2.2 Ising systems which solve boolean circuits

Definition 2.3. Suppose we have a dataset $\mathcal{X} = (\mathbf{x}_1, y_1), ..., (\mathbf{x}_n, y_n)$ where $\mathbf{x}_i$ is a point in $\mathbb{R}^d$ labeled by $y_i \in \{-1, +1\}$. We say that $\mathcal{X}$ is *linearly separable* if there exist $\mathbf{w} \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$y_i(\mathbf{w}^\top \mathbf{x}_i - b) > 0. \tag{2}$$

It turns out that linearly separable labeled datasets correspond precisely to boolean circuits solvable by Ising systems without auxiliaries.

Proposition 2.4. Let (N, M, f) be a circuit with one output; i.e., let $|M| = 1$ so that f is a boolean function. There exists an Ising system (X, H) with $X = N \cup M$ which solves (N, M, f) if and only if the set of true vectors $T = f^{-1}(1)$ is linearly separable from the set of false vectors $F = X \setminus F = f^{-1}(-1)$.

Proof. Set $n = |N|$. Under the identifications $\Sigma^N \cong \{-1, 1\}^n$ and $M = \Sigma = \{-1, 1\}$, we may view Σ^N as the vertices of the n-dimensional hypercube in $\mathbb{R}^n$ and the circuit logic f as a labeling of Σ^N by ± 1. Denote by $T = f^{-1}(1)$ the set of all vertices in Σ^N with label 1 and by $F = f^{-1}(-1)$ the set of all vertices in Σ^N with label -1.

The general form of an Ising Hamiltonian on X is given by

$$H(\mathbf{x}, y) = \sum_{i=1}^{n} h_i x_i \; + \; y h_{n+1} \; + \sum_{i<j\le n} J_{ij}\mathbf{x}_i\mathbf{x}_j \; + \; \sum_{i=1}^{n} J_{i,n+1}\mathbf{x}_i y. \tag{3}$$

for some coefficients h_i and J_{ij}. The Ising system (X, H) solves the circuit (N, M, f) if and only if the following inequalities are satisfied for all $\mathbf{x} \in \Sigma^N$:

$$H(\mathbf{x}, f(\mathbf{x})) < H(\mathbf{x}, -f(\mathbf{x})) \iff H(\mathbf{x}, -f(\mathbf{x})) - H(\mathbf{x}, f(\mathbf{x})) > 0.$$

Plugging in equation 3 and canceling terms yields

$$\begin{aligned} H(\mathbf{x}, -f(\mathbf{x})) - H(\mathbf{x}, f(\mathbf{x})) > 0 &\iff \sum_{i=1}^{n} J_{i,n+1}\mathbf{x}_i(-f(\mathbf{x}) - f(\mathbf{x})) \; + \; h_{n+1}(-f(\mathbf{x}) - f(\mathbf{x})) > 0 \\ &\iff f(\mathbf{x}) \left(\sum_{i=1}^{n} -2J_{i,n+1}\mathbf{x}_i \; - \; 2h_{n+1} \right) > 0. \end{aligned}$$

This is identical to the linear separability condition 2 from Definition 2.3 obtained by setting $\mathbf{w} = -2J_{\bullet,n+1}$ and $b = -2h_{n+1}$. $\square$

Remark 2.5. A boolean function $f : \Sigma^N \longrightarrow \Sigma$ which produces a linearly separable labeling of Σ^N is known as a **threshold function**. Hence Proposition 2.4 can be restated as "a boolean circuit (N, M, f) is solvable if and only if f is a threshold function." Counting the number of threshold functions on the n-dimensional hypercube is a well studied problem; see OEIS sequence A000609 for instance.

Proposition 2.4 gives us a nice way visualizing Ising solvability.

Example 2.6. Let $N = \{1, 2\}$ and $M = \{3\}$. Define two boolean functions $f_1(x_1, x_2) = 4x_1x_2 - 2x_1 - 2x_2 + 1$ and $f_2(x_1, x_2) = -x_1x_2$. Then (N, M, f_1) is the AND circuit and (N, M, f_2) is the XOR circuit, and the labelings they produce on the square $\{-1, 1\}^2$ are seen in Figure 2.

We can use Proposition 2.4 to easily prove that parity check circuits are infeasible without auxiliaries.

Proposition 2.7. Let $N = [n]$, $M = \{n + 1\}$ and $f(x_1, ..., x_n) = x_1x_2 \ldots x_n$. The circuit (N, M, f), called a **n-dimensional parity check**, is infeasible without auxiliaries for $n \geq 2$.

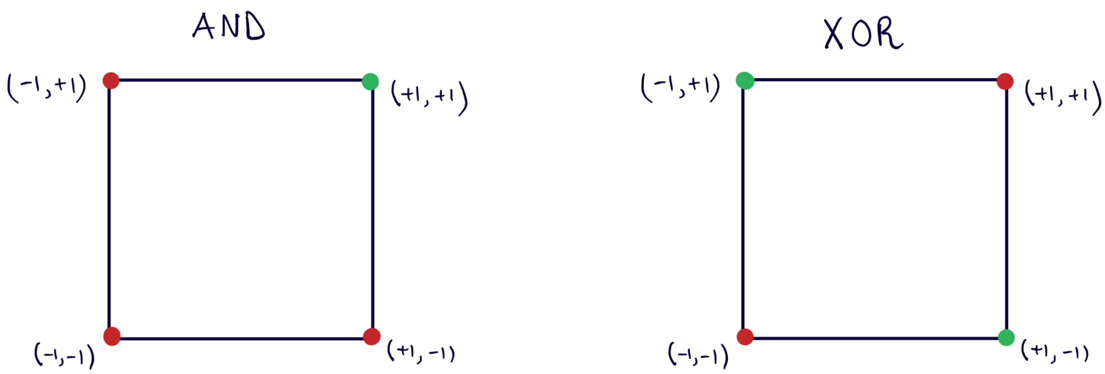

Figure 2: The labelings produced by AND (left) and XOR (right) where green denotes $+1$ and red denotes -1. One can easily draw a line separating the green corner from the three red corners in the AND labeling, but a bit of experimentation should convince the reader that it is impossible to linearly separate the XOR labeling.

Proof. Consider a hyperplane $\mathbf{w}^\top \mathbf{x} - b = 0$ in $\mathbb{R}^n$. If this hyperplane were a linear separation of the n-dimensional hypercube labeled by the parity-check circuit, then it would restrict to a linear separation of the 2-dimensional face $S = \{(x_1, x_2, -1, 1, 1, ..., 1) \in \mathbb{R}^n \;\mid\; x_1, x_2 \in \{-1, +1\}\}$. This is impossible since the parity-check labeling on S is given by $-x_1 x_2$. This is exactly the XOR labeling and is hence infeasible. $\square$