

PART I: NON-RIEMANNIAN HYPERSQUARES

Math 256C: From Schemes to Machinations

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Tutorial

Basics

Here is where lecture content goes, generally a summary or transcription of what is being said or written. Here is a theorem:

Theorem 1.1.1: Kontsevich

The number N_d of rational plane curves of degree d passing through $3d - 1$ points in general position is given recursively by

$$N_d = \sum_{d_A + d_B = d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d - 4}{3d_A - 2} - d_A \binom{3d - 4}{3d_A - 1} \right)$$

The above result, is, of course, thoroughly unrelated to the following fact:

Lemma 1.1.2

In a k -free graph on n vertices, there are at most $\binom{k-1}{r} \left(\frac{n}{k-1}\right)^r$ r -cliques.

Setting $r = 2$ in the above, we recover the following result:

Corollary 1.1.3: Turan's Theorem

In a k -free graph on n vertices, there are at most $\frac{k-2}{k-1} \frac{n^2}{2}$ edges.

You can insert a hyperlinked reference for any theorem box if you add a reference tag (see the \LaTeX code at Corollary 1.1.3 for formatting, and see the style file for the reference prefixes for each theorem style).

PROOF : There is also a proof environment; the proof heading is configured to live in the left margin. ■

Citations live in the right margin¹, but will not work correctly if placed inside a theorem box. The available theorem boxes are `theorem`, `lemma`, `corollary`, `proposition`, `definition`, `example`, `remark`, `question`, `exercise`,

Here is a margin note: I use these generally to annotate my own thoughts or questions during lecture.

You can have multi-paragraph margin notes, which are configured to not have indented paragraphs.

¹ R. Hartshorne. *Algebraic Geometry*. Springer, 1977

and **conjecture**. Unnumbered versions of all the theorem boxes exist:

Proposition: Hurwitz

The group of orientation-preserving conformal automorphisms of a compact Riemann surface of genus $g > 1$ has order at most $84(g - 1)$.

References do not work for unnumbered theorems.

Advanced

Using <https://q.uiver.app/> (whose style file is included in our style file for convenience), we can curve arrows more flexibly in commutative diagrams than `tikz-cd` normally allows:

$$\begin{array}{ccccc}
 & \mathbb{P}^n \setminus \{Q\} & \xrightarrow{\varphi_0} & W_0 & \\
 & \uparrow & & \uparrow & \\
 X & \xrightarrow{\quad} & \mathbb{P}^n \setminus V & \xrightarrow{\varphi_0|_{\mathbb{P}^n \setminus V}} & W_0 \setminus V' \xrightarrow{\varphi'} W \\
 & \searrow & \varphi & \nearrow & \\
 & & & &
 \end{array}$$

Finally, there is (in addition to the default environments provided by `tufte-book`) a two-column environment, which I mostly use to spam examples after a big theorem:

Example 1.1.4

In \mathbb{CP}^2 with its canonical orientation as a complex manifold, we have $H^2 = H_2 = \mathbb{Z}$. Lines L in \mathbb{CP}^2 are embedded copies of $\mathbb{CP}^1 = S^2$, with a single fundamental class $[L] \in H_2$ representing any such line. To compute the self-intersection $[L] \cdot [L]$, choose two representatives L and L' which intersect transversely (in a point, by basic intersection theory); since \mathbb{CP}^1 is canonically oriented as well, and the orientations of L and L' agree with that of \mathbb{CP}^2 (since they are subspaces of \mathbb{CP}^2), we have that $[L] \cdot [L] = 1$.

Example 1.1.5

In $S^2 \times S^2$, the second homology group is \mathbb{Z}^2 , generated by $h = [S^2 \times \{x}]$ and $v = [\{x\} \times S^2]$. $S^2 \times \{x\}$ and $\{x\} \times S^2$ intersect transversely at (x, x) , and, regarding S^2 with its canonical orientation as a complex manifold (\mathbb{CP}^1), the orientations agree to give $h \cdot v = 1$. To compute $h \cdot h$,

intersect representatives $S^2 \times \{x\}$ and $S^2 \times \{y\}$ for $x \neq y$, from which it is set-theoretically clear that $h^2 = 0$ (and similarly for v). Thus, the intersection form as a matrix in the basis (h, v) can be written as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This intersection form is therefore *even*, meaning that for any $k = ah + bv$, $k \cdot k = 2ab$ is even.

References

- B. Fantechi. Stacks for everybody. In C. Casacuberta, R. M. Miró-Roig, J. Verdera, and S. Xambó-Descamps, editors, *European Congress of Mathematics*, pages 349–359, Basel, 2001. Birkhäuser Basel. ISBN 978-3-0348-8268-2.

- R. Hartshorne. *Algebraic Geometry*. Springer, 1977.
- M. Kontsevich and Y. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164:525–562, 1994. DOI: <https://doi.org/10.1007/BF02101490>.
- E. Witten. Two-dimensional gravity and intersection theory on moduli space. *Surveys Diff. Geom.*, 1:243–310, 1991. DOI: [10.4310/SDG.1990.v1.n1.a5](https://doi.org/10.4310/SDG.1990.v1.n1.a5).