mathbb{C} \cdot z^m$$

Now, since D is a Cartier divisor it follows that I is locally principal. In particular it is principal in a neighbourhood of the distinguished point $x_\sigma \in X_\sigma$.

Let $P \lhd \mathbb{C}[S_\sigma]$ be the maximal ideal corresponding to $x_\sigma \in X_\sigma$ and recall that:

(*2) $$P = \bigoplus_{\substack{m \in S_\sigma \\ m \neq 0}} \mathbb{C} \cdot z^m$$

Then I_P is a principal $\mathbb{C}[S_\sigma]_P$-module. Quotienting by the maximal ideal of this local ring, it follows that:

$$I_P / P_P \cdot I_P = I/PI$$

is a 1-dimensional $\mathbb{C}$-vector space.

Using (*1) and (*2), we see that there is a unique monomial

$$z^{m_P} \in I$$

which cannot be written as

$$z^{m_1} z^{m_2}$$

for $m_1 \in S_\sigma \setminus \{0\}$, and $z^{m_2} \in I$.

Finally since σ is full-dimensional it follows that $\sigma^\vee$ is strictly convex. Using this, we can inductively prove that every $z^m \in I$ is divisible by z^{m_D} and hence

$$I = (z^{m_D})$$

as claimed. Finally, if D is not effective then choose

$$m \in (\mathrm{Int}\, \sigma^\vee) \cap M.$$

Then $\langle m, v_\tau \rangle > 0$ for all $\tau \in \sigma(1)$. Hence

$$D + \mathrm{div}(z^{mk})$$

is effective for $k \gg 0$. So:

$$D + \mathrm{div}(z^{mk}) = \mathrm{div}(z^{m'})$$

$$\Rightarrow D = \mathrm{div}(z^{m' - mk}). \qquad \square$$

- we will use the result above

to describe Cartier divisors on an arbitrary toric variety by gluing. For this we require the following generalisation and repackaging:

- <u>Lemma 11.17</u>: Let $\sigma \subseteq N_{\mathbb{R}}$ be a (not-necessarily full-dimensional) cone. Then:

$$\left\{\begin{matrix}T\text{-Cartier} \\ \text{divisors on } X_\sigma\end{matrix}\right\} = M/\underbrace{M(\sigma)}_{\sigma^\perp \cap M}$$

<u>Proof</u>: As usual, split $X_\sigma = X_{\sigma'} \times (\mathbb{C}^*)^{n-\dim\sigma}$.

We again see that every T-Cartier divisor is of the form z^m, and:

(z^m invertible for $m \in M(\sigma)$.)

$\operatorname{div}(z^m) = \operatorname{div}(z^{m'}) \Leftrightarrow m - m' \in M(\sigma)$. □.

- For a cone $\sigma \in \Sigma$, have:

$M/M(\sigma) = \operatorname{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z})$

lattice generated by $\sigma \cap N \subseteq N$.

So we view these as linear maps $N_\sigma \to \mathbb{Z}$.

- <u>Defn 11.18</u>: A <u>linear function</u>

$\varphi : \sigma \to \mathbb{R}$

is the restriction to $\sigma \subseteq N_{\sigma,\mathbb{R}}$ of a $\mathbb{Z}$-linear map $N_\sigma \to \mathbb{Z}$.

So T-Cartier divisors on X_σ correspond to linear functions on σ.

L17 End

L18 Start

- We are now ready to globalise.

- Let Σ be a fan. We know X_Σ has an open covering:

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$$

with $\tau \leq \sigma \Rightarrow X_\tau \overset{\text{open}}{\subseteq} X_\sigma$. To give a Cartier divisor on X_Σ it is equivalent to give Cartier divisors on each X_σ which are compatible with restriction. Thus we obtain:

- <u>Thm 11.19</u>: There is a bijection:

$$\left\{\begin{matrix} \text{T-Cartier} \\ \text{divisors on } X_\Sigma \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{Piecewise-linear} \\ \text{functions } \Sigma \to \mathbb{R} \end{matrix}\right\}.$$

Given $\varphi : \Sigma \to \mathbb{R}$ a PL function, the associated divisor is:

$$D_\varphi = \sum_{\tau \in \Sigma(1)} \varphi(v_\tau) \cdot D_\tau$$

This describes the image of:

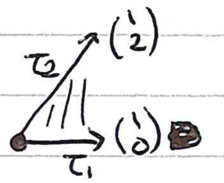
$$\{T\text{-Cartier divisors}\} \hookrightarrow \{T\text{-Weil divisors}\} = \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z} \cdot D_\tau$$

- The relations (given by principal divisors) are the same.

- Th^{m} 11.20: There is a (commuting) diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "Id_M"] & PL(\Sigma, \mathbb{R}) \arrow[r] \arrow[d, hook, "\varphi \mapsto D_\varphi"] & Pic(X_\Sigma) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M \arrow[r] & \bigoplus_{\tau \in \Sigma(1)} \mathbb{Z} \cdot D_\tau \arrow[r] & Cl(X_\Sigma) \arrow[r] & 0 \end{tikzcd}$$

- Example 11.21. Consider the A_1-singularity:



The map $M \to \mathbb{Z} \cdot D_{\tau_1} \oplus \mathbb{Z} \cdot D_{\tau_2}$ is given by:

$$\left.\begin{array}{l} (1,0) \longmapsto D_{\tau_1} + D_{\tau_2} \\ (0,1) \longmapsto 2D_{\tau_2} \end{array}\right\}$$

$\Rightarrow Cl(X_\sigma) = \mathbb{Z}/2\mathbb{Z}$ generated by $D_{\tau_1} = D_{\tau_2}$.

On the other hand the PL functions

are precisely the linear functions, so the group of T-Cartier divisors is generated by:

$$D_{\tau_1}+D_{\tau_2},\ 2D_{\tau_2}.$$

I.e. D_{τ_1} and D_{τ_2} are not Cartier (as we saw before). Hence:

$$\mathrm{Pic}(X_\sigma)=0.$$

This is actually a general fact.

- Lemma 11.22: The Picard group of an affine toric variety is trivial.

  Proof: Immediate consequence of Lemmas 11.16 and 11.17. $\square$

---

§11.C: Intersections on Surfaces

- Let X be a smooth proper surface and

$$C, D \subseteq X$$

be divisors (Weil = Cartier since X smooth). There is a well-defined intersection number:

$$C \cdot D \in \mathbb{Z}$$

with the following properties:

- If $C \cap D$ consists of finitely many points, then $C \cdot D = \#(C \cap D)$ counted with multiplicity.
- If $D_1 \sim D_2$ then $C \cdot D_1 = C \cdot D_2$.

This descends to a bilinear pairing:

$$\left.\begin{array}{l} \operatorname{Pic} X \times \operatorname{Pic} X \longrightarrow \mathbb{Z} \\ (D_1, D_2) \mapsto D_1 \cdot D_2 \end{array}\right\}.$$

- Lemma 11.23: Let X_Σ be a smooth proper toric surface. For $\tau_1 \neq \tau_2 \in \Sigma(1)$:

$$D_{\tau_1} \cdot D_{\tau_2} = \begin{cases} 1 & \text{if } \exists \sigma \in \Sigma \text{ containing } \tau_1 \text{ and } \tau_2 \text{ as faces} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: If such a σ does not exist then $D_{\tau_1} \cap D_{\tau_2} = \emptyset$.

If σ does exist then it is necessarily 2D, hence corresponds to a fixed point $x_\sigma \in X_\Sigma$, and smoothness implies that the intersection

$$D_{\tau_1} \cap D_{\tau_2} = \{x_\sigma\}$$

is transverse (i.e. reduced). $\square$

- This deals with intersections of distinct divisors. There are also <u>self-intersections</u>:

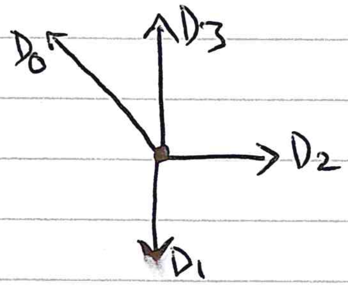
$$D^2 = D \cdot D \in \mathbb{Z}.$$

Whereas intersections of distinct divisors are always ≥ 0, self-intersections can truly be negative. This provides an <u>obstruction to displacing</u> D:

$$D^2 < 0 \implies D \subseteq X \text{ is the only divisor in its linear equivalence class}$$

- Self-intersections of toric divisors can be computed using the relations in $\operatorname{Pic} X_\Sigma$.

- <u>Example 11.24</u>: Consider the fan of $\mathbb{F}_1$:



$$\implies D_3 = D_1 - D_0$$

$$\implies D_3^2 = D_1 D_3 - D_0 D_3 \overset{\text{Lemma 11.23}}{=} -1$$

So D_3 is not displaceable (it's the exceptional divisor of $\mathbb{F}_1 = Bl_p \mathbb{P}^2 \to \mathbb{P}^2$).

- This was intersecting curves on a smooth surface.

The whole thing works more generally: on any variety (not-necessarily smooth) one can intersect:

$D \cdot C$ ← proper curve in X

↑ Cartier divisor in X

If D and C are toric, there is of course a combinatorial formula for this.

We'll skip it. See Fulton §5.1 if you're interested.

---

§11.d: Pullbacks

- Let $f : \Sigma_X \to \Sigma_Y$ be a morphism of fans, corresponding to $f : X \to Y$

Then a PL function $\varphi : \Sigma_Y \to \mathbb{R}$ induces a PL function $\varphi \circ f : \Sigma_X \to \mathbb{R}$.

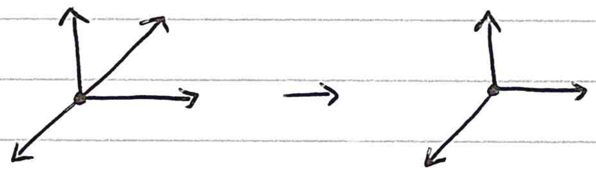
Lemma 11.25: $f^* \mathcal{O}_Y(D_\varphi) = \mathcal{O}_X(D_{\varphi \circ f})$.

- Note: In fact, we have something more:

$$f^{-1}(D_\varphi) = D_{\varphi \circ f}.$$

In general preimages of Cartier divisors may not be Cartier divisors. E.g. if $f(X) \subseteq D$ then $f^{-1}D = X$ is not a divisor. But for toric morphisms this cannot happen, and the preimage is a Cartier divisor.

- Exercise: Consider the blowup of $\mathbb{P}^2$ in a torus-fixed point:



Compute $f^{*}\mathcal{O}_{\mathbb{P}^2}(H_i)$ for each toric divisor $H_i \subseteq \mathbb{P}^2$.

Relate your answer to the relations in $\operatorname{Pic}(\mathrm{Bl}_p \mathbb{P}^2)$.

L18 End

L19 Start

§12: Linear Systems and Projectivity

- In the previous section we saw:

$$\left\{ \begin{matrix} \text{line bundles} \\ \text{on } X_\Sigma \end{matrix} \right\} / \text{iso.} = \left\{ \begin{matrix} \text{PL functions} \\ \text{on } \Sigma \end{matrix} \right\} / M.$$

- we now investigate sections of line bundles.

- <u>Note</u>: For this section I will always assume X_Σ is proper, i.e. $|\Sigma| = N_\mathbb{R}$ (we say "Σ is complete").

  This assumption can be weakened, but I don't want to keep introducing ad hoc hypotheses, and anyway the proper case is the most interesting.

  we do <u>not</u> need any smoothness assumptions.

- <u>Prop 12.1</u>: Let X_Σ be a proper toric variety and let

$$D = \sum_{\tau \in \Sigma(1)} a_\tau \cdot D_\tau$$

be a Cartier divisor. Define the associated polytope to be:

$$\boxed{P_D = \{ m \in M : m(v_\tau) \geq -a_\tau \text{ for all } \tau \in \Sigma(1) \}}$$

Then P_D forms a basis for the space of global sections:

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D} \mathbb{C} \cdot z^m$$

<u>Proof</u>: we can identify $\Gamma(X, \mathcal{O}_X(D))$ with:

$$\Gamma(X, \mathcal{O}_X(D)) = \{ f \in \Gamma(X, K_X^*) : \operatorname{div} f + D \geq 0 \}$$

(effective i.e. all coeffs ≥ 0)

This gives $\operatorname{div} f|_T \geq 0$, so:

$$\Gamma(X, \mathcal{O}_X(D)) \subseteq \mathbb{C}[M].$$

Now, since D is T-invariant it follows that the subspace $\Gamma(X, \mathcal{O}_X(D)) \subseteq \mathbb{C}[M]$ is also, hence it is graded:

$$\Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{\chi^m \in \Gamma(X, \mathcal{O}_X(D))} \mathbb{C} \cdot \chi^m$$

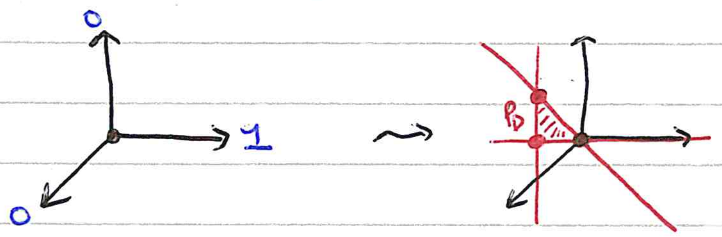
Finally for $f = \chi^m$ we have:

$$\operatorname{div} \chi^m = \sum_{\tau \in \Sigma(1)} m(v_\tau) \cdot D_\tau$$

$$\Rightarrow \operatorname{div} \chi^m + D = \sum_{\tau \in \Sigma(1)} (m(v_\tau) + a_\tau) \cdot D_\tau .$$

So $\operatorname{div} \chi^m + D \geq 0$ iff $m(v_\tau) + a_\tau \geq 0$ for all $\tau \in \Sigma(1)$, i.e. iff $m \in P_D$. $\square$

- <u>Example 12.2</u>: class of a hyperplane in $\mathbb{P}^2$:



$$\Rightarrow P_D = \{ (0,0), (-1,0), (-1,1) \}.$$

$\Rightarrow \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(D)) = \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot t_1^{-1} \oplus \mathbb{C}\cdot t_1^{-1}t_2.$

These are the only rational functions on $\mathbb{P}^2$ with pole locus $D = V(t_1)$.

It looks like $t_1^{-1}t_2^2$ should also work, but this is bogus: we have

$\binom{-1}{2}(-1,-1) = -1$

So $t_1^{-1}t_2^2$ has a pole of order 1 along $D\cdot(-1,-1)$ also.

If x, y, z are homogeneous co-ordinates on $\mathbb{P}^2$ then $t_1 = x/z$, $t_2 = y/z$ and:

$1 = 1, \quad t_1^{-1} = \frac{z}{x}, \quad t_1^{-1}t_2 = \frac{y}{x}$

with $D = V(x)$. Recall that:

$$\mathbb{C}(\mathbb{P}^2) = \left\{ \frac{f(x,y,z)}{g(x,y,z)} : \begin{array}{l} f, g \text{ homogeneous of same degree} \\ g \not\equiv 0 \end{array} \right\}.$$

- <u>Important clarification</u>: There are two ways to think about sections of $\mathcal{O}_X(D)$:

1) Maps $s : X \to \mathrm{Tot}\,\mathcal{O}_X(D)$ with $p\circ s = \mathrm{Id}_X$.

2) Rational functions f with $\mathrm{div} f + D \geq 0$.

I will try to be consistent and use "s" when using 1), and "f" when using 2).

The notion of vanishing (/Pole) set changes depending on which point of view we take. They are related as follows:

$$V(s) = \operatorname{div} f + D.$$

Notice how $V(s)$ is always effective, whereas $\operatorname{div} f$ can have poles along D.

- Defn 12.3: Let L be a line bundle on X and choose a basis

$$s_0, \dots, s_k \in \Gamma(X, L).$$

The Kodaira map is the map:

$$\begin{aligned} X &\dashrightarrow \mathbb{P}^k \\ x &\longmapsto [s_0(x), \dots, s_k(x)]. \end{aligned}$$

- Notes: A couple of things to say about this:

1) We want to assume $k \geq 0$, i.e. $\Gamma(X, L) \neq 0$.

2) The vector $(s_0(x), \dots, s_k(x)) \in \mathbb{A}^{k+1}$

depends on a choice of local trivialisation of L. But $[s_0(x) \ldots s_k(x)] \in \mathbb{P}^k$ does not.

3) The Kodaira map can also be written without choosing a basis, as:

$$\begin{aligned} X &\dashrightarrow \mathbb{P}(\Gamma(X,L)^*) \\ x &\longmapsto [ev_x] \end{aligned}$$

- The Kodaira map is a <u>rational map</u>: it is not defined if $s_0(x) = \cdots = s_k(x) = 0$.

- <u>Defn 12.4</u>: The <u>base locus</u> of L is:

$$B(L) = \{x \in X : s(x) = 0 \text{ for all } s \in \Gamma(X,L)\}$$

$$= \bigcap_{s \in \Gamma(X,L)} V(s)$$

$$= V(s_0) \cap \cdots \cap V(s_k).$$

- The Kodaira map is a morphism:

$$X \setminus B(L) \longrightarrow \mathbb{P}^k.$$

L19 End

L20 Start

- <u>Defn 12.5</u>: A line bundle L is <u>basepoint-free</u> (or <u>globally generated</u>)

iff $B(L) = \emptyset$, i.e. iff the Kodaira map is in fact a morphism $X \to \mathbb{P}^k$.

- Defn 12.6: Recall that toric Cartier divisors $D = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ correspond to PL functions $\varphi_D : |\Sigma| \to \mathbb{R}$ with:

$$\varphi_D(v_\tau) = a_\tau \quad \text{for all } \tau \in \Sigma(1).$$

This is called a support function for D.

- Caution: Many people (e.g. Fulton, CLS) take $\psi_D = -\varphi_D$ to be the support function. It's a convention.

- Thm 12.7: $\mathcal{O}_X(D)$ is basepoint-free iff:

$$\boxed{(-\varphi_D)|_\sigma \in P_D \quad \text{for all } \sigma \in \Sigma(n)}.$$

- Proof: First suppose $(-\varphi_D)|_\sigma \in P_D$ for all σ. Fix $\sigma \in \Sigma(n)$ and let

$$m_\sigma = (-\varphi_D)|_\sigma \in P_D.$$

Then we have a global section:

$$z^{m_\sigma} \in \Gamma(X, \mathcal{O}_X(D)).$$

We claim z^{m_σ} is non-vanishing on the open $X_\sigma \subseteq X_\Sigma$. Indeed:

$$\begin{aligned}
V(z^{m_\sigma})|_{X_\sigma} &= \mathrm{div}(z^{m_\sigma})|_{X_\sigma} + D|_{X_\sigma} \\
&= \sum_{\tau \in \sigma(1)} (m_\sigma(v_\tau) + a_\tau) \cdot D_\tau \\
&= \sum_{\tau \in \sigma(1)} (-\varphi_D(v_\tau) + a_\tau) \cdot D_\tau \\
&= 0
\end{aligned}$$

So for each open X_σ there is a section $z^{m_\sigma} \in \Gamma(X, \mathcal{O}_X(D))$ s.t. $z^{m_\sigma}|_{X_\sigma}$ is non-vanishing.

Since the X_σ cover X, this shows that $\mathcal{O}_X(D)$ is basepoint-free.

Now suppose conversely that there is some $\sigma \in \Sigma(n)$ s.t. $(-\varphi_D)|_\sigma \notin P_D$. Recall:

$$\Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D} \mathbb{C} \cdot z^m$$

$$\Rightarrow B(\mathcal{O}_X(D)) = \bigcap_{m \in P_D} V(z^m).$$

Since $(-\varphi_D)|_\sigma \notin P_D$, it follows that for every $m \in P_D$:

$\exists\, \tau \in \sigma(i)$ s.t. $\overset{m(v_\tau)}{\cancel{\varphi_D(v_\tau)}} > -a_\tau$

$\Rightarrow V(z^m) \supseteq D_\tau \supseteq \bigcap_{\tau \in \sigma(i)} D_\tau.$

Thus we conclude that:

$$B(\mathcal{O}_X(D)) = \bigcap_{m \in P_D} V(z^m) \supseteq \bigcap_{\tau \in \sigma(i)} D_\tau = \{x_\sigma\}$$

where $x_\sigma \in X_\sigma$ is the distinguished torus-fixed point. So $B(\mathcal{O}_X(D)) \neq \emptyset$. $\square$

- Note: The condition $(-\varphi_D)|_\sigma \in P_D$ for $\sigma \in \Sigma(n)$ is equivalent to convexity of $-\varphi_D$. Cf. CLS §6.1.

- BP-free line bundles give morphisms to projective space. These can have various flavours (bundle projections, blow-downs, embeddings).

- Def^n 12.10: A line bundle L on X is very ample iff it is BP free and the Kodaira morphism $X \to \mathbb{P}^k$ is a closed embedding.

A line bundle L is ample iff some power $L^{\otimes k}$ $(k \geq 1)$ is very ample.

- Note: A variety is projective iff it admits an ample line bundle.

  The collection of very ample line bundles controls the possible embeddings of X into projective spaces.

- Thm 12.12: $\mathcal{O}_X(D)$ is ample iff it is BP free and:

  $$\boxed{\varphi_D|_\sigma \neq \varphi_D|_{\sigma'} \text{ for } \sigma \neq \sigma' \in \Sigma(n).}$$

- Proof: [CLS §6.1] $\square$

- Note: Roughly speaking, this means that the polytope P_D is dual to the fan Σ. See CLS §6.2.

- Note: This condition is equivalent to strict convexity of $-\varphi_D$. But be warned that Fulton's exposition is a little confusing. Better to consult CLS §6.2.

- This theorem can be used to construct an example of a smooth proper toric variety which is not projective.

  The idea is to build a fan which

supports no strictly convex PL function see Example sheet 4.

L20 End

L21 Start

§13: Quotients and homogeneous co-ordinates

- we'll see that every simplicial toric variety has a global quotient presentation

$$X_\Sigma = (\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma))/G$$

generalising the description:

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

This will give homogeneous co-ordinates $\{x_\tau : \tau \in \Sigma(1)\}$ on X_Σ.

- For now, let X_Σ be any toric variety with no torus factors.

- Defn 13.1: consider the lattice $\mathbb{Z}^{\Sigma(1)}$ with standard basis $\{e_\tau : \tau \in \Sigma(1)\}$ indexed by the rays of Σ.

For every cone $\sigma \in \Sigma$ we define:

$$\boxed{\tilde{\sigma} = \mathrm{Cone}(e_\tau : \tau \in \sigma(1)) \subseteq \mathbb{R}^{\Sigma(1)}}$$

we then define $\tilde{\Sigma}$ to be the set of all $\tilde{\sigma}$ together with their faces:

$$\boxed{\tilde{\Sigma} = \{\text{faces } \tau \leq \tilde{\sigma} : \sigma \in \Sigma\}}$$

$$= \{\text{faces } \tau \leq \tilde{\sigma} : \sigma \in \Sigma \text{ maximal}\}.$$

Notice that $\tilde{\Sigma}$ captures (some of) the <u>combinatorics</u> of Σ, but forgets all the linear algebra.

- <u>Note</u>: If Σ is simplicial then there is a bijection between faces $\rho \leq \sigma$ and faces $\tilde{\rho} \leq \tilde{\sigma}$, so $\tilde{\Sigma} = \{\tilde{\sigma} : \sigma \in \Sigma\}$.

  But if Σ is not simplicial then there are faces $\tau \leq \tilde{\sigma}$ which are not of the form $\tilde{\rho}$ for any $\rho \leq \sigma$. (Think of the quadric cone)

- $\tilde{\Sigma}$ is a subfan of the fan $\mathbb{R}^{\Sigma(1)}_{\geq 0}$ giving $\mathbb{A}^{\Sigma(1)}$. Hence there is a toric <u>open embedding</u>:

$$X_{\tilde{\Sigma}} \hookrightarrow \mathbb{A}^{\Sigma(1)}.$$

we now describe this.

- Def^n 13.2: A subset $C \subseteq \Sigma(1)$ is Σ-unstable iff $C \not\subseteq \sigma(1)$ for any $\sigma \in \Sigma$.

By the orbit-core correspondence, this is equivalent to:

$$\bigcap_{\tau \in C} D_\tau = \emptyset.$$

- Lemma 13.3: Let $\{x_\tau : \tau \in \Sigma(1)\}$ be the standard co-ordinates on $\mathbb{A}^{\Sigma(1)}$. Then:

$$X_{\tilde{\Sigma}} = \mathbb{A}^{\Sigma(1)} \setminus \bigcup_{C\ \Sigma\text{-unstable}} V(x_\tau : \tau \in C).$$

Clearly it is equivalent to only consider minimal Σ-unstable subsets $C \subseteq \Sigma(1)$. We let:

$$Z(\Sigma) := \bigcup_{C\ \Sigma\text{-unstable}} V(x_\tau : \tau \in C) \subseteq \mathbb{A}^{\Sigma(1)}$$

and refer to it as the unstable locus.

- Proof: The orbit-core correspondence reduces this to combinatorics. The faces of $\mathbb{R}^{\Sigma(1)}_{\geq 0}$ which are "missing" in $\tilde{\Sigma}$ are precisely the cones over Σ-unstable sets $C \subseteq \Sigma(1)$. $\square$

- This $\tilde{X}_{\Sigma}$ is a "combinatorial model" for X_Σ. It is smooth and affine and hence does not have much "geometry".

We now relate $\tilde{X}_{\Sigma}$ to X_Σ. From now on I'll usually write

$$\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)$$

instead of $\tilde{X}_{\Sigma}$, to emphasise its simple structure.

- Consider the lattice map:

$$\left.\begin{aligned} \mathbb{Z}^{\Sigma(1)} &\xrightarrow{P} N \\ e_\tau &\longmapsto v_\tau \end{aligned}\right\} \circledast$$

where v_τ is the primitive generator of τ. This is precisely the map which records the rays of Σ.

- <u>Lemma 13.4:</u> P gives a fan map $\tilde{\Sigma} \to \Sigma$, hence a toric morphism:

$$\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \xrightarrow{\pi} X_\Sigma.$$

- We want to show that π is a quotient. To do this we need to understand P (and $P \otimes \mathbb{C}^\times$) better.

- Recall we have a short exact sequence:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to Cl(X_\Sigma) \to 0$$

Notice that applying $Hom_{\mathbb{Z}}(-,\mathbb{Z})$ to $M \to \mathbb{Z}^{\Sigma(1)}$ produces $P: \mathbb{Z}^{\Sigma(1)} \to N$.

Therefore applying $Hom_{\mathbb{Z}}(-,\mathbb{C}^\times)$ gives $P \otimes \mathbb{C}^\times$ by Hom-Tensor adjunction.

- <u>Def^n-Lemma 13.5</u>: Define G as:

$$\boxed{G = Hom_{\mathbb{Z}}(Cl(X_\Sigma), \mathbb{C}^\times).}$$

Then there is a short exact sequence:

$$\boxed{0 \to G \to (\mathbb{C}^\times)^{\Sigma(1)} \xrightarrow{P \otimes \mathbb{C}^\times} T_N \to 0}$$

and G is isomorphic to the product of an algebraic torus and a finite abelian group.

- <u>Proof</u>: Applying $Hom_{\mathbb{Z}}(-,\mathbb{C}^\times)$ to the original s.e.s gives:

$$0 \to G \to (\mathbb{C}^\times)^{\Sigma(1)} \to T_N$$

This is exact on the right since $\mathbb{C}^\times$ is divisible.

We know $Cl X_\Sigma$ is finitely-generated, So $Cl X_\Sigma \cong \mathbb{Z}^k \times A$ for A a torsion group. Then:

$$G \cong (\mathbb{C}^\times)^k \times A. \qquad \square$$

- <u>Note</u>: If X_Σ is smooth then $Cl X_\Sigma = Pic X_\Sigma$ is free abelian, so:

$$G \cong (\mathbb{C}^\times)^{\operatorname{rk} Pic X_\Sigma}$$

This is perhaps the most intuitive case to keep in mind.

L21 End

L22 Start

- The inclusion $G \hookrightarrow (\mathbb{C}^\times)^{\Sigma(1)}$ induces an action:

$$\boxed{G \curvearrowright \mathbb{A}^{\Sigma(1)}}$$

Since this acts via $(\mathbb{C}^\times)^{\Sigma(1)}$ it preserves co-ordinate subspaces and hence restricts to an action:

$$G \curvearrowright \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma).$$

- <u>Lemma 13.6</u>: The map $\pi : \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$ is constant on G-orbits.

Consequently, every fibre $\pi^{-1}(p)$ is a union of G-orbits.

- Proof: This follows from the equivariance of π and the short exact sequence:

$$0 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{\pi|_{\text{tori}}} T_N \to 0. \qquad \square$$

- Working over the dense torus $T_N \subseteq X_\Sigma$, we see that every fibre of π consists of a single G-orbit.

Does this continue to be true on the boundary?

- Thm 13.7: Assume Σ is simplicial. Then every fibre of π is a single G-orbit, so there is a bijection:

$$\{G\text{-orbits in } \mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)\} \leftrightarrow \{\text{Points in } X_\Sigma\}.$$

We say that X_Σ is a geometric quotient.

- Proof: [CLS, Theorem 5.1.11] $\square$

- Note: This is an "iff": If Σ is not simplicial then there are fibres of π equal to a union of multiple G-orbits.

Try the quadric cone to convince yourself of this.

This has to do with the fact that we had to add extra faces when constructing $\hat{\Sigma}$.

The problem is that certain G-orbits may be non-closed. When this happens, they contain other G-orbits in their closure:

$$\overline{G\cdot x} \supseteq G\cdot y$$

Then G-invariant functions must take the same value on Gx and Gy (by continuity). So the orbits get identified (smooshed together) in the quotient.

- The upshot of Thm 13.7 is that we can write points of X_Σ as:

$$[x_{\tau_1} \ldots x_{\tau_k}]$$

where $[x_{\tau_1} \ldots x_{\tau_k}] = [y_{\tau_1} \ldots y_{\tau_k}]$ if they differ by the action of G.

These are homogeneous co-ordinates on X_Σ.

- They function in many ways like homogeneous co-ordinates on projective space.

  For instance, we can study subvarieties of X_Σ by identifying them with G-invariant subvarieties of $\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)$.

  These will be the vanishing loci of homogeneous ideals

  $$I \triangleleft \mathbb{C}[x_{\tau_1}, \dots, x_{\tau_k}]$$

  where $\mathbb{C}[x_{\tau_1}, \dots, x_{\tau_k}]$ is graded not by $\mathbb{Z}$ but by $\mathrm{Cl}\, X_\Sigma$:

  $$\deg x_{\tau_i} = [D_{\tau_i}] \in \mathrm{Cl}\, X_\Sigma.$$

- For smooth toric varieties X_Σ, there is a description of the functor of points generalising the one for projective space.
- Recall that for S any scheme:

$$\mathrm{Hom}(S, \mathbb{P}^n) = \left\{ \begin{matrix} L \text{ line bundle on } S \\ f_0, \dots, f_n \in \Gamma(S, L) \end{matrix} \right\} / \mathbb{C}^\times$$

where $\mathbb{C}^\times \curvearrowright \Gamma(S, L)$ by scaling.

we also need the condition that $V(f_0) \cap \dots \cap V(f_n) = \emptyset$.

- <u>Thm 13.8</u>: Given X_Σ a smooth toric variety and S any scheme, morphisms $S \to X_\Sigma$ are given by the following data:

  - line bundles L_τ on S for $\tau \in \Sigma(1)$.
  - sections $f_\tau \in \Gamma(S, L_\tau)$. for $\tau \in \Sigma(1)$.
  - isomorphisms $\bigotimes_{\tau \in \Sigma(1)} L_\tau^{\otimes m(v_\tau)} \cong \mathcal{O}_S$ for $m \in M$

  For $C \subseteq \Sigma(1)$ unstable, we need:

  $$\bigcap_{\tau \in C} V(f_\tau) = \emptyset.$$

- Proof: see Cox: "The functor of a smooth toric variety" $\square$

L22: End

L23: Start

§14: Topology

- over $\mathbb{C}$, all algebraic varieties have a classical topology (also called the Euclidean topology).

- Defined on affine opens $U \subseteq X$ be embedding

$$U \subseteq \mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$$

and taking the subspace topology, using Euclidean topology on $\mathbb{C}^n$

Topologies coincide on overlaps, hence glue.

This is the only topology we will use in this section.

- Thm 14.1: Let Σ be a fan in N and let

$$\tilde{N}_\Sigma \subseteq N$$

be the sublattice (i.e. subgroup) generated by all elements of $|\Sigma| \cap N$. Then:

$$\boxed{\pi_1(X_\Sigma) \cong N/N_\Sigma}$$

- Corollary 14.2: X_Σ has no torus factors iff $\pi_1(X_\Sigma)$ is torsion.

- Corollary 14.3: If Σ has an n-dimensional cone then $\pi_1(X_\Sigma) = 0$.

• <u>Proof of Th^m 14.1</u>: I'll only give half the proof. For the full proof see [CLS §12.1] or [Fulton §3.2].

Consider the open subset $T_N \subseteq X_\Sigma$. Since X_Σ is normal, the induced map on fundamental groups is surjective:

$$\pi_1(T_N) \twoheadrightarrow \pi_1(X_\Sigma).$$

(See [CLS §12.1] for a nice discussion of this.) Now, there is a canonical isomorphism:

$$N \xrightarrow{\cong} \pi_1(T_N)$$

given as follows. Recall that $u \in N$ corresponds to a 1-parameter subgroup:

$$\lambda_u : \mathbb{C}^\times \longrightarrow T_N.$$

we get an element of $\pi_1(T_N)$ by restricting λ_u to:

$$S^1 = \{|z| = 1\} \subseteq \mathbb{C}^\times.$$

Now, if $u \in |\Sigma| \cap N$ let $\sigma \in \Sigma$ be a cone containing u. Then we know:

$$\lim_{z \to 0} \lambda_u(z) = x_\sigma \in X_\sigma \subseteq X_\Sigma.$$

This defines a contraction of the loop associated to u: the whole loop shrinks down to the distinguished point x_0.

Here we obtain a surjection:

$N/\tilde{N}_\Sigma \twoheadrightarrow \pi_1(X_\Sigma)$.

The claim is this is also injective. This I won't prove. $\square$

- We now discuss cohomology.

For this (i.e. the rest of the section) I will assume X smooth and proper.

As usual, it is possible to work in a more general context. The price you pay is that the results are less clean. See [CLS §12.4] for example.

- Cohomology groups exist in every degree:

$H^k(X;\mathbb{Z}) \qquad (k\in\mathbb{Z}_{\geq 0})$.

- For smooth compact oriented manifolds (which X is) these can be interpretted as codimension k cycles.

Indeed, Poincaré duality gives:

$$H^k(X;\mathbb{Z}) = H_{2n-k}(X;\mathbb{Z}).$$

"codimension" means "$\mathbb{R}$-codimension" here.

Note that algebraic varieties can have interesting odd cohomology. E.g. if C is a smooth projective curve of genus g then:

$$H^1(C;\mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

- Cohomology forms a graded ring:

$$H^*(X;\mathbb{Z}) = \bigoplus_{k=0}^{2n} H^k(X;\mathbb{Z})$$

The product is called cup product. It is graded:

~~under Poincaré duality~~

$$H^k(X;\mathbb{Z}) \cdot H^l(X;\mathbb{Z}) \subseteq H^{k+l}(X;\mathbb{Z})$$

under Poincaré duality cup product

corresponds to transverse intersection

- Thm 14.4: Let X be a smooth proper toric variety. Then:

$$H^2(X;\mathbb{Z}) = \operatorname{Pic} X \quad (= \operatorname{Cl} X = \mathbb{Z}^{\Sigma(1)}/M.)$$

- Note: There is always a map

$$\operatorname{Pic} X \xrightarrow{c_1} H^2(X;\mathbb{Z}). \quad (\text{"First Chern class"})$$

In general it is neither injective nor surjective. E.g. if $X=C$ is a curve of genus $g \geq 1$, $\operatorname{Pic} C$ is not even finitely-generated.

In general the image is the intersection

$$H^{1,1}(X;\mathbb{C}) \cap H^2(X;\mathbb{Z}) \subseteq H^2(X;\mathbb{C}).$$

This is the Lefschetz theorem on (1,1) classes. The higher-codimension analogue is known as the Hodge conjecture.

If you can prove it, the Clay Institute will pay you \$1,000,000 (which is about enough to buy a 1-bedroom flat in Islington. If you want to pay heating bills you may need to also prove the Riemann hypothesis).

- This can all be studied by looking at the short exact sequence:

$$0 \longrightarrow 2\pi i\, \underline{\mathbb{Z}}_X \rightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

which induces a long exact sequence:

$$0 \rightarrow H^1(X;\mathbb{Z}) \rightarrow H^{0,1}(X) \rightarrow \mathrm{Pic}X \rightarrow$$

$$\rightarrow H^2(X;\mathbb{Z}) \rightarrow H^{0,2}(X) \rightarrow 0$$

See Huybrechts: "Complex Geometry" for a nice treatment.

- This deals with $H^2(X;\mathbb{Z})$. Amazingly, it turns out that for (smooth and proper) toric varieties:

  (i) all cohomology lives in even degree

  (ii) it is all generated by divisor classes.

- <u>Th^m 14.5:</u> As a ring, we have:

$$H^*(X;\mathbb{Z}) = \mathbb{Z}[D_\tau : \tau \in \Sigma(1)]/(I+J)$$

where I, J are the following ideals in the polynomial ring $\mathbb{Z}[D_\tau : \tau \in \Sigma(1)]$

$$I = \left(\prod_{\tau \in C} D_\tau \ : \ \begin{array}{c} C \subseteq \Sigma(1) \\ C \neq \sigma(1) \text{ for} \\ \text{any } \sigma \in \Sigma \end{array} \right)$$ (multiplicative relations)

$$J = \left(\sum_{\tau \in \Sigma(1)} m(v_\tau) D_\tau \ : \ m \in M \right)$$ (additive relations)

- The fact that these relations are necessary should be clear.

  The fact that these give all the relations, and that all classes can be written as polynomials in divisor classes, is less clear.

- The proof uses equivariant cohomology, which is a lovely story in its own right (and one particularly close to my heart). See [CLS §12.4] for details.

L23 End

L24 Start

- Thm 14.6: $\chi(X_\Sigma) = \# \Sigma(n)$.

  Proof: By the localisation theorem in equivariant cohomology, if a torus acts on a manifold

  $T \curvearrowright M$

  cf. Atiyah-Bott.

  such that there are only finitely many T-fixed points, then $\chi(M) =$ number of fixed points. $\square$

- **Thm 14.7:** $\operatorname{rk} H^{2k}(X_\Sigma;\mathbb{Z}) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \cdot \#\Sigma(n-i)$.

  Proof: See [Fulton, §4.5]. The idea is to stratify X_Σ into its torus orbits and exploit the mixed Hodge structure on compactly-supported cohomology. $\square$

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§15: Where do we go from here?

- There is much, much more to be said about toric varieties.
- I have already hinted at some of the further topics. Here are a few more.

- **A: Polytopes and projective toric varieties.**

  We saw that a divisor D on X_Σ defined a polytope

  $P_D \subseteq M$.

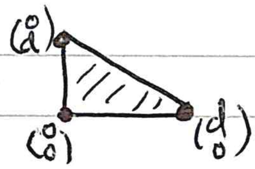
  Moreover we saw, intuitively, that D was ample iff P_D was "dual" to the fan Σ.

This can be made precise, giving a correspondence:

$$\boxed{\left\{\begin{array}{c}\text{full-dimensional} \\ \text{polytopes } P \subseteq M\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Pairs } (X, D) \text{ with } X \\ \text{a proper toric variety} \\ \text{and } D \text{ an ample } T\text{-divisor}\end{array}\right\}}$$

An ample divisor on X is called a <u>polarisation</u> of X. It determines a specific closed embedding of X into projective space.

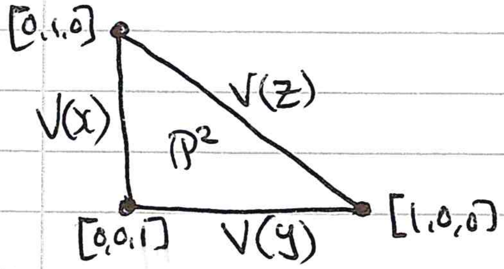
Thus, polytopes correspond to polarised (hence projective) toric varieties.

<u>E.g.</u>: $P =$ 

$\rightsquigarrow (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$.

Cf. [CLS §6.2] and [CLS §2].

- In this formulation of toric geometry, the orbit-cone correspondence is inclusion-<u>preserving</u>:



There is a <u>moment map</u> $X_\Sigma \to P$ with fibres real tori $(S^1)^k$. Cf. [Fulton §4.2].

B: Cohomology and Riemann-Roch

- We've seen a combinatorial description of the global sections of $\mathcal{O}(D)$, via the polytope $P_D \subseteq M$. But recall:

$$\Gamma(X, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D)).$$

- There is also the higher cohomology $H^k(X, \mathcal{O}(D))$.

Unsurprisingly, this also has a fan-theoretic description.

To every $m \in M$ one associates a conical subset

$$Z_D(m) \subseteq |\Sigma|$$

with $Z_D(m) = |\Sigma|$ iff $m \in P_D$. Then:

$$\boxed{H^k(X, \mathcal{O}(D)) = \bigoplus_{m \in M} H^k(|\Sigma|, |\Sigma| \setminus Z_D(m)).}$$

This is close to combinatorial. In practice the RHS can be computed via simplicial cohomology.

Cf. [Fulton §3.5].

- For X smooth and E a vector bundle on X, the Hirzebruch-Riemann-Roch theorem gives:

$$\sum_{k\geq 0} (-1)^k \dim H^k(X,E) = \int_X ch(E) \cdot td(T_X).$$

This is a vast generalisation of RR for line bundles on curves.

- In the toric world, a nice proof of this is given by torus localisation.

Cf. [CLS §13].

- On the other hand (and somewhat remarkably), RR can be applied to deduce point-counting formulae for lattice polytopes, generalising Pick's theorem in 2D.

Cf. [Fulton §5.3].

- C: Toroidal and tropical geometry

- This is closest to my heart. The amazing fact is that toric techniques continue to be useful beyond the world of toric varieties.

- **Defn 15.1**: Let X be a variety and $D \subseteq X$ a hypersurface. Then $(X|D)$ is a toroidal embedding iff it is locally isomorphic to a toric pair, i.e. iff there is an open cover $\{U_i\}$ of X and cones σ_i s.t.:

$$(X|D)|_{U_i} \cong (X_{\sigma_i} | \partial X_{\sigma_i}).$$

- E.g.: Any smooth pair $(X|D)$ is toroidal. More generally, normal crossings pairs $(X|D)$ are toroidal.

- Toroidal embeddings have "fans" obtained by patching together the local cones σ_i.

Because there is no global torus, these "fans" are not embedded in a single vector space: they are abstract cone complexes. We call them "tropicalisations":

$$(X|D) \rightsquigarrow \mathrm{Trop}(X|D).$$

This process forgets a lot of information! E.g. if $(X|D)$ is any smooth pair then $\mathrm{Trop}(X|D) = \mathbb{R}_{\geq 0}$.

But still, we can work with $\mathrm{Trop}(X|D)$ much like the fan of a toric variety. In particular:

- $\{\text{Cones } \sigma \in \mathrm{Trop}(X|D)\} \leftrightarrow \{\text{Strata in } (X|D)\}$
- $\{\text{Subdivisions of } \mathrm{Trop}(X|D)\} \leftrightarrow \{\text{toroidal birational modifications of } X\}$
- $\{\text{PL functions on } \mathrm{Trop}(X|D)\} \leftrightarrow \{\text{Cartier divisors on } X\}$.

This is a tremendously powerful dictionary.

- E.g.: $M_{g,n} = \left\{ \begin{array}{l} \text{Smooth proper curves } C,\ g(C)=g \\ p_1, \dots, p_n \in C \text{ distinct points} \end{array} \right\}$

$$\overline{M}_{g,n} = \left\{ \begin{array}{l} \text{nodal proper curves } C,\ g_a(C)=g \\ p_1, \dots, p_n \in C \text{ distinct smooth pts} \\ \#\mathrm{Aut}(C, p_1, \dots, p_n) < \infty \end{array} \right\}$$

Space of Stable curves / Deligne-Mumford space.

Then $(\overline{M}_{g,n} \,|\, \partial\overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n})$ is a toroidal embedding. Moreover:

$$\mathrm{Trop}(\overline{M}_{g,n} \,|\, \partial\overline{M}_{g,n}) = M^{\mathrm{trop}}_{g,n}$$

moduli space of tropical curves.

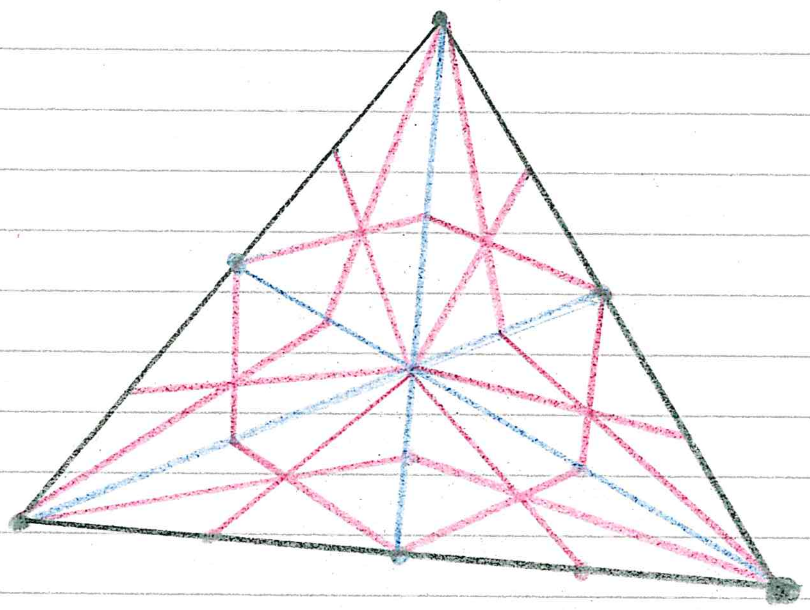
The RHS we understand very well. So can use to study $\overline{M}_{g,n}$. People have!

- A lovely paper to get started in this direction:

Tevelev: "Compactifications of subvarieties of tori."

L24 End

- Toric geometry is hiding everywhere. Go out and find it!



"Don't hate me 'cos I'm beautiful."